Selected chapters from draft of

An Introduction to Game Theory
by
Martin J. Osborne

Please send comments to
Martin J. Osborne
Department of Economics
150 St. George Street
University of Toronto
Toronto, Canada M5S 3G7
email: martin.osborne@utoronto.ca

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Preface

Game theoretic reasoning pervades economic theory and is used widely in other social and behavioral sciences. This book presents the main ideas of game theory and shows how they can be used to understand economic, social, political, and biological phenomena. It assumes no knowledge of economics, political science, or any other social or behavioral science. It emphasizes the ideas behind the theory rather than their mathematical expression, and assumes no specific mathematical knowledge beyond that typically taught in US and Canadian high schools. (Chapter 17 reviews the mathematical concepts used in the book.) In particular, calculus is not used, except in the appendix of Chapter 9 (Section 9.7). Nevertheless, all concepts are defined precisely, and logical reasoning is used extensively. The more comfortable you are with tight logical analysis, the easier you will find the arguments. In brief, my aim is to explain the main ideas of game theory as simply as possible while maintaining complete precision.

The only way to appreciate the theory is to see it in action, or better still to put it into action. So the book includes a wide variety of illustrations from the social and behavioral sciences, and over 200 exercises.

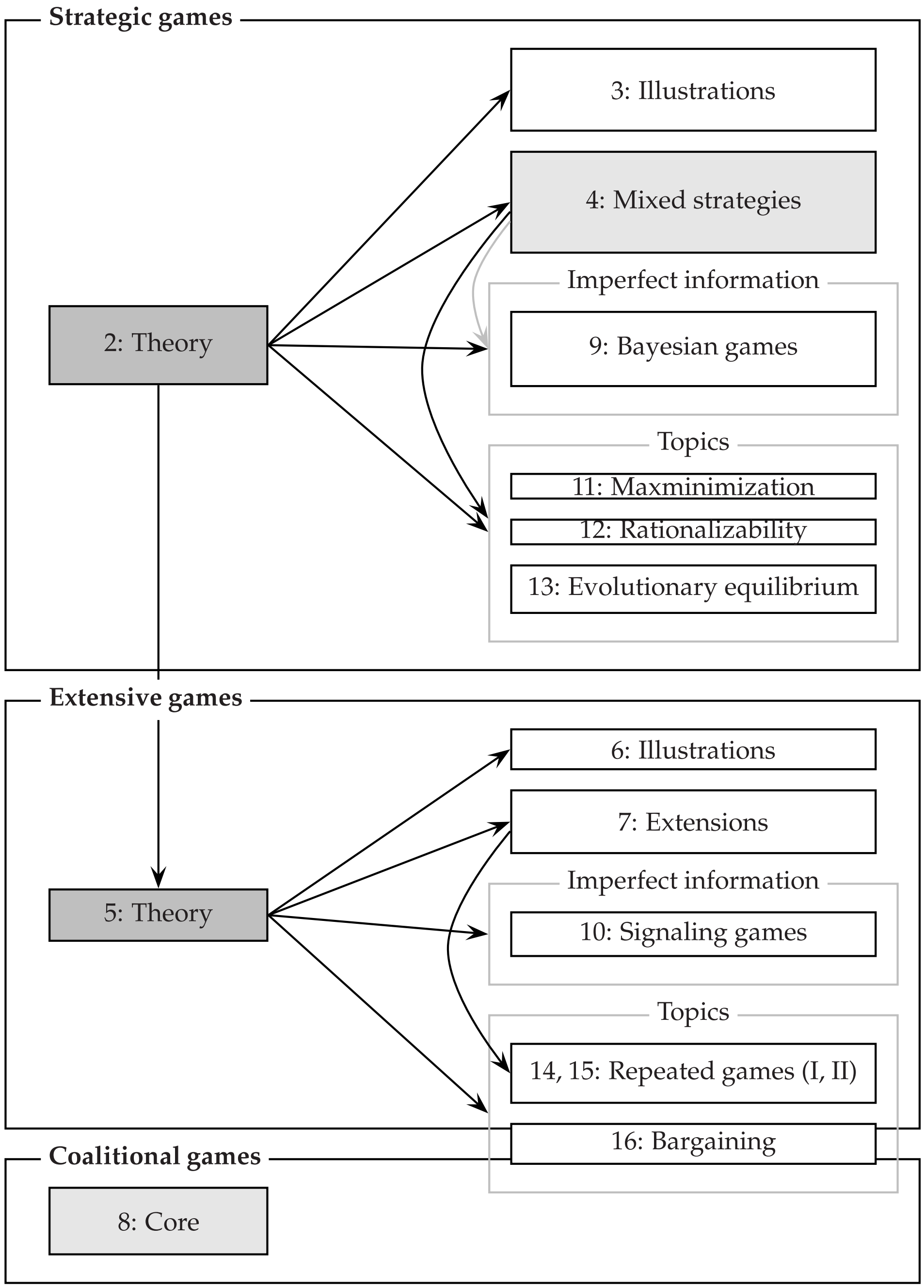
The structure of the book is illustrated in the figure on the next page. The gray boxes indicate core chapters (the darker gray, the more important). An black arrow from Chapter i to Chapter j means that Chapter j depends on Chapter i. The gray arrow from Chapter 4 to Chapter 9 means that the latter depends weakly on the former; for all but Section 9.8 only an understanding of expected payoffs (Section 4.1.3) is required, not a knowledge of mixed strategy Nash equilibrium. (Two chapters are not included in this figure: Chapter 1 reviews the theory of a single rational decision-maker, and Chapter 17 reviews the mathematical concepts used in the book.)

Each topic is presented with the aid of "Examples", which highlight theoretical points, and "Illustrations", which demonstrate how the theory may be used to understand social, economic, political, and biological phenomena. The "Illustrations" for the key models of strategic and extensive games are grouped in separate chapters (3 and 6), whereas those for the other models occupy the same chapters as the theory. The "Illustrations" introduce no new theoretical points, and any or all of them may be skipped without loss of continuity.

The limited dependencies between chapters mean that several routes may be taken through the book.

- At a minimum, you should study Chapters 2 (Nash Equilibrium: Theory) and 5 (Extensive Games with Perfect Information: Theory).

- Optionally you may sample some sections of Chapters 3 (Nash Equilibrium:



xivFigure 0.1 The structure of the book. The area of each box is proportional to the length of the chapter the box represents. The boxes corresponding to the core chapters are shaded gray; the ones shaded dark gray are more central that the ones shaded light gray. An arrow from Chapter i to Chapter j means that Chapter i is a prerequisite for Chapter j. The gray arrow from Chapter 4 to Chapter 9 means that the latter depends only weakly on the former.

Illustrations) and 6 (Extensive Games with Perfect Information: Illustrations).

- You may add to this plan any combination of Chapters 4 (Mixed Strategy Equilibrium), 9 (Bayesian Games, except Section 9.8), 7 (Extensive Games with Perfect Information: Extensions and Discussion), 8 (Coalitional Games and the Core), and 16 (Bargaining).
- If you read Chapter 4 (Mixed Strategy Equilibrium) then you may in addition study any combination of the remaining chapters covering strategic games, and if you study Chapter 7 (Extensive Games with Perfect Information: Extensions and Discussion) then you are ready to tackle Chapters 14 and 15 (Repeated Games).

All the material should be accessible to undergraduate students. A one-semester course for third or fourth year North American economics majors (who have been exposed to a few of the main ideas in first and second year courses) could cover up to about half the material in the book in moderate detail.

Personal pronouns

The lack of a sex-neutral third person singular pronoun in English has led many writers of formal English to use "he" for this purpose. Such usage conflicts with that of everyday speech. People may say "when an airplane pilot is working, he needs to concentrate", but they do not usually say "when a flight attendant is working, he needs to concentrate" or "when a secretary is working, he needs to concetrate". The use of "he" only for roles in which men traditionally predominate in Western societies suggests that women may not play such roles; I find this insinuation unacceptable.

To quote the *New Oxford Dictionary of English*, "[the use of *he* to refer to refer to a person of unspecified sex] has become . . . a hallmark of old-fashioned language or sexism in language." Writers have become sensitive to this issue in the last half century, but the lack of a sex-neutral pronoun "has been felt since at least as far back as Middle English" (*Webster's Dictionary of English Usage*, Merriam-Webster Inc., 1989, p. 499). A common solution has been to use "they", a usage that the *New Oxford Dictionary of English* endorses (and employs). This solution can create ambiguity when the pronoun follows references to more than one person; it also does not always sound natural. I choose a different solution: I use "she" exclusively. Obviously this usage, like that of "he", is not sex-neutral; but its use may do something to counterbalance the widespread use of "he", and does not seem likely to do any harm.

Acknowledgements

I owe a huge debt to Ariel Rubinstein. I have learned, and continue to learn, vastly from him about game theory. His influence on this book will be clear to anyone

familiar with our jointly-authored book *A course in game theory*. Had we not written that book and our previous book *Bargaining and markets*, I doubt that I would have embarked on this project.

Discussions over the years with Jean-Pierre Benoît, Vijay Krishna, Michael Peters, and Carolyn Pitchik have improved my understanding of many game theoretic topics.

Many people have generously commented on all or parts of drafts of the book. I am particularly grateful to Jeffrey Banks, Nikolaos Benos, Ted Bergstrom, Tilman Börgers, Randy Calvert, Vu Cao, Rachel Croson, Eddie Dekel, Marina De Vos, Laurie Duke, Patrick Elias, Mukesh Eswaran, Xinhua Gu, Costas Halatsis, Joe Harrington, Hiroyuki Kawakatsu, Lewis Kornhauser, Jack Leach, Simon Link, Bart Lipman, Kin Chung Lo, Massimo Marinacci, Peter McCabe, Barry O'Neill, Robin G. Osborne, Marco Ottaviani, Marie Rekkas, Bob Rosenthal, Al Roth, Matthew Shum, Giora Slutzki, Michael Smart, Nick Vriend, and Chuck Wilson.

I thank also the anonymous reviewers consulted by Oxford University Press and several other presses; the suggestions in their reviews greatly improved the book.

The book has its origins in a course I taught at Columbia University in the early 1980s. My experience in that course, and in courses at McMaster University, where I taught from early drafts, and at the University of Toronto, brought the book to its current form. The Kyoto Institute of Economic Research at Kyoto University provided me with a splendid environment in which to work on the book during two months in 1999.

References

The "Notes" section at the end of each chapter attempts to assign credit for the ideas in the chapter. Several cases present difficulties. In some cases, ideas evolved over a long period of time, with contributions by many people, making their origins hard to summarize in a sentence or two. In a few cases, my research has led to a conclusion about the origins of an idea different from the standard one. In all cases, I cite the relevant papers without regard to their difficulty.

Over the years, I have taken exercises from many sources. I have attempted to remember where I got them from, and have given credit, but I have probably missed some.

Examples addressing economic, political, and biological issues

The following tables list examples that address economic, political, and biological issues. [SO FAR CHECKED ONLY THROUGH CHAPTER 7.]

Games related to economic issues (THROUGH CHAPTER 7)

Exercise 31.1, Section 2.8.4, Exercise 42.1	Provision of a public good
Section 2.9.4	Collective decision-making
Section 3.1, Exercise 133.1	Cournot's model of oligopoly
Section 3.1.5	Common property
Section 3.2, Exercise 133.2, Exercise 143.2, Exercise 189.1, Exercise 210.1	Bertrand's model of oligopoly
Exercise 75.1	Competition in product characteristics
Section 3.5	Auctions with perfect information
Section 3.6	Accident law
Section 4.6	Expert diagnosis
Exercise 125.2, Exercise 208.1	Price competition between sellers
Section 4.8	Reporting a crime (private provision of a public good)
Example 141.1	All-pay auction with perfect information
Exercise 172.2	Entry into an industry by a financially-constrained challenger
Exercise 175.1	The "rotten kid theorem"
Section 6.2.2	The holdup game
Section 6.3	Stackelberg's model of duopoly
Exercise 207.2	A market game
Section 7.2	Entry into a monopolized industry
Section 7.5	Exit from a declining industry
Example 227.1	Chain-store game

Games related to political issues (THROUGH CHAPTER 7)

Exercise 32.2	Voter participation
Section 2.9.3	Voting
Exercise 47.3	Approval voting
Section 2.9.4	Collective decision-making
Section 3.3, Exercise 193.3, Exercise 193.4, Section 7.3	Hotelling's model of electoral competition

Exercise 73.1	Electoral competition between policy-motivated candidates
Exercise 73.2	Electoral competition between citizen-candidates
Exercise 88.3	Lobbying as an auction
Exercise 115.3	Voter participation
Exercise 139.1	Allocating resources in election campaigns
Section 6.4	Buying votes in a legislature
Section 7.4	Committee decision-making
Exercise 224.1	Cohesion of governing coalitions

Games related to biological issues (THROUGH CHAPTER 7)

Exercise 16.1	Hermaphroditic fish
Section 3.4	War of attrition

Typographic conventions, numbering, and nomenclature

In formal definitions, the terms being defined are set in **boldface**. Terms are set in *italics* when they are defined informally.

Definitions, propositions, examples, and exercises are numbered according to the page on which they appear. If the first such object on page z is an exercise, for example, it is called Exercise z.1; if the next object on that page is a definition, it is called Definition z.2. For example, the definition of a strategic game with ordinal preferences on page 11 is Definition 11.1. This scheme allows numbered items to found rapidly, and also facilitates precise index entries.

Symbol/term	**Meaning**
?	Exercise
??	Hard exercise
►	Definition
■	Proposition
◆	Example: a game that illustrates a game-theoretic point
Illustration	A game, or family of games, that shows how the theory can illuminate observed phenomena

I maintain a website for the book. The current URL is
`http://www.economics.utoronto.ca/osborne/igt/`.

1 Introduction

1.1 What is game theory?

GAME THEORY aims to help us understand situations in which decision-makers interact. A game in the everyday sense—"a competitive activity . . . in which players contend with each other according to a set of rules", in the words of my dictionary—is an example of such a situation, but the scope of game theory is vastly larger. Indeed, I devote very little space to games in the everyday sense; my main focus is the use of game theory to illuminate economic, political, and biological phenomena.

A list of some of the applications I discuss will give you an idea of the range of situations to which game theory can be applied: firms competing for business, political candidates competing for votes, jury members deciding on a verdict, animals fighting over prey, bidders competing in an auction, the evolution of siblings' behavior towards each other, competing experts' incentives to provide correct diagnoses, legislators' voting behavior under pressure from interest groups, and the role of threats and punishment in long-term relationships.

Like other sciences, game theory consists of a collection of models. A model is an abstraction we use to understand our observations and experiences. What "understanding" entails is not clear-cut. Partly, at least, it entails our perceiving relationships between situations, isolating principles that apply to a range of problems, so that we can fit into our thinking new situations that we encounter. For example, we may fit our observation of the path taken by a lobbed tennis ball into a model that assumes the ball moves forward at a constant velocity and is pulled towards the ground by the constant force of "gravity". This model enhances our understanding because it fits well no matter how hard or in which direction the ball is hit, and applies also to the paths taken by baseballs, cricket balls, and a wide variety of other missiles, launched in any direction.

A model is unlikely to help us understand a phenomenon if its assumptions are wildly at odds with our observations. At the same time, a model derives power from its simplicity; the assumptions upon which it rests should capture the essence of the situation, not irrelevant details. For example, when considering the path taken by a lobbed tennis ball we should ignore the dependence of the force of gravity on the distance of the ball from the surface of the earth.

Models cannot be judged by an absolute criterion: they are neither "right" nor "wrong". Whether a model is useful or not depends, in part, on the purpose for which we use it. For example, when I determine the shortest route from Florence to Venice, I do not worry about the projection of the map I am using; I work under the assumption that the earth is flat. When I determine the shortest route from Beijing to Havana, however, I pay close attention to the projection—I assume that the earth is spherical. And were I to climb the Matterhorn I would assume that the earth is neither flat nor spherical!

One reason for improving our understanding of the world is to enhance our ability to mold it to our desires. The understanding that game theoretic models give is particularly relevant in the social, political, and economic arenas. Studying game theoretic models (or other models that apply to human interaction) may also suggest ways in which our behavior may be modified to improve our own welfare. By analyzing the incentives faced by negotiators locked in battle, for example, we may see the advantages and disadvantages of various strategies.

The models of game theory are precise expressions of ideas that can be presented verbally. However, verbal descriptions tend to be long and imprecise; in the interest of conciseness and precision, I frequently use mathematical symbols when describing models. Although I use the language of mathematics, I use few of its concepts; the ones I use are described in Chapter 17. My aim is to take advantage of the precision and conciseness of a mathematical formulation without losing sight of the underlying ideas.

Game-theoretic modeling starts with an idea related to some aspect of the interaction of decision-makers. We express this idea precisely in a model, incorporating features of the situation that appear to be relevant. This step is an art. We wish to put enough ingredients into the model to obtain nontrivial insights, but not so many that we are lead into irrelevant complications; we wish to lay bare the underlying structure of the situation as opposed to describe its every detail. The next step is to analyze the model—to discover its implications. At this stage we need to adhere to the rigors of logic; we must not introduce extraneous considerations absent from the model. Our analysis may yield results that confirm our idea, or that suggest it is wrong. If it is wrong, the analysis should help us to understand why it is wrong. We may see that an assumption is inappropriate, or that an important element is missing from the model; we may conclude that our idea is invalid, or that we need to investigate it further by studying a different model. Thus, the interaction between our ideas and models designed to shed light on them runs in two directions: the implications of models help us determine whether our ideas make sense, and these ideas, in the light of the implications of the models, may show us how the assumptions of our models are inappropriate. In either case, the process of formulating and analyzing a model should improve our understanding of the situation we are considering.

An outline of the history of game theory

Some game-theoretic ideas can be traced to the 18th century, but the major development of the theory began in the 1920s with the work of the mathematician Emile Borel (1871–1956) and the polymath John von Neumann (1903–57). A decisive event in the development of the theory was the publication in 1944 of the book *Theory of games and economic behavior* by von Neumann and Oskar Morgenstern. In the 1950s game-theoretic models began to be used in economic theory and political science, and psychologists began studying how human subjects behave in experimental games. In the 1970s game theory was first used as a tool in evolutionary biology. Subsequently, game theoretic methods have come to dominate microeconomic theory and are used also in many other fields of economics and a wide range of other social and behavioral sciences. The 1994 Nobel prize in economics was awarded to the game theorists John C. Harsanyi (1920–2000), John F. Nash (1928–), and Reinhard Selten (1930–).

John von Neumann

John von Neumann, the most important figure in the early development of game theory, was born in Budapest, Hungary, in 1903. He displayed exceptional mathematical ability as a child (he had mastered calculus by the age of 8), but his father, concerned about his son's financial prospects, did not want him to become a mathematician. As a compromise he enrolled in mathematics at the University of Budapest in 1921, but immediately left to study chemistry, first at the University of Berlin and subsequently at the Swiss Federal Institute of Technology in Zurich, from which he earned a degree in chemical engineering in 1925. During his time in Germany and Switzerland he returned to Budapest to write examinations, and in 1926 obtained a PhD in mathematics from the University of Budapest. He taught in Berlin and Hamburg, and, from 1930 to 1933, at Princeton University. In 1933 he became the youngest of the first six professors of the School of Mathematics at the Institute for Advanced Study in Princeton (Einstein was another).

Von Neumann's first published scientific paper appeared in 1922, when he was 19 years old. In 1928 he published a paper that establishes a key result on strictly competitive games (a result that had eluded Borel). He made many major contributions in pure and applied mathematics and in physics—enough, according to Halmos (1973), "for about three ordinary careers, in pure mathematics alone". While at the Institute for Advanced Study he collaborated with the Princeton economist Oskar Morgenstern in writing *Theory of games and economic behavior*, the book that established game theory as a field. In the 1940s he became increasingly involved in applied work. In 1943 he became a consultant to the Manhattan project, which was developing an atomic bomb. In 1944 he became involved with the development of the first electronic computer, to which he made major contributions. He stayed at Princeton until 1954, when he became a member of the US Atomic Energy Commission. He died in 1957.

1.2 The theory of rational choice

The theory of rational choice is a component of many models in game theory. Briefly, this theory is that a decision-maker chooses the best action according to her preferences, among all the actions available to her. No qualitative restriction is placed on the decision-maker's preferences; her "rationality" lies in the consistency of her decisions when faced with different sets of available actions, not in the nature of her likes and dislikes.

1.2.1 Actions

The theory is based on a model with two components: a set A consisting of all the actions that, under some circumstances, are available to the decision-maker, and a specification of the decision-maker's preferences. In any given situation the decision-maker is faced with a subset[1] of A, from which she must choose a single element. The decision-maker knows this subset of available choices, and takes it as given; in particular, the subset is not influenced by the decision-maker's preferences. The set A could, for example, be the set of bundles of goods that the decision-maker can possibly consume; given her income at any time, she is restricted to choose from the subset of A containing the bundles she can afford.

1.2.2 Preferences and payoff functions

As to preferences, we assume that the decision-maker, when presented with any pair of actions, knows which of the pair she prefers, or knows that she regards both actions as equally desirable (is "indifferent between the actions"). We assume further that these preferences are consistent in the sense that if the decision-maker prefers the action a to the action b, and the action b to the action c, then she prefers the action a to the action c. No other restriction is imposed on preferences. In particular, we do not rule out the possibility that a person's preferences are altruistic in the sense that how much she likes an outcome depends on some other person's welfare. Theories that use the model of rational choice aim to derive implications that do not depend on any qualitative characteristic of preferences.

How can we describe a decision-maker's preferences? One way is to specify, for each possible pair of actions, the action the decision-maker prefers, or to note that the decision-maker is indifferent between the actions. Alternatively we can "represent" the preferences by a *payoff function*, which associates a number with each action in such a way that actions with higher numbers are preferred. More

[1] See Chapter 17 for a description of mathematical terminology.

precisely, the payoff function u represents a decision-maker's preferences if, for any actions a in A and b in A,

$$u(a) > u(b) \text{ if and only if the decision-maker prefers } a \text{ to } b. \qquad (5.1)$$

(A better name than payoff function might be "preference indicator function"; in economic theory a payoff function that represents a consumer's preferences is often referred to as a "utility function".)

◆ EXAMPLE 5.2 (Payoff function representing preferences) A person is faced with the choice of three vacation packages, to Havana, Paris, and Venice. She prefers the package to Havana to the other two, which she regards as equivalent. Her preferences between the three packages are represented by any payoff function that assigns the same number to both Paris and Venice and a higher number to Havana. For example, we can set $u(\text{Havana}) = 1$ and $u(\text{Paris}) = u(\text{Venice}) = 0$, or $u(\text{Havana}) = 10$ and $u(\text{Paris}) = u(\text{Venice}) = 1$, or $u(\text{Havana}) = 0$ and $u(\text{Paris}) = u(\text{Venice}) = -2$.

? EXERCISE 5.3 (Altruistic preferences) Person 1 cares both about her income and about person 2's income. Precisely, the value she attaches to each unit of her own income is the same as the value she attaches to any two units of person 2's income. How do her preferences order the outcomes $(1, 4)$, $(2, 1)$, and $(3, 0)$, where the first component in each case is person 1's income and the second component is person 2's income? Give a payoff function consistent with these preferences.

A decision-maker's preferences, in the sense used here, convey only *ordinal* information. They may tell us that the decision-maker prefers the action a to the action b to the action c, for example, but they do not tell us "how much" she prefers a to b, or whether she prefers a to b "more" than she prefers b to c. Consequently a payoff function that represents a decision-maker's preferences also conveys only ordinal information. It may be tempting to think that the payoff numbers attached to actions by a payoff function convey intensity of preference—that if, for example, a decision-maker's preferences are represented by a payoff function u for which $u(a) = 0$, $u(b) = 1$, and $u(c) = 100$, then the decision-maker likes c a lot more than b but finds little difference between a and b. *But a payoff function contains no such information!* The *only* conclusion we can draw from the fact that $u(a) = 0$, $u(b) = 1$, and $u(c) = 100$ is that the decision-maker prefers c to b to a; her preferences are represented equally well by the payoff function v for which $v(a) = 0$, $v(b) = 100$, and $v(c) = 101$, for example, or any other function w for which $w(a) < w(b) < w(c)$.

From this discussion we see that a decision-maker's preferences are represented by many different payoff functions. Looking at the condition (5.1) under which the payoff function u represents a decision-maker's preferences, we see that if u represents a decision-maker's preferences and the payoff function v assigns a higher number to the action a than to the action b if and only if the payoff function u does so, then v also represents these preferences. Stated more compactly, if u represents a decision-maker's preferences and v is another payoff function for which

$$v(a) > v(b) \text{ if and only if } u(a) > u(b)$$

then v also represents the decision-maker's preferences. Or, more succinctly, if u represents a decision-maker's preferences then any increasing function of u also represents these preferences.

? EXERCISE 6.1 (Alternative representations of preferences) A decision-maker's preferences over the set $A = \{a, b, c\}$ are represented by the payoff function u for which $u(a) = 0$, $u(b) = 1$, and $u(c) = 4$. Are they also represented by the function v for which $v(a) = -1$, $v(b) = 0$, and $v(c) = 2$? How about the function w for which $w(a) = w(b) = 0$ and $w(c) = 8$?

Sometimes it is natural to formulate a model in terms of preferences and then find payoff functions that represent these preferences. In other cases it is natural to start with payoff functions, even if the analysis depends only on the underlying preferences, not on the specific representation we choose.

1.2.3 The theory of rational choice

The theory of rational choice is that in any given situation the decision-maker chooses the member of the available subset of A that is best according to her preferences. Allowing for the possibility that there are several equally attractive best actions, **the theory of rational choice** is:

> *the action chosen by a decision-maker is at least as good, according to her preferences, as every other available action.*

For any action, we can design preferences with the property that no other action is preferred. Thus if we have no information about a decision-maker's preferences, and make no assumptions about their character, any *single* action is consistent with the theory. However, if we assume that a decision-maker who is indifferent between two actions sometimes chooses one action and sometimes the other, not every *collection* of choices for different sets of available actions is consistent with the theory. Suppose, for example, we observe that a decision-maker chooses a whenever she faces the set $\{a, b\}$, but sometimes chooses b when facing the set $\{a, b, c\}$. The fact that she always chooses a when faced with $\{a, b\}$ means that she prefers a to b (if she were indifferent then she would sometimes choose b). But then when she faces the set $\{a, b, c\}$ she must choose either a or c, never b. Thus her choices are inconsistent with the theory. (More concretely, if you choose the same dish from the menu of your favorite lunch spot whenever there are no specials then, regardless of your preferences, it is inconsistent for you to choose some other item *from the menu* on a day when there is an off-menu special.)

If you have studied the standard economic theories of the consumer and the firm, you have encountered the theory of rational choice before. In the economic theory of the consumer, for example, the set of available actions is the set of all bundles of goods that the consumer can afford. In the theory of the firm, the set of available actions is the set of all input-output vectors, and the action a is preferred to the action b if and only if a yields a higher profit than does b.

1.2.4 Discussion

The theory of rational choice is enormously successful; it is a component of countless models that enhance our understanding of social phenomena. It pervades economic theory to such an extent that arguments are classified as "economic" as much because they apply the theory of rational choice as because they involve particularly "economic" variables.

Nevertheless, under some circumstances its implications are at variance with observations of human decision-making. To take a small example, adding an undesirable action to a set of actions sometimes significantly changes the action chosen (see Rabin 1998, 38). The significance of such discordance with the theory depends upon the phenomenon being studied. If we are considering how the markup of price over cost in an industry depends on the number of firms, for example, this sort of weakness in the theory may be unimportant. But if we are studying how advertising, designed specifically to influence peoples' preferences, affects consumers' choices, then the inadequacies of the model of rational choice may be crucial.

No general theory currently challenges the supremacy of rational choice theory. But you should bear in mind as you read this book that the model of choice that underlies most of the theories has its limits; some of the phenomena that you may think of explaining using a game theoretic model may lie beyond these limits. As always, the proof of the pudding is in the eating: if a model enhances our understanding of the world, then it serves its purpose.

1.3 Coming attractions

Part I presents the main models in game theory: a strategic game, an extensive game, and a coalitional game. These models differ in two dimensions. A strategic game and an extensive game focus on the actions of individuals, whereas a coalitional game focuses on the outcomes that can be achieved by groups of individuals; a strategic game and a coalitional game consider situations in which actions are chosen once and for all, whereas an extensive game allows for the possibility that plans may be revised as they are carried out.

The model, consisting of actions and preferences, to which rational choice theory is applied is tailor-made for the theory; if we want to develop another theory, we need to add elements to the model in addition to actions and preferences. The same is not true of most models in game theory: strategic interaction is sufficiently complex that even a relatively simple model can admit more than one theory of the outcome. We refer to a theory that specifies a set of outcomes for a model as a "solution". Chapter 2 describes the model of a strategic game and the solution of Nash equilibrium for such games. The theory of Nash equilibrium in a strategic game has been applied to a vast variety of situations; a handful of some of the most significant applications are discussed in Chapter 3.

Chapter 4 extends the notion of Nash equilibrium in a strategic game to allow for the possibility that a decision-maker, when indifferent between actions, may not always choose the same action, or, alternatively, identical decision-makers facing the same set of actions may choose different actions if more than one is best.

The model of an extensive game, which adds a temporal dimension to the description of strategic interaction captured by a strategic game, is studied in Chapters 5, 6, and 7. Part I concludes with Chapter 8, which discusses the model of a coalitional game and a solution concept for such a game, the core.

Part II extends the models of a strategic game and an extensive game to situations in which the players do not know the other players' characteristics or past actions. Chapter 9 extends the model of a strategic game, and Chapter 10 extends the model of an extensive game.

The chapters in Part III cover topics outside the basic theory. Chapters 11 and 12 examine two theories of the outcome in a strategic game that are alternatives to the theory of Nash equilibrium. Chapter 13 discusses how a variant of the notion of Nash equilibrium in a strategic game can be used to model behavior that is the outcome of evolutionary pressure rather than conscious choice. Chapters 14 and 15 use the model of an extensive game to study long-term relationships, in which the same group of players repeatedly interact. Finally, Chapter 16 uses strategic, extensive, and coalitional models to gain an understanding of the outcome of bargaining.

Notes

Von Neumann and Morgenstern (1944) established game theory as a field. The information about John von Neumann in the box on page 3 is drawn from Ulam (1958), Halmos (1973), Thompson (1987), Poundstone (1992), and Leonard (1995). Aumann (1985), on which I draw in the opening section, contains a very readable discussion of the aims and achievements of game theory. Two papers that discuss the limitations of rational choice theory are Rabin (1998) and Elster (1998).

2 Nash Equilibrium: Theory

Prerequisite: Chapter 1.

2.1 Strategic games

A STRATEGIC GAME is a model of interacting decision-makers. In recognition of the interaction, we refer to the decision-makers as *players*. Each player has a set of possible *actions*. The model captures interaction between the players by allowing each player to be affected by the actions of *all* players, not only her own action. Specifically, each player has *preferences* about the action *profile*—the list of all the players' actions. (See Section 17.5, in the mathematical appendix, for a discussion of profiles.)

More precisely, a strategic game is defined as follows. (The qualification "with ordinal preferences" distinguishes this notion of a strategic game from a more general notion studied in Chapter 4.)

▶ DEFINITION 11.1 (*Strategic game with ordinal preferences*) A **strategic game** (with ordinal preferences) consists of

- a set of **players**
- for each player, a set of **actions**
- for each player, **preferences** over the set of action profiles.

A very wide range of situations may be modeled as strategic games. For example, the players may be firms, the actions prices, and the preferences a reflection of the firms' profits. Or the players may be candidates for political office, the actions campaign expenditures, and the preferences a reflection of the candidates' probabilities of winning. Or the players may be animals fighting over some prey, the actions concession times, and the preferences a reflection of whether an animal wins or loses. In this chapter I describe some simple games designed to capture fundamental conflicts present in a variety of situations. The next chapter is devoted to more detailed applications to specific phenomena.

As in the model of rational choice by a single decision-maker (Section 1.2), it is frequently convenient to specify the players' preferences by giving *payoff functions* that represent them. Bear in mind that these payoffs have only *ordinal* significance. If a player's payoffs to the action profiles a, b, and c are 1, 2, and 10, for example, the only conclusion we can draw is that the player prefers c to b and b to a; the numbers do *not* imply that the player's preference between c and b is stronger than her preference between a and b.

Time is absent from the model. The idea is that each player chooses her action once and for all, and the players choose their actions "simultaneously" in the sense that no player is informed, when she chooses her action, of the action chosen by any other player. (For this reason, a strategic game is sometimes referred to as a "simultaneous move game".) Nevertheless, an action may involve activities that extend over time, and may take into account an unlimited number of contingencies. An action might specify, for example, "if company X's stock falls below \$10, buy 100 shares; otherwise, do not buy any shares". (For this reason, an action is sometimes called a "strategy".) However, the fact that time is absent from the model means that when analyzing a situation as a strategic game, we abstract from the complications that may arise if a player is allowed to change her plan as events unfold: we assume that actions are chosen once and for all.

2.2 Example: the Prisoner's Dilemma

One of the most well-known strategic games is the *Prisoner's Dilemma*. Its name comes from a story involving suspects in a crime; its importance comes from the huge variety of situations in which the participants face incentives similar to those faced by the suspects in the story.

◆ EXAMPLE 12.1 (Prisoner's Dilemma) Two suspects in a major crime are held in separate cells. There is enough evidence to convict each of them of a minor offense, but not enough evidence to convict either of them of the major crime unless one of them acts as an informer against the other (finks). If they both stay quiet, each will be convicted of the minor offense and spend one year in prison. If one and only one of them finks, she will be freed and used as a witness against the other, who will spend four years in prison. If they both fink, each will spend three years in prison.

This situation may be modeled as a strategic game:

Players The two suspects.

Actions Each player's set of actions is $\{Quiet, Fink\}$.

Preferences Suspect 1's ordering of the action profiles, from best to worst, is $(Fink, Quiet)$ (she finks and suspect 2 remains quiet, so she is freed), $(Quiet, Quiet)$ (she gets one year in prison), $(Fink, Fink)$ (she gets three years in prison), $(Quiet, Fink)$ (she gets four years in prison). Suspect 2's ordering is $(Quiet, Fink)$, $(Quiet, Quiet)$, $(Fink, Fink)$, $(Fink, Quiet)$.

We can represent the game compactly in a table. First choose payoff functions that represent the suspects' preference orderings. For suspect 1 we need a function u_1 for which

$$u_1(Fink, Quiet) > u_1(Quiet, Quiet) > u_1(Fink, Fink) > u_1(Quiet, Fink).$$

A simple specification is $u_1(Fink, Quiet) = 3$, $u_1(Quiet, Quiet) = 2$, $u_1(Fink, Fink) = 1$, and $u_1(Quiet, Fink) = 0$. For suspect 2 we can similarly choose the function u_2 for which $u_2(Quiet, Fink) = 3$, $u_2(Quiet, Quiet) = 2$, $u_2(Fink, Fink) = 1$, and $u_2(Fink, Quiet) = 0$. Using these representations, the game is illustrated in Figure 13.1. In this figure the two rows correspond to the two possible actions of player 1, the two columns correspond to the two possible actions of player 2, and the numbers in each box are the players' payoffs to the action profile to which the box corresponds, with player 1's payoff listed first.

		Suspect 2	
		Quiet	*Fink*
Suspect 1	*Quiet*	2, 2	0, 3
	Fink	3, 0	1, 1

Figure 13.1 The *Prisoner's Dilemma* (Example 12.1).

The *Prisoner's Dilemma* models a situation in which there are gains from cooperation (each player prefers that both players choose *Quiet* than they both choose *Fink*) but each player has an incentive to "free ride" (choose *Fink*) whatever the other player does. The game is important not because we are interested in understanding the incentives for prisoners to confess, but because many other situations have similar structures. Whenever each of two players has two actions, say C (corresponding to *Quiet*) and D (corresponding to *Fink*), player 1 prefers (D, C) to (C, C) to (D, D) to (C, D), and player 2 prefers (C, D) to (C, C) to (D, D) to (D, C), the *Prisoner's Dilemma* models the situation that the players face. Some examples follow.

2.2.1 *Working on a joint project*

You are working with a friend on a joint project. Each of you can either work hard or goof off. If your friend works hard then you prefer to goof off (the outcome of the project would be better if you worked hard too, but the increment in its value to you is not worth the extra effort). You prefer the outcome of your both working hard to the outcome of your both goofing off (in which case nothing gets accomplished), and the worst outcome for you is that you work hard and your friend goofs off (you hate to be "exploited"). If your friend has the same preferences then the game that models the situation you face is given in Figure 14.1, which, as you can see, differs from the *Prisoner's Dilemma* only in the names of the actions.

	Work hard	*Goof off*
Work hard	2, 2	0, 3
Goof off	3, 0	1, 1

Figure 14.1 Working on a joint project.

I am *not* claiming that a situation in which two people pursue a joint project *necessarily* has the structure of the *Prisoner's Dilemma*, only that the players' preferences in such a situation *may* be the same as in the *Prisoner's Dilemma*! If, for example, each person prefers to work hard than to goof off when the other person works hard, then the *Prisoner's Dilemma* does *not* model the situation: the players' preferences are different from those given in Figure 14.1.

? EXERCISE 14.1 (Working on a joint project) Formulate a strategic game that models a situation in which two people work on a joint project in the case that their preferences are the same as those in the game in Figure 14.1 except that each person prefers to work hard than to goof off when the other person works hard. Present your game in a table like the one in Figure 14.1.

2.2.2 Duopoly

In a simple model of a duopoly, two firms produce the same good, for which each firm charges either a low price or a high price. Each firm wants to achieve the highest possible profit. If both firms choose *High* then each earns a profit of \$1000. If one firm chooses *High* and the other chooses *Low* then the firm choosing *High* obtains no customers and makes a loss of \$200, whereas the firm choosing *Low* earns a profit of \$1200 (its unit profit is low, but its volume is high). If both firms choose *Low* then each earns a profit of \$600. Each firm cares only about its profit, so we can represent its preferences by the profit it obtains, yielding the game in Figure 14.2.

	High	*Low*
High	1000, 1000	−200, 1200
Low	1200, −200	600, 600

Figure 14.2 A simple model of a price-setting duopoly.

Bearing in mind that what matters are the players' preferences, not the particular payoff functions that we use to represent them, we see that this game, like the previous one, differs from the *Prisoner's Dilemma* only in the names of the actions.

The action *High* plays the role of *Quiet*, and the action *Low* plays the role of *Fink*; firm 1 prefers (*Low*, *High*) to (*High*, *High*) to (*Low*, *Low*) to (*High*, *Low*), and firm 2 prefers (*High*, *Low*) to (*High*, *High*) to (*Low*, *Low*) to (*Low*, *High*).

As in the previous example, I do not claim that the incentives in a duopoly are *necessarily* those in the *Prisoner's Dilemma*; different assumptions about the relative sizes of the profits in the four cases generate a different game. Further, in this case one of the abstractions incorporated into the model—that each firm has only two prices to choose between—may not be harmless; if the firms may choose among many prices then the structure of the interaction may change. (A richer model is studied in Section 3.2.)

2.2.3 The arms race

Under some assumptions about the countries' preferences, an arms race can be modeled as the *Prisoner's Dilemma*. (Because the *Prisoner's Dilemma* was first studied in the early 1950s, when the USA and USSR were involved in a nuclear arms race, you might suspect that US nuclear strategy was influenced by game theory; the evidence suggests that it was not.) Assume that each country can build an arsenal of nuclear bombs, or can refrain from doing so. Assume also that each country's favorite outcome is that it has bombs and the other country does not; the next best outcome is that neither country has any bombs; the next best outcome is that both countries have bombs (what matters is relative strength, and bombs are costly to build); and the worst outcome is that only the other country has bombs. In this case the situation is modeled by the *Prisoner's Dilemma*, in which the action *Don't build bombs* corresponds to *Quiet* in Figure 13.1 and the action *Build bombs* corresponds to *Fink*. However, once again the assumptions about preferences necessary for the *Prisoner's Dilemma* to model the situation may not be satisfied: a country may prefer *not* to build bombs if the other country does not, for example (bomb-building may be very costly), in which case the situation is modeled by a different game.

2.2.4 Common property

Two farmers are deciding how much to allow their sheep to graze on the village common. Each farmer prefers that her sheep graze a lot than a little, regardless of the other farmer's action, but prefers that both farmers' sheep graze a little than both farmers' sheep graze a lot (in which case the common is ruined for future use). Under these assumptions the game is the *Prisoner's Dilemma*. (A richer model is studied in Section 3.1.5.)

2.2.5 Other situations modeled as the Prisoner's Dilemma

A huge number of other situations have been modeled as the *Prisoner's Dilemma*, from mating hermaphroditic fish to tariff wars between countries.

? EXERCISE 16.1 (Hermaphroditic fish) Members of some species of hermaphroditic fish choose, in each mating encounter, whether to play the role of a male or a female. Each fish has a preferred role, which uses up fewer resources and hence allows more future mating. A fish obtains a payoff of H if it mates in its preferred role and L if it mates in the other role, where $H > L$. (Payoffs are measured in terms of number of offspring, which fish are evolved to maximize.) Consider an encounter between two fish whose preferred roles are the same. Each fish has two possible actions: mate in either role, and insist on its preferred role. If both fish offer to mate in either role, the roles are assigned randomly, and each fish's payoff is $\frac{1}{2}(H + L)$ (the average of H and L). If each fish insists on its preferred role, the fish do not mate; each goes off in search of another partner, and obtains the payoff S. The higher the chance of meeting another partner, the larger is S. Formulate this situation as a strategic game and determine the range of values of S, for any given values of H and L, for which the game differs from the *Prisoner's Dilemma* only in the names of the actions.

2.3 Example: Bach or Stravinsky?

In the *Prisoner's Dilemma* the main issue is whether or not the players will cooperate (choose *Quiet*). In the following game the players agree that it is better to cooperate than not to cooperate, but disagree about the best outcome.

◆ EXAMPLE 16.2 (Bach or Stravinsky?) Two people wish to go out together. Two concerts are available: one of music by Bach, and one of music by Stravinsky. One person prefers Bach and the other prefers Stravinsky. If they go to different concerts, each of them is equally unhappy listening to the music of either composer.

We can model this situation as the two-player strategic game in Figure 16.1, in which the person who prefers Bach chooses a row and the person who prefers Stravinsky chooses a column.

	Bach	*Stravinsky*
Bach	2, 1	0, 0
Stravinsky	0, 0	1, 2

Figure 16.1 *Bach or Stravinsky?* (*BoS*) (Example 16.2).

This game is also referred to as the "Battle of the Sexes" (though the conflict it models surely occurs no more frequently between people of the opposite sex than it does between people of the same sex). I refer to the games as *BoS*, an acronym that fits both names. (I assume that each player is indifferent between listening to Bach and listening to Stravinsky when she is alone only for consistency with the standard specification of the game. As we shall see, the analysis of the game remains the same in the absence of this assumption.)

Like the *Prisoner's Dilemma*, *BoS* models a wide variety of situations. Consider, for example, two officials of a political party deciding the stand to take on an issue.

Suppose that they disagree about the best stand, but are both better off if they take the same stand than if they take different stands; both cases in which they take different stands, in which case voters do not know what to think, are equally bad. Then *BoS* captures the situation they face. Or consider two merging firms that currently use different computer technologies. As two divisions of a single firm they will both be better off if they both use the same technology; each firm prefers that the common technology be the one it used in the past. *BoS* models the choices the firms face.

2.4 Example: Matching Pennies

Aspects of both conflict and cooperation are present in both the *Prisoner's Dilemma* and *BoS*. The next game is purely conflictual.

◆ EXAMPLE 17.1 (Matching Pennies) Two people choose, simultaneously, whether to show the Head or the Tail of a coin. If they show the same side, person 2 pays person 1 a dollar; if they show different sides, person 1 pays person 2 a dollar. Each person cares only about the amount of money she receives, and (naturally!) prefers to receive more than less. A strategic game that models this situation is shown in Figure 17.1. (In this representation of the players' preferences, the payoffs are equal to the amounts of money involved. We could equally well work with another representation—for example, 2 could replace each 1, and 1 could replace each -1.)

	Head	*Tail*
Head	$1, -1$	$-1, 1$
Tail	$-1, 1$	$1, -1$

Figure 17.1 *Matching Pennies* (Example 17.1).

In this game the players' interests are diametrically opposed (such a game is called "strictly competitive"): player 1 wants to take the same action as the other player, whereas player 2 wants to take the opposite action.

This game may, for example, model the choices of appearances for new products by an established producer and a new firm in a market of fixed size. Suppose that each firm can choose one of two different appearances for the product. The established producer prefers the newcomer's product to look different from its own (so that its customers will not be tempted to buy the newcomer's product), whereas the newcomer prefers that the products look alike. Or the game could model a relationship between two people in which one person wants to be like the other, whereas the other wants to be different.

? EXERCISE 17.2 (Games without conflict) Give some examples of two-player strategic games in which each player has two actions and the players have the same preferences, so that there is no conflict between their interests. (Present your games as tables like the one in Figure 17.1.)

2.5 Example: the Stag Hunt

A sentence in *Discourse on the origin and foundations of inequality among men* (1755) by the philosopher Jean-Jacques Rousseau discusses a group of hunters who wish to catch a stag. They will succeed if they all remain sufficiently attentive, but each is tempted to desert her post and catch a hare. One interpretation of the sentence is that the interaction between the hunters may be modeled as the following strategic game.

◆ EXAMPLE 18.1 (Stag Hunt) Each of a group of hunters has two options: she may remain attentive to the pursuit of a stag, or catch a hare. If all hunters pursue the stag, they catch it and share it equally; if any hunter devotes her energy to catching a hare, the stag escapes, and the hare belongs to the defecting hunter alone. Each hunter prefers a share of the stag to a hare.

The strategic game that corresponds to this specification is:

Players The hunters.

Actions Each player's set of actions is $\{Stag, Hare\}$.

Preferences For each player, the action profile in which all players choose *Stag* (resulting in her obtaining a share of the stag) is ranked highest, followed by any profile in which she chooses *Hare* (resulting in her obtaining a hare), followed by any profile in which she chooses *Stag* and one or more of the other players chooses *Hare* (resulting in her leaving empty-handed).

Like other games with many players, this game cannot easily be presented in a table like that in Figure 17.1. For the case in which there are two hunters, the game is shown in Figure 18.1.

	Stag	*Hare*
Stag	2, 2	0, 1
Hare	1, 0	1, 1

Figure 18.1 The *Stag Hunt* (Example 18.1) for the case of two hunters.

The variant of the two-player *Stag Hunt* shown in Figure 19.1 has been suggested as an alternative to the *Prisoner's Dilemma* as a model of an arms race, or, more generally, of the "security dilemma" faced by a pair of countries. The game differs from the *Prisoner's Dilemma* in that a country prefers the outcome in which both countries refrain from arming themselves to the one in which it alone arms itself: the cost of arming outweighs the benefit if the other country does not arm itself.

	Refrain	*Arm*
Refrain	3, 3	0, 2
Arm	2, 0	1, 1

Figure 19.1 A variant of the two-player *Stag Hunt* that models the "security dilemma".

2.6 Nash equilibrium

What actions will be chosen by the players in a strategic game? We wish to assume, as in the theory of a rational decision-maker (Section 1.2), that each player chooses the best available action. In a game, the best action for any given player depends, in general, on the other players' actions. So when choosing an action a player must have in mind the actions the other players will choose. That is, she must form a *belief* about the other players' actions.

On what basis can such a belief be formed? The assumption underlying the analysis in this chapter and the next two chapters is that each player's belief is derived from her past experience playing the game, and that this experience is sufficiently extensive that she *knows* how her opponents will behave. No one tells her the actions her opponents will choose, but her previous involvement in the game leads her to be sure of these actions. (The question of *how* a player's experience can lead her to the correct beliefs about the other players' actions is addressed briefly in Section 4.9.)

Although we assume that each player has experience playing the game, we assume that she views each play of the game in isolation. She does not become familiar with the behavior of specific opponents and consequently does not condition her action on the opponent she faces; nor does she expect her current action to affect the other players' future behavior.

It is helpful to think of the following idealized circumstances. For each player in the game there is a population of many decision-makers who may, on any occasion, take that player's role. In each play of the game, players are selected randomly, one from each population. Thus each player engages in the game repeatedly, against ever-varying opponents. Her experience leads her to beliefs about the actions of "typical" opponents, not any specific set of opponents.

As an example, think of the interaction between buyers and sellers. Buyers and sellers repeatedly interact, but to a first approximation many of the pairings may be modeled as random. In many cases a buyer transacts only once with any given seller, or interacts repeatedly but anonymously (when the seller is a large store, for example).

In summary, the solution theory we study has two components. First, each player chooses her action according to the model of rational choice, given her belief about the other players' actions. Second, every player's belief about the other players' actions is correct. These two components are embodied in the following definition.

JOHN F. NASH, JR.

A few of the ideas of John F. Nash Jr., developed while he was a graduate student at Princeton from 1948 to 1950, transformed game theory. Nash was born in 1928 in Bluefield, West Virginia, USA, where he grew up. He was an undergraduate mathematics major at Carnegie Institute of Technology from 1945 to 1948. In 1948 he obtained both a B.S. and an M.S., and began graduate work in the Department of Mathematics at Princeton University. (One of his letters of recommendation, from a professor at Carnegie Institute of Technology, was a single sentence: "This man is a genius" (Kuhn et al. 1995, 282).) A paper containing the main result of his thesis was submitted to the *Proceedings of the National Academy of Sciences* in November 1949, fourteen months after he started his graduate work. ("A fine goal to set ... graduate students", to quote Kuhn! (See Kuhn et al. 1995, 282.)) He completed his PhD the following year, graduating on his 22nd birthday. His thesis, 28 pages in length, introduces the equilibrium notion now known as "Nash equilibrium" and delineates a class of strategic games that have Nash equilibria (Proposition 116.1 in this book). The notion of Nash equilibrium vastly expanded the scope of game theory, which had previously focussed on two-player "strictly competitive" games (in which the players' interests are directly opposed). While a graduate student at Princeton, Nash also wrote the seminal paper in bargaining theory, Nash (1950b) (the ideas of which originated in an elective class in international economics he took as an undergraduate). He went on to take an academic position in the Department of Mathematics at MIT, where he produced "a remarkable series of papers" (Milnor 1995, 15); he has been described as "one of the most original mathematical minds of [the twentieth] century" (Kuhn 1996). He shared the 1994 Nobel prize in economics with the game theorists John C. Harsanyi and Reinhard Selten.

> A *Nash equilibrium* is an action profile a^* with the property that no player i can do better by choosing an action different from a_i^*, given that every other player j adheres to a_j^*.

In the idealized setting in which the players in any given play of the game are drawn randomly from a collection of populations, a Nash equilibrium corresponds to a *steady state*. If, whenever the game is played, the action profile is the same Nash equilibrium a^*, then no player has a reason to choose any action different from her component of a^*; there is no pressure on the action profile to change. Expressed differently, a Nash equilibrium embodies a stable "social norm": if everyone else adheres to it, no individual wishes to deviate from it.

The second component of the theory of Nash equilibrium—that the players' beliefs about each other's actions are correct—implies, in particular, that two players' beliefs about a third player's action are the same. For this reason, the condition is sometimes said to be that the players' "expectations are coordinated".

The situations to which we wish to apply the theory of Nash equilibrium do not in general correspond exactly to the idealized setting described above. For example, in some cases the players do not have much experience with the game; in others they do not view each play of the game in isolation. Whether or not the notion of Nash equilibrium is appropriate in any given situation is a matter of judgment. In some cases, a poor fit with the idealized setting may be mitigated by other considerations. For example, inexperienced players may be able to draw conclusions about their opponents' likely actions from their experience in other situations, or from other sources. (One aspect of such reasoning is discussed in the box on page 30). Ultimately, the test of the appropriateness of the notion of Nash equilibrium is whether it gives us insights into the problem at hand.

With the aid of an additional piece of notation, we can state the definition of a Nash equilibrium precisely. Let a be an action profile, in which the action of each player i is a_i. Let a'_i be any action of player i (either equal to a_i, or different from it). Then (a'_i, a_{-i}) denotes the action profile in which every player j *except* i chooses her action a_j as specified by a, whereas player i chooses a'_i. (The $-i$ subscript on a stands for "except i".) That is, (a'_i, a_{-i}) is the action profile in which all the players other than i adhere to a while i "deviates" to a'_i. (If $a'_i = a_i$ then of course $(a'_i, a_{-i}) = (a_i, a_{-i}) = a$.) If there are three players, for example, then (a'_2, a_{-2}) is the action profile in which players 1 and 3 adhere to a (player 1 chooses a_1, player 3 chooses a_3) and player 2 deviates to a'_2.

Using this notation, we can restate the condition for an action profile a^* to be a Nash equilibrium: no player i has any action a_i for which she prefers (a_i, a^*_{-i}) to a^*. Equivalently, for every player i and every action a_i of player i, the action profile a^* is at least as good for player i as the action profile (a_i, a^*_{-i}).

▸ DEFINITION 21.1 (*Nash equilibrium of strategic game with ordinal preferences*) The action profile a^* in a strategic game with ordinal preferences is a **Nash equilibrium** if, for every player i and every action a_i of player i, a^* is at least as good according to player i's preferences as the action profile (a_i, a^*_{-i}) in which player i chooses a_i while every other player j chooses a^*_j. Equivalently, for every player i,

$$u_i(a^*) \geq u_i(a_i, a^*_{-i}) \text{ for every action } a_i \text{ of player } i, \qquad (21.2)$$

where u_i is a payoff function that represents player i's preferences.

This definition implies neither that a strategic game necessarily has a Nash equilibrium, nor that it has at most one. Examples in the next section show that some games have a single Nash equilibrium, some possess no Nash equilibrium, and others have many Nash equilibria.

The definition of a Nash equilibrium is designed to model a steady state among experienced players. An alternative approach to understanding players' actions in strategic games assumes that the players know each others' preferences, and considers what each player can deduce about the other players' actions from their rationality and their knowledge of each other's rationality. This approach is studied in Chapter 12. For many games, it leads to a conclusion different from that of

Nash equilibrium. For games in which the conclusion is the same the approach offers us an alternative interpretation of a Nash equilibrium, as the outcome of rational calculations by players who do not necessarily have any experience playing the game.

STUDYING NASH EQUILIBRIUM EXPERIMENTALLY

The theory of strategic games lends itself to experimental study: arranging for subjects to play games and observing their choices is relatively straightforward. A few years after game theory was launched by von Neumann and Morgenstern's (1944) book, reports of laboratory experiments began to appear. Subsequently a huge number of experiments have been conducted, illuminating many issues relevant to the theory. I discuss selected experimental evidence throughout the book.

The theory of Nash equilibrium, as we have seen, has two components: the players act in accordance with the theory of rational choice, given their beliefs about the other players' actions, and these beliefs are correct. If every subject understands the game she is playing and faces incentives that correspond to the preferences of the player whose role she is taking, then a divergence between the observed outcome and a Nash equilibrium can be blamed on a failure of one or both of these two components. Experimental evidence has the potential of indicating the types of games for which the theory works well and, for those in which the theory does not work well, of pointing to the faulty component and giving us hints about the characteristics of a better theory. In designing an experiment that cleanly tests the theory, however, we need to confront several issues.

The model of rational choice takes preferences as given. Thus to test the theory of Nash equilibrium experimentally, we need to ensure that each subject's preferences are those of the player whose role she is taking in the game we are examining. The standard way of inducing the appropriate preferences is to pay each subject an amount of money directly related to the payoff given by a payoff function that represents the preferences of the player whose role the subject is taking. Such remuneration works if each subject likes money and cares only about the amount of money she receives, ignoring the amounts received by her opponents. The assumption that people like receiving money is reasonable in many cultures, but the assumption that people care only about their own monetary rewards—are "selfish"—may, in some contexts at least, not be reasonable. Unless we check whether our subjects are selfish in the context of our experiment, we will jointly test two hypotheses: that humans are selfish—a hypothesis not part of game theory—and that the notion of Nash equilibrium models their behavior. In some cases we may indeed wish to test these hypotheses jointly. But in order to test the theory of Nash equilibrium alone we need to ensure that we induce the preferences we wish to study.

Assuming that better decisions require more effort, we need also to ensure that each subject finds it worthwhile to put in the extra effort required to obtain a higher payoff. If we rely on monetary payments to provide incentives, the amount of money a subject can obtain must be sufficiently sensitive to the quality of her decisions to compensate her for the effort she expends (paying a flat fee, for example, is inappropriate). In some cases, monetary payments may not be necessary: under some circumstances, subjects drawn from a highly competitive culture like that of the USA may be sufficiently motivated by the possibility of obtaining a high score, even if that score does not translate into a monetary payoff.

The notion of Nash equilibrium models action profiles compatible with steady states. Thus to study the theory experimentally we need to collect observations of subjects' behavior when they have experience playing the game. But they should not have obtained that experience while knowingly facing the same opponents repeatedly, for the theory assumes that the players consider each play of the game in isolation, not as part of an ongoing relationship. One option is to have each subject play the game against many different opponents, gaining experience about how the other subjects on average play the game, but not about the choices of any other given player. Another option is to describe the game in terms that relate to a situation in which the subjects already have experience. A difficulty with this second approach is that the description we give may connote more than simply the payoff numbers of our game. If we describe the *Prisoner's Dilemma* in terms of cooperation on a joint project, for example, a subject may be biased toward choosing the action she has found appropriate when involved in joint projects, even if the structures of those interactions were significantly different from that of the *Prisoner's Dilemma*. As she plays the experimental game repeatedly she may come to appreciate how it differs from the games in which she has been involved previously, but her biases may disappear only slowly.

Whatever route we take to collect data on the choices of subjects experienced in playing the game, we confront a difficult issue: how do we know when the outcome has converged? Nash's theory concerns only equilibria; it has nothing to say about the path players' choices will take on the way to an equilibrium, and so gives us no guide as to whether 10, 100, or 1,000 plays of the game are enough to give a chance for the subjects' expectations to become coordinated.

Finally, we can expect the theory of Nash equilibrium to correspond to reality only approximately: like all useful theories, it definitely is not *exactly* correct. How do we tell whether the data are close enough to the theory to support it? One possibility is to compare the theory of Nash equilibrium with some other theory. But for many games there is no obvious alternative theory—and certainly not one with the generality of Nash equilibrium. Statistical tests can sometimes aid in deciding whether the data is consistent with the theory, though ultimately we remain the judge of whether or not our observations persuade us that the theory enhances our understanding of human behavior in the game.

2.7 Examples of Nash equilibrium

2.7.1 Prisoner's Dilemma

By examining the four possible pairs of actions in the *Prisoner's Dilemma* (reproduced in Figure 24.1), we see that (*Fink*, *Fink*) is the unique Nash equilibrium.

	Quiet	*Fink*
Quiet	2, 2	0, 3
Fink	3, 0	1, 1

Figure 24.1 The *Prisoner's Dilemma*.

The action pair (*Fink*, *Fink*) is a Nash equilibrium because (*i*) given that player 2 chooses *Fink*, player 1 is better off choosing *Fink* than *Quiet* (looking at the right column of the table we see that *Fink* yields player 1 a payoff of 1 whereas *Quiet* yields her a payoff of 0), and (*ii*) given that player 1 chooses *Fink*, player 2 is better off choosing *Fink* than *Quiet* (looking at the bottom row of the table we see that *Fink* yields player 2 a payoff of 1 whereas *Quiet* yields her a payoff of 0).

No other action profile is a Nash equilibrium:

- (*Quiet*, *Quiet*) does not satisfy (21.2) because when player 2 chooses *Quiet*, player 1's payoff to *Fink* exceeds her payoff to *Quiet* (look at the first components of the entries in the left column of the table). (Further, when player 1 chooses *Quiet*, player 2's payoff to *Fink* exceeds her payoff to *Quiet*: player 2, as well as player 1, wants to deviate. To show that a pair of actions is not a Nash equilibrium, however, it is not necessary to study player 2's decision once we have established that player 1 wants to deviate: it is enough to show that *one* player wishes to deviate to show that a pair of actions is not a Nash equilibrium.)

- (*Fink*, *Quiet*) does not satisfy (21.2) because when player 1 chooses *Fink*, player 2's payoff to *Fink* exceeds her payoff to *Quiet* (look at the second components of the entries in the bottom row of the table).

- (*Quiet*, *Fink*) does not satisfy (21.2) because when player 2 chooses *Fink*, player 1's payoff to *Fink* exceeds her payoff to *Quiet* (look at the first components of the entries in the right column of the table).

In summary, in the only Nash equilibrium of the *Prisoner's Dilemma* both players choose *Fink*. In particular, the incentive to free ride eliminates the possibility that the mutually desirable outcome (*Quiet*, *Quiet*) occurs. In the other situations discussed in Section 2.2 that may be modeled as the *Prisoner's Dilemma*, the outcomes predicted by the notion of Nash equilibrium are thus as follows: both people goof off when working on a joint project; both duopolists charge a low price; both countries build bombs; both farmers graze their sheep a lot. (The overgrazing of a common thus predicted is sometimes called the "tragedy of the commons". The intuition that some of these dismal outcomes may be avoided if the same pair of people play the game repeatedly is explored in Chapter 14.)

In the *Prisoner's Dilemma*, the Nash equilibrium action of each player (*Fink*) is the best action for each player not only if the other player chooses her equilibrium action (*Fink*), but also if she chooses her other action (*Quiet*). The action pair $(Fink, Fink)$ is a Nash equilibrium because if a player believes that her opponent will choose *Fink* then it is optimal for her to choose *Fink*. But in fact it is optimal for a player to choose *Fink regardless* of the action she expects her opponent to choose. In most of the games we study, a player's Nash equilibrium action does not satisfy this condition: the action is optimal if the other players choose their Nash equilibrium actions, but some other action is optimal if the other players choose non-equilibrium actions.

? EXERCISE 25.1 (Altruistic players in the *Prisoner's Dilemma*) Each of two players has two possible actions, *Quiet* and *Fink*; each action pair results in the players' receiving amounts of *money* equal to the numbers corresponding to that action pair in Figure 24.1. (For example, if player 1 chooses *Quiet* and player 2 chooses *Fink*, then player 1 receives nothing, whereas player 2 receives \$3.) The players are not "selfish"; rather, the preferences of each player i are represented by the payoff function $m_i(a) + \alpha m_j(a)$, where $m_i(a)$ is the amount of money received by player i when the action profile is a, j is the other player, and α is a given nonnegative number. Player 1's payoff to the action pair $(Quiet, Quiet)$, for example, is $2 + 2\alpha$.

a. Formulate a strategic game that models this situation in the case $\alpha = 1$. Is this game the *Prisoner's Dilemma*?

b. Find the range of values of α for which the resulting game is the *Prisoner's Dilemma*. For values of α for which the game is not the *Prisoner's Dilemma*, find its Nash equilibria.

? EXERCISE 25.2 (Selfish and altruistic social behavior) Two people enter a bus. Two adjacent cramped seats are free. Each person must decide whether to sit or stand. Sitting alone is more comfortable than sitting next to the other person, which is more comfortable than standing.

a. Suppose that each person cares only about her own comfort. Model the situation as a strategic game. Is this game the *Prisoner's Dilemma*? Find its Nash equilibrium (equilibria?).

b. Suppose that each person is altruistic, ranking the outcomes according to the *other* person's comfort, and, out of politeness, prefers to stand than to sit if the other person stands. Model the situation as a strategic game. Is this game the *Prisoner's Dilemma*? Find its Nash equilibrium (equilibria?).

c. Compare the people's comfort in the equilibria of the two games.

Experimental evidence on the *Prisoner's Dilemma*

The *Prisoner's Dilemma* has attracted a great deal of attention by economists, psychologists, sociologists, and biologists. A huge number of experiments have been conducted with the aim of discovering how people behave when playing the game. Almost all these experiments involve each subject's playing the game repeatedly against an unchanging opponent, a situation that calls for an analysis significantly different from the one in this chapter (see Chapter 14).

The evidence on the outcome of isolated plays of the game is inconclusive. No experiment of which I am aware carefully induces the appropriate preferences and is specifically designed to elicit a steady state action profile (see the box on page 22). Thus in each case the choice of *Quiet* by a player could indicate that she is not "selfish" or that she is not experienced in playing the game, rather than providing evidence against the notion of Nash equilibrium.

In two experiments with very low payoffs, each subject played the game a small number of times against different opponents; between 50% and 94% of subjects chose *Fink*, depending on the relative sizes of the payoffs and some details of the design (Rapoport, Guyer, and Gordon 1976, 135–137, 211–213, and 223-226). A more recent experiment finds that in the last 10 of 20 rounds of play against different opponents, 78% of subjects choose *Fink* (Cooper, DeJong, Forsythe, and Ross 1996). In face-to-face games in which communication is allowed, the incidence of the choice of *Fink* tends to be lower: from 29% to 70% depending on the nature of the communication allowed (Deutsch 1958, and Frank, Gilovich, and Regan 1993, 163–167). (In all these experiments, the subjects were college students in the USA or Canada.)

One source of the variation in the results seems to be that some designs induce preferences that differ from those of the *Prisoner's Dilemma*; no clear answer emerges to the question of whether the notion of Nash equilibrium is relevant to the *Prisoner's Dilemma*. If, nevertheless, one interprets the evidence as showing that some subjects in the *Prisoner's Dilemma* systematically choose *Quiet* rather than *Fink*, one must fault the rational choice component of Nash equilibrium, not the coordinated expectations component. Why? Because, as noted in the text, *Fink* is optimal *no matter* what a player thinks her opponent will choose, so that any model in which the players act according to the model of rational choice, whether or not their expectations are coordinated, predicts that each player chooses *Fink*.

2.7.2 BoS

To find the Nash equilibria of *BoS* (Figure 16.1), we can examine each pair of actions in turn:

- (*Bach*, *Bach*): If player 1 switches to *Stravinsky* then her payoff decreases from 2 to 0; if player 2 switches to *Stravinsky* then her payoff decreases from 1 to 0.

Thus a deviation by either player decreases her payoff. Thus (*Bach*, *Bach*) is a Nash equilibrium.

- (*Bach*, *Stravinsky*): If player 1 switches to *Stravinsky* then her payoff increases from 0 to 1. Thus (*Bach*, *Stravinsky*) is not a Nash equilibrium. (Player 2 can increase her payoff by deviating, too, but to show the pair is not a Nash equilibrium it suffices to show that one player can increase her payoff by deviating.)

- (*Stravinsky*, *Bach*): If player 1 switches to *Bach* then her payoff increases from 0 to 2. Thus (*Stravinsky*, *Bach*) is not a Nash equilibrium.

- (*Stravinsky*, *Stravinsky*): If player 1 switches to *Bach* then her payoff decreases from 1 to 0; if player 2 switches to *Bach* then her payoff decreases from 2 to 0. Thus a deviation by either player decreases her payoff. Thus (*Stravinsky*, *Stravinsky*) is a Nash equilibrium.

We conclude that the game has two Nash equilibria: (*Bach*, *Bach*) and (*Stravinsky*, *Stravinsky*). That is, both of these outcomes are compatible with a steady state; both outcomes are stable social norms. If, in every encounter, both players choose *Bach*, then no player has an incentive to deviate; if, in every encounter, both players choose *Stravinsky*, then no player has an incentive to deviate. If we use the game to model the choices of men when matched with women, for example, then the notion of Nash equilibrium shows that two social norms are stable: both players choose the action associated with the outcome preferred by women, and both players choose the action associated with the outcome preferred by men.

2.7.3 Matching Pennies

By checking each of the four pairs of actions in *Matching Pennies* (Figure 17.1) we see that the game has no Nash equilibrium. For the pairs of actions (*Head*, *Head*) and (*Tail*, *Tail*), player 2 is better off deviating; for the pairs of actions (*Head*, *Tail*) and (*Tail*, *Head*), player 1 is better off deviating. Thus for this game the notion of Nash equilibrium isolates no steady state. In Chapter 4 we return to this game; an extension of the notion of a Nash equilibrium gives us an understanding of the likely outcome.

2.7.4 The Stag Hunt

Inspection of Figure 18.1 shows that the two-player *Stag Hunt* has two Nash equilibria: (*Stag*, *Stag*) and (*Hare*, *Hare*). If one player remains attentive to the pursuit of the stag, then the other player prefers to remain attentive; if one player chases a hare, the other one prefers to chase a hare (she cannot catch a stag alone). (The equilibria of the variant of the game in Figure 19.1 are analogous: (*Refrain*, *Refrain*) and (*Arm*, *Arm*).)

Unlike the Nash equilibria of *BoS*, one of these equilibria is better for both players than the other: each player prefers (*Stag*, *Stag*) to (*Hare*, *Hare*). This fact has no bearing on the equilibrium status of (*Hare*, *Hare*), since the condition for an equilibrium is that a *single* player cannot gain by deviating, *given* the other player's behavior. Put differently, an equilibrium is immune to any *unilateral* deviation; coordinated deviations by groups of players are not contemplated. However, the existence of two equilibria raises the possibility that one equilibrium might more likely be the outcome of the game than the other. I return to this issue in Section 2.7.6.

I argue that the many-player *Stag Hunt* (Example 18.1) also has two Nash equilibria: the action profile (*Stag*, ..., *Stag*) in which every players joins in the pursuit of the stag, and the profile (*Hare*, ..., *Hare*) in which every player catches a hare.

- (*Stag*, ..., *Stag*) is a Nash equilibrium because each player prefers this profile to that in which she alone chooses *Hare*. (A player is better off remaining attentive to the pursuit of the stag than running after a hare if all the other players remain attentive.)
- (*Hare*, ..., *Hare*) is a Nash equilibrium because each player prefers this profile to that in which she alone pursues the stag. (A player is better off catching a hare than pursuing the stag if no one else pursues the stag.)
- No other profile is a Nash equilibrium, because in any other profile at least one player chooses *Stag* and at least one player chooses *Hare*, so that any player choosing *Stag* is better off switching to *Hare*. (A player is better off catching a hare than pursing the stag if at least one other person chases a hare, since the stag can be caught only if everyone pursues it.)

? EXERCISE 28.1 (Variants of the *Stag Hunt*) Consider two variants of the n-hunter *Stag Hunt* in which only m hunters, with $2 \le m < n$, need to pursue the stag in order to catch it. (Continue to assume that there is a single stag.) Assume that a captured stag is shared only by the hunters that catch it.

a. Assume, as before, that each hunter prefers the fraction $1/n$ of the stag to a hare. Find the Nash equilibria of the strategic game that models this situation.

b. Assume that each hunter prefers the fraction $1/k$ of the stag to a hare, but prefers the hare to any smaller fraction of the stag, where k is an integer with $m \le k \le n$. Find the Nash equilibria of the strategic game that models this situation.

The following more difficult exercise enriches the hunters' choices in the *Stag Hunt*. This extended game has been proposed as a model that captures Keynes' basic insight about the possibility of multiple economic equilibria, some undesirable (Bryant 1983, 1994).

?? EXERCISE 28.2 (Extension of the *Stag hunt*) Extend the n-hunter *Stag Hunt* by giving each hunter K (a positive integer) units of effort, which she can allocate between pursuing the stag and catching hares. Denote the effort hunter i devotes to pursuing the stag by e_i, a nonnegative integer equal to at most K. The chance that the stag is caught depends on the smallest of all the hunters' efforts, denoted $\min_j e_j$. ("A chain is as strong as its weakest link.") Hunter i's payoff to the action profile $(e_1, \ldots, e_n)$ is $2 \min_j e_j - e_i$. (She is better off the more likely the stag is caught, and worse off the more effort she devotes to pursuing the stag, which means she catches fewer hares.) Is the action profile $(e, \ldots, e)$, in which every hunter devotes the same effort to pursuing the stag, a Nash equilibrium for any value of e? (What is a player's payoff to this profile? What is her payoff if she deviates to a lower or higher effort level?) Is any action profile in which not all the players' effort levels are the same a Nash equilibrium? (Consider a player whose effort exceeds the minimum effort level of all players. What happens to her payoff if she reduces her effort level to the minimum?)

2.7.5 *Hawk–Dove*

The game in the next exercise captures a basic feature of animal conflict.

? EXERCISE 29.1 (Hawk–Dove) Two animals are fighting over some prey. Each can be passive or aggressive. Each prefers to be aggressive if its opponent is passive, and passive if its opponent is aggressive; given its own stance, it prefers the outcome when its opponent is passive to that in which its opponent is aggressive. Formulate this situation as a strategic game and find its Nash equilibria.

2.7.6 *A coordination game*

Consider two people who wish to go out together, but who, unlike the dissidents in *BoS*, agree on the more desirable concert—say they both prefer *Bach*. A strategic game that models this situation is shown in Figure 29.1; it is an example of a *coordination game*. By examining the four action pairs, we see that the game has two Nash equilibria: (*Bach*, *Bach*) and (*Stravinsky*, *Stravinsky*). In particular, the action pair (*Stravinsky*, *Stravinsky*) in which both people choose their less-preferred concert is a Nash equilibrium.

	Bach	*Stravinsky*
Bach	2, 2	0, 0
Stravinsky	0, 0	1, 1

Figure 29.1 *A coordination game.*

Is the equilibrium in which both people choose *Stravinsky* plausible? People who argue that the technology of Apple computers originally dominated that of IBM computers, and that the Beta format for video recording is better than VHS, would say "yes". In both cases users had a strong interest in adopting the same standard, and one standard was better than the other; in the steady state that emerged in each case, the inferior technology was adopted by a large majority of users.

FOCAL POINTS

In games with many Nash equilibria, the theory isolates more than one pattern of behavior compatible with a steady state. In some games, some of these equilibria seem more likely to attract the players' attentions than others. To use the terminology of Schelling (1960), some equilibria are *focal*. In the coordination game in Figure 29.1, where the players agree on the more desirable Nash equilibrium and obtain the same payoff to every nonequilibrium action pair, the preferable equilibrium seems more likely to be focal (though two examples are given in the text of steady states involving the inferior equilibrium). In the variant of this game in which the two equilibria are equally good (i.e. $(2,2)$ is replaced by $(1,1)$), nothing in the structure of the game gives any clue as to which steady state might occur. In such a game, the names or nature of the actions, or other information, may predispose the players to one equilibrium rather than the other.

Consider, for example, voters in an election. Pre-election polls may give them information about each other's intended actions, pointing them to one of many Nash equilibria. Or consider a situation in which two players independently divide \$100 into two piles, each receiving \$10 if they choose the same divisions and nothing otherwise. The strategic game that models this situation has many Nash equilibria, in each of which both players choose the same division. But the equilibrium in which both players choose the $(\$50, \$50)$ division seems likely to command the players' attentions, possibly for esthetic reasons (it is an appealing division), and possibly because it is a steady state in an unrelated game in which the chosen division determines the players' payoffs.

The theory of Nash equilibrium is neutral about the equilibrium that will occur in a game with many equilibria. If features of the situation not modeled by the notion of a strategic game make some equilibria focal then those equilibria may be more likely to emerge as steady states, and the rate at which a steady state is reached may be higher than it otherwise would have been.

If two people played this game in a laboratory it seems likely that the outcome would be (*Bach*, *Bach*). Nevertheless, (*Stravinsky*, *Stravinsky*) also corresponds to a steady state: if either action pair is reached, there is no reason for either player to deviate from it.

2.7.7 Provision of a public good

The model in the next exercise captures an aspect of the provision of a "public good", like a park or a swimming pool, whose use by one person does not diminish its value to another person (at least, not until it is overcrowded). (Other aspects of public good provision are studied in Section 2.8.4.)

? EXERCISE 31.1 (Contributing to a public good) Each of n people chooses whether or not to contribute a fixed amount toward the provision of a public good. The good is provided if and only if at least k people contribute, where $2 \leq k \leq n$; if it is not provided, contributions are not refunded. Each person ranks outcomes from best to worst as follows: (*i*) any outcome in which the good is provided and she does not contribute, (*ii*) any outcome in which the good is provided and she contributes, (*iii*) any outcome in which the good is not provided and she does not contribute, (*iv*) any outcome in which the good is not provided and she contributes. Formulate this situation as a strategic game and find its Nash equilibria. (Is there a Nash equilibrium in which more than k people contribute? One in which k people contribute? One in which fewer than k people contribute? (Be careful!))

2.7.8 Strict and nonstrict equilibria

In all the Nash equilibria of the games we have studied so far a deviation by a player leads to an outcome *worse* for that player than the equilibrium outcome. The definition of Nash equilibrium (21.1), however, requires only that the outcome of a deviation be *no better* for the deviant than the equilibrium outcome. And, indeed, some games have equilibria in which a player is indifferent between her equilibrium action and some other action, given the other players' actions.

Consider the game in Figure 31.1. This game has a unique Nash equilibrium, namely (T, L). (For every other pair of actions, one of the players is better off changing her action.) When player 2 chooses L, as she does in this equilibrium, player 1 is equally happy choosing T or B; if she deviates to B then she is no worse off than she is in the equilibrium. We say that the Nash equilibrium (T, L) is not a *strict equilibrium*.

	L	M	R
T	$1,1$	$1,0$	$0,1$
B	$1,0$	$0,1$	$1,0$

Figure 31.1 A game with a unique Nash equilibrium, which is not a strict equilibrium.

For a general game, an equilibrium is strict if each player's equilibrium action is *better* than all her other actions, given the other players' actions. Precisely, an action profile a^* is a **strict Nash equilibrium** if for every player i we have $u_i(a^*) > u_i(a_i, a^*_{-i})$ for every action $a_i \neq a^*_i$ of player i. (Contrast the strict inequality in this definition with the weak inequality in (21.2).)

2.7.9 Additional examples

The following exercises are more difficult than most of the previous ones. In the first two, the number of actions of each player is arbitrary, so you cannot mechanically examine each action profile individually, as we did for games in which each player has two actions. Instead, you can consider groups of action profiles that have features in common, and show that all action profiles in any given group are or are not equilibria. Deciding how best to group the profiles into types calls for some intuition about the character of a likely equilibrium; the exercises contain suggestions on how to proceed.

?? EXERCISE 32.1 (Guessing two-thirds of the average) Each of three people announces an integer from 1 to K. If the three integers are different, the person whose integer is closest to $\frac{2}{3}$ of the average of the three integers wins $1. If two or more integers are the same, $1 is split equally between the people whose integer is closest to $\frac{2}{3}$ of the average integer. Is there any integer k such that the action profile (k, k, k), in which every person announces the same integer k, is a Nash equilibrium? (If $k \geq 2$, what happens if a person announces a smaller number?) Is any other action profile a Nash equilibrium? (What is the payoff of a person whose number is the highest of the three? Can she increase this payoff by announcing a different number?)

Game theory is used widely in political science, especially in the study of elections. The game in the following exercise explores citizens' costly decisions to vote.

?? EXERCISE 32.2 (Voter participation) Two candidates, A and B, compete in an election. Of the n citizens, k support candidate A and m $(= n - k)$ support candidate B. Each citizen decides whether to vote, at a cost, for the candidate she supports, or to abstain. A citizen who abstains receives the payoff of 2 if the candidate she supports wins, 1 if this candidate ties for first place, and 0 if this candidate loses. A citizen who votes receives the payoffs $2 - c$, $1 - c$, and $-c$ in these three cases, where $0 < c < 1$.

a. For $k = m = 1$, is the game the same (except for the names of the actions) as any considered so far in this chapter?

b. For $k = m$, find the set of Nash equilibria. (Is the action profile in which everyone votes a Nash equilibrium? Is there any Nash equilibrium in which the candidates tie and not everyone votes? Is there any Nash equilibrium in which one of the candidates wins by one vote? Is there any Nash equilibrium in which one of the candidates wins by two or more votes?)

c. What is the set of Nash equilibria for $k < m$?

If, when sitting in a traffic jam, you have ever thought about the time you might save if another road were built, the next exercise may lead you to think again.

?? EXERCISE 32.3 (Choosing a route) Four people must drive from A to B at the same time. Two routes are available, one via X and one via Y. (Refer to the left panel of Figure 33.1.) The roads from A to X, and from Y to B are both short and narrow; in each case, one car takes 6 minutes, and each additional car increases the travel time *per car* by 3 minutes. (If two cars drive from A to X, for example, *each car* takes 9 minutes.) The roads from A to Y, and from X to B are long and wide; on A to Y one car takes 20 minutes, and each additional car increases the travel time *per car* by 1 minute; on X to B one car takes 20 minutes, and each additional car increases the travel time *per car* by 0.9 minutes. Formulate this situation as a strategic game and find the Nash equilibria. (If all four people take one of the routes, can any of them do better by taking the other route? What if three take one route and one takes the other route, or if two take each route?)

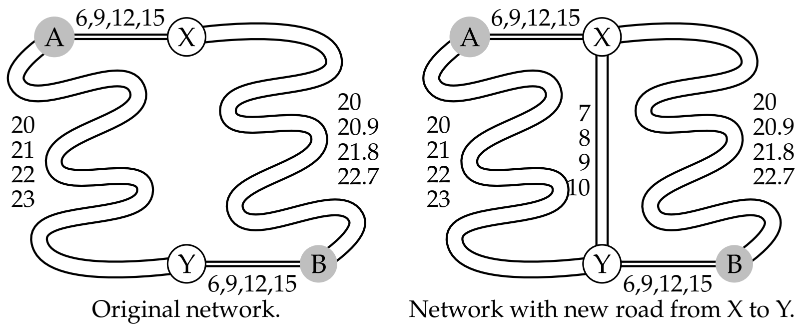


Figure 33.1 Getting from A to B: the road networks in Exercise 32.3. The numbers beside each road are the travel times *per car* when 1, 2, 3, or 4 cars take that road.

Now suppose that a relatively short, wide road is built from X to Y, giving each person four options for travel from A to B: A–X–B, A–Y–B, A–X–Y–B, and A–Y–X–B. Assume that a person who takes A–X–Y–B travels the A–X portion at the same time as someone who takes A–X–B, and the Y–B portion at the same time as someone who takes A–Y–B. (Think of there being constant flows of traffic.) On the road between X and Y, one car takes 7 minutes and each additional car increases the travel time *per car* by 1 minute. Find the Nash equilibria in this new situation. Compare each person's travel time with her travel time in the equilibrium before the road from X to Y was built.

2.8 Best response functions

2.8.1 Definition

We can find the Nash equilibria of a game in which each player has only a few actions by examining each action profile in turn to see if it satisfies the conditions for equilibrium. In more complicated games, it is often better to work with the players' "best response functions".

Consider a player, say player i. For any given actions of the players other than i, player i's actions yield her various payoffs. We are interested in the best actions—those that yield her the highest payoff. In *BoS*, for example, *Bach* is the best action for player 1 if player 2 chooses *Bach*; *Stravinsky* is the best action for player 1 if player 2 chooses *Stravinsky*. In particular, in *BoS*, player 1 has a single best action for each action of player 2. By contrast, in the game in Figure 31.1, both T and B are best actions for player 1 if player 2 chooses L: they both yield the payoff of 1, and player 1 has no action that yields a higher payoff (in fact, she has no other action).

We denote the set of player i's best actions when the list of the other players' actions is a_{-i} by $B_i(a_{-i})$. Thus in *BoS* we have $B_1(\mathit{Bach}) = \{\mathit{Bach}\}$ and $B_1(\mathit{Stravinsky}) = \{\mathit{Stravinsky}\}$; in the game in Figure 31.1 we have $B_1(L) = \{T, B\}$.

Precisely, we define the function B_i by

$$B_i(a_{-i}) = \{a_i \text{ in } A_i : u_i(a_i, a_{-i}) \geq u_i(a'_i, a_{-i}) \text{ for all } a'_i \text{ in } A_i\} :$$

any action in $B_i(a_{-i})$ is at least as good for player i as every other action of player i when the other players' actions are given by a_{-i}. We call B_i the **best response function** of player i.

The function B_i is *set-valued*: it associates a set of actions with any list of the other players' actions. Every member of the set $B_i(a_{-i})$ is a **best response** of player i to a_{-i}: if each of the other players adheres to a_{-i} then player i can do no better than choose a member of $B_i(a_{-i})$. In some games, like *BoS*, the set $B_i(a_{-i})$ consists of a single action for every list a_{-i} of actions of the other players: no matter what the other players do, player i has a *single* optimal action. In other games, like the one in Figure 31.1, $B_i(a_{-i})$ contains more than one action for some lists a_{-i} of actions of the other players.

2.8.2 Using best response functions to define Nash equilibrium

A Nash equilibrium is an action profile with the property that no player can do better by changing her action, given the other players' actions. Using the terminology just developed, we can alternatively define a Nash equilibrium to be an action profile for which every player's action is a best response to the other players' actions. That is, we have the following result.

■ PROPOSITION 34.1 *The action profile a^* is a Nash equilibrium of a strategic game with ordinal preferences if and only if every player's action is a best response to the other players' actions:*

$$a_i^* \text{ is in } B_i(a_{-i}^*) \text{ for every player } i. \tag{34.2}$$

If each player i has a single best response to each list a_{-i} of the other players' actions, we can write the conditions in (34.2) as equations. In this case, for each player i and each list a_{-i} of the other players' actions, denote the single member of $B_i(a_{-i})$ by $b_i(a_{-i})$ (that is, $B_i(a_{-i}) = \{b_i(a_{-i})\}$). Then (34.2) is equivalent to

$$a_i^* = b_i(a_{-i}^*) \text{ for every player } i, \tag{34.3}$$

a collection of n equations in the n unknowns a_i^*, where n is the number of players in the game. For example, in a game with two players, say 1 and 2, these equations are

$$\begin{aligned} a_1^* &= b_1(a_2^*) \\ a_2^* &= b_2(a_1^*). \end{aligned}$$

That is, in a two-player game in which each player has a single best response to every action of the other player, (a_1^*, a_2^*) is a Nash equilibrium if and only if player 1's action a_1^* is her best response to player 2's action a_2^*, and player 2's action a_2^* is her best response to player 1's action a_1^*.

2.8.3 Using best response functions to find Nash equilibria

The definition of a Nash equilibrium in terms of best response functions suggests a method for finding Nash equilibria:

- find the best response function of each player
- find the action profiles that satisfy (34.2) (which reduces to (34.3) if each player has a single best response to each list of the other players' actions).

To illustrate this method, consider the game in Figure 35.1. First find the best response of player 1 to each action of player 2. If player 2 chooses L, then player 1's best response is M (2 is the highest payoff for player 1 in this column); indicate the best response by attaching a star to player 1's payoff to (M, L). If player 2 chooses C, then player 1's best response is T, indicated by the star attached to player 1's payoff to (T, C). And if player 2 chooses R, then both T and B are best responses for player 1; both are indicated by stars. Second, find the best response of player 2 to each action of player 1 (for each row, find highest payoff of player 2); these best responses are indicated by attaching stars to player 2's payoffs. Finally, find the boxes in which both players' payoffs are starred. Each such box is a Nash equilibrium: the star on player 1's payoff means that player 1's action is a best response to player 2's action, and the star on player 2's payoff means that player 2's action is a best response to player 1's action. Thus we conclude that the game has two Nash equilibria: (M, L) and (B, R).

	L	C	R
T	$1, 2^*$	$2^*, 1$	$1^*, 0$
M	$2^*, 1^*$	$0, 1^*$	$0, 0$
B	$0, 1$	$0, 0$	$1^*, 2^*$

Figure 35.1 Using best response functions to find Nash equilibria in a two-player game in which each player has three actions.

? EXERCISE 35.1 (Finding Nash equilibria using best response functions)

a. Find the players' best response functions in the *Prisoner's Dilemma* (Figure 13.1), *BoS* (Figure 16.1), *Matching Pennies* (Figure 17.1), and the two-player *Stag Hunt* (Figure 18.1) (and verify the Nash equilibria of these games).

b. Find the Nash equilibria of the game in Figure 36.1 by finding the players' best response functions.

	L	C	R
T	2, 2	1, 3	0, 1
M	3, 1	0, 0	0, 0
B	1, 0	0, 0	0, 0

Figure 36.1 The game in Exercise 35.1b.

The players' best response functions for the game in Figure 35.1 are presented in a different format in Figure 36.2. In this figure, player 1's actions are on the horizontal axis and player 2's are on the vertical axis. (Thus the columns correspond to choices of player 1, and the rows correspond to choices of player 2, whereas the reverse is true in Figure 35.1. I choose this orientation for Figure 36.2 for consistency with the convention for figures of this type.) Player 1's best responses are indicated by circles, and player 2's by dots. Thus the circle at (T, C) reflects the fact that T is player 1's best response to player 2's choice of C, and the circles at (T, R) and (B, R) reflect the fact that T and B are both best responses of player 1 to player 2's choice of R. Any action pair marked by both a circle and a dot is a Nash equilibrium: the circle means that player 1's action is a best response to player 2's action, and the dot indicates that player 2's action is a best response to player 1's action.

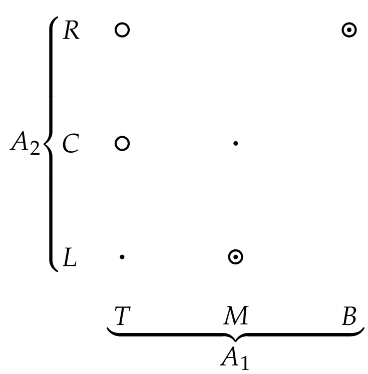


Figure 36.2 The players' best response functions for the game in Figure 35.1. Player 1's best responses are indicated by circles, and player 2's by dots. The action pairs for which there is both a circle and a dot are the Nash equilibria.

? EXERCISE 36.1 (Constructing best response functions) Draw the analogue of Figure 36.2 for the game in Exercise 35.1b.

? EXERCISE 36.2 (Dividing money) Two people have \$10 to divide between themselves. They use the following process to divide the money. Each person names a number of dollars (a nonnegative integer), at most equal to 10. If the sum of the amounts that the people name is at most 10 then each person receives the amount of money she names (and the remainder is destroyed). If the sum of the amounts that the people name exceeds 10 and the amounts named are different then the person who names the smaller amount receives that amount and the other person receives the remaining money. If the sum of the amounts that the people name exceeds 10 and the amounts named are the same then each person receives $5. Determine the best response of each player to each of the other player's actions, plot them in a diagram like Figure 36.2, and thus find the Nash equilibria of the game.

A diagram like Figure 36.2 is a convenient representation of the players' best response functions also in a game in which each player's set of actions is an interval of numbers, as the next example illustrates.

◆ EXAMPLE 37.1 (A synergistic relationship) Two individuals are involved in a synergistic relationship. If both individuals devote more effort to the relationship, they are both better off. For any given effort of individual j, the return to individual i's effort first increases, then decreases. Specifically, an effort level is a nonnegative number, and individual i's preferences (for $i = 1, 2$) are represented by the payoff function $a_i(c + a_j - a_i)$, where a_i is i's effort level, a_j is the other individual's effort level, and $c > 0$ is a constant.

The following strategic game models this situation.

Players The two individuals.

Actions Each player's set of actions is the set of effort levels (nonnegative numbers).

Preferences Player i's preferences are represented by the payoff function $a_i(c + a_j - a_i)$, for $i = 1, 2$.

In particular, each player has infinitely many actions, so that we cannot present the game in a table like those used previously (Figure 36.1, for example).

To find the Nash equilibria of the game, we can construct and analyze the players' best response functions. Given a_j, individual i's payoff is a quadratic function of a_i that is zero when $a_i = 0$ and when $a_i = c + a_j$, and reaches a maximum in between. The symmetry of quadratic functions (see Section 17.4) implies that the best response of each individual i to a_j is

$$b_i(a_j) = \tfrac{1}{2}(c + a_j).$$

(If you know calculus, you can reach the same conclusion by setting the derivative of player i's payoff with respect to a_i equal to zero.)

The best response functions are shown in Figure 38.1. Player 1's actions are plotted on the horizontal axis and player 2's actions are plotted on the vertical axis. Player 1's best response function associates an action for player 1 with every action for player 2. Thus to interpret the function b_1 in the diagram, take a point a_2 on the vertical axis, and go across to the line labeled b_1 (the steeper of the two lines), then read down to the horizontal axis. The point on the horizontal axis that you reach is $b_1(a_2)$, the best action for player 1 when player 2 chooses a_2. Player 2's best response function, on the other hand, associates an action for player 2 with every action of player 1. Thus to interpret this function, take a point a_1 on the horizontal axis, and go up to b_2, then across to the vertical axis. The point on the vertical axis that you reach is $b_2(a_1)$, the best action for player 2 when player 1 chooses a_1.

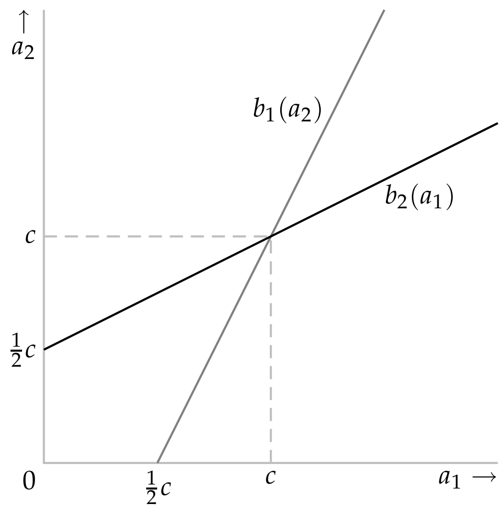


Figure 38.1 The players' best response functions for the game in Example 37.1. The game has a unique Nash equilibrium, $(a_1^*, a_2^*) = (c, c)$.

At a point (a_1, a_2) where the best response functions intersect in the figure, we have $a_1 = b_1(a_2)$, because (a_1, a_2) is on the graph of b_1, player 1's best response function, and $a_2 = b_2(a_1)$, because (a_1, a_2) is on the graph of b_2, player 1's best response function. Thus any such point (a_1, a_2) is a Nash equilibrium. In this game the best response functions intersect at a single point, so there is one Nash equilibrium. In general, they may intersect more than once; every point at which they intersect is a Nash equilibrium.

To find the point of intersection of the best response functions precisely, we can solve the two equations in (34.3):

$$\begin{aligned} a_1 &= \tfrac{1}{2}(c + a_2) \\ a_2 &= \tfrac{1}{2}(c + a_1). \end{aligned}$$

Substituting the second equation in the first, we get $a_1 = \frac{1}{2}(c + \frac{1}{2}(c + a_1)) = \frac{3}{4}c + \frac{1}{4}a_1$, so that $a_1 = c$. Substituting this value of a_1 into the second equation, we get $a_2 = c$. We conclude that the game has a unique Nash equilibrium $(a_1, a_2) = (c, c)$. (To reach this conclusion, it suffices to solve the two equations; we do not have to draw Figure 38.1. However, the diagram shows us at once that the game has a unique equilibrium, in which both players' actions exceed $\frac{1}{2}c$, facts that serve to check the results of our algebra.)

In the game in this example, each player has a unique best response to every action of the other player, so that the best response functions are lines. If a player has many best responses to some of the other players' actions, then her best response function is "thick" at some points; several examples in the next chapter have this property (see, for example, Figure 64.1). Example 37.1 is special also because the game has a unique Nash equilibrium—the best response functions cross once. As we have seen, some games have more than one equilibrium, and others have no equilibrium. A pair of best response functions that illustrates some of the possibilities is shown in Figure 39.1. In this figure the shaded area of player 1's best response function indicates that for a_2 between $\overline{a}_2$ and $\underline{a}_2$, player 1 has a range of best responses. For example, all actions of player 1 from a_1^{**} to a_1^{***} are best responses to the action a_2^{***} of player 2. For a game with these best response functions, the set of Nash equilibria consists of the pair of actions (a_1^*, a_2^*) and all the pairs of actions on player 2's best response function between (a_1^{**}, a_2^{**}) and (a_1^{***}, a_2^{***}).

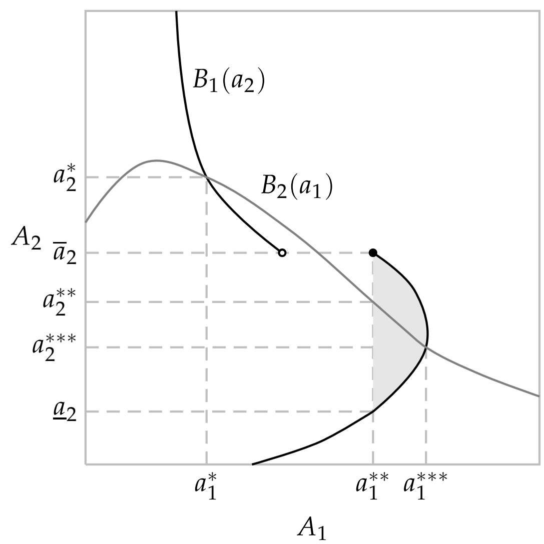


Figure 39.1 An example of the best response functions of a two-player game in which each player's set of actions is an interval of numbers. The set of Nash equilibria of the game consists of the pair of actions (a_1^*, a_2^*) and all the pairs of actions on player 2's best response function between (a_1^{**}, a_2^{**}) and (a_1^{***}, a_2^{***}).

? EXERCISE 39.1 (Strict and nonstrict Nash equilibria) Which of the Nash equilibria of the game whose best response functions are given in Figure 39.1 are strict (see the definition on page 31)?

Another feature that differentiates the best response functions in Figure 39.1 from those in Figure 38.1 is that the best response function b_1 of player 1 is not continuous. When player 2's action is $\overline{a}_2$, player 1's best response is a_1^{**} (indicated by the small disk at $(a_1^{**}, \overline{a}_2)$), but when player 2's action is slightly greater than $\overline{a}_2$, player 1's best response is significantly less than a_1^{**}. (The small circle indicates a point excluded from the best response function.) Again, several examples in the next chapter have this feature. From Figure 39.1 we see that if a player's best response function is discontinuous, then depending on where the discontinuity occurs, the best response functions may not intersect at all—the game may, like *Matching Pennies*, have no Nash equilibrium.

? EXERCISE 40.1 (Finding Nash equilibria using best response functions) Find the Nash equilibria of the two-player strategic game in which each player's set of actions is the set of nonnegative numbers and the players' payoff functions are $u_1(a_1, a_2) = a_1(a_2 - a_1)$ and $u_2(a_1, a_2) = a_2(1 - a_1 - a_2)$.

? EXERCISE 40.2 (A joint project) Two people are engaged in a joint project. If each person i puts in the effort x_i, a nonnegative number equal to at most 1, which costs her $c(x_i)$, the outcome of the project is worth $f(x_1, x_2)$. The worth of the project is split equally between the two people, regardless of their effort levels. Formulate this situation as a strategic game. Find the Nash equilibria of the game when (*a*) $f(x_1, x_2) = 3x_1x_2$ and $c(x_i) = x_i^2$ for $i = 1, 2$, and (*b*) $f(x_1, x_2) = 4x_1x_2$ and $c(x_i) = x_i$ for $i = 1, 2$. In each case, is there a pair of effort levels that yields both players higher payoffs than the Nash equilibrium effort levels?

2.8.4 Illustration: contributing to a public good

Exercise 31.1 models decisions on whether to contribute to the provision of a "public good". We now study a model in which two people decide not only whether to contribute, but also *how much* to contribute.

Denote person i's wealth by w_i, and the amount she contributes to the public good by c_i ($0 \leq c_i \leq w_i$); she spends her remaining wealth $w_i - c_i$ on "private goods" (like clothes and food, whose consumption by one person precludes their consumption by anyone else). The amount of the public good is equal to the sum of the contributions. Each person cares both about the amount of the public good and her consumption of private goods.

Suppose that person i's preferences are represented by the payoff function $v_i(c_1 + c_2) + w_i - c_i$. Because w_i is a constant, person i's preferences are alternatively represented by the payoff function

$$u_i(c_1, c_2) = v_i(c_1 + c_2) - c_i.$$

This situation is modeled by the following strategic game.

Players The two people.

Actions Player i's set of actions is the set of her possible contributions (nonnegative numbers less than or equal to w_i), for $i = 1, 2$.

Preferences Player i's preferences are represented by the payoff function $u_i(c_1, c_2) = v_i(c_1 + c_2) - c_i$, for $i = 1, 2$.

To find the Nash equilibria of this strategic game, consider the players' best response functions. Player 1's best response to the contribution c_2 of player 2 is the value of c_1 that maximizes $v_1(c_1 + c_2) - c_1$. Without specifying the form of the function v_1 we cannot explicitly calculate this optimal value. However, we can determine how it varies with c_2.

First consider player 1's best response to $c_2 = 0$. Suppose that the form of the function v_1 is such that the function $u_1(c_1, 0)$ increases up to its maximum, then decreases (as in Figure 41.1). Then player 1's best response to $c_2 = 0$, which I denote $b_1(0)$, is unique. This best response is the value of c_1 that maximizes $u_1(c_1, 0) = v_1(c_1) - c_1$ subject to $0 \leq c_1 \leq w_1$. Assume that $0 < b_1(0) < w_1$: player 1's optimal contribution to the public good when player 2 makes no contribution is positive and less than her entire wealth.

Now consider player 1's best response to $c_2 = k > 0$. This best response is the value of c_1 that maximizes $u_1(c_1, k) = v_1(c_1 + k) - c_1$. Now, we have

$$u_1(c_1, k) = u_1(c_1 + k, 0) + k.$$

That is, the graph of $u_1(c_1, k)$ as a function of c_1 is the translation to the left k units and up k units of the graph of $u_1(c_1, 0)$ as a function of c_1 (refer to Figure 41.1). Thus if $k \leq b_1(0)$ then $b_1(k) = b_1(0) - k$: if player 2's contribution increases from 0 to k then player 1's best response decreases by k. If $k > b_1(0)$ then, given the form of $u_1(c_1, 0)$, we have $b_1(k) = 0$.

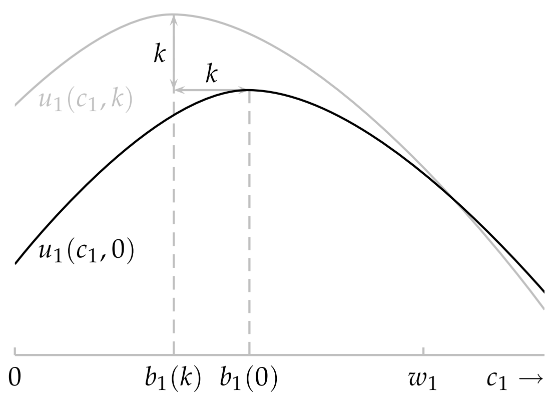


Figure 41.1 The relation between player 1's best responses $b_1(0)$ and $b_1(k)$ to $c_2 = 0$ and $c_2 = k$ in the game of contributing to a public good.

We conclude that if player 2 increases her contribution by k then player 1's best response is to reduce her contribution by k (or to zero, if k is larger than player 1's original contribution)!

The same analysis applies to player 2: for every unit more that player 1 contributes, player 2 contributes a unit less, so long as her contribution is nonnegative. The function v_2 may be different from the function v_1, so that player 1's best contribution $b_1(0)$ when $c_2 = 0$ may be different from player 2's best contribution $b_2(0)$

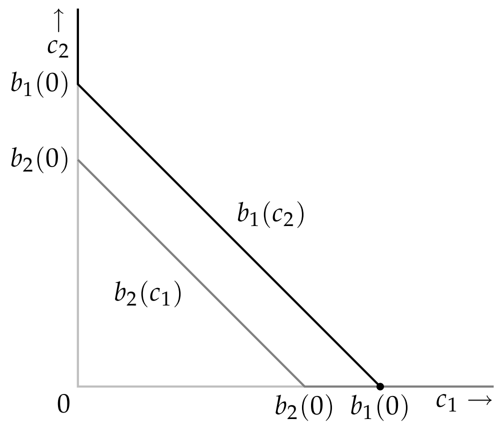


Figure 42.1 The best response functions for the game of contributing to a public good in Section 2.8.4 in a case in which $b_1(0) > b_2(0)$. The best response function of player 1 is the black line; that of player 2 is the gray line.

when $c_1 = 0$. But both best response functions have the same character: the slope of each function is -1 where the value of the function is positive. They are shown in Figure 42.1 for a case in which $b_1(0) > b_2(0)$.

We deduce that if $b_1(0) > b_2(0)$ then the game has a unique Nash equilibrium, $(b_1(0), 0)$: player 2 contributes nothing. Similarly, if $b_1(0) < b_2(0)$ then the unique Nash equilibrium is $(0, b_2(0))$: player 1 contributes nothing. That is, the person who contributes more when the other person contributes nothing is the only one to make a contribution in a Nash equilibrium. Only if $b_1(0) = b_2(0)$, which is not likely if the functions v_1 and v_2 differ, is there an equilibrium in which both people contribute. In this case the downward-sloping parts of the best response functions coincide, so that any pair of contributions (c_1, c_2) with $c_1 + c_2 = b_1(0)$ and $c_i \geq 0$ for $i = 1, 2$ is a Nash equilibrium.

In summary, the notion of Nash equilibrium predicts that, except in unusual circumstances, only one person contributes to the provision of the public good when each person's payoff function takes the form $v_i(c_1 + c_2) + w_i - c_i$, each function $v_i(c_i) - c_i$ increases to a maximum, then decreases, and each person optimally contributes less than her entire wealth when the other person does not contribute. The person who contributes is the one who wishes to contribute more when the other person does not contribute. In particular, the identity of the person who contributes does not depend on the distribution of wealth; any distribution in which each person optimally contributes less than her entire wealth when the other person does not contribute leads to the same outcome.

The next exercise asks you to consider a case in which the amount of the public good affects each person's enjoyment of the private good. (The public good might be clean air, which improves each person's enjoyment of her free time.)

? EXERCISE 42.1 (Contributing to a public good) Consider the model in this section when $u_i(c_1, c_2)$ is the sum of three parts: the amount $c_1 + c_2$ of the public good provided, the amount $w_i - c_i$ person i spends on private goods, and a term $(w_i - c_i)(c_1 + c_2)$ that reflects an interaction between the amount of the public good and her private consumption—the greater the amount of the public good, the more she values her private consumption. In summary, suppose that person i's payoff is $c_1 + c_2 + w_i - c_i + (w_i - c_i)(c_1 + c_2)$, or

$$w_i + c_j + (w_i - c_i)(c_1 + c_2),$$

where j is the other person. Assume that $w_1 = w_2 = w$, and that each player i's contribution c_i may be any number (positive or negative, possibly larger than w). Find the Nash equilibrium of the game that models this situation. (You can calculate the best responses explicitly. Imposing the sensible restriction that c_i lie between 0 and w complicates the analysis, but does not change the answer.) Show that in the Nash equilibrium both players are worse off than they are when they both contribute one half of their wealth to the public good. If you can, extend the analysis to the case of n people. As the number of people increases, how does the total amount contributed in a Nash equilibrium change? Compare the players' equilibrium payoffs with their payoffs when each contributes half her wealth to the public good, as n increases without bound. (The game is studied further in Exercise 358.3.)

2.9 Dominated actions

2.9.1 *Strict domination*

You drive up to a red traffic light. The left lane is free; in the right lane there is a car that may turn right when the light changes to green, in which case it will have to wait for a pedestrian to cross the side street. Assuming you wish to progress as quickly as possible, the action of pulling up in the left lane "strictly dominates" that of pulling up in the right lane. If the car in the right lane turns right then you are much better off in the left lane, where your progress will not be impeded; and even if the car in the right lane does not turn right, you are still better off in the left lane, rather than behind the other car.

In any game, a player's action "strictly dominates" another action if it is superior, *no matter what the other players do*.

► DEFINITION 43.1 (*Strict domination*) In a strategic game with ordinal preferences, player i's action a''_i **strictly dominates** her action a'_i if

$$u_i(a''_i, a_{-i}) > u_i(a'_i, a_{-i}) \text{ for every list } a_{-i} \text{ of the other players' actions,}$$

where u_i is a payoff function that represents player i's preferences.

In the *Prisoner's Dilemma*, for example, the action *Fink* strictly dominates the action *Quiet*: regardless of her opponent's action, a player prefers the outcome when she chooses *Fink* to the outcome when she chooses *Quiet*. In *BoS*, on the other hand, neither action strictly dominates the other: *Bach* is better than *Stravinsky* if the other player chooses *Bach*, but is worse than *Stravinsky* if the other player chooses *Stravinsky*.

