INTRODUCTION TO ALGEBRAIC TOPOLOGY

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2. CW-complexes

2.1. Constructive definition of CW-complexes. CW-complexes are all the spaces which can be obtained by the following construction:

1. We start with a discrete space $X^0$. Single points of $X^0$ are called 0-dimensional cells.

2. Suppose that we have already constructed $X^{n-1}$. For every element $\alpha$ of an index set $J_n$ take a map $f_\alpha : S^{n-1} = \partial D^n_{\alpha} \rightarrow X^{n-1}$ and put

$$X^n = \bigcup_\alpha \left( X^{n-1} \cup f_\alpha D^n_{\alpha} \right).$$

Interiors of discs $D^n_{\alpha}$ are called $n$-dimensional cells and denoted by $e^n_{\alpha}$.

3. We can stop our construction for some $n$ and put $X = X^n$ or we can proceed with $n$ to infinity and put

$$X = \bigcup_{n=0}^{\infty} X^n.$$

In the latter case $X$ is equipped with inductive topology which means that $A \subseteq X$ is closed (open) iff $A \cap X^n$ is closed (open) in $X^n$ for every $n$.

Example A. The sphere $S^n$ is a CW-complex with one cell $e^0$ in dimension 0, one cell $e^n$ in dimension $n$ and the constant attaching map $f : S^{n-1} \rightarrow e^0$.

Example B. The real projective space $\mathbb{RP}^n$ is the space of 1-dimensional linear subspaces in $\mathbb{R}^{n+1}$. It is homeomorphic to

$$S^n / (v \simeq -v) \cong D^n / (w \simeq -w), \quad \text{for } w \in \partial D^n = S^{n-1}.$$

However, $S^{n-1} / (w \simeq -w) \cong \mathbb{RP}^{n-1}$. So $\mathbb{RP}^n$ arises from $\mathbb{RP}^{n-1}$ by attaching one $n$-dimensional cell using the projection $f : S^{n-1} \rightarrow \mathbb{RP}^{n-1}$. Hence $\mathbb{RP}^n$ is a CW-complex with one cell in every dimension from 0 to $n$.

We define $\mathbb{RP}^\infty = \bigcup_{n=1}^{\infty} \mathbb{RP}^n$. It is again a CW-complex.

Example C. The complex projective space $\mathbb{CP}^n$ is the space of complex 1-dimensional linear subspaces in $\mathbb{C}^{n+1}$. It is homeomorphic to

$$S^{2n+1} / (v \simeq \lambda v) \cong \{(w, \sqrt{1-|w|^2}) \in \mathbb{C}^{n+1}; \|w\| \leq 1\} / \{(w, 0) \simeq \lambda (w, 0), \|w\| = 1\}$$

$$\cong D^{2n} / (w \simeq \lambda w; \ w \in \partial D^{2n})$$

for all $\lambda \in \mathbb{C}, |\lambda| = 1$. However, $\partial D^{2n} / (w \simeq \lambda w) \cong \mathbb{CP}^{n-1}$. So $\mathbb{CP}^n$ arises from $\mathbb{CP}^{n-1}$ by attaching one $2n$-dimensional cell using the projection $f : S^{2n-1} = \partial D^{2n} \rightarrow \mathbb{CP}^{n-1}$. Hence $\mathbb{CP}^n$ is a CW-complex with one cell in every even dimension from 0 to $2n$. 

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Define $\mathbb{C}P^\infty = \bigcup_{n=1}^\infty \mathbb{C}P^n$. It is again a CW-complex.

### 2.2. Another definition of CW-complexes.

Sometimes it is advantageous to be able to describe CW-complexes by their properties. We carry it out in this paragraph. Then we show that the both definitions of CW-complexes are equivalent.

**Definition.** A cell complex is a Hausdorff topological space $X$ such that

1. $X$ as a set is a disjoint union of cells $e_\alpha$
   
   $$X = \bigcup_{\alpha \in J} e_\alpha.$$  

2. For every cell $e_\alpha$ there is a number, called dimension.
   
   $$X^n = \bigcup_{\dim e_\alpha \leq n} e_\alpha$$
   
   is the $n$-skeleton of $X$.

3. Cells of dimension 0 are points. For every cell of dimension $\geq 1$ there is a characteristic map
   
   $$\varphi_\alpha : (D^n, S^{n-1}) \to (X, X^{n-1})$$
   
   which is a homeomorphism of $\text{int} D^n$ onto $e_\alpha$.

The cell subcomplex $Y$ of a cell complex $X$ is a union $Y = \bigcup_{\alpha \in K} e_\alpha$, $K \subseteq J$, which is a cell complex with the same characteristic maps as the complex $X$.

A CW-complex is a cell complex satisfying the following conditions:

(C) Closure finite property. The closure of every cell belongs to a finite subcomplex, i.e. subcomplex consisting only from a finite number of cells.

