Příklady na integrování racionálních lomených funkcí s proměnou x a s proměnou $\sin(x)$ a $\cos(x)$, které se substitucí převedopu na předchozí případ, pro cvičení s obecnými poznámkami v úvodu.

Poznámka: komentáře k příkladům byly generovány algoritmicky s použitím programu Maple. Někde se například najde výpočet $\int 0 dx$, ktrý není třeba podrobně studovat ani provádět pomocí substituce uvedené v komentáři. Maple rovněž používá při výpočtech funkce arcus tangens hyperbolický, který my nepoužíváme a rozepisujeme jej pomocí logaritmů. Mocninu funkce, kterou obvykle značíme $f^n(x)$ značí maple $f(x)^n$. Maple také nepíe u primitiovních funkcí integrační konstanty. Já jsme je v tomto textu nedopláäoval, ale vy je, prosím důsledně pište. není dobré je zapomínat, jak ukazují rozdílné tvary výsledku hned v prvním integrálu.

Prostudování následujících příkladů je dosti dobrou průpravou ke zkoušce. Pokud je však vyřešíéte sami a pouze si zkontrolujete výsledky, příp. výpočty, je to průpravou ke zkoušce mnohem lepší.

Tyto příklady vám přenechávámk samostudiu. Ve cvičeních se pak budeme podrobněji zabývat jen těmi příklady u jejichž výpočtů budete mít nejasnosti.

$$\int \frac{1}{(x-\alpha)^n}\,dx$$
$$substituce := x - \alpha = t$$
$$\int t^{(-n)}\,dx$$

$$\int \frac{b\,x+c}{((x-u)^2+v^2)^n}\,dx = \frac{1}{2}\,b\int \frac{2\,x-2\,u}{(x^2-2\,u\,x+u^2+v^2)^n}\,dx + \int \frac{b\,u+c}{((x-u)^2+v^2)^n}\,dt$$

$$\frac{1}{2}\,b\int \frac{2\,x-2\,u}{(x^2-2\,u\,x+u^2+v^2)^n}\,dx$$
$$substituce := x^2-2\,u\,x+u^2+v^2 = t$$
$$\int t^{(-n)}\,dt$$

$$\int \frac{b\,u+c}{((x-u)^2+v^2)^n}\,dt$$
$$substituce := x - u = v\,t$$

převede výpočet na integrál:

$$\int \frac{1}{(t^2+1)^n}\,ddt$$

pro který máme odvozený rekurentní vztah:

$$\mathrm{K}(n) = \frac{1}{2}\,(2\,n-3)\,(n-1)\,\mathrm{K}(n-1) + \frac{1}{2}\,\frac{(n-1)\,t}{(t^2+1)^{(n-1)}},\ \mathrm{K}(1) = \mathrm{arctg}(t)$$

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$$\mathrm{K}(1) = \mathrm{arctg}(t)$$
$$\mathrm{K}(2) = \frac{1}{2}\,\mathrm{arctg}(t) + \frac{1}{2}\,\frac{t}{t^2+1}$$
$$\mathrm{K}(3) = \frac{3}{2}\,\mathrm{arctg}(t) + \frac{3}{2}\,\frac{t}{t^2+1} + \frac{t}{(t^2+1)^2}$$
$$\mathrm{K}(4) = \frac{45}{4}\,\mathrm{arctg}(t) + \frac{45}{4}\,\frac{t}{t^2+1} + \frac{15}{2}\,\frac{t}{(t^2+1)^2} + \frac{3}{2}\,\frac{t}{(t^2+1)^3}$$
$$\mathrm{K}(5) = \frac{315}{2}\,\mathrm{arctg}(t) + \frac{315}{2}\,\frac{t}{t^2+1} + 105\,\frac{t}{(t^2+1)^2} + 21\,\frac{t}{(t^2+1)^3} + 2\,\frac{t}{(t^2+1)^4}$$
$$\vdots$$

$$\int \left(x^2-1\right)^{-1} dx$$

$\left(x^2-1\right)^{-1} = 1/2\ (x-1)^{-1} - 1/2\ (x+1)^{-1}$ [$\int \left(x^2-1\right)^{-1} dx = \int 1/2\ (x-1)^{-1} - 1/2\ (x+1)^{-1} dx = \int 1/2\ (x-1)^{-1} dx + \int -1/2\ (x+1)^{-1} dx, \int 1/2\ (x-1)^{-1} dx,$ ‘ `substituce:` ‘ $, t = 2\,x-2, \int 1/2\,t^{-1}dt = 1/2\ \ln(t) = 1/2\ \ln(2\,x-2), \int -1/2\ (x+1)^{-1} dx,$ ‘ `substituce:` ‘ $, t = 2\,x+2, \int -1/2\,t^{-1}dt = -1/2\ \ln(t) = -1/2\ \ln(2\,x+2)$]

$$\int \left(x^2-1\right)^{-1} dx = 1/2\ \ln(x+1) + 1/2\ \ln(1-x)$$

Pozn.: $\ln(2x+2) = \ln(2(x+1)) = \ln(x+1) + \ln(2)$ a $\int f(x)dx = F(x) + c$, takže $\ln(2)$ lze zahrnout do integrační konstantz a funkce $x \mapsto \ln(2x+2)$ a $x \mapsto \ln(x+1)$ jsou primitivní k téže funkci, což lze ověřit derivováním.