If an action strictly dominates the action a_i, we say that a_i is **strictly dominated**. A strictly dominated action is not a best response to any actions of the other players: whatever the other players do, some other action is better. Since a player's Nash equilibrium action is a best response to the other players' Nash equilibrium actions,

a strictly dominated action is not used in any Nash equilibrium.

When looking for the Nash equilibria of a game, we can thus eliminate from consideration all strictly dominated actions. For example, we can eliminate *Quiet* for each player in the *Prisoner's Dilemma*, leaving (*Fink*, *Fink*) as the only candidate for a Nash equilibrium. (As we know, this action pair is indeed a Nash equilibrium.)

The fact that the action a_i'' strictly dominates the action a_i' of course does *not* imply that a_i'' strictly dominates *all* actions. Indeed, a_i'' may itself be strictly dominated. In the left-hand game in Figure 44.1, for example, M strictly dominates T, but B is better than M if player 2 chooses R. (I give only the payoffs of player 1 in the figure, because those of player 2 are not relevant.) Since T is strictly dominated, the game has no Nash equilibrium in which player 1 uses it; but the game may also not have any equilibrium in which player 1 uses M. In the right-hand game, M strictly dominates T, but is itself strictly dominated by B. In this case, in any Nash equilibrium player 1's action is B (her only action that is not strictly dominated).

	L	R
T	1	0
M	2	1
B	1	3

	L	R
T	1	0
M	2	1
B	3	2

Figure 44.1 Two games in which player 1's action T is strictly dominated by M. (Only player 1's payoffs are given.) In the left-hand game, B is better than M if player 2 chooses R; in the right-hand game, M itself is strictly dominated, by B.

A strictly dominated action is incompatible not only with a steady state, but also with rational behavior by a player who confronts a game for the first time. This fact is the first step in a theory different from Nash equilibrium, explored in Chapter 12.

2.9.2 Weak domination

As you approach the red light in the situation at the start of the previous section, there is a car in *each* lane. The car in the right lane may, or may not, be turning right; if it is, it may be delayed by a pedestrian crossing the side street. The car in the left lane cannot turn right. In this case your pulling up in the left lane "weakly dominates", though does not strictly dominate, your pulling up in the right lane. If the car in the right lane does not turn right, then both lanes are equally good; if it does, then the left lane is better.

In any game, a player's action "weakly dominates" another action if the first action is at least as good as the second action, no matter what the other players do, and is better than the second action for some actions of the other players.

▶ DEFINITION 45.1 (*Weak domination*) In a strategic game with ordinal preferences, player i's action a_i'' **weakly dominates** her action a_i' if

$$u_i(a_i'', a_{-i}) \geq u_i(a_i', a_{-i}) \text{ for every list } a_{-i} \text{ of the other players' actions}$$

and

$$u_i(a_i'', a_{-i}) > u_i(a_i', a_{-i}) \text{ for some list } a_{-i} \text{ of the other players' actions,}$$

where u_i is a payoff function that represents player i's preferences.

For example, in the game in Figure 45.1 (in which, once again, only player 1's payoffs are given), M weakly dominates T, and B weakly dominates M; B strictly dominates T.

	L	R
T	1	0
M	2	0
B	2	1

Figure 45.1 A game illustrating weak domination. (Only player 1's payoffs are given.) The action M weakly dominates T; B weakly dominates M. The action B strictly dominates T.

In a *strict* Nash equilibrium (Section 2.7.8) no player's equilibrium action is weakly dominated: every non-equilibrium action for a player yields her a payoff less than does her equilibrium action, and hence does not weakly dominate the equilibrium action.

Can an action be weakly dominated in a nonstrict Nash equilibrium? Definitely. Consider the games in Figure 46.1. In both games B weakly (but not strictly) dominates C for both players. But in both games (C, C) is a Nash equilibrium: *given* that player 2 chooses C, player 1 cannot do better than choose C, and *given* that player 1 chooses C, player 2 cannot do better than choose C. Both games also have a Nash equilibrium, (B, B), in which neither player's action is weakly dominated. In the left-hand game this equilibrium is better for both players than the equilibrium (C, C) in which both players' actions are weakly dominated, whereas in the right-hand game it is worse for both players than (C, C).

? EXERCISE 45.2 (Strict equilibria and dominated actions) For the game in Figure 46.2, determine, for each player, whether any action is strictly dominated or weakly dominated. Find the Nash equilibria of the game; determine whether any equilibrium is strict.

	B	C
B	$1,1$	$0,0$
C	$0,0$	$0,0$

	B	C
B	$1,1$	$2,0$
C	$0,2$	$2,2$

Figure 46.1 Two strategic games with a Nash equilibrium (C,C) in which both players' actions are weakly dominated.

	L	C	R
T	$0,0$	$1,0$	$1,1$
M	$1,1$	$1,1$	$3,0$
B	$1,1$	$2,1$	$2,2$

Figure 46.2 The game in Exercise 45.2.

? EXERCISE 46.1 (Nash equilibrium and weakly dominated actions) Give an example of a two-player strategic game in which each player has finitely many actions and in the only Nash equilibrium both players' actions are weakly dominated.

2.9.3 Illustration: voting

Two candidates, A and B, vie for office. Each of an odd number of citizens may vote for either candidate. (Abstention is not possible.) The candidate who obtains the most votes wins. (Because the number of citizens is odd, a tie is impossible.) A majority of citizens prefer A to win than B to win.

The following strategic game models the citizens' voting decisions in this situation.

Players The citizens.

Actions Each player's set of actions consists of voting for A and voting for B.

Preferences All players are indifferent between all action profiles in which a majority of players vote for A and between all action profiles in which a majority of players vote for B. Some players (a majority) prefer an action profile of the first type to one of the second type, and the others have the reverse preference.

I claim that a citizen's voting for her less preferred candidate is weakly dominated by her voting for her favorite candidate. Suppose that citizen i prefers candidate A; fix the votes of all citizens other than i. If citizen i switches from voting for B to voting for A then, depending on the other citizens' votes, either the outcome does not change, or A wins rather than B; such a switch cannot cause the winner to change from A to B. That is, citizen i's switching from voting for B to voting for A either has no effect on the outcome, or makes her better off; it cannot make her worse off.

The game has Nash equilibria in which some, or all, citizens' actions are weakly dominated. For example, the action profile in which all citizens vote for B is a Nash equilibrium (no citizen's switching her vote has any effect on the outcome).

? EXERCISE 47.1 (Voting) Find all the Nash equilibria of the game. (First consider action profiles in which the winner obtains one more vote than the loser and at least one citizen who votes for the winner prefers the loser to the winner, then profiles in which the winner obtains one more vote than the loser and all citizens who vote for the winner prefer the winner to the loser, and finally profiles in which the winner obtains three or more votes more than the loser.) Is there any equilibrium in which no player uses a weakly dominated action?

Consider a variant of the game in which the number of candidates is greater than two. A variant of the argument above shows that a citizen's action of voting for her least preferred candidate is weakly dominated by all her other actions. The next exercise asks you to show that no other action is weakly dominated.

? EXERCISE 47.2 (Voting between three candidates) Suppose there are three candidates, A, B, and C. A tie for first place is possible in this case; assume that a citizen who prefers a win by x to a win by y ranks a tie between x and y between an outright win for x and an outright win for y. Show that a citizen's only weakly dominated action is a vote for her least preferred candidate. Find a Nash equilibrium in which some citizen does not vote for her favorite candidate, but the action she takes is not weakly dominated.

? EXERCISE 47.3 (Approval voting) In the system of "approval voting", a citizen may vote for as many candidates as she wishes. If there are two candidates, say A and B, for example, a citizen may vote for neither candidate, for A, for B, or for both A and B. As before, the candidate who obtains the most votes wins. Show that any action that includes a vote for a citizen's least preferred candidate is weakly dominated, as is any action that does not include a vote for her most preferred candidate. More difficult: show that if there are k candidates then for a citizen who prefers candidate 1 to candidate 2 to ... to candidate k the action that consists of votes for candidates 1 and $k-1$ is *not* weakly dominated.

2.9.4 Illustration: collective decision-making

The members of a group of people are affected by a policy, modeled as a number. Each person i has a favorite policy, denoted x_i^*; she prefers the policy y to the policy z if and only if y is closer to x_i^* than is z. The number n of people is odd. The following mechanism is used to choose a policy: each person names a policy, and the policy chosen is the median of those named. (That is, the policies named are put in order, and the one in the middle is chosen. If, for example, there are five people, and they name the policies -2, 0, 0.6, 5, and 10, then the policy 0.6 is chosen.)

What outcome does this mechanism induce? Does anyone have an incentive to name her favorite policy, or are people induced to distort their preferences? We can answer these questions by studying the following strategic game.

Players The n people.

Actions Each person's set of actions is the set of policies (numbers).

Preferences Each person i prefers the action profile a to the action profile a' if and only if the median policy named in a is closer to x_i^* than is the median policy named in a'.

I claim that for each player i, the action of naming her favorite policy x_i^* weakly dominates *all* her other actions. The reason is that relative to the situation in which she names x_i^*, she can change the median only by naming a policy *further* from her favorite policy than the current median; no change in the policy she names moves the median closer to her favorite policy.

Precisely, I show that for each action $x_i \neq x_i^*$ of player i, (a) for *all* actions of the other players, player i is at least as well off naming x_i^* as she is naming x_i, and (b) for *some* actions of the other players she is better off naming x_i^* than she is naming x_i. Take $x_i > x_i^*$.

a. For any list of actions of the players *other than* player i, denote the value of the $\frac{1}{2}(n-1)$th highest action by $\underline{a}$ and the value of the $\frac{1}{2}(n+1)$th highest action by $\overline{a}$ (so that half of the remaining players' actions are at most $\underline{a}$ and half of them are at least $\overline{a}$).

- If $\overline{a} \leq x_i^*$ or $\underline{a} \geq x_i$ then the median policy is the same whether player i names x_i^* or x_i.
- If $\overline{a} > x_i^*$ and $\underline{a} < x_i$ then when player i names x_i^* the median policy is at most the greater of x_i^* and $\underline{a}$ and when player i names x_i the median policy is at least the lesser of x_i and $\overline{a}$. Thus player i is worse off naming x_i than she is naming x_i^*.

b. Suppose that half of the remaining players name policies less than x_i^* and half of them name policies greater than x_i. Then the outcome is x_i^* if player i names x_i^*, and x_i if she names x_i. Thus she is better off naming x_i^* than she is naming x_i.

A symmetric argument applies when $x_i < x_i^*$.

If we think of the mechanism as asking the players to name their favorite policies, then the result is that telling the truth weakly dominates all other actions.

An implication of the fact that player i's naming her favorite policy x_i^* weakly dominates *all* her other actions is that the action profile in which every player names her favorite policy is a Nash equilibrium. That is, truth-telling is a Nash equilibrium, in the interpretation of the previous paragraph.

? EXERCISE 49.1 (Other Nash equilibria of the game modeling collective decision-making) Find two Nash equilibria in which the outcome is the median favorite policy, and one in which it is not.

? EXERCISE 49.2 (Another mechanism for collective decision-making) Consider the variant of the mechanism for collective decision-making described above in which the policy chosen is the *mean*, rather than the median, of the policies named by the players. Does a player's action of naming her favorite policy weakly dominate all her other actions?

2.10 Equilibrium in a single population: symmetric games and symmetric equilibria

A Nash equilibrium of a strategic game corresponds to a steady state of an interaction between the members of several populations, one for each player in the game, each play of the game involving one member of each population. Sometimes we want to model a situation in which the members of a *single* homogeneous population are involved anonymously in a symmetric interaction. Consider, for example, pedestrians approaching each other on a sidewalk or car drivers arriving simultaneously at an intersection from different directions. In each case, the members of each encounter are drawn from the same population: pairs from a single population of pedestrians meet each other, and groups from a single population of car drivers simultaneously approach intersections. And in each case, every participant's role is the same.

I restrict attention here to cases in which each interaction involves two participants. Define a two-player game to be "symmetric" if each player has the same set of actions and each player's evaluation of an outcome depends only on her action and that of her opponent, not on whether she is player 1 or player 2. That is, player 1 feels the same way about the outcome (a_1, a_2), in which her action is a_1 and her opponent's action is a_2, as player 2 feels about the outcome (a_2, a_1), in which *her* action is a_1 and her opponent's action is a_2. In particular, the players' preferences may be represented by payoff functions in which both players' payoffs are the same whenever the players choose the same action: $u_1(a, a) = u_2(a, a)$ for every action a.

▶ DEFINITION 49.3 (*Symmetric two-player strategic game with ordinal preferences*) A two-player strategic game with ordinal preferences is **symmetric** if the players' sets of actions are the same and the players' preferences are represented by payoff functions u_1 and u_2 for which $u_1(a_1, a_2) = u_2(a_2, a_1)$ for every action pair (a_1, a_2).

A two-player game in which each player has two actions is symmetric if the players' preferences are represented by payoff functions that take the form shown in Figure 50.1, where w, x, y, and z are arbitrary numbers. Several of the two-player games we have considered are symmetric, including the *Prisoner's Dilemma*, the two-player *Stag Hunt* (given again in Figure 50.2), and the game in Exercise 36.2. *BoS* (Figure 16.1) and *Matching Pennies* (Figure 17.1) are not symmetric.

	A	B
A	w, w	x, y
B	y, x	z, z

Figure 50.1 A two-player symmetric game.

	Quiet	*Fink*
Quiet	2, 2	0, 3
Fink	3, 0	1, 1

	Stag	*Hare*
Stag	2, 2	0, 1
Hare	1, 0	1, 1

Figure 50.2 Two symmetric games: the *Prisoner's Dilemma* (left) and the two-player *Stag Hunt* (right).

? EXERCISE 50.1 (Symmetric strategic games) Which of the games in Exercises 29.1 and 40.1, Example 37.1, Section 2.8.4, and Figure 46.1 are symmetric?

When the players in a symmetric two-player game are drawn from a single population, nothing distinguishes one of the players in any given encounter from the other. We may call them "player 1" and "player 2", but these labels are only for our convenience. There is only one role in the game, so that a steady state is characterized by a *single* action used by every participant whenever playing the game. An action a^* corresponds to such a steady state if no player can do better by using any other action, given that all the other players use a^*. An action a^* has this property if and only if (a^*, a^*) is a Nash equilibrium of the game. In other words, the solution that corresponds to a steady state of pairwise interactions between the members of a single population is "symmetric Nash equilibrium": a Nash equilibrium in which both players take the same action. The idea of this notion of equilibrium does not depend on the game's having only two players, so I give a definition for a game with any number of players.

► DEFINITION 50.2 (*Symmetric Nash equilibrium*) An action profile a^* in a strategic game with ordinal preferences in which each player has the same set of actions is a **symmetric Nash equilibrium** if it is a Nash equilibrium and a_i^* is the same for every player i.

As an example, consider a model of approaching pedestrians. Each participant in any given encounter has two possible actions—to step to the right, and to step to the left—and is better off when participants both step in the same direction than when they step in different directions (in which case a collision occurs). The resulting symmetric strategic game is given in Figure 51.1. The game has two symmetric Nash equilibria, namely $(Left, Left)$ and $(Right, Right)$. That is, there are two steady states, in one of which every pedestrian steps to the left as she

	Left	*Right*
Left	1,1	0,0
Right	0,0	1,1

Figure 51.1 Approaching pedestrians.

approaches another pedestrian, and in another of which both participants step to the right. (The latter steady state seems to prevail in the USA and Canada.)

A symmetric game may have no symmetric Nash equilibrium. Consider, for example, the game in Figure 51.2. This game has two Nash equilibria, (X, Y) and (Y, X), neither of which is symmetric. You may wonder if, in such a situation, there is a steady state in which each player does not always take the same action in every interaction. This question is addressed in Section 4.7.

	X	Y
X	0,0	1,1
Y	1,1	0,0

Figure 51.2 A symmetric game with no symmetric Nash equilibrium.

? EXERCISE 51.1 (Equilibrium for pairwise interactions in a single population) Find all the Nash equilibria of the game in Figure 51.3. Which of the equilibria, if any, correspond to a steady state if the game models pairwise interactions between the members of a single population?

	A	B	C
A	1,1	2,1	4,1
B	1,2	5,5	3,6
C	1,4	6,3	0,0

Figure 51.3 The game in Exercise 51.1.

Notes

The notion of a strategic game originated in the work of Borel (1921) and von Neumann (1928). The notion of Nash equilibrium (and its interpretation) is due to Nash (1950a). (The idea that underlies it goes back at least to Cournot (1838, Ch. 7).)

The *Prisoner's Dilemma* appears to have first been considered by Melvin Dresher and Merrill Flood, who used it in an experiment at the RAND Corporation in January 1950 (Flood 1958/59, 11–17); it is an example in Nash's PhD thesis, submitted in May 1950. The story associated with it is due to Tucker (1950) (see Straffin 1980). O'Neill (1994, 1010–1013) argues that there is no evidence that game theory (and in particular the *Prisoner's Dilemma*) influenced US nuclear strategists in the 1950s. The idea that a common property will be overused is very old (in Western thought, it goes back at least to Aristotle (Ostrom 1990, 2)); a precise modern analysis was initiated by Gordon (1954). Hardin (1968) coined the phrase "tragedy of the commons".

BoS, like the *Prisoner's Dilemma*, is an example in Nash's PhD thesis; Luce and Raiffa (1957, 90–91) name it and associate a story with it. *Matching Pennies* was first considered by von Neumann (1928). Rousseau's sentence about hunting stags is interpreted as a description of a game by Ullmann-Margalit (1977, 121) and Jervis (1977/78), following discussion by Waltz (1959, 167–169) and Lewis (1969, 7, 47).

The information about John Nash in the box on p. 20 comes from Leonard (1994), Kuhn et al. (1995), Kuhn (1996), Myerson (1996), Nasar (1998), and Nash (1995). *Hawk–Dove* is known also as "Chicken" (two drivers approach each other on a narrow road; the one who pulls over first is "chicken"). It was first suggested (in a more complicated form) as a model of animal conflict by Maynard Smith and Price (1973). The discussion of focal points in the box on p. 30 draws on Schelling (1960, 54–58).

Games modeling voluntary contributions to a public good were first considered by Olson (1965, Section I.D). The game in Exercise 31.1 is studied in detail by Palfrey and Rosenthal (1984). The result in Section 2.8.4 is due to Warr (1983) and Bergstrom, Blume, and Varian (1986).

Game theory was first used to study voting behavior by Farquharson (1969) (whose book was completed in 1958). The system of "approval voting" in Exercise 47.3 was first studied formally by Brams and Fishburn (1978, 1983).

Exercise 16.1 is based on Leonard (1990). Exercise 25.2 is based on Ullmann-Margalit (1977, 48). The game in Exercise 28.2 is taken from Van Huyck, Battalio, and Beil (1990). The game in Exercise 32.1 is taken from Moulin (1986, 72). The game in Exercise 32.2 was first studied by Palfrey and Rosenthal (1983). Exercise 32.3 is based on Braess (1968); see also Murchland (1970). The game in Exercise 36.2 is taken from Brams (1993).

3 Nash Equilibrium: Illustrations

Prerequisite: Chapter 2.

IN THIS CHAPTER I discuss in detail a few key models that use the notion of Nash equilibrium to study economic, political, and biological phenomena. The discussion shows how the notion of Nash equilibrium improves our understanding of a wide variety of phenomena. It also illustrates some of the many forms strategic games and their Nash equilibria can take. The models in Sections 3.1 and 3.2 are related to each other, whereas those in each of the other sections are independent of each other.

3.1 Cournot's model of oligopoly

3.1.1 Introduction

How does the outcome of competition among the firms in an industry depend on the characteristics of the demand for the firms' output, the nature of the firms' cost functions, and the number of firms? Will the benefits of technological improvements be passed on to consumers? Will a reduction in the number of firms generate a less desirable outcome? To answer these questions we need a model of the interaction between firms competing for the business of consumers. In this section and the next I analyze two such models. Economists refer to them as models of "oligopoly" (competition between a small number of sellers), though they involve no restriction on the number of firms; the label reflects the strategic interaction they capture. Both models were studied first in the nineteenth century, before the notion of Nash equilibrium was formalized for a general strategic game. The first is due to the economist Cournot (1838).

3.1.2 General model

A single good is produced by n firms. The cost to firm i of producing q_i units of the good is $C_i(q_i)$, where C_i is an increasing function (more output is more costly to produce). All the output is sold at a single price, determined by the demand for the good and the firms' total output. Specifically, if the firms' total output is Q then the market price is $P(Q)$; P is called the "inverse demand function". Assume that P is a decreasing function when it is positive: if the firms' total output increases, then the price decreases (unless it is already zero). If the output of each firm i is q_i, then the price is $P(q_1 + \cdots + q_n)$, so that firm i's revenue is $q_i P(q_1 + \cdots + q_n)$. Thus firm i's profit, equal to its revenue minus its cost, is

$$\pi_i(q_1, \ldots, q_n) = q_i P(q_1 + \cdots + q_n) - C_i(q_i). \tag{54.1}$$

Cournot suggested that the industry be modeled as the following strategic game, which I refer to as **Cournot's oligopoly game**.

Players The firms.

Actions Each firm's set of actions is the set of its possible outputs (nonnegative numbers).

Preferences Each firm's preferences are represented by its profit, given in (54.1).

3.1.3 Example: duopoly with constant unit cost and linear inverse demand function

For specific forms of the functions C_i and P we can compute a Nash equilibrium of Cournot's game. Suppose there are two firms (the industry is a "duopoly"), each firm's cost function is the same, given by $C_i(q_i) = cq_i$ for all q_i ("unit cost" is constant, equal to c), and the inverse demand function is linear where it is positive, given by

$$P(Q) = \begin{cases} \alpha - Q & \text{if } Q \leq \alpha \\ 0 & \text{if } Q > \alpha, \end{cases} \tag{54.2}$$

where $\alpha > 0$ and $c \geq 0$ are constants. This inverse demand function is shown in Figure 55.1. (Note that the price $P(Q)$ cannot be equal to $\alpha - Q$ for *all* values of Q, for then it would be negative for $Q > \alpha$.) Assume that $c < \alpha$, so that there is some value of total output Q for which the market price $P(Q)$ is greater than the firms' common unit cost c. (If c were to exceed α, there would be no output for the firms at which they could make any profit, because the market price never exceeds α.)

To find the Nash equilibria in this example, we can use the procedure based on the firms' best response functions (Section 2.8.3). First we need to find the firms' payoffs (profits). If the firms' outputs are q_1 and q_2 then the market price $P(q_1 + q_2)$ is $\alpha - q_1 - q_2$ if $q_1 + q_2 \leq \alpha$ and zero if $q_1 + q_2 > \alpha$. Thus firm 1's profit is

$$\begin{aligned} \pi_1(q_1, q_2) &= q_1(P(q_1 + q_2) - c) \\ &= \begin{cases} q_1(\alpha - c - q_1 - q_2) & \text{if } q_1 + q_2 \leq \alpha \\ -cq_1 & \text{if } q_1 + q_2 > \alpha. \end{cases} \end{aligned}$$

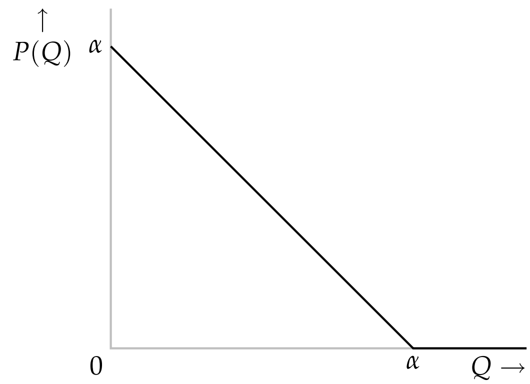


Figure 55.1 The inverse demand function in the example of Cournot's game studied in Section 3.1.3.

To find firm 1's best response to any given output q_2 of firm 2, we need to study firm 1's profit as a function of its output q_1 for given values of q_2. If $q_2 = 0$ then firm 1's profit is $\pi_1(q_1, 0) = q_1(\alpha - c - q_1)$ for $q_1 \leq \alpha$, a quadratic function that is zero when $q_1 = 0$ and when $q_1 = \alpha - c$. This function is the black curve in Figure 56.1. Given the symmetry of quadratic functions (Section 17.4), the output q_1 of firm 1 that maximizes its profit is $q_1 = \frac{1}{2}(\alpha - c)$. (If you know calculus, you can reach the same conclusion by setting the derivative of firm 1's profit with respect to q_1 equal to zero and solving for q_1.) Thus firm 1's best response to an output of zero for firm 2 is $b_1(0) = \frac{1}{2}(\alpha - c)$.

As the output q_2 of firm 2 increases, the profit firm 1 can obtain at any given output decreases, because more output of firm 2 means a lower price. The gray curve in Figure 56.1 is an example of $\pi_1(q_1, q_2)$ for $q_2 > 0$ and $q_2 < \alpha - c$. Again this function is a quadratic up to the output $q_1 = \alpha - q_2$ that leads to a price of zero. Specifically, the quadratic is $\pi_1(q_1, q_2) = q_1(\alpha - c - q_2 - q_1)$, which is zero when $q_1 = 0$ and when $q_1 = \alpha - c - q_2$. From the symmetry of quadratic functions (or some calculus) we conclude that the output that maximizes $\pi_1(q_1, q_2)$ is $q_1 = \frac{1}{2}(\alpha - c - q_2)$. (When $q_2 = 0$, this is equal to $\frac{1}{2}(\alpha - c)$, the best response to an output of zero that we found in the previous paragraph.)

When $q_2 > \alpha - c$, the value of $\alpha - c - q_2$ is negative. Thus for such a value of q_2, we have $q_1(\alpha - c - q_2 - q_1) < 0$ for all positive values of q_1: firm 1's profit is negative for any positive output, so that its best response is to produce the output of zero.

We conclude that the best response of firm 1 to the output q_2 of firm 2 depends on the value of q_2: if $q_2 \leq \alpha - c$ then firm 1's best response is $\frac{1}{2}(\alpha - c - q_2)$, whereas if $q_2 > \alpha - c$ then firm 1's best response is 0. Or, more compactly,

$$b_1(q_2) = \begin{cases} \frac{1}{2}(\alpha - c - q_2) & \text{if } q_2 \leq \alpha - c \\ 0 & \text{if } q_2 > \alpha - c. \end{cases}$$

Because firm 2's cost function is the same as firm 1's, its best response function b_2 is also the same: for any number q, we have $b_2(q) = b_1(q)$. Of course, firm 2's

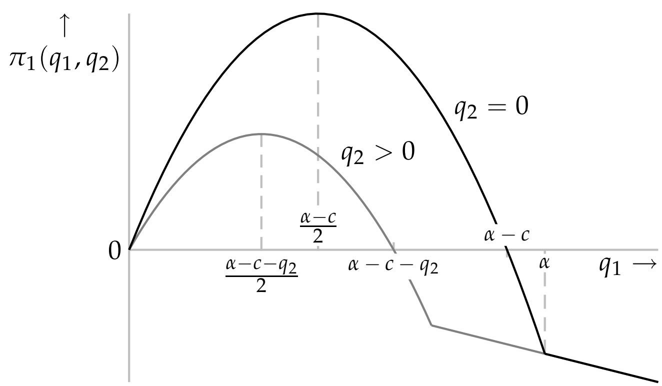


Figure 56.1 Firm 1's profit as a function of its output, given firm 2's output. The black curve shows the case $q_2 = 0$, whereas the gray curve shows a case in which $q_2 > 0$.

best response function associates a value of firm 2's output with every output of firm 1, whereas firm 1's best response function associates a value of firm 1's output with every output of firm 2, so we plot them relative to different axes. They are shown in Figure 56.2 (b_1 is black; b_2 is gray). As for a general game (see Section 2.8.3), b_1 associates each point on the vertical axis with a point on the horizontal axis, and b_2 associates each point on the horizontal axis with a point on the vertical axis.

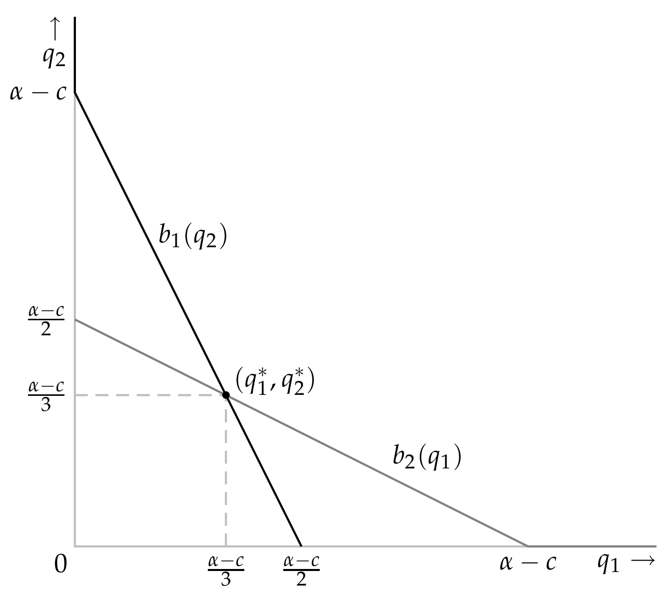


Figure 56.2 The best response functions in Cournot's duopoly game when the inverse demand function is given by (54.2) and the cost function of each firm is cq. The unique Nash equilibrium is $(q_1^*, q_2^*) = (\frac{1}{3}(\alpha - c), \frac{1}{3}(\alpha - c))$.

A Nash equilibrium is a pair (q_1^*, q_2^*) of outputs for which q_1^* is a best response to q_2^*, and q_2^* is a best response to q_1^*:

$$q_1^* = b_1(q_2^*) \quad \text{and} \quad q_2^* = b_2(q_1^*)$$

(see (34.3)). The set of such pairs is the set of points at which the best response functions in Figure 56.2 intersect. From the figure we see that there is exactly one such point, which is given by the solution of the two equations

$$\begin{aligned} q_1 &= \tfrac{1}{2}(\alpha - c - q_2) \\ q_2 &= \tfrac{1}{2}(\alpha - c - q_1). \end{aligned}$$

Solving these two equations (by substituting the second into the first and then isolating q_1, for example) we find that $q_1^* = q_2^* = \frac{1}{3}(\alpha - c)$.

In summary, when there are two firms, the inverse demand function is given by $P(Q) = \alpha - Q$ for $Q \leq \alpha$, and the cost function of each firm is $C_i(q_i) = cq_i$, Cournot's oligopoly game has a unique Nash equilibrium $(q_1^*, q_2^*) = (\frac{1}{3}(\alpha - c), \frac{1}{3}(\alpha - c))$. The total output in this equilibrium is $\frac{2}{3}(\alpha - c)$, so that the price at which output is sold is $P(\frac{2}{3}(\alpha - c)) = \frac{1}{3}(\alpha + 2c)$. As α increases (meaning that consumers are willing to pay more for the good), the equilibrium price and the output of each firm increases. As c (the unit cost of production) increases, the output of each firm falls and the price rises; each unit increase in c leads to a two-thirds of a unit increase in the price.

? EXERCISE 57.1 (Cournot's duopoly game with linear inverse demand and different unit costs) Find the Nash equilibrium of Cournot's game when there are two firms, the inverse demand function is given by (54.2), the cost function of each firm i is $C_i(q_i) = c_i q_i$, where $c_1 > c_2$, and $c_1 < \alpha$. (There are two cases, depending on the size of c_1 relative to c_2.) Which firm produces more output in an equilibrium? What is the effect of technical change that lowers firm 2's unit cost c_2 (while not affecting firm 1's unit cost c_1) on the firms' equilibrium outputs, the total output, and the price?

? EXERCISE 57.2 (Cournot's duopoly game with linear inverse demand and a quadratic cost function) Find the Nash equilibrium of Cournot's game when there are two firms, the inverse demand function is given by (54.2), and the cost function of each firm i is $C_i(q_i) = q_i^2$.

In the next exercise each firm's cost function has a component that is independent of output. You will find in this case that Cournot's game may have more than one Nash equilibrium.

? EXERCISE 57.3 (Cournot's duopoly game with linear inverse demand and a fixed cost) Find the Nash equilibria of Cournot's game when there are two firms, the inverse demand function is given by (54.2), and the cost function of each firm i is given by

$$C_i(q_i) = \begin{cases} 0 & \text{if } q_i = 0 \\ f + cq_i & \text{if } q_i > 0, \end{cases}$$

where $c \geq 0$, $f > 0$, and $c < \alpha$. (Note that the fixed cost f affects only the firm's decision of whether or not to operate; it does not affect the output a firm wishes to produce *if it wishes to operate*.)

So far we have assumed that each firm's objective is to maximize its profit. The next exercise asks you to consider a case in which one firm's objective is to maximize its market share.

? EXERCISE 58.1 (Variant of Cournot's game, with market-share maximizing firms) Find the Nash equilibrium (equilibria?) of a variant of the example of Cournot's duopoly game that differs from the one in this section (linear inverse demand, constant unit cost) only in that one of the two firms chooses its output to maximize its market share subject to not making a loss, rather than to maximize its profit. What happens if *each* firm maximizes its market share?

3.1.4 Properties of Nash equilibrium

Two economically interesting properties of a Nash equilibrium of Cournot's game concern the relation between the firms' equilibrium profits and the profits they could obtain if they acted collusively, and the character of an equilibrium when the number of firms is large.

Comparison of Nash equilibrium with collusive outcomes In Cournot's game with two firms, is there any pair of outputs at which *both* firms' profits exceed their levels in a Nash equilibrium? The next exercise asks you to show that the answer is "yes" in the example considered in the previous section. Specifically, both firms can increase their profits relative to their equilibrium levels by reducing their outputs.

? EXERCISE 58.2 (Nash equilibrium of Cournot's duopoly game and collusive outcomes) Find the total output (call it Q^*) that maximizes the firms' *total* profit in Cournot's game when there are two firms and the inverse demand function and cost functions take the forms assumed Section 3.1.3. Compare $\frac{1}{2}Q^*$ with each firm's output in the Nash equilibrium, and show that each firm's equilibrium profit is less than its profit in the "collusive" outcome in which each firm produces $\frac{1}{2}Q^*$. Why is this collusive outcome not a Nash equilibrium?

The same is true more generally. For nonlinear inverse demand functions and cost functions, the shapes of the firms' best response functions differ, in general, from those in the example studied in the previous section. But for many inverse demand functions and cost functions the game has a Nash equilibrium and, for any equilibrium, there are pairs of outputs in which each firm's output is less than its equilibrium level and each firm's profit exceeds its equilibrium level.

To see why, suppose that (q_1^*, q_2^*) is a Nash equilibrium and consider the set of pairs (q_1, q_2) of outputs at which firm 1's profit is at least its equilibrium profit. The assumption that P is decreasing (higher total output leads to a lower price) implies that if (q_1, q_2) is in this set and $q_2' < q_2$ then (q_1, q_2') is also in the set. (We have $q_1 + q_2' < q_1 + q_2$, and hence $P(q_1 + q_2') > P(q_1 + q_2)$, so that firm 1's profit at (q_1, q_2') exceeds its profit at (q_1, q_2).) Thus in Figure 59.1 the set of pairs of outputs at which firm 1's profit is at least its equilibrium profit lies on or below the line $q_2 = q_2^*$; an example of such a set is shaded light gray. Similarly, the set of pairs of outputs at which firm 2's profit is at least its equilibrium profit lies on or to the left of the line $q_1 = q_1^*$, and an example is shaded light gray.

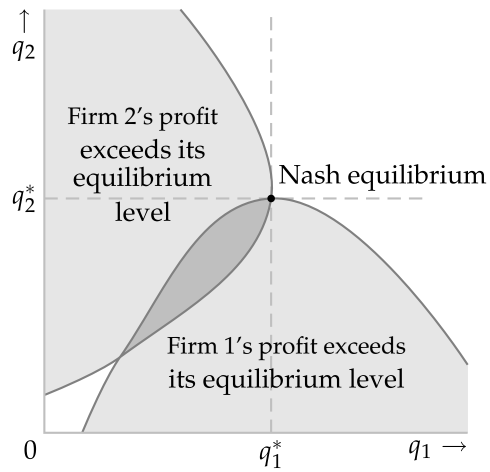


Figure 59.1 The pair (q_1^*, q_2^*) is a Nash equilibrium; along each gray curve one of the firm's profits is constant, equal to its profit at the equilibrium. The area shaded dark gray is the set of pairs of outputs at which both firms' profits exceed their equilibrium levels.

We see that if the parts of the boundaries of these sets indicated by the gray lines in the figure are smooth then the two sets must intersect; in the figure the intersection is shaded dark gray. At every pair of outputs in this area each firm's output is less than its equilibrium level ($q_i < q_i^*$ for $i = 1, 2$) and each firm's profit is higher than its equilibrium profit. That is, *both* firms are better off by restricting their outputs.

Dependence of Nash equilibrium on number of firms How does the equilibrium outcome in Cournot's game depend on the number of firms? If each firm's cost function has the same constant unit cost c, the best outcome for consumers compatible with no firm's making a loss has a price of c and a total output of $\alpha - c$. The next exercise asks you to show that if, for this cost function, the inverse demand function is linear (as in Section 3.1.3), then the price in the Nash equilibrium of Cournot's game decreases as the number of firms increases, approaching c. That is, from the viewpoint of consumers, the outcome is better the larger the number of firms, and when the number of firms is very large, the outcome is close to the best one compatible with nonnegative profits for the firms.

? EXERCISE 59.1 (Cournot's game with many firms) Consider Cournot's game in the case of an arbitrary number n of firms; retain the assumptions that the inverse demand function takes the form (54.2) and the cost function of each firm i is $C_i(q_i) = cq_i$ for all q_i, with $c < \alpha$. Find the best response function of each firm and set up the conditions for $(q_1^*, \ldots, q_n^*)$ to be a Nash equilibrium (see (34.3)), assuming that there is a Nash equilibrium in which all firms' outputs are positive. Solve these equations to find the Nash equilibrium. (For $n = 2$ your answer should be $(\frac{1}{3}(\alpha - c), \frac{1}{3}(\alpha - c))$, the equilibrium found in the previous section. First show that in an equilibrium all firms produce the same output, then solve for that output. If you cannot show that all firms produce the same output, simply assume that they do.) Find the price at which output is sold in a Nash equilibrium and show that this price decreases as n increases, approaching c as the number of firms increases without bound.

The main idea behind this result does not depend on the assumptions on the inverse demand function and the firms' cost functions. Suppose, more generally, that the inverse demand function is any decreasing function, that each firm's cost function is the same, denoted by C, and that there is a single output, say $\underline{q}$, at which the average cost of production $C(q)/q$ is minimal. In this case, any given total output is produced most efficiently by each firm's producing $\underline{q}$, and the lowest price compatible with the firms' not making losses is the minimal value of the average cost. The next exercise asks you to show that in a Nash equilibrium of Cournot's game in which the firms' total output is large relative to $\underline{q}$, this is the price at which the output is sold.

?? EXERCISE 60.1 (Nash equilibrium of Cournot's game with small firms) Suppose that there are infinitely many firms, all of which have the same cost function C. Assume that $C(0) = 0$, and for $q > 0$ the function $C(q)/q$ has a unique minimizer $\underline{q}$; denote the minimum of $C(q)/q$ by $\underline{p}$. Assume that the inverse demand function P is decreasing. Show that in any Nash equilibrium the firms' total output Q^* satisfies

$$P(Q^* + \underline{q}) \leq \underline{p} \leq P(Q^*).$$

(That is, the price is at least the minimal value $\underline{p}$ of the average cost, but is close enough to this minimum that increasing the total output of the firms by $\underline{q}$ would reduce the price to at most $\underline{p}$.) To establish these inequalities, show that if $P(Q^*) < \underline{p}$ or $P(Q^* + \underline{q}) > \underline{p}$ then Q^* is not the total output of the firms in a Nash equilibrium, because in each case at least one firm can deviate and increase its profit.

3.1.5 *A generalization of Cournot's game: using common property*

In Cournot's game, the payoff function of each firm i is $q_i P(q_1 + \cdots + q_n) - C_i(q_i)$. In particular, each firm's payoff depends only on its output and the sum of all the firm's outputs, not on the distribution of the total output among the firms, and decreases when this sum increases (given that P is decreasing). That is, the payoff of each firm i may be written as $f_i(q_i, q_1 + \cdots + q_n)$, where the function f_i is decreasing in its second argument (given the value of its first argument, q_i).

This general payoff function captures many situations in which players compete in using a piece of common property whose value to any one player diminishes as total use increases. The property might be a village green, for example; the higher the total number of sheep grazed there, the less valuable the green is to any given farmer.

The first property of a Nash equilibrium in Cournot's model discussed in the previous section applies to this general model: common property is "overused" in a Nash equilibrium in the sense that every player's payoff increases when every player reduces her use of the property from its equilibrium level. For example, all farmers' payoffs increase if each farmer reduces her use of the village green from its equilibrium level: in an equilibrium the green is "overgrazed". The argument is the same as the one illustrated in Figure 59.1 in the case of two players, because this argument depends only on the fact that each player's payoff function is smooth and is decreasing in the other player's action. (In Cournot's model, the "common property" that is overused is the demand for the good.)

? EXERCISE 61.1 (Interaction among resource-users) A group of n firms uses a common resource (a river or a forest, for example) to produce output. As more of the resource is used, any given firm can produce less output. Denote by x_i the amount of the resource used by firm i $(= 1, \ldots, n)$. Assume specifically that firm i's output is $x_i(1 - (x_1 + \cdots + x_n))$ if $x_1 + \cdots + x_n \leq 1$, and zero otherwise. Each firm i chooses x_i to maximize its output. Formulate this situation as a strategic game. Find values of α and c such that the game is the same as the one studied in Exercise 59.1, and hence find its Nash equilibria. Find an action profile $(x_1, \ldots, x_n)$ at which each firm's output is higher than it is at the Nash equilibrium.

3.2 Bertrand's model of oligopoly

3.2.1 General model

In Cournot's game, each firm chooses an output; the price is determined by the demand for the good in relation to the total output produced. In an alternative model of oligopoly, associated with a review of Cournot's book by Bertrand (1883), each firm chooses a price, and produces enough output to meet the demand it faces, given the prices chosen by all the firms. The model is designed to shed light on the same questions that Cournot's game addresses; as we shall see, some of the answers it gives are different.

The economic setting for the model is similar to that for Cournot's game. A single good is produced by n firms; each firm can produce q_i units of the good at a cost of $C_i(q_i)$. It is convenient to specify demand by giving a "demand function" D, rather than an inverse demand function as we did for Cournot's game. The interpretation of D is that if the good is available at the price p then the total amount demanded is $D(p)$.

Assume that if the firms set different prices then all consumers purchase the good from the firm with the lowest price, which produces enough output to meet this demand. If more than one firm sets the lowest price, all the firms doing so share the demand at that price equally. A firm whose price is not the lowest price receives no demand and produces no output. (Note that a firm does not choose its output strategically; it simply produces enough to satisfy all the demand it faces, given the prices, even if its price is below its unit cost, in which case it makes a loss. This assumption can be modified at the price of complicating the model.)

In summary, **Bertrand's oligopoly game** is the following strategic game.

Players The firms.

Actions Each firm's set of actions is the set of possible prices (nonnegative numbers).

Preferences Firm i's preferences are represented by its profit, equal to $p_i D(p_i)/m - C_i(D(p_i)/m)$ if firm i is one of m firms setting the lowest price ($m = 1$ if firm i's price p_i is lower than every other price), and equal to zero if some firm's price is lower than p_i.

3.2.2 Example: duopoly with constant unit cost and linear demand function

Suppose, as in Section 3.1.3, that there are two firms, each of whose cost functions has constant unit cost c (that is, $C_i(q_i) = cq_i$ for $i = 1, 2$). Assume that the demand function is $D(p) = \alpha - p$ for $p \leq \alpha$ and $D(p) = 0$ for $p > \alpha$, and that $c < \alpha$.

Because the cost of producing each unit is the same, equal to c, firm i makes the profit of $p_i - c$ on every unit it sells. Thus its profit is

$$\pi_i(p_1, p_2) = \begin{cases} (p_i - c)(\alpha - p_i) & \text{if } p_i < p_j \\ \frac{1}{2}(p_i - c)(\alpha - p_i) & \text{if } p_i = p_j \\ 0 & \text{if } p_i > p_j, \end{cases}$$

where j is the other firm ($j = 2$ if $i = 1$, and $j = 1$ if $i = 2$).

As before, we can find the Nash equilibria of the game by finding the firms' best response functions. If firm j charges p_j, what is the best price for firm i to charge? We can reason informally as follows. If firm i charges p_j, it shares the market with firm j; if it charges slightly less, it sells to the entire market. Thus if p_j exceeds c, so that firm i makes a positive profit selling the good at a price slightly below p_j, firm i is definitely better off serving all the market at such a price than serving half of the market at the price p_j. If p_j is very high, however, firm i may be able to do even better: by reducing its price significantly below p_j it may increase its profit, because the extra demand engendered by the lower price may more than compensate for the lower revenue per unit sold. Finally, if p_j is less than c, then firm i's profit is negative if it charges a price less than or equal to p_j, whereas this profit is zero if it charges a higher price. Thus in this case firm i would like to charge any price greater than p_j, to make sure that it gets no customers. (Remember that if customers arrive at its door it is obliged to serve them, whether or not it makes a profit by so doing.)

We can make these arguments precise by studying firm i's payoff as a function of its price p_i for various values of the price p_j of firm j. Denote by p^m the value of p (price) that maximizes $(p-c)(\alpha-p)$. This price would be charged by a firm with a monopoly of the market (because $(p-c)(\alpha-p)$ is the profit of such a firm). Three cross-sections of firm i's payoff function, for different values of p_j, are shown in black in Figure 63.1. (The gray dashed line is the function $(p_i-c)(\alpha-p_i)$.)

- If $p_j < c$ (firm j's price is below the unit cost) then firm i's profit is negative if $p_i \leq p_j$ and zero if $p_i > p_j$ (see the left panel of Figure 63.1). Thus *any* price greater than p_j is a best response to p_j. That is, the set of firm i's best responses is $B_i(p_j) = \{p_i\colon p_i > p_j\}$.

- If $p_j = c$ then the analysis is similar to that of the previous case except that p_j, as well as any price greater than p_j, yields a profit of zero, and hence is a best response to p_j: $B_i(p_j) = \{p_i\colon p_i \geq p_j\}$.

- If $c < p_j \leq p^m$ then firm i's profit increases as p_i increases to p_j, then drops abruptly at p_j (see the middle panel of Figure 63.1). Thus there is no best response: firm i wants to choose a price less than p_j, but is better off the closer that price is to p_j. For any price less than p_j there is a higher price that is also less than p_j, so there is no best price. (I have assumed that a firm can choose *any* number as its price; in particular, it is not restricted to charge an integral number of cents.) Thus $B_i(p_j)$ is empty (has no members).

- If $p_j > p^m$ then p^m is the unique best response of firm i (see the right panel of Figure 63.1): $B_i(p_j) = \{p^m\}$.

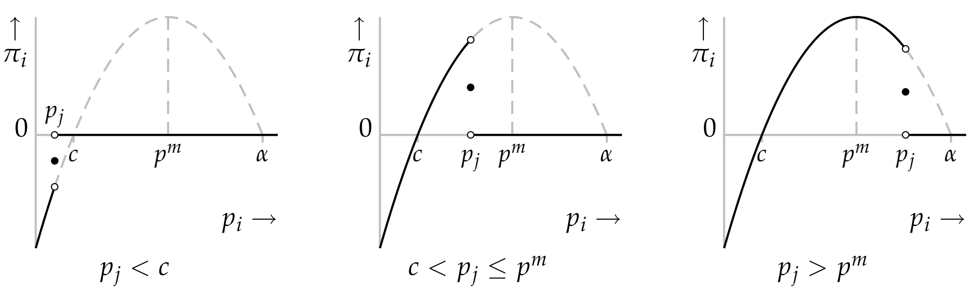


Figure 63.1 Three cross-sections (in black) of firm i's payoff function in Bertrand's duopoly game. Where the payoff function jumps, its value is given by the small disk; the small circles indicate points that are excluded as values of the functions.

In summary, firm i's best response function is given by

$$B_i(p_j) = \begin{cases} \{p_i\colon p_i > p_j\} & \text{if } p_j < c \\ \{p_i\colon p_i \geq p_j\} & \text{if } p_j = c \\ \varnothing & \text{if } c < p_j \leq p^m \\ \{p^m\} & \text{if } p^m < p_j, \end{cases}$$

where ∅ denotes the set with no members (the "empty set"). Note the respects in which this best response function differs qualitatively from a firm's best response function in Cournot's game: for some actions of its opponent, a firm has no best response, and for some actions it has multiple best responses.

The fact that firm i has *no* best response when $c < p_j < p^m$ is an artifact of modeling price as a continuous variable (a firm can choose its price to be any non-negative number). If instead we assume that each firm's price must be a multiple of some indivisible unit ϵ (e.g. price must be an integral number of cents) then firm i's optimal response to a price p_j with $c < p_j < p^m$ is $p_j - \epsilon$. I model price as a continuous variable because doing so simplifies some of the analysis; in Exercise 65.2 you are asked to study the case of discrete prices.

When $p_j < c$, firm i's set of best responses is the set of all prices greater than p_j. In particular, prices between p_j and c are best responses. You may object that setting a price less than c is not very sensible. Such a price exposes firm i to the risk of making a loss (if firm j chooses a higher price) and has no advantage over the price of c, regardless of firm j's price. That is, such a price is *weakly dominated* (Definition 45.1) by the price c. Nevertheless, such a price *is* a best response! That is, it is optimal for firm i to choose such a price, *given* firm j's price: there is no price that yields firm i a higher profit, *given* firm j's price. The point is that when asking if a player's action is a best response to her opponent's action, we do not consider the "risk" that the opponent will take some other action.

Figure 64.1 shows the firms' best response functions (firm 1's on the left, firm 2's on the right). The shaded gray area in the left panel indicates that for a price p_2 less than c, *any* price greater than p_2 is a best response for firm 1. The absence of a black line along the sloping left boundary of this area indicates that only prices p_1 *greater than* (not equal to) p_2 are included. The black line along the top of the area indicates that for $p_2 = c$ any price greater than *or equal to* c is a best response. As before, the dot indicates a point that is included, whereas the small circle indicates a point that is excluded. Firm 2's best response function has a similar interpretation.

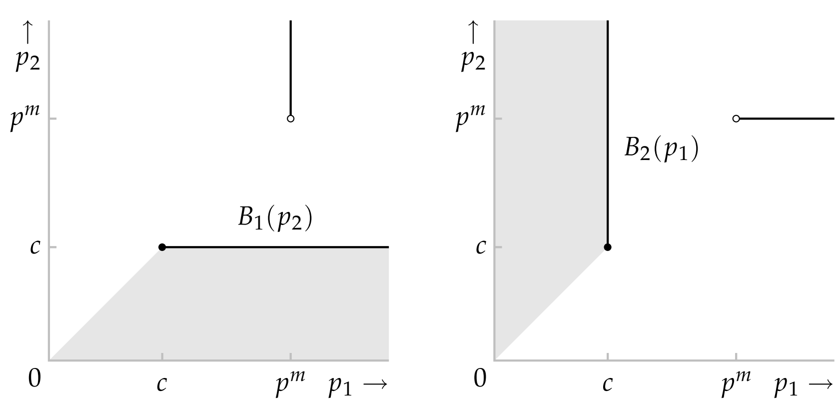


Figure 64.1 The firms' best response functions in Bertrand's duopoly game. Firm 1's best response function is in the left panel; firm 2's is in the right panel.

A Nash equilibrium is a pair (p_1^*, p_2^*) of prices such that p_1^* is a best response to p_2^*, and p_2^* is a best response to p_1^*—that is, p_1^* is in $B_1(p_2^*)$ and p_2^* is in $B_2(p_1^*)$ (see (34.2)). If we superimpose the two best response functions, any such pair is in the intersection of their graphs. If you do so, you will see that the graphs have a single point of intersection, namely $(p_1^*, p_2^*) = (c, c)$. That is, the game has a single Nash equilibrium, in which each firm charges the price c.

The method of finding the Nash equilibria of a game by constructing the players' best response functions is systematic. So long as these functions may be computed, the method straightforwardly leads to the set of Nash equilibria. However, in some games we can make a direct argument that avoids the need to construct the entire best response functions. Using a combination of intuition and trial and error we find the action profiles that seem to be equilibria, then we show precisely that any such profile is an equilibrium and every other profile is not an equilibrium. To show that a pair of actions is not a Nash equilibrium we need only find a *better* response for one of the players—not necessarily the *best* response.

In Bertrand's game we can argue as follows. (*i*) First we show that $(p_1, p_2) = (c, c)$ is a Nash equilibrium. If one firm charges the price c then the other firm can do no better than charge the price c also, because if it raises its price it sells no output, and if it lowers its price it makes a loss. (*ii*) Next we show that no other pair (p_1, p_2) is a Nash equilibrium, as follows.

- If $p_i < c$ for either $i = 1$ or $i = 2$ then the profit of the firm whose price is lowest (or the profit of both firms, if the prices are the same) is negative, and this firm can increase its profit (to zero) by raising its price to c.

- If $p_i = c$ and $p_j > c$ then firm i is better off increasing its price slightly, making its profit positive rather than zero.

- If $p_i > c$ and $p_j > c$, suppose that $p_i \geq p_j$. Then firm i can increase its profit by lowering p_i to slightly below p_j if $D(p_j) > 0$ (i.e. if $p_j < \alpha$) and to p^m if $D(p_j) = 0$ (i.e. if $p_j \geq \alpha$).

In conclusion, both arguments show that when the unit cost of production is a constant c, the same for both firms, and demand is linear, Bertrand's game has a unique Nash equilibrium, in which each firm's price is equal to c.

? EXERCISE 65.1 (Bertrand's duopoly game with constant unit cost) Consider the extent to which the analysis depends upon the demand function D taking the specific form $D(p) = \alpha - p$. Suppose that D is any function for which $D(p) \geq 0$ for all p and there exists $\overline{p} > c$ such that $D(p) > 0$ for all $p \leq \overline{p}$. Is (c, c) still a Nash equilibrium? Is it still the only Nash equilibrium?

? EXERCISE 65.2 (Bertrand's duopoly game with discrete prices) Consider the variant of the example of Bertrand's duopoly game in this section in which each firm is restricted to choose a price that is an integral number of cents. Assume that c is an integral number of cents and that $\alpha > c + 1$. Is (c, c) a Nash equilibrium of this game? Is there any other Nash equilibrium?

3.2.3 Discussion

For a duopoly in which both firms have the same constant unit cost and the demand function is linear, the Nash equilibria of Cournot's and Bertrand's games generate different economic outcomes. The equilibrium price in Bertrand's game is equal to the common unit cost c, whereas the price associated with the equilibrium of Cournot's game is $\frac{1}{3}(\alpha + 2c)$, which exceeds c because $c < \alpha$. In particular, the equilibrium price in Bertrand's game is the lowest price compatible with the firms' not making losses, whereas the price at the equilibrium of Cournot's game is higher. In Cournot's game, the price decreases towards c as the number of firms increases (Exercise 59.1), whereas in Bertrand's game it is c even if there are only two firms. In the next exercise you are asked to show that as the number of firms increases in Bertrand's game, the price remains c.

? EXERCISE 66.1 (Bertrand's oligopoly game) Consider Bertrand's oligopoly game when the cost and demand functions satisfy the conditions in Section 3.2.2 and there are n firms, with $n \geq 3$. Show that the set of Nash equilibria is the set of profiles $(p_1, \ldots, p_n)$ of prices for which $p_i \geq c$ for all i and at least two prices are equal to c. (Show that any such profile is a Nash equilibrium, and that every other profile is not a Nash equilibrium.)

What accounts for the difference between the Nash equilibria of Cournot's and Bertrand's games? The key point is that different strategic variables (output in Cournot's game, price in Bertrand's game) imply different strategic reasoning by the firms. In Cournot's game a firm changes its behavior if it can increase its profit by changing its output, on the assumption that the other firms' outputs will remain the same and the price will adjust to clear the market. In Bertrand's game a firm changes its behavior if it can increase its profit by changing its price, on the assumption that the other firms' prices will remain the same and their outputs will adjust to clear the market. Which assumption makes more sense depends on the context. For example, the wholesale market for agricultural produce may fit Cournot's game better, whereas the retail market for food may fit Bertrand's game better.

Under some variants of the assumptions in the previous section, Bertrand's game has no Nash equilibrium. In one case the firms' cost functions have constant unit costs, and these costs are different; in another case the cost functions have a fixed component. In both these cases, as well as in some other cases, an equilibrium is restored if we modify the way in which consumers are divided between the firms when the prices are the same, as the following exercises show. (We can think of the division of consumers between firms charging the same price as being determined as part of the equilibrium. Note that we retain the assumption that if the firms charge different prices then the one charging the lower price receives all the demand.)

? EXERCISE 66.2 (Bertrand's duopoly game with different unit costs) Consider Bertrand's duopoly game under a variant of the assumptions of Section 3.2.2 in which the firms' unit costs are different, equal to c_1 and c_2, where $c_1 < c_2$. Denote by p_1^m the price that maximizes $(p - c_1)(\alpha - p)$, and assume that $c_2 < p_1^m$ and that the function $(p - c_1)(\alpha - p)$ is increasing in p up to p_1^m.

a. Suppose that the rule for splitting up consumers when the prices are equal assigns all consumers to firm 1 when both firms charge the price c_2. Show that $(p_1, p_2) = (c_2, c_2)$ is a Nash equilibrium and that no other pair of prices is a Nash equilibrium.

b. Show that no Nash equilibrium exists if the rule for splitting up consumers when the prices are equal assigns some consumers to firm 2 when both firms charge c_2.

?? EXERCISE 67.1 (Bertrand's duopoly game with fixed costs) Consider Bertrand's game under a variant of the assumptions of Section 3.2.2 in which the cost function of each firm i is given by $C_i(q_i) = f + cq_i$ for $q_i > 0$, and $C_i(0) = 0$, where f is positive and less than the maximum of $(p - c)(\alpha - p)$ with respect to p. Denote by $\overline{p}$ the price p that satisfies $(p - c)(\alpha - p) = f$ and is less than the maximizer of $(p - c)(\alpha - p)$ (see Figure 67.1). Show that if firm 1 gets all the demand when both firms charge the same price then $(\overline{p}, \overline{p})$ is a Nash equilibrium. Show also that no other pair of prices is a Nash equilibrium. (First consider cases in which the firms charge the same price, then cases in which they charge different prices.)

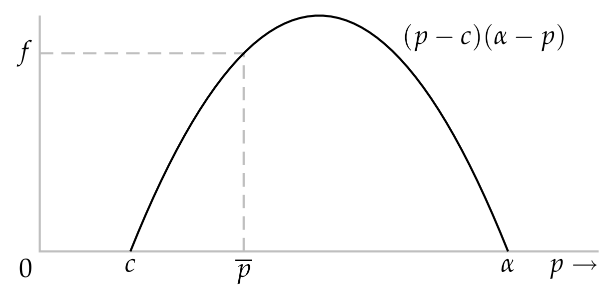


Figure 67.1 The determination of the price $\overline{p}$ in Exercise 67.1.

COURNOT, BERTRAND, AND NASH: SOME HISTORICAL NOTES

Associating the names of Cournot and Bertrand with the strategic games in Sections 3.1 and 3.2 invites two conclusions. First, that Cournot, writing in the first half of the nineteenth century, developed the concept of Nash equilibrium in the context of a model of oligopoly. Second, that Bertrand, dissatisfied with Cournot's game, proposed an alternative model in which price rather than output is the strategic variable. On both points the history is much less straightforward.

Cournot presented his "equilibrium" as the outcome of a dynamic adjustment process in which, in the case of two firms, the firms alternately choose best responses to each other's outputs. During such an adjustment process, each firm, when choosing an output, acts on the assumption that the other firm's output will remain the same, an assumption shown to be incorrect when the other firm subsequently adjusts its output. The fact that the adjustment process rests on the firms' acting on assumptions constantly shown to be false was the subject of criticism in a leading presentation of Cournot's model (Fellner 1949) available at the time Nash was developing his idea.

Certainly Nash did not literally generalize Cournot's idea: the evidence suggests that he was completely unaware of Cournot's work when developing the notion of Nash equilibrium (Leonard 1994, 502–503). In fact, only gradually, as Nash's work was absorbed into mainstream economic theory, was Cournot's solution interpreted as a Nash equilibrium (Leonard 1994, 507–509).

The association of the price-setting model with Bertrand (a mathematician) rests on a paragraph in a review of Cournot's book written by Bertrand in 1883. (Cournot's book, published in 1838, had previously been largely ignored.) The review is confused. Bertrand is under the impression that in *Cournot's* model the firms compete in prices, undercutting each other to attract more business! He argues that there is "no solution" because there is no limit to the fall in prices, a result he says that Cournot's formulation conceals (Bertrand 1883, 503). In brief, Bertrand's understanding of Cournot's work is flawed; he sees that price competition leads each firm to undercut the other, but his conclusion about the outcome is incorrect.

Through the lens of modern game theory we see that the models associated with Cournot and Bertrand are strategic games that differ only in the strategic variable, the solution in both cases being a Nash equilibrium. Until Nash's work, the picture was much murkier.

3.3 Electoral competition

What factors determine the number of political parties and the policies they propose? How is the outcome of an election affected by the electoral system and the voters' preferences among policies? A model that is the foundation for many theories of political phenomena addresses these questions. In the model, each of several candidates chooses a policy; each citizen has preferences over policies and votes for one of the candidates.

A simple version of this model is a strategic game in which the players are the candidates and a policy is a number, referred to as a "position". (The compression of all policy differences into one dimension is a major abstraction, though political positions are often categorized on a left–right axis.) After the candidates have chosen positions, each of a set of citizens votes (nonstrategically) for the candidate whose position she likes best. The candidate who obtains the most votes wins. Each candidate cares only about winning; no candidate has an ideological attachment to any position. Specifically, each candidate prefers to win than to tie for first place (in which case perhaps the winner is determined randomly) than to lose, and if she ties for first place she prefers to do so with as few other candidates as possible.

There is a continuum of voters, each with a favorite position. The distribution of these favorite positions over the set of all possible positions is arbitrary. In particular, this distribution may not be uniform: a large fraction of the voters may have favorite positions close to one point, while few voters have favorite positions close to some other point. A position that turns out to have special significance is the *median* favorite position: the position m with the property that exactly half of the voters' favorite positions are at most m, and half of the voters' favorite positions are at least m. (I assume that there is only one such position.)

Each voter's distaste for any position is given by the distance between that position and her favorite position. In particular, for any value of k, a voter whose favorite position is x^* is indifferent between the positions $x^* - k$ and $x^* + k$. (Refer to Figure 69.1.)

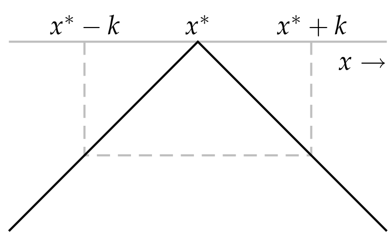


Figure 69.1 The payoff of a voter whose favorite position is x^*, as a function of the winning position, x.

Under this assumption, each candidate attracts the votes of all citizens whose favorite positions are closer to her position than to the position of any other candidate. An example is shown in Figure 70.1. In this example there are three candidates, with positions x_1, x_2, and x_3. Candidate 1 attracts the votes of every citizen whose favorite position is in the interval, labeled "votes for 1", up to the midpoint $\frac{1}{2}(x_1 + x_2)$ of the line segment from x_1 to x_2; candidate 2 attracts the votes of every citizen whose favorite position is in the interval from $\frac{1}{2}(x_1 + x_2)$ to $\frac{1}{2}(x_2 + x_3)$; and candidate 3 attracts the remaining votes. I assume that citizens whose favorite position is $\frac{1}{2}(x_1 + x_2)$ divide their votes equally between candidates 1 and 2, and those whose favorite position is $\frac{1}{2}(x_2 + x_3)$ divide their votes equally between candidates 2 and 3. If two or more candidates take the same position then they share equally the votes that the position attracts.

In summary, I consider the following strategic game, which, in honor of its originator, I call **Hotelling's model of electoral competition**.

Players The candidates.

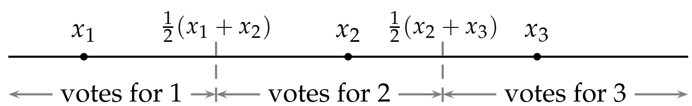


Figure 70.1 The allocation of votes between three candidates, with positions x_1, x_2, and x_3.

Actions Each candidate's set of actions is the set of positions (numbers).

Preferences Each candidate's preferences are represented by a payoff function that assigns n to every terminal history in which she wins outright, k to every terminal history in which she ties for first place with $n - k$ other candidates (for $1 \leq k \leq n - 1$), and 0 to every terminal history in which she loses, where positions attract votes in the way described in the previous paragraph.

Suppose there are two candidates. We can find a Nash equilibrium of the game by studying the players' best response functions. Fix the position x_2 of candidate 2 and consider the best position for candidate 1. First suppose that $x_2 < m$. If candidate 1 takes a position to the left of x_2 then candidate 2 attracts the votes of all citizens whose favorite positions are to the right of $\frac{1}{2}(x_1 + x_2)$, a set that includes the 50% of citizens whose favorite positions are to the right of m, and more. Thus candidate 2 wins, and candidate 1 loses. If candidate 1 takes a position to the right of x_2 then she wins so long as the dividing line between her supporters and those of candidate 2 is less than m (see Figure 70.2). If she is so far to the right that this dividing line lies to the right of m then she loses. She prefers to win than to lose, and is indifferent between all the outcomes in which she wins, so her set of best responses to x_2 is the set of positions that causes the midpoint $\frac{1}{2}(x_1 + x_2)$ of the line segment from x_2 to x_1 to be less than m. (If this midpoint is *equal* to m then the candidates tie.) The condition $\frac{1}{2}(x_1 + x_2) < m$ is equivalent to $x_1 < 2m - x_2$, so candidate 1's set of best responses to x_2 is the set of all positions between x_2 and $2m - x_2$ (excluding the points x_2 and $2m - x_2$).

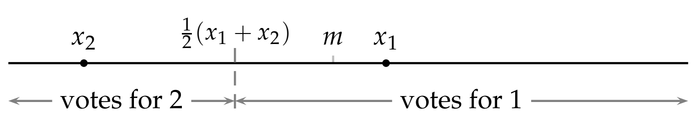


Figure 70.2 An action profile (x_1, x_2) for which candidate 1 wins.

A symmetric argument applies to the case in which $x_2 > m$. In this case candidate 1's set of best responses to x_2 is the set of all positions between $2m - x_2$ and x_2.

Finally consider the case in which $x_2 = m$. In this case candidate 1's unique best response is to choose the *same* position, m! If she chooses any other position then she loses, whereas if she chooses m then she ties for first place.

In summary, candidate 1's best response function is defined by

$$B_1(x_2) = \begin{cases} \{x_1 : x_2 < x_1 < 2m - x_2\} & \text{if } x_2 < m \\ \{m\} & \text{if } x_2 = m \\ \{x_1 : 2m - x_2 < x_1 < x_2\} & \text{if } x_2 > m. \end{cases}$$

Candidate 2 faces exactly the same incentives as candidate 1, and hence has the same best response function. The candidates' best response functions are shown in Figure 71.1.

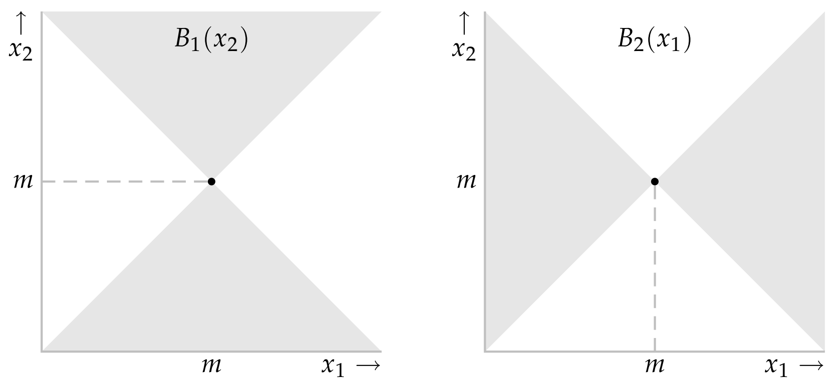


Figure 71.1 The candidates' best response functions in Hotelling's model of electoral competition with two candidates. Candidate 1's best response function is in the left panel; candidate 2's is in the right panel. (The edges of the shaded areas are excluded.)

If you superimpose the two best response functions, you see that the game has a unique Nash equilibrium, in which both candidates choose the position m, the voters' median favorite position. (Remember that the edges of the shaded area, which correspond to pairs of positions that result in ties, are excluded from the best response functions.) The outcome is that the election is a tie.

As in the case of Bertrand's duopoly game in the previous section, we can make a direct argument that (m, m) is the unique Nash equilibrium of the game, without constructing the best response functions. First, (m, m) is an equilibrium: it results in a tie, and if either candidate chooses a position different from m then she loses. Second, no other pair of positions is a Nash equilibrium, by the following argument.

- If one candidate loses then she can do better by moving to m, where she either wins outright (if her opponent's position is different from m) or ties for first place (if her opponent's position is m).
- If the candidates tie (because their positions are either the same or symmetric about m), then either candidate can do better by moving to m, where she wins outright.

Our conclusion is that the competition between the candidates to secure a majority of the votes drives them to select the same position, equal to the median of the citizens' favorite positions. Hotelling (1929, 54), the originator of the model, writes that this outcome is "strikingly exemplified." He continues, "The competition for votes between the Republican and Democratic parties [in the USA] does not lead to a clear drawing of issues, an adoption of two strongly contrasted positions between which the voter may choose. Instead, each party strives to make its platform as much like the other's as possible."

? EXERCISE 72.1 (Electoral competition with asymmetric voters' preferences) Consider a variant of Hotelling's model in which voters's preferences are asymmetric. Specifically, suppose that each voter cares twice as much about policy differences to the left of her favorite position than about policy differences to the right of her favorite position. How does this affect the Nash equilibrium?

In the model considered so far, no candidate has the option of staying out of the race. Suppose that we give each candidate this option; assume that it is better than losing and worse than tying for first place. Then the Nash equilibrium remains as before: both players enter the race and choose the position m. The direct argument differs from the one before only in that in addition we need to check that there is no equilibrium in which one or both of the candidates stays out of the race. If one candidate stays out then, given the other candidate's position, she can enter and either win outright or tie for first place. If both candidates stay out, then either candidate can enter and win outright.

The next exercise asks you to consider the Nash equilibria of this variant of the model when there are three candidates.

? EXERCISE 72.2 (Electoral competition with three candidates) Consider a variant of Hotelling's model in which there are three candidates and each candidate has the option of staying out of the race, which she regards as better than losing and worse than tying for first place. Use the following arguments to show that the game has no Nash equilibrium. First, show that there is no Nash equilibrium in which a single candidate enters the race. Second, show that in any Nash equilibrium in which more than one candidate enters, all candidates that enter tie for first place. Third, show that there is no Nash equilibrium in which two candidates enter the race. Fourth, show that there is no Nash equilibrium in which all three candidates enter the race and choose the same position. Finally, show that there is no Nash equilibrium in which all three candidates enter the race, and do not all choose the same position.

?? EXERCISE 72.3 (Electoral competition in two districts) Consider a variant of Hotelling's model that captures features of a US presidential election. Voters are divided between two districts. District 1 is worth more electoral college votes than is district 2. The winner is the candidate who obtains the most electoral college votes. Denote by m_i the median favorite position among the citizens of district i, for $i = 1$, 2; assume that $m_2 < m_1$. Each of two candidates chooses a single position. Each citizen votes (nonstrategically) for the candidate whose position in closest to her favorite position. The candidate who wins a majority of the votes in a district obtains all the electoral college votes of that district; if the candidates obtain the same number of votes in a district, they each obtain half of the electoral college votes of that district. Find the Nash equilibrium (equilibria?) of the strategic game that models this situation.

So far we have assumed that the candidates care only about winning; they are not at all concerned with the winner's position. The next exercise asks you to consider the case in which each candidate cares *only* about the winner's position, and not at all about winning. (You may be surprised by the equilibrium.)

?? EXERCISE 73.1 (Electoral competition between candidates who care only about the winning position) Consider the variant of Hotelling's model in which the candidates (like the citizens) care about the winner's position, and not at all about winning *per se*. There are two candidates. Each candidate has a favorite position; her dislike for other positions increases with their distance from her favorite position. Assume that the favorite position of one candidate is less than m and the favorite position of the other candidate is greater than m. Assume also that if the candidates tie when they take the positions x_1 and x_2 then the outcome is the compromise policy $\frac{1}{2}(x_1 + x_2)$. Find the set of Nash equilibria of the strategic game that models this situation. (First consider pairs (x_1, x_2) of positions for which either $x_1 < m$ and $x_2 < m$, or $x_1 > m$ and $x_2 > m$. Next consider pairs (x_1, x_2) for which either $x_1 < m < x_2$, or $x_2 < m < x_1$, then those for which $x_1 = m$ and $x_2 \neq m$, or $x_1 \neq m$ and $x_2 = m$. Finally consider the pair (m, m).)

The set of candidates in Hotelling's model is given. The next exercise asks you to analyze a model in which the set of candidates is generated as part of an equilibrium.

?? EXERCISE 73.2 (Citizen-candidates) Consider a game in which the players are the citizens. Any citizen may, at some cost $c > 0$, become a candidate. Assume that the only position a citizen can espouse is her favorite position, so that a citizen's only decision is whether to stand as a candidate. After all citizens have (simultaneously) decided whether to become candidates, each citizen votes for her favorite candidate, as in Hotelling's model. Citizens care about the position of the winning candidate; a citizen whose favorite position is x loses $|x - x^*|$ if the winning candidate's position is x^*. (For any number z, $|z|$ denotes the absolute value of z: $|z| = z$ if $z > 0$ and $|z| = -z$ if $z < 0$.) Winning confers the benefit b. Thus a citizen who becomes a candidate and ties with $k - 1$ other candidates for first place obtains the payoff $b/k - c$; a citizen with favorite position x who becomes a candidate and is not one of the candidates tied for first place obtains the payoff $-|x - x^*| - c$, where x^* is the winner's position; and a citizen with favorite position x who does not become a candidate obtains the payoff $-|x - x^*|$, where x^* is the winner's position. Assume that for every position x there is a citizen for whom x is the favorite position. Show that if $b \leq 2c$ then the game has a Nash equilibrium in which one citizen becomes a candidate. Is there an equilibrium (for any values of b and c) in which two citizens, each with favorite position m, become candidates? Is there an equilibrium in which two citizens with favorite positions different from m become candidates?

Hotelling's model assumes a basic agreement among the voters about the ordering of the positions. For example, if one voter prefers x to y to z and another voter prefers y to z to x, no voter prefers z to x to y. The next exercise asks you to study a model that does not so restrict the voters' preferences.

? EXERCISE 74.1 (Electoral competition for more general preferences) There is a finite number of positions and a finite, odd, number of voters. For any positions x and y, each voter either prefers x to y or prefers y to x. (No voter regards any two positions as equally desirable.) We say that a position x^* is a *Condorcet winner* if for every position y different from x^*, a majority of voters prefer x^* to y.

a. Show that for any configuration of preferences there is at most one Condorcet winner.

b. Give an example in which no Condorcet winner exists. (Suppose there are three positions (x, y, and z) and three voters. Assume that voter 1 prefers x to y to z. Construct preferences for the other two voters such that one voter prefers x to y and the other prefers y to x, one prefers x to z and the other prefers z to x, and one prefers y to z and the other prefers z to y. The preferences you construct must, of course, satisfy the condition that a voter who prefers a to b and b to c also prefers a to c, where a, b, and c are any positions.)

c. Consider the strategic game in which two candidates simultaneously choose positions, as in Hotelling's model. If the candidates choose different positions, each voter endorses the candidate whose position she prefers, and the candidate who receives the most votes wins. If the candidates choose the same position, they tie. Show that this game has a unique Nash equilibrium if the voters' preferences are such that there is a Condorcet winner, and has no Nash equilibrium if the voters' preferences are such that there is no Condorcet winner.

A variant of Hotelling's model of electoral competition can be used to analyze the choices of product characteristics by competing firms in situations in which price is not a significant variable. (Think of radio stations that offer different styles of music, for example.) The set of positions is the range of possible characteristics for the product, and the citizens are consumers rather than voters. Consumers' tastes differ; each consumer buys (at a fixed price, possibly zero) one unit of the product she likes best. The model differs substantially from Hotelling's model of electoral competition in that each firm's objective is to maximize its market share, rather than to obtain a market share larger than that of any other firm. In the next exercise you are asked to show that the Nash equilibria of this game in the case of two or three firms are the same as those in Hotelling's model of electoral competition.

? EXERCISE 75.1 (Competition in product characteristics) In the variant of Hotelling's model that captures competing firms' choices of product characteristics, show that when there are two firms the unique Nash equilibrium is (m, m) (both firms offer the consumers' median favorite product) and when there are three firms there is no Nash equilibrium. (Start by arguing that when there are two firms whose products differ, either firm is better off making its product more similar to that of its rival.)

3.4 The War of Attrition

The game known as the *War of Attrition* elaborates on the ideas captured by the game *Hawk–Dove* (Exercise 29.1). It was originally posed as a model of a conflict between two animals fighting over prey. Each animal chooses the time at which it intends to give up. When an animal gives up, its opponent obtains all the prey (and the time at which the winner intended to give up is irrelevant). If both animals give up at the same time then they each have an equal chance of obtaining the prey. Fighting is costly: each animal prefers as short a fight as possible.

The game models not only such a conflict between animals, but also many other disputes. The "prey" can be any indivisible object, and "fighting" can be any costly activity—for example, simply waiting.

To define the game precisely, let time be a continuous variable that starts at 0 and runs indefinitely. Assume that the value party i attaches to the object in dispute is $v_i > 0$ and the value it attaches to a 50% chance of obtaining the object is $v_i/2$. Each unit of time that passes before the dispute is settled (i.e. one of the parties concedes) costs each party one unit of payoff. Thus if player i concedes first, at time t_i, her payoff is $-t_i$ (she spends t_i units of time and does not obtain the object). If the other player concedes first, at time t_j, player i's payoff is $v_i - t_j$ (she obtains the object after t_j units of time). If both players concede at the same time, player i's payoff is $\frac{1}{2}v_i - t_i$, where t_i is the common concession time. The **War of Attrition** is the following strategic game.

Players The two parties to a dispute.

Actions Each player's set of actions is the set of possible concession times (nonnegative numbers).

Preferences Player i's preferences are represented by the payoff function

$$u_i(t_1, t_2) = \begin{cases} -t_i & \text{if } t_i < t_j \\ \frac{1}{2}v_i - t_i & \text{if } t_i = t_j \\ v_i - t_j & \text{if } t_i > t_j, \end{cases}$$

where j is the other player.

To find the Nash equilibria of this game, we start, as before, by finding the players' best response functions. Intuitively, if player j's intended concession time is early enough (t_j is small) then it is optimal for player i to wait for player j to concede. That is, in this case player i should choose a concession time later than t_j; any such time is equally good. By contrast, if player j intends to hold out for a long time (t_j is large) then player i should concede immediately. Because player i values the object at v_i, the length of time it is worth her waiting is v_i.

To make these ideas precise, we can study player i's payoff function for various fixed values of t_j, the concession time of player j. The three cases that the intuitive argument suggests are qualitatively different are shown in Figure 76.1: $t_j < v_i$ in the left panel, $t_j = v_i$ in the middle panel, and $t_j > v_i$ in the right panel. Player i's best responses in each case are her actions for which her payoff is highest: the set of times after t_j if $t_j < v_i$, 0 and the set of times after t_j if $t_j = v_i$, and 0 if $t_j > v_i$.

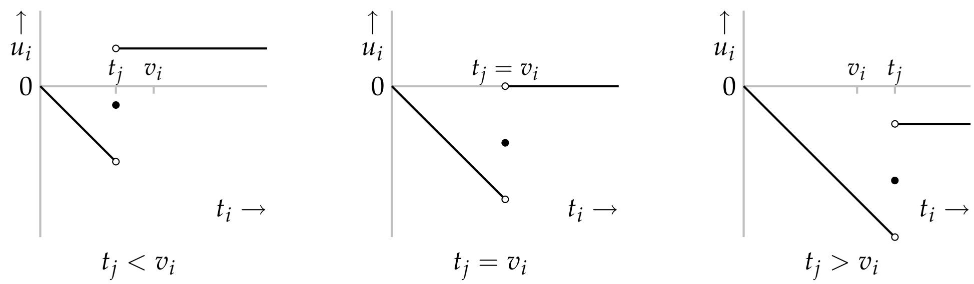


Figure 76.1 Three cross-sections of player i's payoff function in the *War of Attrition*.

In summary, player i's best response function is given by

$$B_i(t_j) = \begin{cases} \{t_i : t_i > t_j\} & \text{if } t_j < v_i \\ \{t_i : t_i = 0 \text{ or } t_i > t_j\} & \text{if } t_j = v_i \\ \{0\} & \text{if } t_j > v_i. \end{cases}$$

For a case in which $v_1 > v_2$, this function is shown in the left panel of Figure 77.1 for $i = 1$ and $j = 2$ (player 1's best response function), and in the right panel for $i = 2$ and $j = 1$ (player 2's best response function).

Superimposing the players' best response functions, we see that there are two areas of intersection: the vertical axis at and above v_1 and the horizontal axis at and to the right of v_2. Thus (t_1, t_2) is a Nash equilibrium of the game if and only if either

$$t_1 = 0 \text{ and } t_2 \geq v_1$$

or

$$t_2 = 0 \text{ and } t_1 \geq v_2.$$

In words, in every equilibrium either player 1 concedes immediately and player 2 concedes at time v_1 or later, or player 2 concedes immediately and player 1 concedes at time v_2 or later.

? EXERCISE 76.1 (Direct argument for Nash equilibria of *War of Attrition*) Give a direct argument, not using information about the entire best response functions, for the set of Nash equilibria of the *War of Attrition*. (Argue that if $t_1 = t_2$, $0 <$

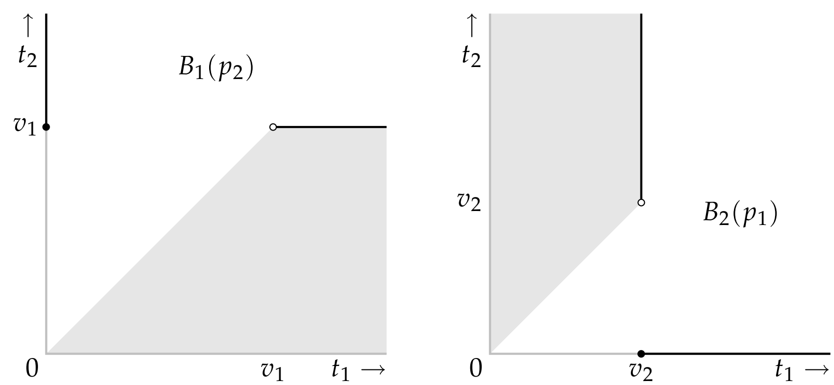


Figure 77.1 The players' best response functions in the *War of Attrition* (for a case in which $v_1 > v_2$). Player 1's best response function is in the left panel; player 2's is in the right panel. (The sloping edges are excluded.)

$t_i < t_j$, or $0 = t_i < t_j < v_i$ (for $i = 1$ and $j = 2$, or $i = 2$ and $j = 1$) then the pair (t_1, t_2) is not a Nash equilibrium. Then argue that any remaining pair is a Nash equilibrium.)

Three features of the equilibria are notable. First, in no equilibrium is there any fight: one player always concedes immediately. Second, either player may concede first, regardless of the players' valuations. In particular, there are always equilibria in which the player who values the object more highly concedes first. Third, the equilibria are *asymmetric* (the players' actions are different), even when $v_1 = v_2$, in which case the game is symmetric—the players' sets of actions are the same and player 1's payoff to (t_1, t_2) is the same as player 2's payoff to (t_2, t_1) (Definition 49.3). Given this asymmetry, the populations from which the two players are drawn must be distinct in order to interpret the Nash equilibria as action profiles compatible with steady states. One player might be the current owner of the object in dispute, and the other a challenger, for example. In this case the equilibria correspond to the two conventions that a challenger always gives up immediately, and that an owner always does so. (Some evidence is discussed in the box on page 379.) If all players—those in the role of player 1 as well as those in the role of player 2—are drawn from a single population, then only symmetric equilibria are relevant (see Section 2.10). The *War of Attrition* has no such equilibria, so the notion of Nash equilibrium makes no prediction about the outcome in such a situation. (A solution that does make a prediction is studied in Example 376.1.)

? EXERCISE 77.1 (Variant of *War of Attrition*) Consider the variant of the *War of Attrition* in which each player attaches no value to the time spent waiting for the other player to concede, but the object in dispute loses value as time passes. (Think of a rotting animal carcass or a melting ice cream cone.) Assume that the value of the object to each player i after t units of time is $v_i - t$ (and the value of a 50% chance of obtaining the object is $\frac{1}{2}(v_i - t)$). Specify the strategic game that models this situation (take care with the payoff functions). Construct the analogue of Figure 76.1, find the players' best response functions, and hence find the Nash equilibria of the game.

The *War of Attrition* is an example of a "game of timing", in which each player's action is a number and each player's payoff depends sensitively on whether her action is greater or less than the other player's action. In many such games, each player's strategic variable is the time at which to act, hence the name "game of timing". The next two exercises are further examples of such games. (In the first the strategic variable is time, whereas in the second it is not.)

? EXERCISE 78.1 (Timing product release) Two firms are developing competing products for a market of fixed size. The longer a firm spends on development, the better its product. But the first firm to release its product has an advantage: the customers it obtains will not subsequently switch to its rival. (Once a person starts using a product, the cost of switching to an alternative, even one significantly better, is too high to make a switch worthwhile.) A firm that releases its product first, at time t, captures the share $h(t)$ of the market, where h is a function that increases from time 0 to time T, with $h(0) = 0$ and $h(T) = 1$. The remaining market share is left for the other firm. If the firms release their products at the same time, each obtains half of the market. Each firm wishes to obtain the highest possible market share. Model this situation as a strategic game and find its Nash equilibrium (equilibria?). (When finding firm i's best response to firm j's release time t_j, there are three cases: that in which $h(t_j) < \frac{1}{2}$ (firm j gets less than half of the market if it is the first to release its product), that in which $h(t_j) = \frac{1}{2}$, and that in which $h(t_j) > \frac{1}{2}$.)

? EXERCISE 78.2 (A fight) Each of two people has one unit of a resource. Each person chooses how much of the resource to use in fighting the other individual and how much to use productively. If each person i devotes y_i to fighting then the total output is $f(y_1, y_2) \geq 0$ and person i obtains the fraction $p_i(y_1, y_2)$ of the output, where

$$p_i(y_1, y_2) = \begin{cases} 1 & \text{if } y_i > y_j \\ \frac{1}{2} & \text{if } y_i = y_j \\ 0 & \text{if } y_i < y_j. \end{cases}$$

The function f is continuous (small changes in y_1 and y_2 cause small changes in $f(y_1, y_2)$), is decreasing in both y_1 and y_2 (the more each player devotes to fighting, the less output is produced), and satisfies $f(1, 1) = 0$ (if each player devotes all her resource to fighting then no output is produced). (If you prefer to deal with a specific function f, take $f(y_1, y_2) = 2 - y_1 - y_2$.) Each person cares only about the amount of output she receives, and prefers to receive as much as possible. Specify this situation as a strategic game and find its Nash equilibrium (equilibria?). (Use a direct argument: first consider pairs (y_1, y_2) with $y_1 \neq y_2$, then those with $y_1 = y_2 < 1$, then those with $y_1 = y_2 = 1$.)

3.5 Auctions

3.5.1 Introduction

In an "auction", a good is sold to the party who submits the highest bid. Auctions, broadly defined, are used to allocate significant economic resources, from works of art to short-term government bonds to offshore tracts for oil and gas exploration to the radio spectrum. They take many forms. For example, bids may be called out sequentially (as in auctions for works of art) or may be submitted in sealed envelopes; the price paid may be the highest bid, or some other price; if more than one unit of a good is being sold, bids may be taken on all units simultaneously, or the units may be sold sequentially. A game-theoretic analysis helps us to understand the consequences of various auction designs; it suggests, for example, the design likely to be the most effective at allocating resources, and the one likely to raise the most revenue. In this section I discuss auctions in which every buyer knows her own valuation and every other buyer's valuation of the item being sold. Chapter 9 develops tools that allow us to study, in Section 9.7, auctions in which buyers are not perfectly informed of each other's valuations.

AUCTIONS FROM BABYLONIA TO EBAY

Auctioning has a very long history. Herodotus, a Greek writer of the fifth century BC who, together with Thucydides, created the intellectual field of history, describes auctions in Babylonia. He writes that the Babylonians' "most sensible" custom was an annual auction in each village of the women of marriageable age. The women most attractive to the men were sold first; they commanded positive prices, whereas men were paid to be matched with the least desirable women. In each auction, bids appear to have been called out sequentially, the man who bid the most winning and paying the price he bid.

Auctions were also used in Athens in the fifth and fourth centuries BC to sell the rights to collect taxes, to dispose of confiscated property, and to lease land and mines. The evidence on the nature of the auctions is slim, but some interesting accounts survive. For example, the Athenian politician Andocides (c. 440–391 BC) reports collusive behavior in an auction of tax-collection rights (see Langdon 1994, 260).

Auctions were frequent in ancient Rome, and continued to be used in medieval Europe after the end of the Roman empire (tax-collection rights were annually auctioned by the towns of the medieval and early modern Low Countries, for example). The earliest use of the English word "auction" given by the *Oxford English Dictionary* dates from 1595, and concerns an auction "when will be sold Slaves, household goods, etc.". Rules surviving from the auctions of this era show that in some cases, at least, bids were called out sequentially, with the bidder remaining at the end obtaining the object at the price she bid (Cassady 1967, 30–31). A variant of this mechanism, in which a time limit is imposed on the bids, is reported by the English diarist and naval administrator Samuel Pepys (1633–1703). The auctioneer lit a short candle, and bids were valid only if made before the flame went out. Pepys reports that a flurry of bidding occurred at the last moment. At an auction on September 3, 1662, a bidder "cunninger than the rest" told him that just as the flame goes out, "the smoke descends", signaling the moment at which one should bid, an observation Pepys found "very pretty" (Pepys 1970, 185–186).

The auction houses of Sotheby's and Christie's were founded in the mid-18th century. At the beginning of the twenty-first century, they are being eclipsed, at least in the value of the goods they sell, by online auction companies. For example, eBay, founded in September 1995, sold US$1.3 billion of merchandise in 62 million auctions during the second quarter of 2000, roughly double the numbers for the second quarter of the previous year; Sotheby's and Christie's together sell around US$1 billion of art and antiques each quarter.

The mechanism used by eBay shares a feature with the ones Pepys observed: all bids must be received before some fixed time. The way in which the price is determined differs. In an eBay auction, a bidder submits a "proxy bid" that is not revealed; the prevailing price is a small increment above the second-highest proxy bid. As in the 17th century auctions Pepys observed, many bidders on eBay act at the last moment—a practice known as "sniping" in the argot of cyberspace. Other online auction houses use different termination rules. For example, Amazon waits ten minutes after a bid before closing an auction. The fact that last-minute bidding is much less common in Amazon auctions than it is in eBay auctions has attracted the attention of game theorists, who have begun to explore models that explain it in terms of the difference in the auctions' termination rules (see, for example, Ockenfels and Roth 2000).

In recent years, many countries have auctioned the rights to the radio spectrum, used for wireless communication. These auctions have been much studied by game theorists; they are discussed in the box on page 298.

3.5.2 Second-price sealed-bid auctions

In a common form of auction, people sequentially submit increasing bids for an object. (The word "auction" comes from the Latin *augere*, meaning "to increase".) When no one wishes to submit a bid higher than the current bid, the person making the current bid obtains the object at the price she bid.

Given that every person is certain of her valuation of the object before the bidding begins, during the bidding no one can learn anything relevant to her actions. Thus we can model the auction by assuming that each person decides, before bidding begins, the most she is willing to bid—her "maximal bid". When the players carry out their plans, the winner is the person whose maximal bid is highest. How much does she need to bid? Eventually only she and the person with the second highest maximal bid will be left competing against each other. In order to win, she therefore needs to bid slightly more than the *second highest* maximal bid. If the bidding increment is small, we can take the price the winner pays to be *equal* to the second highest maximal bid.

Thus we can model such an auction as a strategic game in which each player chooses an amount of money, interpreted as the *maximal* amount she is willing to bid, and the player who chooses the highest amount obtains the object and pays a price equal to the second highest amount.

This game models also a situation in which the people simultaneously put bids in sealed envelopes, and the person who submits the highest bid wins and pays a price equal to the *second* highest bid. For this reason the game is called a *second-price sealed-bid* auction.

To define the game precisely, denote by v_i the value player i attaches to the object; if she obtains the object at the price p her payoff is $v_i - p$. Assume that the players' valuations of the object are all different and all positive; number the players 1 through n in such a way that $v_1 > v_2 > \cdots > v_n > 0$. Each player i submits a (sealed) bid b_i. If player i's bid is higher than every other bid, she obtains the object at a price equal to the second-highest bid, say b_j, and hence receives the payoff $v_i - b_j$. If some other bid is higher than player i's bid, player i does not obtain the object, and receives the payoff of zero. If player i is in a tie for the highest bid, her payoff depends on the way in which ties are broken. A simple (though arbitrary) assumption is that the winner is the player among those submitting the highest bid whose number is smallest (i.e. whose valuation of the object is highest). (If the highest bid is submitted by players 2, 5, and 7, for example, the winner is player 2.) Under this assumption, player i's payoff when she bids b_i and is in a tie for the highest bid is $v_i - b_i$ if her number is lower than that of any other player submitting the bid b_i, and 0 otherwise.

In summary, a **second-price sealed-bid auction** (with perfect information) is the following strategic game.

Players The n bidders, where $n \geq 2$.

Actions The set of actions of each player is the set of possible bids (nonnegative numbers).

Preferences The payoff of any player i is $v_i - b_j$, where b_j is the highest bid submitted by a player other than i if either b_i is higher than every other bid, or b_i is at least as high as every other bid and the number of every other player who bids b_i is greater than i. Otherwise player i's payoff is 0.

This game has many Nash equilibria. One equilibrium is $(b_1, \ldots, b_n) = (v_1, \ldots, v_n)$: each player's bid is equal to her valuation of the object. Because $v_1 > v_2 > \cdots > v_n$, the outcome is that player 1 obtains the object at the price b_2; her payoff is $v_1 - b_2$ and every other player's payoff is zero. This profile is a Nash equilibrium by the following argument.

- If player 1 changes her bid to some other price at least equal to b_2 then the outcome does not change (recall that she pays the *second* highest bid, not the highest bid). If she changes her bid to a price less than b_2 then she loses and obtains the payoff of zero.

- If some other player lowers her bid or raises it to some price at most equal to b_1 then she remains a loser; if she raises her bid above b_1 then she wins but, in paying the price b_1, makes a loss (because her valuation is less than b_1).

Another equilibrium is $(b_1, \ldots, b_n) = (v_1, 0, \ldots, 0)$. In this equilibrium, player 1 obtains the object and pays the price of zero. The profile is an equilibrium because if player 1 changes her bid then the outcome remains the same, and if any of the remaining players raises her bid then either the outcome remains the same (if her new bid is at most v_1) or causes her to obtain the object at a price that exceeds her valuation (if her bid exceeds v_1). (The auctioneer obviously has an incentive for the price to be bid up, but she is not a player in the game!)

In both of these equilibria, player 1 obtains the object. But there are also equilibria in which player 1 does not obtain the object. Consider, for example, the action profile $(v_2, v_1, 0, \ldots, 0)$, in which player 2 obtains the object at the price v_2 and every player (including player 2) receives the payoff of zero. This action profile is a Nash equilibrium by the following argument.

- If player 1 raises her bid to v_1 or more, she wins the object but her payoff remains zero (she pays the price v_1, bid by player 2). Any other change in her bid has no effect on the outcome.

- If player 2 changes her bid to some other price greater than v_2, the outcome does not change. If she changes her bid to v_2 or less she loses, and her payoff remains zero.

- If any other player raises her bid to at most v_1, the outcome does not change. If she raises her bid above v_1 then she wins, but in paying the price v_1 (bid by player 2) she obtains a negative payoff.

? EXERCISE 82.1 (Nash equilibrium of second-price sealed-bid auction) Find a Nash equilibrium of a second-price sealed-bid auction in which player n obtains the object.

Player 2's bid in this equilibrium exceeds her valuation, and thus may seem a little rash: if player 1 were to increase her bid to any value less than v_1, player 2's payoff would be negative (she would obtain the object at a price greater than her valuation). This property of the action profile does not affect its status as an equilibrium, because in a Nash equilibrium a player does not consider the "risk" that another player will take an action different from her equilibrium action; each player simply chooses an action that is optimal, *given* the other players' actions. But the property does suggest that the equilibrium is less plausible as the outcome of the auction than the equilibrium in which every player bids her valuation.

The same point takes a different form when we interpret the strategic game as a model of events that unfold over time. Under this interpretation, player 2's action

v_1 means that she will continue bidding until the price reaches v_1. If player 1 is *sure* that player 2 will continue bidding until the price is v_1, then player 1 rationally stops bidding when the price reaches v_2 (or, indeed, when it reaches any other level at most equal to v_1). But there is little reason for player 1 to believe that player 2 will in fact stay in the bidding if the price exceeds v_2: player 2's action is not credible, because if the bidding were to go above v_2, player 2 would rationally withdraw.

The weakness of the equilibrium is reflected in the fact that player 2's bid v_1 is weakly dominated by the bid v_2. More generally,

> *in a second-price sealed-bid auction (with perfect information), a player's bid equal to her valuation weakly dominates all her other bids.*

That is, for any bid $b_i \neq v_i$, player i's bid v_i is at least as good as b_i, *no matter what the other players bid*, and is better than b_i for some actions of the other players. (See Definition 45.1.) A player who bids less than her valuation stands not to win in some cases in which she could profit by winning (when the highest of the other bids is between her bid and her valuation), and never stands to gain relative to the situation in which she bids her valuation; a player who bids more than her valuation stands to win in some cases in which she obtains a negative payoff by doing so (when the highest of the remaining bids is between her valuation and her bid), and never stands to gain relative to the situation in which she bids her valuation. The key point is that in a second-price auction, a player who changes her bid does not lower the price she pays, but only possibly changes her status from that of a winner into that of a loser, or vice versa.

A precise argument is shown in Figure 84.1, which compares player i's payoffs to the bid v_i with her payoffs to a bid $b_i < v_i$ (top table), and to a bid $b_i < v_i$ (bottom table), as a function of the highest of the other players' bids, denoted $\overline{b}$. In each case, for all bids of the other players, player i's payoffs to v_i are at least as large as her payoffs to the other bid, and for bids of the other players such that $\overline{b}$ is in the middle column of each table, player i's payoffs to v_i are greater than her payoffs to the other bid. Thus player i's bid v_i weakly dominates all her other bids.

In summary, a second-price auction has many Nash equilibria, but the equilibrium $(b_1, \ldots, b_n) = (v_1, \ldots, v_n)$ in which every player's bid is equal to her valuation of the object is distinguished by the fact that every player's action weakly dominates all her other actions.

? EXERCISE 83.1 (Second-price sealed-bid auction with two bidders) Find *all* the Nash equilibria of a second-price sealed-bid auction with two bidders. (Construct the players' best response functions. Apart from a difference in the tie-breaking rule, the game is the same as the one in Exercise 77.1.)

		Highest of other players' bids, $\overline{b}$: $\overline{b} < b_i$ or $\overline{b} = b_i$ & b_i wins	$b_i < \overline{b} < v_i$ or $\overline{b} = b_i$ & b_i loses	$\overline{b} > v_i$
i's bid	$b_i < v_i$	$v_i - \overline{b}$	0	0
	v_i	$v_i - \overline{b}$	$v_i - \overline{b}$	0

		$\overline{b} \leq v_i$	$v_i < \overline{b} < b_i$ or $\overline{b} = b_i$ & b_i wins	$\overline{b} > b_i$ or $\overline{b} = b_i$ & b_i loses
i's bid	v_i	$v_i - \overline{b}$	0	0
	$b_i > v_i$	$v_i - \overline{b}$	$v_i - \overline{b}$ (< 0)	0

Figure 84.1 Player i's payoffs in a second-price sealed-bid auction, as a function of the highest of the other player's bids, denoted $\overline{b}$. The top table gives her payoffs to the bids $b_i < v_i$ and v_i, and the bottom table gives her payoffs to the bids v_i and $b_i > v_i$.

3.5.3 First-price sealed-bid auctions

A first-price auction differs from a second-price auction only in that the winner pays the price she bids, not the second highest bid. Precisely, a **first-price sealed-bid auction** (with perfect information) is defined as follows.

Players The n bidders, where $n \geq 2$.

Actions The set of actions of each player is the set of possible bids (nonnegative numbers).

Preferences The payoff of any player i is $v_i - b_i$ if either b_i is higher than every other bid, or b_i is at least as high as every other bid and the number of every other player who bids b_i is greater than i. Otherwise player i's payoff is 0.

This game models an auction in which people submit sealed bids and the highest bid wins. (You conduct such an auction when you solicit offers for a car you wish to sell, or, as a buyer, get estimates from contractors to fix your leaky basement, assuming in both cases that you do not inform potential bidders of existing bids.) The game models also a dynamic auction in which the auctioneer begins by announcing a high price, which she gradually lowers until someone indicates her willingness to buy the object. (Flowers in the Netherlands are sold in this way.) A bid in the strategic game is interpreted as the price at which the bidder will indicate her willingness to buy the object in the dynamic auction.

One Nash equilibrium of a first-price sealed-bid auction is $(b_1, \ldots, b_n) = (v_2, v_2, v_3, \ldots, v_n)$, in which player 1's bid is player 2's valuation v_2 and every other player's bid is her own valuation. The outcome of this equilibrium is that player 1 obtains the object at the price v_2.

? EXERCISE 84.1 (Nash equilibrium of first-price sealed-bid auction) Show that $(b_1, \ldots, b_n) = (v_2, v_2, v_3, \ldots, v_n)$ is a Nash equilibrium of a first-price sealed-bid auction.

A first-price sealed-bid auction has many other equilibria, but in all equilibria the winner is the player who values the object most highly (player 1), by the following argument. In any action profile $(b_1, \ldots, b_n)$ in which some player $i \neq 1$ wins, we have $b_i > b_1$. If $b_i > v_2$ then i's payoff is negative, so that she can do better by reducing her bid to 0; if $b_i \leq v_2$ then player 1 can increase her payoff from 0 to $v_1 - b_i$ by bidding b_i, in which case she wins. Thus no such action profile is a Nash equilibrium.

? EXERCISE 85.1 (First-price sealed-bid auction) Show that in a Nash equilibrium of a first-price sealed-bid auction the two highest bids are the same, one of these bids is submitted by player 1, and the highest bid is at least v_2 and at most v_1. Show also that any action profile satisfying these conditions is a Nash equilibrium.

In any equilibrium in which the winning bid exceeds v_2, at least one player's bid exceeds her valuation. As in a second-price sealed-bid auction, such a bid seems "risky", because it would yield the bidder a negative payoff if it were to win. In the equilibrium there is no risk, because the bid does not win; but, as before, the fact that the bid has this property reduces the plausibility of the equilibrium.

As in a second-price sealed-bid auction, the potential "riskiness" to player i of a bid $b_i > v_i$ is reflected in the fact that it is weakly dominated by the bid v_i, as shown by the following argument.

- If the other players' bids are such that player i loses when she bids b_i, then the outcome is the same whether she bids b_i or v_i.

- If the other players' bids are such that player i wins when she bids b_i, then her payoff is negative when she bids b_i and zero when she bids v_i (whether or not this bid wins).

However, in a first-price auction, unlike a second-price auction, a bid $b_i < v_i$ of player i is *not* weakly dominated by the bid v_i. If fact, such a bid is not weakly dominated by *any* bid. It is not weakly dominated by a bid $b'_i < b_i$, because if the other players' highest bid is between b'_i and b_i then b'_i loses whereas b_i wins and yields player i a positive payoff. And it is not weakly dominated by a bid $b'_i > b_i$, because if the other players' highest bid is less than b_i then both b_i and b'_i win and b_i yields a lower price.

Further, even though the bid v_i weakly dominates higher bids, this bid is itself weakly dominated, by a lower bid! If player i bids v_i her payoff is 0 regardless of the other players' bids, whereas if she bids less than v_i her payoff is either 0 (if she loses) or positive (if she wins).

In summary,

> *in a first-price sealed-bid auction (with perfect information), a player's bid of at least her valuation is weakly dominated, and a bid of less than her valuation is not weakly dominated.*

An implication of this result is that in *every* Nash equilibrium of a first-price sealed-bid auction at least one player's action is weakly dominated. However, this property of the equilibria depends on the assumption that a bid may be any number. In the variant of the game in which bids and valuations are restricted to be multiples of some discrete monetary unit ϵ (e.g. a cent), an action profile $(v_2 - \epsilon, v_2 - \epsilon, b_3, \ldots, b_n)$ for any $b_j \leq v_j - \epsilon$ for $j = 3, \ldots, n$ is a Nash equilibrium in which no player's bid is weakly dominated. Further, every equilibrium in which no player's bid is weakly dominated takes this form. When ϵ is small, each such equilibrium is close to an equilibrium $(v_2, v_2, b_3, \ldots, b_n)$ (with $b_j \leq v_j$ for $j = 3, \ldots, n$) of the game with unrestricted bids. On this (somewhat *ad hoc*) basis, I select action profiles $(v_2, v_2, b_3, \ldots, b_n)$ with $b_j \leq v_j$ for $j = 3, \ldots, n$ as "distinguished" equilibria of a first-price sealed-bid auction.

One conclusion of this analysis is that while both second-price and first-price auctions have many Nash equilibria, yielding a variety of outcomes, their distinguished equilibria yield the *same* outcome. (Recall that the distinguished equilibrium of a second-price sealed-bid auction is the action profile in which every player bids her valuation.) In every distinguished equilibrium of each game, the object is sold to player 1 at the price v_2. In particular, the auctioneer's revenue is the same in both cases. Thus if we restrict attention to the distinguished equilibria, the two auction forms are "revenue equivalent". The rules are different, but the players' equilibrium bids adjust to the difference and lead to the same outcome:

> *the single Nash equilibrium in which no player's bid is weakly dominated in a second-price auction yields the same outcome as the distinguished equilibria of a first-price auction.*

? EXERCISE 86.1 (Third-price auction) Consider a *third*-price sealed-bid auction, which differs from a first- and a second-price auction only in that the winner (the person who submits the highest bid) pays the third highest price. (Assume that there are at least three bidders.)

a. Show that for any player i the bid of v_i weakly dominates any lower bid, but does not weakly dominate any higher bid. (To show the latter, for any bid $b_i > v_i$ find bids for the other players such that player i is better off bidding b_i than bidding v_i.)

b. Show that the action profile in which each player bids her valuation is not a Nash equilibrium.

c. Find a Nash equilibrium. (There are ones in which every player submits the same bid.)

3.5.4 Variants

Uncertain valuations One respect in which the models in this section depart from reality is in the assumption that each bidder is certain of both her own valuation and every other bidder's valuation. In most, if not all, actual auctions, information is surely less perfect. The case in which the players are uncertain about each other's valuations has been thoroughly explored, and is discussed in Section 9.7. The result that a player's bidding her valuation weakly dominates all her other actions in a second-price auction survives when players are uncertain about each other's valuations, as does the revenue-equivalence of first- and second-price auctions under some conditions on the players' preferences.

