INTEGRÁLNÍ POČET

Tabulka základních vzorců pro integrály

(Integrační konstantu C, není-li uvedena, nutno připočíst.)

$\int 0\,dx = C$	$\int 1\,dx = \int dx = x$
$\int x^k\,dx = \frac{x^{k+1}}{k+1}$, $k \neq -1$	$\int \frac{1}{x}\,dx = \ln\lvert x\rvert$, $x \neq 0$
$\int \sin x\,dx = -\cos x$	$\int \cos x\,dx = \sin x$
$\int \frac{1}{\sin^2 x}\,dx = -\operatorname{cotg} x$	$\int \frac{1}{\cos^2 x}\,dx = \operatorname{tg} x$
$\int e^x\,dx = e^x$	$\int a^x\,dx = \frac{a^x}{\ln a}$, $a > 0$
$\int \sinh x\,dx = \cosh x$	$\int \cosh x\,dx = \sinh x$
$\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\operatorname{arctg}\frac{x}{a}$	$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\ln\frac{x-a}{x+a}$, $x \neq a$, $a > 0$
$\int \frac{dx}{\sqrt{x^2 + b}} = \ln\left\lvert x + \sqrt{x^2 + b}\right\rvert$, $b \neq 0$	$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\frac{x}{a}$, $x \neq a$, $a > 0$
$\int \frac{f'(x)}{f(x)}\,dx = \ln\lvert f(x)\rvert$	$\int f(ax+b)\,dx = \frac{1}{a}F(ax+b)$

Důležité integrály

$$\int \frac{dx}{ax^2 + bx + c} = \begin{cases} -\frac{2}{2ax+b}, & D = b^2 - 4ac = 0 \\ \frac{1}{\sqrt{D}}\ln\frac{2ax+b-\sqrt{D}}{2ax+b+\sqrt{D}}, & D > 0 \\ \frac{2}{\sqrt{-D}}\operatorname{arctg}\frac{2ax+b}{\sqrt{-D}} & D < 0 \end{cases}$$

$$\int \frac{dx}{\sqrt{ax^2 + bx + c}} = \begin{cases} \frac{1}{\sqrt{a}}\ln\left\lvert 2ax + b + 2\sqrt{a(ax^2+bx+c)}\right\rvert, & a > 0 \\ \frac{1}{\sqrt{-a}}\arcsin\frac{-2ax-b}{\sqrt{D}}, & D > 0,\ a < 0 \end{cases}$$

1. Dokažte, že dané funkce jsou primitivní ke stejné funkci, jestliže

a) $f(x) = 2\operatorname{arctg} x$, $g(x) = \arcsin \dfrac{2x}{1+x^2}$, $x \in (-1;1)$

b) $f(x) = e^x . \sinh x$, $g(x) = e^x . \cosh x$

Ř e š e n í :

a) Budeme vycházet přímo z definice primitivní funkce: $f'(x) = 2 . \dfrac{1}{1+x^2}$,

$$g'(x) = \frac{2(1+x^2) - 2x.2x}{(1+x^2)^2} : \sqrt{1 - \frac{4x^2}{(1+x^2)^2}} = \frac{2(1-x^2)}{(1+x^2)^2} :$$

$$: \sqrt{\frac{x^4 + 2x^2 + 1 - 4x^2}{(1+x^2)^2}} = 2 . \frac{1-x^2}{(1+x^2)^2} : \sqrt{\frac{(1-x^2)^2}{(1+x^2)^2}} = \frac{2}{1+x^2} ; \quad \underline{f'(x) = g'(x)}$$

b) Využijeme věty, která říká, že dvě funkce primitivní ke stejné funkci se liší nanejvýš o konstantu: $f(x) = e^x . \dfrac{e^x - e^{-x}}{2} = \dfrac{e^{2x} - 1}{2} = \frac{1}{2}e^{2x} - \frac{1}{2}$

$$g(x) = e^x . \frac{e^x + e^{-x}}{2} = \frac{e^{2x} + 1}{2} = \frac{1}{2}e^{2x} + \frac{1}{2} , \quad \underline{g(x) = f(x) + 1.}$$