$$\int \left(x^2-2\,x+1\right)^{-1} dx$$

$\left(x^2-2\,x+1\right)^{-1} = (x-1)^{-2}$ [$\int (x-1)^{-2} dx,$ ‘ `substituce:` ‘ $, t = x-1, \int t^{-2}dt = -t^{-1} = -(x-1)^{-1}, res_1$]

$$\int \left(x^2-2\,x+1\right)^{-1} dx = -(x-1)^{-1}$$

$$\int \left(x^5-5\,x^4+10\,x^3-10\,x^2+5\,x-1\right)^{-1} dx$$

$\left(x^5-5\,x^4+10\,x^3-10\,x^2+5\,x-1\right)^{-1} = (x-1)^{-5}$ [$\int (x-1)^{-5} dx,$ ‘ `substituce:` ‘ $, t = x-1, \int t^{-5}dt = -1/4\,t^{-4} = -1/4\ (x-1)^{-4}, res_1$]

$$\int \left(x^5-5\,x^4+10\,x^3-10\,x^2+5\,x-1\right)^{-1} dx = -1/4\ (x-1)^{-4}$$

$$\int \left(x^2+1\right)^{-1} dx$$

$\left(x^2+1\right)^{-1} = \left(x^2+1\right)^{-1}$ [$\int \left(x^2+1\right)^{-1} dx = \int 0dx + \int \left(x^2+1\right)^{-1} dx,$ ‘`- -`‘ $, \int 0dx,$ ‘ `substituce:` ‘ $, t = x^2+1, \int 0dt = 0 = 0,$ ‘`- -`‘ $, \int \left(x^2+1\right)^{-1} dx,$ ‘ `substituce` ‘ $, x = t, \int \left(t^2+1\right)^{-1} dt = \arctan(t) = \arctan(x)$]

$$\int \left(x^2+1\right)^{-1} dx = \arctan(x)$$

$$\int \left(x^4+2\,x^2+1\right)^{-1} dx$$

$\left(x^4+2\,x^2+1\right)^{-1} = \left(x^2+1\right)^{-2}$ [$\int \left(x^2+1\right)^{-2} dx = \int 0dx + \int \left(x^2+1\right)^{-2} dx,$ ‘`- -`‘ $, \int 0dx,$ ‘ `substituce:` ‘ $, t = x^2+1, \int 0dt = 0 = 0,$ ‘`- -`‘ $, \int \left(x^2+1\right)^{-2} dx,$ ‘ `substituce` ‘ $, x = t, \int \left(t^2+1\right)^{-2} dt = 1/2\,\frac{t}{t^2+1} + 1/2\ \arctan(t) = 1/2\,\frac{x}{x^2+1} + 1/2\ \arctan(x), res_1$]

$$\int \left(x^4+2\,x^2+1\right)^{-1} dx = 1/2\,\frac{x}{x^2+1} + 1/2\ \arctan(x)$$

$$\int \left(x^2+4\,x+5\right)^{-1} dx$$

$\left(x^2+4\,x+5\right)^{-1} = \left(x^2+4\,x+5\right)^{-1}$ [$\int \left(x^2+4\,x+5\right)^{-1} dx = \int 0dx + \int \left(x^2+4\,x+5\right)^{-1} dx,$ ‘`- -`‘ $, \int 0dx,$ ‘ `substituce:` ‘ $, t = x^2+4\,x+5, \int 0dt = 0 = 0,$ ‘`- -`‘ $, \int \left(x^2+4\,x+5\right)^{-1} dx,$ ‘ `substituce` ‘ $, x = t-2, \int \left(t^2+1\right)^{-1} dt = \arctan(t) = \arctan(x+2), res_1$]

$$\int \left(x^2+4\,x+5\right)^{-1} dx = \arctan(x+2)$$

$$\int \frac{x}{x^2+4\,x+5}dx$$

$\frac{x}{x^2+4\,x+5} = \frac{x}{x^2+4\,x+5}[\int \frac{x}{x^2+4\,x+5}dx = \int 1/2\,\frac{2\,x+4}{x^2+4\,x+5}dx + \int -2\,\left(x^2+4\,x+5\right)^{-1}dx,$ ‘- -‘ $,\int 1/2\,\frac{2\,x+4}{x^2+4\,x+5}dx,$ ‘ substituce: ‘ $,t = x^2+4\,x+5, \int 1/2\,t^{-1}dt = 1/2\,\ln(t) = 1/2\,\ln(x^2+4\,x+5),$ ‘- -‘ $,\int -2\,\left(x^2+4\,x+5\right)^{-1}dx,$ ‘ substituce ‘ $,x = t-2, \int -2\,\left(t^2+1\right)^{-1}dt = -2\,\arctan(t) = -2\,\arctan(x+2), res_1]$

