Proof of 2.33

$\Leftarrow$: Assume $|r| \leq \frac{1}{2}$. For the stationary time series of example (2) in 2.23,
$X_t : Z_t + \Theta Z_{t-1}$, $\{Z_t\} \sim WN(0, \sigma^2)$, it holds:

$\gamma_X(h) = \begin{cases} (1+\theta^2)\sigma^2 & \text{for } h=0 \\ \theta\sigma^2 & \text{for } h = \pm 1 \\ 0 & \text{for } |h|>1 \end{cases}$. It is sufficient to choose θ, σ^2

satisfying: $(1+\theta^2)\sigma^2 = 1$ and $\theta\sigma^2 = r \Rightarrow \sigma^2 = \frac{1}{1+\theta^2}$, $r = \frac{\theta}{1+\theta^2} \equiv$
$\equiv r\theta^2 - \theta + r = 0$. Hence $\theta_{1,2} = \frac{1 \pm \sqrt{1-4r^2}}{2r}$, $\sigma_{1,2}^2 = \frac{r}{\theta_{1,2}}$ are
the desired choices ($|r| \leq \frac{1}{2} \Leftrightarrow \theta_{1,2} \in \mathbb{R} \Leftrightarrow \sigma_{1,2}^2 \in \mathbb{R}$).

$\Rightarrow$: <u>By contradiction</u>: Assume there exists such stat. time series X with $|r| > \frac{1}{2}$

<u>I. Assume $r > \frac{1}{2}$</u>. Choose $n \in \mathbb{N}$, $\underline{c} = [1, -1, 1, -1, \dots]^T$ and put
$\Gamma_n := [\gamma_X(i-j)]_{i,j=1}^n$ where $\rho_X(h) = \gamma_X(h) = \begin{cases} 1 & \text{for } h=0 \\ r & \text{for } h = \pm 1 \\ 0 & \text{for } |h|>1 \end{cases}$. Then

$$\underline{c}^T \Gamma_n \underline{c} = \sum_{\substack{i,j=1 \\ |i-j|\leq 1}}^{n} \gamma_X(i-j) c_i c_j = \underbrace{\sum_{i=1}^{n} 1 \cdot \underset{1}{c_i^2}}_{\text{case } i=j} + \underbrace{\sum_{i=2}^{n} r \cdot \underset{-1}{c_i \cdot c_{i-1}}}_{\text{case } i>j,\ i-j=1} + \underbrace{\sum_{i=1}^{n-1} r \cdot \underset{-1}{c_i \cdot c_{i+1}}}_{\text{case } i<j,\ i-j=-1} =$$

$$= n - 2r(n-1) < 0 \Leftrightarrow n > \frac{2r}{2r-1} \text{ because } 2r-1>0.$$

For such n matrix Γ_n is not non-negatively definite $\overset{\text{Def. 2.25}}{\Rightarrow}$ γ_X is not non-negatively definite $\Rightarrow$ <u>contradiction with 2.29 (3')</u> ($EX_t = 0$ without loss of gen)

<u>II. Assume $r < -\frac{1}{2}$</u>: If there would exist $X = \{X_t\}$ such that $r < -\frac{1}{2}$, X stat.,
then $Y := \{(-1)^t X_t\}$ is clearly stationary as well where $\mu_Y = \mu_X = \text{const.}$
and $\gamma_Y(h) = \begin{cases} 1 & \text{for } h=0 \\ -r & \text{for } h = \pm 1 \\ 0 & \text{for } |h|>1 \end{cases}$, which leads to contradiction as of part I.

Proof of 2.35

Put $y_i := x_i - \hat{\mu}$. Then by Def. 2.34: $\hat{\gamma}(h) = \frac{1}{n} \sum_{j=1}^{n-h} y_{j+h} \cdot y_j$ and
consequently $\hat{\Gamma}_n = \frac{1}{n} Y \cdot Y^T$ where Y is a matrix of size $n \times 2n$
constructed as follows:

$$Y = \left[\begin{array}{cccc|ccccc} 0 & 0 & \cdots & 0 & y_1 & y_2 & \cdots & y_{n-1} & y_n \\ 0 & 0 & \cdots & y_1 & y_2 & y_3 & \cdots & y_n & 0 \\ \vdots & & & & & & & & \\ 0 & y_1 & \cdots & y_{n-1} & y_n & 0 & \cdots & 0 & 0 \end{array}\right].$$

Then for any $\underline{c} \in \mathbb{R}^n$ we have $\sum_{i,j=1}^{n} \hat{\gamma}(i-j) c_i c_j = \underline{c}^T \hat{\Gamma}_n \underline{c} = \frac{1}{n} \underbrace{\underline{c}^T Y}_{(Y^T \underline{c})^T =: \underline{z}^T} \overbrace{Y^T \underline{c}}^{\underline{z}} =$
$= \frac{1}{n} z^T z = \frac{1}{n} \sum_{i=1}^{n} z_i^2 \geq 0$.

We have thus proved that $\hat{\Gamma}_n$ is non-negatively definite.
Symmetry is obvious.