(W) Weak topology property. $F$ is closed in $X$ if and only if $F \cap \bar{e}_\alpha$ is closed for every $\alpha$.

**Example.** Examples of cell complexes which are not CW-complexes:

1. $S^2$ where every point is 0-cell. It does not satisfy property (W).
2. $D^3$ with cells $e^3 = \text{int} B^3$, $e^0_x = \{x\}$ for all $x \in S^2$. It does not satisfy (C).
3. $X = \{1/n; n \geq 1\} \cup \{0\} \subseteq \mathbb{R}$. It does not satisfy (W).
4. $X = \bigcup_{n=1}^\infty \{x \in \mathbb{R}^2; \|x - (1/n, 0)\| = 1/n\} \subseteq \mathbb{R}^2$. If it were a CW-complex, the set $\{(1/n, 0) \in \mathbb{R}^2; n \geq 1\}$ would be closed in $X$, and consequently in $\mathbb{R}^2$.

### 2.3. Equivalence of definitions.

**Proposition.** The definitions 2.1 and 2.2 of CW-complexes are equivalent.

**Proof.** We will show that a space $X$ constructed according to 2.1 satisfies definition 2.2. The proof in the opposite direction is left as an exercise to the reader.

The cells of dimension 0 are points of $X^0$. The cells of dimension $n$ are interiors of discs $D^n_\alpha$ attached to $X^{n-1}$ with characteristic maps

$$\varphi_\alpha : (D^n_\alpha, S^{n-1}) \to (X^{n-1} \cup f_\alpha D^n_\alpha, X^{n-1})$$
induced by identity on $D^n_\alpha$. So $X$ is a cell complex. From the construction 2.1 it follows that $X$ satisfies property (W). It remains to prove property (C). We will carry it out by induction.

Let $n = 0$. Then $e^n_\alpha = e^0_\alpha$.

Let (C) holds for all cells of dimension $\leq n - 1$. $e^n_\alpha$ is a compact set (since it is an image of $D^n_\alpha$). Its boundary $\partial e^n_\alpha$ is compact in $X^{n-1}$. Consider the set of indices

$$K = \{ \beta \in J; \partial e^n_\alpha \cap e_\beta \neq \emptyset \}.$$ 

If we show that $K$ is finite, from the inductive assumption we get that $e^n_\alpha$ lies in a finite subcomplex which is a union of finite subcomplexes for $e_\beta$, $\beta \in K$.

Choosing one point from every intersection $\partial e^n_\alpha \cap e_\beta$, $\beta \in K$ we form a set $A$. $A$ is closed since any intersection with a cell is empty or a one-point set. Simultaneously, it is open, since every its element $a$ forms an open subset (for $A - \{a\}$ is closed). So $A$ is a discrete subset in the compact set $\partial e^n_\alpha$, consequently, it is finite. □

2.4. Compact sets in CW complexes.

**Lemma.** Let $X$ be a CW-complex. Then any compact set $A \subseteq X$ lies in a finite subcomplex, particularly, there is $n$ such that $A \subseteq X^n$.

**Proof.** Consider the set of indices

$$K = \{ \beta \in J; A \cap e_\beta \neq \emptyset \}.$$ 

Similarly as in 2.3 we will show that $K$ is a finite set. Then $A \subseteq \bigcup_{\beta \in K} \bar{e}_\beta$ and every $\bar{e}_\beta$ lies in a finite subcomplexes. Hence $A$ itself is a subset of a finite subcomplex. □

2.5. Cellular maps. Let $X$ and $Y$ be CW-complexes. A map $f : X \to Y$ is called a cellular map if $f(X^n) \subseteq Y^n$ for all $n$. In Section 5 we will prove that every map $g : X \to Y$ is homotopic to a cellular map $f : X \to Y$. If moreover, $g$ restricted to a subcomplex $A \subset X$ is already cellular, $f$ can be chosen in such a way that $f = g$ on $A$.

2.6. Spaces homotopy equivalent to CW-complexes. One can show that every open subset of $\mathbb{R}^n$ is a CW-complex. In [Hatcher], Theorem A.11, it is proved that every retract of a CW-complex is homotopy equivalent to a CW-complex. These two facts imply that every compact manifold with or without boundary is homotopy equivalent to a CW-complex. (See [Hatcher], Corollary A.12.)

2.7. CW complexes and HEP. The most important result of this section is the following theorem:

**Theorem.** Let $A$ be a subcomplex of a CW-complex $X$. Then the pair $(X, A)$ has the homotopy extension property.
Proof. According to the last theorem in Section 1 it is sufficient to prove that $X \times \{0\} \cup A \times I$ is a retract of $X \times I$. We will prove that it is even a deformation retract. There is a retraction $r_n : D^n \times I \to D^n \times \{0\} \cup S^{n-1} \times I$. (See Section 1.) Then $h_n : D^n \times I \times I \to D^n \times I$ defined by

$$h_n(x,s,t) = (1-t)(x,s) + tr_n(x,s)$$

is a deformation retraction, i.e. a homotopy between id and $r_n$.