$$\int \frac{x}{x^2+4\,x+5}dx = 1/2\,\ln(x^2+4\,x+5) - 2\,\arctan(x+2)$$

$$\int \frac{x^2+1}{x^2+4\,x+5}dx$$

$\frac{x^2+1}{x^2+4\,x+5} = 1+\frac{-4-4\,x}{x^2+4\,x+5}[\int \frac{x^2+1}{x^2+4\,x+5}dx = \int 1+\frac{-4-4\,x}{x^2+4\,x+5}dx = \int 1dx + \int \frac{-4-4\,x}{x^2+4\,x+5}dx, \int 1dx = x, \int \frac{-4-4\,x}{x^2+4\,x+5}dx = \int -2\,\frac{2\,x+4}{x^2+4\,x+5}dx + \int 4\,\left(x^2+4\,x+5\right)^{-1}dx,$ ‘- -‘ $,\int -2\,\frac{2\,x+4}{x^2+4\,x+5}dx,$ ‘ substituce: ‘ $,t = x^2+4\,x+5, \int -2\,t^{-1}dt = -2\,\ln(t) = -2\,\ln(x^2+4\,x+5),$ ‘- -‘ $,\int 4\,\left(x^2+4\,x+5\right)^{-1}dx,$ ‘ substituce ‘ $,x = t-2, \int 4\,\left(t^2+1\right)^{-1}dt = 4\,\arctan(t) = 4\,\arctan(x+2)]$

$$\int \frac{x^2+1}{x^2+4\,x+5}dx = x - 2\,\ln(x^2+4\,x+5) + 4\,\arctan(x+2)$$

$$\int \frac{x^3+x^2+1}{x^2+4\,x+5}dx$$

$\frac{x^3+x^2+1}{x^2+4\,x+5} = x-3+\frac{16+7\,x}{x^2+4\,x+5}[\int \frac{x^3+x^2+1}{x^2+4\,x+5}dx = \int x-3+\frac{16+7\,x}{x^2+4\,x+5}dx = \int xdx + \int -3dx + \int \frac{16+7\,x}{x^2+4\,x+5}dx, \int xdx = 1/2\,x^2, \int -3dx = -3\,x, \int \frac{16+7\,x}{x^2+4\,x+5}dx = \int 7/2\,\frac{2\,x+4}{x^2+4\,x+5}dx + \int 2\,\left(x^2+4\,x+5\right)^{-1}dx,$ ‘- -‘ $,\int 7/2\,\frac{2\,x+4}{x^2+4\,x+5}dx,$ ‘ substituce: ‘ $,t = x^2+4\,x+5, \int 7/2\,t^{-1}dt = 7/2\,\ln(t) = 7/2\,\ln(x^2+4\,x+5),$ ‘- -‘ $,\int 2\,\left(x^2+4\,x+5\right)^{-1}dx,$ ‘ substituce ‘ $,x = t-2, \int 2\,\left(t^2+1\right)^{-1}dt = 2\,\arctan(t) = 2\,\arctan(x+2)]$

$$\int \frac{x^3+x^2+1}{x^2+4\,x+5}dx = 1/2\,x^2 - 3\,x + 7/2\,\ln(x^2+4\,x+5) + 2\,\arctan(x+2)$$

$$\int \frac{x^3-9\,x^2+27\,x-27}{x^2-4\,x+4}dx$$

$\frac{x^3-9\,x^2+27\,x-27}{x^2-4\,x+4} = x-5-(x-2)^{-2}+3\,(x-2)^{-1}\,[\int \frac{x^3-9\,x^2+27\,x-27}{x^2-4\,x+4}dx = \int x-5-(x-2)^{-2}+3\,(x-2)^{-1}\,dx = \int xdx + \int -5dx + \int -(x-2)^{-2}\,dx + \int 3\,(x-2)^{-1}\,dx, \int xdx = 1/2\,x^2, \int -5dx = -5\,x, \int -(x-2)^{-2}\,dx,$ ‘ substituce: ‘ $,t = x-2, \int -t^{-2}dt = t^{-1} = (x-2)^{-1}, \int 3\,(x-2)^{-1}\,dx,$ ‘ substituce: ‘ $,t = x-2, \int 3\,t^{-1}dt = 3\,\ln(t) = 3\,\ln(x-2)]$

$$\int \frac{x^3-9\,x^2+27\,x-27}{x^2-4\,x+4}dx = 1/2\,x^2 - 5\,x + (x-2)^{-1} + 3\,\ln(x-2)$$

