Rozklad na parcialni zlomky

Parcialní zlomky jsou:

$$\frac{A}{(x-x_0)^n}, \frac{A\,x+B}{(p\,x^2-q\,x+r)^n}$$

kde $p\,x^2-q\,x+r$ je v $\Re$ ireducibilní, tj. $q^2<4\,p\,r$. Rozložit racionální lomenou funkci na parcilní zlomky znamená napsat ji jako součet polynomu a parciálních zlomků. Příklady:

```
> restart;
PF:=proc(f)
expand(numer(f))/expand(denom(f))=convert(f,parfrac,x);
end;
```

$PF :=$ **proc** (f) expand(numer(f))/expand(denom(f)) = convert(f, $parfrac$, x) **end proc**

```
> PF(2/(x^2-1));
PF(2*x/(x^2-1));
PF(2*x^2/(x^2-1));
PF(2*x^3/(x^2-1));
```

$$\frac{2}{x^2-1}=-\frac{1}{x+1}+\frac{1}{x-1}$$

$$\frac{2\,x}{x^2-1}=\frac{1}{x+1}+\frac{1}{x-1}$$

$$\frac{2\,x^2}{x^2-1}=2-\frac{1}{x+1}+\frac{1}{x-1}$$

$$\frac{2\,x^3}{x^2-1}=2\,x+\frac{1}{x+1}+\frac{1}{x-1}$$

Příklad výpočtu parciálních zlomků:

```
> f:=simplify(2*x^2+(x^5-x^3+x)/expand((x-1)^2*(x+1)*(x^2+1)));
```

$$f:=\frac{x\,(2\,x^6-2\,x^5-3\,x^2+2\,x+x^4+1)}{x^5-x^4-x+1}$$

krok 1.: nejprve najdeme ryzi lomenou cast funkce deleni ciytatele jmenovatelem se zbytkem:

```
>
F:=f=quo(numer(f),denom(f),x)+rem(numer(f),denom(f),x)/denom(f);
 `dale se zabyvame ryzi racionalni lomenou funkci:`;
g[1]:=op(nops(rhs(F)),rhs(F));
```

$$F:=\frac{x\,(2\,x^6-2\,x^5-3\,x^2+2\,x+x^4+1)}{x^5-x^4-x+1}=2\,x^2+1+\frac{-x^3+2\,x-1+x^4}{x^5-x^4-x+1}$$

dale se zabyvame ryzi racionalni lomenou funkci:

$$g_1 := \frac{-x^3 + 2\,x - 1 + x^4}{x^5 - x^4 - x + 1}$$

krok 2.: najdeme (ad hoc) koreny jmenovatele. Zkusime treba doszovat delitele absolutniho clene, tj 1 a -1

```
> g[2]:=g[1]=numer(g[1])/factor(denom(g[1]));
Koreny:=solve(denom(rhs(g[2]))=0,x);
```

$$g_2 := \frac{-x^3 + 2\,x - 1 + x^4}{x^5 - x^4 - x + 1} = \frac{-x^3 + 2\,x - 1 + x^4}{(x-1)^2\,(x+1)\,(x^2+1)}$$

$$Koreny := -1, I, -I, 1, 1$$

Máme dvojnásobny a jednoduchy reálny kořeny kořeny a druhou mocninu ireducibilního kvadratického polynomu. Predpokládáme rozklad na parciální zlomky ve tvaru:

```
> g[3]:=rhs(g[2])=A[1]/(x-1)+A[2]/(x-1)^2+
A[3]/(x+1)#+A[4]/(x+1)^2
+(A[4]*x+A[5])/(x^2+1)#+(A[7]*x+A[8])/(x^2+1)^2
;
```

$$g_3 := \frac{-x^3 + 2\,x - 1 + x^4}{(x-1)^2\,(x+1)\,(x^2+1)} = \frac{A_1}{x-1} + \frac{A_2}{(x-1)^2} + \frac{A_3}{x+1} + \frac{A_4\,x + A_5}{x^2+1}$$

Vynasobime obe strany rovnice jmenovatelem leve:

```
> i:='i':
g[4]:=lhs(g[3]*denom(lhs(g[3])))=
>
add(simplify(op(i,rhs(g[3]))*denom(lhs(g[3]))),
i=1..nops(rhs(g[3])))
;
```

$$g_4 := -x^3 + 2\,x - 1 + x^4 = A_1\,(x^2+1)\,(x^2-1) + A_2\,(x+1)\,(x^2+1) + A_3\,(x-1)^2\,(x^2+1) + (A_4\,x + A_5)\,(x-1)^2\,(x+1)$$

krok 3.: hledame koeficienty A_i tak, aby se oba polynomy rovnaly. Nejprve porovname jejich hodnoty v bodech 1 a -1:

```
> r[1]:=subs(x=1,g[4]);
r[2]:=subs(x=-1,g[4]);
```

$$r_1 := 1 = 4\,A_2$$

$$r_2 := -1 = 8\,A_3$$

Potom porovnáme koeficienty u stejných mocnin:

```
> g[5]:=sort(collect(g[4],x));
r[3]:=subs(x=0,g[5]);
for i from 1 to degree (rhs(g[4]),x) do
r[i+3]:=coeff(lhs(g[5]),x^i)=coeff(rhs(g[5]),x^i);
od;
k:=i+2:
```

$$g_5 := x^4 - x^3 + 2\,x - 1 = (A_1 + A_3 + A_4)\,x^4 + (A_2 - 2\,A_3 - A_4 + A_5)\,x^3$$
$$+ (A_2 + 2\,A_3 - A_4 - A_5)\,x^2 - A_1 + A_2 + A_3 + A_5 + (A_2 - 2\,A_3 + A_4 - A_5)\,x$$