2. Úpravou integrandu a použitím integračních vzorců vypočtěte:

a) $$\int \frac{\sqrt{x^4 + 2 + x^{-4}}}{x^3}\,dx = \int \frac{\sqrt{(x^2 + x^{-2})^2}}{x^3}\,dx = \int \frac{|x^2 + x^{-2}|}{x^3}\,dx = \int \frac{x^2 + x^{-2}}{x^3}\,dx =$$
$$= \int \left(\frac{1}{x} + x^{-5}\right) dx = \ln|x| - \underline{\frac{1}{4x^4}}$$

b) $$\int \frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1-x^4}}\,dx = \int \left[\frac{\sqrt{1+x^2}}{\sqrt{(1+x^2)(1-x^2)}} + \frac{\sqrt{1-x^2}}{\sqrt{(1+x^2)(1-x^2)}}\right] dx =$$
$$= \int \left[\frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1+x^2}}\right] dx = \underline{\underline{\arcsin x + \ln|x + \sqrt{1+x^2}|}} \qquad |x|<1$$

c) $$\int e^x a^x\,dx = \int (ea)^x\,dx = (ea)^x . \frac{1}{\ln(ea)} = \underline{\underline{\frac{e^x a^x}{1 + \ln a}}} \qquad a>0$$

d) $$\int \frac{(2^x + 3^x)^2}{6^x}\,dx = \int \frac{2^{2x} + 2.2^x3^x + 3^{2x}}{3^x.2^x}\,dx = \int \left(\frac{2^x}{3^x} + 2 + \frac{3^x}{2^x}\right) dx =$$
$$= \int \left[\left(\tfrac{2}{3}\right)^x + 2 + \left(\tfrac{3}{2}\right)^x\right] dx = \underline{\underline{\left(\tfrac{2}{3}\right)^x . \frac{1}{\ln\frac{2}{3}} + 2x + \left(\tfrac{3}{2}\right)^x . \frac{1}{\ln\frac{3}{2}}}}$$

e) $$\int \frac{dx}{\sin^2 x . \cos^2 x} = \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x . \cos^2 x}\,dx = \int \left[\frac{1}{\cos^2 x} + \frac{1}{\sin^2 x}\right] dx = \underline{\underline{\operatorname{tg} x - \operatorname{cotg} x}}$$

f) $$\int \frac{1}{\arcsin x . \sqrt{1-x^2}}\,dx = \int \frac{\frac{1}{\sqrt{1-x^2}}}{\arcsin x}\,dx = \int \frac{(\arcsin x)'}{\arcsin x}\,dx = \underline{\ln|\arcsin x|}$$
$|x|<1,\ x \neq 0$

g) $$\operatorname{tg} x\,dx = \frac{\sin x}{\cos x}\,dx = \frac{-(\cos x)'}{\cos x}\,dx = -\ln \cos x + C ;$$

h) $$\cos^2 x\,dx = \frac{1 + \cos 2x}{2}\,dx = \frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right) + C ;$$

i) $$\sin^2 x\,dx = \frac{1 - \cos 2x}{2}\,dx = \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + C ;$$

j) $\int \frac{dx}{\sin x} = \int \frac{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\,dx = \frac{1}{2}\int \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\,dx + \frac{1}{2}\int \frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}\,dx = \underline{\underline{\ln\left|\operatorname{tg}\frac{x}{2}\right|}}$;

k) $\int \frac{dx}{\cos x} = \left| \cos x = \sin\left(x+\frac{\pi}{2}\right) \right| = \underline{\underline{\ln\left|\operatorname{tg}\left(\frac{x}{2}+\frac{\pi}{4}\right)\right|}}$;

l) $\int \sin ax \cos bx\,dx = \left| \sin\alpha\cos\beta = \frac{1}{2}\left(\sin(\alpha+\beta)+\sin(\alpha-\beta)\right) \right| = \underline{\underline{-\frac{1}{2}\left(\cos(a+b)x + \cos(a-b)x\right)}}$

N ě k t e r é   t y p y   i n t e g r á l ů
ř e š i t e l n ý c h   m e t o d o u   p e r   p a r t e s

Je-li $P(x)$ polynom, potom u integrálů

$\int P(x)\ln x\,dx$, $\int P(x)\operatorname{arctg} x\,dx$, $\int P(x)\arcsin x\,dx$ – klademe $u = \ln x$, resp. $u = \operatorname{arctg} x$, resp. $u = \arcsin x$; u' je racionální, resp. iracionální funkce (tedy již ne transcendentní)