$$\int \frac{x^3-6\,x^2+12\,x-8}{x^4-2\,x^2+1}dx$$

$\frac{x^3-6\,x^2+12\,x-8}{x^4-2\,x^2+1} = -1/4\,(x-1)^{-2} + (x-1)^{-1} - \frac{27}{4}\,(x+1)^{-2}\,[\int \frac{x^3-6\,x^2+12\,x-8}{x^4-2\,x^2+1}dx = \int -1/4\,(x-1)^{-2} + (x-1)^{-1} - \frac{27}{4}\,(x+1)^{-2}\,dx = \int -1/4\,(x-1)^{-2}\,dx + \int (x-1)^{-1}\,dx + \int -\frac{27}{4}\,(x+1)^{-2}\,dx, \int -1/4\,(x-1)^{-2}\,dx,$ ‘ substituce: ‘ $,t = 2\,x-2, \int -1/2\,t^{-2}dt = 1/2\,t^{-1} = 1/2\,(2\,x-2)^{-1}, \int (x-1)^{-1}\,dx,$ ‘ substituce: ‘ $,t = x-1, \int t^{-1}dt = \ln(t) = \ln(x-1), \int -\frac{27}{4}\,(x+1)^{-2}\,dx,$ ‘ substituce: ‘ $,t = 2\,x+2, \int -\frac{27}{2}\,t^{-2}dt = \frac{27}{2}\,t^{-1} = \frac{27}{2}\,(2\,x+2)^{-1}]$

$$\int \frac{x^3-6\,x^2+12\,x-8}{x^4-2\,x^2+1}dx = 1/4\,(x-1)^{-1} + \ln(x-1) + \frac{27}{4}\,(x+1)^{-1}$$

$$\int \left(x^2+2\,x+2\right)^{-1}dx$$

$\left(x^2+2\,x+2\right)^{-1}=\left(x^2+2\,x+2\right)^{-1}[\int\left(x^2+2\,x+2\right)^{-1}dx=\int 0dx+\int\left(x^2+2\,x+2\right)^{-1}dx$, `‘- -‘` , $\int 0dx$, `‘ substituce: ‘` , $t=x^2+2\,x+2, \int 0dt=0=0$, `‘- -‘` , $\int\left(x^2+2\,x+2\right)^{-1}dx$, `‘ substituce ‘` , $x=t-1, \int\left(t^2+1\right)^{-1}dt=\arctan(t)=\arctan(x+1), res_1]$

$$\int \left(x^2+2\,x+2\right)^{-1}dx=\arctan(x+1)$$

$$\int \left(x^4+4\,x^3+8\,x^2+8\,x+4\right)^{-1}dx$$

$\left(x^4+4\,x^3+8\,x^2+8\,x+4\right)^{-1}=\left(x^2+2\,x+2\right)^{-2}[\int\left(x^2+2\,x+2\right)^{-2}dx=\int 0dx+\int\left(x^2+2\,x+2\right)^{-2}dx$, `‘- -‘` , $\int 0dx$, `‘ substituce: ‘` , $t=x^2+2\,x+2, \int 0dt=0=0$, `‘- -‘` , $\int\left(x^2+2\,x+2\right)^{-2}dx$, `‘ substituce ‘` , $x=t-1, \int\left(t^2+1\right)^{-2}dt=1/2\,\frac{t}{t^2+1}+1/2\,\arctan(t)=1/2\,\frac{x+1}{(x+1)^2+1}+1/2\,\arctan(x+1), res_1]$

$$\int \left(x^4+4\,x^3+8\,x^2+8\,x+4\right)^{-1}dx=1/4\,\frac{2\,x+2}{x^2+2\,x+2}+1/2\,\arctan(x+1)$$

Cviceni:

$$\int x/(x^4+1)$$

Lépe substitucí $x^2=t$ než rozkladem na parciální zlomky. Porovnejte oba postupy.

Příklady na racionální lomené funkce R v proměnníé sin a cos:

$R(x,-y)=-R(x,y)$ substituce: $\sin(x)=t$

$R(-x,y)=-R(x,y)$ substituce: $\cos(x)=t$

$R(-x,-y)=R(x,y)=R_2(x/y,y^2)$, $R(\sin(x),\cos(x))=R_2(\mathrm{tg}(x),1/(1+\mathrm{tg}^2(x)))$ substituce: $\mathrm{tg}(x)=t$

Univerzální substituce: $\mathrm{tg}(x/2)=t$ neboť pro $x\neq(2k+1)\pi$ je $\cos^2(x/2)=\frac{1}{(1+\mathrm{tg}^2(x/2))}$, $\cos(x)=\cos^2(x/2)-\sin^2(x/2)=2\cos^2(x/2)-1=\frac{1-\mathrm{tg}^2(x)}{1+\mathrm{tg}^2(x/2)}$, $\sin(x)=2\sin(x/2)\cos(x/2)=2\,\mathrm{tg}(x/2)\cos^2(x/2)=\frac{2\,\mathrm{tg}(x/2)}{1+\mathrm{tg}^2(x/2)}$, $1/2\frac{1}{\cos^2(x/2)}dx=dt$.