Common valuations In some auctions the main difference between the bidders is not that the value the object differently but that they have different information about its value. For example, the bidders for an oil tract may put similar values on any given amount of oil, but have different information about how much oil is in the tract. Such auctions involve informational considerations that do not arise in the model we have studied in this section; they are studied in Section 9.7.3.

Multi-unit auctions In some auctions, like those for Treasury Bills (short-term government bonds) in the USA, many units of an object are available, and each bidder may value positively more than one unit. In each of the types of auction described below, each bidder submits a bid for each unit of the good. That is, an action is a list of bids $(b^1, \ldots, b^k)$, where b^1 is the player's bid for the first unit of the good, b^2 is her bid for the second unit, and so on. The player who submits the highest bid for any given unit obtains that unit. The auctions differ in the prices paid by the winners. (The first type of auction generalizes a first-price auction, whereas the next two generalize a second-price auction.)

Discriminatory auction The price paid for each unit is the winning bid for that unit.

Uniform-price auction The price paid for each unit is the same, equal to the highest rejected bid among all the bids for all units.

Vickrey auction A bidder who wins k objects pays the sum of the k highest rejected bids submitted by the *other* bidders.

The next exercise asks you to study these auctions when two units of an object are available.

?? EXERCISE 87.1 (Multi-unit auctions) Two units of an object are available. There are n bidders. Bidder i values the first unit that she obtains at v_i and the second unit at w_i, where $v_i > w_i > 0$. Each bidder submits two bids; the two highest bids win. Retain the tie-breaking rule in the text. Show that in discriminatory and uniform-price auctions, player i's action of bidding v_i and w_i does not dominate all her other actions, whereas in a Vickrey auction it does. (In the case of a Vickrey auction, consider separately the cases in which the other players' bids are such that player i wins no units, one unit, and two units when her bids are v_i and w_i.)

Goods for which the demand exceeds the supply at the going price are sometimes sold to the people who are willing to wait longest in line. We can model such situations as multi-unit auctions in which each person's bid is the amount of time she is willing to wait.

?? EXERCISE 88.1 (Waiting in line) Two hundred people are willing to wait in line to see a movie at a theater whose capacity is one hundred. Denote person i's valuation of the movie in excess of the price of admission, expressed in terms of the amount of time she is willing to wait, by v_i. That is, person i's payoff if she waits for t_i units of time is $v_i - t_i$. Each person attaches no value to a second ticket, and cannot buy tickets for other people. Assume that $v_1 > v_2 > \cdots > v_{200}$. Each person chooses an arrival time. If several people arrive at the same time then their order in line is determined by their index (lower-numbered people go first). If a person arrives to find 100 or more people already in line, her payoff is zero. Model the situation as a variant of a discriminatory multi-unit auction, in which each person submits a bid for only one unit, and find its Nash equilibria. (Look at your answer to Exercise 85.1 before seeking the Nash equilibria.) Arrival times for people at movies do not in general seem to conform with a Nash equilibrium. What feature missing from the model could explain the pattern of arrivals?

The next exercise is another application of a multi-unit auction. As in the previous exercise each person wants to buy only one unit, but in this case the price paid by the winners is the highest losing bid.

? EXERCISE 88.2 (Internet pricing) A proposal to deal with congestion on electronic message pathways is that each message should include a field stating an amount of money the sender is willing to pay for the message to be sent. Suppose that during some time interval, each of n people wants to send one message and the capacity of the pathway is k messages, with $k < n$. The k messages whose bids are highest are the ones sent, and each of the persons sending these messages pays a price equal to the $(k+1)$st highest bid. Model this situation as a multi-unit auction. (Use the same tie-breaking rule as the one in the text.) Does a person's action of bidding the value of her message weakly dominate all her other actions? (Note that the auction differs from those considered in Exercise 87.1 because each person submits only one bid. Look at the argument in the text that in a second-price sealed-bid auction a player's action of bidding her value weakly dominates all her other actions.)

Lobbying as an auction Variants of the models in this section can be used to understand some situations that are not explicitly auctions. An example, illustrated in the next exercise, is the competition between groups pressuring a government to follow policies they favor. This exercise shows also that the outcome of an auction may depend significantly (and perhaps counterintuitively) on the form the auction takes.

? EXERCISE 88.3 (Lobbying as an auction) A government can pursue three policies, x, y, and z. The monetary values attached to these policies by two interest groups, A and B, are given in Figure 89.1. The government chooses a policy in response to the payments the interest groups make to it. Consider the following two mechanisms.

First-price auction Each interest group chooses a policy and an amount of money it is willing to pay. The government chooses the policy proposed by the group willing to pay the most. This group makes its payment to the government, and the losing group makes no payment.

Menu auction Each interest group states, for each policy, the amount it is willing to pay to have the government implement that policy. The government chooses the policy for which the sum of the payments the groups are willing to make is the highest, and *each* group pays the government the amount of money it is willing to pay for that policy.

In each case each interest group's payoff is the value it attaches to the policy implemented minus the payment it makes. Assume that a tie is broken by the government's choosing the policy, among those tied, whose name is first in the alphabet.

	x	y	z
Interest group A	0	3	-100
Interest group B	0	-100	3

Figure 89.1 The values of the interest groups for the policies x, y, and z in Exercise 88.3.

Show that the first-price auction has a Nash equilibrium in which lobby A says it will pay 103 for y, lobby B says it will pay 103 for z, and the government's revenue is 103. Show that the menu auction has a Nash equilibrium in which lobby A announces that it will pay 3 for x, 6 for y, and 0 for z, and lobby B announces that it will pay 3 for x, 0 for y, and 6 for z, and the government chooses x, obtaining a revenue of 6. (In each case the pair of actions given is in fact the unique equilibrium.)

3.6 Accident law

3.6.1 Introduction

In some situations, laws influence the participants' payoffs and hence their actions. For example, a law may provide for the victim of an accident to be compensated by a party who was at fault, and the size of the compensation may affect the care that each party takes. What laws can we expect to produce socially desirable outcomes? A game theoretic analysis is useful in addressing this question.

3.6.2 The game

Consider the interaction between an *injurer* (player 1) and a *victim* (player 2). The victim suffers a loss that depends on the amounts of care taken by both her and the injurer. (How badly you hurt yourself when you fall down on the sidewalk in front of my house depends on both how well I have cleared the ice and how carefully you tread.) Denote by a_i the amount of care player i takes, measured in monetary terms, and by $L(a_1, a_2)$ the loss, also measured in monetary terms, suffered by the victim, as a function of the amounts of care. (In many cases the victim does not suffer a loss with certainty, but only with probability less than one. In such cases we can interpret $L(a_1, a_2)$ as the expected loss—the average loss suffered over many occurrences.) Assume that $L(a_1, a_2) > 0$ for all values of (a_1, a_2), and that more care taken by either player reduces the loss: L is decreasing in a_1 for any fixed value of a_2, and decreasing in a_2 for any fixed value of a_1.

A legal rule determines the fraction of the loss borne by the injurer, as a function of the amounts of care taken. Denote this fraction by $\rho(a_1, a_2)$. If $\rho(a_1, a_2) = 0$ for all (a_1, a_2), for example, the victim bears the entire loss, regardless of how much care she takes or how little care the injurer takes. At the other extreme, $\rho(a_1, a_2) = 1$ for all (a_1, a_2) means that the victim is fully compensated by the injurer no matter how careless she is or how careful the injurer is.

If the amounts of care are (a_1, a_2) then the injurer bears the cost a_1 of taking care and suffers the loss of $L(a_1, a_2)$, of which she bears the fraction $\rho(a_1, a_2)$. Thus the injurer's payoff is

$$-a_1 - \rho(a_1, a_2)L(a_1, a_2).$$

Similarly, the victim's payoff is

$$-a_2 - (1 - \rho(a_1, a_2))L(a_1, a_2).$$

For any given legal rule, embodied in ρ, we can model the interaction between the injurer and victim as the following strategic game.

Players The injurer and the victim.

Actions The set of actions of each player is the set of possible levels of care (nonnegative numbers).

Preferences The injurer's preferences are represented by the payoff function $-a_1 - \rho(a_1, a_2)L(a_1, a_2)$ and the victim's preferences are represented by the payoff function $-a_2 - (1 - \rho(a_1, a_2))L(a_1, a_2)$, where a_1 is the injurer's level of care and a_2 is the victim's level of care.

How do the equilibria of this game depend upon the legal rule? Do any legal rules lead to socially desirable equilibrium outcomes?

I restrict attention to a class of legal rules known as *negligence with contributory negligence*. (This class was established in the USA in the mid-nineteenth century, and prevailed until the mid-1970s.) Each rule in this class requires the injurer to compensate the victim for a loss if and only if *both* the victim is sufficiently careful *and* the injurer is sufficiently careless; the required compensation is the total loss. Rules in the class differ in the standards of care they specify for each party. The rule that specifies the standards of care X_1 for the injurer and X_2 for the victim requires the injurer to pay the victim the entire loss $L(a_1, a_2)$ when $a_1 < X_1$ (the injurer is insufficiently careful) and $a_2 \geq X_2$ (the victim is sufficiently careful), and nothing otherwise. That is, under this rule the fraction $\rho(a_1, a_2)$ of the loss borne by the injurer is

$$\rho(a_1, a_2) = \begin{cases} 1 & \text{if } a_1 < X_1 \text{ and } a_2 \geq X_2 \\ 0 & \text{if } a_1 \geq X_1 \text{ or } a_2 < X_2. \end{cases}$$

Included in this class of rules are those for which X_1 is a positive finite number and $X_2 = 0$ (the injurer has to pay if she is not sufficiently careful, even if the victim takes no care at all), known as rules of *pure negligence*, and that for which X_1 is infinite and $X_2 = 0$ (the injurer has to pay regardless of how careful she is and how careless the victim is), known as the rule of *strict liability*.

3.6.3 Nash equilibrium

Suppose we decide that the pair $(\hat{a}_1, \hat{a}_2)$ of actions is socially desirable. We wish to answer the question: are there values of X_1 and X_2 such that the game generated by the rule of negligence with contributory negligence for (X_1, X_2) has $(\hat{a}_1, \hat{a}_2)$ as its unique Nash equilibrium? If the answer is affirmative, then, assuming the solution concept of Nash equilibrium is appropriate for the situation we are considering, we have found a legal rule that induces the socially desirable outcome.

Specifically, suppose that we select as socially desirable the pair $(\hat{a}_1, \hat{a}_2)$ of actions that maximizes the sum of the players' payoffs. That is,

$$(\hat{a}_1, \hat{a}_2) \text{ maximizes } -a_1 - a_2 - L(a_1, a_2).$$

(For some functions L, this pair $(\hat{a}_1, \hat{a}_2)$ may be a reasonable candidate for a socially desirable outcome; in other cases it may induce a very inequitable distribution of payoff between the players, and thus be an unlikely candidate.)

I claim that the unique Nash equilibrium of the game induced by the legal rule of negligence with contributory negligence for $(X_1, X_2) = (\hat{a}_1, \hat{a}_2)$ is $(\hat{a}_1, \hat{a}_2)$. That is, if the standards of care are equal to their socially desirable levels, then these are the levels chosen by an injurer and a victim in the only equilibrium of the game. The outcome is that the injurer pays no compensation: her level of care is $\hat{a}_1$, just high enough that $\rho(a_1, a_2) = 0$. At the same time the victim's level of care is $\hat{a}_2$, high enough that if the injurer reduces her level of care even slightly then she has to pay full compensation.

I first argue that $(\hat{a}_1, \hat{a}_2)$ is *a* Nash equilibrium of the game, then show that it is the *only* equilibrium. To show that $(\hat{a}_1, \hat{a}_2)$ is a Nash equilibrium, I need to show that the injurer's action $\hat{a}_1$ is a best response to the victim's action $\hat{a}_2$ and *vice versa*.

Injurer's action Given that the victim's action is $\hat{a}_2$, the injurer has to pay compensation if and only if $a_1 < \hat{a}_1$. Thus the injurer's payoff is

$$u_1(a_1, \hat{a}_2) = \begin{cases} -a_1 - L(a_1, \hat{a}_2) & \text{if } a_1 < \hat{a}_1 \\ -a_1 & \text{if } a_1 \geq \hat{a}_1. \end{cases} \quad (91.1)$$

For $a_1 = \hat{a}_1$, this payoff is $-\hat{a}_1$. If she takes more care than $\hat{a}_1$, she is worse off, because care is costly and, beyond $\hat{a}_1$, does not reduce her liability for compensation. If she takes less care, then, given the victim's level of care, she has to pay compensation, and we need to compare the money saved by taking less care with the size of the compensation. The argument is a little tricky. First, by definition,

$$(\hat{a}_1, \hat{a}_2) \text{ maximizes } -a_1 - a_2 - L(a_1, a_2).$$

Hence

$$\hat{a}_1 \text{ maximizes } -a_1 - \hat{a}_2 - L(a_1, \hat{a}_2)$$

(given $\hat{a}_2$). Because $\hat{a}_2$ is a constant, it follows that

$$\hat{a}_1 \text{ maximizes } -a_1 - L(a_1, \hat{a}_2).$$

But from (91.1) we see that $-a_1 - L(a_1, \hat{a}_2)$ is the injurer's payoff $u_1(a_1, \hat{a}_2)$ when her action is $a_1 < \hat{a}_1$ and the victim's action is $\hat{a}_2$. We conclude that the injurer's payoff takes a form like that in the left panel of Figure 92.1. In particular, $\hat{a}_1$ maximizes $u_1(a_1, \hat{a}_2)$, so that $\hat{a}_1$ is a best response to $\hat{a}_2$.

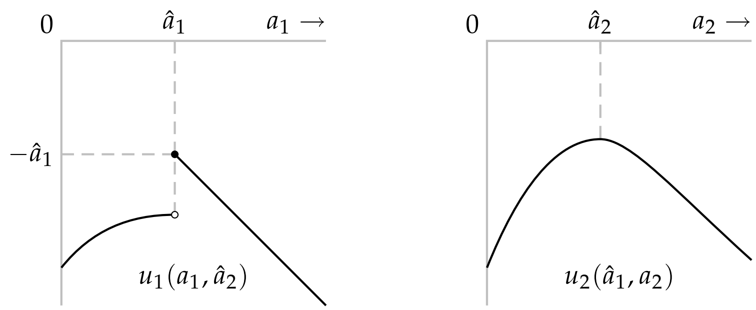


Figure 92.1 Left panel: the injurer's payoff as a function of her level of care a_1 when the victim's level of care is $a_2 = \hat{a}_2$ (see (91.1)). Right panel: the victim's payoff as a function of her level of care a_2 when the injurer's level of care is $a_1 = \hat{a}_1$ (see (92.1)).

Victim's action Given that the injurer's action is $\hat{a}_1$, the victim never receives compensation. Thus her payoff is

$$u_2(\hat{a}_1, a_2) = -a_2 - L(\hat{a}_1, a_2). \tag{92.1}$$

We can argue as we did for the injurer. By definition, $(\hat{a}_1, \hat{a}_2)$ maximizes $-a_1 - a_2 - L(a_1, a_2)$, so

$$\hat{a}_2 \text{ maximizes } -\hat{a}_1 - a_2 - L(\hat{a}_1, a_2)$$

(given $\hat{a}_1$). Because $\hat{a}_1$ is a constant, it follows that

$$\hat{a}_2 \text{ maximizes } -a_2 - L(\hat{a}_1, a_2), \tag{92.2}$$

which is the victim's payoff (see (92.1) and the right panel of Figure 92.1). That is, $\hat{a}_2$ maximizes $u_2(\hat{a}_1, a_2)$, so that $\hat{a}_2$ is a best response to $\hat{a}_1$.

We conclude that $(\hat{a}_1, \hat{a}_2)$ is a Nash equilibrium of the game induced by the legal rule of negligence with contributory negligence when the standards of care are $\hat{a}_1$ for the injurer and $\hat{a}_2$ for the victim.

To show that $(\hat{a}_1, \hat{a}_2)$ is the *only* Nash equilibrium of the game, first consider the injurer's best response function. Her payoff function is

$$u_1(a_1, a_2) = \begin{cases} -a_1 - L(a_1, a_2) & \text{if } a_1 < \hat{a}_1 \text{ and } a_2 \geq \hat{a}_2 \\ -a_1 & \text{if } a_1 \geq \hat{a}_1 \text{ or } a_2 < \hat{a}_2. \end{cases}$$

We can split the analysis into three cases, according to the victim's level of care.

$a_2 < \hat{a}_2$: In this case the injurer does not have to pay any compensation, regardless of her level of care; her payoff is $-a_1$, so that her best response is $a_1 = 0$.

$a_2 = \hat{a}_2$: In this case the injurer's best response is $\hat{a}_1$, as argued when showing that $(\hat{a}_1, \hat{a}_2)$ is a Nash equilibrium.

$a_2 > \hat{a}_2$: In this case the injurer's best response is at most $\hat{a}_1$, because her payoff for larger values of a_1 is equal to $-a_1$, a decreasing function of a_1.

We conclude that the injurer's best response function takes a form like that shown in the left panel of Figure 93.1.

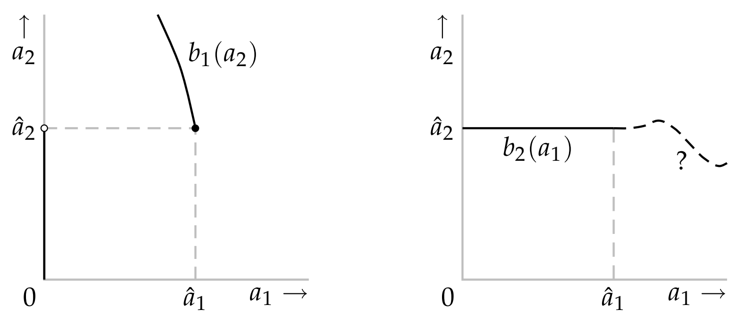


Figure 93.1 The players' best response functions under the rule of negligence with contributory negligence when $(X_1, X_2) = (\hat{a}_1, \hat{a}_2)$. Left panel: the injurer's best response function b_1. Right panel: the victim's best response function b_2. (The position of the victim's best response function for $a_1 > \hat{a}_1$ is not significant, and is not determined in the text.)

Now, given that the injurer's best response to any value of a_2 is never greater than $\hat{a}_1$, in any equilibrium we have $a_1 \leq \hat{a}_1$: any point (a_1, a_2) at which the victim's best response function crosses the injurer's best response function must have $a_1 \leq \hat{a}_1$. (Draw a few possible best response functions for the victim in the left panel of Figure 93.1.) We know that the victim's best response to $\hat{a}_1$ is $\hat{a}_2$ (because $(\hat{a}_1, \hat{a}_2)$ is a Nash equilibrium), so we need to worry only about the victim's best responses to values of a_1 with $a_1 < \hat{a}_1$ (i.e. for cases in which the injurer takes insufficient care).

Let $a_1 < \hat{a}_1$. Then if the victim takes insufficient care she bears the loss; otherwise she is compensated for the loss, and hence bears only the cost a_2 of her taking care. Thus the victim's payoff is

$$u_2(a_1, a_2) = \begin{cases} -a_2 - L(a_1, a_2) & \text{if } a_2 < \hat{a}_2 \\ -a_2 & \text{if } a_2 \geq \hat{a}_2. \end{cases} \tag{94.1}$$

Now, by (92.2) the level of care $\hat{a}_2$ maximizes $-a_2 - L(\hat{a}_1, a_2)$, so that

$$-a_2 - L(\hat{a}_1, a_2) \leq -\hat{a}_2 - L(\hat{a}_1, \hat{a}_2) \text{ for all } a_2.$$

Further, the loss is nonnegative, so $-\hat{a}_2 - L(\hat{a}_1, \hat{a}_2) \leq -\hat{a}_2$. We conclude that

$$-a_2 - L(\hat{a}_1, a_2) \leq -\hat{a}_2 \text{ for all } a_2. \tag{94.2}$$

Finally, the loss increases as the injurer takes less care, so that given $a_1 < \hat{a}_1$ we have $L(a_1, a_2) > L(\hat{a}_1, a_2)$ for all a_2. Thus $-a_2 - L(a_1, a_2) < -a_2 - L(\hat{a}_1, a_2)$ for all a_2, and hence, using (94.2),

$$-a_2 - L(a_1, a_2) < -\hat{a}_2 \text{ for all } a_2.$$

From (94.1) it follows that the victim's best response to any $a_1 < \hat{a}_1$ is $\hat{a}_2$, as shown in the right panel of Figure 93.1.

Combining the two best response functions we see that $(\hat{a}_1, \hat{a}_2)$, the pair of levels of care that maximizes the sum of the players' payoffs, is the unique Nash equilibrium of the game. That is, the rule of negligence with contributory negligence for standards of care equal to $\hat{a}_1$ and $\hat{a}_2$ induces the players to choose these levels of care. If legislators can determine the values of $\hat{a}_1$ and $\hat{a}_2$ then by writing these levels into law they will induce a game that has as its unique Nash equilibrium the socially optimal actions.

Other standards also induce a pair of levels of care equal to $(\hat{a}_1, \hat{a}_2)$, as you are asked to show in the following exercise.

?? EXERCISE 94.3 (Alternative standards of care under negligence with contributory negligence) Show that $(\hat{a}_1, \hat{a}_2)$ is the unique Nash equilibrium for the rule of negligence with contributory negligence for any value of (X_1, X_2) for which *either* $X_1 = \hat{a}_1$ and $X_2 \leq \hat{a}_2$ (including the pure negligence case of $X_2 = 0$), *or* $X_1 \geq M$ and $X_2 = \hat{a}_2$ for sufficiently large M. (Use the lines of argument in the text.)

? EXERCISE 94.4 (Equilibrium under strict liability) Study the Nash equilibrium (equilibria?) of the game studied in the text under the rule of strict liability, in which X_1 is infinite and $X_2 = 0$ (i.e. the injurer is liable for the loss no matter how careful she is and how careless the victim is). How are the equilibrium actions related to $\hat{a}_1$ and $\hat{a}_2$?

Notes

The model in Section 3.1 was developed by Cournot (1838). The model in Section 3.2 is widely credited to Bertrand (1883). The box on p. 67 is based on Leonard (1994) and Magnan de Bornier (1992). The models are discussed in more detail by Shapiro (1989).

The model in Section 3.3 is due to Hotelling (1929) (though the focus of his paper is a model in which the players are firms that choose not only locations, but also prices). Downs (1957, especially Ch. 8) popularized Hotelling's model, using it to gain insights about electoral competition. Shepsle (1991) and Osborne (1995) survey work in the field.

The *War of Attrition* studied in Section 3.4 is due to Maynard Smith (1974); it is a variant of the *Dollar Auction* presented by Shubik (1971).

Vickrey (1961) initiated the formal modeling of auctions, as studied in Section 3.5. The literature is surveyed by Wilson (1992). The box on page 79 draws on Herodotus' *Histories* (Book 1, paragraph 196; see for example Herodotus 1998, 86), Langdon (1994), Cassady (1967, Ch. 3), Shubik (1983), Andreau (1999, 38–39), the website `www.eBay.com`, Ockenfels and Roth (2000), and personal correspondence with Robin G. Osborne (on ancient Greece and Rome) and John H. Munro (on medieval Europe).

The model of accident law discussed in Section 94.3 originated with Brown (1973) and Diamond (1974); the result about negligence with contributory negligence is due to Brown (1973, 340–341). The literature is surveyed by Benoît and Kornhauser (1995).

Novshek and Sonnenschein (1978) study, in a general setting, the issue addressed in Exercise 60.1. A brief summary of the early work on common property is given in the Notes to Chapter 2. The idea of the tie-breaking rule being determined by the equilibrium, used in Exercises 66.2 and 67.1, is due to Simon and Zame (1990). The result in Exercise 73.1 is due to Wittman (1977). Exercise 73.2 is based on Osborne and Slivinski (1996). The notion of a Condorcet winner defined in Exercise 74.1 is associated with Marie-Jean-Antoine-Nicolas de Caritat, marquis de Condorcet (1743–1794), an early student of voting procedures. The game in Exercise 78.1 is a variant of a game studied by Blackwell and Girschick (1954, Example 5 in Ch. 2). It is an example of a *noisy duel* (which models the situation of duelists, each of whom chooses when to fire a single bullet, which her opponent hears, as she gradually approaches her rival). Duels were first modeled as games in the late 1940s by members of the RAND Corporation in the USA; see Karlin (1959b, Ch. 5). Exercise 88.3 is based on Boylan (1997). The situation considered in Exercise 88.1, in which people decide when to join a queue, is studied by Holt and Sherman (1982). Exercise 88.2 is based on MacKie-Mason and Varian (1995).

4 Mixed Strategy Equilibrium

Prerequisite: Chapter 2.

4.1 Introduction

4.1.1 Stochastic steady states

A NASH EQUILIBRIUM of a strategic game is an action profile in which every player's action is optimal given every other player's action (Definition 21.1). Such an action profile corresponds to a steady state of the idealized situation in which for each player in the game there is a population of individuals, and whenever the game is played, one player is drawn randomly from each population (see Section 2.6). In a steady state, every player's behavior is the same whenever she plays the game, and no player wishes to change her behavior, knowing (from her experience) the other players' behavior. In a steady state in which each player's "behavior" is simply an action and within each population all players choose the same action, the outcome of every play of the game is the same Nash equilibrium.

More general notions of a steady state allow the players' choices to vary, as long as the pattern of choices remains constant. For example, different members of a given population may choose different actions, each player choosing the same action whenever she plays the game. Or each individual may, on each occasion she plays the game, choose her action probabilistically according to the same, unchanging distribution. These two more general notions of a steady state are equivalent: a steady state of the first type in which the fraction p of the population representing player i chooses the action a corresponds to a steady state of the second type in which each member of the population representing player i chooses a with probability p. In both cases, in each play of the game the probability that the individual in the role of player i chooses a is p. Both these notions of steady state are modeled by a mixed strategy Nash equilibrium, a generalization of the notion of Nash equilibrium. For expository convenience, in most of this chapter I interpret such an equilibrium as a model of the second type of steady state, in which each player chooses her actions probabilistically; such a steady state is called *stochastic* ("involving probability").

4.1.2 Example: Matching Pennies

An analysis of the game *Matching Pennies* (Example 17.1) illustrates the idea of a stochastic steady state. My discussion focuses on the outcomes of this game, given in Figure 98.1, rather than payoffs that represent the players' preferences, as before.

	Head	*Tail*
Head	$1, −$1	−$1, $1
Tail	−$1, $1	$1, −$1

Figure 98.1 The outcomes of *Matching Pennies*.

As we saw previously, this game has no Nash equilibrium: no pair of actions is compatible with a steady state in which each player's action is the same whenever the game is played. I claim, however, that the game has a *stochastic* steady state in which each player chooses each of her actions with probability $\frac{1}{2}$. To establish this result, I need to argue that if player 2 chooses each of her actions with probability $\frac{1}{2}$, then player 1 optimally chooses each of her actions with probability $\frac{1}{2}$, and vice versa.

Suppose that player 2 chooses each of her actions with probability $\frac{1}{2}$. If player 1 chooses *Head* with probability p and *Tail* with probability $1 - p$ then each outcome $(Head, Head)$ and $(Head, Tail)$ occurs with probability $\frac{1}{2}p$, and each outcome $(Tail, Head)$ and $(Tail, Tail)$ occurs with probability $\frac{1}{2}(1 - p)$. Thus player 1 gains \$1 with probability $\frac{1}{2}p + \frac{1}{2}(1 - p)$, which is equal to $\frac{1}{2}$, and loses \$1 with probability $\frac{1}{2}$. In particular, the probability distribution over outcomes is independent of p! Thus *every* value of p is optimal. In particular, player 1 can do no better than choose *Head* with probability $\frac{1}{2}$ and *Tail* with probability $\frac{1}{2}$. A similar analysis shows that player 2 optimally chooses each action with probability $\frac{1}{2}$ when player 1 does so. We conclude that the game has a stochastic steady state in which each player chooses each action with probability $\frac{1}{2}$.

I further claim that, under a reasonable assumption on the players' preferences, the game has no other steady state. This assumption is that each player wants the probability of her gaining \$1 to be as large as possible. More precisely, if $p > q$ then each player prefers to gain \$1 with probability p and lose \$1 with probability $1 - p$ than to gain \$1 with probability q and lose \$1 with probability $1 - q$.

To show that under this assumption there is no steady state in which the probability of each player's choosing *Head* is different from $\frac{1}{2}$, denote the probability with which player 2 chooses *Head* by q (so that she chooses *Tail* with probability $1 - q$). If player 1 chooses *Head* with probability p then she gains \$1 with probability $pq + (1 - p)(1 - q)$ (the probability that the outcome is either $(Head, Head)$

or $(Tail, Tail)$) and loses \$1 with probability $(1-p)q + p(1-q)$. The first probability is equal to $1 - q + p(2q - 1)$ and the second is equal to $q + p(1 - 2q)$. Thus if $q < \frac{1}{2}$ (player 2 chooses *Head* with probability less than $\frac{1}{2}$), the first probability is decreasing in p and the second is increasing in p, so that the lower is p, the better is the outcome for player 1; the value of p that induces the best probability distribution over outcomes for player 1 is 0. That is, if player 2 chooses *Head* with probability less than $\frac{1}{2}$, then the uniquely best policy for player 1 is to choose *Tail* with certainty. A similar argument shows that if player 2 chooses *Head* with probability greater than $\frac{1}{2}$, the uniquely best policy for player 1 is to choose *Head* with certainty.

Now, if player 1 chooses one of her actions with certainty, an analysis like that in the previous paragraph leads to the conclusion that the optimal policy of player 2 is to choose one of her actions with certainty (*Head* if player 1 chooses *Tail* and *Tail* if player 1 chooses *Head*).

We conclude that there is no steady state in which the probability that player 2 chooses *Head* is different from $\frac{1}{2}$. A symmetric argument leads to the conclusion that there is no steady state in which the probability that player 1 chooses *Head* is different from $\frac{1}{2}$. Thus the only stochastic steady state is that in which each player chooses each of her actions with probability $\frac{1}{2}$.

As discussed in the first section, the stable pattern of behavior we have found can be alternatively interpreted as a steady state in which no player randomizes. Instead, half the players in the population of individuals who take the role of player 1 in the game choose *Head* whenever they play the game and half of them choose *Tail* whenever they play the game; similarly half of those who take the role of player 2 choose *Head* and half choose *Tail*. Given that the individuals involved in any given play of the game are chosen randomly from the populations, in each play of the game each individual faces with probability $\frac{1}{2}$ an opponent who chooses *Head*, and with probability $\frac{1}{2}$ an opponent who chooses *Tail*.

? EXERCISE 99.1 (Variant of Matching Pennies) Find the steady state(s) of the game that differs from *Matching Pennies* only in that the outcomes of (*Head*,*Head*) and of (*Tail*,*Tail*) are that player 1 gains \$2 and player 2 loses \$1.

4.1.3 Generalizing the analysis: expected payoffs

The fact that *Matching Pennies* has only two outcomes for each player (gain \$1, lose \$1) makes the analysis of a stochastic steady state particularly simple, because it allows us to deduce, under a weak assumption, the players' preferences regarding *lotteries* (probability distributions) over outcomes from their preferences regarding deterministic outcomes (outcomes that occur with certainty). If a player prefers the deterministic outcome a to the deterministic outcome b, it is very plausible that if $p > q$ then she prefers the lottery in which a occurs with probability p (and b occurs with probability $1 - p$) to the lottery in which a occurs with probability q (and b occurs with probability $1 - q$).

In a game with more than two outcomes for some player, we cannot extrapolate in this way from preferences regarding deterministic outcomes to preferences regarding lotteries over outcomes. Suppose, for example, that a game has three possible outcomes, a, b, and c, and that a player prefers a to b to c. Does she prefer the deterministic outcome b to the lottery in which a and c each occur with probability $\frac{1}{2}$, or vice versa? The information about her preferences over deterministic outcomes gives us no clue about the answer to this question. She may prefer b to the lottery in which a and c each occur with probability $\frac{1}{2}$, or she may prefer this lottery to b; both preferences are consistent with her preferring a to b to c. In order to study her behavior when she is faced with choices between lotteries, we need to add to the model a description of her preferences regarding lotteries over outcomes.

A standard assumption in game theory restricts attention to preferences regarding lotteries over outcomes that may be represented by the expected value of a payoff function over deterministic outcomes. (See Section 17.7.3 if you are unfamiliar with the notion of "expected value".) That is, for every player i there is a payoff function u_i with the property that player i prefers one lottery over outcomes to another if and only if, according to u_i, the expected value of the first lottery exceeds the expected value of the second lottery.

For example, suppose that there are three outcomes, a, b, and c, and lottery P yields a with probability p_a, b with probability p_b, and c with probability p_c, whereas lottery Q yields these three outcomes with probabilities q_a, q_b, and q_c. Then the assumption is that for each player i there are numbers $u_i(a)$, $u_i(b)$, and $u_i(c)$ such that player i prefers lottery P to lottery Q if and only if $p_a u_i(a) + p_b u_i(b) + p_c u_i(c) > q_a u_i(a) + q_b u_i(b) + q_c u_i(c)$. (I discuss the representation of preferences by the expected value of a payoff function in more detail in Section 4.12, an appendix to this chapter.)

The first systematic investigation of preferences regarding lotteries represented by the expected value of a payoff function over deterministic outcomes was undertaken by von Neumann and Morgenstern (1944). Accordingly such preferences are called **vNM preferences**. A payoff function over deterministic outcomes (u_i in the previous paragraph) whose expected value represents such preferences is called a **Bernoulli payoff function** (in honor of Daniel Bernoulli (1700–1782), who appears to have been one of the first persons to use such a function to represent preferences).

The restrictions on preferences regarding deterministic outcomes required for them to be represented by a payoff function are relatively innocuous (see Section 1.2.2). The same is not true of the restrictions on preferences regarding lotteries over outcomes required for them to be represented by the expected value of a payoff function. (I do not discuss these restrictions, but the box at the end of this section gives an example of preferences that violate them.) Nevertheless, we obtain many insights from models that assume preferences take this form; following standard game theory (and standard economic theory), I maintain the assumption throughout the book.

The assumption that a player's preferences be represented by the expected value of a payoff function does not restrict her attitudes to risk: a person whose preferences are represented by such a function may have an arbitrarily strong like or dislike for risk. Suppose, for example, that a, b, and c are three outcomes, and a person prefers a to b to c. A person who is very averse to risky outcomes prefers to obtain b for sure rather than to face the lottery in which a occurs with probability p and c occurs with probability $1 - p$, even if p is relatively large. Such preferences may be represented by the expected value of a payoff function u for which $u(a)$ is close to $u(b)$, which is much larger than $u(c)$. A person who is not at all averse to risky outcomes prefers the lottery to the certain outcome b, even if p is relatively small. Such preferences are represented by the expected value of a payoff function u for which $u(a)$ is much larger than $u(b)$, which is close to $u(c)$. If $u(a) = 10$, $u(b) = 9$, and $u(c) = 0$, for example, then the person prefers the certain outcome b to any lottery between a and c that yields a with probability less than $\frac{9}{10}$. But if $u(a) = 10$, $u(b) = 1$, and $u(c) = 0$, she prefers any lottery between a and c that yields a with probability greater than $\frac{1}{10}$ to the certain outcome b.

Suppose that the outcomes are amounts of money and a person's preferences are represented by the expected value of a payoff function in which the payoff of each outcome is equal to the amount of money involved. Then we say the person is *risk neutral*. Such a person compares lotteries according to the expected amount of money involved. (For example, she is indifferent between receiving \$100 for sure and the lottery that yields \$0 with probability $\frac{9}{10}$ and \$1000 with probability $\frac{1}{10}$.) On the one hand, the fact that people buy insurance suggests that in some circumstances preferences are *risk averse*: people prefer to obtain \$$z$ with certainty than to receive the outcome of a lottery that yields \$$z$ on average. On the other hand, the fact that people buy lottery tickets that pay, on average, much less than their purchase price, suggests that in other circumstances preferences are *risk preferring*. In both cases, preferences over lotteries are not represented by expected *monetary* values, though they still may be represented by the expected value of a *payoff* function (in which the payoffs to outcome are different from the monetary values of the outcomes).

Any given preferences over deterministic outcomes are represented by many different payoff functions (see Section 1.2.2). The same is true of preferences over lotteries; the relation between payoff functions whose expected values represent the same preferences is discussed in Section 4.12.2 in the appendix to this chapter. In particular, we may choose arbitrary payoffs for the outcomes that are best and worst according to the preferences, as long as the payoff to the best outcome exceeds the payoff to the worst outcome. For example, suppose there are three outcomes, a, b, and c, and a person prefers a to b to c, and is indifferent between b and the lottery that yields a with probability $\frac{1}{2}$ and c with probability $\frac{1}{2}$. Then we may choose $u(a) = 3$ and $u(c) = 1$, in which case $u(b) = 2$; or, for example, we may choose $u(a) = 10$ and $u(c) = 0$, in which case $u(b) = 5$.

SOME EVIDENCE ON EXPECTED PAYOFF FUNCTIONS

Consider the following two lotteries (the first of which is, in fact, deterministic):

Lottery 1 You receive \$2 million with certainty

Lottery 2 You receive \$10 million with probability 0.1, \$2 million with probability 0.89, and nothing with probability 0.01.

Which do you prefer? Now consider two more lotteries:

Lottery 3 You receive \$2 million with probability 0.11 and nothing with probability 0.89

Lottery 4 You receive \$10 million with probability 0.1 and nothing with probability 0.9.

Which do you prefer? A significant fraction of experimental subjects say they prefer lottery 1 to lottery 2, and lottery 4 to lottery 3. (See, for example, Conlisk (1989) and Camerer (1995, 622–623).)

These preferences cannot be represented by an expected payoff function! If they could be, there would exist a payoff function u for which the expected payoff of lottery 1 exceeds that of lottery 2:

$$u(2) > 0.1u(10) + 0.89u(2) + 0.01u(0),$$

where the amounts of money are expressed in millions. Subtracting $0.89u(2)$ and adding $0.89u(0)$ to each side we obtain

$$0.11u(2) + 0.89u(0) > 0.1u(10) + 0.9u(0).$$

But this inequality says that the expected payoff of lottery 3 exceeds that of lottery 4! Thus preferences represented by an expected payoff function that yield a preference for lottery 1 over lottery 2 must also yield a preference for lottery 3 over lottery 4.

Preferences represented by the expected value of a payoff function *are*, however, consistent with a person's being indifferent between lotteries 1 and 2, and between lotteries 3 and 4. Suppose we assume that when a person is almost indifferent between two lotteries, she may make a "mistake". Then a person's expressed preference for lottery 1 over lottery 2 and for lottery 4 over lottery 3 is not directly inconsistent with her preferences being represented by the expected value of a payoff function in which she is almost indifferent between lotteries 1 and 2 and between lotteries 3 and 4. If, however, we add the assumption that mistakes are distributed symmetrically, then the frequency with which people express a preference for lottery 2 over lottery 1 and for lottery 4 over lottery 3 (also inconsistent with preferences represented by the expected value of a payoff function) should be similar to that with which people express a preference for lottery 1 over lottery 2 and for lottery 3 over lottery 4. In fact, however, the second pattern is significantly more common than the first (Conlisk 1989), so that a more significant modification of the theory is needed to explain the observations.

A limitation of the evidence is that it is based on the preferences expressed by people faced with *hypothetical* choices; understandably (given the amounts of money involved), no experiment has been run in which subjects were paid according to the lotteries they chose! Experiments with stakes consistent with normal research budgets show few choices inconsistent with preferences represented by the expected value of a payoff function (Conlisk 1989). This evidence, however, does not contradict the evidence based on hypothetical choices with large stakes: with larger stakes subjects might make choices in line with the preferences they express when asked about hypothetical choices.

In summary, the evidence for an inconsistency with preferences compatible with an expected payoff function is, at a minimum, suggestive. It has spurred the development of alternative theories. Nevertheless, the vast majority of models in game theory (and also in economics) that involve choice under uncertainty currently assume that each decision-maker's preferences are represented by the expected value of a payoff function. I maintain this assumption throughout the book, although many of the ideas I discuss appear not to depend on it.

4.2 Strategic games in which players may randomize

To study stochastic steady states, we extend the notion of a strategic game given in Definition 11.1 by endowing each player with vNM preferences about lotteries over the set of action profiles.

▶ DEFINITION 103.1 A **strategic game** (with vNM preferences) consists of

- a set of **players**
- for each player, a set of **actions**
- for each player, **preferences** regarding lotteries over action profiles that may be represented by the expected value of a ("Bernoulli") payoff function over action profiles.

A two-player strategic game with vNM preferences in which each player has finitely many actions may be presented in a table like those in Chapter 2. Such a table looks exactly the same as it did before, though the interpretation of the numbers in the boxes is different. In Chapter 2 these numbers are values of payoff functions that represent the players' preferences over deterministic outcomes; here they are the values of (Bernoulli) payoff functions whose expected values represent the players' preferences over lotteries.

Given the change in the interpretation of the payoffs, two tables that represent the same strategic game with ordinal preferences no longer necessarily represent the same strategic game with vNM preferences. For example, the two tables in Figure 104.1 represent the same game with ordinal preferences—namely the *Prisoner's Dilemma* (Section 2.2). In both cases the best outcome for each player is that in which she chooses F and the other player chooses Q, the next best outcome is (Q, Q), then comes (F, F), and the worst outcome is that in which she chooses Q and the other player chooses F. However, the tables represent *different* strategic games with vNM preferences. For example, in the left table player 1's payoff to (Q, Q) is the *same* as her expected payoff to the lottery that yields (F, Q) with probability $\frac{1}{2}$ and (F, F) with probability $\frac{1}{2}$ $(2 = \frac{1}{2} \cdot 3 + \frac{1}{2} \cdot 1)$, whereas in the right table her payoff to (Q, Q) is *greater than* her expected payoff to this lottery $(3 > \frac{1}{2} \cdot 4 + \frac{1}{2} \cdot 1)$. Thus the left table represents a situation in which player 1 is indifferent between the deterministic outcome (Q, Q) and the lottery in which (F, Q) occurs with probability $\frac{1}{2}$ and (F, F) occurs with probability $\frac{1}{2}$. In the right table, however, she prefers the deterministic outcome (Q, Q) to the lottery.

	Q	F
Q	2, 2	0, 3
F	3, 0	1, 1

	Q	F
Q	3, 3	0, 4
F	4, 0	1, 1

Figure 104.1 Two tables that represent the same strategic game with ordinal preferences but different strategic games with vNM preferences.

To show, as in this example, that two tables represent different strategic games with vNM preferences we need only find a pair of lotteries whose expected payoffs are ordered differently by the two tables. To show that they represent the *same* strategic game with vNM preferences is more difficult; see Section 4.12.2.

? EXERCISE 104.1 (Extensions of *BoS* with vNM preferences) Construct a table of payoffs for a strategic game with vNM preferences in which the players' preferences over deterministic outcomes are the same as they are in *BoS* (Example 16.2), and their preferences over lotteries satisfy the following condition: each player is indifferent between going to her less preferred concert in the company of the other player and the lottery in which with probability $\frac{1}{2}$ she and the other player go to different concerts and with probability $\frac{1}{2}$ they both go to her more preferred concert. Do the same in the case that each player is indifferent between going to her less preferred concert in the company of the other player and the lottery in which with probability $\frac{3}{4}$ she and the other player go to different concerts and with probability $\frac{1}{4}$ they both go to her more preferred concert. (In each case set each player's payoff to the outcome that she least prefers equal to 0 and her payoff to the outcome that she most prefers equal to 2.)

Despite the importance of saying how the numbers in a payoff table should be interpreted, users of game theory sometimes fail to make the interpretation clear. When interpreting discussions of Nash equilibrium in the literature, a reasonably safe assumption is that if the players are not allowed to choose their actions randomly then the numbers in payoff tables are payoffs that represent the players' ordinal preferences, whereas if the players are allowed to randomize then the numbers are payoffs whose expected values represent the players' preferences regarding lotteries over outcomes.

4.3 Mixed strategy Nash equilibrium

4.3.1 Mixed strategies

In the generalization of the notion of Nash equilibrium that models a stochastic steady state of a strategic game with vNM preferences, we allow each player to choose a probability distribution over her set of actions rather than restricting her to choose a single deterministic action. We refer to such a probability distribution as a **mixed strategy**.

I usually use α to denote a profile of mixed strategies; $\alpha_i(a_i)$ is the probability assigned by player i's mixed strategy α_i to her action a_i. To specify a mixed strategy of player i we need to give the probability it assigns to each of player i's actions. For example, the strategy of player 1 in *Matching Pennies* that assigns probability $\frac{1}{2}$ to each action is the strategy α_1 for which $\alpha_1(\mathit{Head}) = \frac{1}{2}$ and $\alpha_1(\mathit{Tail}) = \frac{1}{2}$. Because this way of describing a mixed strategy is cumbersome, I often use a shorthand for a game that is presented in a table like those in Figure 104.1: I write a mixed strategy as a list of probabilities, one for each action, *in the order the actions are given in the table*. For example, the mixed strategy $(\frac{1}{3}, \frac{2}{3})$ for player 1 in either of the games in Figure 104.1 assigns probability $\frac{1}{3}$ to Q and probability $\frac{2}{3}$ to F.

A mixed strategy may assign probability 1 to a single action: by *allowing* a player to choose probability distributions, we do not prohibit her from choosing deterministic actions. We refer to such a mixed strategy as a **pure strategy**. Player i's choosing the pure strategy that assigns probability 1 to the action a_i is equivalent to her simply choosing the action a_i, and I denote this strategy simply by a_i.

4.3.2 Equilibrium

The notion of equilibrium that we study is called "mixed strategy Nash equilibrium". The idea behind it is the same as the idea behind the notion of Nash equilibrium for a game with ordinal preferences: a mixed strategy Nash equilibrium is a mixed strategy profile α^* with the property that no player i has a mixed strategy α_i such that she prefers the lottery over outcomes generated by the strategy profile $(\alpha_i, \alpha^*_{-i})$ to the lottery over outcomes generated by the strategy profile α^*. The following definition gives this condition using payoff functions whose expected values represent the players' preferences.

▶ DEFINITION 105.1 (*Mixed strategy Nash equilibrium of strategic game with vNM preferences*) The mixed strategy profile α^* in a strategic game with vNM preferences is a **(mixed strategy) Nash equilibrium** if, for each player i and every mixed strategy α_i of player i, the expected payoff to player i of α^* is at least as large as the expected payoff to player i of $(\alpha_i, \alpha^*_{-i})$ according to a payoff function whose expected value represents player i's preferences over lotteries. Equivalently, for each player i,

$$U_i(\alpha^*) \geq U_i(\alpha_i, \alpha^*_{-i}) \text{ for every mixed strategy } \alpha_i \text{ of player } i, \qquad (106.1)$$

where $U_i(\alpha)$ is player i's expected payoff to the mixed strategy profile α.

4.3.3 Best response functions

When studying mixed strategy Nash equilibria, as when studying Nash equilibria of strategic games with ordinal preferences, the players' best response functions (Section 2.8) are often useful. As before, I denote player i's best response function by B_i. For a strategic game with ordinal preferences, $B_i(a_{-i})$ is the set of player i's best actions when the list of the other players' actions is α_{-i}. For a strategic game with vNM preferences, $B_i(\alpha_{-i})$ is the set of player i's best mixed strategies when the list of the other players' mixed strategies is α_{-i}. From the definition of a mixed strategy equilibrium, a profile α^* of mixed strategies is a mixed strategy Nash equilibrium if and only if every player's mixed strategy is a best response to the other players' mixed strategies (cf. Proposition 34.1):

> the mixed strategy profile α^* is a mixed strategy Nash equilibrium if and only if α^*_i is in $B_i(\alpha^*_{-i})$ for every player i.

4.3.4 Best response functions in two-player two-action games

The analysis of *Matching Pennies* in Section 4.1.2 shows that each player's set of best responses to the other player's mixed strategy is either a single pure strategy or the set of *all* mixed strategies. (For example, if player 2's mixed strategy assigns probability less than $\frac{1}{2}$ to *Head* then player 1's unique best response is the pure strategy *Tail*, if player 2's mixed strategy assigns probability greater than $\frac{1}{2}$ to *Head* then player 1's unique best response is the pure strategy *Head*, and if player 2's mixed strategy assigns probability $\frac{1}{2}$ to *Head* then all of player 1's mixed strategies are best responses.)

In any two-player game in which each player has two actions, the set of each player's best responses has a similar character: it consists either of a single pure strategy, or of all mixed strategies. The reason lies in the form of the payoff functions.

Consider a two-player game in which each player has two actions, T and B for player 1 and L and R for player 2. Denote by u_i, for $i = 1, 2$, a Bernoulli payoff function for player i. (That is, u_i is a payoff function over action pairs whose expected value represents player i's preferences regarding lotteries over action pairs.) Player 1's mixed strategy α_1 assigns probability $\alpha_1(T)$ to her action T and probability $\alpha_1(B)$ to her action B (with $\alpha_1(T) + \alpha_1(B) = 1$). For convenience, let $p = \alpha_1(T)$, so that $\alpha_1(B) = 1 - p$. Similarly, denote the probability $\alpha_2(L)$ that player 2's mixed strategy assigns to L by q, so that $\alpha_2(R) = 1 - q$.

We take the players' choices to be independent, so that when the players use the mixed strategies α_1 and α_2, the probability of any action pair (a_1, a_2) is the product of the probability player 1's mixed strategy assigns to a_1 and the probability player 2's mixed strategy assigns to a_2. (See Section 17.7.2 in the mathematical appendix if you are not familiar with the idea of independence.) Thus the probability distribution generated by the mixed strategy pair (α_1, α_2) over the four possible outcomes of the game has the form given in Figure 107.1: (T, L) occurs with probability pq, (T, R) occurs with probability $p(1-q)$, (B, L) occurs with probability $(1-p)q$, and (B, R) occurs with probability $(1-p)(1-q)$.

	$L\ (q)$	$R\ (1-q)$
$T\ (p)$	pq	$p(1-q)$
$B\ (1-p)$	$(1-p)q$	$(1-p)(1-q)$

Figure 107.1 The probabilities of the four outcomes in a two-player two-action strategic game when player 1's mixed strategy is $(p, 1-p)$ and player 2's mixed strategy is $(q, 1-q)$.

From this probability distribution we see that player 1's expected payoff to the mixed strategy pair (α_1, α_2) is

$$pq \cdot u_1(T, L) + p(1-q) \cdot u_1(T, R) + (1-p)q \cdot u_1(B, L) + (1-p)(1-q) \cdot u_1(B, R),$$

which we can alternatively write as

$$p[q \cdot u_1(T, L) + (1-q) \cdot u_1(T, R)] + (1-p)[q \cdot u_1(B, L) + (1-q) \cdot u_1(B, R)].$$

The first term in square brackets is player 1's expected payoff when she uses a *pure* strategy that assigns probability 1 to T and player 2 uses her mixed strategy α_2; the second term in square brackets is player 1's expected payoff when she uses a *pure* strategy that assigns probability 1 to B and player 2 uses her mixed strategy α_2. Denote these two expected payoffs $E_1(T, \alpha_2)$ and $E_1(B, \alpha_2)$. Then player 1's expected payoff to the mixed strategy pair (α_1, α_2) is

$$pE_1(T, \alpha_2) + (1-p)E_1(B, \alpha_2).$$

That is, player 1's expected payoff to the mixed strategy pair (α_1, α_2) is a weighted average of her expected payoffs to T and B when player 2 uses the mixed strategy α_2, with weights equal to the probabilities assigned to T and B by α_1.

In particular, player 1's expected payoff, given player 2's mixed strategy, is a *linear* function of p—when plotted in a graph, it is a straight line. A case in which $E_1(T, \alpha_2) > E_1(B, \alpha_2)$ is illustrated in Figure 108.1.

? EXERCISE 107.1 (Expected payoffs) Construct diagrams like Figure 108.1 for *BoS* (Figure 16.1) and the game in Figure 19.1 (in each case treating the numbers in the tables as Bernoulli payoffs). In each diagram, plot player 1's expected payoff as a function of the probability p that she assigns to her top action in three cases: when the probability q that player 2 assigns to her left action is 0, $\frac{1}{2}$, and 1.

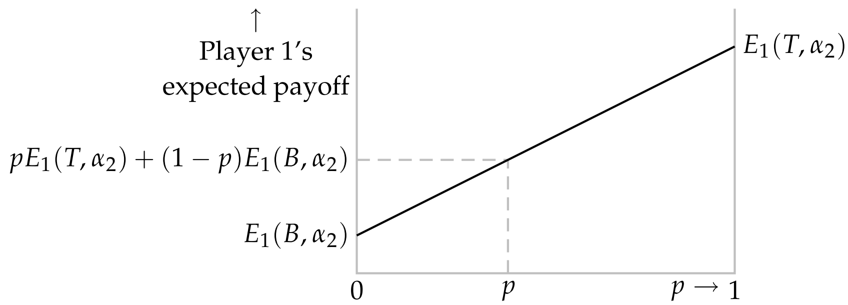


Figure 108.1 Player 1's expected payoff as a function of the probability p she assigns to T in the game in which her actions are T and B, when player 2's mixed strategy is α_2 and $E_1(T, \alpha_2) > E_1(B, \alpha_2)$.

A significant implication of the linearity of player 1's expected payoff is that there are three possibilities for her best response to a given mixed strategy of player 2:

- player 1's unique best response is the pure strategy T (if $E_1(T, \alpha_2) > E_1(B, \alpha_2)$, as in Figure 108.1)
- player 1's unique best response is the pure strategy B (if $E_1(B, \alpha_2) > E_1(T, \alpha_2)$, in which case the line representing player 1's expected payoff as a function of p in the analogue of Figure 108.1 slopes down)
- all mixed strategies of player 1 yield the same expected payoff, and hence all are best responses (if $E_1(T, \alpha_2) = E_1(B, \alpha_2)$, in which case the line representing player 1's expected payoff as a function of p in the analogue of Figure 108.1 is horizontal).

In particular, a mixed strategy $(p, 1 - p)$ for which $0 < p < 1$ is never the *unique* best response; either it is not a best response, or *all* mixed strategies are best responses.

? EXERCISE 108.1 (Best responses) For each game and each value of q in Exercise 107.1, use the graphs you drew in that exercise to find player 1's set of best responses.

4.3.5 Example: Matching Pennies

The argument in Section 4.1.2 establishes that *Matching Pennies* has a unique mixed strategy Nash equilibrium, in which each player's mixed strategy assigns probability $\frac{1}{2}$ to *Head* and probability $\frac{1}{2}$ to *Tail*. I now describe an alternative route to this conclusion that uses the method described in Section 2.8.3, which involves explicitly constructing the players' best response functions; this method may be used in other games.

Represent each player's preferences by the expected value of a payoff function that assigns the payoff 1 to a gain of \$1 and the payoff -1 to a loss of \$1. The resulting strategic game with vNM preferences is shown in Figure 109.1.

	Head	*Tail*
Head	$1, -1$	$-1, \ \ 1$
Tail	$-1, \ \ 1$	$1, -1$

Figure 109.1 *Matching Pennies.*

Denote by p the probability that player 1's mixed strategy assigns to *Head*, and by q the probability that player 2's mixed strategy assigns to *Head*. Then, given player 2's mixed strategy, player 1's expected payoff to the pure strategy *Head* is

$$q \cdot 1 + (1 - q) \cdot (-1) = 2q - 1$$

and her expected payoff to *Tail* is

$$q \cdot (-1) + (1 - q) \cdot 1 = 1 - 2q.$$

Thus if $q < \frac{1}{2}$ then player 1's expected payoff to *Tail* exceeds her expected payoff to *Head*, and hence exceeds also her expected payoff to every mixed strategy that assigns a positive probability to *Head*. Similarly, if $q > \frac{1}{2}$ then her expected payoff to *Head* exceeds her expected payoff to *Tail*, and hence exceeds her expected payoff to every mixed strategy that assigns a positive probability to *Tail*. If $q = \frac{1}{2}$ then both *Head* and *Tail*, and hence all her mixed strategies, yield the same expected payoff. We conclude that player 1's best responses to player 2's strategy are her mixed strategy that assigns probability 0 to *Head* if $q < \frac{1}{2}$, her mixed strategy that assigns probability 1 to *Head* if $q > \frac{1}{2}$, and all her mixed strategies if $q = \frac{1}{2}$. That is, denoting by $B_1(q)$ the set of probabilities player 1 assigns to *Head* in best responses to q, we have

$$B_1(q) = \begin{cases} \{0\} & \text{if } q < \frac{1}{2} \\ \{p: 0 \leq p \leq 1\} & \text{if } q = \frac{1}{2} \\ \{1\} & \text{if } q > \frac{1}{2}. \end{cases}$$

The best response function of player 2 is similar: $B_2(p) = \{1\}$ if $p < \frac{1}{2}$, $B_2(p) = \{q: 0 \leq q \leq 1\}$ if $p = \frac{1}{2}$, and $B_2(p) = \{0\}$ if $p > \frac{1}{2}$. Both best response functions are illustrated in Figure 110.1.

The set of mixed strategy Nash equilibria of the game corresponds (as before) to the set of intersections of the best response functions in this figure; we see that there is one intersection, corresponding to the equilibrium we found previously, in which each player assigns probability $\frac{1}{2}$ to *Head*.

Matching Pennies has no Nash equilibrium if the players are not allowed to randomize. If a game has a Nash equilibrium when randomization is not allowed, is it possible that it has additional equilibria when randomization is allowed? The following example shows that the answer is positive.

4.3.6 Example: BoS

Consider the two-player game with vNM preferences in which the players' preferences over deterministic action profiles are the same as in *BoS* and their prefer-

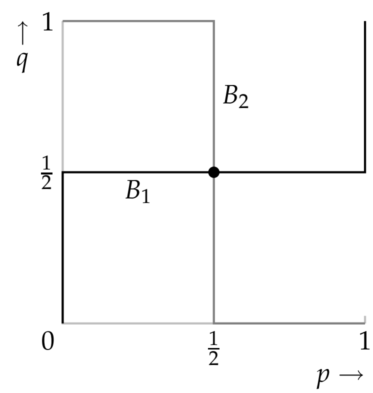


Figure 110.1 The players' best response functions in *Matching Pennies* (Figure 109.1) when randomization is allowed. The probabilities assigned by players 1 and 2 to *Head* are p and q respectively. The best response function of player 1 is black and that of player 2 is gray. The disk indicates the unique Nash equilibrium.

ences over lotteries are represented by the expected value of the payoff functions specified in Figure 110.2. What are the mixed strategy equilibria of this game?

	B	S
B	2, 1	0, 0
S	0, 0	1, 2

Figure 110.2 A version of the game *Bach or Stravinsky?* with vNM preferences.

First construct player 1's best response function. Suppose that player 2 assigns probability q to B. Then player 1's expected payoff to B is $2 \cdot q + 0 \cdot (1-q) = 2q$ and her expected payoff to S is $0 \cdot q + 1 \cdot (1-q) = 1-q$. Thus if $2q > 1-q$, or $q > \frac{1}{3}$, then her unique best response is B, while if $q < \frac{1}{3}$ then her unique best response is S. If $q = \frac{1}{3}$ then both B and S, and hence all player 1's mixed strategies, yield the same expected payoffs, so that every mixed strategy is a best response. In summary, player 1's best response function is

$$B_1(q) = \begin{cases} \{0\} & \text{if } q < \frac{1}{3} \\ \{p : 0 \le p \le 1\} & \text{if } q = \frac{1}{3} \\ \{1\} & \text{if } q > \frac{1}{3}. \end{cases}$$

Similarly we can find player 2's best response function. The best response functions of both players are shown in Figure 111.1.

We see that the game has three mixed strategy Nash equilibria, in which $(p, q) = (0, 0)$, $(\frac{2}{3}, \frac{1}{3})$, and $(1, 1)$. The first and third equilibria correspond to the Nash equilibria of the ordinal version of the game when the players were not allowed to randomize (Section 2.7.2). The second equilibrium is new. In this equilibrium each player chooses both B and S with positive probability (so that each of the four outcomes (B, B), (B, S), (S, B), and (S, S) occurs with positive probability).

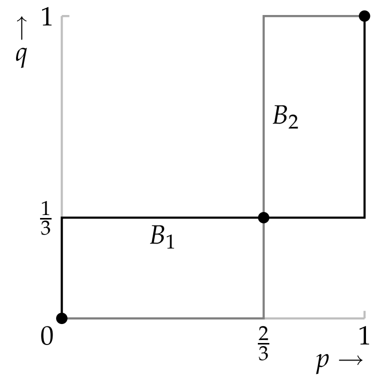


Figure 111.1 The players' best response functions in *BoS* (Figure 110.2) when randomization is allowed. The probabilities assigned by players 1 and 2 to B are p and q respectively. The best response function of player 1 is black and that of player 2 is gray. The disks indicate the Nash equilibria (two pure, one mixed).

? EXERCISE 111.1 (Mixed strategy equilibria of *Hawk–Dove*) Consider the two-player game with vNM preferences in which the players' preferences over deterministic action profiles are the same as in *Hawk–Dove* (Exercise 29.1) and their preferences over lotteries satisfy the following two conditions. Each player is indifferent between the outcome $(Passive, Passive)$ and the lottery that assigns probability $\frac{1}{2}$ to $(Aggressive, Aggressive)$ and probability $\frac{1}{2}$ to the outcome in which she is aggressive and the other player is passive, and between the outcome in which she is passive and the other player is aggressive and the lottery that assigns probability $\frac{2}{3}$ to the outcome $(Aggressive, Aggressive)$ and probability $\frac{1}{3}$ to the outcome $(Passive, Passive)$. Find payoffs whose expected values represent these preferences (take each player's payoff to $(Aggressive, Aggressive)$ to be 0 and each player's payoff to the outcome in which she is passive and the other player is aggressive to be 1). Find the mixed strategy Nash equilibrium of the resulting strategic game.

Both *Matching Pennies* and *BoS* have finitely many mixed strategy Nash equilibria: the players' best response functions intersect at a finite number of points (one for *Matching Pennies*, three for *BoS*). One of the games in the next exercise has a continuum of mixed strategy Nash equilibria because segments of the players' best response functions coincide.

? EXERCISE 111.2 (Games with mixed strategy equilibria) Find all the mixed strategy Nash equilibria of the strategic games in Figure 111.2.

	L	R
T	$6,0$	$0,6$
B	$3,2$	$6,0$

	L	R
T	$0,1$	$0,2$
B	$2,2$	$0,1$

Figure 111.2 Two strategic games with vNM preferences.

? EXERCISE 112.1 (A coordination game) Two people can perform a task if, and only if, they both exert effort. They are both better off if they both exert effort and perform the task than if neither exerts effort (and nothing is accomplished); the worst outcome for each person is that she exerts effort and the other does not (in which case again nothing is accomplished). Specifically, the players' preferences are represented by the expected value of the payoff functions in Figure 112.1, where c is a positive number less than 1 that can be interpreted as the cost of exerting effort. Find all the mixed strategy Nash equilibria of this game. How do the equilibria change as c increases? Explain the reasons for the changes.

	No effort	*Effort*
No effort	$0, 0$	$0, -c$
Effort	$-c, 0$	$1-c, 1-c$

Figure 112.1 The coordination game in Exercise 112.1.

?? EXERCISE 112.2 (Swimming with sharks) You and a friend are spending two days at the beach and would like to go for a swim. Each of you believes that with probability π the water is infested with sharks. If sharks are present, anyone who goes swimming today will surely be attacked. You each have preferences represented by the expected value of a payoff function that assigns $-c$ to being attacked by a shark, 0 to sitting on the beach, and 1 to a day's worth of undisturbed swimming. If one of you is attacked by sharks on the first day then you both deduce that a swimmer will surely be attacked the next day, and hence do not go swimming the next day. If no one is attacked on the first day then you both retain the belief that the probability of the water's being infested is π, and hence swim on the second day only if $-\pi c + 1 - \pi \geq 0$. Model this situation as a strategic game in which you and your friend each decides whether to go swimming on your first day at the beach. If, for example, you go swimming on the first day, you (and your friend, if she goes swimming) are attacked with probability π, in which case you stay out of the water on the second day; you (and your friend, if she goes swimming) swim undisturbed with probability $1 - \pi$, in which case you swim on the second day. Thus your expected payoff if you swim on the first day is $\pi(-c + 0) + (1 - \pi)(1 + 1) = -\pi c + 2(1 - \pi)$, independent of your friend's action. Find the mixed strategy Nash equilibria of the game (depending on c and π). Does the existence of a friend make it more or less likely that you decide to go swimming on the first day? (Penguins diving into water where seals may lurk are sometimes said to face the same dilemma, though Court (1996) argues that they do not.)

4.3.7 A useful characterization of mixed strategy Nash equilibrium

The method we have used so far to study the set of mixed strategy Nash equilibria of a game involves constructing the players' best response functions. Other methods are sometimes useful. I now present a characterization of mixed strategy Nash equilibrium that gives us an easy way to check whether a mixed strategy profile is an equilibrium, and is the basis of a procedure (described in Section 4.10) for finding all equilibria of a game.

The key point is an observation made in Section 4.3.4 for two-player two-action games: a player's expected payoff to a mixed strategy profile is a weighted average of her expected payoffs to her pure strategies, where the weight attached to each pure strategy is the probability assigned to that strategy by the player's mixed strategy. This property holds for any game (with any number of players) in which each player has finitely many actions. We can state it more precisely as follows.

> A player's expected payoff to the mixed strategy profile α is a weighted average of her expected payoffs to all mixed strategy profiles of the type (a_i, α_{-i}), where the weight attached to (a_i, α_{-i}) is the probability $\alpha_i(a_i)$ assigned to a_i by player i's mixed strategy α_i. (113.1)

Symbolically we have

$$U_i(\alpha) = \sum_{a_i \in A_i} \alpha_i(a_i) U_i(a_i, \alpha_{-i}),$$

where A_i is player i's set of actions (pure strategies) and $U_i(a_i, \alpha_{-i})$ is her expected payoff when she uses the pure strategy that assigns probability 1 to a_i and every other player j uses her mixed strategy α_j. (See the end of Section 17.3 in the appendix on mathematics for an explanation of the $\sum$ notation.)

This property leads to a useful characterization of mixed strategy Nash equilibrium. Let α^* be a mixed strategy Nash equilibrium and denote by E_i^* player i's expected payoff in the equilibrium (i.e. $E_i^* = U_i(\alpha^*)$). Because α^* is an equilibrium, player i's expected payoff, given α_{-i}^*, to each of her pure strategies is at most E_i^*. Now, by (113.1), E_i^* is a weighted average of player i's expected payoffs to the pure strategies to which α_i^* assigns positive probability. Thus player i's expected payoffs to these pure strategies are all equal to E_i^*. (If any were smaller then the weighted average would be smaller.) We conclude that the expected payoff to each action to which α_i^* assigns positive probability is E_i^* and the expected payoff to every other action is at most E_i^*. Conversely, if these conditions are satisfied for every player i then α^* is a mixed strategy Nash equilibrium: the expected payoff to α_i^* is E_i^*, and the expected payoff to any other mixed strategy is at most E_i^*, because by (113.1) it is a weighted average of E_i^* and numbers that are at most E_i^*.

This argument establishes the following result.

■ PROPOSITION 113.2 (Characterization of mixed strategy Nash equilibrium of finite game) *A mixed strategy profile α^* in a strategic game with vNM preferences in which each player has finitely many actions is a mixed strategy Nash equilibrium if and only if, for each player i,*

- *the expected payoff, given α_{-i}^*, to every action to which α_i^* assigns positive probability is the same*

- *the expected payoff, given α^*_{-i}, to every action to which α^*_i assigns zero probability is at most the expected payoff to any action to which α^*_i assigns positive probability.*

Each player's expected payoff in an equilibrium is her expected payoff to any of her actions that she uses with positive probability.

The significance of this result is that it gives conditions for a mixed strategy Nash equilibrium in terms of each player's expected payoffs only to her *pure* strategies. For games in which each player has finitely many actions, it allows us easily to check whether a mixed strategy profile is an equilibrium. For example, in *BoS* (Section 4.3.6) the strategy pair $((\frac{2}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3}))$ is a mixed strategy Nash equilibrium because given player 2's strategy $(\frac{1}{3}, \frac{2}{3})$, player 1's expected payoffs to B and S are both equal to $\frac{2}{3}$, and given player 1's strategy $(\frac{2}{3}, \frac{1}{3})$, player 2's expected payoffs to B and S are both equal to $\frac{2}{3}$.

The next example is slightly more complicated.

◆ EXAMPLE 114.1 (Checking whether a mixed strategy profile is a mixed strategy Nash equilibrium) I claim that for the game in Figure 114.1 (in which the dots indicate irrelevant payoffs), the indicated pair of strategies, $(\frac{3}{4}, 0, \frac{1}{4})$ for player 1 and $(0, \frac{1}{3}, \frac{2}{3})$ for player 2, is a mixed strategy Nash equilibrium. To verify this claim, it suffices, by Proposition 113.2, to study each player's expected payoffs to her three pure strategies. For player 1 these payoffs are

T: $\frac{1}{3} \cdot 3 + \frac{2}{3} \cdot 1 = \frac{5}{3}$
M: $\frac{1}{3} \cdot 0 + \frac{2}{3} \cdot 2 = \frac{4}{3}$
B: $\frac{1}{3} \cdot 5 + \frac{2}{3} \cdot 0 = \frac{5}{3}$.

Player 1's mixed strategy assigns positive probability to T and B and probability zero to M, so the two conditions in Proposition 113.2 are satisfied for player 1. The expected payoff to each of player 2's pure strategies is $\frac{5}{2}$ ($\frac{3}{4} \cdot 2 + \frac{1}{4} \cdot 4 = \frac{3}{4} \cdot 3 + \frac{1}{4} \cdot 1 = \frac{3}{4} \cdot 1 + \frac{1}{4} \cdot 7 = \frac{5}{2}$), so the two conditions in Proposition 113.2 are satisfied also for her.

	L (0)	C ($\frac{1}{3}$)	R ($\frac{2}{3}$)
T ($\frac{3}{4}$)	$\cdot, 2$	$3, 3$	$1, 1$
M (0)	$\cdot, \cdot$	$0, \cdot$	$2, \cdot$
B ($\frac{1}{4}$)	$\cdot, 4$	$5, 1$	$0, 7$

Figure 114.1 A partially-specified strategic game, illustrating a method of checking whether a mixed strategy profile is a mixed strategy Nash equilibrium. The dots indicate irrelevant payoffs.

Note that the expected payoff to player 2's action L, which she uses with probability zero, is the *same* as the expected payoff to her other two actions. This equality is consistent with Proposition 113.2, the second part of which requires only that the expected payoffs to actions used with probability zero be *no greater than* the expected payoffs to actions used with positive probability (not that they necessarily be less). Note also that the fact that player 2's expected payoff to L is the same as her expected payoffs to C and R does *not* imply that the game has a mixed strategy Nash equilibrium in which player 2 uses L with positive probability—it may, or it may not, depending on the unspecified payoffs.

? EXERCISE 115.1 (Choosing numbers) Players 1 and 2 each choose a positive integer up to K. If the players choose the same number then player 2 pays \$1 to player 1; otherwise no payment is made. Each player's preferences are represented by her expected monetary payoff.

a. Show that the game has a mixed strategy Nash equilibrium in which each player chooses each positive integer up to K with probability $1/K$.

b. (More difficult.) Show that the game has no other mixed strategy Nash equilibria. (Deduce from the fact that player 1 assigns positive probability to some action k that player 2 must do so; then look at the implied restriction on player 1's equilibrium strategy.)

? EXERCISE 115.2 (Silverman's game) Each of two players chooses a positive integer. If player i's integer is greater than player j's integer and less than three times this integer then player j pays \$1 to player i. If player i's integer is at least three times player j's integer then player i pays \$1 to player j. If the integers are equal, no payment is made. Each player's preferences are represented by her expected monetary payoff. Show that the game has no Nash equilibrium in pure strategies, and that the pair of mixed strategies in which each player chooses 1, 2, and 5 each with probability $\frac{1}{3}$ is a mixed strategy Nash equilibrium. (In fact, this pair of mixed strategies is the unique mixed strategy Nash equilibrium.)

?? EXERCISE 115.3 (Voter participation) Consider the game of voter participation in Exercise 32.2. Assume that $k \leq m$ and that each player's preferences are represented by the expectation of her payoffs given in Exercise 32.2. Show that there is a value of p between 0 and 1 such that the game has a mixed strategy Nash equilibrium in which every supporter of candidate A votes with probability p, k supporters of candidate B vote with certainty, and the remaining $m - k$ supporters of candidate B abstain. How do the probability p that a supporter of candidate A votes and the expected number of voters ("turnout") depend upon c? (Note that if every supporter of candidate A votes with probability p then the probability that exactly $k - 1$ of them vote is $kp^{k-1}(1 - p)$.)

?? EXERCISE 115.4 (Defending territory) General A is defending territory accessible by two mountain passes against an attack by general B. General A has three divisions at her disposal, and general B has two divisions. Each general allocates her divisions between the two passes. General A wins the battle at a pass if and only if she assigns at least as many divisions to the pass as does general B; she successfully defends her territory if and only if she wins the battle at both passes. Formulate this situation as a strategic game and find all its mixed strategy equilibria. (First argue that in every equilibrium B assigns probability zero to the action of allocating one division to each pass. Then argue that in any equilibrium she assigns probability $\frac{1}{2}$ to each of her other actions. Finally, find A's equilibrium strategies.) In an equilibrium do the generals concentrate all their forces at one pass, or spread them out?

An implication of Proposition 113.2 is that a nondegenerate mixed strategy equilibrium (a mixed strategy equilibrium that is not also a pure strategy equilibrium) is never a *strict* Nash equilibrium: every player whose mixed strategy assigns positive probability to more than one action is indifferent between her equilibrium mixed strategy and every action to which this mixed strategy assigns positive probability.

Any equilibrium that is not strict, whether in mixed strategies or not, has less appeal than a strict equilibrium because some (or all) of the players lack a positive incentive to choose their equilibrium strategies, given the other players' behavior. There is no reason for them *not* to choose their equilibrium strategies, but at the same time there is no reason for them not to choose another strategy that is equally good. Many pure strategy equilibria—especially in complex games—are also not strict, but among mixed strategy equilibria the problem is pervasive.

Given that in a mixed strategy equilibrium no player has a positive incentive to choose her equilibrium strategy, what determines how she randomizes in equilibrium? From the examples above we see that a player's equilibrium mixed strategy in a two-player game keeps the *other* player indifferent between a set of her actions, so that *she* is willing to randomize. In the mixed strategy equilibrium of *BoS*, for example, player 1 chooses B with probability $\frac{2}{3}$ so that player 2 is indifferent between B and S, and hence is willing to choose each with positive probability. Note, however, that the theory is *not* that the players consciously choose their strategies with this goal in mind! Rather, the conditions for equilibrium are designed to ensure that it is consistent with a steady state. In *BoS*, for example, if player 1 chooses B with probability $\frac{2}{3}$ and player 2 chooses B with probability $\frac{1}{3}$ then neither player has any reason to change her action. We have not yet studied how a steady state might come about, but have rather simply looked for strategy profiles consistent with steady states. In Section 4.9 I briefly discuss some theories of how a steady state might be reached.

4.3.8 Existence of equilibrium in finite games

Every game we have examined has at least one mixed strategy Nash equilibrium. In fact, every game in which each player has *finitely* many actions has at least one such equilibrium.

■ PROPOSITION 116.1 (Existence of mixed strategy Nash equilibrium in finite games) *Every strategic game with vNM preferences in which each player has finitely many actions has a mixed strategy Nash equilibrium.*

This result is of no help in *finding* equilibria. But it is a useful fact to know: your quest for an equilibrium of a game in which each player has finitely many actions in principle may succeed! Note that the finiteness of the number of actions of each player is only *sufficient* for the existence of an equilibrium, not *necessary*; many games in which the players have infinitely many actions possess mixed strategy Nash equilibria. Note also that a player's mixed strategy in a mixed strategy Nash equilibrium may assign probability 1 to a single action; if every player's strategy does so then the equilibrium corresponds to a ("pure strategy") equilibrium of the associated game with ordinal preferences. Relatively advanced mathematical tools are needed to prove the result; see, for example, Osborne and Rubinstein (1994, 19–20).

4.4 Dominated actions

In a strategic game with ordinal preferences, one action of a player strictly dominates another action if it is superior, no matter what the other players do (see Definition 43.1). In a game with vNM preferences in which players may randomize, we extend this definition to allow an action to be dominated by a *mixed strategy*.

▶ DEFINITION 117.1 (*Strict domination*) In a strategic game with vNM preferences, player i's mixed strategy α_i **strictly dominates** her action a_i' if

$$U_i(\alpha_i, a_{-i}) > u_i(a_i', a_{-i}) \text{ for every list } a_{-i} \text{ of the other players' actions,}$$

where u_i is a payoff function whose expected value represents player i's preferences over lotteries and $U_i(\alpha_i, a_{-i})$ is player i's expected payoff under u_i when she uses the mixed strategy α_i and the actions chosen by the other players are given by a_{-i}.

As before, if a mixed strategy strictly dominates an action, we say that the action is **strictly dominated**. Figure 117.1 (in which only player 1's payoffs are given) shows that an action that is not strictly dominated by any pure strategy (i.e. is not strictly dominated in the sense of Definition 43.1) may be strictly dominated by a mixed strategy. The action T of player 1 is not strictly (or weakly) dominated by either M or B, but it is strictly dominated by the mixed strategy that assigns probability $\frac{1}{2}$ to M and probability $\frac{1}{2}$ to B, because if player 2 chooses L then the mixed strategy yields player 1 the payoff of 2, whereas the action T yields her the payoff of 1, and if player 2 chooses R then the mixed strategy yields player 1 the payoff of $\frac{3}{2}$, whereas the action T yields her the payoff of 1.

	L	R
T	1	1
M	4	0
B	0	3

Figure 117.1 Player 1's payoffs in a strategic game with vNM preferences. The action T of player 1 is strictly dominated by the mixed strategy that assigns probability $\frac{1}{2}$ to M and probability $\frac{1}{2}$ to B.

? EXERCISE 118.1 (Strictly dominated actions) In Figure 117.1, the mixed strategy that assigns probability $\frac{1}{2}$ to M and probability $\frac{1}{2}$ to B is not the only mixed strategy that strictly dominates T. Find all the mixed strategies that do so.

In a Nash equilibrium of a strategic game with ordinal preferences no player uses a strictly dominated action (Section 2.9.1). I now argue that the same is true of a mixed strategy Nash equilibrium of a strategic game with vNM preferences. In fact, I argue that a strictly dominated action is not a best response to any collection of mixed strategies of the other players. Suppose that player i's action a_i' is strictly dominated by her mixed strategy α_i, and the other players' mixed strategies are given by α_{-i}. Player i's expected payoff $U_i(\alpha_i, \alpha_{-i})$ when she uses the mixed strategy α_i and the other players use the mixed strategies α_{-i} is a weighted average of her payoffs $U_i(\alpha_i, a_{-i})$ as a_{-i} varies over all the collections of actions for the other players, with the weight on each a_{-i} equal to the probability with which it occurs when the other players' mixed strategies are α_{-i}. Player i's expected payoff when she uses the action a_i' and the other players use the mixed strategies α_{-i} is a similar weighted average; the weights are the same, but the terms take the form $u_i(a_i', a_{-i})$ rather than $U_i(\alpha_i, a_{-i})$. The fact that a_i' is strictly dominated by α_i means that $U_i(\alpha_i, a_{-i}) > u_i(a_i', a_{-i})$ for *every* collection a_{-i} of the other players' actions. Hence player i's expected payoff when she uses the mixed strategy α_i exceeds her expected payoff when she uses the action a_i', given α_{-i}. Consequently,

> *a strictly dominated action is not used with positive probability in any mixed strategy equilibrium.*

Thus when looking for mixed strategy equilibria we can eliminate from consideration every strictly dominated action.

As before, we can define the notion of weak domination (see Definition 45.1).

▶ DEFINITION 118.2 (*Weak domination*) In a strategic game with vNM preferences, player i's mixed strategy α_i **weakly dominates** her action a_i' if

$$U_i(\alpha_i, a_{-i}) \geq u_i(a_i', a_{-i}) \text{ for every list } a_{-i} \text{ of the other players' actions}$$

and

$$U_i(\alpha_i, a_{-i}) > u_i(a_i', a_{-i}) \text{ for some list } a_{-i} \text{ of the other players' actions,}$$

where u_i is a payoff function whose expected value represents player i's preferences over lotteries and $U_i(\alpha_i, a_{-i})$ is player i's expected payoff under u_i when she uses the mixed strategy α_i and the actions chosen by the other players are given by a_{-i}.

We saw that a weakly dominated action may be used in a Nash equilibrium (see Figure 46.1). Thus a weakly dominated action may be used with positive probability in a mixed strategy equilibrium, so that we *cannot* eliminate *weakly* dominated actions from consideration when finding mixed strategy equilibria!

? EXERCISE 119.1 (Eliminating dominated actions when finding equilibria) Find all the mixed strategy Nash equilibria of the game in Figure 119.1 by first eliminating any strictly dominated actions and then constructing the players' best response functions.

	L	M	R
T	$2,2$	$0,3$	$1,2$
B	$3,1$	$1,0$	$0,2$

Figure 119.1 The strategic game with vNM preferences in Exercise 119.1.

The fact that a player's strategy in a mixed strategy Nash equilibrium may be weakly dominated raises the question of whether a game necessarily has a mixed strategy Nash equilibrium in which no player's strategy is weakly dominated. The following result (which is not easy to prove) shows that the answer is affirmative for a finite game.

■ PROPOSITION 119.2 (Existence of mixed strategy Nash equilibrium with no weakly dominated strategies in finite games) *Every strategic game with vNM preferences in which each player has finitely many actions has a mixed strategy Nash equilibrium in which no player's strategy is weakly dominated.*

4.5 Pure equilibria when randomization is allowed

The analysis in Section 4.3.6 shows that the mixed strategy Nash equilibria of *BoS* in which each player's strategy is pure correspond precisely to the Nash equilibria of the version of the game (considered in Section 2.3) in which the players are not allowed to randomize. The same is true for a general game: equilibria when the players are not allowed to randomize remain equilibria when they are allowed to randomize, and any pure equilibria that exist when they are allowed to randomize are equilibria when they are not allowed to randomize.

To establish this claim, let N be a set of players and let A_i, for each player i, be a set of actions. Consider the following two games.

G: the strategic game with ordinal preferences in which the set of players is N, the set of actions of each player i is A_i, and the preferences of each player i are represented by the payoff function u_i

G': the strategic game with vNM preferences in which the set of players is N, the set of actions of each player i is A_i, and the preferences of each player i are represented by the expected value of u_i.

First I argue that any Nash equilibrium of G corresponds to a mixed strategy Nash equilibrium (in which each player's strategy is pure) of G'. Let a^* be a Nash equilibrium of G, and for each player i let α_i^* be the mixed strategy that assigns probability 1 to a_i^*. Since a^* is a Nash equilibrium of G we know that in G' no player i has an action that yields her a payoff higher than does a_i^* when all the other players adhere to α_{-i}^*. Thus α^* satisfies the two conditions in Proposition 113.2, so that it is a mixed strategy equilibrium of G', establishing the following result.

■ PROPOSITION 120.1 (Pure strategy equilibria survive when randomization is allowed) *Let a^* be a Nash equilibrium of G and for each player i let α_i^* be the mixed strategy of player i that assigns probability one to the action a_i^*. Then α^* is a mixed strategy Nash equilibrium of G'.*

Next I argue that any mixed strategy Nash equilibrium of G' in which each player's strategy is pure corresponds to a Nash equilibrium of G. Let α^* be a mixed strategy Nash equilibrium of G' in which every player's mixed strategy is pure; for each player i, denote by a_i^* the action to which α_i assigns probability one. Then no mixed strategy of player i yields her a payoff higher than does α_i^* when the other players' mixed strategies are given by α_{-i}^*. Hence, in particular, no *pure* strategy of player i yields her a payoff higher than does α_i^*. Thus a^* is a Nash equilibrium of G. In words, if a pure strategy is optimal for a player when she is allowed to randomize then it remains optimal when she is prohibited from randomizing. (More generally, prohibiting a decision-maker from taking an action that is not optimal does not change the set of actions that are optimal.)

■ PROPOSITION 120.2 (Pure strategy equilibria survive when randomization is prohibited) *Let α^* be a mixed strategy Nash equilibrium of G' in which the mixed strategy of each player i assigns probability one to the single action a_i^*. Then a^* is a Nash equilibrium of G.*

4.6 Illustration: expert diagnosis

I seem to confront the following predicament all too frequently. Something about which I am relatively ill-informed (my car, my computer, my body) stops working properly. I consult an expert, who makes a diagnosis and recommends an action. I am not sure if the diagnosis is correct—the expert, after all, has an interest in selling her services. I have to decide whether to follow the expert's advice or to try to fix the problem myself, put up with it, or consult another expert.

4.6.1 Model

A simple model that captures the main features of this situation starts with the assumption that there are two types of problem, *major* and *minor*. Denote the fraction of problems that are major by r, and assume that $0 < r < 1$. An expert knows, on seeing a problem, whether it is *major* or *minor*; a consumer knows only the probability r. (The diagnosis is costly neither to the expert nor to the consumer.) An expert may recommend either a major or a minor repair (regardless of the true nature of the problem), and a consumer may either accept the expert's recommendation or seek another remedy. A major repair fixes both a major problem and a minor one.

Assume that a consumer always accepts an expert's advice to obtain a minor repair—there is no reason for her to doubt such a diagnosis—but may either accept or reject advice to obtain a major repair. Further assume that an expert always recommends a major repair for a major problem—a minor repair does not fix a major problem, so there is no point in an expert's recommending one for a major problem—but may recommend either repair for a minor problem. Suppose that an expert obtains the same profit $\pi > 0$ (per unit of time) from selling a minor repair to a consumer with a minor problem as she does from selling a major repair to a consumer with a major problem, but obtains the profit $\pi' > \pi$ from selling a major repair to a consumer with a minor problem. (The rationale is that in the last case the expert does not in fact perform a major repair, at least not in its entirety.) A consumer pays an expert E for a major repair and $I < E$ for a minor one; the cost she effectively bears if she chooses some other remedy is $E' > E$ if her problem is major and $I' > I$ if it is minor. (Perhaps she consults other experts before proceeding, or works on the problem herself, in either case spending valuable time.) I assume throughout that $E > I'$.

Under these assumptions we can model the situation as a strategic game in which the expert has two actions (recommend a minor repair for a minor problem; recommend a major repair for a minor problem), and the consumer has two actions (accept the recommendation of a major repair; reject the recommendation of a major repair). I name the actions as follows.

Expert *Honest* (recommend a minor repair for a minor problem and a major repair for a major problem) and *Dishonest* (recommend a major repair for both types of problem).

Consumer *Accept* (buy whatever repair the expert recommends) and *Reject* (buy a minor repair but seek some other remedy if a major repair is recommended)

Assume that each player's preferences are represented by her expected monetary payoff. Then the players' payoffs to the four action pairs are as follows; the strategic game is given in Figure 122.1.

(H, A): With probability r the consumer's problem is major, so she pays E, and with probability $1 - r$ it is minor, so she pays I. Thus her expected payoff is $-rE - (1 - r)I$. The expert's profit is π.

(D, A): The consumer's payoff is $-E$. The consumer's problem is major with probability r, yielding the expert π, and minor with probability $1 - r$, yielding the expert π', so that the expert's expected payoff is $r\pi + (1 - r)\pi'$.

(H, R): The consumer's cost is E' if her problem is major (in which case she rejects the expert's advice to get a major repair) and I if her problem is minor, so that her expected payoff is $-rE' - (1-r)I$. The expert obtains a payoff only if the consumer's problem is minor, in which case she gets π; thus her expected payoff is $(1-r)\pi$.

(D, R): The consumer never accepts the expert's advice, and thus obtains the expected payoff $-rE' - (1-r)I'$. The expert does not get any business, and thus obtains the payoff of 0.

		Consumer	
		Accept (q)	*Reject* $(1-q)$
Expert	*Honest* (p)	$\pi, -rE - (1-r)I$	$(1-r)\pi, -rE' - (1-r)I$
	Dishonest $(1-p)$	$r\pi + (1-r)\pi', -E$	$0, -rE' - (1-r)I'$

Figure 122.1 A game between an expert and a consumer with a problem.

4.6.2 *Nash equilibrium*

To find the Nash equilibria of the game we can construct the best response functions, as before. Denote by p the probability the expert assigns to H and by q the probability the consumer assigns to A.

Expert's best response function If $q = 0$ (i.e. the consumer chooses R with probability one) then the expert's best response is $p = 1$ (since $(1-r)\pi > 0$). If $q = 1$ (i.e. the consumer chooses A with probability one) then the expert's best response is $p = 0$ (since $\pi' > \pi$, so that $r\pi + (1-r)\pi' > \pi$). For what value of q is the expert indifferent between H and D? Given q, the expert's expected payoff to H is $q\pi + (1-q)(1-r)\pi$ and her expected payoff to D is $q[r\pi + (1-r)\pi']$, so she is indifferent between the two actions if

$$q\pi + (1-q)(1-r)\pi = q[r\pi + (1-r)\pi'].$$

Upon simplification, this yields $q = \pi/\pi'$. We conclude that the expert's best response function takes the form shown in both panels of Figure 123.1.

Consumer's best response function If $p = 0$ (i.e. the expert chooses D with probability one) then the consumer's best response depends on the relative sizes of E and $rE' + (1-r)I'$. If $E < rE' + (1-r)I'$ then the consumer's best response is $q = 1$, whereas if $E > rE' + (1-r)I'$ then her best response is $q = 0$; if $E = rE' + (1-r)I'$ then she is indifferent between R and A.

If $p = 1$ (i.e. the expert chooses H with probability one) then the consumer's best response is $q = 1$ (given $E < E'$).

We conclude that if $E < rE' + (1-r)I'$ then the consumer's best response to every value of p is $q = 1$, as shown in the left panel of Figure 123.1. If $E > rE' + (1-r)I'$ then the consumer is indifferent between A and R if

$$p[rE + (1-r)I] + (1-p)E = p[rE' + (1-r)I] + (1-p)[rE' + (1-r)I'],$$

which reduces to

$$p = \frac{E - [rE' + (1-r)I']}{(1-r)(E-I')}.$$

In this case the consumer's best response function takes the form shown in the right panel of Figure 123.1.

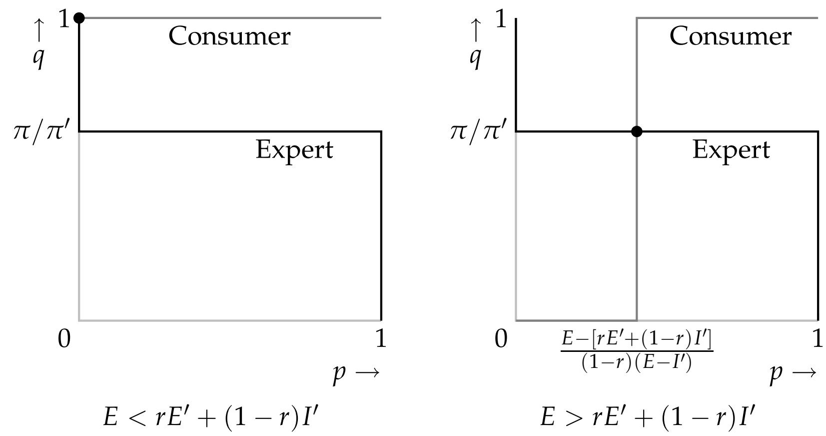


Figure 123.1 The players' best response functions in the game of expert diagnosis. The probability assigned by the expert to H is p and the probability assigned by the consumer to A is q.

Equilibrium Given the best response functions, if $E < rE' + (1-r)I'$ then the pair of pure strategies (D, A) is the unique Nash equilibrium. The condition $E < rE' + (1-r)I'$ says that the cost of a major repair by an expert is less than the *expected* cost of an alternative remedy; the only equilibrium yields the dismal outcome for the consumer in which the expert is always dishonest and the consumer always accepts her advice.

If $E > rE' + (1-r)I'$ then the unique equilibrium of the game is in mixed strategies, with $(p, q) = (p^*, q^*)$, where

$$p^* = \frac{E - [rE' + (1-r)I']}{(1-r)(E-I')} \quad \text{and} \quad q^* = \frac{\pi}{\pi'}.$$

In this equilibrium the expert is sometimes honest, sometimes dishonest, and the consumer sometimes accepts her advice to obtain a major repair, and sometimes ignores such advice.

As discussed in the introduction to the chapter, a mixed strategy equilibrium can be given more than one interpretation as a steady state. In the game we are studying, and the games studied earlier in the chapter, I have focused on the interpretation in which each player chooses her action randomly, with probabilities given by her equilibrium mixed strategy, every time she plays the game. In the game of expert diagnosis a different interpretation fits well: among the population of individuals who may play the role of each given player, every individual chooses the same action whenever she plays the game, but different individuals choose different actions; the fraction of individuals who choose each action is equal to the equilibrium probability that that action is used in a mixed strategy equilibrium. Specifically, if $E > rE' + (1-r)I'$ then the fraction p^* of experts is honest (recommending minor repairs for minor problems) and the fraction $1 - p^*$ is dishonest (recommending major repairs for minor problems), while the fraction q^* of consumers is credulous (accepting any recommendation) and the fraction $1 - q^*$ is wary (accepting only a recommendation of a minor repair). Honest and dishonest experts obtain the same expected payoff, as do credulous and wary consumers.

? EXERCISE 124.1 (Equilibrium in the expert diagnosis game) Find the set of mixed strategy Nash equilibria of the game when $E = rE' + (1-r)I'$.

4.6.3 Properties of the mixed strategy Nash equilibrium

Studying how the equilibrium is affected by changes in the parameters of the model helps us understand the nature of the strategic interaction between the players. I consider the effects of three changes.

Suppose that major problems become less common (cars become more reliable, more resources are devoted to preventive healthcare). If we rearrange the expression for p^* to

$$p^* = 1 - \frac{r(E' - E)}{(1-r)(E - I')},$$

we see that p^* increases as r decreases (the numerator of the fraction decreases and the denominator increases). Thus in a mixed strategy equilibrium, the experts are more honest when major problems are less common. Intuitively, if a major problem is less likely then a consumer has less to lose from ignoring an expert's advice, so that the probability of an expert's being honest has to rise in order that her advice be heeded. The value of q^* is not affected by the change in r: the probability of a consumer's accepting an expert's advice remains the same when major problems become less common. *Given* the expert's behavior, a decrease in r increases the consumer's payoff to rejecting the expert's advice more than it increases her payoff to accepting this advice, so that she prefers to reject the advice. But this partial analysis is misleading: in the equilibrium that exists after r decreases, the consumer is exactly as likely to accept the expert's advice as she was before the change.

Now suppose that major repairs become less expensive relative to minor ones (technological advances reduce the cost of complex equipment). We see that p^* decreases as E decreases (with E' and I' constant): when major repairs are less costly, experts are less honest. As major repairs become less costly, a consumer has more potentially to lose from ignoring an expert's advice, so that she heeds the advice even if experts are less likely to be honest.

Finally, suppose that the profit π' from an expert's fixing a minor problem with an alleged major repair falls (the government requires experts to return replaced parts to the consumer, making it more difficult for an expert to fraudulently claim to have performed a major repair). Then q^* increases—consumers become less wary. Experts have less to gain from acting dishonestly, so that consumers can be more confident of their advice.

? EXERCISE 125.1 (Incompetent experts) Consider a (realistic?) variant of the model, in which the experts are not entirely competent. Assume that each expert always correctly recognizes a major problem but correctly recognizes a minor problem with probability $s < 1$: with probability $1 - s$ she mistakenly thinks that a minor problem is major, and, if the consumer accepts her advice, performs a major repair and obtains the profit π. Maintain the assumption that each consumer believes (correctly) that the probability her problem is major is r. As before, a consumer who does not give the job of fixing her problem to an expert bears the cost E' if it is major and I' if it is minor.

Suppose, for example, that an expert is honest and a consumer rejects advice to obtain a major repair. With probability r the consumer's problem is major, so that the expert recommends a major repair, which the consumer rejects; the consumer bears the cost E'. With probability $1 - r$ the consumer's problem is minor. In this case with probability s the expert correctly diagnoses it as minor, and the consumer accepts her advice and pays I; with probability $1 - s$ the expert diagnoses it as major, and the consumer rejects her advice and bears the cost I'. Thus the consumer's expected payoff in this case is $-rE' - (1 - r)[sI + (1 - s)I']$.

Construct the payoffs for every pair of actions and find the mixed strategy equilibrium in the case $E > rE' + (1 - r)I'$. Does incompetence breed dishonesty? More wary consumers?

? EXERCISE 125.2 (Choosing a seller) Each of two sellers has available one indivisible unit of a good. Seller 1 posts the price p_1 and seller 2 posts the price p_2. Each of two buyers would like to obtain one unit of the good; they simultaneously decide which seller to approach. If both buyers approach the same seller, each trades with probability $\frac{1}{2}$; the disappointed buyer does not subsequently have the option to trade with the other seller. (This assumption models the risk faced by a buyer that a good is sold out when she patronizes a seller with a low price.) Each buyer's preferences are represented by the expected value of a payoff function that assigns the payoff 0 to not trading and the payoff $1 - p$ to purchasing one unit of the good at the price p. (Neither buyer values more than one unit.) For any pair (p_1, p_2) of prices with $0 \leq p_i \leq 1$ for $i = 1, 2$, find the Nash equilibria (in pure and in mixed strategies) of the strategic game that models this situation. (There are three main cases: $p_2 < 2p_1 - 1$, $2p_1 - 1 < p_2 < \frac{1}{2}(1 + p_1)$, and $p_2 > \frac{1}{2}(1 + p_1)$.)

4.7 Equilibrium in a single population

In Section 2.10 I discussed deterministic steady states in situations in which the members of a single population interact. I now discuss stochastic steady states in such situations.

First extend the definitions of a symmetric strategic game and a symmetric Nash equilibrium (Definitions 49.3 and 50.2) to a game with vNM preferences. Recall that a two-player strategic game with ordinal preferences is symmetric if each player has the same set of actions and each player's evaluation of an outcome depends only on her action and that of her opponent, not on whether she is player 1 or player 2. A symmetric game with vNM preferences satisfies the same conditions; its definition differs from Definition 49.3 only because a player's evaluation of an outcome is given by her expected payoff rather than her ordinal preferences.

▶ DEFINITION 126.1 (*Symmetric two-player strategic game with vNM preferences*) A two-player strategic game with vNM preferences is **symmetric** if the players' sets of actions are the same and the players' preferences are represented by the expected values of payoff functions u_1 and u_2 for which $u_1(a_1, a_2) = u_2(a_2, a_1)$ for every action pair (a_1, a_2).

A Nash equilibrium of a strategic game with ordinal preferences in which every player's set of actions is the same is symmetric if all players take the same action. This notion of equilibrium extends naturally to strategic games with vNM preferences. (As before, it does not depend on the game's having only two players, so I define it for a game with any number of players.)

▶ DEFINITION 126.2 (*Symmetric mixed strategy Nash equilibrium*) A profile α^* of mixed strategies in a strategic game with vNM preferences in which each player has the same set of actions is a **symmetric mixed strategy Nash equilibrium** if it is a mixed strategy Nash equilibrium and α_i^* is the same for every player i.

Now consider again the game of approaching pedestrians (Figure 51.1, reproduced in Figure 126.1), interpreting the payoff numbers as Bernoulli payoffs whose expected values represent the players' preferences over lotteries. We found that this game has two deterministic steady states, corresponding to the two symmetric Nash equilibria in pure strategies, $(Left, Left)$ and $(Right, Right)$. The game also has a symmetric mixed strategy Nash equilibrium, in which each player assigns probability $\frac{1}{2}$ to *Left* and probability $\frac{1}{2}$ to *Right*. This equilibrium corresponds to a steady state in which half of all encounters result in collisions! (With probability $\frac{1}{4}$ player 1 chooses *Left* and player 2 chooses *Right*, and with probability $\frac{1}{4}$ player 1 chooses *Right* and player 2 chooses *Left*.)

	Left	*Right*
Left	1, 1	0, 0
Right	0, 0	1, 1

Figure 126.1 Approaching pedestrians.

In this example not only is the game symmetric, but the players' interests coincide. The game in Figure 127.1 is symmetric, but the players prefer to take different actions rather than the same actions. This game has no pure symmetric equilibrium, but has a symmetric mixed strategy equilibrium, in which each player chooses each action with probability $\frac{1}{2}$.

	X	Y
X	$0, 0$	$1, 1$
Y	$1, 1$	$0, 0$

Figure 127.1 A symmetric game.

These two examples show that a symmetric game may have no symmetric *pure* strategy equilibrium. But both games have a symmetric mixed strategy Nash equilibrium, as does any symmetric game in which each player has finitely many actions, by the following result. (Relatively advanced mathematical tools are needed to prove the result.)

■ PROPOSITION 127.1 (Existence of symmetric mixed strategy Nash equilibrium in symmetric finite games) *Every strategic game with vNM preferences in which each player has the same finite set of actions has a symmetric mixed strategy Nash equilibrium.*

? EXERCISE 127.2 (Approaching cars) Members of a single population of car drivers are randomly matched in pairs when they simultaneously approach intersections from different directions. In each interaction, each driver can either stop or continue. The drivers' preferences are represented by the expected value of the payoff functions given in Figure 127.2; the parameter ϵ, with $0 < \epsilon < 1$, reflects the fact that each driver dislikes being the only one to stop. Find the symmetric Nash equilibrium (equilibria?) of the game (find both the equilibrium strategies and the equilibrium payoffs).

	Stop	*Continue*
Stop	$1, 1$	$1 - \epsilon, 2$
Continue	$2, 1 - \epsilon$	$0, 0$

Figure 127.2 The game in Exercise 127.2.

Now suppose that drivers are (re)educated to feel guilty about choosing *Continue*, with the consequence that their payoffs when choosing *Continue* fall by $\delta > 0$. That is, the entry $(2, 1 - \epsilon)$ in Figure 127.2 is replaced by $(2 - \delta, 1 - \epsilon)$, the entry $(1 - \epsilon, 2)$ is replaced by $(1 - \epsilon, 2 - \delta)$, and the entry $(0, 0)$ is replaced by $(-\delta, -\delta)$. Show that all drivers are *better off* in the symmetric equilibrium of this game than they are in the symmetric equilibrium of the original game. Why is the society better off if everyone feels guilty about being aggressive? (The equilibrium of this game, like that of the equilibrium of the game of expert diagnosis in Section 4.6, may attractively be interpreted as representing a steady state in which some members of the population always choose one action, and other members always choose the other action.)

? EXERCISE 128.1 (Bargaining) Pairs of players from a single population bargain over the division of a pie of size 10. The members of a pair simultaneously make demands; the possible demands are the nonnegative *even* integers up to 10. If the demands sum to 10 then each player receives her demand; if the demands sum to less than 10 then each player receives her demand plus half of the pie that remains after both demands have been satisfied; if the demands sum to more than 10 then neither player receives any payoff. Find all the symmetric mixed strategy Nash equilibria in which each player assigns positive probability to at most two demands. (Many situations in which each player assigns positive probability to two actions, say a' and a'', can be ruled out as equilibria because when one player uses such a strategy, some action a''' yields the other player a payoff higher than does a' and/or a''.)

4.8 Illustration: reporting a crime

A crime is observed by a group of n people. Each person would like the police to be informed, but prefers that someone else make the phone call. Specifically, suppose that each person attaches the value v to the police being informed and bears the cost c if she makes the phone call, where $v > c > 0$. Then the situation is modeled by the following strategic game with vNM preferences.

Players The n people.

Actions Each player's set of actions is $\{Call, Don't\ call\}$.

Preferences Each player's preferences are represented by the expected value of a payoff function that assigns 0 to the profile in which no one calls, $v - c$ to any profile in which she calls, and v to any profile in which at least one person calls, but she does not.

This game is a variant of the one in Exercise 31.1, with $k = 1$. It has n pure Nash equilibria, in each of which exactly one person calls. (If that person switches to not calling, her payoff falls from $v - c$ to 0; if any other person switches to calling, her payoff falls from v to $v - c$.) If the members of the group differ in some respect, then these asymmetric equilibria may be compelling as steady states. For example, the social norm in which the oldest person in the group makes the phone call is stable.

If the members of the group either do not differ significantly or are not aware of any differences among themselves—if they are drawn from a single homogeneous population—then there is no way for them to coordinate, and a symmetric equilibrium, in which every player uses the same strategy, is more compelling.

The game has no symmetric pure Nash equilibrium. (If everyone calls, then any person is better off switching to not calling. If no one calls, then any person is better off switching to calling.)

However, it has a symmetric mixed strategy equilibrium in which each person calls with positive probability less than one. In any such equilibrium, each person's expected payoff to calling is equal to her expected payoff to not calling. Each person's payoff to calling is $v - c$, and her payoff to not calling is 0 if no one else calls and v if at least one other person calls, so the equilibrium condition is

$$v - c = 0 \cdot \Pr\{\text{no one else calls}\} + v \cdot \Pr\{\text{at least one other person calls}\},$$

or

$$v - c = v \cdot (1 - \Pr\{\text{no one else calls}\}),$$

or

$$c/v = \Pr\{\text{no one else calls}\}. \quad (129.1)$$

Denote by p the probability with which each person calls. The probability that no one else calls is the probability that every one of the other $n-1$ people does not call, namely $(1-p)^{n-1}$. Thus the equilibrium condition is $c/v = (1-p)^{n-1}$, or

$$p = 1 - (c/v)^{1/(n-1)}.$$

This number p is between 0 and 1, so we conclude that the game has a unique symmetric mixed strategy equilibrium, in which each person calls with probability $1 - (c/v)^{1/(n-1)}$. That is, there is a steady state in which whenever a person is in a group of n people facing the situation modeled by the game, she calls with probability $1 - (c/v)^{1/(n-1)}$.

How does this equilibrium change as the size of the group increases? We see that as n increases, the probability p that any given person calls decreases. (As n increases, $1/(n-1)$ decreases, so that $(c/v)^{1/(n-1)}$ increases.) What about the probability that *at least* one person calls? Fix any player i. Then the event "no one calls" is the same as the event "i does not call and no one *other than i* calls". Thus

$$\Pr\{\text{no one calls}\} = \Pr\{i \text{ does not call}\} \Pr\{\text{no one else calls}\}. \quad (129.2)$$

Now, the probability that any given person calls decreases as n increases, or equivalently the probability that she does not call increases as n increases. Further, from the equilibrium condition (129.1), $\Pr\{\text{no one else calls}\}$ is equal to c/v, *independent of n*. We conclude that the probability that no one calls *increases* as n increases. That is, the larger the group, the *less* likely the police are informed of the crime!

The condition defining a mixed strategy equilibrium is responsible for this result. For any given person to be indifferent between calling and not calling this condition requires that the probability that no one else calls be independent of the size of the group. Thus each person's probability of not calling is larger in a larger group, and hence, by the laws of probability reflected in (129.2), the probability that no one calls is larger in a larger group.

The result that the larger the group, the less likely any given person calls is not surprising. The result that the larger the group, the less likely at least one person calls is a more subtle implication of the notion of equilibrium. In a larger group no individual is any less concerned that the police should be called, but in a steady state the behavior of the group drives down the chance that the police are notified of the crime.

? EXERCISE 130.1 (Contributing to a public good) Consider an extension of the analysis above to the game in Exercise 31.1 for $k \geq 2$. (In this case a player may contribute even though the good is not provided; the player's payoff in this case is $-c$.) Denote by $Q_{n-1,m}(p)$ the probability that exactly m of a group of $n-1$ players contribute when each player contributes with probability p. What condition must be satisfied by $Q_{n-1,k-1}(p)$ in a symmetric mixed strategy equilibrium (in which each player contributes with the same probability)? (When does a player's contribution make a difference to the outcome?) For the case $v = 1$, $n = 4$, $k = 2$, and $c = \frac{3}{8}$ find the equilibria explicitly. (You need to use the fact that $Q_{3,1}(p) = 3p(1-p)^2$, and do a bit of algebra.)