Put $Y^{-1} = A$, $Y^n = X^n \cup A$. Using $h_n$ we can define a deformation retraction $H_n : Y^n \times I \times I \to Y^n \times I$ for the retract $Y^n \times \{0\} \cup Y^{n-1} \times I$ of $Y^n \times I$. Now define the deformation retraction $H : X \times I \times I \to X \times I$ for the retract $X \times \{0\} \cup A \times I$ successively on the subspaces $X \times \{0\} \times I \cup Y^n \times I \times I$ with values in $X \times \{0\} \cup Y^n \times I$. For $n = 0$ put

$$H(x,s,t) = (x,s) \quad \text{for} \quad (x,s) \in X \times \{0\} \text{ or } t \in [0,1/2],$$

$$H(x,s,t) = H_0(x,s,2(t-1/2)) \quad \text{for} \quad x \in Y^0 \text{ and } t \in [1/2,1].$$

Suppose that we have already defined $H$ on $X \times \{0\} \cup Y^{n-1} \times I$. On $X \times \{0\} \cup Y^n \times I$ we put

$$H(x,s,t) = (x,s) \quad \text{for} \quad (x,s) \in X \times \{0\} \text{ or } t \in [0,1/2^{n+1}],$$

$$H(x,s,t) = H_n(x,s,2^{n+1}(t-1/2^{n+1})) \quad \text{for} \quad x \in Y^n \text{ and } t \in [1/2^{n+1},1/2^n],$$

$$H(x,s,t) = H(H(x,s,1/2^n),t) \quad \text{for} \quad x \in Y^n \text{ and } t \in [1/2^n,1].$$

$H : X \times I \times I \to X \times I$ is continuous since so are its restrictions on $X \times \{0\} \times I \cup Y^n \times I \times I$ and the space $X \times I \times I$ is a direct limit of the subspaces $X \times \{0\} \times I \cup Y^n \times I \times I$.

![Figure 2.1. Image of H depending on t](image)

**2.8. First criterion for homotopy equivalence.**

**Proposition.** Suppose that a pair $(X,A)$ has the homotopy extension property and that $A$ is contractible (in $A$). Then the canonical projection $q : X \to X/A$ is a homotopy equivalence.
Proof. Since $A$ is contractible, there is a homotopy $h : A \times I \to A$ between $\text{id}_A$ and constant map. This homotopy together with $\text{id}_X : X \to X$ can be extended to a homotopy $f : X \times I \to X$. Since $f(A, t) \subseteq A$ for all $t \in I$, there is a homotopy $\tilde{f} : X/A \times I \to X/A$ such that the diagram

$$
\begin{array}{ccc}
X \times I & \xrightarrow{f} & X \\
q \downarrow & & \downarrow q \\
X/A \times I & \xrightarrow{\tilde{f}} & X/A
\end{array}
$$

commutes. Define $g : X/A \to X$ by $g([x]) = f(x, 1)$. Then $\text{id}_X \sim g \circ q$ via the homotopy $f$ and $\text{id}_{X/A} \sim q \circ g$ via the homotopy $\tilde{f}$. Hence $X$ is homotopy equivalent to $X/A$. □

Exercise A. Using the previous criterion show that $S^2/S^0 \sim S^2 \vee S^1$.

Exercise B. Using the previous criterion show that the suspension and the reduced suspension of a CW-complex are homotopy equivalent.

2.9. Second criterion for homotopy equivalence.

Proposition. Let $(X, A)$ be a pair of CW-complexes and let $Y$ be a space. Suppose that $f, g : A \to Y$ are homotopic maps. Then $X \cup_f Y$ and $X \cup_g Y$ are homotopy equivalent.

Proof. Let $F : A \times I \to Y$ be a homotopy between $f$ and $g$. We will show that $X \cup_f Y$ and $X \cup_g Y$ are both deformation retracts of $(X \times I) \cup_F Y$. Consequently, they have to be homotopy equivalent.

We construct a deformation retraction in two steps.

1. $(X \times \{0\}) \cup_f Y$ is a deformation retract of $(X \times \{0\} \cup A \times I) \cup_F Y$.
2. $(X \times \{0\} \cup A \times I) \cup_F Y$ is a deformation retract of $(X \times I) \cup_F Y$.

□

Exercise. Let $(X, A)$ be a pair of CW-complexes. Suppose that $A$ is a contractible in $X$, i.e. there is a homotopy $F : A \to X$ between $\text{id}_X$ and const. Using the first criterion show that $X/A \cong X \cup CA/CA \sim X \cup CA$. Using the second criterion prove that $X \cup CA \sim X \vee SA$. Then

$$X/A \sim X \vee SA.$$

Apply it to compute $S^n/S^i$, $i < n$. 

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