$$\bullet\int\frac{1}{\cos(x)}\,dx, :\bullet$$

$$, \textit{substituce} :, \sin(x)=t, \int\frac{1}{\cos(x)}\,dx=\int\frac{1}{1-t^2}\,dt,$$

$$\int\frac{1}{1-x^2}\,dx=\left(\int-\frac12\,\frac{1}{x-1}+\frac12\,\frac{1}{x+1}\,dx=\int-\frac12\,\frac{1}{x-1}\,dx+\int\frac12\,\frac{1}{x+1}\,dx\right), \int-\frac12\,\frac{1}{x-1}\,dx,$$

$$\text{substituce: } t=2\,x-2, \int-\frac12\,\frac1t\,dt=\left(-\frac12\ln(t)=-\frac12\ln(2\,x-2)\right), \int\frac12\,\frac{1}{x+1}\,dx, \text{ substituce:}$$

$$t=2\,x+2, \int\frac12\,\frac1t\,dt=\left(\frac12\ln(t)=\frac12\ln(2\,x+2)\right), \int\frac{1}{1-x^2}\,dx=\mathrm{arctanh}\,(x)=$$

$$=1/2\,\ln(x+1)-1/2\,\ln(1-x)$$

$$\int\frac{1}{\cos(x)}\,dx=\ln\left(\frac{1+\sin(x)}{\cos(x)}\right)$$

Pozn.: $\ln(2x+2)=\ln(2(x+1))=\ln(x+1)+\ln(2)$ a $\int f(x)\,dx=F(x)+c$, takže $\ln(2)$ lze zahrnout do integrační konstantz a funkce $x\mapsto\ln(2x+2)$ a $x\mapsto\ln(x+1)$ jsou primitivní k téže funkci, což lze ověřit derivováním.

$$\bullet\int\frac{\sin(x)+\cos(x)^2}{\cos(x)}\,dx, :\bullet$$

$$, \textit{substituce} :, \sin(x) = t, \int \frac{\sin(x) + \cos(x)^2}{\cos(x)}\, dx = \int \frac{t + 1 - t^2}{1 - t^2}\, dt,$$

$$\int \frac{x + 1 - x^2}{1 - x^2}\, dx = \left(\int 1 - \frac{1}{2}\frac{1}{x-1} - \frac{1}{2}\frac{1}{x+1}\, dx = \int 1\, dx + \int -\frac{1}{2}\frac{1}{x-1}\, dx + \int -\frac{1}{2}\frac{1}{x+1}\, dx\right),$$

$$\int 1\, dx = x, \int -\frac{1}{2}\frac{1}{x-1}\, dx,\ \ \text{substituce: } t = 2\,x - 2, \int -\frac{1}{2}\frac{1}{t}\, dt = \left(-\frac{1}{2}\ln(t) = -\frac{1}{2}\ln(2\,x - 2)\right),$$

$$\int -\frac{1}{2}\frac{1}{x+1}\, dx,\ \ \text{substituce: } t = 2\,x + 2, \int -\frac{1}{2}\frac{1}{t}\, dt = \left(-\frac{1}{2}\ln(t) = -\frac{1}{2}\ln(2\,x + 2)\right),$$

$$\int \frac{x + 1 - x^2}{1 - x^2}\, dx = -\frac{1}{2}\ln(x - 1) + x - \frac{1}{2}\ln(x + 1), \int \frac{\sin(x) + \cos(x)^2}{\cos(x)}\, dx = -\ln(\cos(x)) + \sin(x)$$

$$\bullet \int \frac{\cos(x)}{\sin(x) + \cos(x)^2}\, dx, : \bullet$$

$$, \textit{substituce} :, \sin(x) = t, \int \frac{\cos(x)}{\sin(x) + \cos(x)^2}\, dx = \int \frac{1}{t + 1 - t^2}\, dt, \int -\frac{1}{-x - 1 + x^2}\, dx,$$

$$\text{substituce: } t = \sqrt{-x - 1 + x^2},$$

$$\int 2\,\frac{\operatorname{csgn}(t)}{t\sqrt{5 + 4\,t^2}}\, dt = \left(-\frac{2}{5}\sqrt{5}\operatorname{arctanh}\left(\frac{\sqrt{5}}{\sqrt{5 + 4\,t^2}}\right)\operatorname{csgn}(t) = -\frac{2}{5}\sqrt{5}\operatorname{arctanh}\left(\frac{\sqrt{5}}{\sqrt{1 - 4\,x + 4\,x^2}}\right)\right),$$

$$\int \frac{1}{x + 1 - x^2}\, dx = \frac{2}{5}\sqrt{5}\operatorname{arctanh}\left(\frac{1}{5}\,(-1 + 2\,x)\,\sqrt{5}\right), \int \frac{\cos(x)}{\sin(x) + \cos(x)^2}\, dx =$$

$$-\frac{1}{5}\sqrt{5}\ln\left(\tan\left(\frac{1}{2}\,x\right)^2 + \tan\left(\frac{1}{2}\,x\right) - \sqrt{5}\tan\left(\frac{1}{2}\,x\right) + 1\right)$$

$$+\frac{1}{5}\sqrt{5}\ln\left(\tan\left(\frac{1}{2}\,x\right)^2 + \tan\left(\frac{1}{2}\,x\right) + \sqrt{5}\tan\left(\frac{1}{2}\,x\right) + 1\right)$$

$\operatorname{arctanh}(x) = 1/2\,\ln(x + 1) - 1/2\,\ln(1 - x)$, csgn=signum.