$\int P(x)\cos x\,dx$, $\int P(x)\sin x\,dx$, $\int P(x)\,a^x\,dx$ – klademe $u = P(x)$; a metodu opakujeme tolikrát, jaký je stupeň polynomu

3. Výpočty integrálů metodou per partes:

a) $\int (ax+b)\sin kx\,dx = \left|\begin{matrix} u = ax+b & u' = a \\ v' = \sin kx & v = -\frac{1}{k}\cos kx \end{matrix}\right| = -\frac{ax+b}{k}\cos kx + \frac{a}{k}\int \cos kx\,dx = \underline{-\frac{ax+b}{k}\cos kx + \frac{a}{k^2}\sin kx}$

b) $\int x^3 e^{3x}\,dx = \left|\begin{matrix} u = x^3 & u' = 3x^2 \\ v' = e^{3x} & v = \frac{1}{3}e^{3x} \end{matrix}\right| = \frac{x^3}{3}e^{3x} - \int x^2 e^{3x}\,dx = \left|\begin{matrix} u = x^2 & u' = 2x \\ v' = e^{3x} & v = \frac{1}{3}e^{3x} \end{matrix}\right| =$

$= \frac{x^3}{3}e^{3x} - \frac{x^2}{3}e^{3x} + \frac{2}{3}\int x e^{3x}\,dx = \left|\begin{matrix} u = x & u' = 1 \\ v' = e^{3x} & v = \frac{1}{3}e^{3x} \end{matrix}\right| = \frac{x^3}{3}e^{3x} - \frac{x^2}{3}e^{3x} +$

$+ \frac{2}{9}x e^{3x} - \frac{2}{9}\int e^{3x}\,dx = \underline{\underline{\frac{e^{3x}}{27}\left(9x^3 - 9x^2 + 6x - 2\right)}}$

c) $\int \ln x\,dx = \left|\begin{matrix} u = \ln x & u' = \frac{1}{x} \\ v' = 1 & v = x \end{matrix}\right| = x.\ln x - \int x.\frac{1}{x}\,dx = \underline{x.\ln x - x}$

d) $\int e^{ax}\cos bx\,dx = \left|\begin{matrix} u = \cos bx & u' = -b.\sin bx \\ v' = e^{ax} & v = \frac{1}{a}e^{ax} \end{matrix}\right| = \frac{1}{a}e^{ax}\cos bx + \frac{b}{a}\int e^{ax}\sin bx\,dx$

$= \left|\begin{matrix} u = \sin bx & u' = b.\cos bx \\ v' = e^{ax} & v = \frac{1}{a}e^{ax} \end{matrix}\right| = \frac{1}{a}e^{ax}\cos bx + \frac{b}{a}\left[\frac{1}{a}e^{ax}\sin bx - \frac{b}{a}\int e^{ax}\cos bx\,dx\right] =$

$= \frac{1}{a}e^{ax}\cos bx + \frac{b}{a^2}e^{ax}\sin bx - \frac{b^2}{a^2}\int e^{ax}\cos bx\,dx$. Až na multiplikativní konstantu jsme dostali opět původní integrál. Máme tedy: $I = \frac{1}{a}e^{ax}\cos bx + \frac{b}{a^2}e^{ax}\sin bx - \frac{b^2}{a^2}\cdot I$. Odtud $I + \frac{b^2}{a^2}.I = \frac{1}{a}e^{ax}\cos bx + \frac{b}{a^2}.e^{ax}\sin bx$, tedy

$I \cdot \frac{a^2 + b^2}{a^2} = \frac{1}{a} e^{ax}\cos bx + \frac{b}{a^2} e^{ax}\sin bx$ a

$I = \frac{1}{a^2 + b^2}\left[a \cdot e^{ax}\cos bx + b \cdot e^{ax}\sin bx\right]$

e) $\int \frac{dx}{(a^2 + x^2)^n} = \frac{1}{a^2} \int \frac{a^2}{(a^2 + x^2)^n} dx = \frac{1}{a^2} \int \frac{a^2 + x^2 - x^2}{(a^2 + x^2)^n} dx =$

$= \frac{1}{a^2} \int \frac{1}{(a^2 + x^2)^{n-1}} dx - \frac{1}{a^2} \int x \cdot \frac{x}{(a^2 + x^2)^n} dx = \left| \begin{array}{ll} u = x & u' = 1 \\ v' = \frac{x}{(a^2 + x^2)^n} & v = \end{array} \right|$

