See:

Example: a simple Fourier series

We now use the formulae above to give a Fourier series expansion of a very simple function. Consider a sawtooth function (as depicted in the figure):

$$f(x) = x, \quad \text{for} \quad -\pi < x < \pi,$$
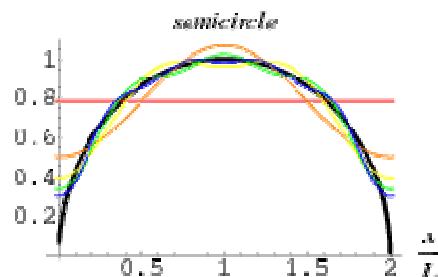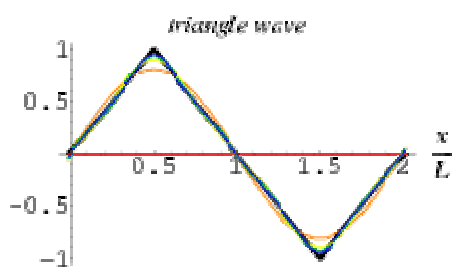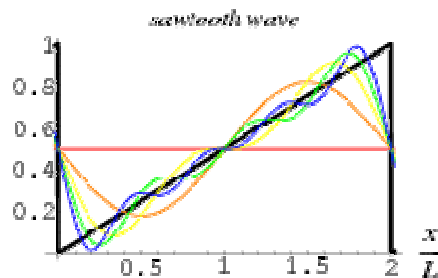$$f(x + 2\pi) = f(x), \quad \text{for} \quad -\infty < x < \infty.$$

In this case, the Fourier coefficients are given by

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos(nx) \, dx = 0.$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin(nx) \, dx = 2\frac{(-1)^{n+1}}{n}.$$

And therefore:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$$

$$= 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx), \quad \text{for} \quad -\infty < x < \infty. \quad (*)$$

In general:

square wave

sawtooth wave

triangle wave

semicircle
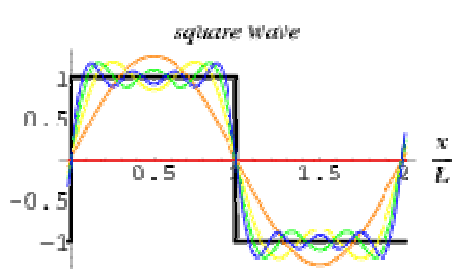
A Fourier series is an expansion of a [periodic function](#) $f(x)$ in terms of an infinite sum of [sines](#) and [cosines](#). Fourier series make use of the [orthogonality](#) relationships of the [sine](#) and [cosine](#) functions. The computation and study of Fourier series is known as [harmonic analysis](#) and is extremely useful as a way to break up an *arbitrary* periodic function into a set of simple terms that can be plugged in, solved individually, and then recombined to obtain the solution to the original problem or an approximation to it to whatever accuracy is desired or practical. Examples of successive approximations to common functions using Fourier series are illustrated above.

In particular, since the [superposition principle](#) holds for solutions of a linear homogeneous [ordinary differential equation](#), if such an equation can be solved in the case of a single sinusoid, the solution for an arbitrary function is immediately available by expressing the original function as a Fourier series and then plugging in the solution for each sinusoidal component. In some special cases where the Fourier series can be summed in closed form, this technique can even yield analytic solutions.

Any set of functions that form a [complete orthogonal system](#) have a corresponding [generalized Fourier series](#) analogous to the Fourier series. For example, using orthogonality of the roots of a [Bessel function of the first kind](#) gives a so-called [Fourier-Bessel series](#).

The computation of the (usual) Fourier series is based on the integral identities

$$\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,dx \quad = \quad \pi\delta_{mn} \tag{1}$$

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx \quad = \quad \pi\delta_{mn} \tag{2}$$

$$\int_{-\pi}^{\pi} \sin(mx)\cos(nx)\,dx \quad = \quad 0 \tag{3}$$

$$\int_{-\pi}^{\pi} \sin(mx)\,dx \quad = \quad 0 \tag{4}$$

$$\int_{-\pi}^{\pi} \cos(mx)\,dx \quad = \quad 0 \tag{5}$$

for $m, n \neq 0$, where δ_{mn} is the [Kronecker delta](#).

Using the method for a [generalized Fourier series](#), the usual Fourier series involving sines and cosines is obtained by taking $f_1(x) = \cos x$ and $f_2(x) = \sin x$. Since these functions form a [complete orthogonal system](#) over $[-\pi, \pi]$, the Fourier series of a function $f(x)$ is given by

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx), \tag{6}$$

where

$$a_0 \quad = \quad \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \tag{7}$$

$$a_n \quad = \quad \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \tag{8}$$

$$b_n \quad = \quad \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx \tag{9}$$

and $n = 1$, 2, 3, ... Note that the coefficient of the constant term a_0 has been written in a special form compared to the general form for a [generalized Fourier series](#) in order to preserve symmetry with the definitions of a_n and b_n.