$$\bullet \int \frac{\sin(x)\,\cos(x)}{1 + \sin(x)^4}\, dx, : \bullet$$

$$, \textit{substituce} :, \sin(x) = t, \int \frac{\sin(x)\,\cos(x)}{1 + \sin(x)^4}\, dx = \int \frac{t}{1 + t^4}\, dt,$$

$$\int \frac{x}{1 + x^4}\, dx = \left(\int \frac{1}{4}\frac{\sqrt{2}}{x^2 - x\sqrt{2} + 1} - \frac{1}{4}\frac{\sqrt{2}}{x^2 + x\sqrt{2} + 1}\, dx = \%2 + \%1\right), \%2 = \int 0\, dx + \%2,$$

$$-\backslash$$

$$-, \int 0\, dx,\ \ \text{substituce: } t = 4\,x^2 - 4\,x\sqrt{2} + 4, \left(\int 0\, dt = 0\right) = 0, -\backslash$$

$$-, \%2,\ \ \textit{substituce}\ , x = \sqrt{-4}\,t + 2\sqrt{2}, \int -\frac{I\sqrt{2}}{8\,t^2 - 12\,I\,t\sqrt{2} - 10}\, dt = ($$

$$-\frac{1}{8}\,I\ln\left(20 + 16\,t^2 - 8\,t\sqrt{2}\right) + \frac{1}{4}\arctan\left(\frac{1}{6}\left(4\,t - \sqrt{2}\right)\sqrt{2}\right) + \frac{1}{8}\,I\ln\left(20 + 16\,t^2 + 8\,t\sqrt{2}\right)$$

$$-\frac{1}{4}\arctan\left(\frac{1}{6}\left(4\,t + \sqrt{2}\right)\sqrt{2}\right) = -\frac{1}{8}\,I\ln\left(20 - 4\left(-x + 2\sqrt{2}\right)^2 - 2\left(-x + 2\sqrt{2}\right)\sqrt{-4}\sqrt{2}\right)$$

$$+\frac{1}{4}\arctan\left(\frac{1}{6}\left(\left(-x + 2\sqrt{2}\right)\sqrt{-4} - \sqrt{2}\right)\sqrt{2}\right)$$

$$+\frac{1}{8}\,I\ln\left(20 - 4\left(-x + 2\sqrt{2}\right)^2 + 2\left(-x + 2\sqrt{2}\right)\sqrt{-4}\sqrt{2}\right)$$

$$-\frac{1}{4}\arctan\left(\frac{1}{6}\left(\left(-x+2\sqrt{2}\right)\sqrt{-4}+\sqrt{2}\right)\sqrt{2}\right),\ \%1=\int 0\,dx+\%1,-\backslash$$

$$-,\ \int 0\,dx,\ \ \text{substituce:}\ \ t=4\,x^2+4\,x\sqrt{2}+4,\ \left(\int 0\,dt=0\right)=0,-\backslash$$

$$-,\ \%1,\ \ \textit{substituce}\ ,\ x=\sqrt{-4}\,t-2\sqrt{2},\ \int\frac{I\sqrt{2}}{8\,t^2+12\,I\,t\sqrt{2}-10}\,dt=\Big($$

$$\frac{1}{8}\,I\ln\left(20+16\,t^2-8\,t\sqrt{2}\right)+\frac{1}{4}\arctan\left(\frac{1}{6}\left(4\,t-\sqrt{2}\right)\sqrt{2}\right)-\frac{1}{8}\,I\ln\left(20+16\,t^2+8\,t\sqrt{2}\right)$$

$$-\frac{1}{4}\arctan\left(\frac{1}{6}\left(4\,t+\sqrt{2}\right)\sqrt{2}\right)=\frac{1}{8}\,I\ln\left(20-4\left(-x-2\sqrt{2}\right)^2-2\left(-x-2\sqrt{2}\right)\sqrt{-4}\sqrt{2}\right)$$

$$+\frac{1}{4}\arctan\left(\frac{1}{6}\left(\left(-x-2\sqrt{2}\right)\sqrt{-4}-\sqrt{2}\right)\sqrt{2}\right)$$

$$-\frac{1}{8}\,I\ln\left(20-4\left(-x-2\sqrt{2}\right)^2+2\left(-x-2\sqrt{2}\right)\sqrt{-4}\sqrt{2}\right)$$

$$-\frac{1}{4}\arctan\left(\frac{1}{6}\left(\left(-x-2\sqrt{2}\right)\sqrt{-4}+\sqrt{2}\right)\sqrt{2}\right),\ \int\frac{x}{1+x^4}\,dx=\frac{1}{2}\arctan\left(x^2\right),$$

$$\int\frac{\sin(x)\,\cos(x)}{1+\sin(x)^4}\,dx=\frac{1}{2}\arctan\left(1-\cos(x)^2\right)$$

$$\%1:=\int-\frac{1}{4}\,\frac{\sqrt{2}}{x^2+x\sqrt{2}+1}\,dx$$

$$\%2:=\int\frac{1}{4}\,\frac{\sqrt{2}}{x^2-x\sqrt{2}+1}\,dx$$

$$\bullet\int\frac{1}{1+3\cos(x)^2}\,dx,\ :\bullet$$

$$,\ \textit{substituce}\ :,\ \mathrm{tg}(x)=t,\ \cos(x)^2=\frac{1}{t^2+1},\ \sin(x)^2=\frac{t^2}{t^2+1},\ dx=\cos(t)^2\ dt,$$