REPORTING A CRIME: SOCIAL PSYCHOLOGY AND GAME THEORY

Thirty-eight people witnessed the brutal murder of Catherine ("Kitty") Genovese over a period of half an hour in New York City in March 1964. During this period, none of them significantly responded to her screams for help; none even called the police. Journalists, psychiatrists, sociologists, and others subsequently struggled to understand the witnesses' inaction. Some ascribed it to apathy engendered by life in a large city: "Indifference to one's neighbor and his troubles is a conditioned reflex of life in New York as it is in other big cities" (Rosenthal 1964, 81–82).

The event particularly interested social psychologists. It led them to try to understand the circumstances under which a bystander would help someone in trouble. Experiments quickly suggested that, contrary to the popular theory, people—even those living in large cities—are not in general apathetic to others' plights. An experimental subject who is the lone witness of a person in distress is very likely to try to help. But as the size of the group of witnesses increases, there is a decline not only in the probability that any given one of them offers assistance, but also in the probability that at least one of them offers assistance. Social psychologists hypothesize that three factors explain these experimental findings. First, "diffusion of responsibility": the larger the group, the lower the psychological cost of not helping. Second, "audience inhibition": the larger the group, the greater the embarrassment suffered by a helper in case the event turns out to be one in which help is inappropriate (because, for example, it is not in fact an emergency). Third, "social influence": a person infers the appropriateness of helping from others' behavior, so that in a large group everyone else's lack of intervention leads any given person to think intervention is less likely to be appropriate.

In terms of the model in Section 4.8, these three factors raise the expected cost and/or reduce the expected benefit of a person's intervening. They all seem plausible. However, they are not needed to explain the phenomenon: our game-theoretic analysis shows that even if the cost and benefit are *independent* of group size, a decrease in the probability that at least one person intervenes is an implication of equilibrium. This game-theoretic analysis has an advantage over the socio-psychological one: it derives the conclusion from the same principles that underlie all the other models studied so far (oligopoly, auctions, voting, and elections, for example), rather than positing special features of the specific environment in which a group of bystanders may come to the aid of a person in distress.

The critical element missing from the socio-psychological analysis is the notion of an *equilibrium*. Whether any given person intervenes depends on the probability she assigns to some other person's intervening. In an equilibrium each person must be indifferent between intervening and not intervening, and as we have seen this condition leads inexorably to the conclusion that an increase in group size reduces the probability that at least one person intervenes.

4.9 The formation of players' beliefs

In a Nash equilibrium, each player chooses a strategy that maximizes her expected payoff, *knowing* the other players' strategies. So far we have not considered how players may acquire such information. Informally, the idea underlying the previous analysis is that the players have learned each other's strategies from their experience playing the game. In the idealized situation to which the analysis corresponds, for each player in the game there is a large population of individuals who may take the role of that player; in any play of the game, one participant is drawn randomly from each population. In this situation, a new individual who joins a population that is in a steady state (i.e. is using a Nash equilibrium strategy profile) can learn the other players' strategies by observing their actions over many plays of the game. As long as the turnover in players is small enough, existing players' encounters with neophytes (who may use nonequilibrium strategies) will be sufficiently rare that their beliefs about the steady state will not be disturbed, so that a new player's problem is simply to learn the other players' actions.

This analysis leaves open the question of what might happen if new players simultaneously join more than one population in sufficient numbers that they have a significant chance of facing opponents who are themselves new. In particular, can we expect a steady state to be reached when no one has experience playing the game?

4.9.1 Eliminating dominated actions

In some games the players may reasonably be expected to choose their Nash equilibrium actions from an introspective analysis of the game. At an extreme, each player's best action may be independent of the other players' actions, as in the *Prisoner's Dilemma* (Example 12.1). In such a game no player needs to worry about the other players' actions. In a less extreme case, some player's best action may depend on the other players' actions, but the actions the other players will choose may be clear because each of these players has an action that strictly dominates all others. For example, in the game in Figure 132.1, player 2's action R strictly dominates L, so that no matter what player 1 thinks player 1 will do, she should choose R. Consequently, player 1, who can deduce by this argument that player 2 will choose R, may reason that she should choose B. That is, even inexperienced players may be led to the unique Nash equilibrium (B, R) in this game.

	L	R
T	$1,2$	$0,3$
B	$0,0$	$1,1$

Figure 132.1 A game in which player 2 has a strictly dominant action whereas player 1 does not.

This line of argument may be extended. For example, in the game in Figure 132.2 player 1's action T is strictly dominated, so player 1 may reason that player 2 will deduce that player 1 will not choose T. Consequently player 1 may deduce that player 2 will choose R, and hence herself may choose B rather than M.

	L	R
T	$0,2$	$0,0$
M	$2,1$	$1,2$
B	$1,1$	$2,2$

Figure 132.2 A game in which player 1 may reason that she should choose B because player 2 will reason that player 1 will not choose T, so that player 2 will choose R.

The set of action profiles that remain at the end of such a reasoning process contains all Nash equilibria; for many games (unlike these examples) it contains many other action profiles. In fact, in many games it does not eliminate any action profile, because no player has a strictly dominated action. Nevertheless, in some classes of games the process is powerful; its logical consequences are explored in Chapter 12.

4.9.2 Learning

Another approach to the question of how a steady state might be reached assumes that each player starts with an unexplained "prior" belief about the other players' actions, and changes these beliefs—"learns"—in response to information she receives. She may learn, for example, from observing the fortunes of other players like herself, from discussing the game with such players, or from her own experience playing the game. Here I briefly discuss two theories in which the same set of participants repeatedly play a game, each participant changing her beliefs about the others' strategies in response to her observations of their actions.

Best response dynamics A particularly simple theory assumes that in each period after the first, each player believes that the other players will choose the actions they chose in the previous period. In the first period, each player chooses a best response to an arbitrary deterministic belief about the other players' actions. In every subsequent period, each player chooses a best response to the other players' actions in the *previous* period. This process is known as *best response dynamics*. An action profile that remains the same from period to period is a pure Nash equilibrium of the game. Further, a pure Nash equilibrium in which each player's action is her only best response to the other players' actions is an action profile that remains the same from period to period.

In some games the sequence of action profiles generated best response dynamics converges to a pure Nash equilibrium, regardless of the players' initial beliefs. The example of Cournot's duopoly game studied in Section 3.1.3 is such a game. Looking at the best response functions in Figure 56.2, you can convince yourself that from arbitrary initial actions, the players' actions approach the Nash equilibrium (q_1^*, q_2^*).

? EXERCISE 133.1 (Best response dynamics in Cournot's duopoly game) Find the sequence of pairs of outputs chosen by the firms in Cournot's duopoly game under the assumptions of Section 3.1.3 if they both initially choose 0. (If you know how to solve a first-order difference equation, find a formula for the outputs in each period; if not, find the outputs in the first few periods.)

? EXERCISE 133.2 (Best response dynamics in Bertrand's duopoly game) Consider Bertrand's duopoly game in which the set of possible prices is discrete, under the assumptions of Exercise 65.2. Does the sequences of prices under best response dynamics converge to a Nash equilibrium when both prices initially exceed $c + 1$? What happens when both prices are initially equal to c?

For other games there are initial beliefs for which the sequence of action profiles generated by the process does not converge. In *BoS* (Example 16.2), for example, if player 1 initially believes that player 2 will choose *Stravinsky* and player 2 initially believes that player 1 will choose *Bach*, then the players' choices will subsequently alternate indefinitely between the action pairs (*Bach*, *Stravinsky*) and (*Stravinsky*, *Bach*). This example highlights the limited extent to which a player is assumed to reason in the model, which does not consider the possibility that she cottons on to the fact that her opponent's action is always a best response to her own previous action.

Fictitious play Under best response dynamics, the players' beliefs are continually revealed to be incorrect unless the starting point is a Nash equilibrium: the players' actions change from period to period. Further, each player believes that every other player is using a pure strategy: a player's belief does not admit the possibility that her opponents' actions are realizations of mixed strategies.

Another theory, known as *fictitious play*, assumes that players consider actions in all the previous periods when forming a belief about their opponents' strategies. They treat these actions as realizations of mixed strategies. Consider a two-player game. Each player begins with an arbitrary probabilistic belief about the other player's action. In the first play of the game she chooses a best response to this belief and observes the other player's action, say A. She then changes her belief to one that assigns probability one to A; in the second period, she chooses a best response to this belief and observes the other player's action, say B. She then changes her belief to one that assigns probability $\frac{1}{2}$ to both A and B, and chooses a best response to this belief. She continues to change her belief each period; in any period she adopts the belief that her opponent is using a mixed strategy in which the probability of each action is proportional to the frequency with which her opponent chose that action in the previous periods. (If, for example, in the first six periods player 2 chooses A twice, B three times, and C once, player 1's belief in period 7 assigns probability $\frac{1}{3}$ to A, probability $\frac{1}{2}$ to B, and probability $\frac{1}{6}$ to C.)

In the game *Matching Pennies* (Example 17.1), reproduced in Figure 134.1, this process works as follows. Suppose that player 1 begins with the belief that player 2's action will be *Tail*, and player 2 begins with the belief that player 1's action will be *Head*. Then in period 1 both players choose *Tail*. Thus in period 2 both players believe that their opponent will choose *Tail*, so that player 1 chooses *Tail* and player 2 chooses *Head*. Consequently in period 3, player 1's belief is that player 2 will choose *Head* with probability $\frac{1}{2}$ and *Tail* with probability $\frac{1}{2}$, and player 2's belief is that player 1 will definitely choose *Tail*. Thus in period 3, both *Head* and *Tail* are best responses of player 1 to her belief, so that she may take either action; the unique best response of player 2 is *Head*. The process continues similarly in subsequent periods.

	Head	*Tail*
Head	$1, -1$	$-1, 1$
Tail	$-1, 1$	$1, -1$

Figure 134.1 *Matching Pennies.*

In two-player games like *Matching Pennies*, in which the players' interests are directly opposed, and in any two-player game in which each player has two actions, this process converges to a mixed strategy Nash equilibrium from any initial beliefs. That is, after a sufficiently large number of periods, the frequencies with which each player chooses her actions are close to the frequencies induced by her mixed strategy in the Nash equilibrium. For other games there are initial beliefs for which the process does not converge. (The simplest example is too complicated to present compactly.)

People involved in an interaction that we model as a game may form beliefs about their opponents' strategies from an analysis of the structure of the players' payoffs, from their observations of their opponents' actions, and from information they obtain from other people involved in similar interactions. The models I have outlined allow us to explore the logical implications of two ways in which players may draw inferences from their opponents' actions. Models that assume the players to be more sophisticated may give more insights into the types of situation in which a Nash equilibrium is likely to be attained; this topic is an active area of current research.

4.10 Extension: Finding all mixed strategy Nash equilibria

We can find all the mixed strategy Nash equilibria of a two-player game in which each player has two actions by constructing the players' best response functions, as we have seen. In more complicated games, this method is usually not practical.

The following systematic method of finding all mixed strategy Nash equilibria of a game is suggested by the characterization of an equilibrium in Proposition 113.2.

- For each player i, choose a subset S_i of her set A_i of actions.
- Check whether there exists a mixed strategy profile α such that (*i*) the set of actions to which each strategy α_i assigns positive probability is S_i and (*ii*) α satisfies the conditions in Proposition 113.2.
- Repeat the analysis for every collection of subsets of the players' sets of actions.

The following example illustrates this method for a two-player game in which each player has two actions.

◆ EXAMPLE 135.1 (Finding all mixed strategy equilibria of a two-player game in which each player has two actions) Consider a two-player game in which each player has two actions. Denote the actions and payoffs as in Figure 136.1. Each player's set of actions has three nonempty subsets: two each consisting of a single action, and one consisting of both actions. Thus there are nine (3×3) pairs of subsets of the players' action sets. For each pair (S_1, S_2), we check if there is a pair (α_1, α_2) of mixed strategies such that each strategy α_i assigns positive probability only to actions in S_i and the conditions in Proposition 113.2 are satisfied.

- Checking the four pairs of subsets in which each player's subset consists of a single action amounts to checking whether any of the four pairs of actions is a pure strategy equilibrium. (For each player, the first condition in Proposition 113.2 is automatically satisfied, because there is only one action in each subset.)
- Consider the pair of subsets $\{T, B\}$ for player 1 and $\{L\}$ for player 2. The second condition in Proposition 113.2 is automatically satisfied for player 1, who has no actions to which she assigns probability 0, and the first condition is automatically satisfied for player 2, because she assigns positive probability to only one action. Thus for there to be a mixed strategy equilibrium in which player 1's probability of using T is p we need $u_{11} = u_{21}$ (player 1's payoffs to her two actions must be equal) and

$$pv_{11} + (1-p)v_{21} \geq pv_{12} + (1-p)v_{22}$$

(L must be at least as good as R, given player 1's mixed strategy). If $u_{11} \neq u_{21}$, or if there is no probability p satisfying the inequality, then there is no equilibrium of this type. A similar argument applies to the three other pairs of subsets in which one player's subset consists of both her actions and the other player's subset consists of a single action.

- To check whether there is a mixed strategy equilibrium in which the subsets are $\{T, B\}$ for player 1 and $\{L, R\}$ for player 2, we need to find a pair of mixed strategies that satisfies the first condition in Proposition 113.2 (the second condition is automatically satisfied because both players assign positive probability to both their actions). That is, we need to find probabilities p and q (if any such exist) for which

$$qu_{11} + (1-q)u_{12} = qu_{21} + (1-q)u_{22} \quad \text{and} \quad pv_{11} + (1-p)v_{21} = pv_{12} + (1-p)v_{22}.$$

	L	R
T	u_{11}, v_{11}	u_{12}, v_{12}
B	u_{21}, v_{21}	u_{22}, v_{22}

Figure 136.1 A two-player strategic game.

For example, in *BoS* we find the two pure equilibria when we check pairs of subsets in which each subset consists of a single action, we find no equilibria when we check pairs in which one subset consists of a single action and the other consists of both actions, and we find the mixed strategy equilibrium when we check the pair $(\{B, S\}, \{B, S\})$.

? EXERCISE 136.1 (Finding all mixed strategy equilibria of two-player games) Use the method described above to find all the mixed strategy equilibria of the games in Figure 111.2.

In a game in which each player has two actions, for any subset of any player's set of actions at most one of the two conditions in Proposition 113.2 is relevant (the first if the subset contains both actions and the second if it contains only one action). When a player has three or more actions and we consider a subset of her set of actions that contains two actions, both conditions are relevant, as the next example illustrates.

◆ EXAMPLE 136.2 (Finding all mixed strategy equilibria of a variant of *BoS*) Consider the variant of *BoS* given in Figure 137.1. First, by inspection we see that the game has two pure strategy Nash equilibria, namely (B, B) and (S, S).

Now consider the possibility of an equilibrium in which player 1's strategy is pure whereas player 2's strategy assigns positive probability to two or more actions. If player 1's strategy is B then player 2's payoffs to her three actions (2, 0, and 1) are all different, so the first condition in Proposition 113.2 is not satisfied. Thus

	B	S	X
B	$4,2$	$0,0$	$0,1$
S	$0,0$	$2,4$	$1,3$

Figure 137.1 A variant of the game *BoS*.

there is no equilibrium of this type. Similar reasoning rules out an equilibrium in which player 1's strategy is S and player 2's strategy assigns positive probability to more than one action, and also an equilibrium in which player 2's strategy is pure and player 1's strategy assigns positive probability to both of her actions.

Next consider the possibility of an equilibrium in which player 1's strategy assigns positive probability to both her actions and player 2's strategy assigns positive probability to two of her three actions. Denote by p the probability player 1's strategy assigns to B. There are three possibilities for the pair of player 2's actions that have positive probability.

B and S: For the conditions in Proposition 113.2 to be satisfied we need player 2's expected payoff to B to be equal to her expected payoff to S and at least her expected payoff to X. That is, we need

$$2p = 4(1-p) \geq p + 3(1-p).$$

The equation implies that $p = \frac{2}{3}$, which does not satisfy the inequality. (That is, if p is such that B and S yield the same expected payoff, then X yields a higher expected payoff.) Thus there is no equilibrium of this type.

B and X: For the conditions in Proposition 113.2 to be satisfied we need player 2's expected payoff to B to be equal to her expected payoff to X and at least her expected payoff to S. That is, we need

$$2p = p + 3(1-p) \geq 4(1-p).$$

The equation implies that $p = \frac{3}{4}$, which satisfies the inequality. For the first condition in Proposition 113.2 to be satisfied for player 1 we need player 1's expected payoffs to B and S to be equal: $4q = 1 - q$, where q is the probability player 2 assigns to B, or $q = \frac{1}{5}$. Thus the pair of mixed strategies $((\frac{3}{4}, \frac{1}{4}), (\frac{1}{5}, 0, \frac{4}{5}))$ is a mixed strategy equilibrium.

S and X: For every strategy of player 2 that assigns positive probability only to S and X, player 1's expected payoff to S exceeds her expected payoff to B. Thus there is no equilibrium of this sort.

The final possibility is that there is an equilibrium in which player 1's strategy assigns positive probability to both her actions and player 2's strategy assigns positive probability to all three of her actions. Let p be the probability player 1's strategy assigns to B. Then for player 2's expected payoffs to her three actions to be equal we need

$$2p = 4(1-p) = p + 3(1-p).$$

For the first equality we need $p = \frac{2}{3}$, violating the second equality. That is, there is no value of p for which player 2's expected payoffs to her three actions are equal, and thus no equilibrium in which she chooses each action with positive probability.

We conclude that the game has three mixed strategy equilibria: $((1,0),(1,0,0))$ (i.e. the pure strategy equilibrium (B,B)), $((0,1),(0,1,0))$ (i.e. the pure strategy equilibrium (S,S)), and $((\frac{3}{4},\frac{1}{4}),(\frac{1}{5},0,\frac{4}{5}))$.

? EXERCISE 138.1 (Finding all mixed strategy equilibria of a two-player game) Use the method described above to find all the mixed strategy Nash equilibria of the strategic game in Figure 138.1.

	L	M	R
T	2, 2	0, 3	1, 3
B	3, 2	1, 1	0, 2

Figure 138.1 The strategic game with vNM preferences in Exercise 138.1.

As you can see from the examples, this method has the disadvantage that for games in which each player has several strategies, or in which there are several players, the number of possibilities to examine is huge. Even in a two-player game in which each player has three actions, each player's set of actions has seven nonempty subsets (three each consisting of a single action, three consisting of two actions, and the entire set of actions), so that there are 49 (7×7) possible collections of subsets to check. In a symmetric game, like the one in the next exercise, many cases involve the same argument, reducing the number of distinct cases to be checked.

? EXERCISE 138.2 (Rock, paper, scissors) Each of two players simultaneously announces either *Rock*, or *Paper*, or *Scissors*. *Paper* beats (wraps) *Rock*, *Rock* beats (blunts) *Scissors*, and *Scissors* beats (cuts) *Paper*. The player who names the winning object receives $1 from her opponent; if both players make the same choice then no payment is made. Each player's preferences are represented by the expected amount of money she receives. (An example of the variant of Hotelling's model of electoral competition considered in Exercise 74.1 has the same payoff structure. Suppose there are three possible positions, A, B, and C, and three citizens, one of whom prefers A to B to C, one of whom prefers B to C to A, and one of whom prefers C to A to B. Two candidates simultaneously choose positions. If the candidates choose different positions each citizen votes for the candidate whose position she prefers; if both candidates choose the same position they tie for first place.)

a. Formulate this situation as a strategic game and find all its mixed strategy equilibria (give both the equilibrium strategies and the equilibrium payoffs).

b. Find all the mixed strategy equilibria of the modified game in which player 1 is prohibited from announcing *Scissors*.

?? EXERCISE 139.1 (Election campaigns) A new political party, A, is challenging an established party, B. The race involves three localities of different sizes. Party A can wage a strong campaign in only one locality; B must commit resources to defend its position in one of the localities, without knowing which locality A has targeted. If A targets district i and B devotes its resources to some other district then A gains a_i votes at the expense of B; let $a_1 > a_2 > a_3 > 0$. If B devotes resources to the district that A targets then A gains no votes. Each party's preferences are represented by the expected number of votes it gains. (Perhaps seats in a legislature are allocated proportionally to vote shares.) Formulate this situation as a strategic game and find its mixed strategy equilibria.

Although games with many players cannot in general be conveniently represented in tables like those we use for two-player games, three-player games can be accommodated. We construct one table for each of player 3's actions; player 1 chooses a row, player 2 chooses a column, and player 3 chooses a *table*. The next exercise is an example of such a game.

? EXERCISE 139.2 (A three-player game) Find the mixed strategy Nash equilibria of the three-player game in Figure 139.1, in which each player has two actions.

	A	B
A	$1,1,1$	$0,0,0$
B	$0,0,0$	$0,0,0$

A

	A	B
A	$0,0,0$	$0,0,0$
B	$0,0,0$	$4,4,4$

B

Figure 139.1 The three-player game in Exercise 139.2.

4.11 Extension: Mixed strategy Nash equilibria of games in which each player has a continuum of actions

In all the games studied so far in this chapter each player has finitely many actions. In the previous chapter we saw that many situations may conveniently be modeled as games in which each player has a continuum of actions. (For example, in Cournot's model the set of possible outputs for a firm is the set of nonnegative numbers, and in Hotelling's model the set of possible positions for a candidate is the set of nonnegative numbers.) The principles involved in finding mixed strategy equilibria of such games are the same as those involved in finding mixed strategy equilibria of games in which each player has finitely many actions, though the techniques are different.

Proposition 113.2 says that a strategy profile in a game in which each player has finitely many actions is a mixed strategy Nash equilibrium if and only if, for each player, (a) every action to which her strategy assigns positive probability yields the same expected payoff, and (b) no action yields a higher expected payoff. Now, a mixed strategy of a player who has a continuum of actions is determined by the probabilities it assigns to sets of actions, not by the probabilities it assigns to single actions (all of which may be zero, for example). Thus (*a*) does not fit such a game. However, the following restatement of the result, equivalent to Proposition 113.2 for a game in which each player has finitely many actions, does fit.

■ PROPOSITION 140.1 (Characterization of mixed strategy Nash equilibrium) *A mixed strategy profile α^* in a strategic game with vNM preferences is a mixed strategy Nash equilibrium if and only if, for each player i,*

- *α_i^* assigns probability zero to the set of actions a_i for which the action profile (a_i, α_{-i}^*) yields player i an expected payoff less than her expected payoff to α^**
- *for no action a_i does the action profile (a_i, α_{-i}^*) yield player i an expected payoff greater than her expected payoff to α^*.*

A significant class of games in which each player has a continuum of actions consists of games in which each player's set of actions is a one-dimensional interval of numbers. Consider such a game with two players; let player i's set of actions be the interval from $\underline{a}_i$ to $\overline{a}_i$, for $i = 1, 2$. Identify each player's mixed strategy with a *cumulative* probability distribution on this interval. (See Section 17.7.4 in the appendix on mathematics if you are not familiar with this notion.) That is, the mixed strategy of each player i is a nondecreasing function F_i for which $0 \leq F_i(a_i) \leq 1$ for every action a_i; the number $F_i(a_i)$ is the probability that player i's action is at most a_i.

The form of a mixed strategy Nash equilibrium in such a game may be very complex. Some such games, however, have equilibria of a particularly simple form, in which each player's equilibrium mixed strategy assigns probability zero except in an interval. Specifically, consider a pair (F_1, F_2) of mixed strategies that satisfies the following conditions for $i = 1, 2$.

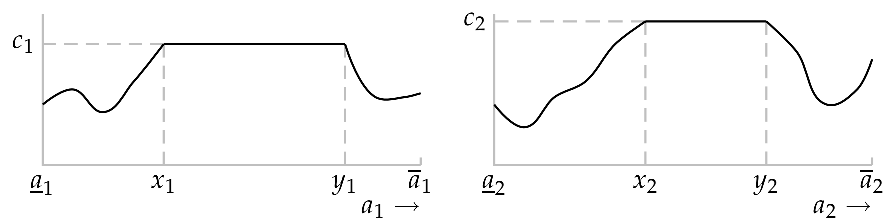
- There are numbers x_i and y_i such that player i's mixed strategy F_i assigns probability zero except in the interval from x_i to y_i: $F_i(z) = 0$ for $z < x_i$, and $F(z) = 1$ for $z \geq y_i$.
- Player i's expected payoff when her action is a_i and the other player uses her mixed strategy F_j takes the form

$$\begin{cases} = c_i & \text{for } x_i \leq a_i \leq y_i \\ \leq c_i & \text{for } a_i < x_i \text{ and } a_i > y_i \end{cases}$$

  where c_i is a constant.

(The second condition is illustrated in Figure 141.1.) By Proposition 140.1, such a pair of mixed strategies, if it exists, is a mixed strategy Nash equilibrium of the game, in which player i's expected payoff is c_i, for $i = 1, 2$.

The next example illustrates how a mixed strategy equilibrium of such a game may be found. The example is designed to be very simple; be warned that in most such games an analysis of the equilibria is, at a minimum, somewhat more



Player 1's expected payoff given F_2 Player 2's expected payoff given F_1

Figure 141.1 If (*i*) F_1 assigns positive probability only to actions in the interval from x_1 to y_1, (*ii*) F_2 assigns positive probability only to the actions in the interval from x_2 to y_2, (*iii*) given player 2's mixed strategy F_2, player 1's expected payoff takes the form shown in the left panel, and (*iv*) given player 1's mixed strategy F_1, player 2's expected payoff takes the form shown in the right panel, then (F_1, F_2) is a mixed strategy equilibrium.

complex. Further, my analysis is not complete: I merely find an equilibrium, rather than studying all equilibria. (In fact, the game has no other equilibria.)

◆ EXAMPLE 141.1 (All-pay auction) Two people submit sealed bids for an object worth \$$K$ to each of them. Each person's bid may be any nonnegative number up to \$$K$. The winner is the person whose bid is higher; in the event of a tie each person receives half of the object, which she values at \$$K/2$. Each person pays her bid, *whether or not she wins*, and has preferences represented by the expected amount of money she receives.

This situation may be modeled by the following strategic game, known as an **all-pay auction**.

Players The two bidders.

Actions Each player's set of actions is the set of possible bids (nonnegative numbers up to K)

Payoff functions Each player i's preferences are represented by the expected value of the payoff function given by

$$u_i(a_1, a_2) = \begin{cases} -a_i & \text{if } a_i < a_j \\ K/2 - a_i & \text{if } a_i = a_j \\ K - a_i & \text{if } a_i > a_j, \end{cases}$$

where j is the other player.

One situation that may be modeled as such an auction is a lobbying process in which each of two interest groups spends resources to persuade a government to carry out the policy it prefers, and the group that spends the most wins. Another situation that may be modeled as such an auction is the competition between two firms to develop a new product by some deadline, where the firm that spends the most develops a better product, which captures the entire market.

An all-pay auction has no pure strategy Nash equilibrium, by the following argument.

- No pair of actions (x, x) with $x < K$ is a Nash equilibrium, because either player can increase her payoff by slightly increasing her bid.
- (K, K) is not a Nash equilibrium, because either player can increase her payoff from $-K/2$ to 0 by reducing her bid to 0.
- No pair of actions (a_1, a_2) with $a_1 \neq a_2$ is a Nash equilibrium because the player whose bid is higher can increase her payoff by reducing her bid (and the player whose bid is lower can, if her bid is positive, increase her payoff by reducing her bid to 0).

Consider the possibility that the game has a mixed strategy Nash equilibrium. Denote by F_i the mixed strategy (i.e. cumulative probability distribution over the interval of possible bids) of player i. I look for an equilibrium in which neither mixed strategy assigns positive probability to any *single* bid. (Remember that there are infinitely many possible bids.) In this case $F_i(a_i)$ is both the probability that player i bids at most a_i and the probability that she bids less than a_i. I further restrict attention to strategy pairs (F_1, F_2) for which, for $i = 1, 2$, there are numbers x_i and y_i such that F_i assigns positive probability only to the interval from x_i to y_i.

To investigate the possibility of such an equilibrium, consider player 1's expected payoff when she uses the action a_1, given player 2's mixed strategy F_2.

- If $a_1 < x_2$ then a_1 is less than player 2's bid with probability one, so that player 1's payoff is $-a_1$.
- If $a_1 > y_2$ then a_1 exceeds player 2's bid with probability one, so that player 1's payoff is $K - a_1$.
- If $x_2 \leq a_1 \leq y_2$ then player 1's expected payoff is calculated as follows. With probability $F_2(a_1)$ player 2's bid is less than a_1, in which case player 1's payoff is $K - a_1$; with probability $1 - F_2(a_1)$ player 2's bid exceeds a_1, in which case player 1's payoff is $-a_1$; and, by assumption, the probability that player 2's bid is exactly equal to a_1 is zero. Thus player 1's expected payoff is

$$(K - a_1)F_2(a_1) + (-a_1)(1 - F_2(a_1)) = KF_2(a_1) - a_1.$$

We need to find values of x_2 and y_2 and a strategy F_2 such that player 1's expected payoff satisfies the condition illustrated in the left panel of Figure 141.1: it is constant on the interval from x_1 to y_1, and less than this constant for $a_1 < x_1$ and $a_1 > y_1$. The constancy of the payoff on the interval from x_1 to y_1 requires that $KF_2(a_1) - a_1 = c_1$ for $x_1 \leq a_1 \leq y_1$, for some constant c_1. We also need $F_2(x_2) = 0$ and $F_2(y_2) = 1$ (because I am restricting attention to equilibria in which neither player's strategy assigns positive probability to any single action), and F_2 must be nondecreasing (so that it is a cumulative probability distribution). Analogous conditions must be satisfied by x_2, y_2, and F_1.

We see that if $x_1 = x_2 = 0$, $y_1 = y_2 = K$, and $F_1(z) = F_2(z) = z/K$ for all z with $0 \leq z \leq K$ then all these conditions are satisfied. Each player's expected payoff is constant, equal to 0 for all her actions a_1.

Thus the game has a mixed strategy Nash equilibrium in which each player randomizes "uniformly" over all her actions. In this equilibrium each player's expected payoff is 0: on average, the amount a player spends is exactly equal to the value of the object. (A more involved argument shows that this equilibrium is the *only* mixed strategy Nash equilibrium of the game.)

?? EXERCISE 143.1 (All-pay auction with many bidders) Consider the generalization of the game considered in the previous example in which there are $n \geq 2$ bidders. Find a mixed strategy Nash equilibrium in which each player uses the same mixed strategy. (If you know how, find each player's mean bid in the equilibrium.)

?? EXERCISE 143.2 (Bertrand's duopoly game) Consider Bertrand's oligopoly game (Section 3.2) when there are two firms. Assume that each firm's preferences are represented by its expected profit. Show that if the function $(p-c)D(p)$ is increasing in p, and increases without bound as p increases without bound, then for every $\underline{p} > c$, the game has a mixed strategy Nash equilibrium in which each firm uses the same mixed strategy F, with $F(\underline{p}) = 0$ and $F(p) > 0$ for $p > \underline{p}$.

In the games in the example and exercises each player's payoff depends only on her action and whether this action is greater than, equal to, or less than the other players' actions. The limited dependence of each player's payoff on the other players' actions makes the calculation of a player's expected payoff straightforward. In many games, each player's payoff is affected more substantially by the other players' actions, making the calculation of expected payoff more complex; more sophisticated mathematical tools are required to analyze such games.

4.12 Appendix: Representing preferences over lotteries by the expected value of a payoff function

4.12.1 Expected payoffs

Suppose that a decision-maker has preferences over a set of deterministic outcomes, and that each of her actions results in a *lottery* (probability distribution) over these outcomes. In order to determine the action she chooses, we need to know her preferences over these lotteries. As argued in Section 4.1.3, we cannot *derive* these preferences from her preferences over deterministic outcomes, but have to specify them as part of the model.

So assume that we are given the decision-maker's preferences over lotteries. As in the case of preferences over deterministic outcomes, under some fairly weak assumptions we can represent these preferences by a payoff function. (Refer to Section 1.2.2.) That is, when there are K deterministic outcomes we can find a function, say U, over lotteries such that

$$U(p_1, \ldots, p_K) > U(p'_1, \ldots, p'_K)$$

if and only if the decision-maker prefers the lottery $(p_1, \ldots, p_K)$ to the lottery $(p'_1, \ldots, p'_K)$ (where $(p_1, \ldots, p_K)$ is the lottery in which outcome 1 occurs with probability p_1, outcome 2 occurs with probability p_2, and so on).

For many purposes, however, we need more structure: we cannot get very far without restricting to preferences for which there is a more specific representation. The standard approach, developed by von Neumann and Morgenstern (1944), is to impose an additional assumption—the "independence axiom"—that allows us to conclude that the decision-maker's preferences can be represented by an *expected payoff function*. More precisely, the independence axiom (which I do not describe) allows us to conclude that there is a payoff function u over *deterministic* outcomes such that the decision-maker's preference relation over lotteries is represented by the function $U(p_1, \ldots, p_K) = \sum_{k=1}^{K} p_k u(a_k)$, where a_k is the kth outcome of the lottery:

$$\sum_{k=1}^{K} p_k u(a_k) > \sum_{k=1}^{K} p'_k u(a_k) \tag{144.1}$$

if and only if the decision-maker prefers the lottery $(p_1, \ldots, p_K)$ to the lottery $(p'_1, \ldots, p'_K)$. That is, the decision-maker evaluates a lottery by its *expected payoff* according to the function u, which is known as the decision-maker's *Bernoulli payoff function*.

Suppose, for example, that there are three possible deterministic outcomes: the decision-maker may receive \$0, \$1, or \$5, and naturally prefers \$5 to \$1 to \$0. Suppose that she prefers the lottery $(\frac{1}{2}, 0, \frac{1}{2})$ to the lottery $(0, \frac{3}{4}, \frac{1}{4})$ (where the first number in each list is the probability of \$0, the second number is the probability of \$1, and the third number is the probability of \$5). This preference is consistent with preferences represented by the expected value of a payoff function u for which $u(0) = 0$, $u(1) = 1$, and $u(5) = 4$, because

$$\tfrac{1}{2} \cdot 0 + \tfrac{1}{2} \cdot 4 > \tfrac{3}{4} \cdot 1 + \tfrac{1}{4} \cdot 4.$$

(Many other payoff functions are consistent with a preference for $(\frac{1}{2}, 0, \frac{1}{2})$ over $(0, \frac{3}{4}, \frac{1}{4})$. Among those in which $u(0) = 0$ and $u(5) = 4$, for example, any function for which $u(1) < \frac{4}{3}$ does the job.) Suppose, on the other hand, that the decision-maker prefers the lottery $(0, \frac{3}{4}, \frac{1}{4})$ to the lottery $(\frac{1}{2}, 0, \frac{1}{2})$. This preference is consistent with preferences represented by the expected value of a payoff function u for which $u(0) = 0$, $u(1) = 3$, and $u(5) = 4$, because

$$\tfrac{1}{2} \cdot 0 + \tfrac{1}{2} \cdot 4 < \tfrac{3}{4} \cdot 3 + \tfrac{1}{4} \cdot 4.$$

? EXERCISE 144.2 (Preferences over lotteries) There are three possible outcomes; in the outcome a_i a decision-maker gains $\$a_i$, where $a_1 < a_2 < a_3$. The decision-maker prefers a_3 to a_2 to a_1 and she prefers the lottery $(0.3, 0, 0.7)$ to $(0.1, 0.4, 0.5)$ to $(0.3, 0.2, 0.5)$ to $(0.45, 0, 0.55)$. Is this information consistent with the decision-maker's preferences being represented by the expected value of a payoff function? If so, find a payoff function consistent with the information. If not, show why not. Answer the same questions when, alternatively, the decision-maker prefers the lottery $(0.4, 0, 0.6)$ to $(0, 0.5, 0.5)$ to $(0.3, 0.2, 0.5)$ to $(0.45, 0, 0.55)$.

Preferences represented by the expected value of a (Bernoulli) payoff function have the great advantage that they are completely specified by that payoff function. Once we know $u(a_k)$ for each possible outcome a_k we know the decision-maker's preferences among all lotteries. This significant advantage does, however, carry with it a small price: it is very easy to confuse a Bernoulli payoff function with a payoff function that represents the decision-maker's preferences over deterministic outcomes.

To describe the relation between the two, suppose that a decision-maker's preferences over lotteries are represented by the expected value of the Bernoulli payoff function u. Then certainly u is a payoff function that represents the decision-maker's preferences over deterministic outcomes (which are special cases of lotteries, in which a single outcome is assigned probability 1). However, the converse is *not* true: if the decision-maker's preferences over deterministic outcomes are represented by the payoff function u (i.e. the decision-maker prefers a to a' if and only if $u(a) > u(a')$), then u is *not* necessarily a Bernoulli payoff function whose expected value represents the decision-maker's preferences over lotteries. For instance, suppose that the decision-maker prefers \$5 to \$1 to \$0, and prefers the lottery $(\frac{1}{2}, 0, \frac{1}{2})$ to the lottery $(0, \frac{3}{4}, \frac{1}{4})$. Then her preferences over deterministic outcomes are consistent with the payoff function u for which $u(0) = 0$, $u(1) = 3$, and $u(5) = 4$. However, her preferences over lotteries are *not* consistent with the expected value of this function (since $\frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 4 < \frac{3}{4} \cdot 3 + \frac{1}{4} \cdot 4$). The moral is that you should be careful to determine the type of payoff function you are dealing with.

4.12.2 Equivalent Bernoulli payoff functions

If a decision-maker's preferences in a deterministic environment are represented by the payoff function u then they are represented also by any payoff function that is an increasing function of u (see Section 1.2.2). The analogous property is not satisfied by Bernoulli payoff functions. Consider the example discussed above. A Bernoulli payoff function u for which $u(0) = 0$, $u(1) = 1$, and $u(5) = 4$ is consistent with a preference for the lottery $(\frac{1}{2}, 0, \frac{1}{2})$ over $(0, \frac{3}{4}, \frac{1}{4})$, but the function $\sqrt{u}$, for which $u(0) = 0$, $u(1) = 1$, and $u(5) = 2$, is not consistent with such a preference ($\frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 2 < \frac{3}{4} \cdot 1 + \frac{1}{4} \cdot 2$), though the square root function is increasing (larger numbers have larger square roots).

Under what circumstances do the expected values of two Bernoulli payoff functions represent the same preferences? The next result shows that they do so if and only if one payoff function is an increasing *linear* function of the other.

■ LEMMA 145.1 (Equivalence of Bernoulli payoff functions) *Suppose there are at least three possible outcomes. The expected values of the Bernoulli payoff functions u and v represent the same preferences over lotteries if and only if there exist numbers η and θ with $\theta > 0$ such that $u(x) = \eta + \theta v(x)$ for all x.*

If the expected value of u represents a decision-maker's preferences over lotteries then so, for example, do the expected values of $2u$, $1+u$, and $-1+4u$; but the expected values of u^2 and of $\sqrt{u}$ do not.

Part of the lemma is easy to establish. Let u be a Bernoulli payoff function whose expected value represents a decision-maker's preferences, and let $v(x) = \eta + \theta u(x)$ for all x, where η and θ are constants with $\theta > 0$. I argue that the expected values of u and of v represent the same preferences. Suppose that the decision-maker prefers the lottery $(p_1, \ldots, p_K)$ to the lottery $(p'_1, \ldots, p'_K)$. Then her expected payoff to $(p_1, \ldots, p_K)$ exceeds her expected payoff to $(p'_1, \ldots, p'_K)$, or

$$\sum_{k=1}^{K} p_k u(a_k) > \sum_{k=1}^{K} p'_k u(a_k) \tag{146.1}$$

(see (144.1)). Now,

$$\sum_{k=1}^{K} p_k v(a_k) = \sum_{k=1}^{K} p_k \eta + \sum_{k=1}^{K} p_k \theta u(a_k) = \eta + \theta \sum_{k=1}^{K} p_k u(a_k),$$

using the fact that the sum of the probabilities p_k is 1. Similarly,

$$\sum_{k=1}^{K} p'_k v(a_k) = \eta + \theta \sum_{k=1}^{K} p'_k u(a_k).$$

Substituting for u in (146.1) we obtain

$$\left(\sum_{k=1}^{K} p_k v(a_k) - \eta\right)/\theta > \left(\sum_{k=1}^{K} p'_k v(a_k) - \eta\right)/\theta,$$

which, given $\theta > 0$, is equivalent to

$$\sum_{k=1}^{K} p_k v(a_k) > \sum_{k=1}^{K} p'_k v(a_k):$$

according to v, the expected payoff of $(p_1, \ldots, p_K)$ exceeds the expected payoff of $(p'_1, \ldots, p'_K)$. We conclude that if u represents the decision-maker's preferences then so does the function v defined by $v(x) = \eta + \theta u(x)$.

I omit the more difficult argument that if the expected values of the Bernoulli payoff functions u and v represent the same preferences over lotteries then $v(x) = \eta + \theta u(x)$ for some constants η and $\theta > 0$.

? EXERCISE 146.2 (Normalized Bernoulli payoff functions) Suppose that a decision-maker's preferences can be represented by the expected value of the Bernoulli payoff function u. Find a Bernoulli payoff function whose expected value represents the decision-maker's preferences and that assigns a payoff of 1 to the best outcome and a payoff of 0 to the worst outcome.

4.12.3 Equivalent strategic games with vNM preferences

Turning to games, consider the three payoff tables in Figure 147.1. All three tables represent the same strategic game with deterministic preferences: in each case, player 1 prefers (B, B) to (S, S) to (B, S), which she regards as indifferent to (S, B), and player 2 prefers (S, S) to (B, B) to (B, S), which she regards as indifferent to (S, B). However, only the left and middle tables represent the same strategic game with vNM preferences. The reason is that the payoff functions in the middle table are linear functions of the payoff functions in the left table, whereas the payoff functions in the right table are not. Specifically, denote the Bernoulli payoff functions of player i in the three games by u_i, v_i, and w_i. Then

$$v_1(a) = 2u_1(a) \text{ and } v_2(a) = -3 + 3u_2(a),$$

so that the left and middle tables represent the same strategic game with vNM preferences. However, w_1 is not a linear function of u_1. If it were, there would exist constants η and $\theta > 0$ such that $w_1(a) = \eta + \theta u_1(a)$ for each action pair a, or

$$\begin{aligned} 0 &= \eta + \theta \cdot 0 \\ 1 &= \eta + \theta \cdot 1 \\ 3 &= \eta + \theta \cdot 2, \end{aligned}$$

but these three equations have no solution. Thus the left and right tables represent different strategic games with vNM preferences. (As you can check, w_2 is not a linear function of u_2 either; but for the games not to be equivalent it is sufficient that *one* player's preferences be different.) Another way to see that player 1's vNM preferences in the left and right games are different is to note that in the left table player 1 is indifferent between the certain outcome (S, S) and the lottery in which (B, B) occurs with probability $\frac{1}{2}$ and (S, B) occurs with probability $\frac{1}{2}$ (each yields an expected payoff of 1), whereas in the right table she prefers the latter (since it yields an expected payoff of 1.5).

	B	*S*
B	2, 1	0, 0
S	0, 0	1, 2

	B	*S*
B	4, 0	0, −3
S	0, −3	2, 3

	B	*S*
B	3, 2	0, 1
S	0, 1	1, 4

Figure 147.1 All three tables represent the same strategic game with ordinal preferences, but only the left and middle games, not the right one, represent the same strategic game with vNM preferences.

? EXERCISE 147.1 (Games equivalent to the *Prisoner's Dilemma*) Which of the tables in Figure 148.1 represents the same strategic game with vNM preferences as the *Prisoner's Dilemma* as specified in the left panel of Figure 104.1, when the numbers are interpreted as Bernoulli payoffs?

	C	D
C	$3,3$	$0,4$
D	$4,0$	$2,2$

	C	D
C	$6,\ \ 0$	$0,\ \ 2$
D	$9,-4$	$3,-2$

Figure 148.1 The payoff tables for Exercise 147.1.

Notes

The ideas behind mixed strategies and preferences represented by expected payoffs date back in Western thought at least to the eighteenth century (see Guilbaud (1961) and Kuhn (1968), and Bernoulli (1738), respectively). The modern formulation of a mixed strategy is due to Borel (1921; 1924, 204–221; 1927); the model of the representation of preferences by an expected payoff function is due to von Neumann and Morgenstern (1944). The model of a mixed strategy Nash equilibrium and Proposition 116.1 on the existence of a mixed strategy Nash equilibrium in a finite game are due to Nash (1950a, 1951). Proposition 119.2 is an implication of the existence of a "trembling hand perfect equilibrium", due to Selten (1975, Theorem 5).

The example in the box on page 102 is taken from Allais (1953). Conlisk (1989) discusses some of the evidence on the theory of expected payoffs; Machina (1987) and Hey (1997) survey the subject. (The purchasing power of the largest prize in Allais' example was roughly US$6.6m in 1989 (the date of Conlisk's paper, in which the prize is US$5m) and roughly US$8m in 1999.) The model in Section 4.6 is due to Pitchik and Schotter (1987). The model in Section 4.8 is a special case of the one in Palfrey and Rosenthal (1984); the interpretation and analysis that I describe is taken from an unpublished 1984 paper of William F. Samuelson. The box on page 130 draws upon Rosenthal (1964), Latané and Nida (1981), Brown (1986), and Aronson (1995). Best response dynamics were first studied by Cournot (1838, Ch. VII), in the context of his duopoly game. Fictitious play was suggested by Brown (1951). Robinson (1951) shows that the process converges to a mixed strategy Nash equilibrium in any two-player game in which the players' interests are opposed; Shapley (1964, Section 5) exhibits a game outside this class in which the process does not converge. Recent work on learning in games is surveyed by Fudenberg and Levine (1998).

The game in Exercise 115.2 is due to David L. Silverman (see Silverman 1981–82 and Heuer 1995). Exercise 115.3 is based on Palfrey and Rosenthal (1983). Exercise 115.4 is taken from Shubik (1982, 226) (who finds only one of the continuum of equilibria of the game).

The model in Exercise 125.2 is taken from Peters (1984). Exercise 127.2 is a variant of an exercise of Moulin (1986, pp. 167, 185). Exercise 130.1 is based on Palfrey and Rosenthal (1984). The game *Rock-Paper-Scissors* (Exercise 138.2) was first studied by Borel (1924) and von Neumann (1928). Exercise 139.1 is based on Karlin (1959a, 92–94), who attributes the game to an unpublished paper by Dresher.

Exercise 143.1 is based on a result in Baye, Kovenock, and de Vries (1996). The mixed strategy Nash equilibria of Bertrand's model of duopoly (Exercise 143.2) are studied in detail by Baye and Morgan (1996).

The method of finding all mixed strategy equilibrium described in Section 4.10 is computationally very intense in all but the simplest games. Some computationally more efficient methods are implemented in the computer program `GAMBIT`, located at `http://www.hss.caltech.edu/\symbol{126}gambit/Gambit.html`.

5 Extensive Games with Perfect Information: Theory

Prerequisite: Chapters 1 and 2.

5.1 Introduction

THE model of a strategic game suppresses the sequential structure of decision-making. When applying the model to situations in which decision-makers move sequentially, we assume that each decision-maker chooses her plan of action once and for all; she is committed to this plan, which she cannot modify as events unfold. The model of an extensive game, by contrast, describes the sequential structure of decision-making explicitly, allowing us to study situations in which each decision-maker is free to change her mind as events unfold.

In this chapter and the next two we study a model in which each decision-maker is always fully informed about all previous actions. In Chapter 10 we study a more general model, which allows each decision-maker, when taking an action, to be imperfectly informed about previous actions.

5.2 Extensive games with perfect information

5.2.1 Definition

To describe an extensive game with perfect information, we need to specify the set of players and their preferences, as for a strategic game (Definition 11.1). In addition, we need to specify the order of the players' moves and the actions each player may take at each point. We do so by specifying the set of all sequences of actions that can possibly occur, together with the player who moves at each point in each sequence. We refer to each possible sequence of actions as a *terminal history* and to the function that gives the player who moves at each point in each terminal history as the *player function*. That is, an extensive game has four components:

- players
- terminal histories

- player function
- preferences for the players.

Before giving precise definitions of these components, I give an example that illustrates them informally.

◆ EXAMPLE 152.1 (Entry game) An incumbent faces the possibility of entry by a challenger. (The challenger may, for example, be a firm considering entry into an industry currently occupied by a monopolist, a politician competing for the leadership of a party, or an animal considering competing for the right to mate with a congener of the opposite sex.) The challenger may enter or not. If it enters, the incumbent may either acquiesce or fight.

We may model this situation as an extensive game with perfect information in which the terminal histories are $(In, Acquiesce)$, $(In, Fight)$, and *Out*, and the player function assigns the challenger to the start of the game and the incumbent to the history *In*.

At the start of an extensive game, and after any sequence of events, a player chooses an action. The sets of actions available to the players are not, however, given explicitly in the description of the game. Instead, the description of the game specifies the set of terminal histories and the player function, from which we can deduce the available sets of actions.

In the entry game, for example, the actions available to the challenger at the start of the game are *In* and *Out*, because these actions (and no others) begin terminal histories, and the actions available to the incumbent are *Acquiesce* and *Fight*, because these actions (and no others) follow *In* in terminal histories. More generally, suppose that (C, D) and (C, E) are terminal histories and the player function assigns player 1 to the start of the game and player 2 to the history C. Then two of the actions available to player 2 after player 1 chooses C at the start of the game are D and E.

The terminal histories of a game are specified as a set of sequences. But not every set of sequences is a legitimate set of terminal histories. If (C, D) is a terminal history, for example, there is no sense in specifying C as a terminal history: the fact that (C, D) is terminal implies that after C is chosen at the start of the game, some player may choose D, so that the action C does not end the game. More generally, a sequence that is a *proper subhistory* of a terminal history cannot itself be a terminal history. This restriction is the only one we need to impose on a set of sequences in order that the set be interpretable as a set of terminal histories.

To state the restriction precisely, define the **subhistories** of a finite sequence $(a^1, a^2, \ldots, a^k)$ of actions to be the empty sequence consisting of no actions, denoted $\varnothing$ (representing the start of the game), and all sequences of the form $(a^1, a^2, \ldots, a^m)$ where $1 \leq m \leq k$. (In particular, the entire sequence is a subhistory of itself.) Similarly, define the **subhistories** of an infinite sequence $(a^1, a^2, \ldots)$ of actions to be the empty sequence $\varnothing$, every sequence of the form $(a^1, a^2, \ldots, a^m)$ where m is a positive integer, and the entire sequence $(a^1, a^2, \ldots)$. A subhistory not equal to the entire sequence is called a **proper subhistory**. A sequence of actions that is a subhistory of some terminal history is called simply a **history**.

In the entry game in Example 152.1, the subhistories of $(In, Acquiesce)$ are the empty history $\varnothing$ and the sequences *In* and $(In, Acquiesce)$; the proper subhistories are the empty history and the sequence *In*.

▶ DEFINITION 153.1 (*Extensive game with perfect information*) An **extensive game with perfect information** consists of

- a set of **players**
- a set of sequences (**terminal histories**) with the property that no sequence is a proper subhistory of any other sequence
- a function (the **player function**) that assigns a player to every sequence that is a proper subhistory of some terminal history
- for each player, **preferences** over the set of terminal histories.

The set of terminal histories is the set of all sequences of actions that may occur; the player assigned by the player function to any history h is the player who takes an action after h.

As for a strategic game, we may specify a player's preferences by giving a payoff function that represents them (see Section 1.2.2). In some situations an outcome is associated with each terminal history, and the players' preferences are naturally defined over these outcomes, rather than directly over the terminal histories. For example, if we are modeling firms choosing prices then we may think in terms of each firm's caring about its profit—the outcome of a profile of prices—rather than directly about the profile of prices. However, any preferences over outcomes (e.g. profits) may be translated into preferences over terminal histories (e.g. sequences of prices). In the general definition, outcomes are conveniently identified with terminal histories and preferences are defined directly over these histories, avoiding the need for an additional element in the specification of the game.

◆ EXAMPLE 153.2 (Entry game) In the situation described in Example 152.1, suppose that the best outcome for the challenger is that it enters and the incumbent acquiesces, and the worst outcome is that it enters and the incumbent fights, whereas the best outcome for the incumbent is that the challenger stays out, and the worst outcome is that it enters and there is a fight. Then the situation may be modeled as the following extensive game with perfect information.

Players The challenger and the incumbent.

Terminal histories $(In, Acquiesce)$, $(In, Fight)$, and *Out*.

Player function $P(\varnothing) = Challenger$ and $P(In) = Incumbent$.

Preferences The challenger's preferences are represented by the payoff function u_1 for which $u_1(In, Acquiesce) = 2$, $u_1(Out) = 1$, and $u_1(In, Fight) = 0$, and the incumbent's preferences are represented by the payoff function u_2 for which $u_2(Out) = 2$, $u_2(In, Acquiesce) = 1$, and $u_2(In, Fight) = 0$.

This game is readily illustrated in a diagram. The small circle at the top of Figure 154.1 represents the empty history (the start of the game). The label above this circle indicates that the challenger chooses an action at the start of the game ($P(\varnothing) = $ *Challenger*). The two branches labeled *In* and *Out* represent the challenger's choices. The segment labeled *In* leads to a small disk, where it is the incumbent's turn to choose an action ($P(In) = $ *Incumbent*) and her choices are *Acquiesce* and *Fight*. The pair of numbers beneath each terminal history gives the players' payoffs to that history, with the challenger's payoff listed first. (The players' payoffs may be given in any order. For games like this one, in which the players move in a well-defined order, I generally list the payoffs in that order. For games in which the players' names are 1, 2, 3, and so on, I list the payoffs in the order of their names.)

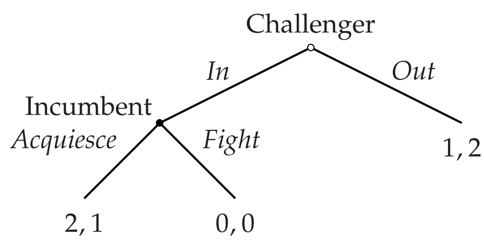


Figure 154.1 The entry game of Example 153.2. The challenger's payoff is the first number in each pair.

Definition 153.1 does not directly specify the sets of actions available to the players at their various moves. As I discussed briefly before the definition, we can deduce these sets from the set of terminal histories and the player function. If, for some nonterminal history h, the sequence (h, a) is a history, then a is one of the actions available to the player who moves after h. Thus the set of all actions available to the player who moves after h is

$$A(h) = \{a\colon (h, a) \text{ is a history}\}. \tag{154.1}$$

For example, for the game in Figure 154.1, the histories are $\varnothing$, *In*, *Out*, (*In*, *Acquiesce*), and (*In*, *Fight*). Thus the set of actions available to the player who moves at the start of the game, namely the challenger, is $A(\varnothing) = \{In, Out\}$, and the set of actions available to the player who moves after the history *In*, namely the incumbent, is $A(In) = \{Acquiesce, Fight\}$.

? EXERCISE 154.2 (Examples of extensive games with perfect information)

a. Represent in a diagram like Figure 154.1 the two-player extensive game with perfect information in which the terminal histories are (C, E), (C, F), (D, G), and (D, H), the player function is given by $P(\varnothing) = 1$ and $P(C) = P(D) = 2$, player 1 prefers (C, F) to (D, G) to (C, E) to (D, H), and player 2 prefers (D, G) to (C, F) to (D, H) to (C, E).

b. Write down the set of players, set of terminal histories, player function, and players' preferences for the game in Figure 158.1.

c. The political figures Rosa and Ernesto each has to take a position on an issue. The options are Berlin (B) or Havana (H). They choose sequentially. A third person, Karl, determines who chooses first. Both Rosa and Ernesto care only about the actions they choose, not about who chooses first. Rosa prefers the outcome in which both she and Ernesto choose B to that in which they both choose H, and prefers this outcome to either of the ones in which she and Ernesto choose different actions; she is indifferent between these last two outcomes. Ernesto's preferences differ from Rosa's in that the roles of B and H are reversed. Karl's preferences are the same as Ernesto's. Model this situation as an extensive game with perfect information. (Specify the components of the game and represent the game in a diagram.)

Definition 153.1 allows terminal histories to be infinitely long. Thus we can use the model of an extensive game to study situations in which the participants do not consider any particular fixed horizon when making decisions. If the length of the longest terminal history is in fact finite, we say that the game has a **finite horizon**.

Even a game with a finite horizon may have infinitely many terminal histories, because some player has infinitely many actions after some history. If a game has a finite horizon *and* finitely many terminal histories we say it is **finite**. Note that a game that is not finite cannot be represented in a diagram like Figure 154.1, because such a figure allows for only finitely many branches.

An extensive game with perfect information models a situation in which each player, when choosing an action, knows all actions chosen previously (has *perfect information*), and always moves alone (rather than simultaneously with other players). Some economic and political situations that the model encompasses are discussed in the next chapter. The competition between interest groups courting legislators is one example. This situation may be modeled as an extensive game in which the groups sequentially offer payments to induce the legislators to vote for their favorite version of a bill (Section 6.4). A race (between firms developing a new technology, or between directors making competing movies, for instance), is another example. This situation is modeled as an extensive game in which the parties alternately decide how much effort to expend (Section 6.5). Parlor games such as chess, ticktacktoe, and go, in which there are no random events, the players move sequentially, and each player always knows all actions taken previously, may also be modeled as extensive games with perfect information (see the box on page 176).

In Section 7.1 I discuss a more general notion of an extensive game in which players may move simultaneously, though each player, when choosing an action, still knows all previous actions. In Chapter 10 I discuss a much more general notion that allows arbitrary patterns of information. In each case I sometimes refer to the object under consideration simply as an "extensive game".

5.2.2 Solutions

In the entry game in Figure 154.1, it seems clear that the challenger will enter and the incumbent will subsequently acquiesce. The challenger can reason that if it enters then the incumbent will acquiesce, because doing so is better for the incumbent than fighting. Given that the incumbent will respond to entry in this way, the challenger is better off entering.

This line of argument is called *backward induction*. Whenever a player has to move, she deduces, for each of her possible actions, the actions that the players (including herself) will subsequently rationally take, and chooses the action that yields the terminal history she most prefers.

While backward induction may be applied to the game in Figure 154.1, it cannot be applied to every extensive game with perfect information. Consider, for example, the variant of this game shown in Figure 156.1, in which the incumbent's payoff to the terminal history $(In, Fight)$ is 1 rather than 0. If, in the modified game, the challenger enters, the incumbent is indifferent between acquiescing and fighting. Backward induction does not tell the challenger what the incumbent will do in this case, and thus leaves open the question of which action the challenger should choose. Games with infinitely long histories present another difficulty for backward induction: they have no end from which to start the induction. The generalization of an extensive game with perfect information that allows for simultaneous moves (studied in Chapter 7) poses yet another problem: when players move simultaneously we cannot in general straightforwardly deduce each player's optimal action. (As in a strategic game, each player's best action depends on the other players' actions.)

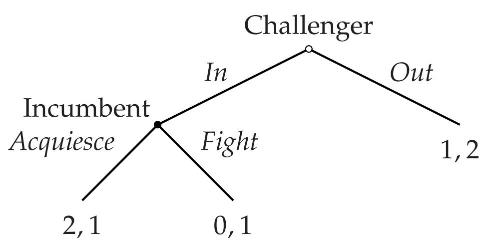


Figure 156.1 A variant of the entry game of Figure 154.1. The challenger's payoff is the first number in each pair.

Another approach to defining equilibrium takes off from the notion of Nash equilibrium. It seeks to model patterns of behavior that can persist in a steady state. The resulting notion of equilibrium applies to all extensive games with perfect information. Because the idea of backward induction is more limited, and the principles behind the notion of Nash equilibrium have been established in previous chapters, I begin by discussing the steady state approach. In games in which backward induction is well-defined, this approach turns out to lead to the backward induction outcome, so that there is no conflict between the two ideas.

5.3 Strategies and outcomes

5.3.1 Strategies

A key concept in the study of extensive games is that of a *strategy*. A player's strategy specifies the action the player chooses for *every* history after which it is her turn to move.

▶ DEFINITION 157.1 (*Strategy*) A **strategy** of player i in an extensive game with perfect information is a function that assigns to each history h after which it is player i's turn to move (i.e. $P(h) = i$, where P is the player function) an action in $A(h)$ (the set of actions available after h).

Consider the game in Figure 157.1.

- Player 1 moves only at the start of the game (i.e. after the empty history), when the actions available to her are C and D. Thus she has two strategies: one that assigns C to the empty history, and one that assigns D to the empty history.
- Player 2 moves after both the history C and the history D. After the history C the actions available to her are E and F, and after the history D the actions available to her are G and H. Thus a strategy of player 2 is a function that assigns either E or F to the history C, and either G or H to the history D. That is, player 2 has *four* strategies, which are shown in Figure 157.2.

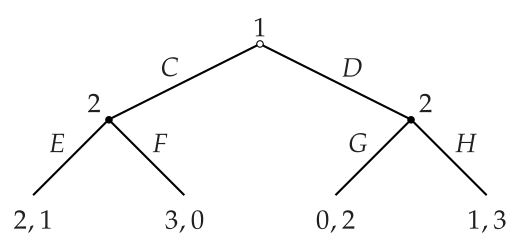


Figure 157.1 An extensive game with perfect information.

	Action assigned to history C	Action assigned to history D
Strategy 1	E	G
Strategy 2	E	H
Strategy 3	F	G
Strategy 4	F	H

Figure 157.2 The four strategies of player 2 in the game in Figure 157.1.

I refer to the strategies of player 1 in this game simply as C and D, and to the strategies of player 2 simply as EG, EH, FG, and FH. For many other finite games I use a similar shorthand: I write a player's strategy as a list of actions, one for each history after which it is the player's turn to move. In general I write the actions in the order in which they occur in the game, and, if they are available at the same "stage", from left to right as they appear in the diagram of the game. When the meaning of a list of actions is unclear, I explicitly give the history after which each action is taken.

Each of player 2's strategies in the game in Figure 157.1 may be interpreted as a plan of action or contingency plan: it specifies what player 2 does *if* player 1 chooses C, *and* what she does *if* player 1 chooses D. In every game, a player's strategy provides sufficient information to determine her *plan of action*: the actions she intends to take, *whatever* the other players do. In particular, if a player appoints an agent to play the game for her, and tells the agent her strategy, then the agent has enough information to carry out her wishes, *whatever* actions the other players take.

In some games some players' strategies are *more* than plans of action. Consider the game in Figure 158.1. Player 1 moves both at the start of the game and after the history (C, E). In each case she has two actions, so she has *four* strategies: CG (i.e. choose C at the start of the game and G after the history (C, E)), CH, DG, and DH. In particular, each strategy specifies an action after the history (C, E) *even if it specifies the action D at the beginning of the game*, in which case the history (C, E) does not occur! The point is that Definition 157.1 requires that a strategy of any player i specify an action for *every* history after which it is player i's turn to move, *even for histories that, if the strategy is followed, do not occur*.

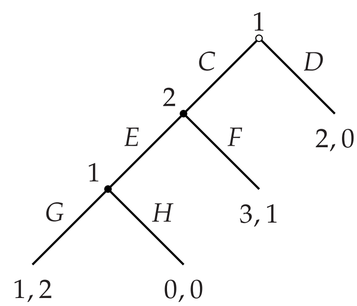


Figure 158.1 An extensive game in which player 1 moves both before and after player 2.

In view of this point and the fact that "strategy" is a synonym for "plan of action" in everyday language, you may regard the word "strategy" as inappropriate for the concept in Definition 157.1. You are right. You may also wonder why we cannot restrict attention to plans of action.

For the purposes of the notion of Nash equilibrium (discussed in the next section), we *could* in fact work with plans of action rather than strategies. But, as we shall see, the notion of Nash equilibrium for an extensive game is not satisfactory; the concept we adopt depends on the players' full strategies. When discussing this concept (in Section 5.5.4) I elaborate on the interpretation of a strategy. At the moment, you may think of a player's strategy as a plan of what to do, whatever the other players do, both if the player carries out her intended actions, and also if she makes mistakes. For example, we can interpret the strategy DG of player 1 in the game in Figure 158.1 to mean "I intend to choose D, but if I make a mistake and choose C instead then I will subsequently choose G". (Because the notion of Nash equilibrium depends only on plans of action, I could delay the definition of a strategy to the start of Section 5.5. I do not do so because the notion of a strategy is central to the study of extensive games, and its precise definition is much simpler than that of a plan of action.)

? EXERCISE 159.1 (Strategies in extensive games) What are the strategies of the players in the entry game (Example 153.2)? What are Rosa's strategies in the game in Exercise 154.2c?

5.3.2 Outcomes

A strategy profile determines the terminal history that occurs. Denote the strategy profile by s and the player function by P. At the start of the game player $P(\varnothing)$ moves. Her strategy is $s_{P(\varnothing)}$, and she chooses the action $s_{P(\varnothing)}(\varnothing)$. Denote this action by a^1. If the history a^1 is not terminal, player $P(a^1)$ moves next. Her strategy is $s_{P(a^1)}$, and she chooses the action $s_{P(a^1)}(a^1)$. Denote this action by a^2. If the history (a^1, a^2) is not terminal, then again the player function specifies whose turn it is to move, and that player's strategy specifies the action she chooses. The process continues until a terminal history is constructed. We refer to this terminal history as the **outcome of** s, and denote it $O(s)$.

In the game in Figure 158.1, for example, the outcome of the strategy pair (DG, E) is the terminal history D, and the outcome of (CH, E) is the terminal history (C, E, H).

Note that the outcome $O(s)$ of the strategy profile s depends only on the players' plans of action, not their full strategies. That is, to determine $O(s)$ we do *not* need to refer to any component of any player's strategy that specifies her actions after histories precluded by that strategy.

5.4 Nash equilibrium

As for strategic games, we are interested in notions of equilibrium that model the players' behavior in a steady state. That is, we look for patterns of behavior with the property that if every player knows every other player's behavior, she has no reason to change her own behavior. I start by defining a Nash equilibrium: a strategy profile from which no player wishes to deviate, given the other players' strategies. The definition is an adaptation of that of a Nash equilibrium in a strategic game (21.1).

► DEFINITION 159.2 (*Nash equilibrium of extensive game with perfect information*) The strategy profile s^* in an extensive game with perfect information is a **Nash equi-**

librium if, for every player i and every strategy r_i of player i, the terminal history $O(s^*)$ generated by s^* is at least as good according to player i's preferences as the terminal history $O(r_i, s^*_{-i})$ generated by the strategy profile (r_i, s^*_{-i}) in which player i chooses r_i while every other player j chooses s^*_j. Equivalently, for each player i,

$$u_i(O(s^*)) \geq u_i(O(r_i, s^*_{-i})) \text{ for every strategy } r_i \text{ of player } i,$$

where u_i is a payoff function that represents player i's preferences and O is the outcome function of the game.

One way to find the Nash equilibria of an extensive game in which each player has finitely many strategies is to list each player's strategies, find the outcome of each strategy profile, and analyze this information as for a strategic game. That is, we construct the following strategic game, known as the **strategic form** of the extensive game.

Players The set of players in the extensive game.

Actions Each player's set of actions is her set of strategies in the extensive game.

Preferences Each player's payoff to each action profile is her payoff to the terminal history generated by that action profile in the extensive game.

From Definition 159.2 we see that

> the set of Nash equilibria of any extensive game with perfect information is the set of Nash equilibria of its strategic form.

◆ EXAMPLE 160.1 (Nash equilibria of the entry game) In the entry game in Figure 154.1, the challenger has two strategies, *In* and *Out*, and the incumbent has two strategies, *Acquiesce* and *Fight*. The strategic form of the game is shown in Figure 160.1. We see that it has two Nash equilibria: (*In*, *Acquiesce*) and (*Out*, *Fight*). The first equilibrium is the pattern of behavior isolated by backward induction, discussed at the start of Section 5.2.2.

		Incumbent	
		Acquiesce	*Fight*
Challenger	*In*	2, 1	0, 0
	Out	1, 2	1, 2

Figure 160.1 The strategic form of the entry game in Figure 154.1.

In the second equilibrium the challenger always chooses *Out*. This strategy is optimal given the incumbent's strategy to fight in the event of entry. Further, the incumbent's strategy *Fight* is optimal given the challenger's strategy: the challenger chooses *Out*, so whether the incumbent plans to choose *Acquiesce* or *Fight* makes no difference to its payoff. Thus neither player can increase its payoff by choosing a different strategy, *given the other player's strategy*.

Thinking about the extensive game in this example raises a question about the Nash equilibrium $(Out, Fight)$ that does not arise when thinking about the strategic form: how does the challenger know that the incumbent will choose *Fight* if it enters? We interpret the strategic game to model a situation in which, whenever the challenger plays the game, it observes the incumbent's action, even if it chooses *Out*. By contrast, we interpret the extensive game to model a situation in which a challenger that always chooses *Out* never observes the incumbent's action, because the incumbent never moves. In a strategic game, the rationale for the Nash equilibrium condition that each player's strategy be optimal given the other players' strategies is that in a steady state, each player's experience playing the game leads her belief about the other players' actions to be correct. This rationale does not apply to the Nash equilibrium $(Out, Fight)$ of the (extensive) entry game, because a challenger who always chooses *Out* never observes the incumbent's action after the history *In*.

We can escape from this difficulty in interpreting a Nash equilibrium of an extensive game by considering a slightly perturbed steady state in which, on rare occasions, nonequilibrium actions are taken (perhaps players make mistakes, or deliberately experiment), and the perturbations allow each player eventually to observe every other player's action after *every* history. Given such perturbations, each player eventually learns the other players' entire strategies.

Interpreting the Nash equilibrium $(Out, Fight)$ as such a perturbed steady state, however, we run into another problem. On those (rare) occasions when the challenger enters, the subsequent behavior of the incumbent to fight is not a steady state in the remainder of the game: if the challenger enters, the incumbent is better off acquiescing than fighting. That is, the Nash equilibrium $(Out, Fight)$ does not correspond to a *robust* steady state of the extensive game.

Note that the extensive game embodies the assumption that the incumbent cannot commit, at the beginning of the game, to fight if the challenger enters; it is free to choose either *Acquiesce* or *Fight* in this event. If the incumbent *could* commit to fight in the event of entry then the analysis would be different. Such a commitment would induce the challenger to stay out, an outcome that the incumbent prefers. In the absence of the possibility of the incumbent's making a commitment, we might think of the its *announcing* at the start of the game that it intends to fight; but such a *threat* is not credible, because after the challenger enters the incumbent's only incentive is to acquiesce.

? EXERCISE 161.1 (Nash equilibria of extensive games) Find the Nash equilibria of the games in Exercise 154.2*a* and Figure 158.1. (When constructing the strategic form of each game, be sure to include *all* the strategies of each player.)

? EXERCISE 161.2 (Voting by alternating veto) Two people select a policy that affects them both by alternately vetoing policies until only one remains. First person 1 vetoes a policy. If more than one policy remains, person 2 then vetoes a policy. If more than one policy still remains, person 1 then vetoes another policy. The process continues until only one policy has not been vetoed. Suppose there are three possible policies, X, Y, and Z, person 1 prefers X to Y to Z, and person 2 prefers Z to Y to X. Model this situation as an extensive game and find its Nash equilibria.

5.5 Subgame perfect equilibrium

5.5.1 Definition

The notion of Nash equilibrium ignores the sequential structure of an extensive game; it treats strategies as choices made once and for all before play begins. Consequently, as we saw in the previous section, the steady state to which a Nash equilibrium corresponds may not be robust.

I now define a notion of equilibrium that models a robust steady state. This notion requires each player's strategy to be optimal, given the other players' strategies, not only at the start of the game, but after every possible history.

To define this concept, I first define the notion of a subgame. For any nonterminal history h, the *subgame* following h is the part of the game that remains after h has occurred. For example, the subgame following the history *In* in the entry game (Example 152.1) is the game in which the incumbent is the only player, and there are two terminal histories, *Acquiesce* and *Fight*.

▶ DEFINITION 162.1 (*Subgame*) Let Γ be an extensive game with perfect information, with player function P. For any nonterminal history h of Γ, the **subgame** $\Gamma(h)$ **following the history** h is the following extensive game.

Players The players in Γ.

Terminal histories The set of all sequences h' of actions such that (h, h') is a terminal history of Γ.

Player function The player $P(h, h')$ is assigned to each proper subhistory h' of a terminal history.

Preferences Each player prefers h' to h'' if and only if she prefers (h, h') to (h, h'') in Γ.

Note that the subgame following the initial history $\varnothing$ is the entire game. Every other subgame is called a *proper subgame*. Because there is a subgame for every nonterminal history, the number of subgames is equal to the number of nonterminal histories.

As an example, the game in Figure 157.1 has three nonterminal histories (the initial history, C, and D), and hence three subgames: the whole game (the part of the game following the initial history), the game following the history C, and the game following the history D. The two proper subgames are shown in Figure 163.1.

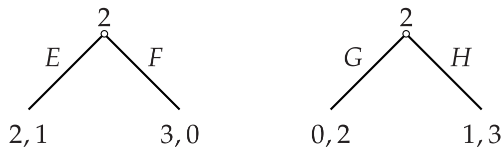


Figure 163.1 The two proper subgames of the extensive game in Figure 157.1.

The game in Figure 158.1 also has three nonterminal histories, and hence three subgames: the whole game, the game following the history C, and the game following the history (C, E). The two proper subgames are shown in Figure 163.2.

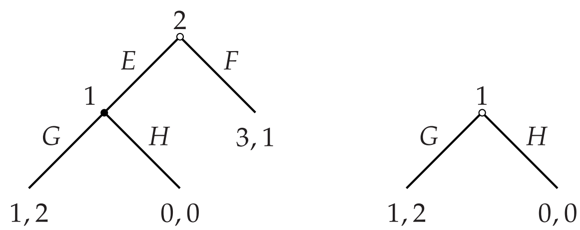


Figure 163.2 The two proper subgames of the extensive game in Figure 158.1.

? EXERCISE 163.1 (Subgames) Find all the subgames of the game in Exercise 154.2*c*.

In an equilibrium that corresponds to a perturbed steady state in which *every* history sometimes occurs, the players' behavior must correspond to a steady state in *every subgame*, not only in the whole game. Interpreting the actions specified by a player's strategy in a subgame to give the player's behavior if, possibly after a series of mistakes, that subgame is reached, this condition is embodied in the following informal definition.

> A *subgame perfect equilibrium* is a strategy profile s^* with the property that in no subgame can any player i do better by choosing a strategy different from s_i^*, given that every other player j adheres to s_j^*.

(Compare this definition with that of a Nash equilibrium of a strategic game, on page 19.)

For example, the Nash equilibrium $(Out, Fight)$ of the entry game (Example 152.1) is not a subgame perfect equilibrium because in the subgame following the history *In*, the strategy *Fight* is not optimal for the incumbent: *in this subgame*, the incumbent is better off choosing *Acquiesce* than it is choosing *Fight*. The Nash equilibrium $(In, Acquiesce)$ *is* a subgame perfect equilibrium: each player's strategy is optimal, given the other player's strategy, both in the whole game, and in the subgame following the history *In*.

To define the notion of subgame perfect equilibrium precisely, we need a new piece of notation. Let h be a history and s a strategy profile. Suppose that h occurs (even though it is not necessarily consistent with s), and *afterwards* the players adhere to the strategy profile s. Denote the resulting terminal history by $O_h(s)$. That is, $O_h(s)$ is the terminal history consisting of h followed by the outcome generated in the subgame following h by the strategy profile induced by s in the subgame. Note that for any strategy profile s, we have $O_\varnothing(s) = O(s)$ (where $\varnothing$, as always, denotes the initial history).

As an example, consider again the entry game. Let s be the strategy profile $(Out, Fight)$ and let h be the history In. If h occurs, and *afterwards* the players adhere to s, the resulting terminal history is $O_h(s) = (In, Fight)$.

▶ DEFINITION 164.1 (*Subgame perfect equilibrium*) The strategy profile s^* in an extensive game with perfect information is a **subgame perfect equilibrium** if, for every player i, every history h after which it is player i's turn to move (i.e. $P(h) = i$), and every strategy r_i of player i, the terminal history $O_h(s^*)$ generated by s^* after the history h is at least as good according to player i's preferences as the terminal history $O_h(r_i, s^*_{-i})$ generated by the strategy profile (r_i, s^*_{-i}) in which player i chooses r_i while every other player j chooses s^*_j. Equivalently, for every player i and every history h after which it is player i's turn to move,

$$u_i(O_h(s^*)) \geq u_i(O_h(r_i, s^*_{-i})) \text{ for every strategy } r_i \text{ of player } i,$$

where u_i is a payoff function that represents player i's preferences and $O_h(s)$ is the terminal history consisting of h followed by the sequence of actions generated by s after h.

The important point in this definition is that each player's strategy is required to be optimal for *every* history after which it is the player's turn to move, not only at the start of the game as in the definition of a Nash equilibrium (159.2).

5.5.2 Subgame perfect equilibrium and Nash equilibrium

In a subgame perfect equilibrium every player's strategy is optimal, in particular, after the initial history (put $h = \varnothing$ in the definition, and remember that $O_\varnothing(s) = O(s)$). Thus:

> Every subgame perfect equilibrium is a Nash equilibrium.

In fact, a subgame perfect equilibrium generates a Nash equilibrium in every subgame: if s^* is a subgame perfect equilibrium then, for any history h and player i, the strategy induced by s^*_i in the subgame following h is optimal given the strategies induced by s^*_{-i} in the subgame. Further, any strategy profile that generates a Nash equilibrium in every subgame is a subgame perfect equilibrium, so that we can give the following alternative definition.

> A *subgame perfect equilibrium* is a strategy profile that induces a Nash equilibrium in every subgame.

In a Nash equilibrium every player's strategy is optimal, given the other players' strategies, in the whole game. As we have seen, it may *not* be optimal in some subgames. I claim, however, that it *is* optimal in any subgame that is reached when the players follow their strategies. Given this claim, the significance of the requirement in the definition of a subgame perfect equilibrium that each player's strategy be optimal after every history, relative to the requirement in the definition of a Nash equilibrium, is that each player's strategy be optimal after histories that do not occur if the players follow their strategies (like the history *In* when the challenger's action is *Out* at the beginning of the entry game).

To show my claim, suppose that s^* is a Nash equilibrium of a game in which you are player i. Then your strategy s_i^* is optimal given the other players' strategies s_{-i}^*. When the other players follow their strategies, there comes a point (possibly the start of the game) when you have to move for the first time. Suppose that at this point you follow your strategy s_i^*; denote the action you choose by C. Now, after having chosen C, should you change your strategy in the rest of the game, given that the other players will continue to adhere to their strategies? No! If you could do better by changing your strategy after choosing C—say by switching to the strategy s_i' in the subgame—then you could have done better at the start of the game by choosing the strategy that chooses C and then follows s_i'. That is, if your plan is optimal, given the other players' strategies, at the start of the game, and you stick to it, then you never want to change your mind after play begins, as long as the other players stick to their strategies. (The general principle is known as the *Principle of Optimality* in dynamic programming.)

5.5.3 Examples

◆ EXAMPLE 165.1 (Entry game) Consider again the entry game of Example 152.1, which has two Nash equilibria, $(In, Acquiesce)$ and $(Out, Fight)$. The fact that the Nash equilibrium $(Out, Fight)$ is not a subgame perfect equilibrium follows from the formal definition as follows. For $s^* = (Out, Fight)$, $i =$ Incumbent, $r_i = Acquiesce$, and $h = In$, we have $O_h(s^*) = (In, Fight)$ and $O_h(r_i, s_{-i}^*) = (In, Acquiesce)$, so that the inequality in the definition is violated: $u_i(O_h(s^*)) = 0$ and $u_i(O_h(r_i, s_{-i}^*)) = 1$.

The Nash equilibrium $(In, Acquiesce)$ *is* a subgame perfect equilibrium because (*a*) it is a Nash equilibrium, so that at the start of the game the challenger's strategy *In* is optimal, given the incumbent's strategy *Acquiesce*, and (*b*) after the history *In*, the incumbent's strategy *Acquiesce* in the subgame is optimal. In the language of the formal definition, let $s^* = (In, Acquiesce)$.

- The challenger moves after one history, namely $h = \varnothing$. We have $O_h(s^*) = (In, Acquiesce)$ and hence for $i =$ challenger we have $u_i(O_h(s^*)) = 2$, whereas for the only other strategy of the challenger, $r_i = Out$, we have $u_i(O_h(r_i, s_{-i}^*)) = 1$.

- The incumbent moves after one history, namely $h = In$. We have $O_h(s^*) = (In, Acquiesce)$ and hence for i = incumbent we have $u_i(O_h(s^*)) = 1$, whereas for the only other strategy of the incumbent, $r_i = Fight$, we have $u_i(O_h(r_i, s^*_{-i})) = 0$.

Every subgame perfect equilibrium is a Nash equilibrium, so we conclude that the game has a unique subgame perfect equilibrium, $(In, Acquiesce)$.

◆ EXAMPLE 166.1 (Variant of entry game) Consider the variant of the entry game in which the incumbent is indifferent between fighting and acquiescing if the challenger enters (see Figure 156.1). This game, like the original game, has two Nash equilibria, $(In, Acquiesce)$ and $(Out, Fight)$. But now *both* of these equilibria are subgame perfect equilibria, because after the history *In* both *Fight* and *Acquiesce* are optimal for the incumbent.

In particular, the game has a steady state in which every challenger always chooses *In* and every incumbent always chooses *Acquiesce*. If you, as the challenger, were playing the game for the first time, you would probably regard the action *In* as "risky", because after the history *In* the incumbent is indifferent between *Acquiesce* and *Fight*, and you prefer the terminal history *Out* to the terminal history $(In, Fight)$. Indeed, as discussed in Section 5.2.2, backward induction does not yield a clear solution of this game. But the subgame perfect equilibrium $(In, Acquiesce)$ corresponds to a perfectly reasonable steady state. If you had played the game hundreds of times against opponents drawn from the same population, and on every occasion your opponent had chosen *Acquiesce*, you could reasonably expect your next opponent to choose *Acquiesce*, and thus optimally choose *In*.

? EXERCISE 166.2 (Checking for subgame perfect equilibria) Which of the Nash equilibria of the game in Figure 158.1 are subgame perfect?

5.5.4 Interpretation

A Nash equilibrium of a strategic game corresponds to a steady state in an idealized setting in which the participants in each play of the game are drawn randomly from a collection of populations (see Section 2.6). The idea is that each player's long experience playing the game leads her to correct beliefs about the other players' actions; given these beliefs her equilibrium action is optimal.

A subgame perfect equilibrium of an extensive game corresponds to a slightly perturbed steady state, in which all players, on rare occasions, take nonequilibrium actions, so that after long experience each player forms correct beliefs about the other players' entire strategies, and thus knows how the other players will behave in every subgame. Given these beliefs, no player wishes to deviate from her strategy either at the start of the game or after *any* history.

This interpretation of a subgame perfect equilibrium, like the interpretation of a Nash equilibrium as a steady state, does not require a player to know the other players' preferences, or to think about the other players' rationality. It entails interpreting a strategy as a plan specifying a player's actions not only after histories consistent with the strategy, but also after histories that result when the player chooses arbitrary alternative actions, perhaps because she makes mistakes or deliberately experiments.

The subgame perfect equilibria of some extensive game can be given other interpretations. In some cases, one alternative interpretation is particularly attractive. Consider an extensive game with perfect information in which each player has a unique best action at every history after which it is her turn to move, and the horizon is finite. In such a game, a player who knows the other players' preferences and knows that the other players are rational can use backward induction to deduce her optimal strategy, as discussed in Section 5.2.2. Thus we can interpret a subgame perfect equilibrium as the outcome of the players' rational calculations about each other's strategies.

This interpretation of a subgame perfect equilibrium entails an interpretation of a strategy different from the one that fits the steady state interpretation. Consider, for example, the game in Figure 158.1. When analyzing this game, player 1 must consider the consequences of choosing C. Thus she must think about player 2's action after the history C, and hence must form a belief about what player 2 thinks she (player 1) will do after the history (C, E). The component of her strategy that specifies her action after this history reflects this belief. For instance, the strategy DG means that player 1 chooses D at the start of the game and believes that were she to choose C, player 2 would believe that after the history (C, E) she would choose G. In an arbitrary game, the interpretation of a subgame perfect equilibrium as the outcome of the players' rational calculations about each other's strategies entails interpreting the components of a player's strategy that assign actions to histories inconsistent with other parts of the strategy as specifying the player's belief about the other players' beliefs about what the player will do if one of these histories occurs.

This interpretation of a subgame perfect equilibrium is not free of difficulties, which are discussed in Section 7.7. Further, the interpretation is not tenable in games in which some player has more than one optimal action after some history, or in the more general extensive games considered in Section 7.1 and Chapter 10. Nevertheless, in some of the games studied in this chapter and the next it is an appealing alternative to the steady state interpretation. Further, an extension of the procedure of backward induction can be used to find all subgame perfect equilibria of finite horizon games, as we shall see in the next section. (This extension cannot be given an appealing behavioral interpretation in games in which some player has more than one optimal action after some history.)

5.6 Finding subgame perfect equilibria of finite horizon games: backward induction

We found the subgame perfect equilibria of the games in Examples 165.1 and 166.1 by finding the Nash equilibria of the games and checking whether each of these equilibria is subgame perfect. In a game with a finite horizon the set of subgame perfect equilibria may be found more directly by using an extension of the procedure of backward induction discussed briefly in Section 5.2.2.

Define the *length of a subgame* to be the length of the longest history in the subgame. (The lengths of the subgames in Figure 163.2, for example, are 2 and 1.) The procedure of backward induction works as follows. We start by finding the optimal actions of the players who move in the subgames of length 1 (the "last" subgames). Then, taking these actions as given, we find the optimal actions of the players who move first in the subgames of length 2. We continue working back to the beginning of the game, at each stage k finding the optimal actions of the players who move at the start of the subgames of length k, given the optimal actions we have found in all shorter subgames.

At each stage k of this procedure, the optimal actions of the players who move at the start of the subgames of length k are easy to determine: they are simply the actions that yield the players the highest payoffs, given the optimal actions in all shorter subgames.

Consider, for example, the game in Figure 168.1.

- First consider subgames of length 1. The game has two such subgames, in both of which player 2 moves. In the subgame following the history C, player 2's optimal action is E, and in the subgame following the history D, her optimal action is H.

- Now consider subgames of length 2. The game has one such subgame, namely the entire game, at the start of which player 1 moves. Given the optimal actions in the subgames of length 1, player 1's choosing C at the start of the game yields her a payoff of 2, whereas her choosing D yields her a payoff of 1. Thus player 1's optimal action at the start of the game is C.

The game has no subgame of length greater than 2, so the procedure of backward induction yields the strategy pair (C, EH).

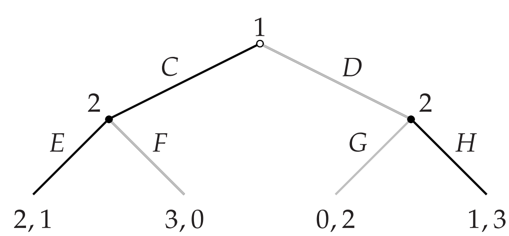


Figure 168.1 A game illustrating the procedure of backward induction. The actions selected by backward induction are indicated in black.

As another example, consider the game in Figure 158.1. We first deduce that in the subgame of length 1 following the history (C, E), player 1 chooses G; then that at the start of the subgame of length 2 following the history C, player 2 chooses E;

then that at the start of the whole game, player 1 chooses D. Thus the procedure of backward induction in this game yields the strategy pair (DG, E).

In any game in which this procedure selects a single action for the player who moves at the start of each subgame, the strategy profile thus selected is the unique subgame perfect equilibrium of the game. (You should find this result very plausible, though a complete proof is not trivial.)

What happens in a game in which at the start of some subgames more than one action is optimal? In such a game an extension of the procedure of backward induction locates all subgame perfect equilibrium. This extension traces back *separately* the implications for behavior in the longer subgames of *every combination* of optimal actions in the shorter subgames.

Consider, for example, the game in Figure 170.1.

- The game has three subgames of length one, in each of which player 2 moves. In the subgames following the histories C and D, player 2 is indifferent between her two actions. In the subgame following the history E, player 2's unique optimal action is K. Thus there are *four* combinations of player 2's optimal actions in the subgames of length 1: FHK, FIK, GHK, and GIK (where the first component in each case is player 2's action after the history C, the second component is her action after the history D, and the third component is her action after the history E).

- The game has a single subgame of length two, namely the whole game, in which player 1 moves first. We now consider player 1's optimal action in this game for *every combination* of the optimal actions of player 2 in the subgames of length 1.

  - For the combinations FHK and FIK of optimal actions of player 2, player 1's optimal action at the start of the game is C.
  - For the combination GHK of optimal actions of player 2, the actions C, D, and E are all optimal for player 1.
  - For the combination GIK of optimal actions of player, player 1's optimal action at the start of the game is D.

Thus the strategy pairs isolated by the procedure are (C, FHK), (C, FIK), (C, GHK), (D, GHK), (E, GHK), and (D, GIK).

The procedure, which for simplicity I refer to simply as **backward induction**, may be described compactly for an arbitrary game as follows.

- Find, for each subgame of length 1, the set of optimal actions of the player who moves first. Index the subgames by j, and denote by $S_j^*(1)$ the set of optimal actions in subgame j. (If the player who moves first in subgame j has a unique optimal action, then $S_j^*(1)$ contains a single action.)

- For each combination of actions consisting of one from each set $S_j^*(1)$, find, for each subgame of length two, the set of optimal actions of the player who

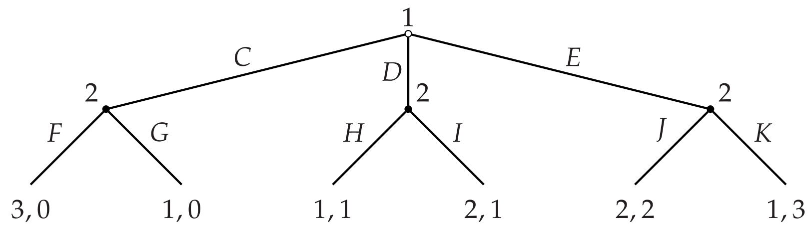


Figure 170.1 A game in which the first-mover in some subgames has multiple optimal actions.

moves first. The result is a set of strategy profiles for each subgame of length two. Denote by $S^*_\ell(2)$ the set of strategy profiles in subgame ℓ.

- Continue by examining successively longer subgames until reaching the start of the game. At each stage k, for each combination of strategy profiles consisting of one from each set $S^*_p(k-1)$ constructed in the previous stage, find, for each subgame of length k, the set of optimal actions of the player who moves first, and hence a set of strategy profiles for each subgame of length k.

The set of strategy profiles that this procedure yields for the whole game is the set of subgame perfect equilibria of the game.

■ PROPOSITION 170.1 (Subgame perfect equilibrium of finite horizon games and backward induction) *The set of subgame perfect equilibria of a finite horizon extensive game with perfect information is equal to the set of strategy profiles isolated by the procedure of backward induction.*

You should find this result, like my claim for games in which the player who moves at the start of every subgame has a single optimal action, very plausible, though again a complete proof is not trivial.

In the terminology of my description of the general procedure, the analysis for the game in Figure 170.1 is as follows. Number the subgames of length one from left to right. Then we have $S^*_1(1) = \{F, G\}$, $S^*_2(1) = \{H, I\}$, and $S^*_3(1) = \{K\}$. There are four lists of actions consisting of one action from each set: *FHK*, *FIK*, *GHK*, and *GIK*. For *FHK* and *FIK*, the action C of player 1 is optimal at the start of the game; for *GHK* the actions C, D, and E are all optimal; and for *GIK* the action D is optimal. Thus the set $S^*(2)$ of strategy profiles consists of (C, FHK), (C, FIK), (C, GHK), (D, GHK), (E, GHK), and (D, GIK). There are no longer subgames, so this set of strategy profiles is the set of subgame perfect equilibria of the game.

Each example I have presented so far in this section is a finite game—that is, a game that not only has a finite horizon, but also a finite number of terminal histories. In such a game, the player who moves first in any subgame has finitely many actions; at least one action is optimal. Thus in such a game the procedure of backward induction isolates at least one strategy profile. Using Proposition 170.1, we conclude that every finite game has a subgame perfect equilibrium.

■ PROPOSITION 171.1 (Existence of subgame perfect equilibrium) *Every finite extensive game with perfect information has a subgame perfect equilibrium.*

Note that this result does *not* claim that a finite extensive game has a *single* subgame perfect equilibrium. (As we have seen, the game in Figure 170.1, for example, has more than one subgame perfect equilibrium.)

A finite horizon game in which some player does not have finitely many actions after some history may or may not possess a subgame perfect equilibrium. A simple example of a game that does not have a subgame perfect equilibrium is the trivial game in which a single player chooses a number *less than* 1 and receives a payoff equal to the number she chooses. There is no greatest number less than one, so the single player has no optimal action, and thus the game has no subgame perfect equilibrium.

? EXERCISE 171.2 (Finding subgame perfect equilibria) Find the subgame perfect equilibria of the games in parts *a* and *c* of Exercise 154.2, and in Figure 171.1.

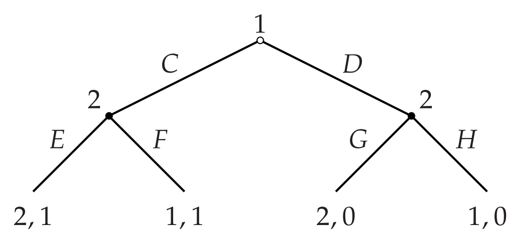


Figure 171.1 One of the games for Exercise 171.2.

? EXERCISE 171.3 (Voting by alternating veto) Find the subgame perfect equilibria of the game in Exercise 161.2. Does the game have any Nash equilibrium that is not a subgame perfect equilibrium? Is any outcome generated by a Nash equilibrium not generated by any subgame perfect equilibrium? Consider variants of the game in which player 2's preferences may be different from those specified in Exercise 161.2. Are there any preferences for which the outcome in a subgame perfect equilibrium of the game in which player 1 moves first differs from the outcome in a subgame perfect equilibrium of the game in which player 2 moves first?

? EXERCISE 171.4 (Burning a bridge) Army 1, of country 1, must decide whether to attack army 2, of country 2, which is occupying an island between the two countries. In the event of an attack, army 2 may fight, or retreat over a bridge to its mainland. Each army prefers to occupy the island than not to occupy it; a fight is the worst outcome for both armies. Model this situation as an extensive game with perfect information and show that army 2 can increase its subgame perfect equilibrium payoff (and reduce army 1's payoff) by burning the bridge to its mainland, eliminating its option to retreat if attacked.

? EXERCISE 172.1 (Sharing heterogeneous objects) A group of n people have to share k objects that the value differently. Each person assigns values to the objects; no one assigns the same value to two different objects. Each person evaluates a set of objects according to the sum of the values she assigns to the objects in the set. The following procedure is used to share the objects. The players are ordered 1 through n. Person 1 chooses an object, then person 2 does so, and so on; if $k > n$, then after person n chooses an object, person 1 chooses a second object, then person 2 chooses a second object, and so on. Objects are chosen until none remain. (In Canada and the USA professional sports teams use a similar procedure to choose new players.) Denote by $G(n,k)$ the extensive game that models this procedure. If $k \leq n$ then obviously $G(n,k)$ has a subgame perfect equilibrium in which each player's strategy is to choose her favorite object among those remaining when her turn comes. Show that if $k > n$ then $G(n,k)$ may have no subgame perfect equilibrium in which person 1 chooses her favorite object on the first round. (You can give an example in which $n = 2$ and $k = 3$.) Now fix $n = 2$. Define x_k to be the object least preferred by the person who does *not* choose at stage k (i.e. who does not choose the last object); define x_{k-1} to be the object, among all those except x_k, least preferred by the person who does *not* choose at stage $k-1$. Similarly, for any j with $2 \leq j \leq k$, given $x_j, \ldots, x_k$, define x_{j-1} to be the object, among all those excluding $\{x_j, \ldots, x_k\}$, least preferred by the person who does *not* choose at stage $j-1$. Show that the game $G(2,3)$ has a subgame perfect equilibrium in which for every $j = 1, \ldots, k$ the object x_j is chosen at stage j. (This result is true for $G(2,k)$ for all values of k.) If $n \geq 3$ then interestingly a person may be better off in all subgame perfect equilibria of $G(n,k)$ when she comes later in the ordering of players. (An example, however, is difficult to construct; one is given in Brams and Straffin (1979).)

The next exercise shows how backward induction can cause a relatively minor change in the way in which a game ends to reverberate to the start of the game, leading to a very different action for the first-mover.

?? EXERCISE 172.2 (An entry game with a financially-constrained firm) An incumbent in an industry faces the possibility of entry by a challenger. First the challenger chooses whether or not to enter. If it does not enter, neither firm has any further action; the incumbent's payoff is TM (it obtains the profit M in each of the following $T \geq 1$ periods) and the challenger's payoff is 0. If the challenger enters, it pays the entry cost $f > 0$, and in each of T periods the incumbent first commits to fight or cooperate with the challenger in that period, then the challenger chooses whether to stay in the industry or to exit. (Note that the order of the firms' moves within a period differs from that in the game in Example 152.1.) If, in any period, the challenger stays in, each firm obtains in that period the profit $-F < 0$ if the incumbent fights and $C > \max\{F, f\}$ if it cooperates. If, in any period, the challenger exits, both firms obtain the profit zero in that period (regardless of the incumbent's action); the incumbent obtains the profit $M > 2C$ and the challenger the profit

0 in every subsequent period. Once the challenger exits, it cannot subsequently re-enter. Each firm cares about the sum of its profits.

a. Find the subgame perfect equilibria of the extensive game that models this situation.

b. Consider a variant of the situation, in which the challenger is constrained by its financial war chest, which allows it to survive at most $T-2$ fights. Specifically, consider the game that differs from the one in part *a* only in that the history in which the challenger enters, in each of the following $T-2$ periods the incumbent fights and the challenger stays in, and in period $T-1$ the incumbent fights, is a terminal history (the challenger has to exit), in which the incumbent's payoff is M (it is the only firm in the industry in the last period) and the challenger's payoff is $-f$. Find the subgame perfect equilibria of this game.

◆ EXAMPLE 173.1 (Dollar auction) Consider an auction in which an object is sold to the highest bidder, but *both* the highest bidder *and* the second highest bidder pay their bids to the auctioneer. When such an auction is conducted and the object is a dollar, the outcome is sometimes that the object is sold at a price *greater* than a dollar. (Shubik writes that "A total of payments between three and five dollars is not uncommon" (1971, 110).) Obviously such an outcome is inconsistent with a subgame perfect equilibrium of an extensive game that models the auction: every participant has the option of not bidding, so that in no subgame perfect equilibrium can anyone's payoff be negative.

Why, then, do such outcomes occur? Suppose that there are two participants, and that both start bidding. If the player making the lower bid thinks that making a bid above the other player's bid will induce the other player to quit, she may be better off doing so than stopping bidding. For example, if the bids are currently \$0.50 and \$0.51, the player bidding \$0.50 is better off bidding \$0.52 *if* doing so induces the other bidder to quit, because she then wins the dollar and obtains a payoff of \$0.48, rather than losing \$0.50. The same logic applies even if the bids are greater than \$1.00, as long as they do not differ by more than \$1.00. If, for example, they are currently \$2.00 and \$2.01, then the player bidding \$2.00 loses only \$1.02 if a bid of \$2.02 induces her opponent to quit, whereas she loses \$2.00 if she herself quits. That is, in subgames in which bids have been made, the player making the second highest bid may optimally beat a bid that exceeds \$1.00, depending on the other players' strategies and the difference between the top two bids. (When discussing outcomes in which the total payment to the auctioneer exceeds \$1, Shubik remarks that "In playing this game, a large crowd is desirable ... the best time is during a party when spirits are high and the propensity to calculate does not settle in until at least two bids have been made" (1971, 109).)

In the next exercise you are asked to find the subgame perfect equilibria of an extensive game that models a simple example of such an auction.

? EXERCISE 173.2 (Dollar auction) An object that two people each value at v (a positive integer) is sold in an auction. In the auction, the people alternately have the opportunity to bid; a bid must be a positive integer greater than the previous bid. (In the situation that gives the game its name, v is 100 cents.) On her turn, a player may pass rather than bid, in which case the game ends and the other player receives the object; *both* players pay their last bids (if any). (If player 1 passes initially, for example, player 2 receives the object and makes no payment; if player 1 bids 1, player 2 bids 3, and then player 1 passes, player 2 obtains the object and pays 3, and player 1 pays 1.) Each person's wealth is w, which exceeds v; neither player may bid more than her wealth. For $v = 2$ and $w = 3$ model the auction as an extensive game and find its subgame perfect equilibria. (A much more ambitious project is to find all subgame perfect equilibria for arbitrary values of v and w.)

In all the extensive games studied so far in this chapter, each player has available finitely many actions whenever she moves. The next example shows how the procedure of backward induction may be used to find the subgame perfect equilibria of games in which a continuum of actions is available after some histories.

◆ EXAMPLE 174.1 (A synergistic relationship) Consider a variant of the situation in Example 37.1, in which two individuals are involved in a synergistic relationship. Suppose that the players choose their effort levels sequentially, rather than simultaneously. First individual 1 chooses her effort level a_1, then individual 2 chooses her effort level a_2. An effort level is a nonnegative number, and individual i's preferences (for $i = 1, 2$) are represented by the payoff function $a_i(c + a_j - a_i)$, where j is the other individual and $c > 0$ is a constant.

To find the subgame perfect equilibria, we first consider the subgames of length 1, in which individual 2 chooses a value of a_2. Individual 2's optimal action after the history a_1 is her best response to a_1, which we found to be $\frac{1}{2}(c + a_1)$ in Example 37.1. Thus individual 2's strategy in any subgame perfect equilibrium is the function that associates with each history a_1 the action $\frac{1}{2}(c + a_1)$.

Now consider individual 1's action at the start of the game. Given individual 2's strategy, individual 1's payoff if she chooses a_1 is $a_1(c + \frac{1}{2}(c + a_1) - a_1)$, or $\frac{1}{2}a_1(3c - a_1)$. This function is a quadratic that is zero when $a_1 = 0$ and when $a_1 = 3c$, and reaches a maximum in between. Thus individual 1's optimal action at the start of the game is $a_1 = \frac{3}{2}c$.

We conclude that the game has a unique subgame perfect equilibrium, in which individual 1's strategy is $a_1 = \frac{3}{2}c$ and individual 2's strategy is the function that associates with each history a_1 the action $\frac{1}{2}(c + a_1)$. The outcome of the equilibrium is that individual 1 chooses $a_1 = \frac{3}{2}c$ and individual 2 chooses $a_2 = \frac{5}{4}c$.

? EXERCISE 174.2 (Firm–union bargaining) A firm's output is $L(100 - L)$ when it uses $L \leq 50$ units of labor, and 2500 when it uses $L > 50$ units of labor. The price of output is 1. A union that represents workers presents a wage demand (a nonnegative number w), which the firm either accepts or rejects. If the firm accepts the demand, it chooses the number L of workers to employ (which you should take to be a continuous variable, not an integer); if it rejects the demand, no production takes place ($L = 0$). The firm's preferences are represented by its profit; the union's preferences are represented by the value of wL.

a. Formulate this situation as an extensive game with perfect information.

b. Find the subgame perfect equilibrium (equilibria?) of the game.

c. Is there an outcome of the game that both parties prefer to any subgame perfect equilibrium outcome?

d. Find a Nash equilibrium for which the outcome differs from any subgame perfect equilibrium outcome.

? EXERCISE 175.1 (The "rotten kid theorem") A child's action a (a number) affects both her own private income $c(a)$ and her parent's income $p(a)$; for all values of a we have $c(a) < p(a)$. The child is selfish: she cares only about the amount of money she has. Her loving parent cares both about how much money she has and how much her child has. Specifically, her preferences are represented by a payoff equal to the smaller of the amount of money she has and the amount of money her child has. The parent may transfer money to the child. First the child takes an action, then the parent decides how much money to transfer. Model this situation as an extensive game and show that in a subgame perfect equilibrium the child takes an action that maximizes the sum of her private income and her parent's income. (In particular, the child's action does not maximize her own private income. The result is not limited to the specific form of the parent's preferences, but holds for any preferences with the property that a parent who is allocating a fixed amount x of money between herself and her child wishes to give more to the child when x is larger.)

? EXERCISE 175.2 (Comparing simultaneous and sequential games) The set of actions available to player 1 is A_1; the set available to player 2 is A_2. Player 1's preferences over pairs (a_1, a_2) are represented by the payoff $u_1(a_1, a_2)$, and player 2's preferences are represented by the payoff $u_2(a_1, a_2)$. Compare the Nash equilibria (in pure strategies) of the strategic game in which the players choose actions simultaneously with the subgame perfect equilibria of the extensive game in which player 1 chooses an action, then player 2 does so. (For each history a_1 in the extensive game, the set of actions available to player 2 is A_2.)

a. Show that if, for every value of a_1, there is a unique member of A_2 that maximizes $u_2(a_1, a_2)$, then in every subgame perfect equilibrium of the extensive game, player 1's payoff is at least equal to her highest payoff in any Nash equilibrium of the strategic game.

b. Show that player 2's payoff in every subgame perfect equilibrium of the extensive game may be higher than her highest payoff in any Nash equilibrium of the strategic game.

c. Show that if for some values of a_1 more than one member of A_2 maximizes $u_2(a_1, a_2)$, then the extensive game may have a subgame perfect equilibrium in which player 1's payoff is less than her payoff in all Nash equilibria of the strategic game.

(For parts *b* and *c* you can give examples in which both A_1 and A_2 contain two actions.)

TICKTACKTOE, CHESS, AND RELATED GAMES

Ticktacktoe, chess, and related games may be modeled as extensive games with perfect information. (A history is a sequence of moves and each player prefers to win than to tie than to lose.) Both ticktacktoe and chess may be modeled as finite games, so by Proposition 171.1 each game has a subgame perfect equilibrium. (The official rules of chess allow indefinitely long sequences of moves, but the game seems to be well modeled by an extensive game in which a draw is declared automatically if a position is repeated three times, rather than a player having the option of declaring a draw in this case, as in the official rules.) The subgame perfect equilibria of ticktacktoe are of course known, whereas those of chess are not (yet).

Ticktacktoe and chess are "strictly competitive" games (Definition 339.1): in every outcome, either one player loses and the other wins, or the players draw. A result in a later chapter implies that for such a game all Nash equilibria yield the same outcome (Corollary 342.1). Further, a player's Nash equilibrium strategy yields *at least* her equilibrium payoff, regardless of the other players' strategies (Proposition 341.1a). (The same is definitely not true for an arbitrary game that is not strictly competitive: look, for example, at the game in Figure 29.1.) Because any subgame perfect equilibrium is a Nash equilibrium, the same is true for subgame perfect equilibrium strategies.

We conclude that in ticktacktoe and chess, either (*a*) one of the players has a strategy that guarantees she wins, or (*b*) each player has a strategy that guarantees at worst a draw.

In ticktacktoe, of course, we know that (*b*) is true. Chess is more subtle. In particular, it is not known whether White has a strategy that guarantees it wins, or Black has a strategy that guarantees it wins, or each player has a strategy that guarantees at worst a draw. The empirical evidence suggests that Black does not have a winning strategy, but this result has not been proved. When will a subgame perfect equilibrium of chess be found? (The answer "never" underestimates human ingenuity!)

? EXERCISE 176.1 (Subgame perfect equilibria of ticktacktoe) Ticktacktoe has subgame perfect equilibria in which the first player puts her first X in a corner. The second player's move is the same in all these equilibria. What is it?

? EXERCISE 176.2 (Toetacktick) Toetacktick is a variant of ticktacktoe in which a player who puts three marks in a line *loses* (rather than wins). Find a strategy of the first-mover that guarantees that she does not lose. (If fact, in all subgame perfect equilibria the game is a draw.)