$v = \frac{1}{2} \int \frac{2x}{(a^2 + x^2)^n} dx = \frac{1}{2} \cdot \frac{-1}{(a^2 + x^2)^{n-1}} \cdot \frac{1}{n - 1} \Big| = \frac{1}{a^2} \int \frac{1}{(a^2 + x^2)^{n-1}} dx -$

$- \frac{1}{a^2}\left[\frac{-x}{2(n-1) \cdot (a^2+x^2)^{n-1}} + \frac{1}{2(n-1)} \int \frac{dx}{(a^2 + x^2)^{n-1}}\right] = \frac{1}{2a^2(n-1)} \cdot \frac{x}{(a^2 + x^2)^{n-1}} +$

$+ \frac{1}{a^2}(1 - \frac{1}{2(n-1)}) \int \frac{dx}{(a^2 + x^2)^{n-1}}$. Takže, označíme-li $\int \frac{dx}{(a^2 + x^2)^n} = K_n$,

odvodili jsme rekurentní vzorec $K_n = \frac{1}{a^2} \cdot \frac{2n-3}{2(n-1)} \cdot K_{n-1} + \frac{1}{2a^2(n-1)} \frac{x}{(a^2 + x^2)^{n-1}} \Rightarrow$

$$\Rightarrow K_n = \frac{1}{2a^2(n-1)} \left[(2n-3)K_{n-1} + \frac{x}{(a^2+x^2)^{n-1}}\right]$$

f) $\int \sin x \cdot \ln(\operatorname{tg} x)\, dx = \left| \begin{array}{ll} u = \ln(\operatorname{tg} x) & u' = \frac{1}{\operatorname{tg} x} \cdot \frac{1}{\cos^2 x} \\ v' = \sin x & v = -\cos x \end{array} \right| = -\cos x \cdot \ln(\operatorname{tg} x) +$

$+ \int \frac{1}{\operatorname{tg} x} \cdot \frac{1}{\cos^2 x} \cdot \cos x\, dx = -\cos x \cdot \ln(\operatorname{tg} x) + \int \frac{1}{\sin x} dx = -\cos x \cdot \ln(\operatorname{tg} x) +$

$+ \int \frac{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2}}{\sin 2 \cdot \frac{x}{2}} dx = -\cos \cdot \ln(\operatorname{tg} x) + \int \frac{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2}}{2\sin\frac{x}{2} \cdot \cos \frac{x}{2}} dx = -\cos x \cdot \ln(\operatorname{tg} x) +$

$+ \int \frac{\frac{1}{2} \sin \frac{x}{2}}{\cos \frac{x}{2}} dx + \int \frac{\frac{1}{2} \cos \frac{x}{2}}{\sin \frac{x}{2}} dx = -\cos x \cdot \ln(\operatorname{tg} x) - \ln \cos \frac{x}{2} + \ln \sin \frac{x}{2} =$

$= -\cos x \cdot \ln \operatorname{tg} x + \ln \operatorname{tg} \frac{x}{2}$. Poznamenejme, že integrál $\int \frac{dx}{\sin x}$ lze najít i jiným způsobem, jak si ukážeme dále.

4. Pomocí vhodně zvolené substituce najděte integrály:

a) $\int \frac{3^x}{\sqrt{1 - 9^x}} dx = \left| \begin{array}{l} 3^x = t \\ 3^x \ln 3\, dx = dt \end{array} \right| = \frac{1}{\ln 3} \int \frac{dt}{\sqrt{1 - t^2}} = \frac{1}{\ln 3} \arcsin t =$

$= \frac{1}{\ln 3} \arcsin 3^x$

b) $\int \frac{1}{x^2} \cdot \sin \frac{1}{x} dx = \left| \begin{array}{l} x^{-1} = t \\ -x^{-2}dx = dt \end{array} \right| = - \int \sin t\, dt = \cos t = \cos \frac{1}{x}$

c) $\int \frac{x^3}{(1 - x)^{25}} dx = \left| \begin{array}{l} 1 - x = t \\ -dx = dt \end{array} \right| = - \int \frac{(1 - t)^3}{t^{25}} dt = \int \frac{t^3 - 3t^2 + 3t - 1}{t^{25}} dt =$