$$\int\frac{1}{1+3\cos(x)^2}\,dx=\int\frac{1}{t^2+4}\,dt,\ \int\frac{1}{x^2+4}\,dx=\int 0\,dx+\int\frac{1}{x^2+4}\,dx,-\backslash$$

$$-,\ \int 0\,dx,\ \ \text{substituce:}\ \ t=x^2+4,\ \left(\int 0\,dt=0\right)=0,-\backslash$$

$$-,\ \int\frac{1}{x^2+4}\,dx,\ \ \textit{substituce}\ ,\ x=\sqrt{4}\,t,\ \int\frac{1}{2\,t^2+2}\,dt=\left(\frac{1}{2}\arctan(t)=\frac{1}{2}\arctan\left(\frac{1}{4}\,x\sqrt{4}\right)\right)$$

$$\int\frac{1}{x^2+4}\,dx=\frac{1}{2}\arctan\left(\frac{1}{2}\,x\right),$$

$$\int\frac{1}{1+3\cos(x)^2}\,dx=\frac{1}{2}\arctan\left(2\tan\left(\frac{1}{2}\,x\right)-\sqrt{3}\right)+\frac{1}{2}\arctan\left(2\tan\left(\frac{1}{2}\,x\right)+\sqrt{3}\right)$$

$$\bullet\int\frac{2-\sin(x)}{2+\cos(x)}\,dx,\ :\bullet$$

$$,\ \textit{substituce}\ :,\ \mathrm{tg}\left(\frac{1}{2}\,x\right)=t,\ \sin(x)=2\,\frac{t}{t^2+1},\ \cos(x)=\frac{1-t^2}{t^2+1},\ dx=2\,\frac{dt}{t^2+1},$$

$$\int\frac{2-\sin(x)}{2+\cos(x)}\,dx=\int 4\,\frac{t^2+1-t}{(t^2+3)\,(t^2+1)}\,dt,$$

$$\int 4\,\frac{x^2+1-x}{(x^2+3)\,(x^2+1)}\,dx=\left(\int 2\,\frac{x+2}{x^2+3}-2\,\frac{x}{x^2+1}\,dx=\int 2\,\frac{x+2}{x^2+3}\,dx+\int-2\,\frac{x}{x^2+1}\,dx\right),$$

$$\int 2\,\frac{x+2}{x^2+3}\,dx=\int 2\,\frac{x}{x^2+3}\,dx+\int\frac{4}{x^2+3}\,dx,-\backslash$$

$$-, \int 2\,\frac{x}{x^2+3}\,dx,\ \ \text{substituce: } t=x^2+3,\ \left(\int \frac{1}{t}\,dt=\ln(t)\right)=\ln\left(x^2+3\right),-\backslash$$

$$-, \int \frac{4}{x^2+3}\,dx,\ \ \textit{substituce}\ ,\ x=\sqrt{3}\,t,$$

$$\int 4\,\frac{\sqrt{3}}{3\,t^2+3}\,dt=\left(\frac{4}{3}\arctan(t)\ \sqrt{3}=\frac{4}{3}\arctan\left(\frac{1}{3}\,x\,\sqrt{3}\right)\ \sqrt{3}\right),$$

$$\int -2\,\frac{x}{x^2+1}\,dx=\int -2\,\frac{x}{x^2+1}\,dx+\int 0\,dx,-\backslash$$

$$-, \int -2\,\frac{x}{x^2+1}\,dx,\ \ \text{substituce: } t=x^2+1,\ \left(\int -\frac{1}{t}\,dt=-\ln(t)\right)=-\ln\left(x^2+1\right),-\backslash$$

$$-, \int 4\,\frac{x^2+1-x}{(x^2+3)\ (x^2+1)}\,dx=\ln\left(x^2+3\right)+\frac{4}{3}\arctan\left(\frac{1}{3}\,x\,\sqrt{3}\right)\ \sqrt{3}-\ln\left(x^2+1\right),$$

$$\int \frac{2-\sin(x)}{2+\cos(x)}\,dx=\ln\left(\tan\left(\frac{1}{2}\,x\right)^2+3\right)+\frac{4}{3}\,\sqrt{3}\arctan\left(\frac{1}{3}\tan\left(\frac{1}{2}\,x\right)\ \sqrt{3}\right)-\ln\left(\tan\left(\frac{1}{2}\,x\right)^2+1\right)$$

$$\bullet \int \frac{1}{1+\cos(x)^2}\,dx,:\bullet$$

$$,\ \textit{substituce}\ :,\ \mathrm{tg}\,(x)=t,\ \cos(x)^2=\frac{1}{t^2+1},\ \sin(x)^2=\frac{t^2}{t^2+1},\ dx=\cos(t)^2\ dt,$$

$$\int \frac{1}{1+\cos(x)^2}\,dx=\int \frac{1}{t^2+2}\,dt,\ \int \frac{1}{x^2+2}\,dx=\int 0\,dx+\int \frac{1}{x^2+2}\,dx,-\backslash$$