? EXERCISE 177.1 (Three Men's Morris, or Mill) The ancient game of "Three Men's Morris" is played on a ticktacktoe board. Each player has three counters. The players move alternately. On each of her first three turns, a player places a counter on an unoccupied square. On each subsequent move, a player may move a counter to an adjacent square (vertically or horizontally, but not diagonally). The first player whose counters are in a row (vertically, horizontally, or diagonally) wins. Find a subgame perfect equilibrium strategy of player 1, and the equilibrium outcome.

Notes

The notion of an extensive game is due to von Neumann and Morgenstern (1944). Kuhn (1950, 1953) suggested the formulation described in this chapter. The description of an extensive game in terms of histories was suggested by Ariel Rubinstein. The notion of subgame perfect equilibrium is due to Selten (1965). Proposition 171.1 is due to Kuhn (1953). The interpretation of a strategy when a subgame perfect equilibrium is interpreted as the outcome of the players' reasoning about each others' rational actions is due to Rubinstein (1991). The principle of optimality in dynamic programming is discussed by Bellman (1957, 83), for example.

The procedure in Exercises 161.2 and 171.3 was first studied by Mueller (1978) and Moulin (1981). The idea in Exercise 171.4 goes back at least to Sun-tzu, who, in *The art of warfare* (probably written between 500BC and 300BC), advises "in surrounding the enemy, leave him a way out; do not press an enemy that is cornered" (end of Ch. 7; see, for example, Sun-tzu (1993, 132)). (That is, if no bridge exists in the situation described in the exercise, army 1 should build one.) Schelling (1966, 45) quotes Sun-tzu and gives examples of the strategy's being used in antiquity. My formulation of the exercise comes from Tirole (1988, 316). The model in Exercise 172.1 is studied by Kohler and Chandrasekaran (1971) and Brams and Straffin (1979). The game in Exercise 172.2 is based on Benoît (1984, Section 1). The dollar auction (Exercise 173.2) was introduced into the literature by Shubik (1971). Some of its subgame perfect equilibria, for arbitrary values of v and w, are studied by O'Neill (1986) and Leininger (1989); see also Taylor (1995, Chs. 1 and 6). Poundstone (1992, 257–272) writes informally about the game and its possible applications. The result in Exercise 175.1 is due to Becker (1974); see also Bergstrom (1989). The first formal study of chess is Zermelo (1913); see Schwalbe and Walker (2000) for a discussion of this paper and related work. Exercises 176.1, 176.2, and 177.1 are taken from Gardner (1959, Ch. 4), which includes several other intriguing examples.

6 Extensive Games with Perfect Information: Illustrations

Prerequisite: Chapter 5.

6.1 Introduction

THE first three sections of this chapter illustrate the notion of subgame perfect equilibrium in games in which the longest history has length two or three. The last section studies a game with an arbitrary finite horizon. Games with infinite horizons are studied in Chapters 16 and 14.

6.2 The ultimatum game and the holdup game

6.2.1 The ultimatum game

Bargaining over the division of a pie may naturally be modeled as an extensive game. Chapter 16 studies several such models. Here I analyze a very simple game that is the basis of one of the richer models studied in the later chapter. The game is so simple, in fact, that you may not initially think of it as a model of "bargaining".

Two people use the following procedure to split $\$c$. Person 1 offers person 2 an amount of money up to $\$c$. If 2 accepts this offer then 1 receives the remainder of the $\$c$. If 2 rejects the offer then *neither* person receives any payoff. Each person cares *only* about the amount of money she receives, and (naturally!) prefers to receive as much as possible.

Assume that the amount person 1 offers can be any number, not necessarily an integral number of cents. Then the following extensive game, known as the **ultimatum game**, models the procedure.

Players The two people.

Terminal histories The set of sequences (x, Z), where x is a number with $0 \leq x \leq c$ (the amount of money that person 1 offers to person 2) and Z is either Y ("yes, I accept") or N ("no, I reject").

Player function $P(\varnothing) = 1$ and $P(x) = 2$ for all x.

Preferences Each person's preferences are represented by payoffs equal to the amounts of money she receives. For the terminal history (x, Y) person 1 receives $c - x$ and person 2 receives x; for the terminal history (x, N) each person receives 0.

This game has a finite horizon, so we can use backward induction to find its subgame perfect equilibria. First consider the subgames of length 1, in which person 2 either accepts or rejects an offer of person 1. For every possible offer of person 1, there is such a subgame. In the subgame that follows an offer x of person 1 for which $x > 0$, person 2's optimal action is to accept (if she rejects, she gets nothing). In the subgame that follows the offer $x = 0$, person 2 is indifferent between accepting and rejecting. Thus in a subgame perfect equilibrium person 2's strategy either accepts all offers (including 0), or accepts all offers $x > 0$ and rejects the offer $x = 0$.

Now consider the whole game. For each possible subgame perfect equilibrium strategy of person 2, we need to find the optimal strategy of person 1.

- If person 2 accepts all offers (including 0), then person 1's optimal offer is 0 (which yields her the payoff $\$c$).
- If person 2 accepts all offers except zero, then *no* offer of person 1 is optimal! No offer $x > 0$ is optimal, because the offer $x/2$ (for example) is better, given that person 2 accept both offers. And an offer of 0 is not optimal because person 2 rejects it, leading to a payoff of 0 for person 1, who is thus better off offering any positive amount less than $\$c$.

We conclude that the only subgame perfect equilibrium of the game is the strategy pair in which person 1 offers 0 and person 2 accepts all offers. In this equilibrium, person 1's payoff is $\$c$ and person 2's payoff is zero.

This one-sided outcome is a consequence of the one-sided structure of the game. If we allow person 2 to make a counteroffer after rejecting person 1's opening offer (and possibly allow further responses by both players), so that the model corresponds more closely to a "bargaining" situation, then under some circumstances the outcome is less one-sided. (An extension of this type is explored in Chapter 16.)

? EXERCISE 180.1 (Nash equilibria of the ultimatum game) Find the values of x for which there is a Nash equilibrium of the ultimatum game in which person 1 offers x.

? EXERCISE 180.2 (Subgame perfect equilibria of the ultimatum game with indivisible units) Find the subgame perfect equilibria of the variant of the ultimatum game in which the amount of money is available only in multiples of a cent.

? EXERCISE 180.3 (Dictator game and impunity game) The "dictator game" differs from the ultimatum game only in that person 2 does not have the option to reject person 1's offer (and thus has no strategic role in the game). The "impunity game" differs from the ultimatum game only in that person 1's payoff when person 2 rejects any offer x is $c - x$, rather than 0. (The game is named for the fact that person 2 is unable to "punish" person 1 for making a low offer.) Find the subgame perfect equilibria of each game.

?? EXERCISE 181.1 (Variants of ultimatum game and impunity game with equity-conscious players) Consider variants of the ultimatum game and impunity game in which each person cares not only about the amount of money she receives, but also about the equity of the allocation. Specifically, suppose that person i's preferences are represented by the payoff function given by $u_i(x_1, x_2) = x_i - \beta_i|x_1 - x_2|$, where x_i is the amount of money person i receives, $\beta_i > 0$, and, for any number z, $|z|$ denotes the absolute value of z (i.e. $|z| = z$ if $z > 0$ and $|z| = -z$ if $z < 0$). Find the set of subgame perfect equilibria of each game and compare them. Are there any values of β_1 and β_2 for which an offer is rejected in equilibrium? (An interesting further variant of the ultimatum game in which person 1 is uncertain about the value of β_2 is considered in Exercise 222.2.)

EXPERIMENTS ON THE ULTIMATUM GAME

The sharp prediction of the notion of subgame perfect equilibrium in the ultimatum game lends itself to experimental testing. The first test was conducted in the late 1970s among graduate students of economics in a class at the University of Cologne (in what was then West Germany). The amount c available varied among the games played; it ranged from 4 DM to 10 DM (around US$2 to US$5 at the time). A group of 42 students was split into two groups and seated on different sides of a room. Each member of one subgroup played the role of player 1 in an ultimatum game. She wrote down on a form the amount (up to c) that she demanded. Her form was then given to a randomly determined member of the other group, who, playing the role of player 2, either accepted what remained of the amount c or rejected it (in which case neither player received any payoff). Each player had 10 minutes to make her decision. The entire experiment was repeated a week later. (Güth, Schmittberger, and Schwarze 1982.)

In the first experiment the average demand by people playing the role of player 1 was $0.65c$, and in the second experiment it was $0.69c$, much less than the amount c or $c - 0.01$ predicted by the notion of subgame perfect equilibrium (0.01DM was the smallest monetary unit; see Exercise 180.2). Almost 20% of offers were rejected over the two experiments, including one of 3DM (out of a pie of 7DM) and five of around 1DM (out of pies of between 4DM and 6DM). Many other experiments, including one in which the amount of money to be divided was much larger (Hoffman, McCabe, and Smith 1996), have produced similar results. In brief, the results do not accord well with the predictions of subgame perfect equilibrium.

Or do they? Each player in the ultimatum game cares only about the amount of money she receives. But an experimental subject may care also about the amount of money her opponent receives. Further, a variant of the ultimatum game in which the players are equity-conscious has subgame perfect equilibria in which offers are significant (as you will have discovered if you did Exercise 181.1).

However, if people are equity-conscious in the strategic environment of the ultimatum game, they should be equity-conscious also in related environments; an explanation of the experimental results in the ultimatum game based on the nature of preferences is not convincing if it applies only to that environment. Several related games have been studied, among them the dictator game and the impunity game (Exercise 180.3). In the subgame perfect equilibria of these games, player 1 offers 0; in a variant in which the players are equity-conscious, player 1's offers are no higher than they are in the analogous variant of the ultimatum game, and, for moderate degrees of equity-conscience, are lower (see Exercise 181.1). These features of the equilibria are broadly consistent with the experimental evidence on dictator, impunity, and ultimatum games (see, for example, Forsythe, Horowitz, Savin, and Sefton 1994, Bolton and Zwick 1995, and Güth and Huck 1997).

One feature of the experimental results is inconsistent with subgame perfect equilibrium even when players are equity-conscious (at least given the form of the payoff functions in Exercise 181.1): positive offers are sometimes rejected. The equilibrium strategy of an equity-conscious player 2 in the ultimatum game rejects inequitable offers, but, knowing this, player 1 does not, in equilibrium, make such an offer. To generate rejections in equilibrium we need to further modify the model by assuming that people differ in their degree of equity-conscience, and that player 1 does not know the degree of equity-conscience of player 2 (see Exercise 222.2).

An alternative explanation of the experimental results focuses on player 2's behavior. The evidence is consistent with player 1's significant offers in the ultimatum game being driven by a fear that player 2 will reject small offers—a fear that is rational, because small offers are often rejected. Why does player 2 behave in this way? One argument is that in our daily lives, we use "rules of thumb" that work well in the situations in which we are typically involved; we do not calculate our rational actions in each situation. Further, we are not typically involved in one-shot situations with the structure of the ultimatum game. Instead, we usually engage in repeated interactions, where it is advantageous to "punish" a player who makes a paltry offer, and to build a reputation for not accepting such offers. Experimental subjects may apply such rules of thumb rather than carefully thinking through the logic of the game, and thus reject low offers in an ultimatum game, but accept them in an impunity game, where rejection does not affect the proposer. The experimental evidence so far collected is broadly consistent with both this explanation and the explanation based on the nature of players' preferences.

? EXERCISE 183.1 (Bargaining over two indivisible objects) Consider a variant of the ultimatum game, with indivisible units. Two people use the following procedure to allocate two desirable identical indivisible objects. One person proposes an allocation (both objects go to person 1, both go to person 2, one goes to each person), which the other person then either accepts or rejects. In the event of rejection, neither person receives either object. Each person cares only about the number of objects she obtains. Construct an extensive game that models this situation and find its subgame perfect equilibria. Does the game have any Nash equilibrium that is not a subgame perfect equilibrium? Is there any outcome that is generated by a Nash equilibrium but not by any subgame perfect equilibrium?

?? EXERCISE 183.2 (Dividing a cake fairly) Two players use the following procedure to divide a cake. Player 1 divides the cake into two pieces, and then player 2 chooses one of the pieces; player 1 obtains the remaining piece. The cake is continuously divisible (no lumps!), and each player likes all parts of it.

a. Suppose that the cake is perfectly homogeneous, so that each player cares only about the size of the piece of cake she obtains. How is the cake divided in a subgame perfect equilibrium?

b. Suppose that the cake is not homogeneous: the players evaluate different parts of it differently. Represent the cake by the set C, so that a piece of the cake is a subset P of C. Assume that if P is a subset of P' not equal to P' (smaller than P') then each player prefers P' to P. Assume also that the players' preferences are continuous: if player i prefers P to P' then there is a subset of P not equal to P that player i also prefers to P'. Let (P_1, P_2) (where P_1 and P_2 together constitute the whole cake C) be the division chosen by player 1 in a subgame perfect equilibrium of the divide-and-choose game, P_2 being the piece chosen by player 2. Show that player 2 is indifferent between P_1 and P_2, and player 1 likes P_1 at least as much as P_2. Give an example in which player 1 prefers P_1 to P_2.

6.2.2 The holdup game

Before engaging in an ultimatum game in which she may accept or reject an offer of person 1, person 2 takes an action that affects the size c of the pie to be divided. She may exert little effort, resulting in a small pie, of size c_L, or great effort, resulting in a large pie, of size c_H. She dislikes exerting effort. Specifically, assume that her payoff is $x - E$ if her share of the pie is x, where $E = L$ if she exerts little effort and $E = H > L$ if she exerts great effort. The extensive game that models this situation is known as the **holdup game**.

? EXERCISE 183.3 (Holdup game) Formulate the holdup game precisely. (Write down the set of players, set of terminal histories, player function, and the players' preferences.)

What is the subgame perfect equilibrium of the holdup game? Each subgame that follows person 2's choice of effort is an ultimatum game, and thus has a unique subgame perfect equilibrium, in which person 1 offers 0 and person 2 accepts all offers. Now consider person 2's choice of effort at the start of the game. If she chooses L then her payoff, given the outcome in the following subgame, is $-L$, whereas if she chooses H then her payoff is $-H$. Consequently she chooses L. Thus the game has a unique subgame perfect equilibrium, in which person 2 exerts little effort and person 1 obtains all of the resulting small pie.

This equilibrium does not depend on the values of c_L, c_H, L, and H (given that $H > L$). In particular, even if c_H is much larger than c_L, but H is only slightly larger than L, person 2 exerts little effort in the equilibrium, although both players could be much better off if person 2 were to exert great effort (which, in this case, is not very great) and person 2 were to obtain some of the extra pie. No such superior outcome is sustainable in an equilibrium because person 2, having exerted great effort, may be "held up" for the entire pie by person 1.

This result does not depend sensitively on the extreme subgame perfect equilibrium outcome of the ultimatum game. In Section 16.3 I analyze a model in which a similar result may emerge when the bargaining following person 2's choice of effort generates a more equal division of the pie.

6.3 Stackelberg's model of duopoly

6.3.1 General model

In the models of oligopoly studied in Sections 3.1 and 3.2, each firm chooses its action not knowing the other firms' actions. How do the conclusions change when the firms move sequentially? Is a firm better off moving before or after the other firms?

In this section I consider a market in which there are two firms, both producing the same good. Firm i's cost of producing q_i units of the good is $C_i(q_i)$; the price at which output is sold when the total output is Q is $P_d(Q)$. (In Section 3.1 I denote this function P; here I add a d subscript to avoid a conflict with the player function of the extensive game.) Each firm's strategic variable is output, as in Cournot's model (Section 3.1), but the firms make their decisions sequentially, rather than simultaneously: one firm chooses its output, then the other firm does so, knowing the output chosen by the first firm.

We can model this situation by the following extensive game, known as **Stackelberg's duopoly game** (after its originator).

Players The two firms.

Terminal histories The set of all sequences (q_1, q_2) of outputs for the firms (where each q_i, the output of firm i, is a nonnegative number).

Player function $P(\varnothing) = 1$ and $P(q_1) = 2$ for all q_1.

Preferences The payoff of firm i to the terminal history (q_1, q_2) is its profit $q_i P(q_1 + q_2) - C_i(q_i)$, for $i = 1, 2$.

Firm 1 moves at the start of the game. Thus a strategy of firm 1 is simply an output. Firm 2 moves after every history in which firm 1 chooses an output. Thus a strategy of firm 2 is a *function* that associates an output for firm 2 with each possible output of firm 1.

The game has a finite horizon, so we may use backward induction to find its subgame perfect equilibria.

- First, for any output of firm 1, we find the outputs of firm 2 that maximize its profit. Suppose that for each output q_1 of firm 1 there is one such output of firm 2; denote it $b_2(q_1)$. Then in any subgame perfect equilibrium, firm 2's strategy is b_2.

- Next, we find the outputs of firm 1 that maximize its profit, *given the strategy of firm 2*. When firm 1 chooses the output q_1, firm 2 chooses the output $b_2(q_1)$, resulting in a total output of $q_1 + b_2(q_1)$, and hence a price of $P_d(q_1 + b_2(q_1))$. Thus firm 1's output in a subgame perfect equilibrium is a value of q_1 that maximizes
$$q_1 P_d(q_1 + b_2(q_1)) - C_1(q_1). \tag{185.1}$$
Suppose that there is one such value of q_1; denote it q_1^*.

We conclude that if firm 2 has a unique best response $b_2(q_1)$ to each output q_1 of firm 1, and firm 1 has a unique best action q_1^*, given firm 2's best responses, then the subgame perfect equilibrium of the game is (q_1^*, b_2): firm 1's equilibrium strategy is q_1^* and firm 2's equilibrium strategy is the function b_2. The output chosen by firm 2, given firm 1's equilibrium strategy, is $b_2(q_1^*)$; denote this output q_2^*.

When firm 1 chooses any output q_1, the outcome, given that firm 2 uses its equilibrium strategy, is the pair of outputs $(q_1, b_2(q_1))$. That is, as firm 1 varies its output, the outcome varies along firm 2's best response function b_2. Thus we can characterize the subgame perfect equilibrium outcome (q_1^*, q_2^*) as the point on firm 2's best response function that maximizes firm 1's profit.

6.3.2 *Example: constant unit cost and linear inverse demand*

Suppose that $C_i(q_i) = cq_i$ for $i = 1, 2$, and
$$P_d(Q) = \begin{cases} \alpha - Q & \text{if } Q \le \alpha \\ 0 & \text{if } Q > \alpha, \end{cases} \tag{185.2}$$
where $c > 0$ and $c < \alpha$ (as in the example of Cournot's duopoly game in Section 3.1.3). We found that under these assumptions firm 2 has a unique best response to each output q_1 of firm 1, given by
$$b_2(q_1) = \begin{cases} \frac{1}{2}(\alpha - c - q_1) & \text{if } q_1 \le \alpha - c \\ 0 & \text{if } q_1 > \alpha - c. \end{cases}$$

Thus in a subgame perfect equilibrium of Stackelberg's game firm 2's strategy is this function b_2 and firm 1's strategy is the output q_1 that maximizes

$$q_1(\alpha - c - (q_1 + \tfrac{1}{2}(\alpha - c - q_1))) = \tfrac{1}{2}q_1(\alpha - c - q_1)$$

(refer to (185.1)). This function is a quadratic in q_1 that is zero when $q_1 = 0$ and when $q_1 = \alpha - c$. Thus its maximizer is $q_1 = \frac{1}{2}(\alpha - c)$.

We conclude that the game has a unique subgame perfect equilibrium, in which firm 1's strategy is the output $\frac{1}{2}(\alpha - c)$ and firm 2's strategy is b_2. The outcome of the equilibrium is that firm 1 produces the output $q_1^* = \frac{1}{2}(\alpha - c)$ and firm 2 produces the output $q_2^* = b_2(q_1^*) = b_2(\frac{1}{2}(\alpha - c)) = \frac{1}{2}(\alpha - c - \frac{1}{2}(\alpha - c)) = \frac{1}{4}(\alpha - c)$. Firm 1's profit is $q_1^*(P(q_1^* + q_2^*) - c) = \frac{1}{8}(\alpha - c)^2$, and firm 2's profit is $q_1^*(P(q_1^* + q_2^*) - c) = \frac{1}{16}(\alpha - c)^2$. By contrast, in the unique Nash equilibrium of Cournot's (simultaneous-move) game under the same assumptions, each firm produces $\frac{1}{3}(\alpha - c)$ units of output and obtains the profit $\frac{1}{9}(\alpha - c)^2$. Thus under our assumptions firm 1 produces more output and obtains more profit in the subgame perfect equilibrium of the sequential game in which it moves first than it does in the Nash equilibrium of Cournot's game, and firm 2 produces less output and obtains less profit.

? EXERCISE 186.1 (Stackelberg's duopoly game with quadratic costs) Find the subgame perfect equilibrium of Stackelberg's duopoly game when $C_i(q_i) = q_i^2$ for $i = 1, 2$, and $P_d(Q) = \alpha - Q$ for all $Q \leq \alpha$ (with $P_d(Q) = 0$ for $Q > \alpha$). Compare the equilibrium outcome with the Nash equilibrium of Cournot's game under the same assumptions (Exercise 57.2).

6.3.3 Properties of subgame perfect equilibrium

First-mover's equilibrium profit In the example just studied, the first-mover is better off in the subgame perfect equilibrium of Stackelberg's game than it is in the Nash equilibrium of Cournot's game. A weak version of this result holds under very general conditions: for any cost and inverse demand functions for which firm 2 has a unique best response to each output of firm 1, firm 1 is at least as well off in any subgame perfect equilibrium of Stackelberg's game as it is in any Nash equilibrium of Cournot's game. This result follows from the general result in Exercise 175.2*a*. The argument is simple. One of firm 1's options in Stackelberg's game is to choose its output in some Nash equilibrium of Cournot's game. If it chooses such an output then firm 2's best action is to choose its output in the same Nash equilibrium, given the assumption that it has a unique best response to each output of firm 1. Thus by choosing such an output, firm 1 obtains its profit at a Nash equilibrium of Cournot's game; by choosing a different output it may possibly obtain a higher payoff.

Equilibrium outputs In the example in the previous section, firm 1 produces more output in the subgame perfect equilibrium of Stackelberg's game than it does in the Nash equilibrium of Cournot's game, and firm 2 produces less. A weak form of this result holds whenever firm 2's best response function is decreasing where it is positive (i.e. a higher output for firm 1 implies a lower optimal output for firm 2).

The argument is illustrated in Figure 187.1. The firms' best response functions are the curves labeled b_1 (dashed) and b_2. The Nash equilibrium of Cournot's game is the intersection $(\overline{q}_1, \overline{q}_2)$ of these curves. Along each gray curve, firm 1's profit is constant; the *lower* curve corresponds to a *higher* profit. (For any given value of firm 1's output, a reduction in the output of firm 2 increases the price and thus increases firm 1's profit.) Each constant-profit curve of firm 1 is horizontal where it crosses firm 1's best response function, because the best response is precisely the output that maximizes firm 1's profit, given firm 2's output. (Cf. Figure 59.1.) Thus the subgame perfect equilibrium outcome—the point on firm 2's best response function that yields the highest profit for firm 1—is the point (q_1^*, q_2^*) in the figure. In particular, given that the best response function of firm 2 is downward-sloping, firm 1 produces at least as much, and firm 2 produces at most as much, in the subgame perfect equilibrium of Stackelberg's game as in the Nash equilibrium of Cournot's game.

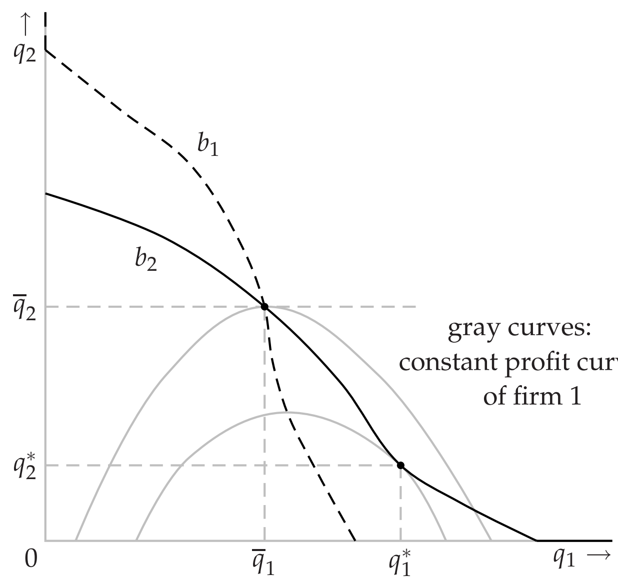


Figure 187.1 The subgame perfect equilibrium outcome (q_1^*, q_2^*) of Stackelberg's game and the Nash equilibrium $(\overline{q}_1, \overline{q}_2)$ of Cournot's game. Along each gray curve, firm 1's profit is constant; the lower curve corresponds to higher profit than does the upper curve. Each curve has a slope of zero where it crosses firm 1's best response function b_1.

For some cost and demand functions, firm 2's output in a subgame perfect equilibrium of Stackelberg's game is zero. An example is shown in Figure 188.1. The discontinuity in firm 2's best response function at q_1^* in this example may arise because firm 2 incurs a "fixed" cost—a cost independent of its output—when it produces a positive output (see Exercise 57.3). When firm 1's output is q_1^*, firm 2's maximal profit is zero, which it obtains both when it produces no output (and does not pay the fixed cost) and when it produces the output $\hat{q}_2$. When firm 1 produces less than q_1^*, firm 2's maximal profit is positive, and firm 2 optimally produces a positive output; when firm 1 produces more than q_1^*, firm 2 optimally produces no output. Given this form of firm 2's best response function and the form of firm 1's constant profit curves shown in the figure, the point on firm 2's best response function that yields firm 1 the highest profit is $(q_1^*, 0)$.

I claim that this example has a unique subgame perfect equilibrium, in which firm 1 produces q_1^* and firm 2's strategy coincides with its best response function except at q_1^*, where the strategy specifies the output 0. The output firm 2's equilibrium strategy specifies after each history must be a best response to firm 1's output, so the only question regarding firm 2's strategy is whether it specifies an output of 0 or $\hat{q}_2$ when firm 1's output is q_1^*. The argument that there is no subgame perfect equilibrium in which firm 2's strategy specifies the output $\hat{q}_2$ is similar to the argument that there is no subgame perfect equilibrium in the ultimatum game in which person 2 rejects the offer 0. If firm 2 produces the output $\hat{q}_2$ in response to firm 1's output q_1^* then firm 1 has no optimal output: it would like to produce a little more than q_1^*, inducing firm 2 to produce zero, but is better off the closer its output is to q_1^*. Because there is no smallest output greater than q_1^*, no output is *optimal* for firm 1 in this case. Thus the game has no subgame perfect equilibrium in which firm 2's strategy specifies the output $\hat{q}_2$ in response to firm 1's output q_1^*.

Note that if firm 2 were entirely absent from the market, firm 1 would produce $\hat{q}_1$, less than q_1^*. Thus firm 2's presence affects the outcome, even though it produces no output.

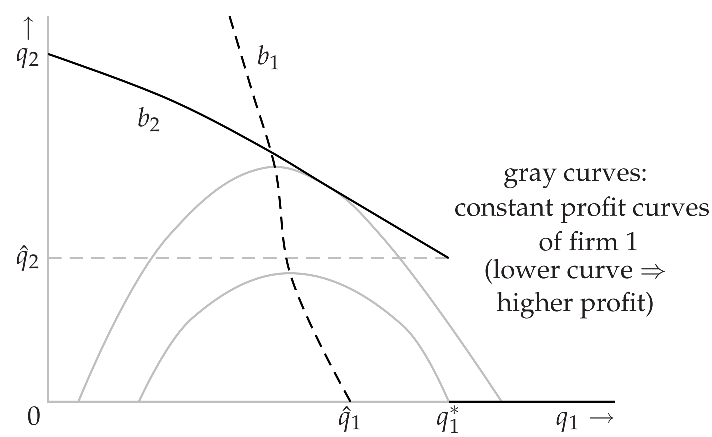


Figure 188.1 The subgame perfect equilibrium output q_1^* of firm 1 in Stackelberg's sequential game when firm 2 incurs a fixed cost. Along each gray curve, firm 1's profit is constant; the lower curve corresponds to higher profit than does the upper curve.

? EXERCISE 188.1 (Stackelberg's duopoly game with fixed costs) Suppose that the inverse demand function is given by (185.2) and the cost function of each firm i is given by

$$C_i(q_i) = \begin{cases} 0 & \text{if } q_i = 0 \\ f + cq_i & \text{if } q_i > 0, \end{cases}$$

where $c \geq 0$, $f > 0$, and $c < \alpha$, as in Exercise 57.3. Show that if $c = 0$, $\alpha = 12$, and $f = 4$, Stackelberg's game has a unique subgame perfect equilibrium, in which firm 1's output is 8 and firm 2's output is zero. (Use your results from Exercise 57.3).

The value of commitment Firm 1's output in a subgame perfect equilibrium of Stackelberg's game is *not* in general a best response to firm 2's output: if firm 1 could adjust its output after firm 2 has chosen its output, then it would do so! (In the case shown in Figure 187.1, it would reduce its output.) However, if firm 1 had this opportunity, and firm 2 knew that it had the opportunity, then firm 2 would choose a different output. Indeed, if we simply add a third stage to the game, in which firm 1 chooses an output, then the first stage is irrelevant, and *firm 2* is effectively the first-mover; in the subgame perfect equilibrium firm 1 is worse off than it is in the Nash equilibrium of the simultaneous-move game. (In the example in the previous section, the unique subgame perfect equilibrium has firm 2 choose the output $(\alpha - c)/2$ and firm 1 choose the output $(\alpha - c)/4$.) In summary, even though firm 1 can increase its profit by changing its output after firm 2 has chosen its output, in the game in which it has this opportunity it is worse off than it is in the game in which it must choose its output before firm 2 and cannot subsequently modify this output. That is, firm 1 prefers to be *committed* not to change its mind.

? EXERCISE 189.1 (Sequential variant of Bertrand's duopoly game) Consider the variant of Bertrand's duopoly game (Section 3.2) in which first firm 1 chooses a price, then firm 2 chooses a price. Assume that each firm is restricted to choose a price that is an integral number of cents (as in Exercise 65.2), that each firm's unit cost is constant, equal to c (an integral number of cents), and that the monopoly profit is positive.

a. Specify an extensive game with perfect information that models this situation.

b. Give an example of a strategy of firm 1 and an example of a strategy of firm 2.

c. Find the subgame perfect equilibria of the game.

6.4 Buying votes

A legislature has k members, where k is an odd number. Two rival bills, X and Y, are being considered. The bill that attracts the votes of a majority of legislators will pass. Interest group X favors bill X, whereas interest group Y favors bill Y. Each group wishes to entice a majority of legislators to vote for its favorite bill. First interest group X gives an amount of money (possibly zero) to each legislator, then interest group Y does so. Each interest group wishes to spend as little as possible. Group X values the passing of bill X at $\$V_X > 0$ and the passing of bill Y

at zero, and group Y values the passing of bill Y at $\$V_Y > 0$ and the passing of bill X at zero. (For example, group X is indifferent between an outcome in which it spends V_X and bill X is passed and one in which it spends nothing and bill Y is passed.) Each legislator votes for the favored bill of the interest group that offers her the most money; a legislator to whom both groups offer the same amount of money votes for bill Y (an arbitrary assumption that simplifies the analysis without qualitatively changing the outcome). For example, if $k = 3$, the amounts offered to the legislators by group X are $x = (100, 50, 0)$, and the amounts offered by group Y are $y = (100, 0, 50)$, then legislators 1 and 3 vote for Y and legislator 2 votes for X, so that Y passes. (In some legislatures the inducements offered to legislators are more subtle than cash transfers.)

We can model this situation as the following extensive game.

Players The two interest groups, X and Y.

Terminal histories The set of all sequences (x, y), where x is a list of payments to legislators made by interest group X and y is a list of payments to legislators made by interest group Y. (That is, both x and y are lists of k nonnegative integers.)

Player function $P(\varnothing) = X$ and $P(x) = Y$ for all x.

Preferences The preferences of interest group X are represented by the payoff function

$$\begin{cases} V_X - (x_1 + \cdots + x_k) & \text{if bill } X \text{ passes} \\ -(x_1 + \cdots + x_k) & \text{if bill } Y \text{ passes,} \end{cases}$$

where bill Y passes after the terminal history (x, y) if and only if the number of components of y that are at least equal to the corresponding components of x is at least $\frac{1}{2}(k+1)$ (a bare majority of the k legislators). The preferences of interest group Y are represented by the analogous function (where V_Y replaces V_X, y replaces x, and Y replaces X).

Before studying the subgame perfect equilibria of this game for arbitrary values of the parameters, consider two examples. First suppose that $k = 3$ and $V_X = V_Y = 300$. Under these assumptions, the most group X is willing to pay to get bill X passed is 300. For any payments it makes to the three legislators that sum to at most 300, two of the payments sum to at most 200, so that if group Y matches these payments it spends less than V_Y (= 300) and gets bill Y passed. Thus in any subgame perfect equilibrium group X makes no payments, group Y makes no payments, and (given the tie-breaking rule) bill Y is passed.

Now suppose that $k = 3$, $V_X = 300$, and $V_Y = 100$. In this case by paying each legislator more than 50, group X makes matching payments by group Y unprofitable: only by spending more than V_Y (= 100) can group Y cause bill Y to be passed. However, there is no subgame perfect equilibrium in which group X pays each legislator more than 50, because it can always pay a little less (as long as the payments still exceed 50) and still prevent group Y from profitably matching. In the only subgame perfect equilibrium group X pays each legislator exactly 50, and group Y makes no payments. Given group X's action, group Y is indifferent between matching X's payments (so that bill Y is passed), and making no payments. However, there is no subgame perfect equilibrium in which group Y matches group X's payments, because if this were group Y's response then group X could increase its payments a little, making matching payments by group Y unprofitable.

For arbitrary values of the parameters the subgame perfect equilibrium outcome takes one of the forms in these two examples: either no payments are made and bill Y is passed, or group X makes payments that group Y does not wish to match, group Y makes no payments, and bill X is passed.

To find the subgame perfect equilibria in general, we may use backward induction. First consider group Y's best response to an arbitrary strategy x of group X. Let $\mu = \frac{1}{2}(k+1)$, a bare majority of k legislators, and denote by m_x the sum of the smallest μ components of x—the total payments Y needs to make to buy off a bare majority of legislators.

- If $m_x < V_Y$ then group Y can buy off a bare majority of legislators for less than V_Y, so that its best response to x is to match group X's payments to the μ legislators to whom group X's payments are smallest; the outcome is that bill Y is passed.

- If $m_x > V_Y$ then the cost to group Y of buying off any majority of legislators exceeds V_Y, so that group Y's best response to x is to make no payments; the outcome is that bill X is passed.

- If $m_x = V_Y$ then both the actions in the previous two cases are best responses by group Y to x.

We conclude that group Y's strategy in a subgame perfect equilibrium has the following properties.

- After a history x for which $m_x < V_Y$, group Y matches group X's payments to the μ legislators to whom X's payments are smallest.

- After a history x for which $m_x > V_Y$, group Y makes no payments.

- After a history x for which $m_x = V_Y$, group Y either makes no payments or matches group X's payments to the μ legislators to whom X's payments are smallest.

Given that group Y's subgame perfect equilibrium strategy has these properties, what should group X do? If it chooses a list of payments x for which $m_x < V_Y$ then group Y matches its payments to a bare majority of legislators, and bill Y passes. If it reduces all its payments, the same bill is passed. Thus the only list of payments x with $m_x < V_Y$ that may be optimal is $(0, \ldots, 0)$. If it chooses a list of payments x with $m_x > V_Y$ then group Y makes no payments, and bill X passes. If it reduces all its payments a little (keeping the payments to every bare majority greater than V_Y), the outcome is the same. Thus no list of payments x for which $m_x > V_Y$ is optimal.

We conclude that in any subgame perfect equilibrium we have either $x = (0, \ldots, 0)$ (group X makes no payments) or $m_x = V_Y$ (the smallest sum of group X's payments to a bare majority of legislators is V_Y). Under what conditions does each case occur? If group X needs to spend more than V_X to deter group Y from matching its payments to a bare majority of legislators, then its best strategy is to make no payments ($x = (0, \ldots, 0)$). How much does it need to spend to deter group Y? It needs to pay more than V_Y to every bare majority of legislators, so it needs to pay each legislator more than V_Y/μ, in which case its total payment is more than kV_Y/μ. Thus if $V_X < kV_Y/\mu$, group X is better off making no payments than getting bill X passed by making payments large enough to deter group Y from matching its payments to a bare majority of legislators.

If $V_X > kV_Y/\mu$, on the other hand, group X can afford to make payments large enough to deter group Y from matching. In this case its best strategy is to pay each legislator V_Y/μ, so that its total payment to every bare majority of legislators is V_Y. Given this strategy, group Y is indifferent between matching group X's payments to a bare majority of legislators and making no payments. I claim that the game has no subgame perfect equilibrium in which group Y matches. The argument is similar to the argument that the ultimatum game has no subgame perfect equilibrium in which person 2 rejects the offer 0. Suppose that group Y matches. Then group X can increase its payoff by increasing its payments a little (keeping the total less than V_X), thereby deterring group Y from matching, and ensuring that bill X passes. Thus in any subgame perfect equilibrium group Y makes no payments in response to group X's strategy.

In conclusion, if $V_X \neq kV_Y/\mu$ then the game has a unique subgame perfect equilibrium, in which group Y's strategy is

- match group X's payments to the μ legislators to whom X's payments are smallest after a history x for which $m_x < V_Y$
- make no payments after a history x for which $m_x \geq V_Y$

and group X's strategy depends on the relative sizes of V_X and V_Y:

- if $V_X < kV_Y/\mu$ then group X makes no payments;
- if $V_X > kV_Y/\mu$ then group X pays each legislator V_Y/μ.

If $V_X < kV_Y/\mu$ then the outcome is that neither group makes any payment, and bill Y is passed; if $V_X > kV_Y/\mu$ then the outcome is that group X pays each legislator V_Y/μ, group Y makes no payments, and bill X is passed. (If $V_X = kV_Y/\mu$ then the analysis is more complex.)

Three features of the subgame perfect equilibrium are significant. First, the outcome favors the second-mover in the game (group Y): only if $V_X > kV_Y/\mu$, which is close to $2V_Y$ when k is large, does group X manage to get bill X passed. Second, group Y never makes any payments! According to its equilibrium strategy it is prepared to make payments in response to certain strategies of group X, but given group X's *equilibrium* strategy it spends not a cent. Third, if group X makes any payments (as it does in the equilibrium for $V_X > kV_Y/\mu$) then it makes a payment to *every* legislator. If there were no competing interest group but nonetheless each legislator would vote for bill X only if she were paid at least some amount, then group X would make payments to only a bare majority of legislators; if it were to act in this way in the presence of group Y it would supply group Y with almost a majority of legislators who could be induced to vote for bill Y at no cost.

? EXERCISE 193.1 (Three interest groups buying votes) Consider a variant of the model in which there are *three* bills, X, Y, and Z, and *three* interest groups, X, Y, and Z, who choose lists of payments sequentially. Ties are broken in favor of the group moving later. Find the bill that is passed in any subgame perfect equilibrium when $k = 3$ and (*a*) $V_X = V_Y = V_Z = 300$, (*b*) $V_X = 300$, $V_Y = V_Z = 100$, and (*c*) $V_X = 300$, $V_Y = 202$, $V_Z = 100$. (You may assume that in each case a subgame perfect equilibrium exists; note that you are not asked to find the subgame perfect equilibria themselves.)

? EXERCISE 193.2 (Interest groups buying votes under supermajority rule) Consider an alternative variant of the model in which a supermajority is required to pass a bill. There are two bills, X and Y, and a "default outcome". A bill passes if and only if it receives at least $k^* > \frac{1}{2}(k+1)$ votes; if neither bill passes the default outcome occurs. There are two interest groups. Both groups attach value 0 to the default outcome. Find the bill that is passed in any subgame perfect equilibrium when $k = 7$, $k^* = 5$, and (*a*) $V_X = V_Y = 700$ and (*b*) $V_X = 750$, $V_Y = 400$. In each case, would the legislators be better off or worse off if a simple majority of votes were required to pass a bill?

? EXERCISE 193.3 (Sequential positioning by two political candidates) Consider the variant of Hotelling's model of electoral competition in Section 3.3 in which the n candidates choose their positions sequentially, rather than simultaneously. Model this situation as an extensive game. Find the subgame perfect equilibrium (equilibria?) when $n = 2$.

?? EXERCISE 193.4 (Sequential positioning by three political candidates) Consider a further variant of Hotelling's model of electoral competition in which the n candidates choose their positions sequentially and each candidate has the option of staying out of the race. Assume that each candidate prefers to stay out than to enter and lose, prefers to enter and tie with any number of candidates than to stay out, and prefers to tie with as few other candidates as possible. Model the situation as an extensive game and find the subgame perfect equilibrium outcomes when $n = 2$ (easy) and when $n = 3$ and the voters' favorite positions are distributed uniformly from 0 to 1 (i.e. the fraction of the voters' favorite positions less than x is x) (hard).

6.5 A race

6.5.1 General model

Firms compete with each other to develop new technologies; authors compete with each other to write books and film scripts about momentous current events; scientists compete with each other to make discoveries. In each case the winner enjoys a significant advantage over the losers, and each competitor can, at a cost, increase her pace of activity. How do the presence of competitors and size of the prize affect the pace of activity? How does the identity of the winner of the race depend on the each competitor's initial distance from the finish line?

We can model a race as an extensive game with perfect information in which the players alternately choose how many "steps" to take. Here I study a simple example of such a game, with two players.

Player i is initially $k_i > 0$ steps from the finish line, for $i = 1, 2$. On each of her turns, a player can either not take any steps (at a cost of 0), or can take one step, at a cost of $c(1)$, or two steps, at a cost of $c(2)$. The first player to reach the finish line wins a prize, worth $v_i > 0$ to player i; the losing player's payoff is 0. To make the game finite, I assume that if, on successive turns, neither player takes any step, the game ends and neither player obtains the prize.

I denote the game in which player i moves first by $G_i(k_1, k_2)$. The game $G_1(k_1, k_2)$ is defined precisely as follows.

Players The two parties.

Terminal histories The set of sequences of the form $(x^1, y^1, x^2, y^2, \ldots, x^T)$ or $(x^1, y^1, x^2, y^2, \ldots, y^T)$ for some integer T, where each x^t (the number of steps taken by player 1 on her tth turn) and each y^t (the number of steps taken by player 2 on her tth turn) is 0, 1, or 2, there are never two successive 0's except possibly at the end of a sequence, and either $x^1 + \cdots + x^T = k_1$ and $y^1 + \cdots + y^T < k_2$ (player 1 reaches the finish line first), or $x^1 + \cdots + x^T < k_1$ and $y^1 + \cdots + y^T = k_2$ (player 2 reaches the finish line first).

Player function $P(\varnothing) = 1$, $P(x^1) = 2$ for all x^1, $P(x^1, y^1) = 1$ for all (x^1, y^1), $P(x^1, y^1, x^2) = 2$ for all (x^1, y^1, x^2), and so on.

Preferences For a terminal history in which player i loses, her payoff is the negative of the sum of the costs of all her moves; for a terminal history in which she wins it is v_i minus the sum of these costs.

6.5.2 Subgame perfect equilibria of an example

A simple example illustrates the features of the subgame perfect equilibria of this game. Suppose that both v_1 and v_2 are between 6 and 7 (their exact values do not affect the equilibria), the cost $c(1)$ of a single step is 1, and the cost $c(2)$ of two steps is 4. (Given that $c(2) > 2c(1)$, each player, in the absence of a competitor, would like to take one step at a time.)

The game has a finite horizon, so we may use backward induction to find its subgame perfect equilibria. Each of its subgames is either a game $G_i(m_1, m_2)$ with $i = 1$ or $i = 2$ and $0 < m_1 \leq k_1$ and $0 < m_2 \leq k_2$, or, if the last player to move before the subgame took no steps, a game that differs from $G_i(m_1, m_2)$ only in that it ends if player i initially takes no steps (i.e. the only terminal history starting with 0 consists only of 0).

First consider the very simplest game, $G_1(1, 1)$, in which each player is initially one step from the finish line. If player 1 takes one step, she wins; if she does not move then player 2 optimally takes one step (if she does not, the game ends) and wins. We conclude that the game has a unique subgame perfect equilibrium, in which player 1 initially takes one step and wins.

A similar argument applies to the game $G_1(1, 2)$. If player 1 does not move then player 2 has the option of taking one or two steps. If she takes one step then play moves to a subgame identical $G_1(1, 1)$, in which we have just concluded that player 1 wins. Thus player 2 takes two steps, and wins, if player 1 does not move at the start of $G_1(1, 2)$. We conclude that the game has a unique subgame perfect equilibrium, in which player 1 initially takes one step and wins.

Now consider player 1's options in the game $G_1(2, 1)$.

Player 1 takes two steps: She wins, and obtains a payoff of at least $6 - 4 = 2$ (her valuation is more than 6, and the cost of two steps is 4).

Player 1 take one step: Play moves to a subgame identical to $G_2(1, 1)$; we know that in the equilibrium of this subgame player 2 initially takes one step and wins.

Player 1 does not move: Play moves to a subgame in which player 2 is the first-mover and is one step from the finish line, and, if player 2 does not move, the game ends. In an equilibrium of this subgame player 2 takes one step and wins.

We conclude that the game $G_1(2, 1)$ has a unique subgame perfect equilibrium, in which player 1 initially takes two steps and wins.

I have spelled out the details of the analysis of these cases to show how we use the result for the game $G_1(1, 1)$ to find the equilibria of the games $G_1(1, 2)$ and $G_1(2, 1)$. In general, the equilibria of the games $G_i(k_1, k_2)$ for all values of k_1 and k_2 up to $\overline{k}$ tell us the consequences of player 1's taking one or two steps in the game $G_1(\overline{k} + 1, \overline{k})$.

? EXERCISE 195.1 (The race $G_1(2, 2)$) Show that the game $G_1(2, 2)$ has a unique subgame perfect equilibrium outcome, in which player 1 initially takes two steps, and wins.

So far we have concluded that in any game in which each player is initially at most two steps from the finish line, the first-mover takes enough steps to reach the finish line, and wins.

Now suppose that player 1 is at most two steps from the finish line, but player 2 is three steps away. Suppose that player 1 takes only *one* step (even if she is initially two steps from the finish line). Then if player 2 takes either one or two steps, play moves to a subgame in which player 1 (the first-mover) wins. Thus player 2 is better off not moving (and not incurring any cost), in which case player 1 takes one step on her next turn, and wins. (Player 1 prefers to move one step at a time than to move two steps initially, because the former costs her 2 whereas the latter costs her 4.) We conclude that the outcome of a subgame perfect equilibrium in the game $G_1(2,3)$ is that player 1 takes one step on her first turn, then player 2 does not move, and then player 1 takes another step, and wins.

By a similar argument, in a subgame perfect equilibrium of any game in which player 1 is at most two steps from the finish line and player 2 is three or more steps away, player 1 moves one step at a time, and player 2 does not move; player 1 wins. Symmetrically, in a subgame perfect equilibrium of any game in which player 1 is three or more steps from the finish line and player 2 is at most two steps away, player 1 does not move, and player 2 moves one step at a time, and wins.

Our conclusions so far are illustrated in Figure 197.1. In this figure, player 1 moves to the left, and player 2 moves down. The values of (k_1, k_2) for which the subgame perfect equilibrium outcome has been determined so far are labeled. The label "1" means that, regardless of who moves first, in a subgame perfect equilibrium player 1 moves one step on each turn, and player 2 does not move; player 1 wins. Similarly, the label "2" means that, regardless of who moves first, player 2 moves one step on each turn, and player 1 does not move; player 2 wins. The label "f" means that the first player to move takes enough steps to reach the finish line, and wins.

Now consider the game $G_1(3,3)$. If player 1 takes one step, we reach the game $G_2(2,3)$. From Figure 197.1 we see that in the subgame perfect equilibrium of this game player 1 wins, and does so by taking one step at a time (the point $(2,3)$ is labeled "1"). If player 1 takes two steps, we reach the game $G_2(1,3)$, in which player 1 also wins. Player 1 prefers not to take two steps unless she has to, so in the subgame perfect equilibrium of $G_1(3,3)$ she takes one step at a time, and wins, and player 2 does not move. Similarly, in a subgame perfect equilibrium of $G_2(3,3)$, player 2 takes one step at a time, and wins, and player 1 does not move.

A similar argument applies to each of the games $G_i(3,4)$, $G_i(4,3)$, and $G_i(4,4)$ for $i = 1, 2$. The argument differs only if the first-mover is four steps from the finish line, in which case she initially takes two steps in order to reach a game in which she wins. (If she initially takes only one step, the other player wins.)

Now consider the game $G_i(3,5)$ for $i = 1, 2$. By taking one step in $G_1(3,5)$, player 1 reaches a game in which she wins by taking one step at a time. The cost of her taking three steps is less than v_1, so in a subgame perfect equilibrium of $G_1(3,5)$ she takes one step at a time, and wins, and player 2 does not move.

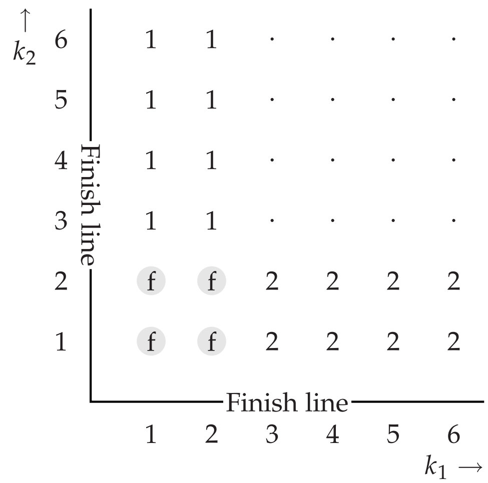


Figure 197.1 The subgame perfect equilibrium outcomes of the race $G_i(k_1, k_2)$. Player 1 moves to the left, and player 2 moves down. The values of (k_1, k_2) for which the subgame perfect equilibrium outcome has been determined so far are labeled; dots represent cases that have not yet been studied. The labels are explained in the text.

If player 2 takes either one or two steps in $G_2(3,5)$, she reaches a game (either $G_1(3,4)$ or $G_1(3,3)$) in which player 1 wins. Thus whatever she does, she loses, so that in a subgame perfect equilibrium she does not move and player 1 moves one step at a time. We conclude that in a subgame perfect equilibrium of both $G_1(3,5)$ and $G_2(3,5)$, player 1 takes one step on each turn and player 2 does not move; player 1 wins.

A similar argument applies to any game in which one player is initially three or four steps from the finish line, and the other player is five or more steps from the finish line. We have now made arguments to justify the labeling in Figure 198.1. In this figure the labels have the same meaning as in the previous figure, except that "f" means that the first player to move takes enough steps to reach the finish line *or* to reach the closest point labeled with her name, whichever is closer.

A feature of the subgame perfect equilibrium of the game $G_1(4,4)$ is noteworthy. Suppose that, as planned, player 1 takes two steps, but then player 2 deviates from her equilibrium strategy and takes two steps (rather than not moving). According to our analysis, player 1 should take two steps, to reach the finish line. If she does so, her payoff is negative (less than $7 - 4 - 4 = -1$). Nevertheless she should definitely take the two steps: if she does not, her payoff is even smaller (-4), because player 2 wins. The point is that the cost of her first move is "sunk"; her decision after player 2 deviates must be based on her options from that point on.

The analysis of the games in which each player is initially either 5 or 6 steps from the finish line involves arguments similar to those used in the previous cases, with one amendment. A player who is initially 6 steps from the finish line is better

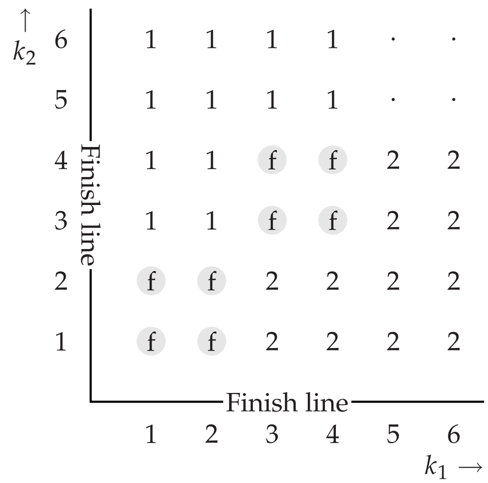


Figure 198.1 The subgame perfect equilibrium outcomes of the race $G_i(k_1, k_2)$. Player 1 moves to the left, and player 2 moves down. The values of (k_1, k_2) for which the subgame perfect equilibrium outcome has been determined so far are labeled; dots represent cases that have not yet been studied. The labels are explained in the text.

off not moving at all (and obtaining the payoff 0) than she is moving two steps on any turn (and obtaining a negative payoff). An implication is that in the game $G_1(6,5)$, for example, player 1 does not move: if she takes only one step then player 2 becomes the first-mover and, by taking a single step, moves the play to a game that she wins. We conclude that the first-mover wins in the games $G_i(5,5)$ and $G_i(6,6)$, whereas player 2 wins in $G_i(6,5)$ and player 1 wins in $G_i(5,6)$, for $i = 1, 2$.

A player who is initially more than six steps from the finish line obtains a negative payoff if she moves, even if she wins, so in any subgame perfect equilibrium she does not move. Thus our analysis of the game is complete. The subgame perfect equilibrium outcomes are indicated in Figure 199.1, which shows also the steps taken in the equilibrium of each game when player 1 is the first-mover.

? EXERCISE 198.1 (A race in which the players' valuations of the prize differ) Find the subgame perfect equilibrium outcome of the game in which player 1's valuation of the prize is between 6 and 7, and player 2's valuation is between 4 and 5.

In both of the following exercises, inductive arguments on the length of the game, like the one for $G_i(k_1, k_2)$, can be used.

? EXERCISE 198.2 (Removing stones) Two people take turns removing stones from a pile of n stones. Each person may, on each of her turns, remove either one stone or two stones. The person who takes the last stone is the winner; she gets \$1 from her opponent. Find the subgame perfect equilibria of the games that model this situation for $n = 1$ and $n = 2$. Find the winner in each subgame perfect

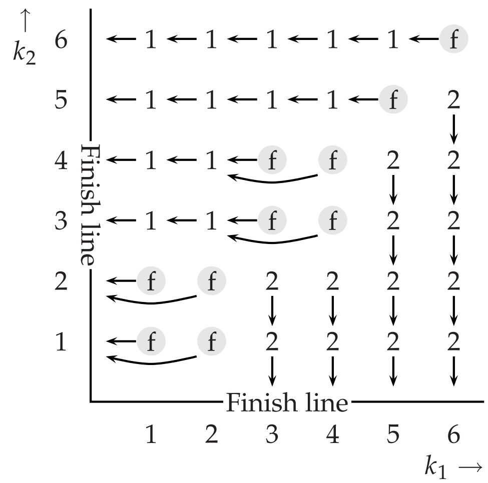


Figure 199.1 The subgame perfect equilibrium outcomes of the race $G_i(k_1, k_2)$. Player 1 moves to the left, and player 2 moves down. The arrows indicate the steps taken in the subgame perfect equilibrium outcome of the games in which player 1 moves first. The labels are explained in the text.

equilibrium for $n = 3$, using the fact that the subgame following player 1's removal of one stone is the game for $n = 2$ in which player 2 is the first-mover, and the subgame following player 1's removal of two stones is the game for $n = 1$ in which player 2 is the first mover. Use the same technique to find the winner in each subgame perfect equilibrium for $n = 4$, and, if you can, for an arbitrary value of n.

?? EXERCISE 199.1 (Hungry lions) The members of a hierarchical group of hungry lions face a piece of prey. If lion 1 does not eat the prey, the game ends. If it eats the prey, it becomes fat and slow, and lion 2 can eat it. If lion 2 does not eat lion 1, the game ends; if it eats lion 1 then it may be eaten by lion 3, and so on. Each lion prefers to eat than to be hungry, but prefers to be hungry than to be eaten. Find the subgame perfect equilibrium (equilibria?) of the extensive game that models this situation for any number n of lions.

6.5.3 General lessons

Each player's equilibrium strategy involves a "threat" to speed up if the other player deviates. Consider, for example, the game $G_1(3, 3)$. Player 1's equilibrium strategy calls for her to take one step at a time, and player 2's equilibrium strategy calls for her not to move. Thus along the equilibrium path player 1's debt climbs to 3 (the cost of her three single steps) before she reaches the finish line.

Now suppose that after player 1 takes her first step, player 2 deviates and takes a step. In this case, player 1's strategy calls for her to take two steps. If she does so, her debt climbs to 5. If at no stage can her debt exceed 3 (its maximal level on the equilibrium path) then her strategy cannot embody such threats.

The general point is that a limit on the debt a player can accumulate may affect the outcome even if it exceeds the player's debt along the equilibrium path in the absence of any limits. You are asked to study an example in the next exercise.

? EXERCISE 200.1 (A race with a liquidity constraint) Find the subgame perfect equilibrium of the variant of the game $G_1(3,3)$ in which player 1's debt may never exceed 3.

In the subgame perfect equilibrium of every game $G_i(k_1, k_2)$, only one player moves; her opponent "gives up". This property of equilibrium holds in more general games. What added ingredient might lead to an equilibrium in which both players are active? A player's uncertainty about the other's characteristics would seem to be such an ingredient: if a player does not know the cost of its opponent's moves, it may assign a positive probability less than one to its winning, at least until it has accumulated some evidence of its opponent's behavior, and while it is optimistic it may be active even though its rival is also active. To build such considerations into the model we need to generalize the model of an extensive game to encompass imperfect information, as we do in Chapter 10.

Another feature of the subgame perfect equilibrium of $G_i(k_1, k_2)$ that holds in more general games is that the presence of a competitor has little effect on the speed of the player who moves. A lone player would move one step at a time. When there are two players, for most starting points the one that moves does so at the same leisurely pace. Only for a small number of starting points, in all of which the players' initial distances from the starting line are similar, does the presence of a competitor induce the active player to hasten its progress, and then only in the first period.

Notes

The first experiment on the ultimatum game is reported in Güth, Schmittberger, and Schwarze (1982). Grout (1984) is an early analysis of a holdup game. The model in Section 6.3 is due to von Stackelberg (1934). The vote-buying game in Section 6.4 is taken from Groseclose and Snyder (1996). The model of a race in Section 6.5 is a simplification suggested by Vijay Krishna of a model of Harris and Vickers (1985).

For more discussion of the experimental evidence on the ultimatum game (discussed in the box on page 181), see Roth (1995). Bolton and Ockenfels (2000) study the implications of assuming that players are equity-conscious, and relate these implications to the experimental outcomes in various games. The explanation of the experimental results in terms of rules of thumb is discussed by Aumann (1997, 7–8). The problem of fair division, an example of which is given in Exercise 183.2, is studied in detail by Brams and Taylor (1996), who trace the idea of divide-and-choose back to antiquity (p. 10). I have been unable to find the origin of the idea in Exercise 199.1; Barton Lipman suggested the formulation in the exercise.

7 Extensive Games with Perfect Information: Extensions and Discussion

Prerequisite: Chapter 5.

7.1 Allowing for simultaneous moves

7.1.1 Definition

THE model of an extensive game with perfect information (Definition 153.1) assumes that after every sequences of events, a single decision-maker takes an action, knowing every decision-maker's previous actions. I now describe a more general model that allows us to study situations in which, after some sequences of events, the members of a group of decision-makers choose their actions "simultaneously", each member knowing every decision-maker's *previous* actions, but not the contemporaneous actions of the other members of the group.

In the more general model, a terminal history is a sequence of *lists* of actions, each list specifying the actions of a set of players. (A game in which each set contains a single player is an extensive game with perfect information as defined previously.) For example, consider a situation in which player 1 chooses either C or D, then players 2 and 3 simultaneously take actions, each choosing either E or F. In the extensive game that models this situation, $(C, (E, E))$ is a terminal history, in which first player 1 chooses C, and then players 2 and 3 both choose E. In the general model, the player function assigns a *set* of players to each nonterminal history. In the example just described, this set consists of the single player 1 for the initial history, and consists of players 2 and 3 for the history C.

An extensive game with perfect information (Definition 153.1) does not specify explicitly the sets of actions available to the players. However, we may derive the set of actions of the player who moves after any nonterminal history from the set of terminal histories and the player function (see (154.1)). When we allow simultaneous moves, the players' sets of actions are conveniently specified in the definition of a game. In the example of the previous paragraph, for instance, we specify the game by giving the eight possible terminal histories (C or D followed by one of the four pairs (E, E), (E, F), (F, E), and (F, F)), the player function defined by $P(\varnothing) = 1$ and $P(C) = P(D) = \{2, 3\}$, the sets of actions $\{C, D\}$ for player 1 at the start of the game and $\{E, F\}$ for both player 2 and player 3 after the histories C and D, and each player's preferences over terminal histories.

In any game, the set of terminal histories, player function, and sets of actions for the players must be consistent: the list of actions that follows a subhistory of any terminal history must be a list of actions of the players assigned by the player function to that subhistory. In the game described above, for example, the list of actions following the subhistory C of the terminal history $(C, (E, E))$ is (E, E), which is a pair of actions for the players (2 and 3) assigned by the player function to the history C.

Precisely, an extensive game with perfect information and simultaneous moves is defined as follows.

▶ DEFINITION 202.1 An **extensive game with perfect information and simultaneous moves** consists of

- a set of **players**
- a set of sequences (**terminal histories**) with the property that no sequence is a proper subhistory of any other sequence
- a function (the **player function**) that assigns a set of players to every sequence that is a proper subhistory of some terminal history
- for each proper subhistory h of each terminal history and each player i that is a member of the set of players assigned to h by the player function, a set $A_i(h)$ (the set of **actions** available to player i after the history h)
- for each player, **preferences** over the set of terminal histories

such that the set of terminal histories, player function, and sets of actions are consistent in the sense that h is a terminal history if and only if either (i) h takes the form $(a^1, \ldots, a^k)$ for some integer k, the player function is not defined at h, and for every $\ell = 0, \ldots, k-1$, $a^{\ell+1}$ is a list of actions of the players assigned by the player function to $(a^1, \ldots, a^\ell)$ (the empty history if $\ell = 0$), or (ii) h takes the form $(a^1, a^2, \ldots)$ and for every $\ell = 0, 1, \ldots$, $a^{\ell+1}$ is a list of actions of the players assigned by the player function to $(a^1, \ldots, a^\ell)$ (the empty history if $\ell = 0$).

This definition encompasses both extensive games with perfect information as in Definition 153.1 and, in a sense, strategic games. An extensive game with perfect information is an extensive game with perfect information and simultaneous moves in which the set of players assigned to each history consists of exactly one member. (The definition of an extensive game with perfect information and simultaneous moves includes the players' actions, whereas the definition of an extensive game with perfect information does not. However, actions may be derived from the terminal histories and player function of the latter.)

For any strategic game there is an extensive game with perfect information and simultaneous moves in which every terminal history has length one that models the same situation. In this extensive game, the set of terminal histories is the set of action profiles in the strategic game, the player function assigns the set of all players to the initial history, and the single set $A_i(\varnothing)$ of actions of each player i is the set of actions of player i in the strategic game.

◆ EXAMPLE 203.1 (Variant of *BoS*) First, person 1 decides whether to stay home and read a book or to attend a concert. If she reads a book, the game ends. If she decides to attend a concert then, as in *BoS*, she and person 2 independently choose whether to sample the aural delights of Bach or Stravinsky, not knowing the other person's choice. Both people prefer to attend the concert of their favorite composer in the company of the other person to the outcome in which person 1 stays home and reads a book, and prefer this outcome to attending the concert of their less preferred composer in the company of the other person; the worst outcome for both people is that they attend different concerts.

The following extensive game with perfect information and simultaneous moves models this situation.

Players The two people (1 and 2).

Terminal histories $Book$, $(Concert, (B, B))$, $(Concert, (B, S))$, $(Concert, (S, B))$, $(Concert, (S, S))$.

Player function $P(\varnothing) = 1$ and $P(Concert) = \{1, 2\}$.

Actions The set of player 1's actions at the initial history $\varnothing$ is $A_1(\varnothing) = \{Concert, Book\}$ and the set of her actions after the history *Concert* is $A_1(Concert) = \{B, S\}$; the set of player 2's actions after the history *Concert* is $A_2(Concert) = \{B, S\}$.

Preferences Player 1 prefers $(Concert, (B, B))$ to *Book* to $(Concert, (S, S))$ to $(Concert, (B, S))$, which she regards as indifferent to $(Concert, (S, B))$. Player 2 prefers $(Concert, (S, S))$ to *Book* to $(Concert, (B, B))$ to $(Concert, (B, S))$, which she regards as indifferent to $(Concert, (S, B))$.

This game is illustrated in Figure 204.1, in which I represent the simultaneous choices between B and S in the way that I previously represented a strategic game. (Only a game in which all the simultaneous moves occur at the end of terminal histories may be represented in a diagram like this one. For most other games no convenient diagrammatic representation exists.)

7.1.2 Strategies and Nash equilibrium

As in a game without simultaneous moves, a player's strategy specifies the action she chooses for every history after which it is her turn to move. Definition 157.1 requires only minor rewording to allow for the possibility that players may move simultaneously.

► DEFINITION 203.2 A **strategy** of player i in an extensive game with perfect information and simultaneous moves is a function that assigns to each history h after

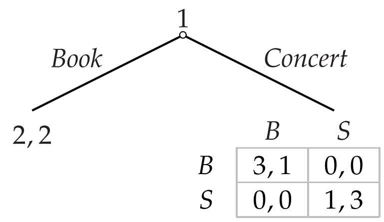


Figure 204.1 The variant of *BoS* described in Example 204.1.

which i is one of the players whose turn it is to move (i.e. i is a member of $P(h)$, where P is the player function of the game) an action in $A_i(h)$ (the set of actions available to player i after h).

The definition of a *Nash equilibrium* of an extensive game with perfect information and simultaneous moves is exactly the same as the definition for a game with no simultaneous moves (Definition 159.2): a Nash equilibrium is a strategy profile with the property that no player can induce a better outcome for herself by changing her strategy, given the other players' strategies. Also as before, the *strategic form* of a game is the strategic game in which the players' actions are their strategies in the extensive game (see Section 5.4), and a strategy profile is a Nash equilibrium of the extensive game if and only if it is a Nash equilibrium of the strategic form of the game.

◆ EXAMPLE 204.1 (Nash equilibria of a variant of *BoS*) In the game in Example 203.1, a strategy of player 1 specifies her actions at the start of the game and after the history *Concert*; a strategy of player 2 specifies her action after the history *Concert*. Thus player 1 has four strategies, $(Concert, B)$, $(Concert, S)$, $(Book, B)$, and $(Book, S)$, and player 2 has two strategies, B and S. (Remember that a player's strategy is more than a plan of action; it specifies an action for *every* history after which the player moves, even histories that it precludes. For example, player 1's strategy specifies her action after the history *Concert* even if it specifies that she choose *Book* at the beginning of the game.)

The strategic form of the game is given in Figure 204.2. We see that the game has three pure Nash equilibria: $((Concert, B), B)$, $((Book, B), S)$, and $((Book, S), S)$.

	B	S
$(Concert, B)$	3, 1	0, 0
$(Concert, S)$	0, 0	1, 3
$(Book, B)$	2, 2	2, 2
$(Book, S)$	2, 2	2, 2

Figure 204.2 The strategic form of the game in Example 203.1.

Every extensive game has a unique strategic form. However, some strategic games are the strategic forms of more than one extensive game. Consider, for example, the strategic game in Figure 205.1. This game is the strategic form of the extensive game with perfect information and simultaneous moves in which the two players choose their actions simultaneously; it is also the strategic form of the entry game in Figure 154.1.

	L	R
T	1, 2	1, 2
B	0, 0	2, 0

Figure 205.1 A strategic game that is the strategic form of more than one extensive game.

7.1.3 Subgame perfect equilibrium

As for a game in which one player moves after each history, the subgame following the history h of an extensive game with perfect information and simultaneous moves is the extensive game "starting at h". (The formal definition is a variant of Definition 162.1.)

For instance, the game in Example 203.1 has two subgames: the whole game, and the game in which the players engage after player 1 chooses *Concert*. In the second subgame, the terminal histories are (B, B), (B, S), (S, B), and (S, S), the player function assigns the set $\{1, 2\}$ consisting of both players to the initial history (the only nonterminal history), the set of actions of each player at the initial history is $\{B, S\}$, and the players' preferences are represented by the payoffs in the table in Figure 204.1. (This subgame models the same situation as *BoS*.)

A subgame perfect equilibrium is defined as before: a *subgame perfect equilibrium* of an extensive game with perfect information and simultaneous moves is a strategy profile with the property that in no subgame can any player increase her payoff by choosing a different strategy, given the other players' strategies. The formal definition differs from the definition of a subgame perfect equilibrium of a game without simultaneous moves (164.1) only in that the meaning of "it is player i's turn to move" is that i is a member of $P(h)$, rather than $P(h) = i$.

To find the set of subgame perfect equilibria of an extensive game with perfect information and simultaneous moves that has a finite horizon, we can, as before, use backward induction. The only wrinkle is that some (perhaps all) of the situations we need to analyze are not single-person decision problems, as they are in the absence of simultaneous moves, but problems in which several players choose actions simultaneously. We cannot simply find an optimal action for the player whose turn it is to move at the start of each subgame, given the players' behavior in the remainder of the game. We need to find a *list* of actions for the players who move at the start of each subgame, with the property that each player's action is optimal given the other players' simultaneous actions and the players' behavior in the remainder of the game. That is, the argument we need to make is the same as the one we make when finding a Nash equilibrium of a strategic game. This argument may use any of the techniques discussed in Chapter 2: it may check each action profile in turn, it may construct and study the players' best response functions, or it may show directly that an action profile we have obtained by a combination of intuition and trial and error is an equilibrium.

◆ EXAMPLE 206.1 (Subgame perfect equilibria of a variant of *BoS*) Consider the game in Figure 204.1. Backward induction proceeds as follows.

- In the subgame that follows the history *Concert*, there are two Nash equilibria (in pure strategies), namely (S, S) and (B, B), as we found in Section 2.7.2.
- If the outcome in the subgame that follows *Concert* is (S, S) then the optimal choice of player 1 at the start of the game is *Book*.
- If the outcome in the subgame that follows *Concert* is (B, B) then the optimal choice of player 1 at the start of the game is *Concert*.

We conclude that the game has two subgame perfect equilibria: $((Book, S), S)$ and $((Concert, B), B)$.

Every finite extensive game with perfect information has a (pure) subgame perfect equilibrium (Proposition 171.1). The same is not true of a finite extensive game with perfect information and simultaneous moves because, as we know, a finite strategic game (which corresponds to an extensive game with perfect information and simultaneous moves of length one) may not possess a pure strategy Nash equilibrium. (Consider *Matching pennies* (Example 17.1).) If you have studied Chapter 4, you know that some strategic games that lack a pure strategy Nash equilibrium have a "mixed strategy Nash equilibrium", in which each player randomizes. The same is true of extensive games with perfect information and simultaneous moves. However, in this chapter I restrict attention almost exclusively to pure strategy equilibria; the only occasion on which mixed strategy Nash equilibrium appears is Exercise 208.1.

? EXERCISE 206.2 (Extensive game with simultaneous moves) Find the subgame perfect equilibria of the following game. First player 1 chooses either A or B. After either choice, she and player 2 simultaneously choose actions. If player 1 initially chooses A then she and player 2 subsequently each choose either C or D; if player 1 chooses B initially then she and player 2 subsequently each choose either E or F. Among the terminal histories, player 1 prefers $(A, (C, C))$ to $(B, (E, E))$ to $(A, (D, D))$ to $(B, (F, F))$, and prefers all these to $(A, (C, D))$, $(A, (D, C))$, $(B, (E, F))$, and $(B, (F, E))$, between which she is indifferent. Player 2 prefers $(A, (D, D))$ to $(B, (F, F))$ to $(A, (C, C))$ to $(B, (E, E))$, and prefers all these to $(A, (C, D))$, $(A, (D, C))$, $(B, (E, F))$, and $(B, (F, E))$, between which she is indifferent.

? EXERCISE 206.3 (Two-period *Prisoner's Dilemma*) Two people simultaneously choose actions; each person chooses either Q or F (as in the *Prisoner's Dilemma*). Then they simultaneously choose actions again, once again each choosing either Q or F. Each person's preferences are represented by the payoff function that assigns to the terminal history $((W, X), (Y, Z))$ (where each component is either Q or F)

a payoff equal to the sum of the person's payoffs to (W, X) and to (Y, Z) in the *Prisoner's Dilemma* given in Figure 13.1. Specify this situation as an extensive game with perfect information and simultaneous moves and find its subgame perfect equilibria.

? EXERCISE 207.1 (Timing claims on an investment) An amount of money is accumulating; in period t $(= 1, 2, \ldots, T)$ its size is $\$2t$. In each period two people simultaneously decide whether to claim the money. If only one person does so, she gets all the money; if both people do so, they split the money equally; and if neither person does so, both people have the opportunity to do so in the next period. If neither person claims the money in period T, each person obtains $\$T$. Each person cares only about the amount of money she obtains. Formulate this situation as an extensive game with perfect information and simultaneous moves, and find its subgame perfect equilibria). (Start by considering the cases $T = 1$ and $T = 2$.)

? EXERCISE 207.2 (A market game) A seller owns one indivisible unit of a good, which she does not value. Several potential buyers, each of whom attaches the same positive value v to the good, simultaneously offer prices they are willing to pay for the good. After receiving the offers, the seller decides which, if any, to accept. If she does not accept any offer, then no transaction takes place, and all payoffs are 0. Otherwise, the buyer whose offer the seller accepts pays the amount p she offered and receives the good; the payoff of the seller is p, the payoff of the buyer who obtained the good is $v - p$, and the payoff of every other buyer is 0. Model this situation as an extensive game with perfect information and simultaneous moves and find its subgame perfect equilibria. (Use a combination of intuition and trial and error to find a strategy profile that appears to be an equilibrium, then argue directly that it is. The incentives in the game are closely related to those in Bertrand's oligopoly game (see Exercise 66.1), with the roles of buyers and sellers reversed.) Show, in particular, that in every subgame perfect equilibrium every buyer's payoff is zero.

MORE EXPERIMENTAL EVIDENCE ON SUBGAME PERFECT EQUILIBRIUM

Experiments conducted in 1989 and 1990 among college students (mainly taking economics classes) show that the subgame perfect equilibria of the game in Exercise 207.2 correspond closely to experimental outcomes (Roth, Prasnikar, Okuno-Fujiwara, and Zamir 1991), in contrast to the subgame perfect equilibrium of the ultimatum game (see the box on page 181).

In experiments conducted at four locations (Jerusalem, Ljubljana, Pittsburgh, and Tokyo), nine "buyers" simultaneously bid for the rough equivalent (in terms of local purchasing power) of US$10, held by a "seller". Each experiment involved a group of 20 participants, which was divided into two markets, each with one seller and nine buyers. Each participant was involved in ten rounds of the market; in each round the sellers and buyers were assigned anew, and in any given round no participant knew who, among the other participants, were sellers and buyers, and who was involved in her market. In every session of the experiment the maximum proposed price was accepted by the seller, and by the seventh round of every experiment the highest bid was at least (the equivalent of) US$9.95.

Experiments involving the ultimatum game, run in the same locations using a similar design, yielded results similar to those of previous experiments (see the box on page 181): proposers kept considerably less than 100% of the pie, and nontrivial offers were rejected.

The box on page 181 discusses two explanations for the experimental results in the ultimatum game. Both explanations are consistent with the results in the market game. One explanation is that people are concerned not only with their own monetary payoffs, but also with other people's payoffs. At least some specifications of such preferences do not affect the subgame perfect equilibria of a market game with many buyers, which still all yield every buyer the payoff of zero. (When there are many buyers, even a seller who cares about the other players' payoffs accepts the highest price offered, because accepting a lower price has little impact on the distribution of monetary payoffs, all but two of which remain zero.) Thus such preferences are consistent with both sets of experimental outcomes. Another explanation is that people incorrectly recognize the ultimatum game as one in which the rule of thumb "don't be a sucker" is advantageously invoked, and thus reject a poor offer, "punishing" the person who makes such an offer. In the market game, the players treated poorly in the subgame perfect equilibrium are the buyers, who have no opportunity to punish any other player, because they move first. Thus the rule of thumb is not relevant in this game, so that this explanation is also consistent with both sets of experimental outcomes.

In the next exercise you are asked to investigate subgame perfect equilibria in which some players use mixed strategies (discussed in Chapter 4).

?? EXERCISE 208.1 (Price competition) Extend the model in Exercise 125.2 by having the sellers simultaneously choose their prices before the buyers simultaneously choose which seller to approach. Assume that each seller's preferences are represented by the expected value of a Bernoulli payoff function in which the payoff to not trading is 0 the payoff to trading at the price p is p. Formulate this model precisely as an extensive game with perfect information and simultaneous moves. Show that for every $p \geq \frac{1}{2}$ the game has a subgame perfect equilibrium in which each seller announces the price p. (You may use the fact that if seller j's price is at least $\frac{1}{2}$, seller i's payoff in the mixed strategy equilibrium of the subgame in which the buyers choose which seller to approach is decreasing in her price p_i when $p_i > p_j$.)

7.2 Illustration: entry into a monopolized industry

7.2.1 General model

An industry is currently monopolized by a single firm (the "incumbent"). A second firm (the "challenger") is considering entry, which entails a positive cost f in addition to its production cost. If the challenger stays out then its profit is zero, whereas if it enters, the firms simultaneously choose outputs (as in Cournot's model of duopoly (Section 3.1)). The cost to firm i of producing q_i units of output is $C_i(q_i)$. If the firms' total output is Q then the market price is $P_d(Q)$. (As in Section 6.3, I add a subscript to P to avoid a clash with the player function of the game.)

We can model this situation as the following extensive game with perfect information and simultaneous moves, illustrated in Figure 209.1.

Players The two firms: the incumbent (firm 1) and the challenger (firm 2).

Terminal histories $(In, (q_1, q_2))$ for any pair (q_1, q_2) of outputs (nonnegative numbers), and (Out, q_1) for any output q_1.

Player function $P(\varnothing) = \{2\}$, $P(In) = \{1, 2\}$, and $P(Out) = \{1\}$.

Actions $A_2(\varnothing) = \{In, Out\}$; $A_1(In)$, $A_1(Out)$, and $A_2(In)$ are all equal to the set of possible outputs (nonnegative numbers).

Preferences Each firm's preferences are represented by its profit, which for a terminal history $(In, (q_1, q_2))$ is $q_1 P_d(q_1 + q_2) - C_1(q_1)$ for the incumbent and $q_2 P_d(q_1 + q_2) - C_2(q_2) - f$ for the challenger, and for a terminal history (Out, q_1) is $q_1 P_d(q_1) - C_1(q_1)$ for the incumbent and 0 for the challenger.

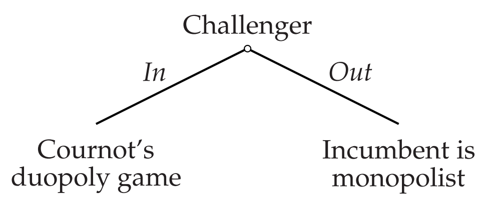


Figure 209.1 An entry game.

7.2.2 Example

Suppose that $C_i(q_i) = cq_i$ for all q_i ("unit cost" is constant, equal to c), and the inverse demand function is linear where it is positive, given by $P_d(Q) = \alpha - Q$ for $Q \leq \alpha$, as in Section 3.1.3. To find the subgame perfect equilibria, first consider the subgame that follows the history In. The strategic form of this subgame is the same as the example of Cournot's duopoly game studied in Section 3.1.3, except that the payoff of the challenger is reduced by f (the fixed cost of entry) regardless of the challenger's output. Thus the subgame has a unique Nash equilibrium, in which the output of each firm is $\frac{1}{3}(\alpha - c)$; the incumbent's profit is $\frac{1}{9}(\alpha - c)^2$, and the challenger's profit is $\frac{1}{9}(\alpha - c)^2 - f$.

Now consider the subgame that follows the history *Out*. In this subgame the incumbent chooses an output. The incumbent's profit when it chooses the output q_1 is $q_1(\alpha - q_1) - cq_1 = q_1(\alpha - c - q_1)$. This function is a quadratic that increases and then decreases as q_1 increases, and is zero when $q_1 = 0$ and when $q_1 = \alpha - c$. Thus the function is maximized when $q_1 = \frac{1}{2}(\alpha - c)$. We conclude that in any subgame perfect equilibrium the incumbent chooses $q_1 = \frac{1}{2}(\alpha - c)$ in the subgame following the history *Out*.

Finally, consider the challenger's action at the start of the game. If the challenger stays out then its profit is 0, whereas if it enters then, given the actions chosen in the resulting subgame, its profit is $\frac{1}{9}(\alpha - c)^2 - f$. Thus in any subgame perfect equilibrium the challenger enters if $\frac{1}{9}(\alpha - c)^2 > f$ and stays out if $\frac{1}{9}(\alpha - c)^2 < f$. If $\frac{1}{9}(\alpha - c)^2 = f$ then the game has two subgame perfect equilibria, in one of which the challenger enters and in the other of which it does not.

In summary, the set of subgame perfect equilibria depend on the value of f. In all equilibria the incumbent's strategy is to produce $\frac{1}{3}(\alpha - c)$ if the challenger enters and $\frac{1}{2}(\alpha - c)$ if it does not, and the challenger's strategy involves its producing $\frac{1}{3}(\alpha - c)$ if it enters.

- If $f < \frac{1}{9}(\alpha - c)^2$ there is a unique subgame perfect equilibrium, in which the challenger enters. The outcome is that the challenger enters and each firm produces the output $\frac{1}{3}(\alpha - c)$.
- If $f > \frac{1}{9}(\alpha - c)^2$ there is a unique subgame perfect equilibrium, in which the challenger stays out. The outcome is that the challenger stays out and the incumbent produces $\frac{1}{2}(\alpha - c)$.
- If $f = \frac{1}{9}(\alpha - c)^2$ the game has two subgame perfect equilibria: the one for the case $f < \frac{1}{9}(\alpha - c)^2$ and the one for the case $f > \frac{1}{9}(\alpha - c)^2$.

Why, if f is small, does the game have no subgame perfect equilibrium in which the incumbent floods the market if the challenger enters, so that the challenger optimally stays out and the incumbent obtains a profit higher than its profit if the challenger enters? Because the action this strategy prescribes after the history in which the challenger enters is not the incumbent's action in a Nash equilibrium of the subgame: the subgame has a unique Nash equilibrium, in which each firm produces $\frac{1}{3}(\alpha - c)$. Put differently, the incumbent's "threat" to flood the market if the challenger enters is not credible.

? EXERCISE 210.1 (Bertrand's duopoly game with entry) Find the subgame perfect equilibria of the variant of the game studied in this section in which the post-entry competition is a game in which each firm chooses a price, as in the example of Bertrand's duopoly game studied in Section 3.2.2, rather than an output.

7.3 Illustration: electoral competition with strategic voters

The voters in Hotelling's model of electoral competition (Section 3.3) are not players in the game: each citizen is assumed simply to vote for the candidate whose position she most prefers. How do the conclusions of the model change if we assume that each citizen *chooses* the candidate for whom to vote?

Consider the extensive game in which the candidates first simultaneously choose actions, then the citizens simultaneously choose how to vote. As in the variant of Hotelling's game considered on page 72, assume that each candidate may either choose a position (as in Hotelling's original model) or choose to stay out of the race, an option she is assumed to rank between losing and tying for first place with all the other candidates.

Players The candidates and the citizens.

Terminal histories All sequences (x, v) where x is a list of the candidates' actions, each component of which is either a position (a number) or *Out*, and v is a list of voting decisions for the citizens (i.e. a list of candidates, one for each citizen).

Player function $P(\varnothing)$ is the set of all the candidates, and $P(x)$, for any list x of positions for the candidates, is the set of all citizens.

Actions The set of actions available to each candidate at the start of the game consists of *Out* and the set of possible positions. The set of actions available to each citizen after a history x is the set of candidates.

Preferences Each candidate's preferences are represented by a payoff function that assigns n to every terminal history in which she wins outright, k to every terminal history in which she ties for first place with $n-k$ other candidates (for $1 \leq k \leq n-1$), 0 to every terminal history in which she stays out of the race, and -1 to every terminal history in which she loses, where n is the number of candidates. Each citizen's preferences are represented by a payoff function that assigns to each terminal history the average distance from the citizen's favorite position of the set of winning candidates in that history.

First consider the game in which there are two candidates (and an arbitrary number of citizens). Every subgame following choices of positions by the candidates has many Nash equilibria (as you know if you solved Exercise 47.1). For example, any action profile in which *all* citizens vote for the same candidate is a Nash equilibrium. (A citizen's switching her vote to another candidate has no effect on the outcome.)

This plethora of Nash equilibria allows us to construct, for *every* pair of positions, a subgame perfect equilibrium in which the candidates choose those positions! Consider the strategy profile in which the candidates choose the positions x_1 and x_2, and

- all citizens vote for candidate 1 after a history (x_1', x_2') in which $x_1' = x_1$
- all citizens vote for candidate 2 after a history (x_1', x_2') in which $x_1' \neq x_1$.

The outcome is that the candidates choose the positions x_1 and x_2 and candidate 1 wins. The strategy profile is a subgame perfect equilibrium because for every history (x_1, x_2) the profile of the citizens' actions is a Nash equilibrium, and neither candidate can induce an outcome she prefers by deviating: a deviation by candidate 1 to a position different from x_1 leads her to lose, and a deviation by candidate 2 has no effect on the outcome.

However, most of the Nash equilibria of the voting subgames are fragile (as you know if you solved Exercise 47.1): a citizen's voting for her less preferred candidate is weakly dominated (Definition 45.1) by her voting for her favorite candidate. (A citizen who switches from voting for her less preferred candidate to voting for her favorite candidate either does not affect the outcome (if her favorite candidate was three or more votes behind) or causes her favorite candidate either to tie for first place rather than lose, or to win rather than tie.) Thus in the only Nash equilibrium of a voting subgame in which no citizen uses a weakly dominated action, each citizen votes for the candidate whose position is closest to her favorite position.

Hotelling's model (Section 3.3) *assumes* that each citizen votes for the candidate whose position is closest to her favorite position; in its unique Nash equilibrium, each candidate's position is the median of the citizens' favorite positions. Combining this result with the result of the previous paragraph, we conclude that the game we are studying has only one subgame perfect equilibrium in which no player's strategy is weakly dominated: each candidate chooses the median of the citizens' favorite positions, and for every pair of the candidates' positions, each citizen votes for her favorite candidate.

In the game with three or more candidates, not only do many of the voting subgames have many Nash equilibria, with a variety of outcomes, but restricting to voting strategies that are not weakly dominated does not dramatically affect the set of equilibria: a citizen's only weakly dominated strategy is a vote for her least preferred candidate (see Exercise 47.2).

However, the set of equilibrium outcomes is dramatically restricted by the assumption that each candidate prefers to stay out of the race than to enter and lose, as the next two exercises show. The result in the first exercise is that the game has a subgame perfect equilibrium in which no citizen's strategy is weakly dominated and every candidate enters and chooses as her position the median of the citizens' favorite positions. The result in the second exercise is that under an assumption that makes the citizens averse to ties and an assumption that there exist citizens with extreme preferences, in *every* subgame perfect equilibrium all candidates who enter do so at the median of the citizens' favorite positions. The additional assumptions about the citizens' preferences are much stronger that necessary; they are designed to make the argument relatively easy.

? EXERCISE 212.1 (Electoral competition with strategic voters) Assume that there are $n \geq 3$ candidates and q citizens, where $q \geq 2n$ is odd (so that the median of the voters' favorite positions is well-defined) and divisible by n. Show that the game has a subgame perfect equilibrium in which no citizen's strategy is weakly dominated and every candidate enters the race and chooses the median of the citizens' favorite positions. (You may use the fact that every voting subgame has a (pure) Nash equilibrium in which no citizen's action is weakly dominated.)

?? EXERCISE 213.1 (Electoral competition with strategic voters) Consider the variant of the game in this section in which (*i*) the set of possible positions is the set of numbers x with $0 \leq x \leq 1$, (*ii*) the favorite position of at least one citizen is 0 and the favorite position of at least one citizen is 1, and (*iii*) each citizen's preferences are represented by a payoff function that assigns to each terminal history the distance from the citizen's favorite position to the position of the candidate in the set of winners whose position is *furthest* from her favorite position. Under the other assumptions of the previous exercise, show that in every subgame perfect equilibrium in which no citizen's action is weakly dominated, the position chosen by every candidate who enters is the median of the citizens' favorite positions. To do so, first show that in any equilibrium each candidate that enters is in the set of winners. Then show that in any Nash equilibrium of any voting subgame in which there are more than two candidates and not all candidates' positions are the same, some candidate loses. (Argue that if all candidates tie for first place, some citizen can increase her payoff by changing her vote.) Finally, show that in any subgame perfect equilibrium in which either only two candidates enter, or all candidates who enter choose the same position, every entering candidates chooses the median of the citizens' favorite positions.

7.4 Illustration: committee decision-making

How does the procedure used by a committee affect the decision it makes? One approach to this question models a decision-making procedure as an extensive game with perfect information and simultaneous moves in which a sequence of ballots are taken, in each of which the committee members vote simultaneously, and the result of each ballot determines the choices on the next ballot, or, eventually, the decision to be made.

Fix a set of committee members and a set of *alternatives* over which each member has strict preferences (no member is indifferent between any two alternatives). Assume that the number of committee members is odd, to avoid ties in votes. If there are two alternatives, the simplest committee procedure is that in which the members vote simultaneously for one of the alternatives. (We may interpret the game in Section 2.9.3 as a model of this procedure.) In the procedure illustrated in Figure 214.1, there are three alternatives, x, y, and z. The committee first votes whether to choose x (option "a") or to eliminate it from consideration (option "b"). If it votes to eliminate x, it subsequently votes between y and z.

In these procedures, each vote is between two options. Such procedures are

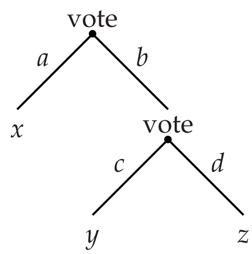


Figure 214.1 A voting procedure, or "binary agenda".

called *binary agendas*. We may define a binary agenda with the aid of an auxiliary one-player extensive game with perfect information in which the set $A(h)$ of actions following any nonterminal history h contains two elements, and the number of terminal histories is at least the number of alternatives. We associate with every terminal history h of this auxiliary game an alternative $\alpha(h)$ in such a way that each alternative is associated with at least one terminal history.

In the binary agenda associated with the auxiliary game G, all players vote simultaneously whenever the player in G takes an action. The options on the ballot following the nonterminal history in which a majority of committee members choose option a^1 at the start of the game, then option a^2, and so on, are the members of the set $A(a^1, \ldots, a^k)$ of actions of the player in G after the history $(a^1, \ldots, a^k)$. The alternative selected after the terminal history in which the majority choices are $a^1, \ldots, a^k$ is the alternative $\alpha(a^1, \ldots, a^k)$ associated with $(a^1, \ldots, a^k)$ in G. For example, in the auxiliary one-person game that defines the structure of the agenda in Figure 214.1, the single player first chooses a or b; if she chooses a the game ends, whereas if she chooses b, she then chooses between c and d. The alternative x is associated with the terminal history a, y is associated with (b, c), and z is associated with (b, d).

Precisely, the **binary agenda** associated with the auxiliary game G is the extensive game with perfect information and simultaneous moves defined as follows.

Players The set of committee members.

Terminal histories A sequence $(v^1, \ldots, v^k)$ of action profiles (in which each v^j is a list of the players' votes) is a terminal history if and only if there is a terminal history $(a^1, \ldots, a^k)$ of G such that for every $j = 0, \ldots, k-1$, every element of v^{j+1} is a member of $A(a^1, \ldots, a^j)$ ($A(\varnothing)$ if $j = 0$) and a majority of the players' actions in v^{j+1} are equal to a^{j+1}.

Player function For every nonterminal history h, $P(h)$ is the set of all players.

Actions For every player i and every nonterminal history $(v^1, \ldots, v^j)$, player i's set of actions is $A(a^1, \ldots, a^j)$, where $(a^1, \ldots, a^j)$ is the history of G in which, for all ℓ, a^ℓ is the action chosen by the majority of players in v^ℓ.

Preferences The rank each player assigns to the terminal history $(v^1, \dots, v^k)$ is equal to the rank she assigns to the alternative $\alpha(a^1, \dots, a^k)$ associated with the terminal history $(a^1, \dots, a^k)$ of G in which, for all j, a^j is the action chosen by a majority of players in v^j.

Every binary agenda, like every voting subgame of the model in the previous section, has many subgame perfect equilibria. In fact, in any binary agenda, *every* alternative is the outcome of some subgame perfect equilibrium, because if, in every vote, every player votes for the same option, no player can affect the outcome by changing her strategy. However, if we restrict attention to weakly undominated strategies, we greatly reduce the set of equilibria. As we saw before (Section 2.9.3), in a ballot with two options, a player's action of voting for the option she prefers weakly dominates the action of voting for the other option. Thus in a subgame perfect equilibrium of a binary agenda in which every player's vote on every ballot is weakly undominated, on each ballot every player votes for the option that leads, ultimately (given the outcomes of the later ballots), to the alternative she prefers. The alternative associated with the terminal history generated by such a subgame perfect equilibrium is said to be the outcome of *sophisticated voting*.

Which alternatives are the outcomes of sophisticated voting in binary agendas? Say that alternative x *beats* alternative y if a majority of committee members prefer x to y. An alternative that beats every other alternative is called a *Condorcet winner*. For any preferences, there is either one Condorcet winner or no Condorcet winner (see Exercise 74.1).

First suppose that the players' preferences are such that some alternative, say x^*, is a Condorcet winner. I claim that x^* is the outcome of sophisticated voting in *every* binary agenda. The argument, using backward induction, is simple. First consider a subgame of length 1 in which one option leads to the alternative x^*. In this subgame a majority of the players vote for the option that leads to x^*, because a majority prefers x^* to every other alternative, and each player's only weakly undominated strategy is to vote for the option that leads to the alternative she prefers. Thus in at least one subgame of length 2, at least one option leads ultimately to the decision x^* (given the players' votes in the subgames of length 1). In this subgame, by the same argument as before, the winning option leads to x^*. Continuing backwards, we conclude that at least one option on the first ballot leads ultimately to x^*, and that consequently the winning option on this ballot leads to x^*.

Thus if the players' preferences are such that a Condorcet winner exists, the agenda does not matter: the outcome of sophisticated voting is always the Condorcet winner. If the players' preferences are such that no alternative is a Condorcet winner, the outcome of sophisticated voting depends on the agenda. Consider, for example, a committee with three members facing three alternatives. Suppose that one member prefers x to y to z, another prefers y to z to x, and the third prefers z to x to y. For these preferences, no alternative is a Condorcet winner. The outcome of sophisticated voting in the binary agenda in Figure 214.1 is the alternative x. (Use backward induction: y beats z, and x beats y.) If the positions of x

and y are interchanged then the outcome is y, and if the positions of x and z are interchanged then the outcome is z. Thus in this case, for *every* alternative there is a binary agenda for which that alternative is the outcome of sophisticated voting.

Which alternatives are the outcomes of sophisticated voting in binary agendas when no alternative is a Condorcet winner? Consider a committee with arbitrary preferences (not necessarily the ones considered in the previous paragraph), using the agenda in Figure 214.1. In order for x to be the outcome of sophisticated voting it must beat the winner of y and z. It may not beat both y and z directly, but it must beat them both at least "indirectly": either x beats y beats x, or x beats z beats y. Similarly, if y or z is the outcome of sophisticated voting then it must beat both of the other alternatives at least indirectly.

Precisely, say that alternative x *indirectly beats* alternative y if for some $k \geq 1$ there are alternatives $u_1, \ldots, u_k$ such that x beats u_1, u_j beats u_{j+1} for $j = 1, \ldots, k-1$, and u_k beats y. The set of alternatives x such that x beats every other alternative either directly or indirectly is called the *top cycle set*. (Note that if alternative x beats any alternative indirectly, it beats at least one alternative directly.) If there is a Condorcet winner, then the top cycle set consists of this single alternative. If there is no Condorcet winner, then the top cycle set contains more than one alternative.

? EXERCISE 216.1 (Top cycle set) A committee has three members.

a. Suppose that there are three alternatives, x, y, and z, and that one member prefers x to y to z, another prefers y to z to x, and the third prefers z to x to y. Find the top cycle set.

b. Suppose that there are four alternatives, w, x, y, and z, and that one member prefers w to z to x to y, one member prefers y to w to z to x, and one member prefers x to y to w to z. Find the top cycle set. Show, in particular, that z is in the top cycle set even though *all* committee members prefer w.

Rephrasing my conclusion for the agenda in Figure 214.1, if an alternative is the outcome of sophisticated voting, then it is in the top cycle set. The argument for this conclusion extends to any binary agenda. In every subgame, the outcome of sophisticated voting must beat the alternative that will be selected if it is rejected. Thus by backward induction, the outcome of sophisticated voting in the whole game must beat every other alternative either directly or indirectly: the outcome of sophisticated voting in any binary agenda is in the top cycle set.

Now consider a converse question: for any given alternative x in the top cycle set, is there a binary agenda for which x is the outcome of sophisticated voting? The answer is affirmative. The idea behind the construction of an appropriate agenda is illustrated by a simple example. Suppose that there are three alternatives, x, y, and z, and x beats y beats z. Then the agenda in Figure 214.1 is one for which x is the outcome of sophisticated voting. Now suppose there are two additional alternatives, u and w, and x beats u beats w. Then we can construct a larger agenda in which x is the outcome of sophisticated voting by replacing the alternative x in Figure 214.1 with a subgame in which a vote is taken for or against

x, and, if x is rejected, a vote is subsequently taken between u and w. If there are other chains through which x beats other alternatives, we can similarly add further subgames.

? EXERCISE 217.1 (Designing agendas) A committee has three members; there are five alternatives. One member prefers x to y to v to w to z, another prefers z to x to v to w to y, and the third prefers y to z to w to v to x. Find the top cycle set, and for each alternative a in the set design a binary agenda for which a is the outcome of sophisticated voting. Convince yourself that for no binary agenda is the outcome of sophisticated voting outside the top cycle set.

? EXERCISE 217.2 (An agenda that yields an undesirable outcome) Design a binary agenda for the committee in Exercise 216.1 for which the outcome of sophisticated voting is z (which is worse for all committee members than w).

In summary, (*i*) for any binary agenda, the alternative generated by the subgame perfect equilibrium in which no citizen's action in any ballot is weakly dominated is in the top cycle set, and (*ii*) for every alternative in the top cycle set, there is a binary agenda for which that alternative is generated by the subgame perfect equilibrium in which no citizen's action in any ballot is weakly dominated. In particular, the extent to which the procedure used by a committee affects its decision depends on the nature of the members' preferences. At one extreme, for preferences such that some alternative is a Condorcet winner, the agenda is irrelevant. At another extreme, for preferences for which every alternative is in the top cycle set, the agenda is instrumental in determining the decision. Further, for some preferences there are agendas for which the subgame perfect equilibrium yields an alternative that is unambiguously undesirable in the sense that there is another alternative that *all* committee members prefer.

7.5 Illustration: exit from a declining industry

An industry currently consists of two firms, one with a large capacity, and one with a small capacity. Demand for the firms' output is declining steadily over time. When will the firms leave the industry? Which firm will leave first? Do the firms' financial resources affect the outcome? The analysis of a model that answers these questions illustrates a use of backward induction more sophisticated than that in the previous sections of this chapter.

7.5.1 A model

Take time to be a discrete variable, starting in period 1. Denote by $P_t(Q)$ the market price in period t when the firms' total output is Q, and assume that this price is declining over time: for every value of Q, we have $P_{t+1}(Q) < P_t(Q)$ for all $t \geq 1$. (See Figure 219.1.) We are interested in the firms' decisions to exit, rather than their decisions of how much to produce in the event they stay in the market, so we assume that firm i's only decision is whether to produce some fixed output, denoted k_i, or to produce no output. (You may think of k_i as firm i's capacity.) Once a firm stops production, it cannot start up again. Assume that $k_2 < k_1$ (firm 2 is smaller than firm 1) and that each firm's cost of producing q units of output is cq.

The following extensive game with simultaneous moves models this situation.

Players The two firms.

Terminal histories All sequences $(X^1, \ldots, X^t)$ for some $t \geq 1$, where $X^s = (Stay, Stay)$ for $1 \leq s \leq t-1$ and $X^t = (Exit, Exit)$ (both firms exit in period t), or $X^s = (Stay, Stay)$ for all s with $1 \leq s \leq r-1$ for some r, $X^r = (Stay, Exit)$ or $(Exit, Stay)$, $X^s = Stay$ for all s with $r+1 \leq s \leq t-1$, and $X^t = Exit$ (one firm exits in period r and the other exits in period t), and all infinite sequences $(X^1, X^2, \ldots)$ where $X^r = (Stay, Stay)$ for all r (neither firm ever exits).

Player function $P(h) = \{1, 2\}$ after any history h in which neither firm has exited; $P(h) = 1$ after any history h in which only firm 2 has exited; and $P(h) = 2$ after any history h in which only firm 1 has exited.

Actions Whenever a firm moves, its set of actions is $\{Stay, Exit\}$.

Preferences Each firm's preferences are represented by a payoff function that associates with each terminal history the firm's total profit, where the profit of firm i (= 1, 2) in period t is $(P_t(k_i) - c)k_i$ if the other firm has exited and $(P_t(k_1 + k_2) - c)k_i$ if the other firm has not exited.

7.5.2 Subgame perfect equilibrium

In a period in which $P_t(k_i) < c$, firm i makes a loss even if it is the only firm remaining (the market price for its output is less than its unit cost). Denote by t_i the last period in which firm i is profitable if it is the only firm in the market. That is, t_i is the largest value of t for which $P_t(k_i) \geq c$. (Refer to Figure 219.1.) Because $k_1 > k_2$, we have $t_1 \leq t_2$: the time at which the large firm becomes unprofitable as a loner is no later than the time at which the small firm becomes unprofitable as a loner.

The game has an infinite horizon, but after period t_i firm i's profit is negative even if it is the only firm remaining in the market. Thus if firm i is in the market in any period after t_i, it chooses *Exit* in that period in every subgame perfect equilibrium. In particular, both firms choose *Exit* in every period after t_2. We can use backward induction from period t_2 to find the firms' subgame perfect equilibrium actions in earlier periods.

If firm 1 (the larger firm) is in the market in any period from t_1 on, it should exit, whether or not firm 2 is still operating. As a consequence, if firm 2 is still

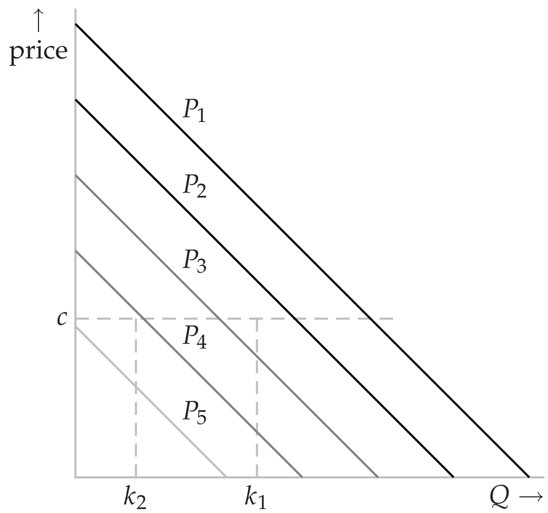


Figure 219.1 The inverse demand curves in a declining industry. In this example, t_1 (the last period in which firm 1 is profitable if it is the only firm in the market) is 2, and t_2 is 4.

operating in any period from $t_1 + 1$ to t_2 it should stay: firm 1 will exit in any such period, and in its absence firm 2's profit is positive.

So far we have concluded that in every subgame perfect equilibrium, firm 1's strategy is to exit in every period from $t_1 + 1$ on if it has not already done so, and firm 2's strategy is to exit in every period from $t_2 + 1$ on if it has not already done so.

Now consider period t_1, the last period in which firm 1's profit is positive if firm 2 is absent. If firm 2 exits, its profit from then on is zero. If it stays and firm 1 exits then it earns a profit from period t_1 to period t_2, after which it leaves. If both firms stay, firm 2 sustains a loss in period t_1 but earns a profit in the subsequent periods up to t_2, because in every subgame perfect equilibrium firm 1 exits in period $t_1 + 1$. Thus if firm 2's one-period loss in period t_1 when firm 1 stays in that period is less than the sum of its profits from period $t_1 + 1$ on, then *regardless of whether firm 1 stays or exits in period* t_1, firm 2 stays in every subgame perfect equilibrium. In period $t_1 + 1$, when firm 1 is absent from the industry, the price is relatively high, so that the assumption that firm 2's one-period loss is less than its subsequent multi-period profit is valid for a significant range of parameters. From now on, I assume that this condition holds.

We conclude that in every subgame perfect equilibrium firm 2 stays in period t_1, so that firm 1 optimally exits. (It definitely exits in the next period, and if it stays in period t_1 it makes a loss, because firm 2 stays.)

Now continue to work backwards. If firm 2 stays in period $t_1 - 1$ it earns a profit in periods t_1 through t_2, because in every subgame perfect equilibrium firm 1 exits in period t_1. It may make a loss in period $t_1 - 1$ (if firm 1 stays in that period), but this loss is less than the loss it makes in period t_1 in the company of firm 1, which we have assumed is outweighed by its subsequent profit. Thus regardless of firm 1's action in period $t_1 - 1$, firm 2's best action is to stay in that period. If $t_2 < t_1 - 1$ then firm 1 makes a loss in period $t_1 - 1$ in the company of firm 2, and so should exit.

The same logic applies to all periods back to the first period in which the firms cannot profitably co-exist in the industry: in every such period, in every subgame perfect equilibrium firm 1 exits if it has not already done so. Denote by t_0 the last period in which both firms can profitably co-exist in the industry: that is, t_0 is the largest value of t for which $P_t(k_1 + k_2) \geq c$.

We conclude that if firm 2's loss in period t_1 when both firms are active is less than the sum of its profits in periods $t_1 + 1$ through t_2 when it alone is active, then the game has a unique subgame perfect equilibrium, in which the large firm exits in period $t_0 + 1$, the first period in which both firms cannot profitably co-exist in the industry, and the small firm continues operating until period t_2, after which it alone becomes unprofitable.

? EXERCISE 220.1 (Exit from a declining industry) Assume that $c = 10$, $k_1 = 40$, $k_2 = 20$, and $P_t(Q) = 100 - t - Q$ for all values of t and Q for which $100 - t - Q > 0$, otherwise $P_t(Q) = 0$. Find the values of t_1 and t_2 and check whether firm 2's loss in period t_1 when both firms are active is less than the sum of its profits in periods $t_1 + 1$ through t_2 when it alone is active.

7.5.3 The effect of a constraint on firm 2's debt

When the firms follow their subgame perfect equilibrium strategies, each firm's profit is nonnegative in every period. However, the equilibrium depends on firm 2's ability to go into debt. Firm 2's strategy calls for it to stay in the market if firm 1, contrary to its strategy, does not exit in the first period in which the market cannot profitably sustain both firms. This feature of firm 2's strategy is essential to the equilibrium. If such a deviation by firm 1 induces firm 2 to exit, then firm 1's strategy of exiting may not be optimal, and the equilibrium may consequently fall apart.

Consider an extreme case, in which firm 2 can never go into debt. We can incorporate this assumption into the model by making firm 2's payoff a large negative number for any terminal history in which its profit in any period is negative. (The size of firm 2's profit depends on the contemporaneous action of firm 1, so we cannot easily incorporate the assumption by modifying the choices available to firm 2.) Consider a history in which firm 1 stays in the market after the last period in which the market can profitably sustain both firms. After such a history firm 2's best action is no longer to stay: if it does so it profit is negative, whereas if it exits its profit is zero. Thus if firm 1 deviates from its equilibrium strategy in the absence of a borrowing constraint for firm 2, and stays in the first period in which it is supposed to exit, then firm 2 optimally exits, and firm 1 reaps positive profits for several periods, as the lone firm in the market. Consequently in this case firm 2 exits first; firm 1 stays in the market until period t_1.