$$r_3 := -1 = -A_1 + A_2 + A_5 + A_3$$

$$r_4 := 2 = A_2 - 2\,A_3 + A_4 - A_5$$

$$r_5 := 0 = A_2 + 2\,A_3 - A_4 - A_5$$

$$r_6 := -1 = A_2 - 2\,A_3 - A_4 + A_5$$

$$r_7 := 1 = A_1 + A_3 + A_4$$

Vybereme si ovsem pouze 4 rovnic, 7 je zbytecne mnoho. ty vyresime:

```
> i:='i':
koef:=op(solve({r[i] $i=1..k}));
>
```

$$koef := A_1 = \frac{3}{8},\, A_5 = \frac{-3}{4},\, A_4 = \frac{3}{4},\, A_3 = \frac{-1}{8},\, A_2 = \frac{1}{4}$$

a mame:

```
> subs(koef,g[3]);
```

$$\frac{x^4 - x^3 + 2\,x - 1}{(x-1)^2\,(x+1)\,(x^2+1)} = \frac{3}{8\,(x-1)} + \frac{1}{4\,(x-1)^2} - \frac{1}{8\,(x+1)} + \frac{\frac{3\,x}{4} - \frac{3}{4}}{x^2+1}$$

a tedy:

```
>
> PF(f);
```

$$\frac{2\,x^7 - 2\,x^6 - 3\,x^3 + 2\,x^2 + x^5 + x}{x^5 - x^4 - x + 1} =$$
$$2\,x^2 + 1 + \frac{3\,x - 3}{4\,(x^2+1)} + \frac{1}{4\,(x-1)^2} - \frac{1}{8\,(x+1)} + \frac{3}{8\,(x-1)}$$

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$$\mathrm{PF}\left(\frac{1}{x^2-1}\right); \quad \mathrm{PF}\left(\frac{x^2}{x^2+1}\right); \quad \mathrm{PF}\left(\frac{4\,x}{(x^2-1)^2}\right); \quad \mathrm{PF}\left(\frac{x^2+1}{(x^2-2)^2}\right);$$
$$\mathrm{PF}\left(\frac{x}{x^2-1}\right); \quad \mathrm{PF}\left(\frac{4}{(x^2-1)^2}\right); \quad \mathrm{PF}\left(\frac{1}{(x^2+1)\,(x^2-1)}\right); \quad \mathrm{PF}\left(\frac{x^5+1}{x^4-x^2}\right)$$

$$\frac{1}{x^2-1} = -\frac{1}{2\,(x+1)} + \frac{1}{2\,(x-1)}$$

$$\frac{x}{x^2-1} = \frac{1}{2\,(x+1)} + \frac{1}{2\,(x-1)}$$

$$\frac{x^2}{x^2+1} = 1 - \frac{1}{x^2+1}$$

$$\frac{4}{x^4-2x^2+1} = \frac{1}{(x+1)^2} + \frac{1}{(x-1)^2} + \frac{1}{x+1} - \frac{1}{x-1}$$

$$\frac{4x}{x^4-2x^2+1} = -\frac{1}{(x+1)^2} + \frac{1}{(x-1)^2}$$

$$\frac{1}{x^4-1} = -\frac{1}{2(x^2+1)} - \frac{1}{4(x+1)} + \frac{1}{4(x-1)}$$

$$\frac{x^2+1}{x^4-4x^2+4} = \frac{1}{x^2-2} + \frac{3}{(x^2-2)^2}$$

$$\frac{x^5+1}{x^4-x^2} = x - \frac{1}{x^2} + \frac{1}{x-1}$$

>

```
> f:=x^12/expand((x^2-1)^1*(x^2+1)^1);
```

$$f := \frac{x^{12}}{x^4-1}$$

```
> ParcFrac(f);
```

$\frac{x^{12}}{x^4-1} = x^8 + x^4 + 1 + \frac{1}{x^4-1}$,

dale se zabyvame ryzi racionalni lomenou funkci: ,

, $\frac{1}{x^4-1}$,

, $\frac{1}{x^4-1} = \frac{1}{(x-1)(x+1)(x^2+1)}$,

, $\frac{1}{(x-1)(x+1)(x^2+1)} = \frac{A_1 x + A_2}{x^2+1} + \frac{A_3}{x-1} + \frac{A_4}{x+1}$,

resime rovnici: ,

, $1 = (A_1 x + A_2)(x^2-1) + A_3(x+1)(x^2+1) + A_4(x-1)(x^2+1)$,

dosadime koreny ,

, $1 = -A_2 + A_3 - A_4$, $0 = -A_1 + A_3 + A_4$,

porovname koeficienty u stejnych mocnin: ,

, $1 = (A_1 + A_3 + A_4)x^3 + (A_2 + A_3 - A_4)x^2 - A_2 + A_3 - A_4 + (-A_1 + A_3 + A_4)x$, *dostaneme rovnice:* ,

, $0 = A_2 + A_3 - A_4$, $0 = A_1 + A_3 + A_4$,

reseni je, $A_1 = 0$, $A_4 = \frac{-1}{4}$, $A_2 = \frac{-1}{2}$, $A_3 = \frac{1}{4}$,

a mame: , $\frac{1}{(x-1)(x+1)(x^2+1)} = -\frac{1}{2(x^2+1)} + \frac{1}{4(x-1)} - \frac{1}{4(x+1)}$,

a tedy: , $\frac{x^{12}}{x^4-1} = x^8 + x^4 + 1 - \frac{1}{2(x^2+1)} + \frac{1}{4(x-1)} - \frac{1}{4(x+1)}$