A Fourier series converges to the function $\bar{f}$ (equal to the original function at points of continuity or to the average of the two limits at points of discontinuity)

$$\bar{f} \equiv \begin{cases} \frac{1}{2} \left| \lim_{x \to x_0^-} f(x) + \lim_{x \to x_0^+} f(x) \right| & \text{for } -\pi < x_0 < \pi \\ \frac{1}{2} \left| \lim_{x \to -\pi^+} f(x) + \lim_{x \to \pi^-} f(x) \right| & \text{for } x_0 = -\pi, \pi \end{cases} \tag{10}$$

if the function satisfies so-called [Dirichlet conditions](#).
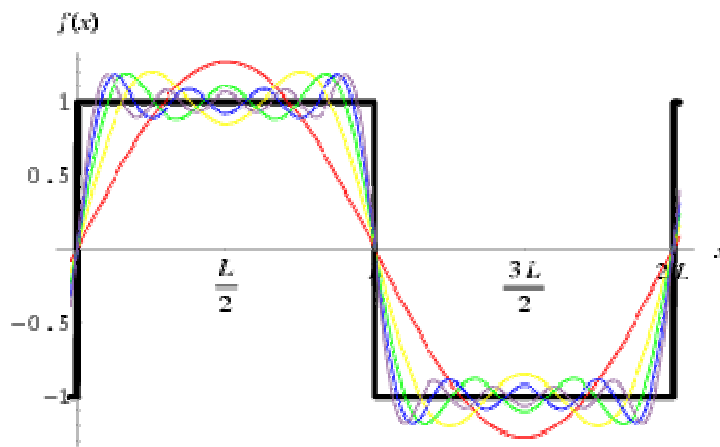
As a result, near points of discontinuity, a "ringing" known as the [Gibbs phenomenon](#), illustrated above, can occur.

For a function $f(x)$ periodic on an interval $[-L, L]$ instead of $[-\pi, \pi]$, a simple change of variables can be used to transform the interval of integration from $[-\pi, \pi]$ to $[-L, L]$. Let

$$x \equiv \frac{\pi x'}{L} \tag{11}$$

$$dx = \frac{\pi\, dx'}{L}. \tag{12}$$

Solving for x' gives $x' = Lx/\pi$, and plugging this in gives

$$f(x') = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x'}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x'}{L}\right). \tag{13}$$

Therefore,

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x')\, dx' \tag{14}$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x') \cos\left(\frac{n\pi x'}{L}\right) dx' \tag{15}$$

$$b_n \quad = \quad \frac{1}{L} \int_{-L}^{L} f(x') \sin\left(\frac{n \pi x'}{L}\right) dx'. \tag{16}$$

Similarly, the function is instead defined on the interval $[0, 2L]$, the above equations simply become

$$a_0 \quad = \quad \frac{1}{L} \int_{0}^{2L} f(x') dx' \tag{17}$$

$$a_n \quad = \quad \frac{1}{L} \int_{0}^{2L} f(x') \cos\left(\frac{n \pi x'}{L}\right) dx' \tag{18}$$

$$b_n \quad = \quad \frac{1}{L} \int_{0}^{2L} f(x') \sin\left(\frac{n \pi x'}{L}\right) dx'. \tag{19}$$

In fact, for $f(x)$ periodic with period $2L$, *any* interval $(x_0, x_0 + 2L)$ can be used, with the choice being one of convenience or personal preference (Arfken 1985, p. 769).

The coefficients for Fourier series expansions of a few common functions are given in Beyer (1987, pp. 411-412) and Byerly (1959, p. 51). One of the most common functions usually analyzed by this technique is the square wave. The Fourier series for a few common functions are summarized in the table below.

function	$f(x)$	Fourier series
Fourier series--sawtooth wave	$\frac{x}{2L}$	$\frac{1}{2} - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi x}{L}\right)$
Fourier series--square wave	$2\left[H\left(\frac{x}{L}\right) - H\left(\frac{x}{L} - 1\right)\right] - 1$	$\frac{4}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi x}{L}\right)$
Fourier series--triangle wave	$T(x)$	$\frac{8}{\pi^2} \sum_{n=1,3,5,\ldots}^{\infty} \frac{(-1)^{(n-1)/2}}{n^2} \sin\left(\frac{n\pi x}{L}\right)$

If a function is even so that $f(x) = f(-x)$, then $f(x)\sin(n x)$ is odd. (This follows since $\sin(n x)$ is odd and an even function times an odd function is an odd function.) Therefore, $b_n = 0$ for all n. Similarly, if a function is odd so that $f(x) = -f(-x)$, then $f(x)\cos(n x)$ is odd. (This follows since $\cos(n x)$ is even and an even function times an odd function is an odd function.) Therefore, $a_n = 0$ for all n.