How much debt does firm 2 need to be able to bear in order that the game has a subgame perfect equilibrium in which firm 1 exits in period t_0 and firm 2 stays until period t_2? Suppose that firm 2 can sustain losses from period $t_0 + 1$ through period $t_0 + k$, but no longer, when both firms stay in the market. In order for firm 1 to optimally exit in period $t_0 + 1$, the consequence of its staying in the market must be that firm 2 also stays. Suppose that firm 2's strategy is to stay through period $t_0 + k$, but no longer, if firm 1 does so. Which strategy is best for firm 1 in the subgame starting in period $t_0 + 1$? If it exits, its payoff is zero. If it stays through period $t_0 + k$, its payoff is negative (it makes a loss in every period). If it stays beyond period $t_0 + k$ (when firm 2 exits), it should stay until period t_1, when its payoff is the sum of profits that are negative from period $t_0 + 1$ through period $t_0 + k$ and then positive through period t_1. (See Figure 221.1.) If this payoff is positive it should stay through period t_1; otherwise it should exit immediately.

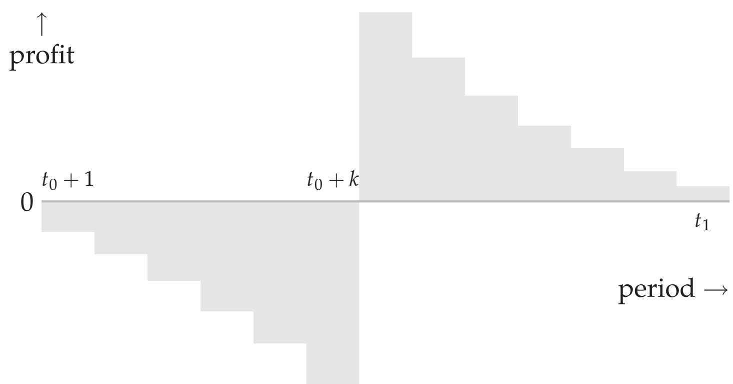


Figure 221.1 Firm 1's profits starting in period $t_0 + 1$ when firm 2 stays in the market until period $t_1 + k$ and firm 1 stays until period t_1.

We conclude that in order for firm 1 to exit in period $t_0 + 1$, the period $t_0 + k$ until which firm 2 can sustain losses must be large enough that firm 1's total profit from period $t_0 + 1$ through period t_1 if it shares the market with firm 2 until period $t_0 + k$, then has the market to itself, is nonpositive. This value of k determines the debt that firm 2 must be able to accumulate: the requisite debt equals its total loss when it remains in the market with firm 1 from period $t_0 + 1$ through period $t_0 + k$.

? EXERCISE 221.1 (Effect of borrowing constraint of firms' exit decisions in declining industry) Under the assumptions of Exercise 220.1, how much debt does firm 2 need to be able to bear in order for the subgame perfect equilibrium outcome in the absence of a debt constraint to remain a subgame perfect equilibrium outcome?

7.6 Allowing for exogenous uncertainty

7.6.1 General model

The model of an extensive game with perfect information (with or without simultaneous moves) does not allow random events to occur during the course of play. However, we can easily extend the model to cover such situations. The definition of an **extensive game with perfect information and chance moves** is a variant of the definition of an extensive game with perfect information (153.1) in which

- the player function assigns "chance", rather than a set of players, to some histories
- the probabilities that chance uses after any such history are specified
- the players' preferences are defined over the set of lotteries over terminal histories (rather than simply over the set of terminal histories).

(We may similarly add chance moves to an extensive game with perfect information and simultaneous moves by modifying Definition 202.1.) To keep the analysis simple, assume that the random event after any given history is independent of the random event after any other history. (That is, the realization of any random event is not affected by the realization of any other random event.)

The definition of a player's strategy remains the same as before. The outcome of a strategy profile is now a probability distribution over terminal histories. The definition of subgame perfect equilibrium remains the same as before.

◆ EXAMPLE 222.1 (Extensive game with chance moves) Consider a situation involving two players in which player 1 first chooses A or B. If she chooses A the game ends, with (Bernoulli) payoffs $(1, 1)$. If she chooses B then with probability $\frac{1}{2}$ the game ends, with payoffs $(3, 0)$, and with probability $\frac{1}{2}$ player 2 gets to choose between C, which yields payoffs $(0, 1)$ and D, which yields payoffs $(1, 0)$. An extensive game with perfect information and chance moves that models this situation is shown in Figure 223.1. The label c denotes chance; the number beside each action of chance is the probability with which that action is chosen.

We may use backward induction to find the subgame perfect equilibria of this game. In any equilibrium, player 2 chooses C. Now consider the consequences of player 1's actions. If she chooses A then she obtains the payoff 1. If she chooses B then she obtains 3 with probability $\frac{1}{2}$ and 0 with probability $\frac{1}{2}$, yielding an expected payoff of $\frac{3}{2}$. Thus the game has a unique subgame perfect equilibrium, in which player 1 choose B and player 2 chooses C.

? EXERCISE 222.2 (Variant of ultimatum game with equity-conscious players) Consider a variant of the game in Exercise 181.1 in which $\beta_1 = 0$, and the person 2 whom person 1 faces is drawn randomly from a population in which the fraction p have $\beta_2 = 0$ and the remaining fraction $1 - p$ have $\beta_2 = 1$. When making her

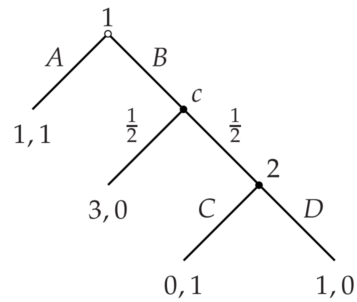


Figure 223.1 An extensive game with perfect information and chance moves. The label c denotes chance; the number beside each action of chance is the probability with which that action is chosen.

offer, person 1 knows only that her opponent's characteristic is $\beta_2 = 0$ with probability p and $\beta_2 = 1$ with probability $1 - p$. Model this situation as an extensive game with perfect information and chance moves in which person 1 makes an offer, then chance determines the type of person 2, and finally person 2 accepts or rejects person 1's offer. opponent Find the subgame perfect equilibria of this game. (Use the fact that if $\beta_2 = 0$, then in any subgame perfect equilibrium of the game in Exercise 181.1 person 2 accepts all offers $x > 0$, rejects all offers $x < 0$, and may accept or reject the offer 0, and if $\beta_2 = 1$ then she accepts all offers $x > \frac{1}{3}$, may accept or reject the offer $\frac{1}{3}$, and rejects all offers $x < \frac{1}{3}$.) Are there any values of p for which an offer is rejected in equilibrium?

? EXERCISE 223.1 (Sequential duel) In a sequential duel, two people alternately have the opportunity to shoot each other; each has an infinite supply of bullets. On each of her turns, a person may shoot, or refrain from doing so. Each of person i's shots hits (and kills) its intended target with probability p_i (independently of whether any other shots hit their targets). (If you prefer to think about a less violent situation, interpret the players as political candidates who alternately may launch attacks, which may not be successful, against each other.) Each person cares only about her probability of survival (not about the other person's survival). Model this situation as an extensive game with perfect information and chance moves. Show that the strategy pairs in which neither person ever shoots and in which each person always shoots are both subgame perfect equilibria. (Note that the game does not have a finite horizon, so backward induction cannot be used.)

?? EXERCISE 223.2 (Sequential truel) Each of persons A, B, and C has a gun containing a single bullet. Each person, as long as she is alive, may shoot at any surviving person. First A can shoot, then B (if still alive), then C (if still alive). (As in the previous exercise, you may interpret the players as political candidates. In this exercise, each candidate has a budget sufficient to launch a negative campaign to discredit exactly one of its rivals.) Denote by p_i the probability that player i hits her intended target; assume that $0 < p_i < 1$. Assume that each player wishes to maximize her probability of survival; among outcomes in which her survival probability is the same, she wants the danger posed by any other survivors to be as small as possible. (The last assumption is intended to capture the idea that there is some chance that further rounds of shooting may occur, though the possibility of such rounds is not incorporated explicitly into the game.) Model this situation as an extensive game with perfect information and chance moves. (Draw a diagram. Note that the subgames following histories in which A misses her intended target are the same.) Find the subgame perfect equilibria of the game. (Consider only cases in which p_A, p_B, and p_C are all different.) Explain the logic behind A's equilibrium action. Show that "weakness is strength" for C: she is better off if $p_C < p_B$ than if $p_C > p_B$.

Now consider the variant in which each player, on her turn, has the additional option of shooting into the air. Find the subgame perfect equilibria of this game when $p_A < p_B$. Explain the logic behind A's equilibrium action.

?? EXERCISE 224.1 (Cohesion in legislatures) The following pair of games is designed to study the implications of different legislative procedures for the cohesion of a governing coalition. In both games a legislature consists of three members. Initially a governing coalition, consisting of two of the legislators, is given. There are two periods. At the start of each period a member of the governing coalition is randomly chosen (i.e. each legislator is chosen with probability $\frac{1}{2}$) to propose a bill, which is a partition of one unit of payoff between the three legislators. Then the legislators simultaneously cast votes; each legislator votes either for or against the bill. If two or more legislators vote for the bill, it is accepted. Otherwise the course of events differs between the two games. In a game that models the current US legislature, rejection of a bill in period t leads to a given partition d^t of the pie, where $0 < d_i^t < \frac{1}{2}$ for $i = 1, 2, 3$; the governing coalition (the set from which the proposer of a bill is drawn) remains the same in period 2 following a rejection in period 1. In a game that models the current UK legislature, rejection of a bill brings down the government; a new governing coalition is determined randomly, and no legislator receives any payoff in that period. Specify each game precisely and find its subgame perfect equilibrium outcomes. Study the degree to which the governing coalition is cohesive (i.e. all its members vote in the same way).

7.6.2 Using chance moves to model mistakes

A game with chance moves may be used to model the possibility that players make mistakes. Suppose, for example, that two people simultaneously choose actions. Each person may choose either A or B. Absent the possibility of mistakes, suppose that the situation is modeled by the strategic game in Figure 225.1, in which the numbers in the boxes are Bernoulli payoffs. This game has two Nash equilibria, (A, A) and (B, B).

Now suppose that each person may make a mistake. With probability $1 - p_i > \frac{1}{2}$ the action chosen by person i is the one she intends, and with probability $p_i < \frac{1}{2}$ it is her other action. We can model this situation as the following extensive game

	A	B
A	$1,1$	$0,0$
B	$0,0$	$0,0$

Figure 225.1 The players' Bernoulli payoffs to the four pairs of actions in the game studied in Section 7.6.2.

with perfect information, simultaneous moves, and chance moves.

Players The two people.

Terminal histories All sequences of the form $((W, X), Y, Z)$, where W, X, Y, and Z are all either A or B; in the history $((W, X), Y, Z)$ player 1 chooses W, player 2 chooses X, and then chance chooses Y for player 1 and Z for player 2.

Player function $P(\varnothing) = \{1, 2\}$ (both players move simultaneously at the start of the game), and $P(W, X) = P((W, X), Y) = \{c\}$ (chance moves twice after the players have acted, first selecting player 1's action and then player 2's action).

Actions The set of actions available to each player at the start of the game, and to chance at each of its moves, is $\{A, B\}$.

Chance probabilities After any history (W, X), chance chooses W with probability $1 - p_1$ and player 1's other action with probability p_1. After any history $((W, X), Y)$, chance chooses X with probability $1 - p_2$ and player 2's other action with probability p_2.

Preferences Each player's preferences are represented by the expected value of a Bernoulli payoff function that assigns 1 to any history $((W, X), A, A)$ (in which chance chooses the action A for each player), and 0 to any other history.

The players in this game move simultaneously, so that the subgame perfect equilibria of the game are its Nash equilibria. To find the Nash equilibria we construct the strategic form of the game. Suppose that each player chooses the action A. Then the outcome is (A, A) with probability $(1 - p_1)(1 - p_2)$ (the probability that neither player makes a mistake). Thus each player's expected payoff is $(1 - p_1)(1 - p_2)$. Similarly, if player 1 chooses A and player 2 chooses B then the outcome is (A, A) with probability $(1 - p_1)p_2$ (the probability that player 1 does not make a mistake, whereas player 2 does). Making similar computations for the other two cases yields the strategic form in Figure 226.1.

For $p_1 = p_2 = 0$, this game is the same as the original game (Figure 225.1); it has two Nash equilibria, (A, A) and (B, B). If at least one of the probabilities is positive then only (A, A) is a Nash equilibrium: if $p_i > 0$ then $(1 - p_j)p_i > p_j p_i$

	A	B
A	$(1-p_1)(1-p_2), (1-p_1)(1-p_2)$	$(1-p_1)p_2, (1-p_1)p_2$
B	$p_1(1-p_2), p_1(1-p_2)$	p_1p_2, p_1p_2

Figure 226.1 The strategic form of the extensive game with chance moves that models the situation in which with probability p_i each player i in the game in Figure 225.1 chooses an action different from the one she intends.

(given that each probability is less than $\frac{1}{2}$). That is, only the equilibrium (A, A) of the original game is robust to the possibility that the players make small mistakes.

In the original game each player's action B is weakly dominated (Definition 45.1). Introducing the possibility of mistakes captures the fragility of the equilibrium (B, B): B is optimal for a player only if she is absolutely certain that the other player will choose B also. The slightest chance that the other player will choose A is enough to make A unambiguously the best choice.

We may use the idea that an equilibrium should survive when the players may make small mistakes to discriminate among the Nash equilibria of any strategic game. For two-player games we are led to the set of Nash equilibria in which no player's action is weakly dominated, but for games with more than two players we are led to a smaller set of equilibria, as the following exercise shows.

? EXERCISE 226.1 (Nash equilibria when players may make mistakes) Consider the three-player game in Figure 226.2. Show that (A, A, A) is a Nash equilibrium in which no player's action is weakly dominated. Now modify the game by assuming that the outcome of any player i's choosing an action X is that X occurs with probability $1 - p_i$ and the player's other action occurs with probability $p_i > 0$. Show that (A, A, A) is not a Nash equilibrium of the modified game when $p_i < \frac{1}{2}$ for $i = 1, 2, 3$.

	A	B
A	$1,1,1$	$0,0,1$
B	$1,1,1$	$1,0,1$

A

	A	B
A	$0,1,0$	$1,0,0$
B	$1,1,0$	$0,0,0$

B

Figure 226.2 A three-player strategic game in which each player has two actions. Player 1 chooses a row, player 2 chooses a column, and player 3 chooses a table.

7.7 Discussion: subgame perfect equilibrium and backward induction

Some of the situations we have studied do not fit well into the idealized setting for the steady state interpretation of a subgame perfect equilibrium discussed in Section 5.5.4, in which each player repeatedly engages in the same game with a variety of randomly selected opponents. In some cases an alternative interpretation fits better: each player deduces her optimal strategy from an analysis of the other players' best actions, given her knowledge of their preferences. Here I discuss a difficulty with this interpretation.

Consider the game in Figure 227.1, in which player 1 moves both before and after player 2. This game has a unique subgame perfect equilibrium, in which player 1's strategy is (B, F) and player 2's strategy is C. Consider player 2's analysis of the game. If she deduces that the only rational action for player 1 at the start of the game is B, then what should she conclude if player 1 chooses A? It seems that she must conclude that something has "gone wrong": perhaps player 1 has made a "mistake", or she misunderstands player 1's preferences, or player 1 is not rational. If she is convinced that player 1 simply made a mistake, then her analysis of the rest of the game should not be affected. However, if player 1's move induces her to doubt player 1's motivation, she may need to reconsider her analysis of the rest of the game. Suppose, for example, that A and E model similar actions; specifically, suppose that they both correspond to player 1's moving left, whereas B and F both involve her moving right. Then player 1's choice of A at the start of the game may make player 2 wonder whether player 1 confuses left and right, and therefore may choose E after the history (A, C). If so, player 2 should choose D rather than C after player 1 chooses A, giving player 1 an incentive to choose A rather than B at the start of the game.

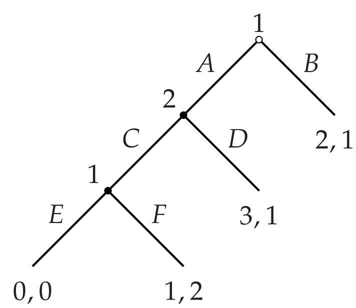


Figure 227.1 An extensive game in which player 1 moves both before and after player 2.

The next two examples are richer games that more strikingly manifest the difficulty with the alternative interpretation of subgame perfect equilibrium. The first example is an extension of the entry game in Figure 154.1.

◆ EXAMPLE 227.1 (Chain-store game) A chain-store operates in K markets. In each market a single challenger must decide whether to compete with it. The challengers make their decisions sequentially. If any challenger enters, the chain-store may acquiesce to its presence (A) or fight it (F). Thus in each period k the outcome is either *Out* (challenger k does not enter), (In, A) (challenger k enters and the chain-store acquiesces), or (In, F) (challenger k enters and is fought). When taking an action, any challenger knows all the actions previously chosen. The profits of challenger k and the chain-store in market k are shown in Figure 228.1 (cf. Figure 154.1); the chain-store's profit in the whole game is the sum of its profits in the

K markets.

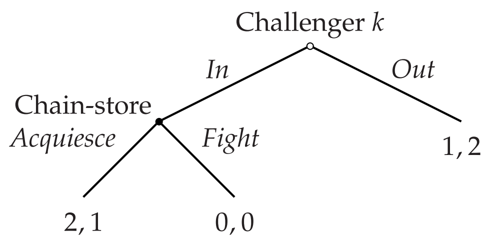


Figure 228.1 The structure of the players' choices in market k in the chain-store game. The first number in each pair is challenger k's profit and the second number is the chain-store's profit.

We can model this situation as the following extensive game with perfect information.

Players The chain-store and the K challengers.

Terminal histories The set of all sequences $(e_1, \ldots, e_K)$, where each e_j is either *Out*, (In, A), or (In, F).

Player function The chain-store is assigned to every history that ends with *In*, challenger 1 is assigned to the initial history, and challenger k (for $k = 2, \ldots, K$) is assigned to every history $(e_1, \ldots, e_{k-1})$, where each e_j is either *Out*, (In, A), or (In, F).

Preferences Each player's preferences are represented by its profits.

This game has a finite horizon, so we may find its subgame perfect equilibria by using backward induction. Every subgame at the start of which challenger K moves resembles the game in Figure 228.1 for $k = K$; it differs only in that the chain-store's profit after each of the three terminal histories is greater by an amount equal to its profit in the previous $K - 1$ markets. Thus in a subgame perfect equilibrium challenger K chooses *In* and the incumbent chooses A in market K.

Now consider the subgame faced by challenger $K - 1$. We know that the outcome in market K is independent of the actions of challenger $K - 1$ and the chain-store in market $K - 1$: whatever they do, challenger K enters and the chain-store acquiesces to its entry. Thus the chain-store should choose its action in market $K - 1$ on the basis of its payoffs in that market alone. We conclude that the chain-store's optimal action in market $K - 1$ is A, and challenger $K - 1$'s optimal action is *In*.

We have now concluded that in any subgame perfect equilibrium, the outcome in each of the last two markets is (In, A), regardless of the history. Continuing to work backwards to the start of the game we see that the game has a unique subgame perfect equilibrium, in which every challenger enters and the chain-store always acquiesces to entry.

? EXERCISE 228.1 (Nash equilibria of chain-store game) Find the set of Nash equilibrium outcomes of the game for an arbitrary value of K. (First think about the case $K = 1$, then generalize your analysis.)

? EXERCISE 229.1 (Subgame perfect equilibrium of chain-store game) Consider the following strategy pair in the game for $K = 100$. For $k = 1, \ldots, 90$, challenger k stays out after any history in which every previous challenger that entered was fought (or no challenger entered), and otherwise enters; challengers 91 through 100 enter. The chain-store fights every challenger up to challenger 90 that enters after a history in which it fought every challenger that entered (or no challenger entered), acquiesces to any of these challengers that enters after any other history, and acquiesces to challengers 91 through 100 regardless of the history. Find the players' payoffs in this strategy pair. Show that the strategy pair is not a subgame perfect equilibrium: find a player who can increase her payoff in some subgame. How much can the deviant increase its payoff?

Suppose that $K = 100$. You are in charge of challenger 21. You observe, contrary to the subgame perfect equilibrium, that every previous challenger entered and that the chain-store fought each one. What should you do? According to the subgame perfect equilibrium, the chain-store will acquiesce to your entry. But should you really regard the chain-store's 19 previous decisions as "mistakes"? You might instead read some logic into the chain-store's *deliberately* fighting the first 20 entrants: if, by doing so, it persuades more than 20 of the remaining challengers to stay out, then its profit will be higher than it is in the subgame perfect equilibrium. That is, you may imagine that the chain-store's aggressive behavior in the earlier markets is an attempt to establish a reputation for being a fighter, which, if successful, will make it better off. By such reasoning you may conclude that your best strategy is to stay out.

Thus, a deviation from the subgame perfect equilibrium by the chain-store in which it engages in a long series of fights may not be dismissed by challengers as a series of mistakes, but rather may cause them to doubt the chain-store's future behavior. This doubt may lead a challenger who is followed by enough future challengers to stay out.

◆ EXAMPLE 229.2 (Centipede game) The two-player game in Figure 230.1 is known as a "centipede game" because of its shape. (The game, like the arthropod, may have fewer than 100 legs.) The players move alternately; on each move a player can stop the game (S) or continue (C). On any move, a player is better off stopping the game than continuing if the other player stops immediately afterwards, but is worse off stopping than continuing if the other player continues, regardless of the subsequent actions. After k periods, the game ends.

This game has a finite horizon, so we may find its subgame perfect equilibria by using backward induction. The last player to move prefers to stop the game than to continue. Given this player's action, the player who moves before her also prefers to stop the game than to continue. Working backwards, we conclude that the game has a unique subgame perfect equilibrium, in which each player's strategy is to stop the game whenever it is her turn to move. The outcome is that player 1 stops the game immediately.

? EXERCISE 229.3 (Nash equilibria of the centipede game) Show that the outcome

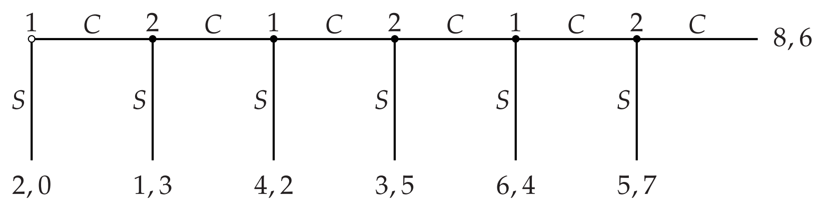


Figure 230.1 A 6-period centipede game.

of every Nash equilibrium of this game is the same as the outcome of the unique subgame perfect equilibrium (i.e. player 1 stops the game immediately).

The logic that in the only steady state player 1 stops the game immediately is unassailable. Yet this pattern of behavior is intuitively unappealing, especially if the number k of periods is large. The optimality of player 1's choosing to stop the game depends on her believing that if she continues, then player 2 will stop the game in period 2. Further, player 2's decision to stop the game in period 2 depends on her believing that if she continues then player 1 will stop the game in period 3. Each decision to stop the game is based on similar considerations. Consider a player who has to choose an action in period 21 of a 100-period game, after each player has continued in the first 20 periods. Is she likely to consider the first 20 decisions—half of which were hers—"mistakes"? Or will these decisions induce her to doubt that the other player will stop the game in the next period? These questions have no easy answers; some experimental evidence is discussed in the accompanying box.

EXPERIMENTAL EVIDENCE ON THE CENTIPEDE GAME

In experiments conducted in the USA in 1989, each of 58 student subjects played a game with the monetary payoffs (in US$) shown in Figure 231.1 (McKelvey and Palfrey 1992). Each subject played the game 9 or 10 times, facing a different opponent each time; in each play of the game, each subject had previously played the same number of games. Each subject knew in advance how many times she would play the game, and knew that she would not play against the same opponent more than once. If each subject cared only about her own monetary payoff, the game induced by the experiment was a 6-period centipede.

The fraction of plays of the game that ended in each period is shown in Figure 231.2. Results are broken down according to the players' experience (first 5 rounds, last 5 rounds). The game ended earlier when the participants were experienced, but even among experienced participants the outcomes are far from the Nash equilibrium outcome, in which the game ends in period 1.

Ten plays of the game may not be enough to achieve convergence to a steady state. But putting aside this limitation of the data, and supposing that convergence

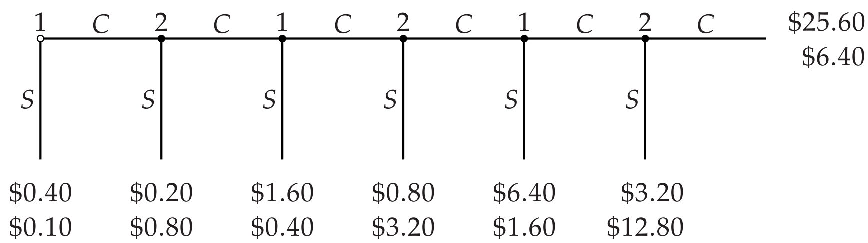


Figure 231.1 The game in McKelvey and Palfrey's (1992) experiment. The payoff of player 1 is written above the payoff of player 2.

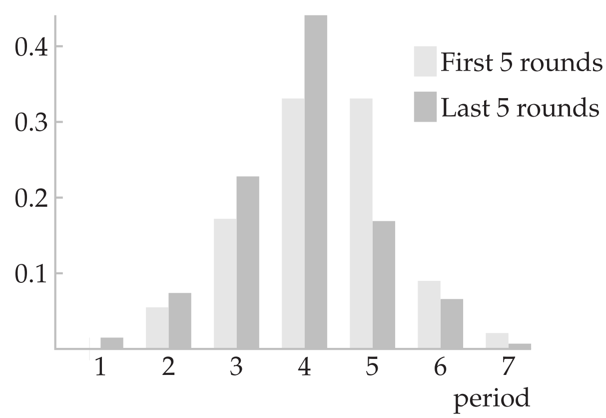


Figure 231.2 Fraction of games ending in each period of McKelvey and Palfrey's experiments on the six-period centipede game. (A game is counted as ending in period 7 if the last player to move chose *C*.) Computed from McKelvey and Palfrey (1992, Table IIIA).

was in fact achieved at the end of 10 rounds, how far does the observed behavior differ from a Nash equilibrium (maintaining the assumption that each player cares only about her own monetary payoff)?

The theory of Nash equilibrium has two components: each player optimizes, given her beliefs about the other players, and these beliefs are correct. Some decisions in McKelvey and Palfrey's experiment were patently suboptimal, regardless of the subjects' beliefs: a few subjects in the role of player 2 chose to continue in period 6, obtaining $6.40 with certainty instead of $12.80 with certainty. To assess the departure of the other decisions from optimality we need to assign the subjects beliefs (which were not directly observed). An assumption consistent with the steady state interpretation of Nash equilibrium is that a player's belief is based on her observations of the other players' actions. Even in round 10 of the experiment each player had only 9 observations on which to base her belief, and could have used these data in various ways. But suppose that, somehow, at the end of round 4, each player correctly inferred the distribution of her opponents' strategies in the next 5 rounds. What strategy should she subsequently have used? From Palfrey and McKelvey (1992, Table IIIB) we may deduce that the optimal strategy of player 1 stops in period 5 and that of player 2 stops in period 6. That is, each player's best response to the empirical distribution of the other players' strategies differs dramatically from her subgame perfect equilibrium strategy. Other assumptions about the subjects' beliefs rationalize other strategies; the data seem too limited to conclude that the subjects were not optimizing given beliefs they might reasonably have held, given their experience. That is, the experimental data are not strongly inconsistent with the theory of Nash equilibrium as a steady state.

Are the data inconsistent with the theory that rational players, even those with no experience playing the game, will deduce their opponents' rational actions from an analysis of the game using backward induction? This theory predicts that the first player immediately stops the game, so certainly the data are inconsistent with it. How inconsistent? One way to approach this question is to consider the implications of each player's thinking that the others are *likely* to be rational, but are not *certainly* so. If, in any period, player 1 thinks that the probability that player 2 will stop the game in the next period is less than $\frac{6}{7}$, continuing yields a higher expected payoff than stopping. Given the limited time the subjects had to analyze the game (and the likelihood that they had never before thought about any related game), even those who understood the implications of backward induction may reasonably have entertained the relatively small doubt about the other players' cognitive abilities required to make stopping the game immediately an unattractive option. Or, alternatively, a player confident of her opponents' logical abilities may have doubted her opponents' assessment of *her own* analytical skills. If player 1 believes that player 2 thinks that the probability that player 1 will continue in period 3 is greater than $\frac{1}{7}$, then she should continue in period 1, because player 2 will continue in period 2. That is, relatively minor departures from the theory yield outcomes close to those observed.

Notes

The idea of regarding games with simultaneous moves as games with perfect information is due to Dubey and Kaneko (1984).

The model in Section 7.3 was first studied by Ledyard (1981, 1984). The approach to voting in committees in Section 7.4 was initiated by Farquharson (1969). (The publication of Farquharson's book was delayed; the book was completed in 1958.) The top cycle set was first defined by Ward (1961) (who called it the "majority set"). The characterization of the outcomes of sophisticated voting in binary agendas in terms of the top cycle set is due to Miller (1977) (who calls the top cycle set the "Condorcet set") and McKelvey and Niemi (1978). Miller (1995) surveys the field. The model in Section 7.5 is taken from Nalebuff and Ghemawat (1985); the idea is closely related to that of Benoît (1984, Section 1) (see Exercise 172.2). My discussion draws on an unpublished exposition of the model by Vijay Krishna. The idea of discriminating among Nash equilibria by considering the possibility that players make mistakes, briefly discussed in Section 7.6.2, is due to Selten (1975). The chain-store game in Example 227.1 is due to Selten (1978). The centipede game in Example 229.2 is due to Rosenthal (1981).

The experimental results discussed in the box on page 207 are due to Roth, Prasnikar, Okuno-Fujiwara, and Zamir (1991). The subgame perfect equilibria of a variant of the market game in which each player's payoff depends on the other players' monetary payoffs are analyzed by Bolton and Ockenfels (2000). The model in Exercise 208.1 is taken from Peters (1984). The results in Exercises 212.1 and 213.1 are due to Feddersen, Sened, and Wright (1990). The game in Exercise 223.2 is a simplification of an example due to Shubik (1954); the main idea appears in Phillips (1937, 159) and Kinnaird (1946, 246), both of which consist mainly of puzzles previously published in newspapers. Exercise 224.1 is based on Diermeier and Feddersen (1996). The experiment discussed in the box on page 230 is reported in McKelvey and Palfrey (1992).

8 Coalitional Games and the Core

Prerequisite: Chapter 1.

8.1 Coalitional games

A COALITIONAL GAME is a model of interacting decision-makers that focuses on the behavior of groups of players. It associates a set of actions with every group of players, not only with individual players, like the models of a strategic game (Definition 11.1) and extensive game (Definition 153.1)). We call each group of players a *coalition*, and the coalition of *all* the players the *grand coalition*.

An outcome of a coalitional game consists of a partition of the set of players into groups, together with an action for each group in the partition. (See Section 17.3 if you are not familiar with the notion of a "partition" of a set.) At one extreme, each group in the partition may consist of a single player, who acts on her own; at another extreme, the partition may consist of a single group containing all the players. The most general model of a coalitional game allows players to care about the action chosen by each group in the partition that defines the outcome. I discuss only the widely-studied class of games in which each player cares only about the action chosen by the member of the partition to which she belongs. In such games, each player's preferences rank the actions of all possible groups of players that contain her.

▶ DEFINITION 235.1 (*Coalitional game*) A **coalitional game** consists of

- a set of **players**
- for each coalition, a set of **actions**
- for each player, **preferences** over the set of all actions of all coalitions of which she is a member.

I usually denote the grand coalition (the set of all the players) by N and an arbitrary coalition by S. As before, we may conveniently specify a player's preferences by giving a payoff function that represents them.

In several of the examples that I present, each coalition controls some quantity of a good, which may be distributed among its members. Each action of a coalition S in such a game is a distribution among the members of S of the good that S controls, which I refer to as an S-**allocation** of the good. I refer to an N-allocation simply as an **allocation**.

Note that the definition of a coalitional game does not relate the actions of a coalition to the actions of the members of the coalition. The coalition's actions are simply taken as given; they are not derived from the individual players' actions.

A coalitional game is designed to model situations in which players can beneficially form groups, rather than acting individually. Most of the theory is oriented to situations in which the incentive to coalesce is extreme, in the sense that there is no disadvantage to the formation of the *single* group consisting of all the players. In considering the action that this single group takes in such a situation, we need to consider the possibility that smaller groups break away on their own; but when looking for "equilibria" we can restrict attention to outcomes in which all the players coalesce. Such situations are modeled as games in which the grand coalition can achieve outcomes at least as desirable for every player as those achievable by any partition of the players into subgroups. We call such games "cohesive", defined precisely as follows.

► DEFINITION 236.1 (*Cohesive coalitional game*) A coalitional game is **cohesive** if, for every partition $\{S_1, \ldots, S_k\}$ of the set of all players and every combination $(a_{S_1}, \ldots, a_{S_k})$ of actions, one for every coalition in the partition, the grand coalition N has an action that is at least as desirable for every player i as the action a_{S_j} of the member S_j of the partition to she player i belongs.

The concepts I subsequently describe may be applied to any game, cohesive and not, but have attractive interpretations only for cohesive games.

◆ EXAMPLE 236.2 (Two-player unanimity game) Two people can together produce one unit of output, which they may share in any way they wish. Neither person by herself can produce any output. Each person cares only about the amount of output she receives, and prefers more to less. The following coalitional game models this situation.

Players The two people (players 1 and 2).

Actions Each player by herself has a single action, which yields her no output. The set of actions of the coalition $\{1, 2\}$ of both players is the set of all pairs (x_1, x_2) of nonnegative numbers such that $x_1 + x_2 = 1$ (the set of divisions of one unit of output between the two players).

Preferences Each player's preferences are represented by the amount of output she obtains.

The possible partitions of the set of players are $\{\{1,2\}\}$, consisting of the single coalition of both players, and $\{\{1\},\{2\}\}$, in which each player acts alone. The latter has only one combination of actions available to it, which produces not output. Thus the game is cohesive.

In the next example the opportunities for producing output are richer and the participants are not all symmetric.

◆ EXAMPLE 237.1 (Landowner and workers) A landowner's estate, when used by k workers, produces the output $f(k+1)$ of food, where f is a increasing function for which $f(0) = 0$. The total number of workers is m. The landowner and each worker care only about the amount of output she receives, and prefer more to less. The following coalitional game models this situation.

Players The landowner and the m workers.

Actions A coalition consisting solely of workers has a single action in which no member receives any output. The set of actions of a coalition S consisting of the landowner and k workers is the set of all S-allocations of the output $f(k+1)$ among the members of S.

Preferences Each player's preferences are represented by the amount of output she obtains.

This game is cohesive because the grand coalition produces more output than any other coalition, and, for any partition of the set of all the players, only one coalition produces any output.

◆ EXAMPLE 237.2 (Three-player majority game) Three people have access to one unit of output. Any majority—two or three people—may control the allocation of this output. Each person cares only about the amount of output she obtains.

We may model this situation as the following coalitional game.

Players The three people.

Actions Each coalition consisting of a single player has a single action, which yields the player no output. The set of actions of each coalition S with two or three players is the set of S-allocations of one unit of output.

Preferences Each player's preferences are represented by the amount of output she obtains.

This game is cohesive because every partition of the set of players contains at most one majority coalition, and for every action of such a coalition there is an action of the grand coalition that yields each player as least as much output.

In these examples the set of actions of each coalition S is the set of S-allocations of the output that S can obtain, and each player's preferences are represented by the amount of output she obtains. Thus we can summarize each coalition's set of actions by a single number, equal to the total output it can obtain, and can interpret this number as the total "payoff" that may be distributed among the members of the coalition. A coalitional game in which the set of payoff distributions resulting from each coalition's actions may be represented in this way is said to have **transferable payoff**.

We refer to the total payoff of any coalition S in a game with transferable payoff as the **worth** of S, and denote it $v(S)$. Such a game is thus specified by its set of players N and its worth function.

For the two-player unanimity game, for example, we have $N = \{1, 2\}$, $v(\{1\}) = v(\{2\}) = 0$, and $v(\{1, 2\}) = 1$. For the landlord–worker game we have $N = \{1, \ldots, m+1\}$ (where 1 is the landowner and $2, \ldots, m$ are the workers) and

$$v(S) = \begin{cases} 0 & \text{if 1 is not a member of } S \\ f(k) & \text{if } S \text{ consists of 1 and } k \text{ workers.} \end{cases}$$

For the three-player majority game we have $N = \{1, 2, 3\}$, $v(\{i\}) = 0$ for $i = 1, 2, 3$, and $v(S) = 1$ for every other coalition S.

In the next two examples, payoff is not transferable.

◆ EXAMPLE 238.1 (House allocation) Each member of a group of n people has a single house. Any subgroup may reallocate its members' houses in any way it wishes (one house to each person). (Time-sharing and other devices to evade the indivisibility of a house are prohibited.) The values assigned to houses vary among the people; each person cares only about the house she obtains. The following coalitional game models this situation.

Players The n people.

Actions The set of actions of a coalition S is the set of all assignments to members of S of the houses originally owned by members of S.

Preferences Each player prefers one outcome to another according to the house she is assigned.

This game is cohesive because any allocation of the houses that can be achieved by the coalitions in any partition of the set of players can also be achieved by the set of all players. It does not have transferable payoff. For example, a coalition of players 1 and 2 can achieve only the two payoff distributions (v_1, w_2) and (v_2, w_1), where v_i is the payoff to player 1 of the house owned by player i and w_i is the payoff to player 2 of the house owned by player i.

◆ EXAMPLE 238.2 (Marriage market) A group of men and a group of women may be matched in pairs. Each person cares about her partner. A *matching* of the members of a coalition S is a partition of the members of S into male-female pairs and singles. The following coalitional game models this situation.

Players The set of all the men and all the women.

Actions The set of actions of a coalition S is the set of all matchings of the members of S.

Preferences Each player prefers one outcome to another according to the partner she is assigned.

This game is cohesive because the matching of the members of the grand coalition induced by any collection of actions of the coalitions in a partition can be achieved by some action of the grand coalition.

8.2 The core

Which action may we expect the grand coalition to choose? We seek an action compatible with the pressures imposed by the opportunities of each coalition, rather than simply those of individual players as in the models of a strategic game (Chapter 2) and an extensive game (Chapter 5). We define an action of the grand coalition to be "stable" if no coalition can break away and choose an action that all its members prefer. The set of all stable actions of the grand coalition is called the *core*, defined precisely as follows.

▶ DEFINITION 239.1 (*Core*) The **core** of a coalitional game is the set of actions a_N of the grand coalition N such that no coalition has an action that all its members prefer to a_N.

If a coalition S has an action that all its members prefer to some action a_N of the grand coalition, we say that S can **improve upon** a_N. Thus we may alternatively define the core to be the set of all actions of the grand coalition upon which no coalition can improve.

Note that the core is defined as a *set* of actions, so it always *exists*; a game cannot fail to have a core, though it may be the empty set, in which case no action of the grand coalition is immune to deviations.

We have restricted attention to games in which, when evaluating an outcome, each player cares only about the action chosen by the coalition in the partition of which she is a member. Thus the members of a coalition do not need to speculate about the remaining players' behavior when considering a deviation. Consequently an interpretation of the core does not require us to assume that the players are experienced; the concept makes sense even for naïve players with no experience in the game. (By contrast, the main interpretations of Nash equilibrium and subgame perfect equilibrium require the players to have experience playing the game.)

In a game with transferable payoff, a coalition S can improve upon an action a_N of the grand coalition if and only if its worth $v(S)$ (i.e. the total payoff it can achieve by itself) exceeds the total payoff of its members in a_N. That is, a_N is in the core if and only if for every coalition S the total payoff $x_S(a_N)$ it yields the members of S is at least $v(S)$:

$$x_S(a_N) \geq v(S) \text{ for every coalition } S.$$

To find the core of a coalitional game we need to find the set of all actions of the grand coalition upon which no coalition can improve. In the next example, no coalition can improve upon any action of the grand coalition, so the core consists of *all* actions of the grand coalition.

◆ EXAMPLE 240.1 (Two-player unanimity game) Consider the two-player unanimity game in Example 236.2. An action of the grand coalition is a pair (x_1, x_2) with $x_1 + x_2 = 1$ and $x_i \geq 0$ for $i = 1, 2$ (a division of the one unit of output between the two players). I claim that the core consists of *all* possible divisions:

$$\{(x_1, x_2) : x_1 + x_2 = 1 \text{ and } x_i \geq 0 \text{ for } i = 1, 2\}.$$

Any such division is in the core because if a single player deviates she obtains no output, and if the grand coalition chooses a different division then one player is worse off.

In this example no coalition has any action that imposes any restriction on the action of the grand coalition. In most other games the coalitions' opportunities constrain the actions of the grand coalition.

One way to find the core is to check each action of the grand coalition in turn. For each action and each coalition S, we impose the condition that S cannot make all its members better off; an action is a member of the core if and only if it satisfies these conditions.

Consider, for example, a variant of the two-player unanimity game in which player 1, by herself, can obtain p units of output, and player 2, by herself, can obtain q units of output. Then the condition that the coalition consisting of player 1 not be able to improve upon the action (x_1, x_2) of the grand coalition is $x_1 \geq p$, and the condition that the coalition consisting of player 2 not be able to improve upon this action is $x_2 \geq q$. As in the original game, the coalition of both players cannot improve upon any action (x_1, x_2), so the core is

$$\{(x_1, x_2) : x_1 + x_2 = 1, x_1 \geq p, \text{ and } x_2 \geq q\}.$$

(An implication is that if $p + q > 1$—in which case the game is not cohesive—the core is empty.)

An example of the landowner–worker game further illustrates this method of finding the core.

◆ EXAMPLE 240.2 (Landowner–worker game with two workers) Consider the game in Example 237.1 in which there are two workers ($k = 2$). Let (x_1, x_2, x_3) be an action of the grand coalition. That is, let (x_1, x_2, x_3) be an allocation of the output $f(3)$ among the three players. The only coalitions that can obtain a positive amount of output are that consisting of the landowner (player 1), which can obtain the output $f(1)$, those consisting of the landowner and a worker, which can obtain $f(2)$, and the grand coalition. Thus (x_1, x_2, x_3) is in the core if and only if

$$\begin{aligned} x_1 &\geq f(1) \\ x_2 &\geq 0 \\ x_3 &\geq 0 \\ x_1 + x_2 &\geq f(2) \\ x_1 + x_3 &\geq f(2) \\ x_1 + x_2 + x_3 &= f(3), \end{aligned}$$

where the last condition ensures that (x_1, x_2, x_3) is an allocation of $f(3)$.

From the last condition we have $x_1 = f(3) - x_2 - x_3$, so that we may rewrite the conditions as

$$\begin{aligned} 0 \leq x_2 &\leq f(3) - f(2) \\ 0 \leq x_3 &\leq f(3) - f(2) \\ x_2 + x_3 &\leq f(3) - f(1) \\ x_1 + x_2 + x_3 &= f(3). \end{aligned}$$

That is, in an action in the core, each worker obtains at most the extra output $f(3) - f(2)$ produced by the third player, and the workers together obtain at most the extra output $f(3) - f(1)$ produced by the second and third players together.

? EXERCISE 241.1 (Three-player majority game) Show that the core of the three-player majority game (Example 237.2) has an empty core. Find the core of the variant of this game in which player 1 has three votes (and player 2 and player 3 each has one vote, as in the original game).

The next example introduces a class of games that model the market for an economic good.

◆ EXAMPLE 241.2 (Market with one owner and two buyers) A person holds one indivisible unit of a good and each of two (potential) buyers has a large amount of money. The owner values money but not the good; each buyer values both money and the good and regards the good as equivalent to one unit of money. Each coalition may assign the good (if owned by one of its members) to any of its members and allocate its members' money in any way it wishes among its members.

We may model this situation as the following coalitional game.

Players The owner and the two buyers.

Actions The set of actions of each coalition S is the set of S-allocations of the money and good (if any) owned by S.

Preferences The owner's preferences are represented by the amount of money she obtains; each buyer's preferences are represented by the amount of the good (either 0 or 1) she obtains plus the amount of money she holds.

I claim that for any action in the core, the owner does not keep the good. Let a_N be an action of the grand coalition in which the owner keeps the good, and let m_i be the amount of money transferred from potential buyer i to the owner in this action. (Transfers of money from the buyers to the owner when the owner keeps the good may not sound sensible, but they are feasible, so that we need to consider them.) Consider the alternative action a'_N of the grand coalition in which the good is allocated to buyer 1, who transfers $m_1 + 2\epsilon$ money to the owner, and buyer 2 transfers $m_2 - \epsilon$ money to the owner, where $0 < \epsilon < \frac{1}{2}$. We see that all the players' payoffs are higher in a'_N than they are in a_N. (The owner's payoff is ϵ

higher, buyer 1's payoff is $1 - 2\epsilon$ higher, and buyer 2's payoff is ϵ higher.) Thus a_N is not in the core.

Consider an action a_N in the core in which buyer 1 obtains the good. I claim that in a_N buyer 1 pays one unit of money to the owner and buyer 2 pays no money to the owner. If buyer 2 pays a positive amount she can improve upon a_N by acting by herself (and making no payment). If buyer 1 pays more than one unit of money to the owner she too can improve upon a_N by acting by herself. Finally, suppose buyer 1 pays $m_1 < 1$ to the owner. Then the owner and buyer 2 can improve upon a_N by allocating the good to buyer 2 and transferring $\frac{1}{2}(1 + m_1)$ units of money from buyer 2 to the owner, yielding the owner a payoff greater than m_1 and buyer 2 a positive payoff.

We conclude that the core contains exactly two actions, in each of which the good is allocated to a buyer and one unit of the buyer's money is allocated to the owner. That is, the good is sold to a buyer at the price of 1, yielding the buyer who obtains the good the same payoff that she obtains if she does not trade. This extreme outcome is a result of the competition between the buyers for the good: any outcome in which the owner trades with buyer i at a price less than 1 can be improved upon by the coalition consisting of the owner and the *other* buyer, who is willing to pay a little more for the good than does buyer i.

? EXERCISE 242.1 (Market with one owner and two heterogeneous buyers) Consider the variant of the game in the previous example in which buyer 1's valuation of the good is 1 and buyer 2's valuation is $v < 1$ (i.e. buyer 2 is indifferent between owning the good and owning v units of money). Find the core the game that models this situation.

In the next exercise, the grand coalition has finitely many actions; one way of finding the core is to check each one in turn.

? EXERCISE 242.2 (Vote trading) A legislature with three members decides, by majority vote, the fate of three bills, A, B, and C. Each legislator's preferences are represented by the sum of the values she attaches to the bills that pass. The value attached by each legislator to each bill is indicated in Figure 242.1. For example, if bills A and B pass and C fails, then the three legislators' payoffs are 1, 3, and 0 respectively. Each majority coalition can achieve the passage of any set of bills, whereas each minority is powerless.

	A	B	C
Legislator 1	2	−1	1
Legislator 2	1	2	−1
Legislator 3	−1	1	2

Figure 242.1 The legislators' payoffs to the three bills in Exercise 242.2.

a. Find the core of the coalitional game that models this situation.

b. Find the core of the game in which the values the legislators attach to the payoff of each bill differ from those in Figure 242.1 only in that legislator 3 values the passage of bill C at 0.

c. Find the core of the game in which the values the legislators attach to the payoff of each bill differ from those in Figure 242.1 only in that each 1 is replaced by -1.

8.3 Illustration: ownership and the distribution of wealth

In economies dominated by agriculture, the distribution and institutions of land ownership differ widely. By studying the cores of coalitional games that model various institutions, we can gain an understanding of the implications of these institutions for the distribution of wealth.

A group of $n \geq 3$ people may work land to produce food. Denote the output of food when k people work all the land by $f(k)$. Assume that f is an increasing function, $f(0) = 0$, and the output produced by an additional person decreases as the number of workers increases: $f(k) - f(k-1)$ is decreasing in k. An example of such a function f is shown in Figure 243.1. In all the games that I study the set of players is the set of the n people and each person cares only about the amount of food she obtains.

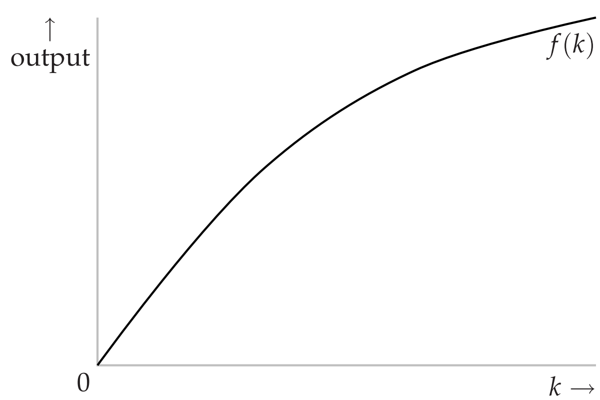


Figure 243.1 The output of food as a function of the number k of workers, under the assumption that the output of an additional worker decreases as the number of workers increases.

8.3.1 Single landowner and landless workers

First suppose that the land is owned by a single person, the *landowner*. I refer to the other people as *workers*. In this case we obtain the game in Example 237.1. In this game the action a_N of the grand coalition in which the landowner obtains all the output $f(n)$ is in the core: all coalitions that can produce any output include the landowner, and none of these coalitions has any action that makes her better off than she is in a_N.

Are the workers completely powerless, or does the core contain actions in which they receive some output? The workers need the landowner to produce any output, but the landowner also needs the workers to produce more than $f(1)$, so there is reason to think that stable actions of the grand coalition exist in which the workers receive some output. Take the landowner to be player 1, and consider the action a_N of the grand coalition in which each player i obtains the output x_i, where $x_1 + \cdots + x_n = f(n)$. Under what conditions on $(x_1, \ldots, x_n)$ is a_N in the core? Because of my assumption on the shape of the function f, the coalitions most capable of profitably deviating from a_N consist of the landowner and every worker but one. Such a coalition can, by itself, produce $f(n-1)$, and may distribute this output in any way among its members. Thus for a deviation by such a coalition not to be profitable, the sum of x_1 and any collection of $n-2$ other x_i's must be at least $f(n-1)$. That is, $(x_1 + \cdots + x_n) - x_j \geq f(n-1)$ for every $j = 2, \ldots, n$. Because $x_1 + \cdots + x_n = f(n)$, we conclude that $x_j \leq f(n) - f(n-1)$ for every player j with $j \geq 2$ (i.e. every worker). That is, if a_N is in the core then $0 \leq x_j \leq f(n) - f(n-1)$ for every player $j \geq 2$. If fact, every such action is in the core, as you are asked to verify in the following exercise.

? EXERCISE 244.1 (Core of landowner–worker game) Check that no coalition can improve upon any action of the grand coalition in which the output received by every worker is nonnegative and at most $f(n) - f(n-1)$. (Use the fact that the form of f implies that $f(n) - f(k) \geq (n-k)(f(n) - f(n-1))$ for every $k \leq n$.)

We conclude that the core of the game is the set of all actions of the grand coalition in which the output x_i obtained by each worker i satisfies $0 \leq x_i \leq f(n) - f(n-1)$ and the output obtained by the landowner is the difference between $f(n)$ and the sum of the workers' shares. In economic jargon, $f(n) - f(n-1)$ is a worker's "marginal product". Thus in any action in the core, each worker obtains at most her marginal product.

The workers' shares of output are driven down to at most $f(n) - f(n-1)$ by competition between coalitions consisting of the landowner and workers. If the output received by any worker exceeds $f(n) - f(n-1)$ then the other workers, in cahoots with the landowner, can deviate and increase their share of output. That is, each worker's share of output is limited by her comrades' attempts to obtain more output.

The fact that each worker's share of output is held down by inter-worker competition suggests that if the workers were to agree not to join deviating coalitions *except as a group* then they might be better off. You are asked to check this idea in the following exercise.

? EXERCISE 244.2 (Unionized workers in landowner–worker game) Formulate as a coalitional game the variant of the landowner–worker game in which any group of fewer than $n-1$ workers refuses to work with the landowner, and find its core.

The core of the original game is closely related to the outcomes predicted by the economic notion of "competitive equilibrium". Suppose that the landowner believes she can hire any number of workers at the fixed wage w (given as an amount of output), and every worker believes that she can obtain employment at this wage. If $w \geq 0$ then every worker wishes to work, and if $w \leq f(n) - f(n-1)$ the landowner wishes to employ all $n-1$ workers. (Reducing the number of workers by one reduces the output by $f(n) - f(n-1)$; further reducing the number of workers reduces the output by successively larger amounts, given the shape of f.) If $w > f(n) - f(n-1)$ then the landowner wishes to employ fewer than $n-1$ workers, because the wage exceeds the increase in the total output that results when the $(n-1)$th worker is employed. Thus the demand for workers is equal to the supply if and only if $0 \leq w \leq f(n) - f(n-1)$; every such wage w is a "competitive equilibrium".

A different assumption about the form of f yields a different conclusion about the core. Suppose that each additional worker produces *more* additional output than the previous one. An example of a function f with this form is shown in Figure 245.1. Under this assumption the economy has no competitive equilibrium: for any wage, the landowner wishes to employ an indefinitely large number of workers. The next exercise asks you to study the core of the induced coalitional game.

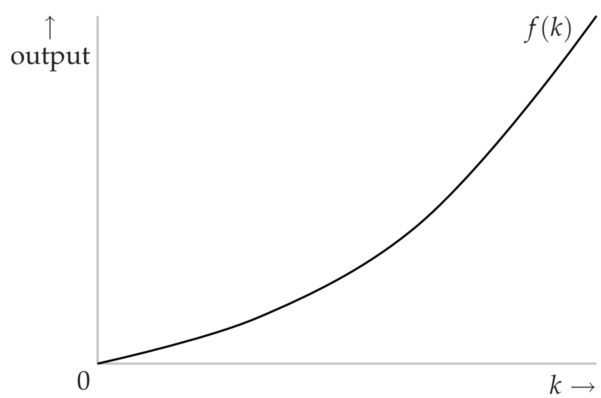


Figure 245.1 The output of food as a function of the number k of workers, under the assumption that the output of an additional worker increases as the number of workers increases.

? EXERCISE 245.1 (Landowner–worker game with increasing marginal products) Consider the variant of the landowner–worker game in which each additional worker produces more additional output than the previous one. (That is, $f(k)/k < f(k+1)/(k+1)$ for all k.) Show that the core of this game contains the action of the grand coalition in which each player obtains an equal share of the total output.

8.3.2 Small landowners

Suppose that the land is distributed equally between all n people, rather than being concentrated in the hands of a single landowner. Assume that a group of k people who pool their land and work together produce $(k/n)f(n)$ units of output. (The output produced by half the people working half the land, for example, is half the output produced by all the people working all the land.)

The following specification of the set of actions available to each coalition models this situation.

Actions The set of actions of a coalition S consisting of k players is the set of all S-allocations of the output $(k/n)f(n)$ between the members of S.

As you might expect, one action in the core of this game is that in which every player obtains an equal share of the total output—that is, $f(n)/n$ units. Under this action, the total amount received by each coalition is precisely the total amount the coalition produces. In fact, no other action is in the core. In any other action, some player receives less than $f(n)/n$, and hence can improve upon the action alone (obtaining $f(n)/n$ for herself). That is, the core consists of the single action in which every player obtains $f(n)/n$ units of output.

8.3.3 Collective ownership

Suppose that the land is owned collectively and the distribution of output is determined by majority voting. Assume that any majority may distribute the output in any way it wishes; any majority may, in particular, take all the output for itself. In this case the set of actions available to each coalition are given as follows.

Actions The set of actions of a coalition S consisting of more than $n/2$ players is the set of all S-allocations of the output $f(n)$ between the members of S. The set of actions of a coalition S consisting of at most $n/2$ players is the single S-allocation in which no player in S receives any output.

The core of the coalitional game defined by this assumption is empty. For every action of the grand coalition, at least one player obtains a positive amount of output. But if player i obtains a positive amount of output then the coalition of the remaining players, which is a majority, may improve upon the action, distributing the output $f(n)$ among its members (so that player i gets nothing). Thus every action of the grand coalition may be improved upon by some coalition; no distribution of output is "stable".

The core of this game is empty because of the extreme power of every majority coalition. If any majority coalition may control how the land is used, but every player owns a "share" that entitles her to the fraction $1/n$ of the output, then a majority coalition with k members can lay claim to only the fraction k/n of the total output, and a stable distribution of output may exist. This alternative ownership institution, which tempers the power of majority coalitions, does not have interesting implications in the model in this section because the control of land use vested in a majority coalition is inconsequential—only one sensible pattern of use exists (all the players work!). If choices exist—if, for example, different crops may be grown, and people differ in their preferences for these crops—then collective ownership in which each player is entitled to an equal share of the output may yield a different outcome from individual ownership.

8.4 Illustration: exchanging homogeneous horses

Markets may be modeled as coalitional games in which the set of actions of each coalition S is the set of S-allocations of the good initially owned by the members of S. The core of such a game is the set of allocations of the goods available in the economy that are robust to the trading opportunities of all possible groups of participants: if a_N is in the core then no group of agents can secede from the economy, trade among themselves, and produce an outcome they all prefer to a_N.

In this section I describe a simple example of a market, in which there is money and a single homogeneous good (all units of which are identical). In the next section I describe a market in which there is a single *heterogeneous* good. In both case the core makes a very precise prediction about the outcome.

8.4.1 Model

Some people own one unit of an indivisible good, whereas others possess only money. Some non-owners value a unit of the good more highly than some owners, so that mutually beneficial trades exist. Which allocation of goods and money will result?

We may address this question with the help of a coalitional game that generalizes the one in Example 241.2. I refer to the goods as "horses" (following the literature on the model, which takes off from an analysis by Eugen von Böhm-Bawerk (1851–1914)). Call each person who owns a horse simply an *owner*, and every other person a *nonowner*. Assume that all horses are identical, and that no one wishes to own more than one. People value a horse differently; denote player i's valuation by v_i. Assume that there are at least two owners and two nonowners, and that some owner's valuation is less than some nonowner's valuation (i.e. for some owner i and nonowner j have $v_i < v_j$), so that some trade is mutually desirable. Assume also, to avoid some special cases, that some nonowner's valuation is less than some owner's valuation (i.e. for some nonowner i and owner j we have $v_i < v_j$) and that no two players have the same valuation. Further assume that every person has enough money to fully compensate the owner who values a horse most highly, so that no one's behavior is constrained by her cash balance.

As to preferences, assume that each person cares only about the amount of money she has and whether or not she has a horse. (In particular, no one cares about any other person's holdings.) Specifically, assume that each player i's preferences are represented by the payoff function

$$\begin{cases} v_i + r & \text{if she has a horse and \$}r\text{ more money than she had originally} \\ r & \text{if she has no horse and \$}r\text{ more money than she had originally.} \end{cases}$$

(This assumption does not mean that people do not value the money they have initially. Equivalently we could represent player i's preferences by the functions $v_i + r + m_i$ if she has a horse and $r + m_i$ if she does not, where m_i is the amount of money she has initially.)

The following coalitional game models, which I call a **horse trading game**, models the situation.

Players The group of people (owners and nonowners).

Actions The set of actions of each coalition S is the set of S-allocations of the horses and the total amount of money owned by S in which each player obtains at most one horse.

Preferences Each player's preferences are represented by the payoff function described above.

This game incorporates no restriction on the way in which a coalition may distribute its money and horses. In particular, players are not restricted to bilateral trades of money for horses. A coalition of two owners and two nonowners, for example, may, if it wishes, allocate each of the owners' horses to a nonowner and transfer money from *both* nonowners to only one owner, or from one nonowner to the other.

8.4.2 The core

Number the owners in ascending order and the nonowners in descending order of the valuations they attach to a horse. Figure 249.1 illustrates the valuations, ordered in this way. (This diagram should be familiar—perhaps it is a little too familiar—if you have studied economics.) Denote owner i's valuation σ_i and nonowner i's valuation β_i. Denote by k^* the largest number i such that $\beta_i > \sigma_i$ (so that among the owners and nonowners whose indices are k^* or less, every nonowner's valuation is greater than every owner's valuation).

Let a_N be an action in the core. Denote by L^* the set of owners who have no horse in a_N (the set of *sellers*) and by B^* the set of nonowners who have a horse in a_N (the set of *buyers*). These two sets must have the same number of members (by the law of conservation of horses). Denote by r_i the amount of money received by owner i and by p_j the amount paid by nonowner j in a_N.

I claim that $p_j = 0$ for every nonowner j not in B^*. (That is, no nonowner who does not acquire a horse either pays or receives any money.)

- If $p_j > 0$ for some nonowner j not in B^* then her payoff is negative, and she can unilaterally improve upon a_N by retaining her original money.

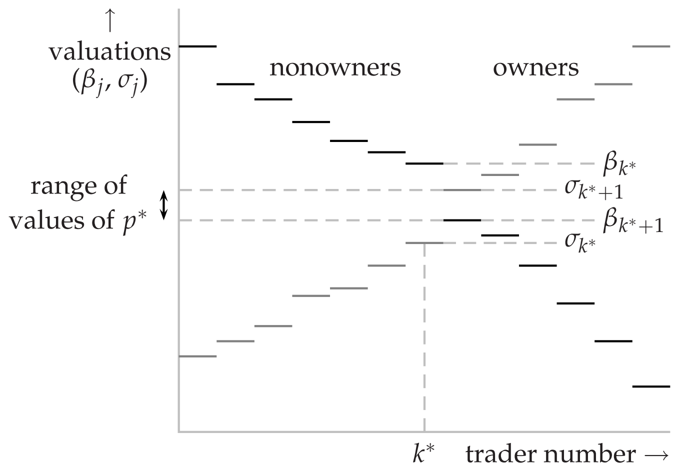


Figure 249.1 An example of the players' valuations in a market with an indivisible good. The buyers' valuations are given in black, and the sellers' in gray.

- If $p_j < 0$ for some nonowner j not in B^* then the coalition of all players *other than* j has p_j less money than it owned initially, and the same number of horses. Thus this coalition can improve upon a_N by assigning horses in the same way as they are assigned in a_N and giving each of its members $p_j/(n-1)$ more units of money than she gets in a_N (where n is the total number of players).

By a similar argument, $r_i = 0$ for every owner not in L^* (an owner who does not sell her horse neither pays nor receives any money.)

I now argue that in a_N every seller (member of L^*) receives the same amount of money, every buyer (member of B^*) pays the same amount of money, and these amounts are equal: $r_i = p_j$ for every seller i and buyer j. That is, all trades occur at the same price.

Suppose that $r_i < p_j$ for seller i and buyer j. I argue that the coalition $\{i, j\}$ can improve upon a_N: i can sell her horse to j at a price between r_i and p_j. Under a_N, seller i's payoff is r_i and buyer j's payoff is $\beta_j - p_j$. If i sells her horse to j at the price $\frac{1}{2}(r_i + p_j)$ then her payoff is $\frac{1}{2}(r_i + p_j) > r_i$ and j's payoff is $\beta_j - \frac{1}{2}(r_i + p_j) > \beta_j - p_j$, so that both i and j are better off than they are in a_N. Thus $r_i \geq p_j$ for every seller i and every buyer j.

Now, the sum of all the amounts r_i received by sellers is equal to the sum of all the amounts p_j paid by buyers (by the law of conservation of money), and L^* and B^* have the same number of members. Thus we have $r_i = p_j$ for every seller i in L^* and buyer j in B^*.

In summary,

> for every action a_N in the core there exists p^* such that $r_i = p_i = p^*$ for every owner i in L^* and every nonowner j in B^*, and $r_i = p_i = 0$ for

every owner not in L^* and every nonowner j not in B^*.

I now argue that the common price p^* at which all trades take place lies in a narrow range.

In a_N, every owner i whose valuation of a horse is less than p^* must sell her horse: if she did not then the coalition consisting of herself and any nonowner j who buys a horse in a_N could improve upon a_N by taking the action in which j buys i's horse at a price between the owner's valuation and p^*. Also, no owner whose valuation exceeds p^* trades, because her payoff from doing so is negative. Similarly, every nonowner whose valuation is greater than p^* buys a horse, and no nonowner whose valuation is less than p^* does so.

? EXERCISE 250.1 (Range of prices in horse market) Show that the requirement that the number of owners who sell their horses must equal the number of nonowners who buy horses, together with the arguments above, implies that the common trading price p^* is at least σ_{k^*}, at least β_{k^*+1}, at most β_{k^*}, and at most σ_{k^*+1}. That is, $p^* \geq \max\{\sigma_{k^*}, \beta_{k^*+1}\}$ and $p^* \leq \min\{\beta_{k^*}, \sigma_{k^*+1}\}$.

Finally, I argue that in any action in the core a player whose valuation is equal to p^* trades. Suppose nonowner i's valuation is equal to p^*. Then owner i's valuation is less than p^* and owner $i+1$'s valuation is greater than p^* (given my assumption that no two players have the same valuation), so that exactly i owners trade. Thus exactly i nonowners must trade, implying that nonowner i trades. Symmetrically, a owner whose valuation is equal to p^* trades.

In summary, in every action in the core of a horse trading game,

- every nonowner pays the same price for a horse
- the common price is at least $\max\{\sigma_{k^*}, \beta_{k^*+1}\}$ and at most $\min\{\beta_{k^*}, \sigma_{k^*+1}\}$
- every owner whose valuation is at most the price trades her horse
- every nonowner whose valuation is at least the price obtains a horse.

(250.2)

The action satisfying these conditions for the price p^* yields the payoffs

$$\begin{cases} \max\{v_i, p^*\} & \text{for every owner } i \\ \max\{v_i, p^*\} - p^* & \text{for every nonowner } i. \end{cases}$$

The core does not impose any additional restrictions on the actions of the grand coalition: every action that satisfies these conditions is in the core. To establish this result, I need to show that for any action a_N that satisfies the conditions, no coalition has an action that is better for all its members. When a coalition deviates, which of its actions has the best chance of improving upon a_N? The optimal action definitely assigns the coalition's horses to the members who value a horse most highly. (If $v_i < v_j$ then the transfer of a horse from i to j, accompanied by the transfer from j to i of an amount of money between v_i and v_j makes both i and j better off.) No transfer of money makes anyone better off without making someone worse off, so in order for a coalition to improve upon a_N there must be some distribution of the total amount of money it owns that, given the optimal distribution of horses, makes all its members better off than they are in a_N. For every distribution of a coalition's money the total payoff of the members of the coalition is the same. Thus a coalition can improve upon a_N if and only if the total payoff of its members under a_N is less than its total payoff when it assigns its horses optimally.

Consider an arbitrary coalition S. Denote by ℓ the total number of owners in S, by b the total number of nonowners in S, and by S^* the set of ℓ members of S whose valuations are highest. Then S's total payoff when it assigns its horses optimally is

$$\sum_{i \in S^*} v_i,$$

whereas its total payoff under a_N is

$$\sum_{i \in S} \max\{v_i, p^*\} - bp^* = \sum_{i \in S^*} \max\{v_i, p^*\} + \sum_{i \in S \setminus S^*} \max\{v_i, p^*\} - bp^*,$$

where $S \setminus S^*$ is the set of members of S *not* in S^*. The former is never higher than the latter because $S \setminus S^*$ has b members, so that $\sum_{i \in S \setminus S^*} \max\{v_i, p^*\} - bp^* \geq 0$.

In summary, the core of a horse trading game is the set of actions of the grand coalition that satisfies the four conditions in (250.2).

? EXERCISE 251.1 (Horse trading game with single seller) Find the core of the variant of the horse trading game in which there is a single owner, whose valuation is less than the highest valuation of the nonowners.

If you have studied economics you know that this outcome is the same as the "competitive equilibrium". The theories differ, however. The theory of competitive equilibrium *assumes* that all trades take place at the same price. It defines an equilibrium price to be one at which "demand" (the total number of nonowners whose valuations exceed the price) is equal to "supply" (the total number of owners whose valuations are less than the price). This equilibrium may be justified by the argument that if demand exceeds supply then the price will tend to rise, and if supply exceeds demand it will tend to fall. Thus in this theory, "market pressures" generate an equilibrium price; no agent in the market chooses a price.

By contrast, the coalitional game we have studied models the players' actions explicitly; each group may exchange its horses and money in any way it wishes. The core is the set of actions of all players that survives the pressures imposed by the trading opportunities of each possible group of players. A uniform price is not assumed, but is shown to be a necessary property of any action in the core.

? EXERCISE 251.2 (Horse trading game with large seller) Consider the variant of the horse trading game in which there is a single owner who has two horses. Assume that the owner's payoff is $\sigma_1 + r$ if she keeps one of her horses and $2\sigma_1 + r$ if she keeps both of them, where r is the amount of money she receives. Assume that there are at least two nonowners, both of whose values of a horse exceed σ_1. Find the core of this game. (Do all trades take place at the same price, as they do in a competitive equilibrium?)

8.5 Illustration: exchanging heterogeneous houses

8.5.1 Model

Each member of a group of n people owns an indivisible good—call it a house. Houses, unlike the horses of the previous section, differ. Any subgroup may reallocate its members' houses in any way it wishes (one house to each person). (Time-sharing and other devices to evade the indivisibility of a house are prohibited.) Each person cares only about the house she obtains, and has a strict ranking of the houses (she is not indifferent between any two houses).

Which assignments of houses to people are stable? You may think that without imposing any restrictions on the nature or diversity of preferences, this question is hard to answer, and that for some sufficiently conflicting configurations of preferences no assignment is stable. If so, you are wrong on both counts, at least as far as the core is concerned; remarkably, for *any* preferences, a slight variant of the core yields a *unique* stable outcome.

The following coalitional game, which I call a **house exchange game**, models the situation.

Players The n people.

Actions The set of actions of a coalition S is the set of all assignments to members of S of the houses originally owned by members of S.

Preferences Each player prefers one outcome to another according to the house she is assigned.

8.5.2 The top trading cycle procedure and the core

One property of an action in the core is immediate: any player who initially owns her favorite house obtains that house in any assignment in the core, because every player has the option of simply keeping the house she initially owns.

This property allows us to completely analyze the simplest nontrivial example of the game, with two people. Denote the person who initially owns player i's favorite house by $o(i)$.

- If at least one person initially owns her favorite house (i.e. if $o(1) = 1$ or $o(2) = 2$), then the core contains the single assignment in which each person keeps the house she owns.

- If each person prefers the house owned by the other person (i.e. if $o(1) = 2$ and $o(2) = 1$), then the core contains the single assignment in which the two people exchange houses.

In the second case we say that "12 is a 2-cycle". When there are more players, longer cycles are possible. For example, if there are three or more players and $o(i) = j$, $o(j) = k$, and $o(k) = i$, then we say that "ijk is a 3-cycle". (If $o(i) = i$, we can think of i as a "1-cycle".)

The case in which there are three people raises some new possibilities.

- If at least two people initially own their favorite houses, then the core contains the single assignment in which each person keeps the house she initially owns.
- If exactly one person, say player i, initially owns her favorite house, then in any assignment in the core, that person keeps her house. Whether the other two people exchange their houses depends on their preferences over these houses, ignoring player i's house (which has already been assigned); the analysis is the same as that for the two-player game.
- If no person initially owns her favorite house, there are two cases.
  - If there is a 2-cycle (i.e. if there exist persons i and j such that j initially owns i's favorite house and i initially owns j's favorite house), then the only assignment in the core is that in which i and j swap houses and the remaining player keeps the house she owns initially.
  - Otherwise, suppose that $o(i) = j$. Then $o(j) = k$, where k is the third player (otherwise ij is a 2-cycle), and $o(k) = i$ (otherwise kj is a 2-cycle.) That is, ijk is a 3-cycle. Consider the assignment in which i gets j's house, j gets k's house, and k gets i's house. Every player is assigned her favorite house, so the assignment is in the core. (This argument does not show that the core contains no other assignments.)

This construction of an assignment in the core can be extended to games with any number of players. First we look for cycles among the houses at the top of the players' rankings, and assign to each member of each cycle her favorite house. (If there are at most three players, only one cycle containing more than one player may exist, but if there are more players, many cycles may exist.) Then we eliminate from consideration the players involved in these cycles and the houses they are allocated, look for any cycles at the top of the remains of the players' rankings, and assign to each member of each of these cycles her favorite house among those remaining. We continue in the same manner until all players are assigned houses. This procedure is called the *top trading cycle procedure*.

To illustrate the procedure, consider the game with four players whose preferences satisfy the specification in Figure 254.1. In this figure, h_i denotes the house owned by player i and the players' rankings are listed from best to worst, starting at the top (player 3 prefers player 1's house to player 2's house to player 4's house, for example). Hyphens indicate irrelevant parts of the rankings. We see that 12 is a 2-cycle, so at the first step players 1 and 2 are assigned their favorite houses (h_2 and h_1 respectively). After eliminating these players and their houses, 34 becomes a 2-cycle, so that player 3 is assigned h_4 and player 4 is assigned h_3. If player 3's ranking of h_3 and h_4 were reversed then at the second stage 3 would be a one-cycle, so that player 3 would be assigned h_3, and then at the third stage player 4 would be assigned h_4.

Player 1	Player 2	Player 3	Player 4
h_2	h_1	h_1	h_3
-	-	h_2	h_2
-	-	h_4	h_4
-	-	h_3	-

Figure 254.1 A partial specification of the players' preferences in a game with four players, illustrating the top trading cycle procedure. Each player's ranking is given from best to worst, reading from top to bottom. Hyphens indicate irrelevant parts of the rankings.

? EXERCISE 254.1 (House assignment with identical preferences) Find all the assignments in the core of the n-player game in which every player ranks the houses in the same way.

I now argue that

> for any (strict preferences), the core of a house exchange game contains the assignment induced by the top trading cycle procedure.

The following argument establishes this result. Every player assigned a house in the first round receives her favorite house, so that no coalition containing such a player can make all its members better off than they are in a_N. Now consider a coalition that contains players assigned houses in the second round, but no players assigned houses in the first round. Such a coalition does not own any of the houses assigned on the first round, so that its members who were assigned in the second round obtain their favorite houses *among the houses it owns*. Thus such a coalition has no action that makes all its members better off than they are in a_N. A similar argument applies to coalitions containing players assigned in later rounds.

8.5.3 The strong core

I remarked that my analysis of a three-player game does not establish the existence of a *unique* assignment in the core. Indeed, consider the preferences in Figure 255.1. We see that 123 is a 3-cycle, so that the top cycle trading procedure generates the assignment in which each player receives her favorite house.

Player 1	Player 2	Player 3
h_3	h_1	h_2
h_2	h_2	h_3
h_1	h_3	h_1

Figure 255.1 The players' preferences in a game with three players. Each player's ranking is given from best to worst, reading from top to bottom.

I claim that the alternative assignment a'_N, in which player 1 obtains h_2, player 2 obtains h_1, and player 3 obtains h_3 is also in the core. Player 2 obtains her favorite house, so no coalition containing her can improve upon a'_N. Neither player 1 nor player 3 alone can improve upon a'_N because player 1 prefers h_2 to h_1 and player 3 obtains the house she owns. The only remaining coalition is $\{1,3\}$, which owns h_1 and h_3. If it deviates and assigns h_1 to player 1 then she is worse off than she is in a'_N, and if it deviates and assigns h_1 to player 3 then she is worse off than she is in a'_N. Thus no coalition can improve upon a'_N.

Although no coalition S can achieve any S-allocation that makes all of its members better off than they are in a'_N, the coalition N of all three players *can* make two of its members (players 1 and 3) better off, while keeping the remaining member (player 2) with the same house. That is, it can "weakly" improve upon a'_N.

This example suggests that if we modify the definition of the core so that actions upon which any coalition can weakly improve are eliminated, we might reduce the core to a single assignment.

Define the *strong core* of any game to be the set of actions a_N of the grand coalition N such that no coalition S has an action a_S that some of its members prefer to a_N and all of its members regard to be at least as good as a_N.

The argument I have given shows that the action a'_N is not in the strong core of the game in which the players' preferences are given in Figure 255.1, though it is in the core. In fact,

> for any (strict) preferences, the strong core of a house exchange game consists of the single assignment defined by the top cycle trading procedure.

I omit details of the argument for this result. The result shows that the (strong) core is a highly successful solution for house exchange games; for *any* (strict) preferences, it pinpoints a *single* stable assignment, which is the outcome of a simple, intuitively appealing, procedure.

Unfortunately, the strengthening of the definition of the core has a side effect: if we depart from the assumption that all preferences are strict, and allow players to be indifferent between houses, then the core may be empty. The next exercise gives an example.

? EXERCISE 255.1 (Emptiness of the strong core when preferences are not strict) Suppose that some players are indifferent between some pairs of houses. Specifically, suppose there are three players, whose preferences are given in Figure 256.1. Find the core and show that the strong core is empty.

Player 1	Player 2	Player 3
h_2	h_1, h_3	h_2
h_1, h_3	h_2	h_1, h_3

Figure 256.1 The players' preferences in the game in Exercise 255.1. A cell containing two houses indicates indifference between these two houses.

8.6 Illustration: voting

A group of people chooses a policy by majority voting. How does the chosen policy depend on their preferences? In Chapter 2 we studied a strategic game that models this situation and found that the notion of Nash equilibrium admits a very wide range of stable outcomes. In a Nash equilibrium no single player, by changing her vote, can improve the outcome for herself, but a group of players, by coordinating their votes, may be able to do so. By modeling the situation as a coalitional game and using the notion of the core to isolate stable outcomes, we can find the implications of group deviations for the outcome.

To model voting as a coalitional game, the specification I have given of such a game needs to be slightly modified. Recall that an outcome of a coalitional game is a partition of the set of players and an aciton for each coalition in the partition. So far I have assumed that each player cares only about the action chosen by the coalition in the partition to which she belongs. This assumption means that the payoff of a coalition that deviates from an outcome is determined independently of the action of any other coalition; when deviating, a coalition does not have to consider the action that any other coalition takes. In the situation I now present, a different constellation of conditions has the same implication: only coalitions containing a majority of the players have more than one possible action, and every player cares only about the action chosen by the majority coalition (of which there is at most one) in the outcome partition. In brief, any majority may choose an action that affects everyone, and every minority is powerless.

Precisely, assume that there is an odd number of players, each of whom has preferences over a set of *policies* and prefers the outcome x to the outcome y if and only if either there are majority coalitions in the partitions associated with both x and y and she prefers the action chosen by the majority coalition in x to the action chosen by the majority coalition in y, or there is a majority coalition in x by not in y. (If there is a majority coalition in neither x nor y, she is indifferent between x and y.) The set of actions available to any coalition containing a majority of the players is the set of all policies; every other coalition has a single action.

The definition of the core of this variant of a coalitional game is the natural variant of Definition 239.1: the set of actions a_N of the grand coalition N such that no majority coalition has an action that all its members prefer to a_N.

Suppose that the policy x is in the core of this game. Then no policy is preferred to x by a coalition consisting of a majority of the players. Equivalently, for every policy $y \neq x$, the set of players who either prefer x to y or regard x and y to be equally good is a majority. If we assume that every player's preferences are strict—no player is indifferent between any two policies—then for every policy $y \neq x$, the set of players who prefer x to y is a majority. That is, x is a Condorcet winner (see Exercise 74.1). For any preferences, there is at most one Condorcet winner, so we have established that

> if every player's preferences are strict, the core of a majority voting game is empty if there is no Condorcet winner, and otherwise is the set consisting of the single Condorcet winner.

How does the existence and character of a Condorcet winner depend on the players' preferences? First suppose that a policy is a number. Assume that each player i has a favorite policy x_i^*, and that her preferences are *single-peaked*: if x and x' are policies for which $x < x' < x_i^*$ or $x_i^* < x' < x$ then she prefers x' to x. Then the median of the players' favorite positions is the Condorcet winner, as you are asked to show in the next exercise, and hence the unique member of the core of the voting game. (The median is well-defined because the number of players is odd.)

? EXERCISE 257.1 (Median voter theorem) Show that when the policy space is one-dimensional and the players' preferences are single-peaked the unique Condorcet winner is the median of the players' favorite positions. (This result is known as the *median voter theorem*.)

A one-dimensional space captures some policy choices, but in other situations a higher dimensional space is needed. For example, a government has to choose the amounts to spend on health care and defense, and not all citizens' preferences are aligned on these issues. Unfortunately, for most configurations of the players' preferences, a Condorcet winner does not exist in a policy space of two or more dimensions, so that the core is empty.

To see why this claim is plausible, suppose the policy space is two-dimensional and there are three players. Place the players' favorite positions at three arbitrary points, like x_1^*, x_2^*, and x_3^* in Figure 258.1. Assume that each player i's distaste for a position x different from her favorite position x_i^* is exactly the distance between x and x_i^*, so that for any value of r she is indifferent between all policies on the circle with radius r centered at x_i^*.

Now choose any policy and ask if it is a Condorcet winner. The policy x in the figure is not, because any policy in the shaded area is preferred to x by players 1 and 2, who constitute a majority. The policy x is also beaten in a majority vote by any policy in either of the other lens-shaped areas defined by the intersection of the circles centered at x_1^*, x_2^*, and x_3^*. Is there any policy for which no such lens-shaped

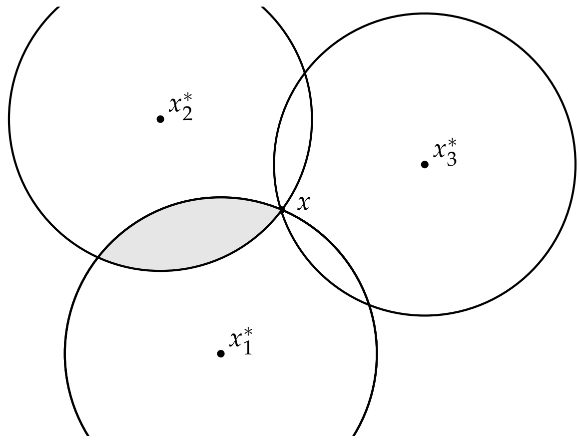


Figure 258.1 A two-dimensional policy space with three players. The point x_i^* is the favorite position of player i for $i = 1, 2, 3$. Every policy in the shaded lens is preferred by players 1 and 2 to x.

area is created? By checking a few other policies you can convince yourself that there is no such policy. That is, no policy is a Condorcet winner, so that the core of the game is empty.

For *some* configurations of the players' favorite positions a Condorcet winner exists. For example, if the positions lie on a straight line then the middle one is a Condorcet winner. But only very special configurations yield a Condorcet winner—in general there is none, so that the core is empty, and our analysis suggests that no policy is stable under majority rule when the policy space is multidimensional.

In some situations in which policies are determined by a vote, a decision requires a positive vote by more than a simple majority. For example, some jury verdicts in the USA require unanimity, and changes in some organizations' and countries' constitutions require a two-thirds majority. To study the implications of these alternative voting rules, fix q with $n/2 \leq q \leq n$ and consider a variant of the majority-rule game that I call the *q-rule game*, in which the only coalitions that can choose policies are those containing at least q players. Roughly, the larger is the value of q, the larger is the core. You are invited to explore some examples in the next exercise.

? EXERCISE 258.1 (Cores of q-rule games)

a. Suppose that the set of policies is one-dimensional and that each player's preferences are single-peaked. Find the core of the q-rule game for any value of q with $n/2 \leq q \leq n$.

b. Find the core of the q-rule game when $q = 3$ in the example in Figure 258.1 (with a two-dimensional policy space and three players).

8.7 Illustration: matching

Applicants must be matched with universities, workers with firms, and football players with teams. Do stable matchings exist? If so, what are their properties, and which institutions generate them?

In this section I analyze a model of two-sided one-to-one matching: each party on one side must be matched with exactly *one* party on the other side. Most of the main ideas that emerge apply also to many-to-one matching problems.

The model I analyze is sometimes referred to as one of "marriage", though of course it captures only one dimension of matrimony. Some of the language I use is taken from this interpretation of the model.

8.7.1 Model

I refer to the two sides as X's and Y's. Each X may be matched with at most one Y, and each Y may be matched with at most one X; staying single is an option for each individual. A *matching* of any set of individuals thus splits the set into pairs, each consisting of an X and a Y, and single individuals. I denote the partner of any player i under the matching μ by $\mu(i)$. If i and j are matched, we thus have $\mu(i) = j$ and $\mu(j) = i$; if i is single then $\mu(i) = i$. Each person cares only about her partner, not about anyone else's partner. Assume that every person's preferences are *strict*: no person is indifferent between any two partners. I refer to the set of partners that i prefers to the option of remaining single as the set of i's *acceptable* partners. The following coalitional game, which I refer to as a **two-sided one-to-one matching game**, models this situation.

Players The set of all X's and all Y's.

Actions The set of actions of a coalition S is the set of all matchings of the members of S.

Preferences Each player prefers one outcome to another according to the partner she is assigned.

An example of possible preferences is given in Figure 260.1. For instance, player x_1 ranks y_2 first, then y_1, and finds y_3 unacceptable.

8.7.2 The core and the deferred acceptance procedure

A matching in the core of a two-sided one-to-one matching game has the property that no group of players may, by rearranging themselves, produce a matching that they all like better. I claim that when looking for matchings in the core, we may restrict attention to coalitions consisting either of a single individual or of one X and one Y. Precisely, a matching is in the core if and only if

a. each player prefers her partner to being single

X's			Y's		
x_1	x_2	x_3	y_1	y_2	y_3
y_2	y_1	y_1	x_1	x_2	x_1
y_1	y_2	y_2	x_3	x_1	x_3
	y_3		x_2	x_3	x_2

Figure 260.1 An example of the players' preferences in a two-sided one-to-one matching game. Each column gives one player's ranking (from best to worst) of all the players of the other type that she finds acceptable.

b. for no pair (i, j) consisting of an X and a Y is it the case that i prefers j to $\mu(i)$ and j prefers i to $\mu(j)$.

The following argument establishes this claim. First, any matching μ that does not satisfy the conditions is not in the core: if (*a*) is violated then some player can improve upon μ by staying single, and if (*b*) is violated then some pair of players can improve upon μ by matching with each other. Second, suppose that μ is not in the core. Then for some coalition S there is a matching μ' of its members for which every member i prefers $\mu'(i)$ to $\mu(i)$. If S consists of a single individual, then (*a*) is violated. Otherwise suppose that i is a member of S, and let $j = \mu'(i)$, so that $i = \mu'(j)$. Then i prefers j to $\mu(i)$ and j prefers i to $\mu(j)$. Thus (*b*) is violated.

In the game in which the players' preferences are those given in Figure 260.1, for example, the matching μ in which $\mu(x_1) = y_1$, $\mu(x_2) = y_2$, $\mu(x_3) = x_3$, and $\mu(y_3) = y_3$ (i.e. x_3 and y_3 stay single) is in the core, by the following argument. No single player can improve upon it, because every matched player's partner is acceptable to her. Now consider pairs of players. No pair containing x_3 or y_3 can improve upon the matching, because x_1 and x_2 are matched with partners they prefer to y_3, and y_1 and y_2 are matched with partners they prefer to x_3. A matched pair cannot improve upon the matching either, so the only pairs to consider are $\{x_1, y_2\}$ and $\{x_2, y_1\}$. The first cannot improve upon μ because y_2 prefers x_2, with whom she is matched, to x_1; the second cannot upon μ because y_1 prefers x_1, with whom she is matched, to x_2.

How may matchings in the core be found? As in the case of the market for houses studied in Section 8.5, one member of the core is generated by an intuitively appealing procedure. (In contrast to the core of the house market, however, the core of a two-sided one-to-one matching game may contain more than one action, as we shall see.)

The procedure comes in two flavors, one in which proposals are made by X's, and one in which they are made by Y's. The *deferred acceptance procedure with proposals by* X's is defined as follows. Initially, each X proposes to her favorite Y, and each Y either rejects all the proposals she receives, if none is from an X acceptable to her, or rejects all but the best proposal (according to her preferences). Each proposal that is not rejected results in a tentative match between an X and a Y. If every offer is accepted, the process ends, and the tentative matches become definite. Otherwise, there is a second stage in which each X whose proposal was rejected in the first stage proposes to the Y she ranks second, and each Y chooses among the set of X's who proposed to her *and* the one with whom she was tentatively matched in the first stage, rejecting all but her favorite among these X's. Again, if every offer is accepted, the process ends, and the tentative matches become definite, whereas if some offer is rejected, there is another round of proposals.

Precisely, each stage has two steps, as follows.

1. Each X (*a*) whose offer was rejected at the previous stage and (*b*) for whom some Y is acceptable, proposes to her top-ranked Y out of those who have not previously rejected an offer from her.

2. Each Y rejects the proposal of any X who is unacceptable to her, and is "engaged" to the X she likes best in the set consisting of all those who proposed to her and the one to whom she was previously engaged.

The procedure stops when the proposal of no X is rejected or when every X whose offer was rejected has run out of acceptable Y's.

Consider, for example, the preferences in Figure 260.1. The progress of the procedure is shown in Figure 261.1, in which "→" stands for "proposes to". First x_1 proposes to y_2 and both x_2 and x_3 propose to y_1; y_1 rejects x_2's proposal. Then x_2 proposes to y_2, so that y_2 may choose between x_2 and x_1 (with whom she was tentatively matched at the first stage). Player y_2 chooses x_2, and rejects x_1, who then proposes to y_1. Player y_1 now chooses between x_1 and x_3 (with whom she was tentatively matched at the first stage), and rejects x_3. Finally, x_3 proposes to y_2, who rejects her offer. The final matching is thus (x_1, y_1), (x_2, y_2), x_3 (alone), and y_3 (alone).

	Stage 1		Stage 2		Stage 3		Stage 4	
x_1:	$\to y_2$			reject	$\to y_1$			
x_2:	$\to y_1$	reject	$\to y_2$					
x_3:	$\to y_1$					reject	$\to y_2$	reject

Figure 261.1 The progress of the deferred acceptance procedure with proposals by X's when the players' preferences are those given in Figure 260.1. Each row gives the proposals of one X.

For any preferences, the procedure eventually stops, because there are finitely many players. To show that the matching μ it produces is in the core we need to consider deviations by coalitions of only one or two players, by an earlier argument.

- No single player may improve upon μ because no X ever proposes to an unacceptable Y, and every Y always rejects every unacceptable X.

- Consider a coalition $\{i, j\}$ of two players, where i is an X and j is a Y. If i prefers j to $\mu(i)$, she must have proposed to j, and been rejected, before proposing to $\mu(i)$. The fact that j rejected her proposal means that j obtained a more desirable proposal. Thus j prefers $\mu(j)$ to i, so that $\{i, j\}$ cannot improve upon μ.

The analogous procedure in which proposals are made by Y's generates a matching in the core, by the same argument. For some preferences the matchings produced by the two procedures are the same, whereas for others they are different.

? EXERCISE 262.1 (Deferred acceptance procedure with proposals by Y's) Find the matching produced by the deferred acceptance procedure with proposals by Y's for the preferences given in Figure 260.1.

In particular, the core may contain more than one matching. It can be shown that the matching generated by the deferred acceptance procedure with proposals by X's yields each X her most preferred partner among all her partners in matchings in the core, and yields each Y her least preferred partner among all her partners in matchings in the core. Similarly, the matching generated by the deferred acceptance procedure with proposals by Y's yields each Y her most preferred partner among all her partners in matchings in the core, and yields each X her least preferred partner among all her partners in matchings in the core.

? EXERCISE 262.2 (Example of deferred acceptance procedure) Find the matchings produced by the deferred acceptance procedure both with proposals by X's and with proposals by Y's for the preferences given in Figure 262.1. Verify the results in the previous paragraph. (Argue that the only matchings in the core are the two generated by the procedures.)

x_1	x_2	x_3	y_1	y_2	y_3
y_1	y_1	y_1	x_1	x_1	x_1
y_2	y_2	y_3	x_2	x_3	x_2
y_3	y_3	y_2	x_3	x_2	x_3

Figure 262.1 The players' preferences in the game in Exercise 262.2.

In summary, every two-sided one-to-one matching game has a nonempty core, which contains the matching generated by each deferred acceptance procedure. The matching generated by the procedure is the best one in the core for the side making proposals, and the worst one in the core for the other side.

8.7.3 Variants

Strategic behavior So far, I have considered the deferred acceptance procedures only as algorithms that an administrator who *knows* the participants' preferences may use to find matchings in the core. Suppose the participants' preferences are *not* known. We may use the tools developed in Chapter 2 to study whether the participants' interests are served by revealing their true preferences. Consider the strategic game in which each player names a ranking of her possible partners and the outcome is the matching produced by the deferred acceptance procedure with proposals by X's, given the announced rankings. One can show that in this game each X's naming her true ranking is a dominant action, and although the equilibrium actions of Y's may *not* be their true rankings, the equilibrium of the game is in the core of the coalitional game defined by the players' *true* rankings.

? EXERCISE 263.1 (Strategic behavior under a deferred acceptance procedure) Consider the preferences in Figure 263.1. Find the matchings produced by the deferred acceptance procedures, and show that the core contains no other matchings. Consider the strategic game described in the previous paragraph that is induced by the procedure with proposals by X's. Take as given that each X's naming her true ranking is a dominant strategy. Show that the game has a Nash equilibrium in which y_1 names the ranking (x_1, x_2, x_3) and every other player names her true ranking.

x_1	x_2	x_3	y_1	y_2	y_3
y_2	y_1	y_1	x_1	x_3	x_1
y_1	y_3	y_2	x_3	x_1	x_3
y_3	y_2	y_3	x_2	x_2	x_2

Figure 263.1 The players' preferences in the game in Exercise 263.1.

Other matching problems I motivated the topic of matching by citing the problems of matching applicants with universities, workers with firms, and football players with teams. All these problems are many-to-one rather than one-to-one. Under mild assumptions about the players' preferences, the results I have presented for one-to-one matching games hold, with minor changes, for many-to-one matching games. In particular, the strong core (defined on page 255) is nonempty, and a variant of the deferred acceptance procedure generates matchings in it.

At this point you may suspect that the nonemptiness of the core in matching games is a very general result. If so, the next exercise shows that your suspicion is incorrect—at least, if "very general" includes the "roommate problem".

? EXERCISE 263.2 (Empty core in roommate problem) An even number of people have to be split into pairs; any person may be matched with any other person. (The matching problem is "one-sided".) Consider an example in which there are four people, i, j, k, and ℓ. Show that if the preferences of i, j, and k are those given in Figure 264.1 then for any preferences of ℓ the core is empty. (Notice that ℓ is the least favorite roommate of every other player.)

i	j	k
j	k	i
k	i	j
ℓ	ℓ	ℓ

Figure 264.1 The preferences of players i, j, and k in the game in Exercise 263.2.

?? EXERCISE 264.1 (Spatial preferences in roommate problem) An even number of people have to be split into pairs. Each person's characteristic is a number; no two characteristics are the same. Each person would like to have a roommate whose characteristic is as close as possible to her own, and prefers to be matched with the most remote partner to remaining single. Find the set of matchings in the core.

MATCHING DOCTORS WITH HOSPITALS

Around 1900, newly-trained doctors in the USA were first given the option of working as "interns" (now called "residents") in hospitals, where they gain experience in clinical medicine. Initially, hospitals advertised positions, for which newly-trained doctors applied. The number of positions exceeded the supply of doctors, and the competition between hospitals for interns led the date at which agreements were finalized to retreat. By 1944, student doctors were finalizing agreements two full years before their internships were to begin. Making agreements at such an early date was undesirable for hospitals, who at that point lacked extensive information about the students.

The American Association of Medical Colleges attempted to solve the problem by having its members agree not to release any information about students before the end of their third year (of a four-year program). This change prevented hospitals from making earlier appointments, but in doing so brought to the fore the problem of coordinating offers and acceptances. Hospitals wanted their first-choice students to accept quickly, but students wanted to delay as much as possible, hoping to receive better offers. In 1945, hospitals agreed to give students 10 days to consider offers. But there was pressure to reduce this period. In 1949 a 12-hour period was rejected by the American Hospital Association as too long; it was agreed that all offers be made at 12:01AM on November 15, and hospitals could insist on a response within any period. Forcing students to make decisions without having a chance to collect offers from hospitals whose first-choice students had rejected them obviously led to inefficient matches.

These difficulties with efficiently matching doctors with hospitals led to the design of a centralized matching procedure that combines hospitals' rankings of students and students' rankings of hospitals to produce an assignment of students to hospitals. It can be shown that this procedure, designed ten years before Gale and Shapley's work on the deferred acceptance procedure, generates a matching in the core for any stated preferences! It differs from the natural generalization of Gale and Shapley's deferred acceptance procedure to a many-to-one matching problem, but generates precisely the same matching, namely the one in the core that is best for the hospitals. (Gale and Shapley, and the designers of the student–hospital matching procedure were not aware of each other's work until the mid-1970s, when a physician heard Gale speak on his work.)

In the early years of operation of the procedure, over 95% of students and hospitals participated. In the mid-1970s the participation rate fell to around 85%. Many nonparticipants were married couples both members of which wished to obtain positions in the same city. The matching procedure contained a mechanism for dealing with married couples, but, unlike the mechanism for single students, it could lead to a matching upon which some couple could improve. The difficulty is serious: when couples exist who restrict themselves to accept positions in the same city, for some preferences the core of the resulting game is empty—no matching is stable.

Further problems arose. In the 1990s, associations of medical students began to argue that changes were needed because the procedure was favorable to hospitals, and possibilities for strategic behavior on the part of students existed. The game theorist Alvin E. Roth was retained by the "National Resident Matching Program" to design a new procedure to generate stable matchings that are as favorable as possible to applicants. The new procedure was first used in 1998; it matches around 20,000 new doctors with hospitals each year.

8.8 Discussion: other solution concepts

In replacing the requirement of a Nash equilibrium that no individual player may profitably deviate with the requirement that no *group* of players may profitably deviate, the notion of the core makes an assumption that is unnecessary when interpreting a Nash equilibrium. A single player who deviates from an action profile in a strategic game can be *sure* of her deviant action, because she unilaterally chooses it. But a member of a group of players that chooses a deviant action must assume that no subgroup of her comrades will deviate further, or, at least, she will remain better off if they do.

Consider, for example, the three-player majority game (Example 237.2 and Exercise 241.1). The action $(\frac{1}{2}, \frac{1}{2}, 0)$ of the grand coalition in this game is not in the core because, for example, the coalition consisting of players 1 and 3 can take an action that gives player 1 an amount x with $\frac{1}{2} < x < 1$ and player 3 the amount $1 - x$, which leads to the payoff profile $(x, 0, 1 - x)$. But this profile itself is not stable—the coalition consisting of players 2 and 3, for example, has an action that generates the payoff profile $(0, y, 1 - y)$, where $0 < y < x$, in which both of them are better off than they are in $(x, 0, 1 - x)$. The fact that player 3 will be tempted by an offer of player 2 to deviate from $(x, 0, 1-x)$ may dampen player 1's enthusiasm for joining player 3 in the deviation from $(\frac{1}{2}, \frac{1}{2}, 0)$. For similar reasons, player 2 may be reluctant to join in a deviation from this action.

Several solution concepts that take into account these considerations have been suggested. None has so far had anything like the success of the core in illuminating social and economic phenomena, however.

Notes

The notion of a coalitional game is due to von Neumann and Morgenstern (1944). Shapley and Shubik (1953), Luce and Raiffa (1957, 234–235), and Aumann and Peleg (1960) generalized von Numann and Morgenstern's notion. The notion of the core was introduced in the early 1950s by Gillies as a tool to study another solution concept (his work is published in Gillies 1959); Shapley and Shubik developed it as a solution concept.

Edgeworth (1881, 35–39) pointed out a connection between the competitive equilibria of a market model and the set of outcomes we now call the core. von Neumann and Morgensten (1944, 583–584) first suggested modeling markets as coalitional games; Shubik (1959a) recognized the game-theoretic content of Edgeworth's arguments and, together with Shapley (1959), developed the analysis. Section 8.3 is based on Shapley and Shubik (1967). The core of the market studied in Section 8.4 was first studied by Shapley and Shubik (1971/72). My discussion owes a debt to Moulin (1995, Section 2.3).

Voting behavior in committees was first studied formally by Black (1958) (written in the mid-1940s), Black and Newing (1951), and Arrow (1951). Black used the core as the solution (before it had been defined generally) and established the median voter theorem (Exercise 257.1). He also noticed that in policy spaces of dimension greater than 1 a Condorcet winner is not likely to exist, a result extended by Plott (1967) and refined by Banks (1995) and others, who find conditions relating the number of voters, the dimension of the policy space, and the value of q for which the core of the q-rule game is generally empty; see Austen-Smith and Banks (1999, Secton 6.1) for details.

The model and result on the nonemptiness of the core in Section 8.5 are due to Shapley and Scarf (1974), who credit David Gale with the top trading cycle procedure. The result that the strong core contains a single action is due to Roth and Postlewaite (1977). The model is discussed in detail by Moulin (1995, Section 3.2).

The model and main results in Section 8.7 are due to Gale and Shapley (1962). The result about the strategic properties of the deferred acceptance procedures at the end of the section is a combination of results due to Dubins and Freedman (1981) and Roth (1982), and to Roth (1984a). Exercise 263.1 is based on an example in Moulin (1995, 113 and 116). Exercise 263.2 is taken from Gale and Shapley (1962, Example 3). For a comprehensive presentation of results on two-sided matching, see Roth and Sotomayor (1990). The box on page 264 is based on Roth (1984b), Roth and Sotomayor (1990, Section 5.4), and Roth and Peranson (1999).

9 Bayesian Games

Prerequisite: Chapter 2 and Section 4.1.3; Section 9.8 requires Chapter 4.

9.1 Introduction

An assumption underlying the notion of Nash equilibrium is that each player holds the correct belief about the other players' actions. To do so, a player must know the game she is playing; in particular, she must know the other players' preferences. In many situations the participants are not perfectly informed about their opponents' characteristics: bargainers may not know each others' valuations of the object of negotiation, firms may not know each others' cost functions, combatants may not know each others' strengths, and jurors may not know their colleagues' interpretations of the evidence in a trial. In some situations, a participant may be well informed about her opponents' characteristics, but may not know how well these opponents are informed about her own characteristics. In this chapter I describe the model of a "Bayesian game", which generalizes the notion of a strategic game to allows us to analyze any situation in which each player is imperfectly informed about some aspect of her environment relevant to her choice of an action.

9.2 Motivational examples

I start with two examples that illustrate the main ideas in the model of a Bayesian game. I define the notion of Nash equilibrium separately for each game. In the next section I define the general model of a Bayesian game and the notion of Nash equilibrium for such a game.

◆ Example 271.1 (Variant of *BoS* with imperfect information) Consider a variant of the situation modeled by *BoS* (Figure 16.1) in which player 1 is unsure whether player 2 prefers to go out with her or prefers to avoid her, whereas player 2, as before, knows player 1's preferences. Specifically, suppose player 1 thinks that with probability $\frac{1}{2}$ player 2 wants to got out with her, and with probability $\frac{1}{2}$ player 2 wants to avoid her. (Presumably this assessment comes from player 1's experience: half of the time she is involved in this situation she faces a player who wants to go out with her, and half of the time she faces a player who wants to avoid her.) That is, player 1 thinks that with probability $\frac{1}{2}$ she is playing the game on the left of Figure 272.1 and with probability $\frac{1}{2}$ she is playing the game on the right. Because probabilities are involved, an analysis of the situation requires us to know the players' preferences over lotteries, even if we are interested only in pure strategy equilibria; thus the numbers in the tables are Bernoulli payoffs.

prob. $\frac{1}{2}$ — 1 — prob. $\frac{1}{2}$

2

	B	*S*
B	2, 1	0, 0
S	0, 0	1, 2

2 wishes to meet 1

2

	B	*S*
B	2, 0	0, 2
S	0, 1	1, 0

2 wishes to avoid 1

Figure 272.1 A variant of *BoS* in which player 1 is unsure whether player 2 wants to meet her or to avoid her. The frame labeled 2 enclosing each table indicates that player 2 knows the relevant table. The frame labeled 1 enclosing both tables indicates that player 1 does not know the relevant table; the probabilities she assigns to the two tables are printed on the frame.

We can think of there being two *states*, one in which the players' Bernoulli payoffs are given in the left table and one in which these payoffs are given in the right table. Player 2 knows the state—she knows whether she wishes to meet or avoid player 2—whereas player 1 does not; player 1 assigns probability $\frac{1}{2}$ to each state.

The notion of Nash equilibrium for a strategic game models a steady state in which each player's beliefs about the other players' actions are correct, and each player acts optimally, given her beliefs. We wish to generalize this notion to the current situation.

From player 1's point of view, player 2 has two possible *types*, one whose preferences are given in the left table of Figure 272.1, and one whose preferences are given in the right table. Player 1 does not know player 2's type, so to choose an action rationally she needs to form a belief about the action of each type. Given these beliefs and her belief about the likelihood of each type, she can calculate her expected payoff to each of her actions. For example, if she thinks that the type who wishes to meet her will choose *B* and the type who wishes to avoid her will choose *S*, then she thinks that *B* will yield her a payoff of 2 with probability $\frac{1}{2}$ and a payoff of 0 with probability $\frac{1}{2}$, so that her expected payoff is $\frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 0 = 1$, and *S* will yield her an expected payoff of $\frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 1 = \frac{1}{2}$. Similar calculations for the other combinations of actions for the two types of player 2 yield the expected payoffs in Figure 273.1. Each column of the table is a pair of actions for the two types of player 2, the first member of each pair being the action of the type who wishes to meet player 1 and the second member being the action of the type who wishes to avoid player 1.

	(B, B)	(B, S)	(S, B)	(S, S)
B	2	1	1	0
S	0	$\frac{1}{2}$	$\frac{1}{2}$	1

Figure 273.1 The expected payoffs of player 1 for the four possible pairs of actions of the two types of player 2 in Example 271.1.

For this situation we define a pure strategy *Nash equilibrium* to be a triple of actions, one for player 1 and one for each type of player 2, with the property that

- the action of player 1 is optimal, given the actions of the two types of player 2 (and player 1's belief about the state)
- the action of each type of player 2 is optimal, given the action of player 1.

That is, we treat the two types of player 2 as separate players, and analyze the situation as a three-player strategic game in which player 1's payoffs as a function of the actions of the two other players (i.e. the two types of player 2) are given in Figure 273.1, and the payoff of each type of player 2 is independent of the actions of the other type and depends on the action of player 1 as given in the tables in Figure 272.1 (the left table for the type who wishes to meet player 1, and the right table for the type who wishes to avoid player 1). In a Nash equilibrium, player 1's action is a best response in Figure 273.1 to the pair of actions of the two types of player 2, the action of the type of player 2 who wishes to meet player 1 is a best response in the left table of Figure 272.1 to the action of player 1, and the action of the type of player 2 who wishes to avoid player 1 is a best response in the right table of Figure 272.1 to the action of player 1.

Why should player 2, who knows whether she wants to meet or avoid player 1, have to plan what to do in both cases? She does not have to do so! But we, as analysts, need to consider what she does in both cases, because player 1, who does not know player 2's type, needs to think about the action each type would take; we would like to impose the condition that player 1's beliefs are correct, in the sense that for each type of player 2 they specify a best response to player 1's equilibrium action.

I claim that $(B, (B, S))$, where the first component is the action of player 1 and the other component is the pair of actions of the two types of player 2, is a Nash equilibrium. Given that the actions of the two types of player 2 are (B, S), player 1's action B is optimal, from Figure 273.1; given that player 1 chooses B, B is optimal for the type who wishes to meet player 2 and S is optimal for the type who wishes to avoid player 2, from Figure 272.1. Suppose that in fact player 2 wishes to meet player 1. Then we interpret the equilibrium as follows. Both player 1 and player 2 chooses B; player 1, who does not know if player 2 wants to meet her or avoid her, believes that if player 2 wishes to meet her she will choose B, and if she wishes to avoid her she will choose S.