$$-, \int 0\,dx,\ \ \text{substituce: } t=x^2+2,\ \left(\int 0\,dt=0\right)=0,-\backslash$$

$$-, \int \frac{1}{x^2+2}\,dx,\ \ \textit{substituce}\ ,\ x=t\,\sqrt{2},\ \int \frac{\sqrt{2}}{2\,t^2+2}\,dt=\left(\frac{1}{2}\arctan(t)\ \sqrt{2}=\frac{1}{2}\arctan\left(\frac{1}{2}\,x\,\sqrt{2}\right)\ \sqrt{2}\right),$$

$$\int \frac{1}{x^2+2}\,dx=\frac{1}{2}\arctan\left(\frac{1}{2}\,x\,\sqrt{2}\right)\ \sqrt{2},\int \frac{1}{1+\cos(x)^2}\,dx=\frac{1}{8}\,\sqrt{2}\ln\left(\frac{\tan\left(\frac{1}{2}\,x\right)^2+\%1+1}{\tan\left(\frac{1}{2}\,x\right)^2-\%1+1}\right)$$

$$+\frac{1}{2}\,\sqrt{2}\arctan(\%1+1)+\frac{1}{2}\,\sqrt{2}\arctan(\%1-1)+\frac{1}{8}\,\sqrt{2}\ln\left(\frac{\tan\left(\frac{1}{2}\,x\right)^2-\%1+1}{\tan\left(\frac{1}{2}\,x\right)^2+\%1+1}\right)$$

$$\%1:=\tan\left(\frac{1}{2}\,x\right)\ \sqrt{2}$$

$$\bullet \int \frac{\sin(x)^3}{2+\cos(x)}\,dx,:\bullet$$

$$,\ \textit{substituce}\ :,\ \cos(x)=t,\ \int \frac{\sin(x)^3}{2+\cos(x)}\,dx=\int \frac{1-t^2}{t+2}\,dt,$$

$$\int \frac{1-x^2}{x+2}\,dx=\left(\int -x+2-\frac{3}{x+2}\,dx=\int -x\,dx+\int 2\,dx+\int -\frac{3}{x+2}\,dx\right),\ \int -x\,dx=-\frac{1}{2}\,x^2,$$

$$\int 2\,dx=2\,x,\ \int -\frac{3}{x+2}\,dx,\ \ \textit{substituce}\ :\ ,\ t=x+2,\ \int -\frac{3}{t}\,dt=(-3\ln(t)=-3\ln(x+2)),$$

$$\int \frac{1-x^2}{x+2}\,dx=-\frac{1}{2}\,x^2+2\,x-3\ln(x+2),\int \frac{\sin(x)^3}{2+\cos(x)}\,dx=\left(3\ln\left(\tan\left(\frac{1}{2}\,x\right)^2+3\right)\ \tan\left(\frac{1}{2}\,x\right)^4\right.$$

$$+ 6\ln\left(\tan\left(\frac{1}{2}x\right)^2 + 3\right)\tan\left(\frac{1}{2}x\right)^2 + 3\ln\left(\tan\left(\frac{1}{2}x\right)^2 + 3\right) - 3\ln\left(\tan\left(\frac{1}{2}x\right)^2 + 1\right)\tan\left(\frac{1}{2}x\right)^4$$

$$- 6\ln\left(\tan\left(\frac{1}{2}x\right)^2 + 1\right)\tan\left(\frac{1}{2}x\right)^2 - 3\ln\left(\tan\left(\frac{1}{2}x\right)^2 + 1\right) - 6\tan\left(\frac{1}{2}x\right)^2 - 4/.\left(\tan\left(\frac{1}{2}x\right)^2 + 1\right)^2$$

$$\bullet \int \frac{1 + \sin(x)^2}{\cos(x)}\,dx, : \bullet$$

$$, \textit{substituce} :, \sin(x) = t, \int \frac{1 + \sin(x)^2}{\cos(x)}\,dx = \int \frac{t^2 + 1}{1 - t^2}\,dt,$$

$$\int \frac{x^2 + 1}{1 - x^2}\,dx = \left(\int -1 - \frac{1}{x-1} + \frac{1}{x+1}\,dx = \int -1\,dx + \int -\frac{1}{x-1}\,dx + \int \frac{1}{x+1}\,dx\right), \int -1\,dx = -x,$$

$$\int -\frac{1}{x-1}\,dx, \text{ substituce: } t = x - 1, \int -\frac{1}{t}\,dt = (-\ln(t) = -\ln(x-1)), \int \frac{1}{x+1}\,dx, \text{ substituce:}$$

$$t = x + 1, \int \frac{1}{t}\,dt = (\ln(t) = \ln(x+1)), \int \frac{x^2 + 1}{1 - x^2}\,dx = -x - \ln(x-1) + \ln(x+1),$$

$$\int \frac{1 + \sin(x)^2}{\cos(x)}\,dx = 2\ln\left(\frac{1 + \sin(x)}{\cos(x)}\right) - \sin(x)$$