The notion of a Fourier series can also be extended to complex coefficients. Consider a real-valued function $f(x)$. Write

$$f(x) = \sum_{n=-\infty}^{\infty} A_n e^{inx}. \tag{20}$$

Now examine

$$\int_{-\pi}^{\pi} f(x) e^{-imx} dx \quad = \quad \int_{-\pi}^{\pi} \left(\sum_{n=-\infty}^{\infty} A_n e^{inx} \right) e^{-imx} dx \tag{21}$$

$$= \quad \sum_{n=-\infty}^{\infty} A_n \int_{-\pi}^{\pi} e^{i(n-m)x} dx \tag{22}$$

$$= \quad \sum_{n=-\infty}^{\infty} A_n \int_{-\pi}^{\pi} \{\cos[(n-m)x] + i\sin[(n-m)x]\} dx \tag{23}$$

$$= \sum_{n=-\infty}^{\infty} A_n \, 2\pi \delta_{mn} \tag{24}$$

$$= 2\pi A_m, \tag{25}$$

so

$$A_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, e^{-inx} \, dx. \tag{26}$$

The [coefficients](#) can be expressed in terms of those in the Fourier series

$$A_n \quad = \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \left[\cos(nx) - i\sin(nx) \right] dx \tag{27}$$

$$= \begin{cases} \dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} f(x) \left[\cos(nx) + i\sin(nx) \right] dx & n < 0 \\[2mm] \dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} f(x) \, dx & n = 0 \\[2mm] \dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} f(x) \left[\cos(nx) - i\sin(nx) \right] dx & n > 0 \end{cases} \tag{28}$$

$$\tag{29}$$

$$= \begin{cases} \dfrac{1}{2}(a_n + i b_n) & \text{for } n < 0 \\[2mm] \dfrac{1}{2} a_0 & \text{for } n = 0 \\[2mm] \dfrac{1}{2}(a_n - i b_n) & \text{for } n > 0. \end{cases} \tag{30}$$

For a function periodic in $[-L/2, L/2]$, these become

$$f(x) \quad = \quad \sum_{n=-\infty}^{\infty} A_n e^{i(2\pi nx/L)} \tag{31}$$

$$A_n \quad = \quad \frac{1}{L} \int_{-L/2}^{L/2} f(x) \, e^{-i(2\pi nx/L)} \, dx. \tag{32}$$

These equations are the basis for the extremely important [Fourier transform](#), which is obtained by transforming A_n from a discrete variable to a continuous one as the length $L \to \infty$.

SEE ALSO: [Complete Set of Functions](#), [Dirichlet Fourier Series Conditions](#), [Fourier-Bessel Series](#), [Fourier Cosine Series](#), [Fourier-Legendre Series](#), [Fourier Series--Power](#), [Fourier Series--Sawtooth Wave](#), [Fourier Series--Semicircle](#), [Fourier Series--Square Wave](#), [Fourier Series--Triangle Wave](#), [Fourier Sine Series](#), [Fourier Transform](#), [Generalized Fourier Series](#), [Gibbs Phenomenon](#), [Harmonic Addition Theorem](#), [Harmonic Analysis](#), [Lacunary Fourier Series](#), [Lebesgue Constants](#), [Power Spectrum](#), [Riesz-Fischer Theorem](#), [Simple Harmonic Motion](#), [Superposition Principle](#)

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Maple code:

```
Four:=proc(N,f)
local x,a,b,n,i,xxx,M;
a:=(f,n)->1/Pi*evalf(Int(f(x)*cos(n*x),x=-Pi..Pi));
b:=(f,n)->1/Pi*evalf(Int(f(x)*sin(n*x),x=-Pi..Pi));
if N mod 2 = 1 then
    M:=(N-1)/2;
    xxx:=a(f,0)/2+sum(a(f,i)*cos(i*x)+b(f,i)*sin(i*x),i=1..M);
  else
    M:=(N-2)/2;

xxx:=a(f,0)/2+sum(a(f,i)*cos(i*x)+b(f,i)*sin(i*x),i=1..M)+a(f,
N/2)*cos(N/2*x);
  fi;
unapply(xxx,x)
end;
```

example:

```
Four(5,exp)(z);
```

$11.54873936/\text{Pi}-3.676077913*\cos(z)+3.676077910*\sin(z)+1.470431167*\cos(2.*z)-2.940862328*\sin(2.*z)$