? EXERCISE 274.1 (Equilibria of a variant of *BoS* with imperfect information) Show that there is no pure strategy Nash equilibrium of this game in which player 1 chooses S. If you have studied mixed strategy Nash equilibrium (Chapter 4), find the mixed strategy Nash equilibria of the game. (First check whether there is an equilibrium in which both types of player 2 use pure strategies, then look for equilibria in which one or both of these types randomize.)

We can interpret the actions of the two types of player 2 to reflect player 2's intentions in the hypothetical situation *before* she knows the state. We can tell the following story. Initially player 2 does not know the state; she is informed of the state by a *signal* that depends on the state. Before receiving this signal, she plans an action for each possible signal. After receiving the signal she carries out her planned action for that signal. We can tell a similar story for player 1. To be consistent with her not knowing the state when she takes an action, her signal must be uninformative: it must be the same in each state. Given her signal, she is unsure of the state; when choosing an action she takes into account her belief about the likelihood of each state, given her signal. The framework of states, beliefs, and signals is unnecessarily baroque in this simple example, but comes into its own in the analysis of more complex situations.

◆ EXAMPLE 274.2 (Variant of *BoS* with imperfect information) Consider another variant of the situation modeled by *BoS*, in which neither player knows whether the other wants to go out with her. Specifically, suppose that player 1 thinks that with probability $\frac{1}{2}$ player 2 wants to go out with her, and with probability $\frac{1}{2}$ player 2 wants to avoid her, and player 2 thinks that with probability $\frac{2}{3}$ player 1 wants to go out with her and with probability $\frac{1}{3}$ player 1 wants to avoid her. As before, assume that each player knows her own preferences.

We can model this situation by introducing four states, one for each of the possible configurations of preferences. I refer to these states as yy (each player wants to go out with the other), yn (player 1 wants to go out with player 2, but player 2 wants to avoid player 1), ny, and nn.

The fact that player 1 does not know player 2's preferences means that she cannot distinguish between states yy and yn, or between states ny and nn. Similarly, player 2 cannot distinguish between states yy and ny, and between states yn and nn. We can model the players' information by assuming that each player receives a *signal* before choosing an action. Player 1 receives the same signal, say y_1, in states yy and yn, and a different signal, say n_1, in states ny and nn; player 2 receives the same signal, say y_2, in states yy and ny, and a different signal, say n_2, in states yn and nn. After player 1 receives the signal y_1, she is referred to as *type* y_1 of player 1 (who wishes to go out with player 2); after she receives the signal n_1 she is referred to as *type* n_1 of player 1 (who wishes to avoid player 2). Similarly, player 2 has two *types*, y_2 and n_2.

Type y_1 of player 1 believes that the probability of each of the states yy and yn is $\frac{1}{2}$; type n_1 of player 1 believes that the probability of each of the states ny and nn is $\frac{1}{2}$. Similarly, type y_2 of player 2 believes that the probability of state yy is $\frac{2}{3}$ and that of state ny is $\frac{1}{3}$; type n_2 of player 2 believes that the probability of state yn is $\frac{2}{3}$ and that of state nn is $\frac{1}{3}$. This model of the situation is illustrated in Figure 275.1.

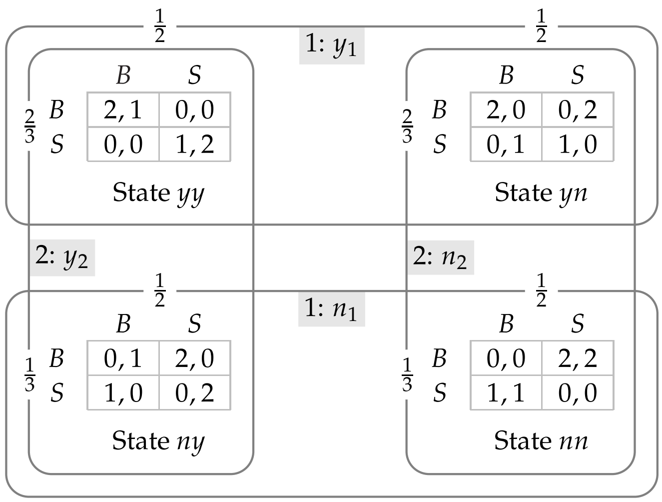


Figure 275.1 A variant of *BoS* in which each player is unsure of the other player's preferences. The frame labeled i: x encloses the states that generate the signal x for player i; the numbers printed over this frame next to each table are the probabilities that type x of player i assigns to each state that she regards to be possible.

As in the previous example, to study the equilibria of this model we consider the players' plans of action before they receive their signals. That is, each player plans an action for each of the two possible signals she may receive. We may think of there being four players: the two types of player 1 and the two types of player 2. A *Nash equilibrium* consists of four actions, one for each of these players, such that the action of each type of each original player is optimal, given her belief about the state after observing her signal, and given the actions of each type of the other original player.

Consider the payoffs of type y_1 of player 1. She believes that with probability $\frac{1}{2}$ she faces type y_2 of player 2, and with probability $\frac{1}{2}$ she faces type n_2. Suppose that type y_2 of player 2 chooses B and type n_2 chooses S. Then if type y_1 of player 1 chooses B, her expected payoff is $\frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 0 = 1$, and if she chooses S, her expected payoff is $\frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 1 = \frac{1}{2}$. Her expected payoffs for all four pairs of actions of the two types of player 2 are given in Figure 276.1.

? EXERCISE 275.1 (Expected payoffs in a variant of *BoS* with imperfect information) Construct tables like the one in Figure 276.1 for type n_1 of player 1, and for types y_2 and n_2 of player 2.

I claim that $((B, B), (B, S))$ and $((S, B), (S, S))$ are Nash equilibria of the game, where in each case the first component gives the actions of the two types of player 1

	(B,B)	(B,S)	(S,B)	(S,S)
B	2	1	1	0
S	0	$\frac{1}{2}$	$\frac{1}{2}$	1

Figure 276.1 The expected payoffs of type y_1 of player 1 in Example 274.2. Each row corresponds to a pair of actions for the two types of player 2; the action of type y_2 is listed first, that of type n_2 second.

and the second component gives the actions of the two types of player 2. Using Figure 276.1 you may verify that B is a best response of type y_1 of player 1 to the pair (B,S) of actions of player 2, and S is a best response to the pair of actions (S,S). You may use your answer to Exercise 275.1 to verify that in each of the claimed Nash equilibria the action of type n_1 of player 1 and the action of each type of player 2 is a best response to the other players' actions.

In each of these examples a Nash equilibrium is a list of actions, one for each type of each player, such that the action of each type of each player is a best response to the actions of all the types of the other player, given the player's beliefs about the state after she observes her signal. The actions planned by the various types of player i are not relevant to the decision problem of any type of player i, but there is no harm in taking them, as well as the actions of the types of the *other* player, as given when player i is choosing an action. Thus we may define a Nash equilibrium in each example to be a Nash equilibrium of the strategic game in which the set of players is the set of all types of all players in the original situation.

In the next section I define the general notion of a Bayesian game, and the notion of Nash equilibrium in such a game. These definitions require significant theoretical development. If you find the theory in the next section heavy-going, you may be able to skim the section and then study the subsequent illustrations, relying on the intuition developed in the examples in this section, and returning to the theory only as necessary for clarification.

9.3 General definitions

9.3.1 Bayesian games

A strategic game with imperfect information is called a "Bayesian game". (The reason for this nomenclature will become apparent.) As in a strategic game, the decision-makers are called *players*, and each player is endowed with a set of *actions*.

A key component in the specification of the imperfect information is the set of *states*. Each state is a complete description of one collection of the players' relevant characteristics, including both their preferences and their information. For every collection of characteristics that some player believes to be possible, there must be a state. For instance, suppose in Example 271.1 that player 2 wishes to meet player 1. In this case, the reason for including in the model the state in which player 2 wishes to avoid player 1 is that player 1 believes such a preference to be possible.

At the start of the game a state is realized. The players do not observe this state. Rather, each player receives a *signal* that may give her some information about the state. Denote the signal player i receives in state ω by $\tau_i(\omega)$. The function τ_i is called player i's *signal function*. (Note that the signal is a *deterministic* function of the state: for each state a definite signal is received.) The states that generate any given signal t_i are said to be *consistent* with t_i. The sizes of the sets of states consistent with each of player i's signals reflect the quality of player i's information. If, for example, $\tau_i(\omega)$ is different for each value of ω, then player i knows, given her signal, the state that has occurred; after receiving her signal, she is perfectly informed about all the players' relevant characteristics. At the other extreme, if $\tau_i(\omega)$ is the same for all states, then player i's signal conveys no information about the state. If $\tau_i(\omega)$ is constant over some subsets of the set of states, but is not the same for all states, then player i's signal conveys partial information. For example, if there are three states, ω_1, ω_2, and ω_3, and $\tau_i(\omega_1) \neq \tau_i(\omega_2) = \tau_i(\omega_3)$, then when the state is ω_1 player i knows that it is ω_1, whereas when it is either ω_2 or ω_3 she knows only that it is one of these two states.

We refer to player i in the event that she receives the signal t_i as *type* t_i of player i. Each type of each player holds a *belief* about the likelihood of the states consistent with her signal. If, for example, $t_i = \tau_i(\omega_1) = \tau_i(\omega_2)$, then type t_i of player i assigns probabilities to ω_1 and ω_2. (A player who receives a signal consistent with only one state naturally assigns probability 1 to that state.)

Each player may care about the actions chosen by the other players, as in a strategic game with perfect information, and also about the state. The players may be uncertain about the state, so we need to specify their preferences regarding probability distributions over pairs (a, ω) consisting of an action profile a and a state ω. I assume that each player's preferences over such probability distributions are represented by the expected value of a *Bernoulli payoff function*. Thus I specify each player i's preferences by giving a Bernoulli payoff function u_i over pairs (a, ω). (Note that in both Example 271.1 and Example 274.2, both players care only about the other player's action, not independently about the state.)

In summary, a Bayesian game is defined as follows.

► DEFINITION 277.1 A **Bayesian game** consists of

- a set of **players**
- a set of **states**

and for each player

- a set of **actions**
- a set of **signals** that she may receive and a **signal function** that associates a signal with each state

- for each signal that she may receive, a **belief** about the states consistent with the signal (a probability distribution over the set of states with which the signal is associated)
- a **Bernoulli payoff function** over pairs (a, ω), where a is an action profile and ω is a state, the expected value of which represents the player's preferences among lotteries over the set of such pairs.

The eponymous Thomas Bayes (1702–61) first showed how probabilities should be changed in the light of new information. His formula (discussed in Section 17.7.5) is needed when working with a variant of Definition 277.1 in which each player is endowed with a "prior" belief about the states, from which the belief of each of her types is derived. For the purposes of this chapter, the belief of each type of each player is more conveniently taken as a primitive, rather than being derived from a prior belief.

The game in Example 271.1 fits into this general definition as follows.

Players The pair of people.

States The set of states is $\{meet, avoid\}$.

Actions The set of actions of each player is $\{B, S\}$.

Signals Player 1 may receive a single signal, say z; her signal function τ_1 satisfies $\tau_1(meet) = \tau_1(avoid) = z$. Player 2 receives one of two signals, say m and v; her signal function τ_2 satisfies $\tau_2(meet) = m$ and $\tau_2(avoid) = v$.

Beliefs Player 1 assigns probability $\frac{1}{2}$ to each state after receiving the signal z. Player 2 assigns probability 1 to the state *meet* after receiving the signal m, and probability 1 to the state *avoid* after receiving the signal v.

Payoffs The payoffs $u_i(a, meet)$ of each player i for all possible action pairs are given in the left panel of Figure 272.1, and the payoffs $u_i(a, avoid)$ are given in the right panel.

Similarly, the game in Example 274.2 fits into the definition as follows.

Players The pair of people.

States The set of states is $\{yy, yn, ny, nn\}$.

Actions The set of actions of each player is $\{B, S\}$.

Signals Player 1 receives one of two signals, y_1 and n_1; her signal function τ_1 satisfies $\tau_1(yy) = \tau_1(yn) = y_1$ and $\tau_1(ny) = \tau_1(nn) = n_1$. Player 2 receives one of two signals, y_2 and n_2; her signal function τ_2 satisfies $\tau_2(yy) = \tau_2(ny) = y_2$ and $\tau_2(yn) = \tau_2(nn) = n_2$.

Beliefs Player 1 assigns probability $\frac{1}{2}$ to each of the states yy and yn after receiving the signal y_1 and probability $\frac{1}{2}$ to each of the states ny and nn after receiving the signal n_1. Player 2 assigns probability $\frac{2}{3}$ to the state yy and probability $\frac{1}{3}$ to the state ny after receiving the signal y_2, and probability $\frac{2}{3}$ to the state yn and probability $\frac{1}{3}$ to the state nn after receiving the signal n_2.

Payoffs The payoffs $u_i(a, \omega)$ of each player i for all possible action pairs and states are given in Figure 275.1.

9.3.2 Nash equilibrium

In a strategic game, each player chooses an action. In a Bayesian game, each player chooses a collection of actions, one for each signal she may receive. That is, in a Bayesian game each type of each player chooses an action. In a Nash equilibrium of such a game, the action chosen by each type of each player is optimal, given the actions chosen by every type of every other player. (In a steady state, each player's experience teaches her these actions.) Any given type of player i is not affected by the actions chosen by the other types of player i, so there is no harm in thinking that player i takes as given these actions, as well as those of the other players. Thus we may define a Nash equilibrium of a Bayesian game to be a Nash equilibrium of a strategic game in which each player is one type of one of the players in the Bayesian game. What is each player's payoff function in this strategic game?

Consider type t_i of player i. For each state ω she knows every other player's type (i.e. she knows the signal received by every other player). This information, together with her belief about the states, allows her to calculate her expected payoff for each of her actions and each collection of actions for the various types of the other players. For instance, in Example 271.1, player 1's belief is that the probability of each state is $\frac{1}{2}$, and she knows that player 2 is type m in the state *meet* and type v in the state *avoid*. Thus if type m of player 2 chooses B and type v of player 2 chooses S, player 1 thinks that if she chooses B then her expected payoff is

$$\tfrac{1}{2}u_1(B, B, \mathit{meet}) + \tfrac{1}{2}u_1(B, S, \mathit{avoid}),$$

where u_1 is her payoff function in the Bayesian game. (In general her payoff may depend on the state, though in this example it does not.) The top box of the second column in Figure 273.1 gives this payoff; the other boxes give player 1's payoffs for her other action and the other combinations of actions for the two types of player 2.

In a general game, denote the probability assigned by the belief of type t_i of player i to state ω by $\Pr(\omega|t_i)$. Denote the action taken by each type t_j of each player j by $a(j, t_j)$. Player j's signal in state ω is $\tau_j(\omega)$, so her action in state ω is $a(j, \tau_j(\omega))$. For each state ω, denote by $\hat{a}(\omega)$ the action profile in which each player j chooses the action $a(j, \tau_j(\omega))$. Then the expected payoff of type t_i of player i when she chooses the action a_i is

$$\sum_{\omega \in \Omega} \Pr(\omega \mid t_i) u_i((a_i, \hat{a}_{-i}(\omega)), \omega), \tag{279.1}$$

where Ω is the set of states and $(a_i, \hat{a}_{-i}(\omega))$ is the action profile in which player i chooses the action a_i and every other player j chooses $\hat{a}_j(\omega)$. (Note that this expected payoff does not depend on the actions of any other types of player i, but only on the actions of the various types of the *other* players.)

We may now define precisely a Nash equilibrium of a Bayesian game.

▶ DEFINITION 280.1 A **Nash equilibrium of a Bayesian game** is a Nash equilibrium of the strategic game (with vNM preferences) defined as follows.

Players The set of all pairs (i, t_i) where i is a player in the Bayesian game and t_i is one of the signals that i may receive.

Actions The set of actions of each player (i, t_i) is the set of actions of player i in the Bayesian game.

Preferences The Bernoulli payoff function of each player (i, t_i) is given by (279.1).

? EXERCISE 280.2 (A fight with imperfect information about strengths) Two people are involved in a dispute. Person 1 does not know whether person 2 is strong or weak; she assigns probability α to person 2's being strong. Person 2 is fully informed. Each person can either fight or yield. Each person's preferences are represented by the expected value of a Bernoulli payoff function that assigns the payoff of 0 if she yields (regardless of the other person's action) and a payoff of 1 if she fights and her opponent yields; if both people fight then their payoffs are $(-1, 1)$ if person 2 is strong and $(1, -1)$ if person 2 is weak. Formulate this situation as a Bayesian game and find its Nash equilibria if $\alpha < \frac{1}{2}$ and if $\alpha > \frac{1}{2}$.

? EXERCISE 280.3 (An exchange game) Each of two individuals receives a ticket on which there is an integer from 1 to m indicating the size of a prize she may receive. The individuals' tickets are assigned randomly and independently; the probability of an individual's receiving each possible number is positive. Each individual is given the option to exchange her prize for the other individual's prize; the individuals are given this option simultaneously. If both individuals wish to exchange then the prizes are exchanged; otherwise each individual receives her own prize. Each individual's objective is to maximize her expected monetary payoff. Model this situation as a Bayesian game and show that in any Nash equilibrium the highest prize that either individual is willing to exchange is the smallest possible prize.

? EXERCISE 280.4 (Adverse selection) Firm A (the "acquirer") is considering taking over firm T (the "target"). It does not know firm T's value; it believes that this value, when firm T is controlled by its own management, is at least \$0 and at most \$100, and assigns equal probability to each of the 101 dollar values in this range. Firm T will be worth 50% more under firm A's management than it is under its own management. Suppose that firm A bids y to take over firm T, and firm T is worth x (under its own management). Then if T accepts A's offer, A's payoff is $\frac{3}{2}x - y$ and T's payoff is y; if T rejects A's offer, A's payoff is 0 and T's payoff is

x. Model this situation as a Bayesian game in which firm A chooses how much to offer and firm T decides the lowest offer to accept. Find the Nash equilibrium (equilibria?) of this game. Explain why the logic behind the equilibrium is called *adverse selection*.

9.4 Two examples concerning information

The notion of a Bayesian game may be used to study how information patterns affect the outcome of strategic interaction. Here are two examples.

9.4.1 More information may hurt

A decision-maker in a single-person decision problem cannot be worse off if she has more information: if she wishes, she can ignore the information. In a game the same is not true: if a player has more information and the other players know that she has more information then she may be worse off.

Consider, for example, the two-player Bayesian game in Figure 281.1, where $0 < \epsilon < \frac{1}{2}$. In this game there are two states, and neither player knows the state. Player 2's unique best response to every strategy of player 1 is L (which yields the expected payoff $2 - 2(1-\epsilon)p$, whereas M and R both yield $\frac{3}{2} - \frac{3}{2}(1-\epsilon)p$, where p is the probability player 1 assigns to T), and player 1's unique best response to L is B. Thus (B, L) is the unique Nash equilibrium of the game, yielding each player a payoff of 2.

Player 1: $\frac{1}{2}$ (State ω_1), $\frac{1}{2}$ (State ω_2); Player 2: $\frac{1}{2}$ (State ω_1), $\frac{1}{2}$ (State ω_2)

State ω_1

	L	M	R
T	$1, 2\epsilon$	$1, 0$	$1, 3\epsilon$
B	$2, 2$	$0, 0$	$0, 3$

State ω_2

	L	M	R
T	$1, 2\epsilon$	$1, 3\epsilon$	$1, 0$
B	$2, 2$	$0, 3$	$0, 0$

Figure 281.1 The first Bayesian game considered in Section 9.4.1.

Now consider the variant of this game in which player 2 is informed of the state: player 2's signal function τ_2 satisfies $\tau_2(\omega_1) \neq \tau_2(\omega_2)$. In this game $(T, (R, M))$ is the unique Nash equilibrium. (Each type of player 2 has a strictly dominant action, to which T is player 1's unique best response.)

Player 2's payoff in the unique Nash equilibrium of the original game is 2, whereas her payoff in the unique Nash equilibrium of the game in which she knows the state is 3ϵ in each state. Thus she is worse off when she knows the state than when she does not.

Player 2's action R is good only in state ω_1 whereas her action M is good only in state ω_2. When she does not know the state she optimally chooses L, which is better than the average of R and M whatever player 1 does. Her choice induces player 1 to choose B. When player 2 is fully informed she optimally tailors her action to the state, which induces player 1 to choose T. There is no steady state in which she ignores her information and chooses L because this action leads player 1 to choose B, making R better for player 2 in state ω_1 and M better in state ω_2.

9.4.2 Infection

The notion of a Bayesian game may be used to model not only situations in which players are uncertain about each others' preferences, but also situations in which they are uncertain about each others' *knowledge*. Consider, for example, the Bayesian game in Figure 282.1.

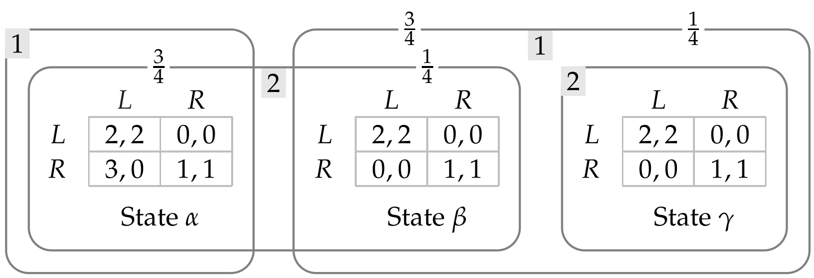


Figure 282.1 The first Bayesian game in Section 9.4.2. In the unique Nash equilibrium of this game, each type of each player chooses R.

Notice that player 2's preferences are the same in all three states, and player 1's preferences are the same in states β and γ. In particular, in state γ, each player knows the other player's preferences, and player 2 knows that player 1 knows her preferences. The shortcoming in the players' information in state γ is that player 1 does not know that player 2 knows her preferences: player 1 knows only that the state is either β or γ, and in state β player 2 does not know whether the state is α or β, and hence does not know player 1's preferences (because player 1's preferences in these two states differ).

This imperfection in player 1's knowledge of player 2's information significantly affects the equilibria of the game. If information were perfect in state γ, then both (L, L) and (R, R) would be Nash equilibria. However, the whole game has a *unique* Nash equilibrium, in which the outcome in state γ is (R, R), as you are asked to show in the next exercise. The argument shows that the incentives faced by player 1 in state α "infect" the remainder of the game.

? EXERCISE 282.1 (Infection) Show that the Bayesian game in Figure 282.1 has a unique Nash equilibrium, in which each player chooses R regardless of her signal. (Start by considering player 1's action in state α. Next consider player 2's action when she gets the signal that the state is α or β. Then consider player 1's action when she gets the signal that the state is β or γ. Finally consider player 2's action in state γ.)

Now extend the game as in Figure 283.1. Consider state δ. In this state, player 2 knows player 1's preferences (because she knows that the state is either γ or δ, and in both states player 1's preferences are the same). What player 2 does not know is whether player 1 knows that player 2 knows player 1's preferences. The reason is that player 2 does not know whether the state is γ or δ; and in state γ player 1 does not know that player 2 knows her preferences, because she does not know whether the state is β or γ, and in state β player 2 (who does not know whether the state is α or β) does not know her preferences. Thus the level of the shortcoming in the players' information is higher than it is in the game in Figure 282.1. Nevertheless, the incentives faced by player 1 in state α again "infect" the remainder of the game, and in the only Nash equilibrium every type of each player chooses R.

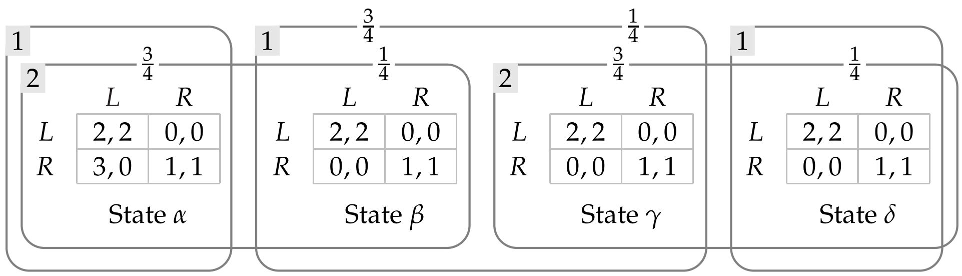


Figure 283.1 The second Bayesian game in Section 9.4.2.

The game may be further extended. As it is extended, the level of the imperfection in the players' information in the last state increases. When the number of states is large, the players' information in the last state is only very slightly imperfect. Nevertheless, the incentives of player 1 in state α still cause the game to have a unique Nash equilibrium, in which every type of each player chooses R.

In each of these examples, the equilibrium induces an outcome in every state that is worse for both players than another outcome (namely (L, L)); in all states but the first, the alternative outcome is a Nash equilibrium in the game with perfect information. For some other specifications of the payoffs in state α and the players' beliefs, the game has a unique equilibrium in which the "good" outcome (L, L) occurs in every state; the point is only that one of the two Nash equilibria are selected, not that the "bad" equilibrium is necessarily selected. (Modify the payoffs of player 1 in state α so that L strictly dominates R, and change the beliefs to assign probability $\frac{1}{2}$ to each state compatible with each signal.)

9.5 Illustration: Cournot's duopoly game with imperfect information

9.5.1 Imperfect information about cost

Two firms compete in selling a good; one firm does not know the other firm's cost function. How does the imperfect information affect the firms' behavior?

Assume that both firms can produce the good at constant unit cost. Assume also that they both know that firm 1's unit cost is c, but only firm 2 knows its own unit cost; firm 1 believes that firm 2's cost is c_L with probability θ and c_H with probability $1 - \theta$, where $0 < \theta < 1$ and $c_L < c_H$.

We may model this situation as a Bayesian game that is a variant of Cournot's game (Section 3.1).

Players Firm 1 and firm 2.

States $\{L, H\}$.

Actions Each firm's set of actions is the set of its possible outputs (nonnegative numbers).

Signals Firm 1's signal function τ_1 satisfies $\tau_1(H) = \tau_2(L)$ (its signal is the same in both states); firm 2's signal function τ_2 satisfies $\tau_2(H) \neq \tau_2(L)$ (its signal is perfectly informative of the state).

Beliefs The single type of firm 1 assigns probability θ to state L and probability $1 - \theta$ to state H. Each type of firm 2 assigns probability 1 to the single state consistent with its signal.

Payoff functions The firms' Bernoulli payoffs are their profits; if the actions chosen are (q_1, q_2) and the state is I (either L or H) then firm 1's profit is $q_1(P(q_1 + q_2) - c)$ and firm 2's profit is $q_2(P(q_1 + q_2) - c_I)$, where $P(q_1 + q_2)$ is the market price when the firms' outputs are q_1 and q_2.

The information structure in this game is similar to that in Example 271.1; it is illustrated in Figure 284.1.

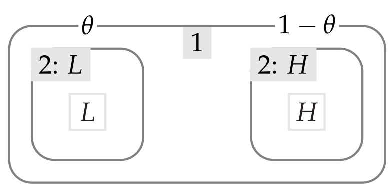


Figure 284.1 The information structure for the model in the variant of Cournot's model in Section 9.5.1, in which firm 1 does not know firm 2's cost. The frame labeled 2: x, for $x = L$ and $x = H$, encloses the state that generates the signal x for firm 2.

A Nash equilibrium of this game is a triple (q_1^*, q_L^*, q_H^*), where q_1^* is the output of firm 1, q_L^* is the output of type L of firm 2 (i.e. firm 2 when it receives the signal $\tau_2(L)$), and q_H^* is the output of type H of firm 2 (i.e. firm 2 when it receives the signal $\tau_2(H)$), such that

- q_1^* maximizes firm 1's profit given the output q_L^* of type L of firm 2 and the output q_H^* of type H of firm 2
- q_L^* maximizes the profit of type L of firm 2 given the output q_1^* of firm 1

- q_H^* maximizes the profit of type H of firm 2 given the output q_1^* of firm 1.

To find an equilibrium, we first find the firms' best response functions. Given firm 1's posterior beliefs, its best response $b_1(q_L, q_H)$ to (q_L, q_H) solves

$$\max_{q_1} \left[\theta(P(q_1 + q_L) - c)q_1 + (1-\theta)(P(q_1 + q_H) - c)q_1\right].$$

Firm 2's best response $b_L(q_1)$ to q_1 when its cost is c_L solves

$$\max_{q_L} \left[(P(q_1 + q_L) - c_L)q_L\right],$$

and its best response $b_H(q_1)$ to q_1 when its cost is c_H solves

$$\max_{q_H} \left[(P(q_1 + q_H) - c_H)q_H\right].$$

A Nash equilibrium is a triple (q_1^*, q_L^*, q_H^*) such that

$$q_1^* = b_1(q_L^*, q_H^*),\ q_L^* = b_L(q_1^*), \text{ and } q_H^* = b_H(q_1^*).$$

? EXERCISE 285.1 (Cournot's duopoly game with imperfect information) Consider the game when the inverse demand function is given by $P(Q) = \alpha - Q$ for $Q \leq \alpha$ and $P(Q) = 0$ for $Q > \alpha$ (see (54.2)). For values of c_H and c_L close enough that there is a Nash equilibrium in which all outputs are positive, find this equilibrium. Compare this equilibrium with the Nash equilibrium of the game in which firm 1 knows that firm 2's unit cost is c_L, and with the Nash equilibrium of the game in which firm 1 knows that firm 2's unit cost is c_H.

9.5.2 *Imperfect information about both cost and information*

Now suppose that firm 2 does not know whether firm 1 knows its cost. That is, suppose that one circumstance that firm 2 believes to be possible is that firm 1 knows it cost (although in fact it does not). Because firm 2 thinks this circumstance to be possible, we need *four* states to model the situation, which I call $L0$, $H0$, $L1$, and $H1$, with the following interpretations.

$L0$: firm 2's cost is low and firm 1 does not know whether it is low or high

$H0$: firm 2's cost is high and firm 1 does not know whether it is low or high

$L1$: firm 2's cost is low and firm 1 knows it is low

$H1$: firm 2's cost is high and firm 1 knows it is high.

Firm 1 receives one of three possible signals, 0, L, and H. The states $L0$ and $H0$ generate the signal 0 (firm 1 does not know firm 2's cost), the state $L1$ generates the signal L (firm 1 knows firm 2's cost is low), and the state $H1$ generates the signal H (firm 1 knows firm 2's cost is high). Firm 2 receives one of two possible signals, L, in states $L0$ and $L1$, and H, in states $H0$ and $H1$. Denote by θ (as before) the probability assigned by type 0 of firm 1 to firm 2's cost being c_L, and by π the probability assigned by each type of firm 2 to firm 1's knowing firm 2's cost. (The case $\pi = 0$ is equivalent to the one considered in the previous section.) A Bayesian game that models the situation is defined as follows.

Players Firm 1 and firm 2.

States $\{L0, L1, H0, H1\}$, where the first letter in the name of the state indicates firm 2's cost and the second letter indicates whether (1) or not (0) firm 1 knows firm 2's cost.

Actions Each firm's set of actions is the set of its possible outputs (nonnegative numbers).

Signals Firm 1 gets one of the signals 0, L, and H, and her signal function τ_1 satisfies $\tau_1(L0) = \tau_1(H0) = 0$, $\tau_1(L1) = L$, and $\tau_1(H1) = H$. Firm 2 gets the signal L or H and her signal function τ_2 satisfies $\tau_2(L0) = \tau_2(L1) = L$ and $\tau_2(H0) = \tau_2(H1) = H$.

Beliefs Firm 1: type 0 assigns probability θ to state $L0$ and probability $1 - \theta$ to state $H0$; type L assigns probability 1 to state $L1$; type H assigns probability 1 to state H. Firm 2: type L assigns probability π to state $L1$ and probability $1 - \pi$ to state $L0$; type H assigns probability π to state $H1$ and probability $1 - \pi$ to state $H0$.

Payoff functions The firms' Bernoulli payoffs are their profits; if the actions chosen are (q_1, q_2), then firm 1's profit is $q_1(P(q_1 + q_2) - c)$ and firm 2's profit is $q_2(P(q_1 + q_2) - c_L)$ in states $L0$ and $L1$, and $q_2(P(q_1 + q_2) - c_L)$ in states $H0$ and $H1$.

The information structure in this game is illustrated in Figure 287.1. You are asked to investigate its Nash equilibria in the following exercise.

? EXERCISE 286.1 (Cournot's duopoly game with imperfect information) Write down the maximization problems that determine the best response function each type of each player. (Denote by q_0, q_ℓ, and q_h the outputs of types 0, ℓ, and h of firm 1, and by q_L and q_H the outputs of types L and H of firm 2.) Now suppose that the inverse demand function is given by $P(Q) = \alpha - Q$ for $Q \leq \alpha$ and $P(Q) = 0$ for $Q > \alpha$. For values of c_H and c_L close enough that there is a Nash equilibrium in which all outputs are positive, find this equilibrium. Check that when $\pi = 0$ the equilibrium output of type 0 of firm 1 is equal to the equilibrium output of firm 1 you found in Exercise 285.1, and that the equilibrium outputs of the two types of firm 2 are the same as the ones you found in that exercise. Check also that when $\pi = 1$ the equilibrium outputs of type ℓ of firm 1 and type L of firm 2 are the same as the equilibrium outputs when there is perfect information and the costs are c and c_L, and that the equilibrium outputs of type h of firm 1 and type H of firm 2 are the same as the equilibrium outputs when there is perfect information and the

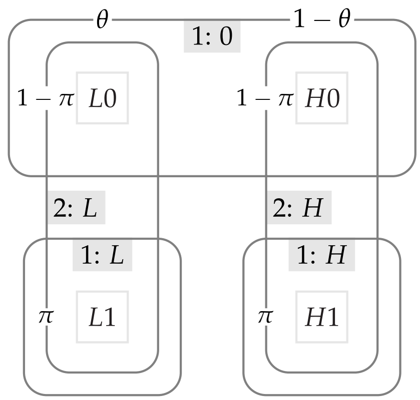


Figure 287.1 The information structure for the model in Section 9.5.2, in which firm 2 does not know whether firm 1 knows its cost. The frame labeled i: x encloses the states that generates the signal x for firm i.

costs are c and c_H. Show that for $0 < \pi < 1$, the equilibrium outputs of types L and H of firm 2 lie between their values when $\pi = 0$ and when $\pi = 1$.

9.6 Illustration: providing a public good

Suppose that a public good is provided to a group of people if at least one person is willing to pay the cost of the good (as in the model of crime-reporting in Section 4.8). Assume that the people differ in their valuations of the good, and each person knows only her own valuation. Who, if anyone, will pay the cost?

Denote the number of individuals by n, the cost of the good by $c > 0$, and individual i's payoff if the good is provided by v_i. If the good is not provided then each individual's payoff is 0. Each individual i knows her own valuation v_i. She does not know anyone else's valuation, but knows that all valuations are at least $\underline{v}$ and at most $\overline{v}$, where $0 \leq \underline{v} < c < \overline{v}$. She believes that the probability that any one individual's valuation is at most v is $F(v)$, independent of all other individuals' valuations, where F is a continuous increasing function. The fact that F is increasing means that the individual does not assign zero probability to any range of values between $\underline{v}$ and $\overline{v}$; the fact that it is continuous means that she does not assign positive probability to any single valuation. (An example of the function F is shown in Figure 288.1.)

The following mechanism determines whether the good is provided. All n individuals simultaneously submit envelopes; the envelope of any individual i may contain either a contribution of c or nothing (no intermediate contributions are allowed). If all individuals submit 0 then the good is not provided and each individual's payoff is 0. If at least one individual submits c then the good is provided, each individual i who submits c obtains the payoff $v_i - c$, and each individual i who submits 0 obtains the payoff v_i. (The pure strategy Nash equilibria of a vari-

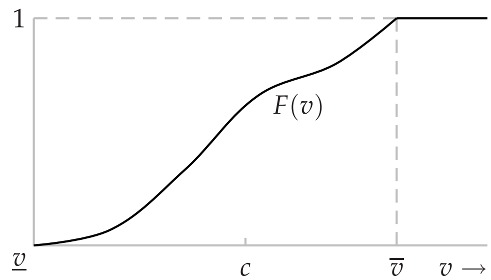


Figure 288.1 An example of the function F for the model in Section 9.6. For each value of v, $F(v)$ is the probability that any given individual's valuation is at most v.

ant of this model, in which more than one contribution is needed to provide the good, are considered in Exercise 31.1.)

We can formulate this situation as a Bayesian game as follows.

Players The set of n individuals.

States The set of all profiles $(v_1, \ldots, v_n)$ of valuations, where $0 \leq v_i \leq \overline{v}$ for all i.

Actions Each player's set of actions is $\{0, c\}$.

Signals The set of signals that each player may observe is the set of possible valuations. The signal function τ_i of each player i is given by $\tau_i(v_1, \ldots, v_n) = v_i$ (each player knows her own valuation).

Beliefs Each type of player i assigns probability $F(v_1)F(v_2) \cdots F(v_{i-1})F(v_{i+1}) \cdots F(v_n)$ to the event that the valuation of every other player j is at most v_j.

Payoff functions Player i's Bernoulli payoff in state $(v_1, \ldots, v_n)$ is

$$\begin{cases} 0 & \text{if no one contributes} \\ v_i & \text{if } i \text{ does not contribute but some other player does} \\ v_i - c & \text{if } i \text{ contributes.} \end{cases}$$

? EXERCISE 288.1 (Nash equilibria of game of contributing to a public good) Find conditions under which for each value of i this game has a pure strategy Nash equilibrium in which each type v_i of player i with $v_i \geq c$ contributes, whereas every other type of player i, and all types of every other player, do not contribute.

In addition to the Nash equilibria identified in this exercise, the game has a symmetric Nash equilibrium in which every player contributes if and only if her valuation exceeds some critical amount v^*. For such a strategy profile to be an equilibrium, a player whose valuation is less than v^* must optimally not contribute, and a player whose valuation is at least v^* must optimally contribute. Consider player i. Suppose that every other player contributes if and only if her valuation is at least v^*. The probability that at least one of the other players contributes is the probability that at least one of the other players' valuations is at least v^*, which is $1 - (F(v^*))^{n-1}$. (Note that $(F(v^*))^{n-1}$ is the probability that all the other valuations are at most v^*.) Thus if player i's valuation is v_i, her expected payoff is $(1 - (F(v^*))^{n-1})v_i$ if she does not contribute and $v_i - c$ if she does contribute. Hence the conditions for player i to optimally not contribute when $v_i < v^*$ and optimally contribute when $v_i \geq v^*$ are $(1 - (F(v^*))^{n-1})v_i \geq v_i - c$ if $v_i < v^*$, and $(1 - (F(v^*))^{n-1})v_i \leq v_i - c$ if $v_i \geq v^*$, or equivalently

$$\begin{aligned} v_i(F(v^*))^{n-1} &\leq c \quad \text{if } v_i < v^* \\ v_i(F(v^*))^{n-1} &\geq c \quad \text{if } v_i \geq v^*. \end{aligned} \tag{289.1}$$

If these inequalities are satisfied then

$$v^*(F(v^*))^{n-1} = c. \tag{289.2}$$

Conversely, if v^* satisfies (289.2) then it satisfies the two equations in (289.1). Thus the game has a Nash equilibrium in which every player contributes whenever her valuation is at least v^* if and only if v^* satisfies (289.2).

Note that because $F(v) = 1$ only if $v \geq \overline{v}$, and $\overline{v} > c$, we have $v^* > c$. That is, every player's cutoff for contributing exceeds the cost of the public good. When at least one player's valuation exceeds c, all players are better off if the public good is provided and the high-valuation player contributes than if the good is not provided. But in the equilibrium, the good is provided only if at least one player's valuation exceeds v^*, which exceeds c.

As the number of individuals increases, is the good more or less likely to be provided in this equilibrium? The probability that the good is provided is the probability that at least one player's valuation is at least v^*, which is equal to $1 - (F(v^*))^n$. (Note that $(F(v^*))^n$ is the probability that every player's valuation is less than v^*.) From (289.2) this probability is equal to $1 - cF(v^*)/v^*$. How does v^* vary with n? As n increases, for any given value of v^* the value of $(F(v^*))^{n-1}$ decreases, and thus the value of $v^*(F(v^*))^{n-1}$ decreases. Thus to maintain the equality (289.2), the value of v^* must increase as n increases. We conclude that as n increases the change in the probability that the good is provided depends on the change in $F(v^*)/v^*$ as v^* increases: the probability increases if $F(v^*)/v^*$ is a decreasing function of v^*, whereas it decreases if $F(v^*)/v^*$ is an increasing function of v^*. If F is uniform and $\underline{v} > 0$, for example, $F(v^*)/v^*$ is a decreasing function of v^*, so that the probability that the good is provided increases as the population size increases.

The notion of a Bayesian game may be used to model a situation in which each player is uncertain of the number of other players. In the next exercise you are asked to study another variant of the crime-reporting model of Section 4.8 in which each of the two players does not know whether she is the only witness or whether there is another witness (in which case she knows that witness's valuation). (The exercise requires a knowledge of mixed strategy Nash equilibrium (Chapter 4).)

? EXERCISE 290.1 (Reporting a crime with an unknown number of witnesses) Consider the variant of the model of Section 4.8 in which each of two players does not know whether she is the only witness, or whether there is another witness. Denote by π the probability each player assigns to being the sole witness. Model this situation as a Bayesian game with three states: one in which player 1 is the only witness, one in which player 2 is the only witness, and one in which both players are witnesses. Find a condition on π under which the game has a pure Nash equilibrium in which each player chooses *Call* (given the signal that she is a witness). When the condition is violated, find the symmetric mixed strategy Nash equilibrium of the game, and check that when $\pi = 0$ this equilibrium coincides with the one found in Section 4.8 for $n = 2$.

9.7 Illustration: auctions

9.7.1 Introduction

In the analysis of auctions in Section 3.5, every bidder knows every other bidder's valuation of the object for sale. Here I use the notion of a Bayesian game to analyze auctions in which bidders are not perfectly informed about each others' valuations.

Assume that a single object is for sale, and that each bidder independently receives some information—a "signal"—about the value of the object to her. If each bidder's signal is simply her valuation of the object, as assumed in Section 3.5, we say that the bidders' valuations are *private*. If each bidder's valuation depends on other bidders' signals as well as her own, we say that the valuations are *common*.

The assumption of private values is appropriate, for example, for a work of art whose beauty rather than resale value interests the buyers. Each bidder knows her valuation of the object, but not that of any other bidder; the other bidders' valuations have no bearing on her valuation. The assumption of common values is appropriate, for example, for an oil tract containing unknown reserves on which each bidder has conducted a test. Each bidder i's test result gives her some information about the size of the reserves, and hence her valuation of these reserves, but the other bidders' test results, if known to bidder i, would typically improve this information.

As in the analysis of auctions in which the bidders are perfectly informed about each others' valuations, I study models in which bids for a single object are submitted simultaneously (bids are *sealed*), and the participant who submits the highest bid obtains the object. As before I consider both *first-price* auctions, in which the winner pays the price she bid, and *second-price* auctions, in which the winner pays the highest of the remaining bids.

(In Section 3.5 I argue that the first-price rule models an open descending ("Dutch") auction, and the second-price rule models an open ascending ("English") auction. Note that the argument that the second-price rule corresponds to an open ascending auction depends upon the bidders' valuations being private. If a bidder is uncertain of her valuation, which is related to that of other bidders, then in an open ascending auction she may obtain information about her valuation from other participants' bids, information not available in a sealed-bid auction.)

I first consider the case in which the bidders' valuations are private, then the case in which they are common.

9.7.2 Independent private values

In the case in which the bidders' valuations are private, the assumptions about these valuations are similar to those in the previous section (on the provision of a public good). Each bidder knows that all other bidders' valuations are at least $\underline{v}$, where $\underline{v} \geq 0$, and at most $\overline{v}$. She believes that the probability that any given bidder's valuation is at most v is $F(v)$, independent of all other bidders' valuations, where F is a continuous increasing function (as in Figure 288.1).

The preferences of a bidder whose valuation is v are represented by the expected value of the Bernoulli payoff function that assigns 0 to the outcome in which she does not win the object and $v - p$ to the outcome in which she wins the object and pays the price p. (That is, each bidder is risk neutral.) I assume that the expected payoff of a bidder whose bid is tied for first place is $(v - p)/m$, where m is the number of tied winning bids. (The assumption about the outcome when bids are tied for first place has mainly "technical" significance; in Section 3.5, it was convenient to make an assumption different from the one here.)

Denote by $P(b)$ the price paid by the winner of the auction when the profile of bids is b. For a first-price auction $P(b)$ is the winning bid (the largest b_i), whereas for a second-price auction it is the highest bid made by a bidder different from the winner. Given the appropriate specification of P, the following Bayesian game models **first- and second-price auctions with independent private valuations** (and imperfect information about valuations).

Players The set of bidders, say $1, \ldots, n$.

States The set of all profiles $(v_1, \ldots, v_n)$ of valuations, where $\underline{v} \leq v_i \leq \overline{v}$ for all i.

Actions Each player's set of actions is the set of possible bids (nonnegative numbers).

Signals The set of signals that each player may observe is the set of possible valuations. The signal function τ_i of each player i is given by $\tau_i(v_1, \ldots, v_n) = v_i$ (each player knows her own valuation).

Beliefs Each type of player i assigns probability $F(v_1)F(v_2) \cdots F(v_{i-1})F(v_{i+1}) \cdots F(v_n)$ to the event that the valuation of every other player j is at most v_j.

Payoff functions Player i's Bernoulli payoff in state $(v_1, \ldots, v_n)$ is 0 if her bid b_i is not the highest bid, and $(v_i - P(b))/m$ if no bid is higher than b_i and m

bids (including b_i) are equal to b_i:

$$u_i(b,(v_1,\ldots,v_n)) = \begin{cases} (v_i - P(b))/m & \text{if } b_j \leq b_i \text{ for all } j \neq i \text{ and} \\ & \qquad b_j = b_i \text{ for } m \text{ players} \\ 0 & \text{if } b_j > b_i \text{ for some } j \neq i. \end{cases}$$

Nash equilibrium in a second-price sealed-bid auction As in a second-price sealed-bid auction in which every bidder knows every other bidder's valuation,

> *in a second-price sealed-bid auction with imperfect information about valuations, a player's bid equal to her valuation weakly dominates all her other bids.*

Precisely, consider some type v_i of some player i, and let b_i be a bid not equal to v_i. Then for all bids by all types of all the other players, the expected payoff of type v_i of player i is at least as high when she bids v_i as it is when she bids b_i, and for some bids by the various types of the other players, her expected payoff is greater when she bids v_i than it is when she bids b_i.

The argument for this result is similar to the argument in Section 3.5.2 in the case in which the players know each others' valuations. The main difference between the arguments arises because in the case in which the players do not know each others' valuations, any given bids for every type of every player but i leave player i uncertain about the highest of the remaining bids, because she is uncertain of the other players' types. (The difference in the tie-breaking rules between the two cases also necessitates a small change in the argument.) In the next exercise you are asked to fill in the details.

? EXERCISE 292.1 (Weak domination in second-price sealed-bid action) Show that for each type v_i of each player i in a second-price sealed-bid auction with imperfect information about valuations the bid v_i weakly dominates all other bids.

We conclude, in particular, that a second-price sealed-bid auction with imperfect information about valuations has a Nash equilibrium in which every type of every player bids her valuation. The game has also other equilibria, some of which you are asked to find in the next exercise.

? EXERCISE 292.2 (Nash equilibria of a second-price sealed-bid auction) For every player i, find a Nash equilibrium of a second-price sealed-bid auction in which player i wins. (Think about the Nash equilibria when the players know each others' valuations, studied in Section 3.5.)

Nash equilibrium in a first-price sealed-bid auction As when the players are perfectly informed about each others' valuations, the bid of v_i by type v_i of player i weakly dominates any bid greater than v_i, but does not weakly dominate bids less than v_i, and is itself weakly dominated by any such lower bid. (If type v_i of player i bids v_i, her payoff is certainly 0 (either she wins and pays her valuation, or she loses), whereas if she bids less than v_i, she may win and obtain a positive payoff.)

These facts suggest that the game may have a Nash equilibrium in which each player bids less than her valuation. An analysis of the game for an arbitrary distribution F of valuations requires calculus, and is relegated to an appendix (Section 9.9). Here I consider the case in which there are two bidders and each player's valuation is distributed "uniformly" between 0 and 1. This assumption on the distribution of valuations means that the fraction of valuations less than v is exactly v, so that $F(v) = v$ for all v with $0 \leq v \leq 1$.

Denote by $\beta_i(v)$ the bid of type v of player i. I claim that if there are two bidders and the distribution of valuations is uniform between 0 and 1, the game has a (symmetric) Nash equilibrium in which the function β_i is the same for both players, with $\beta_i(v) = \frac{1}{2}v$ for all v. That is, each type of each player bids exactly half her valuation.

To verify this claim, suppose that each type of player 2 bids in this way. Then as far as player 1 is concerned, player 2's bids are distributed uniformly between 0 and $\frac{1}{2}$. Thus if player 1 bids more than $\frac{1}{2}$ she surely wins, whereas if she bids $b_1 \leq \frac{1}{2}$ the probability that she wins is the probability that player 2's valuation is less than $2b_1$ (in which case player 2 bids less than b_1), which is $2b_1$. Consequently her payoff as a function of her bid b_1 is

$$\begin{cases} 2b_1(v_1 - b_1) & \text{if } 0 \leq b_1 \leq \frac{1}{2} \\ v_1 - b_1 & \text{if } b_1 > \frac{1}{2}. \end{cases}$$

This function is shown in Figure 293.1. Its maximizer is $\frac{1}{2}v_1$ (see Exercise 446.1), so that player 1's optimal bid is half her valuation. Both players are identical, so this argument shows also that given $\beta_1(v) = \frac{1}{2}v$, player 2's optimal bid is half her valuation. Thus, as claimed, the game has a Nash equilibrium in which each type of each player bids half her valuation.

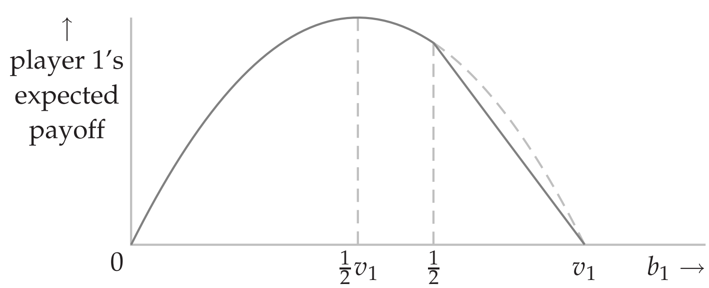


Figure 293.1 Player 1's expected payoff as a function of its bid in a first-price sealed-bid auction in which there are two bidders and the valuations are uniformly distributed from 0 to 1, given that player 2 bids $\frac{1}{2}v_2$.

When the number n of bidders exceeds two, a similar analysis shows that the game has a (symmetric) Nash equilibrium in which every player bids the fraction $1 - 1/n$ of her valuation: $\beta_i(v) = (1 - 1/n)v$ for every player i and every valuation v. (You are asked to verify a claim more general than this one in Exercise 295.1.)

In this example—and, it turns out, for any distribution F satisfying the conditions in Section 9.7.2—the players' common bidding function in a symmetric Nash equilibrium may be given an illuminating interpretation. Choose $n-1$ valuations randomly and independently, each according to the cumulative distribution function F. The highest of these $n-1$ valuations is a "random variable": its value depends on the $n-1$ valuations that were chosen. Denote it by $\mathbf{X}$. Fix a valuation v. Some values of $\mathbf{X}$ are less than v; others are greater than v. Consider the distribution of $\mathbf{X}$ in those cases in which it is less than v. The expected value of this distribution is denoted $\mathrm{E}[\mathbf{X} \mid \mathbf{X} < v]$: the expected value of $\mathbf{X}$ conditional on $\mathbf{X}$ being less than v. We may prove the following result. (A proof is given in the appendix, Section 9.9.)

> *For a distribution of valuations satisfying the conditions in Section 9.7.2, a first-price sealed-bid auction with imperfect information about valuations has a (symmetric) Nash equilibrium in which each type v of each player bids $\mathrm{E}[\mathbf{X} \mid \mathbf{X} < v]$, the expected value of the highest of the other players' bids conditional on v being higher than all the other valuations.*

Put differently, each bidder asks the following question: Over all the cases in which my valuation is the highest, what is the expectation of the highest of the other players' valuations? This expectation is the amount she bids.

In the case considered above in which F is uniform from 0 to 1 and $n = 2$, we may verify that indeed the equilibrium we found may be expressed in this way. For any valuation v of player 1, the cases in which player 2's valuation is less than v are distributed uniformly from 0 to v, so that the expected value of player 2's valuation conditional on its being less than v is $\frac{1}{2}v$, which is equal to the equilibrium bidding function that we found.

Comparing equilibria of first- and second-price auctions At the end of Section 3.5.3 we saw that first- and second-price auctions are "revenue equivalent" when the players know each others' valuations: their distinguished equilibria yield the same outcome. The same is true when the players are uncertain of each others' valuations.

Consider the equilibrium of a second-price auction in which every player bids her valuation. In this equilibrium, the expected price paid by a bidder with valuation v who wins is the expectation of the highest of the other $n-1$ valuations, conditional on this maximum being less than v, or, in the notation above, $\mathrm{E}[\mathbf{X} \mid \mathbf{X} < v]$. We have just seen that a first-price auction has a symmetric Nash equilibrium in which this amount is precisely the bid of a player with valuation v, and hence the amount paid by such a player. Thus in the equilibria of both auctions the expected price paid by a winning bidder is the same. In both cases, the player with the highest valuation submits the winning bid, so both auctions yield the same revenue for the auctioneer:

> *if each bidder is risk neutral and the distribution of valuations satisfies the conditions in Section 9.7.2, then the Nash equilibrium of a second-price sealed-bid*

auction with independent private valuations (and imperfect information about valuations) in which each player bids her valuation yields the same revenue as the symmetric Nash equilibrium of the corresponding first-price sealed-bid auction.

This result depends on the assumption that each player's preferences are represented by the expected value of a risk neutral Bernoulli payoff function. The next exercise asks you to study an example in which each player is risk averse. (See page 101 for a discussion of risk neutrality and risk aversion.)

?? EXERCISE 295.1 (Auctions with risk averse bidders) Consider a variant of the Bayesian game defined in Section 9.7.2 in which the players are risk averse. Specifically, suppose each of the n players' preferences are represented by the expected value of the Bernoulli payoff function $x^{1/m}$, where x is the player's monetary payoff and $m > 1$. Suppose also that each player's valuation is distributed uniformly between 0 and 1, as in the example in Section 9.7.2. Show that the Bayesian game that models a first-price sealed-bid auction under these assumptions has a (symmetric) Nash equilibrium in which each type v_i of each player i bids $(1 - 1/[m(n-1)+1])v_i$. (You need to use the mathematical fact that the solution of the problem $\max_b [b^k(v - b)^\ell]$ is $kv/(k+\ell)$.) Compare the auctioneer's revenue in this equilibrium with her revenue in the symmetric Nash equilibrium of a second-price sealed-bid auction in which each player bids her valuation. (Note that the equilibrium of the second-price auction does not depend on the players' payoff functions.)

9.7.3 Common valuations

In an auction with common valuations, each player's valuation depends on the other players' signals as well as her own. (As before, I assume that the players' signals are independent.) I denote the function that gives player i's valuation by g_i, and assume that it is increasing in all the signals. Given the appropriate specification of the function P that determines the price $P(b)$ paid by the winner as a function of the profile b of bids, the following Bayesian game models **first- and second-price auctions with common valuations** (and imperfect information about valuations).

Players The set of bidders, say $\{1, \ldots, n\}$.

States The set of all profiles $(t_1, \ldots, t_n)$ of signals that the players may receive.

Actions Each player's set of actions is the set of possible bids (nonnegative numbers).

Signals The signal function τ_i of each player i is given by $\tau_i(t_1, \ldots, t_n) = t_i$ (each player observes her own signal).

Beliefs Each type of each player believes that the signal of every type of every other player is independent of all the other players' signals.

Payoff functions Player i's Bernoulli payoff in state $(t_1, \ldots, t_n)$ is 0 if her bid b_i is not the highest bid, and $(g_i(t_1, \ldots, t_n) - P(b))/m$ if no bid is higher than b_i and m bids (including b_i) are equal to b_i:

$$u_i(b, (t_1, \ldots, t_n)) = \begin{cases} (g_i(t_1, \ldots, t_n) - P(b))/m & \text{if } b_j \leq b_i \text{ for all } j \neq i \text{ and} \\ & \quad b_j = b_i \text{ for } m \text{ players} \\ 0 & \text{if } b_j > b_i \text{ for some } j \neq i. \end{cases}$$

Nash equilibrium in a second-price sealed-bid auction The main ideas in the analysis of sealed-bid common value auctions are illustrated by an example in which there are two bidders, each bidder's signal is uniformly distributed from 0 to 1, and the valuation of each bidder i is given by $v_i = \alpha t_i + \gamma t_j$, where j is the other player and $\alpha \geq \gamma \geq 0$. The case in which $\alpha = 1$ and $\gamma = 0$ is exactly the one studied in Section 9.7.2: in this case, the bidders' valuations are private. If $\alpha = \gamma$ then for any given signals, each bidder's valuation is the same—a case of "pure common valuations". If, for example, the signal t_i is the number of barrels of oil in a tract, then the expected valuation of a bidder i who knows the signals t_i and t_j is $p \cdot \frac{1}{2}(t_i + t_j)$, where p is the monetary worth of a barrel of oil. Our assumption, of course, is that a bidder does *not* know any other player's signal. However, a key point in the analysis of common value auctions is that the other players' bids contain *some* information about the other players' signals—information that may profitably be used.

I claim that under these assumptions a second-price sealed-bid auction has a Nash equilibrium in which each type t_i of each player i bids $(\alpha + \gamma)t_i$.

To verify this claim, suppose that each type of player 2 bids in this way and type t_1 of player 1 bids b_1. To determine the expected payoff of type t_1 of player 1, we need to find the probability with which she wins, and both the expected price she pays and the expected value of player 2's signal if she wins.

Probability that player 1 wins: Given that player 2's bidding function is $(\alpha + \gamma)t_2$, player 1's bid of b_1 wins only if $b_1 \geq (\alpha + \gamma)t_2$, or if $t_2 \leq b_1/(\alpha + \gamma)$. Now, t_2 is distributed uniformly from 0 to 1, so the probability that it is at most $b_1/(\alpha + \gamma)$ is $b_1/(\alpha + \gamma)$. Thus a bid of b_1 by player 1 wins with probability $b_1/(\alpha + \gamma)$.

Expected price player 1 pays if she wins: The price she pays is equal to player 2's bid, which, *conditional on its being less than* b_1, is distributed uniformly from 0 to b_1. Thus the expected value of player 2's bid, *given that it is less than* b_1, is $\frac{1}{2}b_1$.

Expected value of player 2's signal if player 1 wins: Player 2's bid, given her signal t_2, is $(\alpha + \gamma)t_2$, so that the expected value of signals that yield a bid of less than b_1 is $\frac{1}{2}b_1/(\alpha + \gamma)$ (because of the uniformity of the distribution of t_2).

Now, player 1's expected payoff if she bids b_1 is the difference between her expected valuation, given her signal t_1 and the fact that she wins, and the expected price she pays, multiplied by her probability of winning. Combining the calculations above, player 1's expected payoff if she bids b_1 is thus $(\alpha t_1 + \frac{1}{2}\gamma b_1/(\alpha+\gamma) - \frac{1}{2}b_1)b_1/(\alpha+\gamma)$, or

$$\frac{\alpha}{2(\alpha+\gamma)^2} \cdot (2(\alpha+\gamma)t_1 - b_1)b_1.$$

This function is maximized at $b_1 = (\alpha+\gamma)t_1$. That is, if each type t_2 of player 2 bids $(\alpha+\gamma)t_2$, any type t_1 of player 1 optimally bids $(\alpha+\gamma)t_1$. Symmetrically, if each type t_1 of player 1 bids $(\alpha+\gamma)t_1$, any type t_2 of player 2 optimally bids $(\alpha+\gamma)t_2$. Hence, as claimed, the game has a Nash equilibrium in which each type t_i of each player i bids $(\alpha+\gamma)t_i$.

? EXERCISE 297.1 (Asymmetric Nash equilibria of second-price sealed-bid common value auctions) Show that when $\alpha = \gamma = 1$, for *any* value of $\lambda > 0$ the game studied above has an (asymmetric) Nash equilibrium in which each type t_1 of player 1 bids $(1+\lambda)t_1$ and each type t_2 of player 2 bids $(1+1/\lambda)t_2$.

Note that when player 1 calculates her expected value of the object, she finds the expected value of player 2's signal *given that her bid wins*. If her bid is low then she is unlikely to be the winner, but if she *is* the winner, player 2's signal must be low, and so she should impute a low value to the object. She should not base her bid simply on an estimate of the valuation derived from her own signal and the (unconditional) expectation of the other player's signal. If she does so, then over all the cases in which she wins, she more likely than not overvalues the object. A bidder who incorrectly behaves in this way is said to suffer the *winner's curse*. (Bidders in real auctions know this problem: when a contractor gives you a quotation to renovate your house, she does not base her price simply on an unbiased estimate out how much it will cost her to do the job, but takes into account that you will select her only if her competitors' estimates are all be higher than hers, in which case her estimate may be suspiciously low.)

Nash equilibrium in a first-price sealed-bid auction I claim that under the assumptions on the players' signals and valuations in the previous section, a first-price sealed-bid auction has a Nash equilibrium in which each type t_i of each player i bids $\frac{1}{2}(\alpha+\gamma)t_i$. This claim may be verified by arguments like those in the previous section. In the next exercise, you are asked to supply the details.

? EXERCISE 297.2 (First-price sealed-bid auction with common valuations) Verify that under the assumptions on signals and valuations in the previous section, a first-price sealed-bid auction has a Nash equilibrium in which the bid of each type t_i of each player i is $\frac{1}{2}(\alpha+\gamma)t_i$.

Comparing equilibria of first- and second-price auctions We see that the revenue equivalence of first- and second-price auctions that holds when valuations are private hold also for the symmetric equilibria of the examples above in which the valuations are common. That is, the expected price paid by a player of any given type is the same in the symmetric equilibrium of the first-price auction as it is in the symmetric equilibrium of the second-price auction: in each case type t_i of player i pays $\frac{1}{2}(\alpha + \gamma)t_i$ if she wins, and wins with the same probability.

In fact, the revenue equivalence principle holds much more generally. Whenever each bidder is risk neutral and independently receives a signal that the same distribution, which satisfies the conditions on the distribution of valuations in Section 9.7.2, the expected payment of a bidder of any given type is the same in the symmetric Nash equilibrium of a second-price sealed-bid auction revenue-equivalent as it is in the symmetric Nash equilibrium of a first-price sealed-bid auction. Further, this revenue equivalence is not restricted to first- and second-price auctions; a general result, encompassing a wider range of auction forms, is stated at the end of the appendix (Section 9.9).

AUCTIONS OF THE RADIO SPECTRUM

In the 1990s several countries started auctioning the right to use parts of the radio spectrum used for wireless communication (by mobile telephones, for example). Spectrum licenses in the USA were originally allocated on the basis of hearings by the Federal Communications Commission (FCC). This procedure was time-consuming, and a large backlog developed, prompting a switch to lotteries. Licenses awarded by the lotteries could be re-sold at high prices, attracting many participants. In one case that drew attention, the winner of a license to run cellular telephones in Cape Cod sold it to Southwestern Bell for US$41.5 million (*New York Times*, May 30, 1991, p. A1). In the early 1990s, the US government was persuaded that auctioning licenses would allocate them more efficiently and might raise nontrivial revenue.

For each interval of the spectrum, many licenses were available, each covering a geographic area. A buyer's valuation of a license could be expected to depend on the other licenses it owned, so many interdependent goods were for sale. In designing an auction mechanism, the FCC had many choices: for example, the bidding could be open, or it could be sealed, with the price equal to either the highest bid or the second-highest bid; the licenses could be sold sequentially, or simultaneously, in which case participants could submit bids for individual licenses, or for combinations of licenses. Experts in auction theory were consulted on the design of the mechanism. John McMillan (who advised the FCC), writes that "When theorists met the policy-makers, concepts like Bayes-Nash equilibrium, incentive-compatibility constraints, and order-statistic theorems came to be discussed in the corridors of power" (1994, 146). No theoretical analysis fitted the environment of the auction well, but the experts appealed to some principles from the existing theory, the results of laboratory experiments, and experience in auctions held in New Zealand and Australia in the early 1990s in making their recommendations. The mechanism adopted in 1994 was an open ascending auction for which bids were accepted simultaneously for all licenses in each round. Experts argued that the open (as opposed to sealed-bid) format and the simultaneity of the auctions promoted an efficient outcome because at each stage the bidders could see their rivals' previous bids for all licenses.

The FCC has conducted several auctions, starting with "narrowband" licenses (each covering a sliver of the spectrum, used by paging services) and continuing with "broadband" licenses (used for voice and data communications). These auctions have provided more employment for game theorists, many of whom have advised the companies bidding for licenses. In response to growing congestion of the airwaves and the expectation that a significant part of the rapidly growing Internet traffic will move to wireless devices, in 2000 the US president Bill Clinton ordered further auctions of large parts of the spectrum (*New York Times*, October 14, 2000). Whether the auctions that have been held have allocated licenses efficiently is hard to tell, though it appears that the winners were able to obtain the sets of licenses they wanted. Certainly the auctions have been successful in generating revenue: the first four generated over US$18 billion.

9.8 Illustration: juries

9.8.1 Model

In a trial, jurors are presented with evidence concerning the guilt or innocence of a defendant. They may interpret the evidence differently. On the basis of her interpretation, each juror votes either to convict or acquit the defendant. Assume that a unanimous verdict is required for conviction: the defendant is convicted if and only if every juror votes to convict her. (This rule is used in the USA and Canada, for example.) What can we say about the chances of an innocent defendant's being convicted and a guilty defendant's being acquitted?

In deciding how to vote, each juror must consider the costs of convicting an innocent person and of acquitting a guilty person. She must consider also the likely effect of her vote on the outcome, which depends on the other jurors' votes. For example, a juror who thinks that at least one of her colleagues is likely to vote for acquittal may act differently from one who is sure that all her colleagues will vote for conviction. Thus an answer to the question requires us to consider the strategic interaction between the jurors, which we may model as a Bayesian game.

Assume that each juror comes to the trial with the belief that the defendant is guilty with probability π (the same for every juror), a belief modified by the evidence presented. We model the possibility that jurors interpret the evidence differently by assuming that for each of the defendant's true statuses (guilty and innocent), each juror interprets the evidence to point to guilt with positive probability, and to innocence with positive probability, and that the jurors' interpretations are independent (no juror's interpretation depends on any other juror's interpretation). I assume that the probabilities are the same for all jurors, and denote the probability of any given juror's interpreting the evidence to point to guilt when the defendant is guilty by p, and the probability of her interpreting the evidence to point to innocence when the defendant is innocent by q. I assume also that a juror is more likely than not to interpret the evidence correctly, so that $p > \frac{1}{2}$ and $q > \frac{1}{2}$, and hence in particular $p > 1 - q$.

Each juror wishes to convict a guilty defendant and acquit an innocent one. She is indifferent between these two outcomes, and prefers each of them to one in which an innocent defendant is convicted or a guilty defendant is acquitted. Assume specifically that each juror's Bernoulli payoffs are:

$$\begin{cases} 0 & \text{if guilty defendant convicted or innocent defendant acquitted} \\ -z & \text{if innocent defendant convicted} \\ -(1-z) & \text{if guilty defendant acquitted.} \end{cases} \tag{300.1}$$

The parameter z may be given an appealing interpretation. Denote by r the probability a juror assigns to the defendant's being guilty, given all her information. Then her expected payoff if the defendant is acquitted is $-r(1-z) + (1-r) \cdot 0 = -r(1-z)$ and her expected payoff if the defendant is convicted is $r \cdot 0 - (1-r)z = -(1-r)z$. Thus she prefers the defendant to be acquitted if $-r(1-z) > -(1-r)z$, or $r < z$, and convicted if $r > z$. That is, z is equal to the probability of guilt required for the juror to want the defendant to be convicted. Put differently, for any juror

$$\begin{array}{l} \text{acquittal is at least as good as conviction if and only if} \\ \Pr(\text{defendant is guilty, given juror's information}) \leq z. \end{array} \tag{300.2}$$

We may now formulate a Bayesian game that models the situation. The players are the jurors, and each player's action is a vote to convict (C) or to acquit (Q). We need one state for each configuration of the players' preferences and information. Each player's preferences depend on whether the defendant is guilty or innocent, and each player's information consists of her interpretation of the evidence. Thus we define a state to be a list $(X, s_1, \ldots, s_n)$, where X denotes the defendant's true status, guilty (G) or innocent (I), and s_i represents player i's interpretation of the evidence, which may point to guilt (g) or innocence (b). (I do not use i for "innocence" because I use it to index the players; b stands for "blameless".) The signal that each player i receives is her interpretation of the evidence, s_i. In any state in which $X = G$ (i.e. the defendant is guilty), each player assigns the probability p to any other player's receiving the signal g, and the probability $1 - p$ to her receiving the signal b, independently of all other players' signals. Similarly, in any state in which $X = I$ (i.e. the defendant is innocent), each player assigns the probability q to any other player's receiving the signal b, and the probability $1 - q$ to her receiving the signal g, independently of all other players' signals.

Each player cares about the verdict, which depends on the players' actions, and the defendant's true status. Given the assumption that unanimity is required to convict the defendant, only the action profile $(C, \ldots, C)$ leads to conviction. Thus (300.1) implies that player i's payoff function in the Bayesian game is defined as follows.

$$u_i(a,\omega) = \begin{cases} 0 & \text{if } a \neq (C,\ldots,C) \text{ and } \omega_1 = I \text{ or} \\ & \quad \text{if } a = (C,\ldots,C) \text{ and } \omega_1 = G \\ -z & \text{if } a = (C,\ldots,C) \text{ and } \omega_1 = I \\ -(1-z) & \text{if } a \neq (C,\ldots,C) \text{ and } \omega_1 = G, \end{cases} \tag{301.1}$$

where ω_1 is the first component of the state, giving the defendant's true status.

In summary, the following Bayesian game models the situation.

Players A set of n jurors.

States The set of states is the set of all lists $(X, s_1, \ldots, s_n)$ where $X \in \{G, I\}$ and $s_j \in \{g, b\}$ for every juror j, where $X = G$ if the defendant is guilty, $X = I$ if she is innocent, $s_j = g$ if player j receives the signal that she is guilty, and $s_j = b$ if player j receives the signal that she is innocent.

Actions The set of actions of each player is $\{C, Q\}$, where C means vote to convict, and Q means vote to acquit.

Signals The set of signals that each player may receive is $\{g, b\}$ and player j's signal function is defined by $\tau_j(X, s_1, \ldots, s_n) = s_j$ (each juror is informed only of her own signal).

Beliefs Type g of any player i believes that the state is $(G, s_1, \ldots, s_n)$ with probability $\pi p^{k-1}(1-p)^{n-k}$ and $(I, s_1, \ldots, s_n)$ with probability $(1-\pi)(1-q)^{k-1}q^{n-k}$, where k is the number of players j (including i) for whom $s_j = g$ in each case. Type b of any player i believes that the state is $(G, s_1, \ldots, s_n)$ with probability $\pi p^k(1-p)^{n-k-1}$ and $(I, s_1, \ldots, s_n)$ with probability $(1-\pi)(1-q)^{k-1}q^{n-k-1}$, where k is the number of players j for whom $s_j = g$ in each case.

Payoff functions The Bernoulli payoff function of each player i is given in (301.1).

9.8.2 Nash equilibrium

One juror Start by considering the very simplest case, in which there is a single juror. Suppose that her signal is b. To determine whether she prefers conviction or acquittal we need to find the probability she assigns to the defendant's being guilty, given her signal. We can find this probability, denoted $\Pr(G \mid b)$, by using Bayes' Rule (see Section 17.7.5, in particular (454.2)), as follows.

$$\begin{aligned} \Pr(G \mid b) &= \frac{\Pr(b \mid G)\Pr(G)}{\Pr(b \mid G)\Pr(G) + \Pr(b \mid I)\Pr(I)} \\ &= \frac{(1-p)\pi}{(1-p)\pi + q(1-\pi)} \end{aligned}$$

Thus by (300.2), acquittal yields an expected payoff at least as high as does conviction if and only if

$$z \geq \frac{(1-p)\pi}{(1-p)\pi + q(1-\pi)}.$$

That is, after getting the signal that the defendant is innocent, the juror chooses acquittal as long as z is not too small—as long as she is too concerned about acquitting a guilty defendant. If her signal is g then a similar calculation leads to the conclusion that conviction yields an expected payoff at least as high as does acquittal if

$$z \leq \frac{p\pi}{p\pi + (1-q)(1-\pi)}.$$

Thus if

$$\frac{(1-p)\pi}{(1-p)\pi + q(1-\pi)} \leq z \leq \frac{p\pi}{p\pi + (1-q)(1-\pi)} \tag{302.1}$$

then the juror optimally acts according to her signal, acquitting the defendant when her signal is b and convicting her when it is g. (A bit of algebra shows that the term on the left of (302.1) is less than the term on the right, given $p > 1 - q$.)

Two jurors Now suppose there are two jurors. Are there values for z such that the game has a Nash equilibrium in which each juror votes according to her signal? Suppose that juror 2 acts in this way: type b votes to acquit, and type g votes to convict. Consider type b of juror 1. If juror 2's signal is b, juror 1's vote has no effect on the outcome, because juror 2 votes to acquit and unanimity is required for conviction. Thus when deciding how to vote, juror 1 should ignore the possibility that juror 2's signal is b, and assume it is g. That is, juror 1 should take as evidence her signal and the fact that juror 2's signal is g. Hence, given (300.2), for type b of juror 1 acquittal is at least as good as conviction if the probability that the defendant is guilty, given juror 1's signal is b and juror 2's signal is g, is at most z. This probability is

$$\begin{aligned}\Pr(G \mid b, g) &= \frac{\Pr(b, g \mid G)\Pr(G)}{\Pr(b, g \mid G)\Pr(G) + \Pr(b, g \mid I)\Pr(I)} \\ &= \frac{(1-p)p\pi}{(1-p)p\pi + q(1-q)(1-\pi)}.\end{aligned}$$

Thus type b of juror 1 optimally votes for acquittal if

$$z \geq \frac{(1-p)p\pi}{(1-p)p\pi + q(1-q)(1-\pi)}.$$

By a similar argument, for type g of juror 1 conviction is at least as good as acquittal if

$$z \leq \frac{p^2\pi}{p^2\pi + (1-q)^2(1-\pi)}.$$

Thus when there are two jurors, the game has a Nash equilibrium in which each juror acts according to her signal, voting to acquit the defendant when her signal is b and to convict her when it is g, if

$$\frac{(1-p)p\pi}{(1-p)p\pi + q(1-q)(1-\pi)} \leq z \leq \frac{p^2\pi}{p^2\pi + (1-q)^2(1-\pi)}. \tag{302.2}$$

Consider the expressions on the left of (302.1) and (302.2). Divide the numerator and denominator of the expression on the left of (302.1) by $1 - p$ and the numerator and denominator of the expression on the left of (302.2) by $(1-p)p$. Then, given $p > 1 - q$, we see that the expression on the left of (302.2) is greater than the expression on the left of (302.1). That is, the lowest value of z for which an equilibrium exists in which each juror votes according to her signal is higher when there are two jurors than when there is only one juror. Why? Because a juror who receives the signal b, knowing that her vote makes a difference only if the other juror votes to convict, makes her decision on the assumption that the other juror's signal is g, and so is less worried about convicting an innocent defendant than is a single juror in isolation.

Many jurors Now suppose the number of jurors is arbitrary, equal to n. Suppose that every juror other than juror 1 votes to acquit when her signal is b and to convict when her signal is g. Consider type b of juror 1. As in the case of two jurors, juror 1's vote has no effect on the outcome unless every other juror's signal is g. Thus when deciding how to vote, juror 1 should assume that all the other signals are g. Hence, given (300.2), for type b of juror 1 acquittal is at least as good as conviction if the probability that the defendant is guilty, given juror 1's signal is b and every other juror's signal is g, is at most z. This probability is

$$\begin{aligned}\Pr(G \mid b, g, \ldots, g) &= \frac{\Pr(b, g, \ldots, g \mid G)\Pr(G)}{\Pr(b, g, \ldots, g \mid G)\Pr(G) + \Pr(b, g, \ldots, g \mid I)\Pr(I)} \\ &= \frac{(1-p)p^{n-1}\pi}{(1-p)p^{n-1}\pi + q(1-q)^{n-1}(1-\pi)}.\end{aligned}$$

Thus type b of juror 1 optimally votes for acquittal if

$$\begin{aligned}z &\geq \frac{(1-p)p^{n-1}\pi}{(1-p)p^{n-1}\pi + q(1-q)^{n-1}(1-\pi)} \\ &= \frac{1}{1 + \dfrac{q}{1-p}\left(\dfrac{1-q}{p}\right)^{n-1}\dfrac{1-\pi}{\pi}}.\end{aligned}$$

Now, given that $p > 1 - q$, the denominator decreases to 1 as n increases. Thus the lower bound on z for which type b of juror 1 votes for acquittal approaches 1 as n increases. (You may check that if $p = q = 0.8$, $\pi = 0.5$, and $n = 12$, the lower bound on z exceeds 0.999999.) In particular, in a large jury, if jurors care even slightly about acquitting a guilty defendant then a juror who interprets the evidence to point to innocence will nevertheless vote for conviction. The reason is that the vote of a juror who interprets the evidence to point to innocence makes a difference to the outcome only if every other juror interprets the evidence to point to guilt, in which case the probability that the defendant is in fact guilty is very high.

We conclude that the model of a large jury in which the jurors are concerned about acquitting a guilty defendant has no Nash equilibrium in which every juror votes according to her signal. What *are* its equilibria? You are asked to find the conditions for two equilibria in the next exercise.

? EXERCISE 304.1 (Signal-independent equilibria in a model of a jury) Find conditions under which the game, for an arbitrary number of jurors, has a Nash equilibrium in which every juror votes for acquittal regardless of her signal, and conditions under which every juror votes for conviction regardless of her signal.

Under some conditions on z the game has in addition a symmetric mixed strategy Nash equilibrium in which each type g juror votes for conviction, and each type b juror votes for acquittal and conviction each with positive probability. Denote by β the mixed strategy of each juror of type b. As before, a juror's vote affects the outcome only if all other jurors vote for conviction, so when choosing an action a juror should assume that all other jurors vote for conviction.

Each type b juror must be indifferent between voting for conviction and voting for acquittal, because she takes each action with positive probability. By (300.2) we thus need the mixed strategy β to be such that the probability that the defendant is guilty, given that all other jurors vote for conviction, is equal to z. Now, the probability of any given juror's voting for conviction is $p + (1-p)\beta(C)$ if the defendant is guilty and $1 - q + q\beta(C)$ if she is innocent. Thus

$$\begin{aligned}
&\Pr(G \mid \text{signal } b \text{ and } n-1 \text{ votes for } C)\\
&\quad= \frac{\Pr(b \mid G)(\Pr(\text{vote for } C \mid G))^{n-1}\Pr(G)}{\Pr(b \mid G)(\Pr(\text{vote for } C \mid G))^{n-1}\Pr(G) + \Pr(b \mid I)(\Pr(\text{vote for } C \mid I))^{n-1}\Pr(I)}\\
&\quad= \frac{(1-p)(p+(1-p)\beta(C))^{n-1}\pi}{(1-p)(p+(1-p)\beta(C))^{n-1}\pi + q(1-q+q\beta(C))^{n-1}(1-\pi)}.
\end{aligned}$$

The condition that this probability equals z implies

$$(1-p)(p+(1-p)\beta(C))^{n-1}\pi(1-z) = q(1-q+q\beta(C))^{n-1}(1-\pi)z \qquad (304.2)$$

and hence

$$\beta(C) = \frac{pX-(1-q)}{q-(1-p)X},$$

where $X = [\pi(1-p)(1-z)/((1-\pi)qz)]^{1/(n-1)}$. For a range of parameter values, $0 \le \beta(C) \le 1$, so that $\beta(C)$ is indeed a probability. Notice that when n is large, X is close to 1, and hence $\beta(C)$ is close to 1: a juror who interprets the evidence as pointing to innocence very likely nonetheless votes for conviction.

Each type g juror votes for conviction, and so must get an expected payoff at least as high from conviction as from acquittal. From an analysis like that for each type b juror, this condition is

$$p(p+(1-p)\beta(C))^{n-1}\pi(1-z) \ge (1-q)(1-q+q\beta(C))^{n-1}(1-\pi)z.$$

Given $p > \frac{1}{2}$ and $q > \frac{1}{2}$, this condition follows from (304.2).

An interesting property of this equilibrium is that the probability that an innocent defendant is convicted *increases* as n increases: the larger the jury, the *more* likely an innocent defendant is to be convicted. (The proof of this result is not simple.)

Variants The key point behind the results is that under unanimity rule a juror's vote makes a difference to the outcome only if every other juror votes for conviction. Consequently, a juror, when deciding how to vote, rationally assesses the defendant's probability of guilt under the assumption that every other juror votes for conviction. The fact that this implication of unanimity rule drives the results suggests that the Nash equilibria might be quite different if less than unanimity were required for conviction. The analysis of such rules is difficult, but indeed the Nash equilibria they generate differ significantly from the Nash equilibria under unanimity rule. In particular, the analog of the mixed strategy Nash equilibria considered above generate a probability that an innocent defendant is convicted that approaches zero as the jury size increases, as Feddersen and Pesendorfer (1998) show.

The idea behind the equilibria of the model in the next exercise is related to the ideas in this section, though the model is different.

? EXERCISE 305.1 (Swing voter's curse) Whether candidate 1 or candidate 2 is elected depends on the votes of two citizens. The economy may be in one of two states, A and B. The citizens agree that candidate 1 is best if the state is A and candidate 2 is best if the state is B. Each citizen's preferences are represented by the expected value of a Bernoulli payoff function that assigns a payoff of 1 if the best candidate for the state wins (obtains more votes than the other candidate), a payoff of 0 if the other candidate wins, and payoff of $\frac{1}{2}$ if the candidates tie. Citizen 1 is informed of the state, whereas citizen 2 believes it is A with probability 0.9 and B with probability 0.1. Each citizen may either vote for candidate 1, vote for candidate 2, or not vote.

a. Formulate this situation as a Bayesian game. (Construct the table of payoffs for each state.)

b. Show that the game has exactly two pure Nash equilibria, in one of which citizen 2 does not vote and in the other of which she votes for 1.

c. Show that one of the player's actions in the second of these equilibria is weakly dominated.

d. Why is the "swing voter's curse" an appropriate name for the determinant of citizen 2's decision in the second equilibrium?

9.9 Appendix: Analysis of auctions for an arbitrary distribution of valuations

9.9.1 *First-price sealed-bid auctions*

In this section I construct a symmetric equilibrium of a first-price sealed-bid auction for an arbitrary distribution F of valuations that satisfies the assumptions in Section 9.7.2. (Unlike the remainder of the book, the section uses calculus.)

The method I use to find the equilibrium is the same as the one used previously: first I find conditions satisfied by the players' best response functions, then impose the equilibrium condition that the bid of each type of each player be a best response to the bids of each type of every other player.

As before, denote the bid of type v_i of player i (i.e. player i when her valuation is v_i) by $\beta_i(v_i)$. In a symmetric equilibrium we have $\beta_i = \beta$ for every player i. A reasonable guess is that in an equilibrium the common bidding function β is increasing: bidders with higher valuations bid more. I start by making this assumption. After finding a possible equilibrium, I check that in fact the bidding function has this property.

Each player is uncertain about the other players' valuations, and hence is uncertain about the bids they will make, even though she knows the bidding function β. Denote by $G_\beta(b)$ the probability that, given β, any given player's bid is at most b. Under my assumption that β is increasing, a player's bid is at most b if and only if her valuation is at most $\beta^{-1}(b)$ (where β^{-1} is the inverse of β). Thus

$$G_\beta(b) = \Pr\{v \leq \beta^{-1}(b)\} = F(\beta^{-1}(b)).$$

Now, the expected payoff of a player with valuation v who bids b when all other players act according to the bidding function β is

$$(v - b)\Pr\{\text{Highest bid is } b\}. \tag{306.1}$$

The probability $\Pr\{\text{Highest bid is } b\}$ is equal to the probability that all the valuations of the other $n-1$ bidders are less than b, which is $(G_\beta(b))^{n-1}$. Thus the expected payoff in (306.1) is

$$(v - b)(G_\beta(b))^{n-1}. \tag{306.2}$$

Consider the best response function of each type of an arbitrary player. Denote the optimal bid by a player with valuation v, given that all the other players use the bidding function β, by $B_v(\beta)$. This bid maximizes the expected payoff in (306.2), and thus satisfies the condition that the derivative of this payoff with respect to b is zero:

$$-(G_\beta(B_v(\beta)))^{n-1} + (v - B_v(\beta))(n-1)(G_\beta(B_v(\beta)))^{n-2}G'_\beta(B_v(\beta)) = 0. \tag{306.3}$$

For $(\beta^*, \ldots, \beta^*)$ to be a Nash equilibrium, we need

$$B_v(\beta^*) = \beta^*(v) \text{ for all } v.$$

That is, for every valuation v, the best response of a player with valuation v when every other player acts according to β^* must be precisely $\beta^*(v)$.

Now, from the definition of G_β we have $G_{\beta^*}(B_v(\beta^*)) = F(\beta^{*-1}(\beta^*(v)) = F(v)$, and, for any β,

$$G'_\beta(b) = F'(\beta^{-1}(b))(\beta^{-1})'(b) = \frac{F'(\beta^{-1}(b))}{\beta'(\beta^{-1}(b))}.$$

Hence $G'_{\beta^*}(\beta^*(v)) = F'(v)/\beta^{*\prime}(v)$. Thus we deduce from (306.3) that an equilibrium bidding function β^* satisfies

$$-(F(v))^{n-1} + (v - \beta^*(v))(n-1)(F(v))^{n-2}F'(v)/\beta^{*\prime}(v) = 0,$$

or

$$\beta^{*\prime}(v)(F(v))^{n-1} + (n-1)\beta^*(v)(F(v))^{n-2}F'(v) = (n-1)v(F(v))^{n-2}F'(v).$$

We may solve this differential equation by noting that the left-hand side is precisely the derivative with respect to v of $\beta^*(v)(F(v))^{n-1}$. Thus integrating both sides we obtain

$$\begin{aligned}\beta^*(v)(F(v))^{n-1} &= \int_{\underline{v}}^{v} (n-1)x(F(x))^{n-2}F'(x)\,dx \\ &= v(F(v))^{n-1} - \int_{\underline{v}}^{v} (F(x))^{n-1}\,dx\end{aligned}$$

(using integration by parts to obtain the second line). Hence

$$\beta^*(v) = v - \frac{\int_{\underline{v}}^{v}(F(x))^{n-1}\,dx}{(F(v))^{n-1}}. \tag{307.1}$$

⁇ EXERCISE 307.2 (Properties of the bidding function in a first-price auction) Show that the bidding function defined in (307.1) is increasing in v for $v > \underline{v}$. Show also that a bidder with the lowest possible valuation bids her valuation, whereas a bidder with any other valuation bids less than her valuation: $\beta^*(\underline{v}) = \underline{v}$ and $\beta^*(v) < v$ for all $v > \underline{v}$ (use L'Hôpital's rule).

? EXERCISE 307.3 (Example of Nash equilibrium in a first-price auction) Verify that for the distribution F uniform from 0 to 1 the bidding function defined by (307.1) is $(1 - 1/n)v$.

The alternative expression for the Nash equilibrium bidding function discussed in the text may be derived as follows. As before, denote by $\mathbf{X}$ the random variable equal to the highest of $n-1$ independent valuations, each with cumulative distribution function F. The cumulative distribution function of $\mathbf{X}$ is H defined by $H(x) = (F(x))^{n-1}$. Thus the expected value of $\mathbf{X}$, conditional on its being less than v, is

$$\begin{aligned}\mathrm{E}[\mathbf{X} \mid \mathbf{X} < v] &= \frac{\int_{\underline{v}}^{v} xH'(x)\,dx}{H(v)} \\ &= \frac{\int_{\underline{v}}^{v}(n-1)x(F(x))^{n-2}F'(x)\,dx}{(F(v))^{n-1}},\end{aligned}$$

which is precisely $\beta^*(v)$. (Integrating the numerator by parts.) That is, $\beta^*(v) = \mathrm{E}[\mathbf{X} \mid \mathbf{X} < v]$.

9.9.2 Revenue equivalence of auctions

I argued in the text that the expected price paid by the winner of a first-price auction is the same as the expected price paid by the winner of a second-price auction. A much more general result may be established.

Suppose that n risk neutral bidders are involved in a sealed-bid auction in which the price is an arbitrary function of the bids (not necessarily the highest, or second highest). Each player's bid affects the probability p that she wins and the expected amount $e(p)$ that she pays. Thus we can think of each bidder's choosing a value of p, and can formulate the problem of a bidder with valuation v as

$$\max_p (p \cdot v - e(p)).$$

Denote the solution of this problem by $p^*(v)$. Assuming that e is differentiable, the first-order condition for this problem implies that

$$v = e'(p^*(v)) \text{ for all } v.$$

Integrating both sides of this equation we have

$$e(p^*(v)) = e(p^*(\underline{v})) + \int_{\underline{v}}^{v} x \, dp^*(x). \tag{308.1}$$

Now consider an equilibrium with the property that the object is sold to the bidder with the highest valuation, so that $p^*(v) = \Pr\{\mathbf{X} < v\}$, and the expected payoff $e(p^*(\underline{v})) = 0$ of a bidder with the lowest possible valuation is zero. In any such equilibrium, (308.1) implies that the expected payment $e(p^*(v))$ of a bidder with any given valuation v is independent of the price-determination rule in the auction, equal to $\Pr(\mathbf{X} < v)\mathrm{E}[\mathbf{X} \mid \mathbf{X} < v]$.

This result generalizes the earlier observation that the expected payments of bidders in the Nash equilibria of first- and second-price auctions in which the bidders' valuations are independent and private are the same. It is a special case of the more general *revenue equivalence principle*, which applies to a class of common value auctions, as well as private value auctions, and may be stated as follows.

> *Suppose that each bidder (i) is risk neutral, (ii) independently receives a signal from the same distribution, which satisfies the conditions on the distribution of valuations in Section 9.7.2, and (iii) has a valuation that may depend on all the bidders' signals. Consider auction mechanisms in the symmetric Nash equilibria of which the object is sold to the bidder with the highest signal and the expected payoff of a bidder with the lowest possible valuation is zero. In the symmetric Nash equilibrium of any such mechanism the expected payment of a bidder of any given type is the same, and hence the auctioneer's expected revenue is the same.*

Notes

The notion of a general Bayesian game was defined and studied by Harsanyi (1967/68). The formulation I describe here is taken (with a minor change) from Osborne and Rubinstein (1994, Section 2.6).

The origin of the observation that more information may hurt (Section 9.4.1) is unclear. The idea of "infection" in Section 9.4.2 was first studied by Rubinstein (1989). The game in Figure 282.1 is a variant suggested by Eddie Dekel of the one analyzed by Morris, Rob, and Shin (1995).

Games modeling voluntary contributions to a public good were first considered by Olson (1965, Section I.D), and have been subsequently much studied. The model in Section 9.6 is a variant of one in an unpublished paper of William F. Samuelson dated 1984.

Vickrey (1961) initiated the study of auctions described in Section 9.7. First-price common value auctions (Section 9.7.3) were first studied by Wilson (1967, 1969, 1977). The "winner's curse" appears to have been first articulated by Capen, Clapp, and Campbell (1971). The general revenue equivalence principle at the end of Section 9.9.2 is due to Myerson (1981) and Riley and Samuelson (1981); their results are generalized by Bulow and Klemperer (1996, Lemma 3). The equilibria in Exercise 297.1 are described by Milgrom (1981, Theorem 6.3). The literature is surveyed by Klemperer (1999). The box on spectrum auctions on page 298 is based on McMillan (1994), Cramton (1995, 1997, 1998), and McAfee and McMillan (1996).

Section 9.8 is based on Austen-Smith and Banks (1996) and Feddersen and Pesendorfer (1996).

Exercise 280.2 was suggested by Ariel Rubinstein. Exercise 280.3 is based on Brams, Kilgour, and Davis (1993). A model of adverse selection was first studied by Akerlof (1974); the model in Exercise 280.4 is taken from Samuelson and Bazerman (1985). Exercise 305.1 is based on Feddersen and Pesendorfer (1996).

11 Strictly Competitive Games and Maxminimization

Prerequisite: Chapters 2 and 4.

11.1 Introduction

THE NOTION of Nash equilibrium (studied in Chapters 2, 3, and 4) models a steady state. The idea is that each player, through her experience playing the game against various opponents, knows the actions that the other players in the game will take, and chooses her action in light of this knowledge.

In this chapter and the next, we study the likely outcome of a game from a different angle. We consider the implications of each player's forming a belief about the other players' actions not from her experience, but from her analysis of the game.

In this chapter we focus on two-player strictly competitive games, in which the players' interests are diametrically opposed. In such games a simple decision-making procedure leads each player to choose a Nash equilibrium action.

11.2 Definitions and examples

You are confronted with a game for the first time; you have no idea what actions your opponents will take. How should you choose your action? A conservative criterion entails your working under the assumption that whatever you do, your opponents will take the worst possible action for you. For each of your actions, you look at all the outcomes that can occur, as the other players choose different actions, and find the one that gives you the lowest payoff. Then you choose the action for which this lowest payoff is largest. This procedure for choosing an action is called **maxminimization**.

Many of the interesting examples of this procedure involve mixed strategies, so from the beginning I define the concepts for a strategic game with vNM preferences (Definition 103.1), though the ideas do not depend upon the players' randomizing. Let U_i be an expected payoff function that represents player i's preferences on lotteries over action profiles in a strategic game. For any given mixed strategy α_i of player i, the lowest payoff that she obtains, for any possible vector α_{-i}

of mixed strategies of the other players, is

$$\min_{\alpha_{-i}} U_i(\alpha_i, \alpha_{-i}).$$

A maxminimizing mixed strategy for player i is a mixed strategy that maximizes this minimal payoff.

▶ DEFINITION 336.1 A **maxminimizing mixed strategy** for player i in a strategic game (with vNM payoffs) is a mixed strategy α_i^* that solves the problem

$$\max_{\alpha_i} \min_{\alpha_{-i}} U_i(\alpha_i, \alpha_{-i}),$$

where U_i is player i's vNM payoff function.

In words, a maxminimizing strategy for player i maximizes her payoff under the (pessimistic) assumption that whatever she does the other players will act in such a way as to minimize her expected payoff.

A different way of looking at a maxminimizing strategy is useful. Say that a mixed strategy α_i **guarantees** player i the payoff $\overline{u}_i$ if, no matter what mixed strategies α_{-i} the other players use, i's payoff is at least $\overline{u}_i$:

$$u_i(\alpha_i, \alpha_{-i}) \geq \overline{u}_i \text{ for every list } \alpha_{-i} \text{ of the other players' mixed strategies.}$$

A maxminimizing mixed strategy maximizes the payoff that a player can guarantee: if α_i^* is a maxminimizer then

$$\min_{\alpha_{-i}} u_i(\alpha_i^*, \alpha_{-i}) \geq \min_{\alpha_{-i}} u_i(\alpha_i, \alpha_{-i}) \text{ for every mixed strategy } \alpha_i \text{ of player } i.$$

◆ EXAMPLE 336.2 (Maxminimizers in a bargaining game) Consider the game in Exercise 36.2, restricting attention to pure strategies (actions). If you demand any amount x up to \$5 then your payoff is x regardless of the other player's action. If you demand \$6 then you may get \$6 (if the other player demands \$4 or less, or \$7 or more), but you may get only \$5 (if the other player demands \$5 or \$6). If you demand $x \geq \$7$ then you may get x (if the other player demands at most $\$(10-x)$), but you may get only $\$(11-x)$ (if the other player demands $x-1$). For each amount that you can demand, the smallest amount that you may get is given in Figure 337.1. Maxminimization in this game thus leads each player to demand either \$5 or \$6 (for both of which the worst possible outcome is that the player receives \$5).

Why should you assume that the other players will take actions that minimize your payoff? In some games such an assumption is not sensible. But if you have only one opponent and her interests in the game are diametrically opposed to yours—in which case we call the game *strictly competitive*—then the assumption may be reasonable. In fact, it turns out that in such games there is a very close relationship between the outcome that occurs if each player maxminimizes and

Amount demanded	0	1	2	3	4	5	6	7	8	9	10
Smallest amount obtained	0	1	2	3	4	5	5	4	3	2	1

Figure 337.1 The lowest payoffs that a player receives in the game in Exercise 36.2 for each of her possible actions, as the other player's action varies.

the Nash equilibrium outcome. Another reason that you may be attracted to a maxminimizing action is that such an action maximizes the payoff that you can guarantee: there is no other action that yields a higher payoff no matter what the other players do.

In the game in Example 336.2 we restricted attention to pure strategies. The following example shows that a player may be able to guarantee a higher payoff by using a mixed strategy, and illustrates how a maxminimizing mixed strategy may be found.

◆ EXAMPLE 337.1 (Example of maxminimizers) Consider the game in Figure 337.2. If player 1 chooses T then the worse that can happen is that player 2 chooses R; if player 1 chooses B then the worst that can happen is that player 2 chooses L. In both cases player 1's payoff is -1, so that if player 1 is restricted to choose either T or B then there is nothing to choose between them; both guarantee her a payoff of -1.

	L	R
T	$2, -2$	$-1, 1$
B	$-1, 1$	$1, -1$

Figure 337.2 The game in Example 337.1.

However, player 1 can do better if she randomizes between T and B. Let p be the probability she assigns to T. To find her maxminimizing mixed strategy it is helpful to refer to Figure 338.1. The upward-sloping line indicates player 1's expected payoff, as p varies, if player 2 chooses the action L; the downward-sloping line indicates player 1's expected payoff, as p varies, if player 2 chooses R. Player 1's expected payoff if player 2 randomizes lies between the two lines; in particular it lies above the lower line. Thus for each value of p, the lower of the two lines indicates the lowest payoff that player 1 can obtain if she chooses that value of p. That is, the lowest payoff that player 1 can obtain for each value of p is indicated by the heavy inverted V; the maxminimizing mixed strategy of player 1 is thus $p = \frac{2}{5}$, which yields her a payoff of $\frac{1}{5}$.

The maxminimizing mixed strategy of player 1 in this example has the property that it yields player 1 the same payoff whether player 2 chooses L or R. Note that the indifference here is different from that in a Nash equilibrium, in which player 1's mixed strategy yields *player 2* the same payoff to each of her actions.

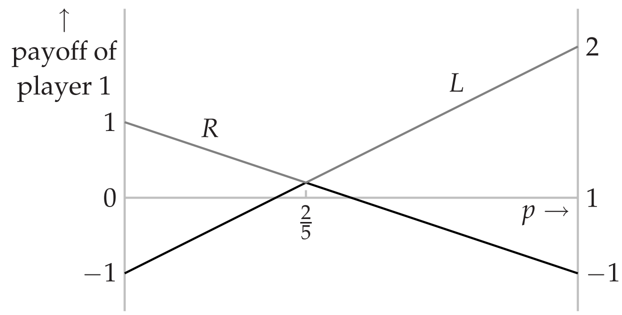


Figure 338.1 The expected payoff of player 1 in the game in Figure 337.2 for each of player 2's actions, as a function of the probability p that player 1 assigns to T.

What is the relation between Nash equilibrium strategies and maxminimizers? In the next section I show that for the class of strictly competitive games the relation is very close. In an *arbitrary* game, whether strictly competitive or not, a player's Nash equilibrium payoff is *at least* her maxminimized payoff.

■ LEMMA 338.1 *The payoff of each player in any Nash equilibrium of a strategic game is at least equal to her maxminimized payoff.*

Proof. Let (α_1^*, α_2^*) be a Nash equilibrium. Consider player 1. First note that by the definition of a Nash equilibrium,

$$U_1(\alpha_1^*, \alpha_2^*) \geq U_1(\alpha_1, \alpha_2^*) \text{ for every mixed strategy } \alpha_1 \text{ of player 1,}$$

so that

$$U_1(\alpha_1^*, \alpha_2^*) \geq \min_{\alpha_2} U_1(\alpha_1, \alpha_2) \text{ for every mixed strategy } \alpha_1 \text{ of player 1.}$$

Since the inequality holds for every mixed strategy α_1 of player 1, we conclude that

$$U_1(\alpha_1^*, \alpha_2^*) \geq \max_{\alpha_1} \min_{\alpha_2} U_1(\alpha_1, \alpha_2),$$

as required. □

? EXERCISE 338.2 (Nash equilibrium payoffs and maxminimized payoffs) Give an example of a game with a unique Nash equilibrium in which each player's Nash equilibrium payoff exceeds her maxminimized payoff.

11.3 Strictly competitive games

A strictly competitive game is a strategic game in which there are two players, whose preferences are diametrically opposed: whenever one player prefers some outcome a to another outcome b, the other players prefers b to a. Assume for convenience that the players' names are "1" and "2". If we restrict attention to pure strategies then we have the following definition.

▶ DEFINITION 339.1 (*Strictly competitive strategic game with ordinal preferences*) A strategic game with ordinal preferences is **strictly competitive** if it has two players and

$$(a_1, a_2) \succsim_1 (b_1, b_2) \text{ if and only if } (b_1, b_2) \succsim_2 (a_1, a_2),$$

where (a_1, a_2) and (b_1, b_2) are pairs of actions.

Note that it follows from this definition that in a strictly competitive game we have $(a_1, a_2) \sim_1 (b_1, b_2)$ if and only if $(a_1, a_2) \sim_2 (b_1, b_2)$ (since $(a_1, a_2) \sim_2 (b_1, b_2)$ implies both $(a_1, a_2) \succsim_1 (b_1, b_2)$ and $(b_1, b_2) \succsim_1 (a_1, a_2)$) and $(a_1, a_2) \succ_1 (b_1, b_2)$ if and only if $(b_1, b_2) \succ_2 (a_1, a_2)$.

Note also that there are payoff functions representing the players' preferences in a strictly competitive game with the property that the sum of the players' payoffs is zero for every action profile. (For example, we can assign payoffs as follows: 0 to both players for the worst outcome for player 1, 1 to player 1 and -1 to player 2 for the next worst outcome for player 1, and so on.) For this reason a strictly competitive game is sometimes referred to as a **zerosum** game.

The *Prisoner's Dilemma* (Figure 13.1) is not strictly competitive since both players prefer $(Quiet, Quiet)$ to $(Fink, Fink)$. *BoS* (Figure 16.1) is not strictly competitive either, since (for example) both players prefer (B, B) to (S, B). *Matching Pennies* (Figure 17.1), on the other hand, *is* strictly competitive: player 1's preference ordering over the four outcomes is precisely the reverse of player 2's. The game in Figure 339.1 is also strictly competitive: player 1's preference ordering is $(B, R) \succ_1 (T, L) \succ_1 (B, L) \succ_1 (T, R)$, the reverse of player 2's ordering $(T, R) \succ_1 (B, L) \succ_1 (T, L) \succ_1 (B, R)$.

	L	R
T	2, 1	0, 5
B	1, 3	5, 0

Figure 339.1 A strategic game. If attention is restricted to pure strategies then the game is strictly competitive. If mixed strategies are considered, however, it is not.

If we consider mixed strategies, then the appropriate definition of a strictly competitive game is the following.

▶ DEFINITION 339.2 (*Strictly competitive strategic game with vNM preferences*) A strategic game with vNM preferences is **strictly competitive** if it has two players and

$$U_1(\alpha_1, \alpha_2) \geq U_1(\beta_1, \beta_2) \text{ if and only if } U_2(\beta_1, \beta_2) \geq U_2(\alpha_1, \alpha_2),$$

where (α_1, α_2) and (β_1, β_2) are pairs of mixed strategies and U_i is player i's expected payoff as a function of the pair of mixed strategies (her vNM payoff function).

As for the case of games in which we restrict attention to pure strategies, there are payoff functions representing the players' preferences in a strictly competitive game with the property that the sum of the players' payoffs is zero for every action profile. To see this, let u_i, for each player i, represent i's preferences in a strictly competitive game. Denote by $\overline{a}$ and $\underline{a}$ the best and worst outcomes respectively for player 1. Now choose another representation v_i with the property that $v_1(\overline{a}) = 1$ and $v_1(\underline{a}) = 0$, and $v_2(\overline{a}) = -1$ and $v_2(\underline{a}) = 0$. (Why is it possible to do this?) Let a be any outcome and let $p = u_1(a)$. Then $u_1(a) = pu_1(\overline{a}) + (1-p)u_1(\underline{a})$. But since the game is strictly competitive we have $u_2(a) = pu_2(\underline{a}) + (1-p)u_2(\overline{a}) = -p$. Hence $u_1(a) + u_2(a) = 0$. Thus if player 2's preferences are not represented by the payoff function $-u_1$ then we know that the game is not strictly competitive.

Any game that is strictly competitive when we allow mixed strategies is clearly strictly competitive when we restrict attention to pure strategies, but the converse is not true. Consider, for example, the game in Figure 339.1, interpreting the numbers in the boxes as vNM payoffs. In this game player 1 is indifferent between the outcome (T, L) and the lottery in which (T, R) occurs with probability $\frac{3}{5}$ and (B, R) occurs with probability $\frac{2}{5}$ (since $\frac{3}{5} \cdot 0 + \frac{2}{5} \cdot 5 = 2$), but player 2 is not indifferent between these two outcomes (her payoff to (T, L) is 1, while her expected payoff to the lottery is $\frac{3}{5} \cdot 5 + \frac{2}{5} \cdot 0 = 3$).

? EXERCISE 340.1 (Determining strict competitiveness) Are either of the two games in Figure 340.1 strictly competitive (a) if we restrict attention to pure strategies and (b) if we allow mixed strategies?

	L	R
U	$1, -1$	$3, -5$
D	$2, -3$	$1, -1$

	L	R
U	$1, -1$	$3, -6$
D	$2, -3$	$1, -1$

Figure 340.1 The games in Exercise 340.1.

We saw above that in any game a player's Nash equilibrium payoff is at least her maxminimized payoff. I now show that for a strictly competitive game that possesses a Nash equilibrium, the two payoffs are the same: a pair of actions is a Nash equilibrium if and only if the action of each player is a maxminimizer. Denote player i's vNM payoff function by U_i and assume, without loss of generality, that $U_2 = -U_1$.

Though the proof may look complicated, the ideas it entails are very simple; the arguments involve no more than the manipulation of inequalities. The following fact is used in the argument. The maximum of any function f is equal to the negative of the minimum of $-f$: $\max_x f(x) = -\min_x(-f(x))$. It follows that

$$\begin{aligned}\max_{\alpha_2} \min_{\alpha_1} U_2(\alpha_1, \alpha_2) &= \max_{\alpha_2} \min_{\alpha_1}(-U_1(\alpha_1, \alpha_2)) \\ &= \max_{\alpha_2}(-\max_{\alpha_1} U_1(\alpha_1, \alpha_2))\end{aligned}$$

so that

$$\max_{\alpha_2} \min_{\alpha_1} U_2(\alpha_1, \alpha_2) = -\min_{\alpha_2} \max_{\alpha_1} U_1(\alpha_1, \alpha_2). \tag{340.2}$$

■ PROPOSITION 341.1 (Nash equilibrium strategies and maxminimizers of strictly competitive games) *Consider a strictly competitive strategic game with vNM preferences. Denote the vNM payoff function of each player i by* U_i.

a. *If* (α_1^*, α_2^*) *is a Nash equilibrium then* α_1^* *is a maxminimizer for player 1,* α_2^* *is a maxminimizer for player 2, and* $\max_{\alpha_1} \min_{\alpha_2} U_1(\alpha_1, \alpha_2) = \min_{\alpha_2} \max_{\alpha_1} U_1(\alpha_1, \alpha_2) = U_1(\alpha_1^*, \alpha_2^*)$.

b. *If* α_1^* *is a maxminimizer for player 1,* α_2^* *is a maxminimizer for player 2, and* $\max_{\alpha_1} \min_{\alpha_2} U_1(\alpha_1, \alpha_2) = \min_{\alpha_2} \max_{\alpha_1} U_1(\alpha_1, \alpha_2)$ *(and thus, in particular, if the game has a Nash equilibrium (see part a)), then* (α_1^*, α_2^*) *is a Nash equilibrium.*

Proof. I first prove part a. By the definition of Nash equilibrium we have

$$U_2(\alpha_1^*, \alpha_2^*) \geq U_2(\alpha_1^*, \alpha_2) \text{ for every mixed strategy } \alpha_2 \text{ of player 2}$$

or, since $U_2 = -U_1$,

$$U_1(\alpha_1^*, \alpha_2^*) \leq U_1(\alpha_1^*, \alpha_2) \text{ for every mixed strategy } \alpha_2 \text{ of player 2}.$$

Hence

$$U_1(\alpha_1^*, \alpha_2^*) = \min_{\alpha_2} U_1(\alpha_1^*, \alpha_2).$$

Now, the function on the right hand side of this equality is evaluated at the specific strategy α_1^*, so that its value is not more than the maximum as we vary α_1, namely $\max_{\alpha_1} \min_{\alpha_2} U_1(\alpha_1, \alpha_2)$. Thus we conclude that

$$U_1(\alpha_1^*, \alpha_2^*) \leq \max_{\alpha_1} \min_{\alpha_2} U_1(\alpha_1, \alpha_2).$$

Now, from Lemma 338.1 we have the opposite inequality: a player's Nash equilibrium payoff is at least her maxminimized payoff. Thus $U_1(\alpha_1^*, \alpha_2^*) = \max_{\alpha_1} \min_{\alpha_2} U_1(\alpha_1, \alpha_2)$, so that α_1^* is a maxminimizer for player 1.

An analogous argument for player 2 establishes that α_2^* is a maxminimizer for player 2 and $U_2(\alpha_1^*, \alpha_2^*) = \max_{\alpha_2} \min_{\alpha_1} U_2(\alpha_1, \alpha_2)$. From (340.2) we deduce that $U_1(\alpha_1^*, \alpha_2^*) = \min_{\alpha_2} \max_{\alpha_1} U_1(\alpha_1, \alpha_2)$, completing the proof of part *a*.

To prove part *b*, let

$$v^* = \max_{\alpha_1} \min_{\alpha_2} U_1(\alpha_1, \alpha_2) = \min_{\alpha_2} \max_{\alpha_1} U_1(\alpha_1, \alpha_2).$$

From (340.2) we have $\max_{\alpha_2} \min_{\alpha_1} U_2(\alpha_1, \alpha_2) = -v^*$. Since α_1^* is a maxminimizer for player 1 we have $U_1(\alpha_1^*, \alpha_2) \geq v^*$ for every mixed strategy a_2 of player 2; since α_2^* is a maxminimizer for player 2 we have $U_2(\alpha_1, \alpha_2^*) \geq -v^*$ for every mixed strategy α_1 of player 1. Letting $\alpha_2 = \alpha_2^*$ and $\alpha_1 = \alpha_1^*$ in these two inequalities we obtain $U_1(\alpha_1^*, \alpha_2^*) \geq v^*$ and $U_2(\alpha_1^*, \alpha_2^*) \geq -v^*$, or $U_1(\alpha_1^*, \alpha_2^*) \leq v^*$, so that $U_1(\alpha_1^*, \alpha_2^*) = v^*$. Thus

$$U_1(\alpha_1^*, \alpha_2) \geq U_1(\alpha_1^*, \alpha_2^*) \text{ for every mixed strategy } \alpha_2 \text{ of player 2,}$$

or

$$U_2(\alpha_1^*, \alpha_2) \leq U_2(\alpha_1^*, \alpha_2^*) \text{ for every mixed strategy } \alpha_2 \text{ of player 2.}$$

Similarly,

$$U_2(\alpha_1, \alpha_2^*) \geq U_2(\alpha_1^*, \alpha_2^*) \text{ for every mixed strategy } \alpha_1 \text{ of player 1,}$$

or

$$U_1(\alpha_1, \alpha_2^*) \leq U_1(\alpha_1^*, \alpha_2^*) \text{ for every mixed strategy } \alpha_1 \text{ of player 1,}$$

so that (α_1^*, α_2^*) is a Nash equilibrium of the game. □

This result is of interest not only because it shows the close relation between the Nash equilibria and maxminimizers in a strictly competitive game, but also because it reveals properties of Nash equilibria in a strictly competitive game that are independent of the notion of maxminimization.

First, part *a* of the result implies that the Nash equilibrium payoff of each player in a strictly competitive game is unique.

■ COROLLARY 342.1 *Every Nash equilibrium of a strictly competitive game yields the same pair of payoffs.*

As we have seen, this property of Nash equilibria is not necessarily satisfied in games that are not strictly competitive (consider *BoS* (Figure 16.1), for example).

Second, the result implies that a Nash equilibrium of a strictly competitive game can be found by solving the problem $\max_{\alpha_1} \min_{\alpha_2} U_1(\alpha_1, \alpha_2)$. Further, if we know player 1's equilibrium payoff then any mixed strategy that yields this payoff when player 2 uses any of her pure strategies solves the maxminimization problem, and hence is an equilibrium mixed strategy of player 1. This fact is sometimes useful when calculating the mixed strategy equilibria of a game when we know the equilibrium payoffs before we have found the equilibrium strategies (see, for example, Exercise 344.2).

Third, suppose that (α_1, α_2) and (α_1', α_2') are Nash equilibria of a strictly competitive game. Then by part *a* of the result the strategies α_1 and α_1' are maxminimizers for player 1, the strategies α_2 and α_2' are maxminimizers for player 2, and

$$\max_{\alpha_1} \min_{\alpha_2} U_1(\alpha_1, \alpha_2) = \min_{\alpha_2} \max_{\alpha_1} U_1(\alpha_1, \alpha_2) = U_1(\alpha_1, \alpha_2) = U_1(\alpha_1', \alpha_2').$$

But then by part *b* of the result both (α_1, α_2') and (α_1', α_2) are Nash equilibria of the game. That is, the result implies that Nash equilibria of a strictly competitive game have the following property.

■ COROLLARY 342.2 *The Nash equilibria of a strictly competitive game are interchangeable: if (α_1, α_2) and (α_1', α_2') are Nash equilibria then so are (α_1, α_2') and (α_1', α_2).*

The game *BoS* shows that the Nash equilibria of a game that is not strictly competitive are not necessarily interchangeable.

Part *a* of Proposition 341.1 shows that for any strictly competitive game that has a Nash equilibrium we have

$$\max_{\alpha_1} \min_{\alpha_2} U_1(\alpha_1, \alpha_2) = \min_{\alpha_2} \max_{\alpha_1} U_1(\alpha_1, \alpha_2).$$

Note that the inequality

$$\max_{\alpha_1} \min_{\alpha_2} U_1(\alpha_1, \alpha_2) \leq \min_{\alpha_2} \max_{\alpha_1} U_1(\alpha_1, \alpha_2)$$

holds more generally: for any α_1' we have $U_1(\alpha_1', \alpha_2) \leq \max_{\alpha_1} U_1(\alpha_1, \alpha_2)$ for all α_2, so that $\min_{\alpha_2} U_1(\alpha_1', \alpha_2) \leq \min_{\alpha_2} \max_{\alpha_1} U_1(\alpha_1, \alpha_2)$. Thus in *any* game (whether or not it is strictly competitive) the payoff that player 1 can guarantee herself is *at most* the amount that player 2 can hold her down to.

If $\max_{\alpha_1} \min_{\alpha_2} U_1(\alpha_1, \alpha_2) = \min_{\alpha_2} \max_{\alpha_1} U_1(\alpha_1, \alpha_2)$ then we say that this payoff, the equilibrium payoff of player 1, is the **value** of the game. An implication of Proposition 341.1 is that any equilibrium strategy of player 1 *guarantees* that her payoff is *at least* v^*, and any equilibrium strategy of player 2 *guarantees* that player 1's payoff is *at most* v^*.

■ COROLLARY 343.1 *Any Nash equilibrium strategy of player 1 in a strictly competitive game guarantees that her payoff is at least the value of the game, and any Nash equilibrium strategy of player 2 guarantees that player 1's payoff is at most the value.*

Proof. For $i = 1, 2$, let α_i^* be an equilibrium strategy of player i and let v^* be the value of the game. By Proposition 341.1*a*, α_1^* is a maxminimizer, so that it guarantees that player 1's payoff is at least v^*:

$$U_1(\alpha_1^*, \alpha_2) \geq \min_{\alpha_2} U_1(\alpha_1^*, \alpha_2) = \max_{\alpha_1} \min_{\alpha_2} U_1(\alpha_1^*, \alpha_2) = v^*.$$

Similarly, any equilibrium strategy of player 2 guarantees that her payoff is at least her equilibrium payoff $-v^*$; or, equivalently, any equilibrium strategy of player 2 guarantees that player 1's payoff is at most v^*. □

In a game that is not strictly competitive a player's equilibrium strategy does not in general have these properties, as the following exercise shows.

? EXERCISE 343.2 (Maxminimizers in *BoS*) For the game *BoS* (Figure 16.1) find the maxminimizer of each player. Show for each equilibrium, the strategy of neither player guarantees her equilibrium payoff.

? EXERCISE 343.3 (Increasing payoffs and eliminating actions in strictly competitive games) Let G be a strictly competitive game that has a Nash equilibrium.

a. Show that if some of player 1's payoffs in G are increased in such a way that the resulting game G' is strictly competitive then G' has no equilibrium in which player 1 is worse off than she was in an equilibrium of G. (Note that G' may have no equilibrium at all.)

b. Show that the game that results if player 1 is prohibited from using one of her actions in G does not have an equilibrium in which player 1's payoff is higher than it is in an equilibrium of G.

c. Give examples to show that neither of the above properties necessarily holds for a game that is not strictly competitive.

? EXERCISE 344.1 (Equilibrium in strictly competitive games) Either prove or give a counterexample to the claim that if the equilibrium payoff of player 1 in a strictly competitive game is v then any strategy pair that gives player 1 a payoff of v is an equilibrium.

? EXERCISE 344.2 (Guessing Morra) In the two-player game "Guessing Morra", each player simultaneously holds up one or two fingers and also guesses the total shown. If exactly one player guesses correctly then the other player pays her the amount of her guess (in $, say). If either both players guess correctly or neither does so then no payments are made.

a. Specify this situation as a strategic game.

b. Use the symmetry of the game to show that the unique equilibrium payoff of each player is 0.

c. Find the mixed strategies of player 1 that guarantee that her payoff is at least 0, and hence find all the mixed strategy equilibria of the game.

? EXERCISE 344.3 (O'Neill's game) Consider the game in Figure 344.1.

a. Find a completely mixed Nash equilibrium in which each player assigns the same probability to the actions 1, 2, and 3.

b. Use the facts that in a strictly competitive game the players' equilibrium payoffs are unique and each player's equilibrium strategy guarantees her payoff is at least her equilibrium payoff to show that the equilibrium you found in part *a* is the only equilibrium of the game.

	1	2	3	J
1	$-1, 1$	$1, -1$	$1, -1$	$-1, 1$
2	$1, -1$	$-1, 1$	$1, -1$	$-1, 1$
3	$1, -1$	$1, -1$	$-1, 1$	$-1, 1$
J	$-1, 1$	$-1, 1$	$-1, 1$	$1, -1$

Figure 344.1 The game in Exercise 344.3.

MAXMINIMIZATION: SOME HISTORY

The theory of maxminimization in general strictly competitive games was developed by John von Neumann in the late 1920's. However, the idea of maxminimization in the context of a specific game appeared two centuries earlier. In 1713 or 1714

Pierre Rémond de Montmort, a Frenchman who "devoted himself to religion, philosophy, and mathematics" (Todhunter (1865, p. 78)) published *Essay d'analyse sur les jeux de hazard* (Analytical essay on games of chance), in which he reported correspondence with Nikolaus Bernoulli (a member of the Swiss family of scientists and mathematicians). Among the correspondence is a letter in which Montmort describes a letter (dated November 13, 1713) he received from "M. de Waldegrave" (probably Baron Waldegrave of Chewton, a British noble born and educated in France). Montmort, Bernoulli, and Waldegrave had been corresponding about the two-player card game *le Her* ("the gentleman").

This two player game uses an ordinary deck of cards. Each player is first dealt a single card, which she alone sees. The object is to hold a card with a higher value than your opponent, with the ace counted as 1 and the jack, queen, and king counted as 11, 12, and 13 respectively. After each player has received her card, player 1 can, if she wishes, exchange her card with that of player 2, who must make the exchange unless she holds a king, in which case she is automatically the winner. Then, whether or not player 1 exchanges her card, player 2 has the option of exchanging hers for a card randomly selected from the remaining cards in the deck; if the randomly selected card is a king she automatically loses, and otherwise she makes the exchange. Finally, the players compare their cards and the one whose card has the higher value wins; if both cards have the same value then player 2 wins.

We can view this situation as a strategic game in which an action for player 1 is a rule that says, for each possible card that she may receive, whether she *keeps* or *exchanges* the card. For example, one possible action is to *exchange* any card with value up to 5 and to *keep* any card with higher value; another possible action is to *exchange* any even card and to *keep* any odd card. Since there are 13 different values of cards, player 1 has 2^{13} actions. If player 1 exchanges her card then player 2 knows both cards being held, and she should clearly exchange with a random card from the deck if and only if the card she hold would otherwise lose. If player 1 does not exchange her card then player 2's decision of whether to exchange or not is not as clear. As for player 1 at the start of the game, an action of player 2 is a rule that says, for each possible card that she holds, whether to *keep* or *exchange* the card. Like player 1, player 2 has 2^{13} actions.

Montmort, Bernoulli, and Waldegrave had argued that the only actions that could possibly be optimal are "*exchange* up to 6 and *keep* 7 and over" or "*exchange* up to 7 and *keep* 8 and over" for player 1, and "*exchange* up to 7 and *keep* 8 and over" or "*exchange* up to 8 and *keep* 9 and over" for player 2. When the players are restricted to use only these actions the game is equivalent to

0, 0	5, −5
3, −3	0, 0

The three scholars had corresponded about which of these actions is best. As you can see, the best action for each player depends on the other player's action, and the game has no pure strategy Nash equilibrium. Waldegrave made the key conceptual leap of considering the possibility that the players randomize. He observed that if player 1 uses the mixed strategy $(\frac{3}{8}, \frac{5}{8})$ then her payoff is the same regardless of player 2's action, and guarantees her a payoff of $\frac{15}{8}$, and that if player 2 uses the mixed strategy $(\frac{5}{8}, \frac{3}{8})$ then she ensures that player 1's payoff is no more than $\frac{15}{8}$.

That is, Waldegrave found the maxminimizers for each player and appreciated their significance; Montmort wrote to Bernoulli that "it seems to me that [Waldegrave's letter] exhausts everything that one can say on [the players' behavior in *le Her*]".

The decision criterion of maxminimization seems to be conservative. In particular, in any game, a player's Nash equilibrium payoff is at least her maxminimized payoff. We have seen that in strictly competitive games the two are equal, and the notions of Nash equilibrium and maxminimizing yield the same predictions. In some games that are not strictly competitive the two payoffs are also equal. The next example gives such an example, in which the notions of Nash equilibrium and maxminimization do not yield the same outcome and, from a decision-theoretic viewpoint, a maxminimizer seems preferable to a Nash equilibrium strategy.

◆ EXAMPLE 346.1 (Maxminimizers *vs*. Nash equilibrium actions) The game in Figure 346.1 has a unique Nash equilibrium, in which player 1's strategy is $(\frac{1}{4}, \frac{3}{4})$ and player 2's strategy is $(\frac{2}{3}, \frac{1}{3})$. In this equilibrium player 1's payoff is 4.

	L	R
T	$6,0$	$0,6$
B	$3,2$	$6,0$

Figure 346.1 A strategic game.

Now consider the maxminimizer for player 1. Player 1's payoff as a function of the probability that she assigns to T is shown in Figure 347.1. We see that the maxminimizer for player 1 is $(\frac{1}{3}, \frac{2}{3})$, and this strategy guarantees player 1 a payoff of 4.

Thus in this game player 1's maxminimizer guarantees that she obtain her payoff in the unique equilibrium, while her equilibrium strategy does not. If player 1 is certain that player 2 will adhere to the equilibrium then her equilibrium strategy yields her equilibrium payoff of 4, but if player 2 chooses a different strategy then player 1's payoff may be less than 4 (it also may be greater than 4). Player 1's maxminimizer, on the other hand, guarantees a payoff of 4 regardless of player 2's behavior.

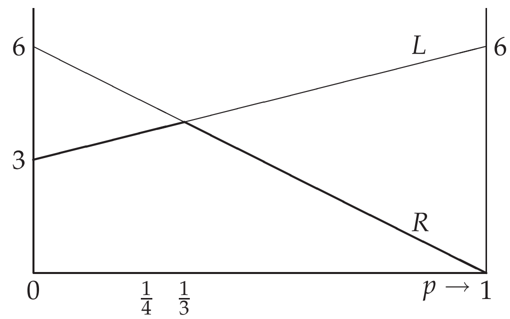


Figure 347.1 The expected payoff of player 1 in the game in Figure 346.1 for each of player 2's actions, as a function of the probability p that player 1 assigns to T.

TESTING THE THEORY OF NASH EQUILIBRIUM IN STRICTLY COMPETITIVE GAMES

The theory of maxminimization makes a sharp prediction about the outcome of a strictly competitive game. Does human behavior correspond to this prediction?

In designing an experiment, we face the problem in a general game of inducing the appropriate preferences. We can avoid this problem by working with games with only two outcomes. In such games the players' preferences are represented by the monetary payoffs of the outcomes, so that we do not need to control for subjects' risk attitudes.

O'Neill (1987) conducted an experiment with such a game. He confronted people with the game in Exercise 344.3, in which each player's equilibrium strategy is $(0.2, 0.2, 0.2, 0.4)$. In order to collect a large amount of data he had each of 25 pairs of people play the game 105 times. This design raises two issues. First, when players confront each other repeatedly, strategic possibilities that are absent from a one-shot game emerge: each player may condition her current action on her opponent's past actions. However, an analysis that takes into account these strategic options leads to the conclusion that, for the game used in the experiment, the players will eschew them. Second, in the experiment, each subject faced more than two possible outcomes. However, under the hypothesis that each player's preferences are separable between the different trials, these preferences in any trial are still represented by the expected monetary payoffs.

Each subject was given US\$2.50 in cash at the start of the game, was paid US\$0.05 for every win, and paid her opponent US\$0.05 for every loss. On average, subjects in the role of player 1 chose the actions with probabilities $(0.221, 0.215, 0.203, 0.362)$ and subjects in the role of player 2 chose them with probabilities $(0.226, 0.179, 0.169, 0.426)$. These observed frequencies seem fairly close to those predicted by the theory of maxminimization. But how can we measure closeness? A standard statistical test (χ^2) asks the question: if each player used exactly her equilibrium strategy, what is the probability of the observed frequencies deviating at least as much from the predicted ones? Applying this test to the aggregate data on the frequencies of the 16 possible outcomes of the game leads to minmax behavior being decisively rejected (the probability of a deviation from the prediction at least as large as that observed is less than 1 in 1,000). Other tests on O'Neill's data also reject the minmax hypothesis (Brown and Rosenthal (1990)).

In a variant of O'Neill's experiment, with considerably higher stakes and a somewhat more complicated game, the evidence also does not support maxminimization, although maxminimization explains the data better than two alternative theories (Rapoport and Boebel (1992)). (Of course, it's relatively easy to design a theory that works well in *one particular* game; in order to "understand" behavior we want a theory that works well in a large class of games.) In summary, the evidence so far tends not to support the theory of maxminimization, although no other theory is systematically superior.

Notes

The material in the box on page 344 is based on Todhunter (1865) and Kuhn (1968). Guilbaud (1961) rediscovered Monmort's report of Waldegrave's work.

12 Rationalizability

Prerequisite: Chapters 2 and 4.

12.1 Introduction

WHAT outcomes in a strategic game are consistent with the players' analyses of each others' rational behavior? The main solution notion we have studied so far, Nash equilibrium, is not designed to address this question, but rather models a steady state in which each player has learned the other players' actions from her long experience playing the game. In this chapter I discuss an approach to the question that considers players who carefully study a game, deducing their opponents' rational actions from their knowledge of their opponents' preferences and analyses of their opponents' reasoning about their rational actions.

Suppose that we model each player's decision problem as follows. She forms a probabilistic belief about the other players' actions, and chooses her action (or mixed strategy) to maximize her expected payoff given this probabilistic belief. We say that a player who behaves in this way is *rational*. Precisely, suppose that player i's preferences are represented by the expected value of the Bernoulli payoff function u_i. Denote by μ_i her probabilistic *belief* about the other players' actions: $\mu_i(a_{-i})$ is the probability she assigns to the collection a_{-i} of the other players' actions. Denote by $U_i(\alpha_i, a_{-i})$ player i's expected payoff when she uses the mixed strategy α_i and the other players' actions are given by a_{-i}.

► DEFINITION 349.1 A **belief** of player i about the other players' actions is a probability distribution over A_{-i}. Player i is **rational** if she chooses her mixed strategy α_i to solve the problem

$$\max_{\alpha_i} \sum_{a_{-i}} \mu_i(a_{-i}) U_i(\alpha_i, a_{-i}),$$

where μ_i is her belief about the other players' actions.

Suppose that each player's belief is correct—that is, the probability that it assigns to each collection of actions of the other players is the probability implied by their mixed strategies. Then a solution of each player's maximization problem is her Nash equilibrium strategy. That is, if each player's belief about the other players' behavior is correct then her equilibrium action is optimal for her. (Note, however, that some nonequilibrium actions may be optimal too.)

The assumption that each player's belief about the other players is correct is not very appealing if we imagine a player confronting a game in which she has little or no experience. In such a case the most that we might reasonably assume is that she knows (or at least assumes) that the other players are rational—that is, that the other players, like her, have beliefs and choose their actions to maximize their expected payoffs given these beliefs.

To think about the consequences of this assumption, consider a variant of the game in Exercise 36.2.

◆ EXAMPLE 350.1 (Rationalizable actions in a bargaining game) Two players split \$4 using the following procedure. Each announces an integral number of dollars. If the sum of the amounts named is at most \$4 then each player receives the amount she names. If the sum of the amounts named exceeds \$4 and both players name the same amount then each receives \$2. If the sum of the amounts named exceeds \$4 and the players name different amounts then the player who names the smaller amount receives that amount plus a small amount proportional to the difference between the amounts, and the other player receives the balance of the \$4. (That is, there is a small penalty for making a demand that is "excessive" relative to that of the other player.) In summary, the payoff of each player i is given by

$$\begin{cases} a_i & \text{if } a_1 + a_2 \leq 4 \\ 2 & \text{if } a_1 + a_2 > 4 \text{ and } a_i = a_j \\ 4 - a_j - (a_i - a_j)\epsilon & \text{if } a_1 + a_2 > 4 \text{ and } a_i > a_j, \\ a_i + (a_j - a_i)\epsilon & \text{if } a_1 + a_2 > 4 \text{ and } a_i < a_j, \end{cases}$$

where $\epsilon > 0$ is a small amount (less than 30 cents); the payoffs are shown in Figure 350.1.

	0	1	2	3	4
0	$0,0$	$0,1$	$0,2$	$0,3$	$0,4$
1	$1,0$	$1,1$	$1,2$	$1,3$	$1+3\epsilon, 3-3\epsilon$
2	$2,0$	$2,1$	$2,2$	$2+\epsilon, 2-\epsilon$	$2+2\epsilon, 2-2\epsilon$
3	$3,0$	$3,1$	$2-\epsilon, 2+\epsilon$	$2,2$	$3+\epsilon, 1-\epsilon$
4	$4,0$	$3-3\epsilon, 1+3\epsilon$	$2-2\epsilon, 2+2\epsilon$	$1-\epsilon, 3+\epsilon$	$2,2$

Figure 350.1 The players' payoffs in the game in Example 350.1.

Suppose that you, as a player in this game, hold a probabilistic belief about your opponent's action and choose an action that maximizes your expected payoff given this belief. I claim that, whatever your belief, you will not demand \$0. Why? Because if you do so then you receive \$0 *whatever* amount the other player names, while if instead you name \$1 then you receive at least \$1 *whatever* amount the other player names. Thus for *no* belief about the other player's behavior is it optimal for you to demand \$0. Without considering whether your belief about the other player's behavior is consistent with her being rational, we can conclude that if you maximize your payoff given some belief about the other player then you will not demand \$0. We say that a demand of \$0 is a *never best response*.

By a similar argument we can conclude that you will not demand \$1, whatever your belief. But you might demand \$2. Why? Because you might believe, for example, that the other player is sure to demand \$2 (that is, you might assign probability 1 to the other player's demanding \$2), in which case your best action is to demand \$2 (if you demand more than \$2 then you obtain less than \$2, since you pay a small penalty for making an excessive demand).

Is there any belief under which it is optimal for you to demand \$3? Yes: if you are sure that the other player will demand \$1 then it is optimal to demand \$3 (if you demand less then the sum of the demands will be less than \$4 and you will receive what you demand, while if you demand more then the sum of the demands will exceed \$4 and you will receive \$3 minus a small penalty). Similarly, if you are sure that the other player will demand \$0 then it is optimal for you to demand \$4.

In summary, any demand of at least \$2 is consistent with your choosing an action to maximize your expected payoff given some belief, while any smaller demand is not. Or, more succinctly,

> the only demands consistent with your being rational are \$2, \$3, and \$4.

Now take the argument one step further. Suppose that you work under the assumption that your adversary is rational. Then you can conclude that she will not demand less than \$2: for *any* belief that *she* holds about *you*, it is not optimal for her to demand less than \$2 (just as it is not optimal for you to demand less than \$2 if you are rational). But if she demands at least \$2 then it is not optimal for you to demand \$4, whatever belief you hold about her demand: you are better off demanding \$2 or \$3 than you are demanding \$4, whether you think your adversary will demand \$2, \$3, or \$4. On the other hand, the demands of \$2 and \$3 are both optimal for some belief that assigns positive probability only to your adversary demanding \$2, \$3, or \$4: if you are sure that the other player will demand \$4, for example, it is optimal for you to demand \$3.

We have now argued that only the demands \$2 and \$3 are consistent with your choosing an action to maximize your expected payoff given some belief about the other player's actions that is consistent with her being rational in the sense that for each action to which it assigns positive probability there is a belief that *she* can hold about *your* behavior that makes that action optimal for her:

> only the demands \$2 and \$3 are consistent with your being rational *and* your assuming that the other player is rational.

We can take the argument yet another step. What if you assume not only that your opponent is rational but that she assumes that you are rational? Then each of the actions to which each of her beliefs about you assigns positive probability should in turn be justified by a possible belief of yours about her. The only demands consistent with your rationality are those at least equal to \$2, as we saw above. Thus if she assumes that you are rational then each of her beliefs about you must assign positive probability only to demands of at least \$2. But then, by the last argument above, the belief that you hold must assign positive probability only to demands of \$2 or \$3. Finally, referring to Figure 350.1 you can see that if you hold such a belief you will not demand \$3: a demand of \$2 generates a higher payoff for you, whether your opponent demands \$2 or \$3. To summarize:

> only the demand of \$2 is consistent with your rationality, your assuming that your opponent is rational, and your assuming that your opponent assumes that you are rational.

The line of reasoning can be taken further: we can consider the consequence of your assuming that your opponent assumes that you assume that she is rational. However, such reasoning eliminates no more actions: a demand of \$2 survives every additional level, since a demand of \$2 is optimal for a player who is sure that her opponent will demand \$2. (That is, $(\$2, \$2)$ is a Nash equilibrium of the game.)

In summary, in this game we conclude that

- if you are rational you will demand either \$2, \$3, or \$4
- if you assume that your opponent is rational you will demand either \$2 or \$3
- if you assume that your opponent assumes that you are rational then you will demand \$2.

The general structure of this argument is illustrated in Figure 353.1. (I restrict the informal discussion, though not the definitions and results, to two-player games.) The rectangles represent the sets A_i and A_j of players i and j in the game. Assume that the action a_i^* is consistent with player i's acting rationally. Then there is a belief of player i about player j's actions under which a_i^* is optimal. Let μ_i^1 be one such belief, and let the set of actions to which this belief assigns positive probability be the shaded set on the right, which I denote X_j^1. In the example, if $a_1^* = \$0$ or \$1 then there is no such belief. If $a_1^* = \$2$, \$3, or \$4 there are such beliefs; if $a_1^* = \$4$, for example, then all such beliefs assign relatively high probability to \$0.

Now further assume that a_i^* is consistent with player i's assuming that player j is rational. Then for some belief of player i about player j's actions that makes a_i^* optimal—say μ_i^1—each action in X_j^1 (the set of actions to which μ_i^1 assigns positive probability) must be optimal for player j under some belief about player i's action. For the two actions a_j' and a_j'' in X_j^1 the beliefs $\mu_j^2(a_j')$ and $\mu_j^2(a_j'')$ under which the actions are optimal are indicated in the figure, together with the sets of actions of player i to which they assign positive probability. The shaded set on the left is the set of actions of player i to which some belief $\mu_j^2(a_j)$ of player j for a_j in X_j^1 assigns positive probability. (The action a_i^* may or may not be a member of X_i^2; in the figure it is not.)

Note that we do not require that for *every* belief of player i under which a_i^* is optimal the actions of player j to which that belief assigns positive probability be

Figure 353.1 An illustration of the argument that an action is rationalizable.

optimal given some belief of player j about player i; rather, we require only that *there exists* a belief of player i under which a_i^* is optimal with this property. In the *Prisoner's Dilemma*, for example, the belief of player 1 that assigns probability 1 to player 2's choosing *Fink* has the properties that if player 1 holds this belief then it is optimal for her to choose *Fink*, and there is some belief of player 2 under which the action that player 1's belief assigns positive probability is optimal for player 2. It is also optimal for player 1 to choose *Fink* if she holds a belief that assigns positive probability to player 2's choosing *Quiet*. However, such a belief cannot play the role of μ_1^1 in the argument above, since there is no belief of player 2 under which the action *Quiet* of player 2, to which the belief assigns positive probability, is optimal. That is, if we start off by letting μ_1^1 be a belief of player 1 that assigns positive probability to both *Fink* and *Quiet* then we get stuck at the next round: there is no belief that justifies *Quiet*. On the other hand, if we start off by letting μ_1^1 be the belief of player 1 that assigns probability 1 to player 2's choosing *Fink* then we can continue the argument.

The next step of the argument requires that every action a_i in X_i^2 be optimal for player i given some belief $\mu_i^3(a_i)$ about player j; denote the set of actions to which $\mu_i^3(a_i)$ assigns positive probability for some a_i in X_i^2 by X_j^3. Subsequent steps are similar: at each step every action in X_k^t has to be optimal for some belief about the other player and the set of actions of the other player (say ℓ) to at least one of these beliefs in this set assigns positive probability is the new set X_ℓ^{t+1}.

If we can continue the process indefinitely then we say that the action a_i^* is *rationalizable*. If we cannot—that is, if there is a stage t at which some action in the set X_k^t is not justified by *any* belief of player k—then a_i^* is not rationalizable.

Under what circumstances can we continue the argument indefinitely? Certainly we can do so if there are sets Z_1 and Z_2 of actions of player 1 and player 2 respectively such that Z_i contains a_i^*, every action in Z_1 is a best response to a belief of player 1 on Z_2 (i.e. a belief that assigns positive probability only to actions in Z_2), and every action in Z_2 is a best response to a belief of player 2 on Z_1. Conversely, suppose that it can be continued indefinitely. For player i let Z_i be the union of $\{a_i^*\}$ with the union of the sets X_i^t for all even values of t and let Z_j be the union of the sets X_j^t for all odd values of t. Then for $i = 1, 2$, every action in Z_i is a best response to a belief on Z_j. Thus we can define an action to be rationalizable as follows, where Z_{-i} denotes the set of all collections a_{-i} of actions for the players other than i for which $a_j \in Z_j$ for all j.

▶ DEFINITION 354.1 The action a_i^* of player i in a strategic game is **rationalizable** if for each player j there exists a set Z_j of actions such that

- Z_i contains a_i^*
- for every player j, every action a_j in Z_j is a best response to a belief of player j on Z_{-j}.

Suppose that a^* is a pure strategy Nash equilibrium. Then for each player i the action a_i^* is a best response to a belief that assigns probability one to the other players' choosing a_{-i}^*. Setting $Z_i = \{a_i^*\}$ for each i, we see that a^* is rationalizable. In fact, we have the following stronger result.

■ PROPOSITION 354.2 *Every action used with positive probability in some mixed strategy Nash equilibrium is rationalizable.*

Proof. For each player i, let Z_i be the set of actions to which player i's equilibrium mixed strategy assigns positive probability. Then every action in Z_i is a best response to the belief of player i that coincides with the probability distribution over the other players' actions that is generated by their mixed strategies (which by definition assigns positive probability only to collections of actions in Z_{-i}). Hence every action in Z_i is rationalizable. □

In many games, actions not used with positive probability in some Nash equilibrium are rationalizable. Consider, for example, the game in Figure 355.1, which has a unique Nash equilibrium (M, C).

? EXERCISE 354.3 (Mixed strategy equilibrium of game in Figure 355.1) Show that the game in Figure 355.1 has no nondegenerate mixed strategy equilibrium.

Each action of each player is a best response to some action of the other player (for example, T is a best response of player 1 to R, M is a best response to C, and B is a best response to L). Thus, setting $Z_1 = \{T, M, B\}$ and $Z_2 = \{L, C, R\}$ we see that every action of each player is rationalizable. In particular the actions T and B of player 1 are rationalizable, even though they are not used with positive probability in any Nash equilibrium. The argument for player 1's choosing T, for example, is that player 2 might choose R, which is rational for her if she thinks player 1 will choose B, and it is reasonable for player 2 to so think since B is optimal for player 1 if she thinks that player 2 will choose L, which in turn is rational for player 2 if she thinks that player 1 will choose T, and so on.

	L	C	R
T	$0,7$	$2,5$	$7,0$
M	$5,2$	$3,3$	$5,2$
B	$7,0$	$2,5$	$0,7$

Figure 355.1 A game in which the actions T and B or player 1 and L and R of player 2 are not used with positive probability in any Nash equilibrium, but are rationalizable.

Even in games in which every rationalizable action is used with positive probability in some Nash equilibrium, the predictions of the notion of rationalizability are weaker than those of Nash equilibrium. The reason is that the notion of Nash equilibrium makes a prediction about the *profile* of chosen actions, while the notion of rationalizability makes a prediction about the actions chosen by each player. In a game with more than one Nash equilibrium these two predictions may differ. Consider, for example, the game in Figure 355.2. The notion of Nash equilibrium predicts that the outcome will be either (T, L) or (B, R) in this game, while the notion of rationalizability does not restrict the outcome at all: both T and B are rationalizable for player 1 and both L and R are rationalizable for player 2, so the outcome could be any of the four possible pairs of actions.

	L	R
T	$2,2$	$1,0$
B	$0,1$	$1,1$

Figure 355.2 A game with two Nash equilibria, (T, L) and (B, R).

12.2 Iterated elimination of strictly dominated actions

The notion of rationalizability, in requiring that a player act rationally, starts by restricting attention to actions that are best responses to some belief. That is, it eliminates from consideration actions that are not best responses to any belief: *never best responses*.

▶ DEFINITION 355.1 A player's action is a **never best response** if it is not a best response to any belief about the other players' actions.

Another criterion that we might use to eliminate an action from consideration is *domination*. Define an action a_i of player i to be *strictly dominated* if there is a mixed strategy of player i that yields her a higher payoff than does a_i *regardless* of the other players' behavior.

▶ DEFINITION 355.2 An action a_i of player i in a strategic game is **strictly dominated** if there is a mixed strategy α_i of player i for which

$$U_i(\alpha_i, a_{-i}) > u_i(a_i, a_{-i}) \text{ for all } a_{-i}.$$

(As before, $U_i(\alpha_i, a_{-i})$ is the expected payoff of player i when she uses the mixed strategy α_i and the collection of actions chosen by the other players is a_{-i}.)

In the *Prisoner's Dilemma*, for example, the action *Quiet* is strictly dominated by the action *Fink*: whichever action the other player chooses, *Fink* yields a higher payoff than does *Quiet*. In the game in Figure 356.1, no action of either player is strictly dominated by another action, but the action R of player 2 is strictly dominated by the mixed strategy that assigns probability $\frac{1}{2}$ to L and probability $\frac{1}{2}$ to C: the action R yields player 2 a payoff of 1 regardless of how player 1 behaves, while the mixed strategy yields her a payoff of $\frac{3}{2}$ regardless of how player 1 behaves. (The action R is dominated by other mixed strategies too: a mixed strategy that assigns probability q to L and probability $1-q$ to C yields the payoff $3q$ if player 1 chooses T and $3(1-q)$ if player 1 chooses B, and hence strictly dominates R whenever $3q > 1$ and $3(1-q) > 1$, or whenever $\frac{1}{3} < q < \frac{2}{3}$.)

	L	C	R
T	1, 3	0, 0	1, 1
B	0, 0	1, 3	0, 1

Figure 356.1 A strategic game in which the action R of player 2 is strictly dominated by the mixed strategy that assigns probability $\frac{1}{2}$ to each of the actions L and C.

If an action is strictly dominated then it is a never best response by the following argument. Suppose that a_i^* is strictly dominated by the mixed strategy α_i and let μ_i be a belief of player i about the other players' actions. Then since $U_i(\alpha_i, a_{-i}) > u_i(a_i^*, a_{-i})$ for all a_{-i} we have

$$\sum_{a_{-i}} \mu_i(a_{-i}) U_i(\alpha_i, a_{-i}) > \sum_{a_{-i}} \mu_i(a_{-i}) u_i(a_i^*, a_{-i}).$$

Hence a_i^* is not a best response to μ_i; since μ_i is arbitrary, a_i^* is a never best response. In fact, the converse is also true: if an action is a never best response then it is strictly dominated. Although it is easy to convince oneself that this result is reasonable, the proof is not trivial. In summary, we have the following.

■ LEMMA 356.1 *A player's action in a finite strategic game is a never best response if and only if it is strictly dominated.*

Now reconsider the argument behind the rationalizability of an action of player i. First we argued that player i will not use a never best response, or equivalently, a strictly dominated action. Then we argued that if she works under the assumption that her opponent is rational then her belief should not assign positive probability to any action of her opponent that is a never best response. That is, she should not choose an action that is strictly dominated in the game that results when we eliminate all her opponent's strictly dominated actions. At the next step we argued that if player i works under the assumption that her opponent assumes that she is rational then she will assume that the action chosen by her opponent is a best response to some belief that assigns positive probability to actions of player i that are best responses to beliefs of player i. That is, in this case player i will assume that her opponent's action is not strictly dominated in the game that results when all of player i's strictly dominated actions are eliminated. Thus player i will choose an action that is not strictly dominated in the game that results when first all of player i's strictly dominated actions are eliminated, then all of player j's strictly dominated actions are eliminated.

We see that each step in the argument is equivalent to one more round of elimination of strictly dominated strategies in the game; the actions that remain no matter how many rounds of elimination we perform are the rationalizable actions. That is, rationalizability is equivalent to *iterative elimination of strictly dominated actions*.

In fact, we do not have to remove *all* the strictly dominated actions of one of the players at each stage: the set of action profiles that remain if we keep eliminating strictly dominated actions until we are left with a game in which no action of any player is strictly dominated does not depend on the order in which we perform the elimination or the number of actions that we eliminate at each stage; the surviving set is always the set of rationalizable action profiles. We now state this result precisely.

▶ DEFINITION 357.1 Suppose that for each player i in a strategic game and each $t = 1, \ldots, T$ there is a set X_i^t of actions of player i such that

- $X_i^1 = A_i$ (we start with the set of all possible actions).
- X_i^{t+1} is a subset of X_i^t for each $t = 1, \ldots, T-1$ (at each stage we may eliminate some actions).
- For each $t = 0, \ldots, T-1$ every action of player i in X_i^t that is not in X_i^{t+1} is strictly dominated in the game in which the set of actions of each player j is X_j^t (we eliminate only strictly dominated actions)
- No action in X_i^T is strictly dominated in the game in which the set of actions of each player j is X_j^T (at the end of the process no action of any player is strictly dominated).

Then the set of action profiles a such that $a_i \in X_i^T$ for every player i **survives iterated elimination of strictly dominated actions**.

Then we can show the following.

■ PROPOSITION 357.2 *For any finite strategic game, there is a unique set of action profiles that survives iterated elimination of strictly dominated actions, and this set coincides with the set of profiles of rationalizable actions.*

◆ EXAMPLE 357.3 (Rationalizable actions in an extension of *BoS*) Consider the game in Figure 358.1. The action B of player 2 is strictly dominated by *Book*. In the game obtained by eliminating B for player 2 the action B of player 1 is strictly dominated. Finally, in the game obtained by eliminating B for player 1 the action

	B	*S*	*Book*
B	3, 1	0, 0	−1, 2
S	0, 0	1, 3	0, 2

Figure 358.1 Bach, Stravinsky, or a book.

Book for player 2 is strictly dominated. We conclude that the only rationalizable action for each player is *S*.

? EXERCISE 358.1 (Finding rationalizable actions) Find the set of rationalizable actions of each player in the game in Figure 358.2.

	L	*C*	*R*
T	2, 1	1, 4	0, 3
B	1, 8	0, 2	1, 3

Figure 358.2 The game in Exercise 358.1

? EXERCISE 358.2 (Rationalizable actions in Guessing Morra) Find the rationalizable actions of each player in the game *Guessing Morra* (Exercise 344.2).

? EXERCISE 358.3 (Rationalizable actions in a public good game) (More difficult, but also more interesting.) Show the following results for the variant of the game in Exercise 42.1in which contributions are restricted to be nonnegative.

a. Any contribution of more than $w_i/2$ is strictly dominated for player i.

b. If $n = 3$ and $w_1 = w_2 = w_3 = w$ then every contribution of at most $w/2$ is rationalizable. [Show that every such contribution is a best response to a belief that assigns probability one to each of the other players' contributing some amount at most equal to $w/2$.]

c. If $n = 3$ and $w_1 = w_2 < \frac{1}{3}w_3$ then the unique rationalizable contribution of players 1 and 2 is 0 and the unique rationalizable contribution of player 3 is w_3. [Eliminate strictly dominated actions iteratively. After eliminating a contribution of more than $w_i/2$ for each player i (by part *a*), you can eliminate small contributions by player 3; subsequently you can eliminate any positive contribution by players 1 and 2.]

? EXERCISE 358.4 (Rationalizable actions in Hotelling's spatial model) Consider a variant of the game in Section 3.3 in which there are two players, the distribution of the citizens' favorite positions is uniform [not needed?, but makes things easier to talk about?], and each player is restricted to choose a position of the form ℓ/m for some integer ℓ between 0 and m, where m is even (or to stay out of the competition). Show that the unique rationalizable action of each player is the position $\frac{1}{2}$.

12.3 Iterated elimination of weakly dominated actions

A strictly dominated action is clearly unattractive to a rational player. Now consider an action a_i that is *weakly dominated* in the sense that there is another action that yields *at least as high* a payoff as does a_i whatever the other players choose and yields a higher payoff than does a_i for some choice of the other players. In the game in Figure 359.1, for example, the action T of player 1 weakly (though not strictly) dominates B.

	L	R
T	$1,1$	$0,0$
B	$0,0$	$0,0$

Figure 359.1 A game in which the action B for player 1 and the action R for player 2 are weakly, but not strictly, dominated.

▶ DEFINITION 359.1 The action a_i of player i in a strategic game is **weakly dominated** if there is a mixed strategy α_i of player i such that

$$U_i(\alpha_i, a_{-i}) \geq u_i(a_i, a_{-i}) \text{ for all } a_{-i} \in A_{-i}$$

and

$$U_i(\alpha_i, a_{-i}) > u_i(a_i, a_{-i}) \text{ for some } a_{-i} \in A_{-i}.$$

A weakly dominated action that is not strictly dominated, unlike a strictly dominated one, is not an unambiguously poor choice: by Lemma 356.1 such an action is a best response to *some* belief. For example, in the game in Figure 359.1, if player 1 is *sure* that player 2 will choose R then B is an optimal choice for her. However, the rationale for choosing a weakly dominated action is very weak: there is no advantage to a player's choosing a weakly dominated action, whatever her belief. For example, if player 1 in the game in Figure 359.1 has the the slightest suspicion that player 2 might choose L then T is better than B, and even if player 2 chooses R, T is no worse than B.

If we argue that it is unreasonable for a player to choose a weakly dominated action then we can argue also that each player should work under the assumption that her opponents will not choose weakly dominated actions, and they will assume that she does not do so, and so on. Thus, as in the case of strictly dominated actions, we can argue that weakly dominated actions should be removed *iteratively* from the game. That is, first we should mark actions of player 1 that are weakly dominated; then, without removing these actions of player 1, mark actions of player 2 that are weakly dominated, and proceed similarly with the other players. Then we should remove all the marked actions, and again mark weakly dominated actions for every player. Once again, having marked weakly dominated actions for every player, we should remove all the actions and go through the process again. We should repeat the process until no more actions can be eliminated for any player. This procedure, however, is less compelling than the iterative removal of strictly dominated actions since the set of actions that survive may depend on whether we remove *all* the weakly dominated actions at each round, or only some of them, as the two-player game in Figure 360.1 shows. The sequence in which we first eliminate L (weakly dominated by C) and then T (weakly dominated by B) leads to an outcome in which player 1 chooses B and the payoff profile is $(1,2)$. On the other hand, the sequence in which we first eliminate R (weakly dominated by C) and then B (weakly dominated by T) leads to an outcome in which player 1 chooses T and the payoff profile is $(1,1)$.

	L	C	R
T	1,1	1,1	0,0
B	0,0	1,2	1,2

Figure 360.1 A two-player game in which the set of actions that survive iterated elimination of weakly dominated actions depends on the order in which actions are eliminated.

◆ EXAMPLE 360.1 (A card game) A set of n cards consists of one with "1" on one side and "2" on the other side, one with "2" on one side and "3" on the other side, and so on. A card is selected at random; player 1 sees one side (determined randomly) and player 2 sees the other side. Each player can either *veto* the card, or *accept* it. If at least one player vetoes a card, the players tie; if both players accept it, the one who sees the higher number wins (and the other player loses).

We can model this situation as a strategic game in which a player's action is the set of numbers she accepts. If $n = 2$, for example, each player has 8 actions: $\varnothing$ (accept no number), $\{1\}$, $\{2\}$, $\{3\}$, $\{1,2\}$, $\{1,3\}$, $\{2,3\}$, and $\{1,2,3\}$. A player's payoff is her probability of winning minus her probability of losing. If $n = 2$ and player 1's strategy is $\{3\}$ and player 2's action is $\{2,3\}$, for example, then if the card $1 - -2$ is selected one player vetoes it, while if the card $2 - -3$ is selected player 1 vetoes it if she sees "2" and both players accept it if player 1 sees "3", in which case player 1 wins. Thus player 1's payoff is $\frac{1}{4}$ and player 2's payoff is $-\frac{1}{4}$.

I claim that only the pairs of actions in which each player either accepts only $n+1$ or does not accept any number survive iterated elimination of weakly dominated actions.

I first argue that any action a_i that accepts 1 is weakly dominated by the action a_i' that differs only in that it vetoes 1. Given any action of the other player, a_i and a_i' lead to possibly different outcomes only if the player sees the number 1, in which case a_i either loses (if the other player's action accepts 2) or ties, while a_i' is guaranteed to tie.

Now eliminate all actions of each player that accept 1. I now argue that any action a_i that accepts 2 is weakly dominated by the action a_i' that differs only in that it vetoes 2. Given any action of the other player, a_i and a_i' lead to possibly different outcomes only if the player sees the number 2, in which case a_i never wins, because all remaining actions of the other player veto 1. Thus a_i either loses (if the other player's action accepts 3) or ties, while a_i' is guaranteed to tie.

Continuing the argument, we eliminate all actions that accept any number up to n. The only pairs of actions that remain are those in which each player either accepts only $n+1$ or accepts no number. These two actions yield the same payoffs, given the other player's remaining actions (all payoffs are 0), so neither action can be eliminated.

Now consider the special case in which *all* weakly dominated actions of each player are eliminated at each step. If all the players are indifferent between all action profiles that survive when we perform such iterated elimination then we say that the game is *dominance solvable*. candidates,

? EXERCISE 361.1 (Dominance solvability) Find the set of Nash equilibria (mixed as well as pure) of the game in Figure 361.1. Show that the game is dominance solvable; find the pair of payoffs that survives. Find an order of elimination such that more than one outcome survives.

	L	C	R
T	$2,2$	$0,2$	$0,1$
M	$2,0$	$1,1$	$0,2$
B	$1,0$	$2,0$	$0,0$

Figure 361.1 The game for Exercise 361.1.

? EXERCISE 361.2 (Dominance solvability) Show that the variant of the game in Example 350.1 in which $\epsilon = 0$ is dominance solvable and find the set of surviving outcomes.

? EXERCISE 361.3 (Dominance solvability in Bertrand's duopoly game) Consider the variant of Bertrand's duopoly game in Exercise 65.2, in which each firm is restricted to choose prices that are integral numbers of cents. Assume that the profit function $(p-c)D(p)$ has a single local maximum. Show that the game is dominance solvable and find the set of surviving outcomes.

Notes

[Highly incomplete.]

The notion of rationalizability is due to Bernheim (1984) and Pearce (1984). Example 360.1 is taken from Littlewood (1953, 4). (Whether Littlewood is the originator or not is unclear. He presents the situation as a good example of "mathematics with minimum 'raw material' ".)

13 Evolutionary equilibrium

Monomorphic pure strategy equilibrium · Mixed strategies and polymorphic equilibrium · Asymmetric equilibria · Extensive games · Illustrations: sibling behavior; nesting behavior of wasps. *Prerequisite*: Chapters 2, 4, and 5.

13.1 Introduction

According to the Darwinian theory of evolution the modes of behavior that survive are those that are most successful in producing offspring. In an environment in which organisms interact, the reproductive success of a mode of behavior may depend on the modes of behavior followed by all the organisms in the population. For example, if all organisms act aggressively, then an organism may be able to survive only if it is aggressive; if all organisms are passive, then an organism's reproductive success may be greater if the organism acts passively than if it acts aggressively. Game theory provides tools with which to study evolution in such an environment.

In the games studied in this chapter, the players are representatives from an evolving population of organisms (humans, animals, plants, bacteria, ...). Each player's payoffs measure the increments in the player's biological *fitness*, or reproductive success (e.g. expected number of healthy offspring), associated with the possible outcomes, rather than indicating the player's subjective feelings about the outcomes. Each player's actions are modes of behavior that the player is programmed to follow.

The players do not make conscious choices. Rather, each player's mode of behavior comes from one of two sources: with high probability it is inherited from the player's parent (or parents), and with low (but positive) probability it is assigned to the player as the result of a mutation. For most of the models in this chapter, inheritance is conceived very simply: each player has a single parent, and, unless it is a mutant, simply takes the same action as does its parent. This model of inheritance captures the essential features of both genetic inheritance and social inheritance: players either follow the programs encoded in their genes, which come from their parents, or learn how to behave by imitating their parents. The distinction between genetic and social evolution may be significant if we wish to change society, but is insignificant for most of the models considered in this chapter.

We choose each player's set of actions to consist of all the modes of behavior that will, eventually, be generated by mutation (that is, we assume that for each action a, mutation eventually produces an organism that follows a). If, given the modes of behavior of all other organisms, the increment to biological fitness associated with the action a exceeds that associated with the action a' for some player, then adherents of a reproduce faster than adherents of a', and hence come to dominate the population. Very roughly, adherents of actions that are not best responses to the environment are eventually overwhelmed by adherents of better actions. The population from which each player in the game is drawn is subject to the same selective pressure, so this argument suggests that outcomes that are evolutionarily stable are related to Nash equilibria of the game. In this chapter we study the relation precisely.

The theory finds many applications in which the organisms are animals or plants. However, human behavior also can sometimes insightfully be modeled as the outcome of an evolutionary process: some human action, at least, appear to be more the result of inherited behavior than the outcome of reasoned choice.

13.2 Monomorphic pure strategy equilibrium

13.2.1 Introduction

Members of a single large population of organisms are repeatedly randomly matched in pairs. The set of possible modes of behavior of each member of any pair is the same, and the consequence of an interaction for an organism depends only on the actions of the organism and its opponent, not on its name. As an example, think of a population of identical animals, pairs of which periodically are engaged in conflicts (over prey, for example). The actions available to each animal may correspond to various degrees of aggression, and the outcome for each animal depends only on its degree of aggression and that of its opponent. Each organism produces offspring (reproduction is asexual), to each of whom, with high probability, it passes on its mode of behavior; with low probability, each offspring is a mutant that adopts some other mode of behavior.

We can model the interaction between each pair of organisms as a symmetric strategic game (Definition 48.1) in which the payoff $u(a, a')$ of an organism that takes the action a when its opponent takes the action a' measures its expected number of offspring. We assume that the adherents of each mode of behavior multiply at a rate proportional to their payoff, and look for a configuration of modes of behavior in the population that is stable in the sense that in the event that the population contains a small fraction of mutants taking the same action, every mutant obtains an expected payoff lower than that of any nonmutant. (We ignore the case in which mutants taking different actions are present in the population at the same time.)

In this section I restrict attention to situations in which all organisms (except those thrown up by mutation) follow the *same* mode of behavior, which has no random component. That is, I consider only *monomorphic pure strategy* equilibria ("monomorphic" = "one form").

13.2.2 Examples

To get an idea of the implications of evolutionary stability, consider two examples. First suppose that the game between each pair of organisms is the one in the left panel of Figure 281.1. Suppose that every organism normally takes the action X. If

	X	Y
X	$2, 2$	$0, 0$
Y	$0, 0$	$1, 1$

	X	Y
X	$2, 2$	$0, 0$
Y	$0, 0$	$0, 0$

Figure 281.1 Two strategic games, illustrating the idea of an evolutionarily stable strategy.

the population contains the small fraction ϵ of mutants who take the action Y, then a normal organism has as its opponent another normal organism with probability $1-\epsilon$ and a mutant with probability ϵ. (The population is large, so that we can treat the fraction of mutants in the *rest* of the population as equal to the fraction of mutants in the entire population.) Thus the expected payoff of a normal organism is

$$2 \cdot (1-\epsilon) + 0 \cdot \epsilon = 2(1-\epsilon).$$

Similarly, the expected payoff of a mutant is

$$0 \cdot (1-\epsilon) + 1 \cdot \epsilon = \epsilon.$$

If ϵ is small enough then the first payoff exceeds the second, so that the entry of a *small* fraction of mutants leads to a situation in which the expected payoff (fitness) of every mutant is lower than the payoff of every normal organism. We conclude that the action X is evolutionarily stable.

Now suppose that every organism normally takes the action Y. Then, making a similar calculation, the expected payoff of a normal organism is $1-\epsilon$, while the expected payoff of a mutant is 2ϵ. Mutants who meet each other obtain a payoff higher than that of normal organisms who meet each other. But when ϵ is small mutants are usually paired with normal organisms, in which case their expected payoff is 0 and, as in the previous case, the first payoff exceeds the second, so that the action Y is evolutionarily stable. The value of ϵ for which a normal organism does better than a mutant is smaller in this case than it is in the case that the normal action is X. However, in both cases, if ϵ is *sufficiently* small then mutants cannot invade. Since we wish to capture the idea that mutation is extremely rare, relative to normal behavior, we are satisfied with this existence of *some* value of ϵ that prevents invasion by mutants; we do not attach significance to the size of the critical value of ϵ.

Now consider the game in the right panel of Figure 281.1. By an argument like those above, the action X is evolutionarily stable. Is the action Y also evolutionarily stable? In a population containing the fraction ϵ of mutants choosing X, the expected payoff of a normal organism is 0 (it obtains 0 whether its opponent is normal or a mutant) while the expected payoff of a mutant is 2ϵ (it obtains 2 against another mutant and 0 against a normal organism). Thus the action Y is *not* evolutionarily stable: for *any* value of ϵ the expected payoff of a mutant exceeds that of a normal organism.

In both games, both (X, X) and (Y, Y) are Nash equilibria, but while X is an evolutionarily stable action in both games, Y is evolutionarily stable only in the left game. What is the essential difference between the games? If the normal action is Y then in the left game a mutant who chooses X is worse off than a normal organism in encounters with normal organisms, while in the right game a mutant that chooses X obtains the *same* expected payoff as does a normal organism in encounters with normal organisms. In the left game, there is always a value of ϵ small enough that the gain (relative to the payoff of a normal organism) that a mutant obtains with probability ϵ when it faces another mutant does not cancel out the loss it obtains with probability $1 - \epsilon$ when it faces a normal organism. In the right game, however, a mutant loses nothing relative to a normal organism, so no matter how small ϵ is, a mutant is better off than a normal organism. That is, the essential difference between the games is that $u(X, Y) < u(Y, Y)$ in the left game, but $u(X, Y) = u(Y, Y)$ in the right game.

13.2.3 General definitions

Consider now an arbitrary symmetric strategic game in which each player has finitely many actions. Under what circumstances is the action a^* evolutionarily stable?

Suppose that a small group of mutants choosing the action b different from a^* enters the population. The notion of stability that we consider requires that each such mutant obtain an expected payoff less than that of each normal organism, so that the mutants die out. (If the mutants obtained a payoff higher than that of the normal organisms then they would eventually come to dominate the population; if they obtained the same payoff as that of the normal organisms then they would neither multiply nor decline. Our notion of stability excludes the latter case: it is a strong notion that requires that mutants be driven out of the population.)

Denote the fraction of mutants in the population by ϵ. First consider a mutant, which adopts the action b. In a random encounter, the probability that it faces an organism that adopts the action a^* is approximately $1 - \epsilon$ (the population is large, so that the fraction in the *rest* of the population is close to the fraction in the entire population), while the probability that it faces a mutant, which adopts b, is approximately ϵ. Thus its expected payoff is

$$(1 - \epsilon)u(b, a^*) + \epsilon u(b, b).$$

Similarly, the expected payoff of an organism that adopts the action a^* is

$$(1 - \epsilon)u(a^*, a^*) + \epsilon u(a^*, b).$$

In order that any mutation be driven out of the population, we need the expected payoff of any mutant to be less than the expected payoff of a normal organism:

$$(1-\epsilon)u(a^*, a^*) + \epsilon u(a^*, b) > (1-\epsilon)u(b, a^*) + \epsilon u(b, b) \text{ for all } b \neq a^*. \qquad (283.1)$$

To capture the idea that mutation is extremely rare, the notion of evolutionary stability requires only that there is *some* (small) number $\overline{\epsilon}$ such that the inequality holds whenever $\epsilon < \overline{\epsilon}$. That is, we can make the following definition:

> The action a^* is *evolutionarily stable* if there exists $\overline{\epsilon} > 0$ such that a^* satisfies (283.1) for all $\epsilon < \overline{\epsilon}$.

Intuitively, the larger is $\overline{\epsilon}$, the "more stable" is the action a^*, since larger mutations are resisted. However, in the current discussion we do not attach any significance to the value of $\overline{\epsilon}$; in order that a^* be evolutionarily stable we require only that there is *some* size for $\overline{\epsilon}$ such that all smaller mutations are resisted.

The condition in this definition of evolutionary stability is a little awkward to work with, since whenever we apply it we need to check whether we can find a suitable value of $\overline{\epsilon}$. I now reformulate the condition in a way that avoids the variable $\overline{\epsilon}$.

I first claim that

> if there exists $\overline{\epsilon} > 0$ such that a^* satisfies (283.1) for all $\epsilon < \overline{\epsilon}$ then (a^*, a^*) is a Nash equilibrium.

To reach this conclusion, suppose that (a^*, a^*) is not a Nash equilibrium. Then there exists an action b such that $u(b, a^*) > u(a^*, a^*)$. Hence (283.1) is strictly violated when $\epsilon = 0$, and thus remains violated for all sufficiently small positive values of ϵ. (If $w < x$ and y and z are any numbers, then $(1-\epsilon)w + \epsilon y < (1-\epsilon)x + \epsilon z$ whenever ϵ is small enough.) Thus there is no $\overline{\epsilon}$ such that the inequality holds whenever $\epsilon < \overline{\epsilon}$. Our conclusion is that a *necessary* condition for an action a^* to be evolutionarily stable is that (a^*, a^*) be a Nash equilibrium.

Similar considerations lead to the conclusion that

> if (a^*, a^*) is a *strict* Nash equilibrium then there exists $\overline{\epsilon} > 0$ such that a^* satisfies (283.1) for all $\epsilon < \overline{\epsilon}$.

The argument is that if (a^*, a^*) is a strict Nash equilibrium then $u(b, a^*) < u(a^*, a^*)$ for all b, so that the strict inequality in (283.1) is satisfied for $\epsilon = 0$; hence it is also satisfied for sufficiently small positive values of ϵ. That is, we conclude that a *sufficient* condition for a^* to be evolutionarily stable is that (a^*, a^*) be a strict Nash equilibrium.

What happens if (a^*, a^*) is a Nash equilibrium, but is not strict? Suppose that $b \neq a^*$ is a best response to a^*: $u(b, a^*) = u(a^*, a^*)$. Then (283.1) reduces to the condition $u(a^*, b) > u(b, b)$, so that a^* is evolutionarily stable if and only if this condition is satisfied.

We conclude that *necessary and sufficient* conditions for the action a^* to be evolutionarily stable are that (i) (a^*, a^*) is a Nash equilibrium, and (ii) $u(a^*, b) > u(b, b)$ for every $b \neq a^*$ that is a best response to a^*. Intuitively, in order that mutant behavior die out it must be that (i) no mutant does better than a^* in encounters with organisms using a^* and (ii) any mutant that does as well as a^* in such encounters must do worse than a^* in encounters with mutants.

To summarize, the definition of evolutionary stability given above is equivalent to the following definition (which is much easier to work with).

▶ DEFINITION 284.1 An action a^* of a player in a symmetric two-player game is **evolutionarily stable** with respect to mutants using pure strategies if

- (a^*, a^*) is a Nash equilibrium, and
- $u(b, b) < u(a^*, b)$ for every best response b to a^* for which $b \neq a^*$,

where u is each player's payoff function.

As I argued above, if (a^*, a^*) is a *strict* Nash equilibrium then a^* is evolutionarily stable. This fact follows from the definition, since if (a^*, a^*) is a strict Nash equilibrium then the only best response to a^* is a^*, so that the second condition in the definition is vacuously satisfied.

Note that the inequality in the second condition is strict. If it were an equality then we would include as stable situations in which mutants neither multiply nor die out, but reproduce at the same rate as the normal population.

13.2.4 Examples

Both of the symmetric pure Nash equilibria of the left game in Figure 281.1 are strict, so that both X and Y are evolutionarily stable (confirming our previous analysis). In the right game in Figure 281.1, (X, X) and (Y, Y) are symmetric pure Nash equilibria also. But in this case (X, X) is strict while (Y, Y) is not. Further, since $u(X, X) > u(Y, X)$, the second condition in the definition of evolutionary stability is not satisfied by Y. Thus in this game only X is evolutionarily stable (again confirming our previous analysis).

The *Prisoner's Dilemma* (Figure 13.1) has a unique symmetric Nash equilibrium (D, D), and this Nash equilibrium is strict. Thus the action D is the only evolutionarily stable action. The game *BoS* (Figure 16.1) has no symmetric pure Nash equilibrium, and hence no evolutionarily stable action. (I consider mixed strategies in the next section.)

The following game, which generalizes the ideas on the game in Exercise 28.3, presents a richer range of possibilities for evolutionarily stable actions.

◆ EXAMPLE 284.2 (Hawk–Dove) Two animals of the same species compete for a resource (e.g. food, or a good nesting site) whose value (in units of "fitness") is $v > 0$. (That is, v measures the increase in the expected number of offspring brought by control of the resource.) Each animal can be either *aggressive* or *passive*. If both

Evolutionary game theory: some history

In his book *The Descent of Man*, Charles Darwin gave a game-theoretic argument that in sexually-reproducing species, the only evolutionarily stable sex ratio is 50:50 (1871, Vol. I, 316). Darwin's argument is game-theoretic in appealing to the fact that the number of an animal's descendants depends on the "behavior" of the other members of the population (the sex ratio of their offspring; see Exercise 303.1). Coming as it did 50 years before the language and methods of game theory began to develop, however, it is not couched in game-theoretic terms. In the late 1960s, two decades after the appearance of von Neumann and Morgenstern's (1944) seminal book, Hamilton (1967) proposed an explicitly game theoretic model of sex ratio evolution that applies to situations more general than that considered by Darwin.

But the key figure in the application of game theory to evolutionary biology is John Maynard Smith. Maynard Smith (1972a) and Maynard Smith and Price (1973) propose the notion of an evolutionarily stable strategy, and Maynard Smith's subsequent research develops the field in many directions. (Maynard Smith gives significant credit to Price: he writes that he would probably not have had the idea of using game theory had he not seen unpublished work by Price; "[u]nfortunately", he writes, "Dr. Price is better at having ideas than at publishing them" (1972b, vii).)

In the last two decades evolutionary game theory has blossomed. Biological models abound, and the methods of the theory have made their way into economics.

animals are aggressive they fight until one is seriously injured; the winner obtains the resource without sustaining any injury, while the loser suffers a loss of c. Each animal is equally likely to win, so each animal's expected payoff is $\frac{1}{2}v + \frac{1}{2}(-c)$. If both animals are passive then each obtains the resource with probability $\frac{1}{2}$, without a fight. Finally, if one animal is aggressive while the other is passive then the aggressor obtains the resource without a fight. The game is shown in Figure 285.1.

	A	P
A	$\frac{1}{2}(v-c), \frac{1}{2}(v-c)$	$v, 0$
P	$0, v$	$\frac{1}{2}v, \frac{1}{2}v$

Figure 285.1 The game *Hawk–Dove*.

If $v > c$ then the game has a unique Nash equilibrium (A, A), which is strict, so that A is the unique evolutionarily stable action.

If $v = c$ then also the game has a unique Nash equilibrium (A, A). But in this case the equilibrium is not strict: against an opponent that chooses A, a player obtains the same payoff whether it chooses A or P. However, the second condition in Definition 284.1 is satisfied: $v/2 = u(P, P) < u(A, P) = v$. Thus A is the unique evolutionarily stable action in this case also.

In both of these cases, a population of passive players can be invaded by aggressive players: an aggressive mutant does better than a passive player when its opponent is passive, and at least as well as a passive player when its opponent is aggressive.

If $v < c$ then the game has no symmetric Nash equilibrium in pure strategies: neither (A, A) nor (P, P) is a Nash equilibrium. Thus in this case the game has no evolutionarily stable action. (The game has only *asymmetric* Nash equilibria in this case.)

? EXERCISE 286.1 (Evolutionary stability and weak domination) Let a^* be an evolutionarily stable action. Does a^* necessarily weakly dominate every other action? Is it possible that some other action weakly dominates a^*?

? EXERCISE 286.2 (Example of evolutionarily stable actions) Pairs of members of a single population engage in the following game. Each player has three actions, corresponding to demands of 1, 2, or 3 units of payoff. If both players in a pair make the same demand, each player obtains her demand. Otherwise the player who demands less obtains the amount demanded by her opponent, while the player who demands more obtains $a\delta$, where a is her demand and δ is a number less than $\frac{1}{3}$. Find the set of pure strategy symmetric Nash equilibria of the game, and the set of pure evolutionarily stable strategies. What happens if each player has n actions, corresponding to demands of $1, 2, \ldots, n$ units of payoff (and $\delta < 1/n$)?

To gain an understanding of the outcome that evolutionary pressure might induce in games that have no evolutionarily stable action (e.g. *BoS*, and *Hawk–Dove* when $v < c$) we can take several routes. One is to consider mixed strategies as well as pure strategies; another is to allow for the possibility of several types of behavior coexisting in the population; a third is to consider interpretations of the asymmetric equilibria. I begin by discussing the first two approaches; in the following section I consider the third approach.

13.3 Mixed strategies and polymorphic equilibrium

13.3.1 Definition

So far we have considered only situations in which both "normal" organisms and mutants use pure strategies. If we assume that mixed strategies, as well as pure strategies, are passed on from parents to offspring, and may be thrown up by mutation, then an argument analogous to the one in the previous section leads to the conclusion that an evolutionarily stable mixed strategy satisfies conditions like those in Definition 284.1. Precisely, we can define an evolutionarily stable (mixed) strategy, known briefly as an ESS, as follows.

► DEFINITION 286.3 An **evolutionarily stable strategy** (**ESS**) in a symmetric two-player game is a mixed strategy α^* such that

- (α^*, α^*) is a Nash equilibrium
- $U(\beta, \beta) < U(\alpha^*, \beta)$ for every best response β to α^* for which $\beta \neq \alpha^*$,

where $U(\alpha, \alpha')$ is the expected payoff of a player using the mixed strategy α when its opponent uses the mixed strategy α'.

(If you do not believe that animals can randomize, you may be persuaded by an argument of Maynard Smith:

> "If it were selectively advantageous, a randomising device could surely evolve, either as an entirely neuronal process or by dependence on functionally irrelevant external stimuli. Perhaps the one undoubted example of a mixed ESS is the production of equal numbers of X and Y gametes by the heterogametic sex: if the gonads can do it, why not the brain?" (1982, 76).

Or you may be convinced by the evidence presented by Brockman et al. (1979) indicating that certain wasps pursue mixed strategies. (For a discussion of Brockman et al.'s model, see Section 13.6.))

13.3.2 Pure strategies and mixed strategies

Of course, Definition 286.3 does not preclude the use of pure strategies: every pure strategy is a special case of a mixed strategy. Suppose that a^* is an evolutionarily stable action in the sense of the definition in the previous section (284.1), and let α^* be the mixed strategy that assigns probability 1 to the action a^*. Since a^* is evolutionarily stable, (a^*, a^*) is a Nash equilibrium, so (α^*, α^*) is a mixed strategy Nash equilibrium (see Proposition 116.2). Is α^* necessarily an ESS (in the sense of the definition just given)? No: the second condition in the definition of an ESS may be violated. That is, a pure strategy may be immune to invasion by mutants that follow *pure* strategies, but may not be immune to invasion by mutants that follow some *mixed* strategy. Stated briefly, though a pure strategy Nash equilibrium is a mixed strategy Nash equilibrium, an action that is evolutionarily stable in the sense of Definition 284.1 is *not* necessarily an ESS in the sense of Definition 286.3.

	X	Y	Z
X	$2,2$	$1,2$	$1,2$
Y	$2,1$	$0,0$	$3,3$
Z	$2,1$	$3,3$	$0,0$

Figure 287.1 A game illustrating the difference between Definitions 284.1 and 286.3. The action X is an evolutionarily stable action in the sense of the first definition, but not in the sense of the second.

The game in Figure 287.1 illustrates this point. In studying this game, it may help to think of pairs of players working on a project. Two type X's work well together, and both a type Y and a type Z work well with an X, although the X

suffers a bit in each case. However, two type Y's are a disaster working together, as are two type Z's; but a Y and a Z make a great combination.

The action X is evolutionarily stable in the sense of Definition 284.1: (X, X) is a Nash equilibrium, and the two actions Y and Z different from X that are best responses to X satisfy $u(Y, Y) = 0 < 1 = u(X, Y)$ and $u(Z, Z) = 0 < 1 = u(X, Z)$. However, the action X is *not* an ESS in the sense of Definition 286.3. Precisely, the mixed strategy α^* that assigns probability 1 to X is not an ESS. To establish this claim we need only find a mixed strategy β that is a best response to α^* and satisfies $U(\beta, \beta) \geq U(\alpha^*, \beta)$ (in which case a mutant that uses the mixed strategy β will not die out of the population). Let β be the mixed strategy that assigns probability $\frac{1}{2}$ to Y and probability $\frac{1}{2}$ to Z. Since both Y and Z are best responses to X, so is β. Further, $U(\alpha^*, \beta) = 1 < \frac{3}{2} = U(\beta, \beta)$ (when both players use β the outcome is (Y, Y) with probability $\frac{1}{4}$, (Y, Z) with probability $\frac{1}{4}$, (Z, Y) with probability $\frac{1}{4}$, and (Z, Z) with probability $\frac{1}{4}$). Thus α^* is not a mixed strategy ESS: even though a population of adherents to α^* cannot be invaded by any mutant using a pure strategy, it *can* be invaded by mutants using the mixed strategy β. The point is that Y types do poorly against each other and so do Z types, but the match of a Y and a Z is very productive. Thus if all mutants either invariantly choose Y or invariantly choose Z then they fare badly when they meet each other; but if all mutants follow the mixed strategy that chooses Y and Z with equal probability then with probability $\frac{1}{2}$ two mutants that are matched are of different types, and are very productive.

13.3.3 Strict equilibria

We saw in the previous section that a *strict* pure Nash equilibrium is evolutionarily stable. Any strict Nash equilibrium is also an ESS, since the second condition in Definition 286.3 is then vacuously satisfied. However, this fact is of no help when we consider truly mixed strategies, since no mixed strategy Nash equilibrium in which positive probability is assigned to two or more actions is strict. Why not? Since if (α^*, α^*) is a mixed strategy equilibrium then, as we saw in Chapter 4, every action to which α^* assigns positive probability is a best response to α^*, and so too is any mixed strategy that assigns positive probability to the same pure strategies as does α^* (Proposition 111.1). Thus the second condition in the definition of an ESS is never vacuously satisfied for any mixed strategy equilibrium (α^*, α^*) that is not pure: when considering the possibility that a mixed equilibrium strategy is an ESS, at a minimum we need to check that $U(\beta, \beta) < U(\alpha^*, \beta)$ for every mixed strategy β that assigns positive probability to the same set of actions as does α^*.

13.3.4 Polymorphic steady states

A mixed strategy ESS corresponds to a monomorphic steady state in which each organism randomly chooses an action in each play of the game, according to the probabilities in the mixed strategy. Alternatively, it corresponds to a *polymorphic* steady state, in which a variety of pure strategies is in use in the population, the fraction of the population using each pure strategy being given by the probability the mixed strategy assigns to that pure strategy. (Cf. one of the interpretations of a mixed strategy equilibrium discussed in Section 4.1.) In Section 13.2.3 I argue that, in the case of a monomorphic steady state in which each player's strategy is pure, the two conditions in the definition of an ESS are equivalent to the requirement that any mutant die out. The same argument applies also to the case of a monomorphic steady state in which every player's strategy is mixed, but does not apply directly to the case of a polymorphic steady state. However, a different argument, based on similar ideas, shows that in this case too the conditions in the definition of an ESS are necessary and sufficient for the stability of a steady state (see Hammerstein and Selten (1994, 948–951)): mutations that change the fractions of the population using each pure strategy generate changes in payoffs that cause the fractions to return to their equilibrium values.

13.3.5 Examples

◆ EXAMPLE 289.1 (Bach or Stravinsky?) The members of a single population are randomly matched in pairs, and play *BoS*, with payoffs given in Figure 289.1. This

	L	D
L	$0,0$	$2,1$
D	$1,2$	$0,0$

Figure 289.1 The game *BoS*.

game has no symmetric pure strategy equilibrium. It has a unique symmetric mixed strategy equilibrium, in which the strategy α^* of each player assigns probability $\frac{2}{3}$ to L. As for any mixed strategy equilibrium, any mixed strategy that assigns positive probabilities to the same pure strategies as does α^* are best responses to α^*. Let $\beta = (p, 1-p)$ be such a mixed strategy. In order that α^* be an ESS we need $U(\beta,\beta) < U(\alpha^*,\beta)$ whenever $\beta \neq \alpha^*$. The payoffs in the game are low when the players choose the same action, so it seems possible that this condition is satisfied. To check the condition precisely, we need to find $U(\beta,\beta)$ and $U(\alpha^*,\beta)$. If both players use the strategy β then the outcome is (L,L) with probability p^2, (L,D) and (D,L) each with probability $p(1-p)$, and (D,D) with probability $(1-p)^2$. Thus $U(\beta,\beta) = 3p(1-p)$. Similarly, $U(\alpha^*,\beta) = \frac{4}{3} - p$. Thus for α^* to be an ESS we need

$$3p(1-p) < \tfrac{4}{3} - p$$

for all $p \neq \frac{2}{3}$. This inequality is equivalent to $(p-\frac{2}{3})^2 > 0$, so the strategy $\alpha^* = (\frac{2}{3}, \frac{1}{3})$ is an ESS.

◆ EXAMPLE 289.2 (A coordination game) The members of a single population are randomly matched in pairs, and play the game in Figure 290.1. In this game both (X,X) and (Y,Y) are strict pure Nash equilibria (as we noted previously), so that

	X	Y
X	$2,2$	$0,0$
Y	$0,0$	$1,1$

Figure 290.1 The game in Example 289.2.

both X and Y are ESSs. The game also has a symmetric mixed strategy equilibrium (α^*, α^*), in which $\alpha^* = (\frac{1}{3}, \frac{2}{3})$. Since every mixed strategy $\beta = (p, 1-p)$ is a best response to α^*, we need $U(\beta, \beta) < U(\alpha^*, \beta)$ whenever $\beta \neq \alpha^*$ in order that α^* be an ESS. In this game the players are better off choosing the same action as each other than they are choosing different actions, so it is plausible that this condition is not satisfied. The β that seems most likely to violate the condition is the pure strategy X (i.e. $\beta = (1,0)$). In this case we have $U(\beta, \beta) = 2$ and $U(\alpha^*, \beta) = \frac{2}{3}$, so indeed the condition is violated. Thus the game has no mixed strategy ESS.

The intuition for this result is that a mutant that uses the pure strategy X is better off than a normal organism that uses the mixed strategy $(\frac{1}{3}, \frac{2}{3})$ both when it encounters a mutant, and when it encounters a normal organism. Thus such mutants will invade a population of organisms using the mixed strategy $(\frac{1}{3}, \frac{2}{3})$. (In fact, a mutant following *any* strategy different from α^* invades the population, as you can easily verify.)

◆ EXAMPLE 290.1 (Mixed strategies in Hawk–Dove) Consider again the game *Hawk–Dove* (Example 284.2). If $v > c$ then the only symmetric Nash equilibrium is the strict pure equilibrium (A, A), so that the only ESS is A.

If $v \leq c$ the game has a unique symmetric mixed strategy equilibrium, in which the strategy of each player is $(v/c, 1-v/c)$. To see whether this strategy is an ESS we need to check the second condition in the definition of an ESS. Let $\beta = (p, 1-p)$ be any mixed strategy. We need to determine whether $U(\beta, \beta) < U(\alpha^*, \beta)$ for $\beta \neq \alpha^*$, where $\alpha^* = (v/c, 1-v/c)$. If each player uses the strategy β then the outcome is (A, A) with probability p^2, (A, P) and (P, A) each with probability $p(1-p)$, and (P, P) with probability $(1-p)^2$. Thus

$$U(\beta, \beta) = p^2 \cdot \tfrac{1}{2}(v-c) + p(1-p) \cdot v + p(1-p) \cdot 0 + (1-p)^2 \cdot \tfrac{1}{2}v.$$

Similarly, if a player uses the strategy α^* and its opponent uses the strategy β then its expected payoff is

$$U(\alpha^*, \beta) = (v/c)p \cdot \tfrac{1}{2}(v-c) + (v/c)(1-p) \cdot v + (1-v/c)(1-p) \cdot \tfrac{1}{2}v.$$

Upon simplification we find that $U(\alpha^*, \beta) - U(\beta, \beta) = \frac{1}{2}c(v/c-p)^2$, which is positive if $p \neq v/c$. Thus $U(\beta, \beta) < U(\alpha^*, \beta)$ for any $\beta \neq \alpha^*$. We conclude that if $v \leq c$ then the game has a unique ESS, namely the mixed strategy $\alpha^* = (v/c, 1-v/c)$.

To summarize, if injury is not costly ($c \leq v$) then only aggression survives. In this case, a passive mutant is doomed: it is worse off than an aggressive organism in encounters with other mutants and does no better than an aggressive organism in encounters with aggressive organisms. If injury costs more than the value of the resource ($c > v$) then aggression is not universal in an ESS. A population containing exclusively aggressive organisms is not evolutionarily stable in this case, since passive mutants do better than aggressive organisms against aggressive opponent. Nor is a population containing exclusively passive organisms evolutionarily stable, since aggressive pays against a passive opponent. The only ESS is a mixed strategy, which may be interpreted as corresponding to a situation in which the fraction v/c of organisms are aggressive and the fraction $1 - v/c$ are passive. As the cost of injury increases the fraction of aggressive organisms declines; the incidence of fights decreases, and an increasing number of encounters end without a fight (the dispute is settled "conventionally", in the language of biologists).

? EXERCISE 291.1 (Hawk–Dove–Retaliator) Consider the variant of *Hawk–Dove* in which a third strategy is available: "retaliator", which fights only if the opponent does so. Assume that a retaliator has a slight advantage over a passive animal against a passive opponent. The game is shown in Figure 291.1; assume $\delta < \frac{1}{2}v$. Find the ESSs.

	A	P	R
A	$\frac{1}{2}(v-c), \frac{1}{2}(v-c)$	$v, 0$	$\frac{1}{2}(v-c), \frac{1}{2}(v-c)$
P	$0, v$	$\frac{1}{2}v, \frac{1}{2}v$	$\frac{1}{2}v - \delta, \frac{1}{2}v + \delta$
R	$\frac{1}{2}(v-c), \frac{1}{2}(v-c)$	$\frac{1}{2}v + \delta, \frac{1}{2}v - \delta$	$\frac{1}{2}v, \frac{1}{2}v$

Figure 291.1 The game *Hawk–Dove–Retaliator*.

? EXERCISE 291.2 (Variant of *BoS*) Find all the ESSs, in pure and mixed strategies, of the game

	A	B	C
A	$0, 0$	$3, 1$	$0, 0$
B	$1, 3$	$0, 0$	$0, 0$
C	$0, 0$	$0, 0$	$1, 1$

? EXERCISE 291.3 (Bargaining) Pairs of players bargain over the division of a pie of size 10. The members of a pair simultaneously make demands; the possible demands are the nonnegative even integers up to 10. If the demands sum to 10 then each player receives her demand; if the demands sum to less than 10 then each player receives her demand plus half of the pie that remains after both demands have been satisfied; if the demands sum to more than 10 then no player receives any payoff. Show that the game has an ESS that assigns positive probability only to the demands 2 and 8 and also has an ESS that assigns positive probability only to the demands 4 and 6.

The next example reexamines the *War of attrition*, studied previously in Section 3.4 (pure equilibria). The game entered the literature as a model of animal conflicts. The actions of each player are the lengths of time the animal displays; the animal that displays longest wins.

◆ EXAMPLE 292.1 (War of attrition) Consider the *War of attrition* introduced in Section 3.4. If $v_1 = v_2$ then the game is symmetric. We found that even in this case the game has no symmetric pure strategy equilibrium. The only symmetric equilibrium is a mixed strategy equilibrium, in which each player's mixed strategy has the probability distribution function

$$F(t) = 1 - e^{-t/v},$$

where v is the common valuation.

Is this equilibrium strategy an ESS? Since the strategy assigns positive probability to every interval of actions, *every* strategy is a best response to it. Thus it is an ESS if and only if $U(G,G) < U(F,G)$ for every strategy $G \neq F$. To show this inequality is difficult. Here I show only that the inequality holds whenever G is a *pure* strategy. Let G be the pure strategy that assigns probability 1 to the action a. Then $U(G,G) = \frac{1}{2}v - a$ and

$$U(F,G) = \int_0^a (-s)F'(s)ds + (1 - F(a))(v - a) = v(2e^{-a/v} - 1)$$

(substituting for F and performing the integrations). Thus

$$U(F,G) - U(G,G) = 2ve^{-a/v} - \tfrac{3}{2}v + a,$$

which is positive for all values of a (find the minimum (by setting the derivative equal to zero) and show it is positive). Thus no mutant using a pure strategy can invade a population of players using the strategy F.

13.3.6 Games that have no ESS

Every game we have studied so far possesses an ESS. But there are games that do not. A very simple example is the trivial game shown in Figure 292.1. Let α be

	X	Y
X	$1,1$	$1,1$
Y	$1,1$	$1,1$

Figure 292.1

any mixed strategy. Then the strategy pair (α, α) is a Nash equilibrium. However, since $U(X,X) = 1 = U(\alpha, X)$, the mixed strategy α does not satisfy the second condition in the definition of an ESS. In a population in which all players use α, a mutant who uses X reproduces at the same rate as the other players (its fitness is the same), and thus does not die out. At the same time, such a mutant does not come to dominate the population. Thus, although the game has no ESS, every mixed strategy is neutrally stable.

However, we can easily give an example of a game in which there is not even any mixed strategy that is neutrally stable. Consider, for example, the game in Figure 293.1 with $\gamma > 0$. (If γ were zero then the game would be *Rock, paper, scissors* (Exercise 125.2).) This game has a unique symmetric Nash equilibrium, in

	A	B	C
A	$\gamma,\ \ \gamma$	$-1,\ \ 1$	$1, -1$
B	$1, -1$	$\gamma,\ \ \gamma$	$-1,\ \ 1$
C	$-1,\ \ 1$	$1, -1$	$\gamma,\ \ \gamma$

Figure 293.1 A game that has no ESS. In the unique symmetric Nash equilibrium of this game each player's mixed strategy is $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$; this strategy is not an ESS.

which each player's mixed strategy is $\alpha^* = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. To see that this strategy is not an ESS, let a be a pure strategy. Every pure strategy is a best response to α^* and $U(a, a) = \gamma > \gamma/3 = U(\alpha^*, a)$, strictly violating the second requirement for an ESS. Thus the game not only lacks an ESS; since the violation of the second requirement of an ESS is strict, it also lacks a neutrally stable strategy. The only candidate for a stable strategy is the unique symmetric mixed equilibrium strategy, but if all members of the population use this strategy then a mutant using any of the three pure strategy invades the population. Put differently, the notion of evolutionary stability—even in a weak form—makes no prediction about the outcome of this game.

13.4 Asymmetric equilibria

13.4.1 Introduction

So far we have studied the case of a homogeneous population, in which all organisms are identical, so that only symmetric equilibria are relevant: the players' roles are the same, so that a player cannot condition its behavior on whether it is player 1 or player 2. If the population is heterogeneous—if the players differ by size, by weight, by their current ownership status, or by any other observable characteristic—then even if the differences among players do not affect the payoffs, asymmetric equilibria may be relevant. I restrict attention to an example that illustrates some of the main ideas.

13.4.2 Example: Hawk–Dove

Consider a variant of *Hawk–Dove* (Example 284.2), in which the resource being contested is a nesting site, and one animal is the (current) owner while the other is an intruder. An individual will sometimes be an owner and sometimes be an intruder; its strategy specifies its action in each case. Thus we can describe the situation as a (symmetric) strategic game in which each player has *four* strategies: AA, AP, PA, and PP, where XY means that the player uses X when it is an owner and Y when it is an intruder. Since in each encounter there is one owner and one intruder, it is natural to assume that the probability that any given animal has each role is $\frac{1}{2}$.

Assume that the value of the nesting site may be different for the owner and the intruder; denote it by V for the owner and by v for the intruder. Assume also that $v < c$ and $V < c$, where c (as before) measures the loss suffered by a loser. (Recall that in the case $v < c$ there is no symmetric pure strategy equilibrium in the original version of the game.) Then in an encounter between an animal using the strategy AA and an animal using the strategy AP, for example, with probability $\frac{1}{2}$ the first animal is the owner and the second is the intruder, and the owner obtains the payoff V (the pair of actions chosen in the interaction being (A, P)), and with probability $\frac{1}{2}$ the first animal is the intruder and the second is the owner, and the intruder obtains the payoff $\frac{1}{2}(v-c)$ (the pair of actions chosen in the interaction being (A, A)). Thus in this case the expected payoff of the first animal is $\frac{1}{2}V + \frac{1}{4}(v-c) = \frac{1}{4}(2V + v - c)$. The payoffs to all strategy pairs are given in Figure 294.1; for convenience they are multiplied by four, and player 1's payoff is displayed above, not beside, player 2's.

	AA	AP	PA	PP
AA	$V+v-2c$ $V+v-2c$	$2V+v-c$ $V-c$	$V+2v-c$ $v-c$	$2V+2v$ 0
AP	$V-c$ $2V+v-c$	$2V$ $2V$	$V+v-c$ $V+v-c$	$2V+v$ V
PA	$v-c$ $V+2v-c$	$V+v-c$ $V+v-c$	$2v$ $2v$	$V+2v$ v
PP	0 $2V+2v$	V $2V+v$	v $V+2v$	$V+v$ $V+v$

Figure 294.1 A variant of *Hawk–Dove*, in which one player in each encounter is an owner and the other is an intruder. The payoffs are multiplied by four and player 1's is shown above, not beside, player 2's (for convenience in presentation). The strategy XY means take the action X when an owner and the action Y when an intruder.

The strategy pairs (AP, AP) and (PA, PA) are symmetric pure strategy equilibria of the game. Both of these equilibria are strict, so both AP and PA are ESSs (*regardless* of the relative sizes of v and V).

Now consider the possibility that the game has a mixed strategy ESS, say α^*. Then (α^*, α^*) is a mixed strategy equilibrium. I now argue that α^* does not assign positive probability to either of the actions AP or PA. If α^* assigns positive probability to AP then AP is a best response to α^* (since (α^*, α^*) is a Nash equilibrium), so that for α^* to be an ESS we need $U(AP,AP) < U(\alpha^*,AP)$. But this inequality contradicts the fact that (AP,AP) is a Nash equilibrium. Hence α^* does not assign positive probability to AP. An analogous argument shows that α^* does not assign positive probability to PA. In the following exercise you are asked to show that the game has no symmetric mixed strategy equilibrium (α^*, α^*) in which α^* assigns positive probability only to the actions AA and PP. We conclude that the game has no mixed ESS.

? EXERCISE 295.1 (Nash equilibrium in an asymmetric variant of *Hawk–Dove*) Let β be a mixed strategy that assigns positive probability only to the actions AA and PP in the game in Figure 294.1. Show that in order that AA and PP yield a player the same expected payoff when her opponent uses the strategy β, we need β to assign probability $(V+v)/2c$ to AA. Show further that when her opponent uses this strategy β, a player obtains a higher expected payoff from the action AP than she does from the action AA, so that (β, β) is not a Nash equilibrium.

? EXERCISE 295.2 (ESSs and mixed strategy equilibria) Generalize the argument that no ESS in the game in Figure 294.1 assigns positive probability to AP or to PA, to show the following result. Let (α^*, α^*) be a mixed strategy equilibrium; denote the set of actions to which α^* assigns positive probability by A^*. Then the only strategy assigning positive probability to every action in A^* that can be an ESS is α^*.

In summary, this analysis of *Hawk–Dove* for the case in which $v < c$ and $V < c$ leads to the conclusion that there are two evolutionarily stable strategies. In one, a player is aggressive when it is an owner and passive when it is an intruder, and in the other a player is passive when it is an owner and aggressive when it is an intruder. In both cases the dispute is resolved without a fight. The first strategy, in which an intruder concedes to an owner without a fight, is known as the *bourgeois strategy*; the second, in which the owner concedes to an intruder, is known as the *paradoxical strategy*. There are many examples in nature of the bourgeois strategy. The paradoxical strategy gets its name from the fact that it leads the members of a population to constantly change roles: whenever there is an encounter, the intruder becomes the owner, and the owner becomes a potential intruder. One example of this convention is described in the box on p. 295.

EXPLAINING THE OUTCOMES OF CONTESTS IN NATURE

[Note: this box is rough.] *Hawk–Dove* and its variants give us insights into the way in which animal conflicts are resolved. Before the development of evolutionary game theory, one explanation of the observation that conflicts are often settled without a fight was that it is not in the interest of a species for its members to be killed or injured. This theory is not explicit about how evolution could generate a situation in which individual members of a species act in a way that benefits the species as a whole. Further, by no means all animal conflicts are resolved peacefully, and the theory has nothing to say about the conditions under which peaceful resolution is likely to be the norm. As we have seen, a game theoretic analysis in which the unit of analysis is the individual member of the species suggests that in a symmetric contest the relation between the value of the resource under contention and the cost of an escalated contest determines the incidence of escalation. In an asymmetric contest the theory predicts that no escalation will occur, regardless of the value of the resource and the cost of injury. In particular, the convention that the owner always wins (the *bourgeois strategy*) is evolutionarily stable. (Classical theory appealed to an unexplained bias towards the owner, or to behavior that, in the context of the game theoretic models, is not rational.)

Biologists have studied behavior in many species in order to determine whether the predictions of the theory correspond to observed outcomes. Maynard Smith motivated his models by facts about conflicts between baboons. An example of more recent work concerns the behavior of the funnel web spider *Agelenopsis aperta* in New Mexico. Spiders differ in weight and web sites differ greatly in their desirability (some offer much more prey). At an average web site a confrontation usually ends without a fight. If the weights of the owner and intruder are similar, the dispute is usually settled in favor of the owner; if the weights are significantly different then the heavier spider wins. Hammerstein and Riechert (1988) estimate from field observations the fitness associated with various events and conclude that the ESS yields good predictions.

? EXERCISE 296.1 (Variant of *BoS*) Members of a population are randomly matched and play the game *BoS*. Each player in any given match can condition her action on whether she was the first to suggest getting together. Assume that for any given player the probability of being the first is one half. Find the ESSs of this game.

13.5 Variation on a theme: sibling behavior

The models of the previous sections are simple examples illustrating the main ideas of evolutionary game theory. In this section and the next I describe more detailed models that illustrate how these ideas may be applied in specific contexts.

Consider the interaction of siblings. The models in the previous sections assume that each player is equally likely to encounter any other player in the population. If we wish to study siblings' behavior toward each other we need to modify this assumption. I retain the other assumptions made previously: players interact pairwise, and payoffs measure fitness (reproductive success). I restrict attention throughout to pure strategies.

13.5.1 Asexual reproduction

The analysis in the previous sections rests on a simple model of reproduction, in which each organism, on its own, produces offspring. Before elaborating upon this model, consider its implications for the evolution of intrasibling behavior. Suppose that every organism in the population originally uses the action a^* when interacting with its siblings, obtaining the payoff $u(a^*, a^*)$. If a mutant using the action b appears, then, assuming that it has *some* offspring, all these offspring inherit the same behavior (ignoring further mutations). Thus the payoff (fitness) of each of these offspring in its interactions with its siblings is $u(b,b)$. All the descendants of any of these offspring also obtain the payoff $u(b,b)$ in their interaction with each other, so that the mutant behavior b invades the population if and only if $u(b,b) > u(a^*,a^*)$; it is driven out of the population if and only if $u(b,b) < u(a^*,a^*)$. We conclude that

> if an action a^* is evolutionarily stable then $u(a^*,a^*) \geq u(b,b)$ for every action b; if $u(a^*,a^*) > u(b,b)$ for every action b then a^* is evolutionarily stable.

When studying the behavior of one member of a population in interactions with another arbitrary member of the population, we found that a necessary condition for an action a^* to be evolutionarily stable is that (a^*,a^*) be a Nash equilibrium of the game. In intrasibling interaction, however, no such requirement appears: only actions a^* for which $u(a^*,a^*)$ is as high as possible can be evolutionarily stable. For example, if the game the siblings play is the *Prisoner's Dilemma* (Figure 297.1), then the only evolutionarily stable action is C; if this game is played between unrelated organisms then the only evolutionarily stable action is D.

	C	D
C	$2,2$	$0,3$
D	$3,0$	$1,1$

Figure 297.1 The *Prisoner's Dilemma*.

We can think of an evolutionarily stable action as follows. A player assumes that whatever action it takes, its sibling will take the same action. An action is evolutionarily stable if, under this assumption, the action maximizes the player's payoff. An important assumption in reaching this conclusion is that reproduction is asexual. As Bergstrom (1995, 61) succinctly puts it, "Careful observers of human siblings will not be surprised to find that in sexually reproducing species, equilibrium behavior is not so perfectly cooperative".

13.5.2 Sexual reproduction

This model, like the ones in the previous sections, incorporates an unrefined model of reproduction and inheritance. We have assumed that each organism by itself produces offspring, which inherit their parent's behavior. For species (like humans) in which offspring are the result of two animals mating, this assumption is only a rough approximation. I now describe a model in which each player has two parents. We need to specify how behavior is inherited: what behavior does the offspring of parents with different modes of behavior inherit?

The model I describe goes back to the level of individual genes in order to answer this question. Each animal carries two genes. Each offspring of a pair of animals inherits one randomly chosen gene from each of its parents; the pair of genes that it carries is its *genotype*. Denote by a the action that an animal of genotype xx

(i.e. with two x genes) is programmed to take and denote by b the action that an animal of genotype XX is programmed to take. Suppose that $a \neq b$, and that an animal of genotype xx mates with an animal of genotype XX. All the offspring have genotype Xx, and there are two possibilities for the action taken by these offspring: a and b. If the offspring are programmed to take the action b, we say that X is *dominant* and x is *recessive*, and if they are programmed to take the action a then X is recessive and x is dominant.

Assume that all mating is monogamous: all siblings share the same two parents. Reproductive success depends on both parents' characteristics; it simplifies the discussion to assume that animals differ not in their fecundity, but in their chance of surviving to adulthood (the age at which they start reproducing).

Under what circumstances is a population of animals of genotype xx, each choosing the action a^*, evolutionarily stable? Genes are now the basic unit of analysis, from which behavior is derived, so we need to consider whether any mutant gene, say X, can invade the population. That is, we need to consider the consequences of an animal of genotype Xx being produced. There are two cases to consider: X may be dominant or recessive.

Invasion by dominant genes

First consider the case in which X is dominant. Denote the action taken by animals of genotype XX and Xx by b, and assume that $b \neq a^*$. (If $b = a^*$ then the mutation is inconsequential for behavior.) Since almost all animals have genotype xx, almost every mutant (of genotype Xx) mates with an animal of genotype xx. Each of the offspring of such a pair inherits an x gene from her xx parent, and a second gene from her genotype Xx parent that is x with probability $\frac{1}{2}$ and X with probability $\frac{1}{2}$. Thus each offspring has genotype xx with probability $\frac{1}{2}$ and genotype Xx with probability $\frac{1}{2}$.

We now need to compare the payoffs of mutants and normal animals. We are assuming that the mutation is rare, so every mutant Xx has one Xx parent and one xx parent. Thus in its random matchings with its siblings, such a mutant faces an Xx with probability $\frac{1}{2}$ and an xx with probability $\frac{1}{2}$. Hence its expected payoff is

$$\tfrac{1}{2}u(b, a^*) + \tfrac{1}{2}u(b, b).$$

Normal xx animals are present both in ("normal") families with two xx parents and in families with one xx parent and one Xx parent; the vast majority are in normal families. Thus to determine whether Xx's come to dominate the population we need to consider only the payoff (survival probability) of an Xx relative to that of an xx in a normal family. All the siblings of an xx in a normal family have genotype xx, and hence obtain the payoff

$$u(a^*, a^*).$$

We conclude that no dominant mutant gene can invade the population if

$$\tfrac{1}{2}u(b, a^*) + \tfrac{1}{2}u(b, b) < u(a^*, a^*) \text{ for every action } b.$$

Conversely, a dominant mutant gene *can* invade if the inequality is reversed for any action b.

If we define the function v by

$$v(b, a) = \tfrac{1}{2}u(b, a) + \tfrac{1}{2}u(b, b),$$

then, noting that $v(a, a) = u(a, a)$ for any action a, we can rewrite the sufficient condition for a^* to be evolutionarily stable as

$$v(b, a^*) < v(a^*, a^*) \text{ for every action } b.$$

That is, (a^*, a^*) is a strict Nash equilibrium *of the game with payoff function* v. If the inequality is reversed for any action b then a^* is not evolutionarily stable, so that a necessary condition for a^* to be evolutionarily stable is that (a^*, a^*) be a Nash equilibrium of the game with payoff function v.

In summary, a sufficient condition for a^* to be evolutionarily stable is that (a^*, a^*) be a strict Nash equilibrium of the game with payoff function v, in which a player's payoff is the average of its payoff in the original game and the payoff it obtains if its sibling mimics its behavior; a necessary condition is that (a^*, a^*) be a Nash equilibrium of this game.

Invasion by recessive genes

Now consider the case in which X is recessive. An animal of genotype Xx choose the same action a^* as does an animal of genotype xx in this case. In a family in which one parent has genotype xx and the other has genotype Xx, half the offspring have genotype xx and half have genotype Xx, and hence all take the action a^* and receive the payoff $u(a^*, a^*)$ in interactions with each other. Thus on this account the X gene neither invades the population nor is eliminated from it. To determine the fate of mutants, we need to consider the outcome of the interaction between siblings in families that constitute an even smaller fraction of the population.

The next smallest group of families are those in which the genotype of both parents is Xx, in which case one fourth of the offspring have genotype XX. Suppose that an animal of genotype XX takes the action $b \neq a^*$. If, in interactions with its siblings, such an animal is more successful than animals of genotypes xx or Xx then the mutant gene X, though starting from a very small base, can invade the population.

In families with two Xx parents, the genotypes of the offspring are distributed as follows: one fourth are xx, one half are Xx, and one fourth are XX. To find the expected payoff of an X gene in the offspring of such families, we need to consider each possible pair of siblings in turn. The analysis is somewhat complicated; I omit the details. The conclusion is that the expected payoff to an X gene is

$$\tfrac{1}{8}u(b, b) + \tfrac{1}{8}u(a^*, b) + \tfrac{3}{8}u(b, a^*) + \tfrac{3}{8}u(a^*, a^*).$$

The expected payoff of the "normal" gene x (which initially dominates the population, in families in which both parents are xx) is $u(a^*, a^*)$, so the mutant gene cannot invade the population if

$$\tfrac{1}{8}u(b,b) + \tfrac{1}{8}u(a^*,b) + \tfrac{3}{8}u(b,a^*) + \tfrac{3}{8}u(a^*,a^*) < u(a^*,a^*).$$

or

$$\tfrac{1}{5}u(b,b) + \tfrac{1}{5}u(a^*,b) + \tfrac{3}{5}u(b,a^*) < u(a^*,a^*).$$

If we define the function w by

$$w(a,b) = \tfrac{1}{5}u(a,a) + \tfrac{1}{5}u(b,a) + \tfrac{3}{5}u(a,b),$$

then the sufficient condition for evolutionary stability can be rewritten as

$$w(b,a^*) < w(a^*,a^*) \text{ for every action } b.$$

That is, (a^*,a^*) is a strict Nash equilibrium *of the game with payoff function* w. As before, a necessary condition for a^* to be evolutionarily stable is that (a^*,a^*) be a Nash equilibrium of this game.

Evolutionary stability

In order that a^* be evolutionarily stable, it must resist invasion by both dominant *and* recessive genes. Thus we have the following conclusion.

> If (a^*,a^*) is a strict Nash equilibrium of the game with payoff function v *and* a strict Nash equilibrium of the game with payoff function w then a population of players of genotype xx, choosing a^*, is evolutionarily stable. If (a^*,a^*) is not a Nash equilibrium of both these games then a^* is not evolutionarily stable.

Consider the implications for the *Prisoner's Dilemma*. The evolutionarily stable action depends on the relative magnitudes of the payoffs corresponding to each outcome. First consider the case of the payoff function in the left of Figure 300.1. In the middle and right figures the games with payoff functions v and w are shown.

	C	D
C	$5,5$	$0,6$
D	$6,0$	$2,2$

u

	C	D
C	$5,5$	$\frac{5}{2},4$
D	$4,\frac{5}{2}$	$2,2$

v

	C	D
C	$5,5$	$\frac{11}{5},4$
D	$4,\frac{11}{5}$	$2,2$

w

Figure 300.1 A *Prisoner's Dilemma*. On the left is the basic game, with payoff function u. In the middle is the game with payoff function v, and on the right is the game with payoff function w.

We see that (C,C) is a Nash equilibrium for both of the payoff functions v and w, and (D,D) is not a Nash equilibrium for either one. Hence in this case C is the only evolutionarily stable strategy in the game between siblings.

Now consider the case of the payoff function in the left of Figure 301.1. We see that (D,D) is a Nash equilibrium for both of the payoff function v and w, while *u*:

	C	D
C	$3, 3$	$0, 6$
D	$6, 0$	$2, 2$

v:

	C	D
C	$3, 3$	$\frac{3}{2}, 4$
D	$4, \frac{3}{2}$	$2, 2$

w:

	C	D
C	$3, 3$	$\frac{9}{5}, 4$
D	$4, \frac{9}{5}$	$2, 2$

Figure 301.1 A version of the *Prisoner's Dilemma*. On the left is the basic game, with payoff function u. In the middle is the game with payoff function v, and on the right is the game with payoff function w.

(C, C) is not a Nash equilibrium of either game. Hence in this case D is the only evolutionarily stable strategy in the game between siblings.

Thus in the *Prisoner's Dilemma*, whether or not siblings in a sexually reproducing species are cooperative or not depends on the gain to be had from being uncooperative. When this gain is small, the cooperative outcome is evolutionarily stable. Even though purely selfish behavior fails to sustain cooperation, the genetic similarity of siblings causes cooperative behavior to be evolutionarily stable. When the gain is large enough, however, the relatedness of siblings is not enough to overcome the pressure to defect, and the only evolutionarily stable outcome is joint defection.

? EXERCISE 301.1 (A coordination game between siblings) Consider the game in Figure 301.2. For what values of $x > 1$ is X the unique evolutionarily stable action when the game is played between siblings?

	X	Y
X	x, x	$0, 0$
Y	$0, 0$	$1, 1$

Figure 301.2 The game in Exercise 301.1.

13.6 Variation on a theme: nesting behavior of wasps

In all the situations I have analyzed so far, the players interact in pairs. In many situations the result of a player's action depends on the behavior of all the other players, not only on the action of one of these players; pairwise interactions cannot be identified. In this section I consider such a situation; the analysis illustrates how the methods of the previous sections can be generalized.

Female great golden digger wasps (*Sphex ichneumoneus*) lay their eggs in burrows, which must be stocked with katydids for the larva to feed on when they hatch. In a simple model, each wasp decides, when ready to lay an egg, whether to dig a burrow or to invade an existing burrow. A wasp that invades a burrow fights with the occupant, losing with probability π. If invading is less prevalent than digging then not all diggers are invaded, so that while digging takes time, it offers the possibility of laying an egg without a fight. The higher the proportion of invaders, the worse off is a wasp that digs its own burrow, since it is more likely to be invaded.

Each wasp's fitness is measured by the number of eggs it lays. Assuming that the length of a wasp's life is independent of its behavior, we can work with payoffs equal to the number of eggs laid per unit time. Let T_d be the time it takes for a wasp to build a burrow and stock it with katydids; let T_i be the time spent on a nest by an invader (T_i is not zero, since fighting takes time) and assume $T_i < T_d$. Assume that all wasps lay the same number of eggs in a nest, and choose the units in which eggs are measured so that this number is 1.

Suppose that the fraction of the population that digs is p and the fraction that invades is $1 - p$. In order to determine the probability that a digger is invaded, we need to take into account the fact that since invading takes less time than digging, an invader can invade more than one nest in the time that it takes a digger to dig. If invading takes half the time of digging, for example, and there are only half as many invaders as there are diggers in the population, then *all* diggers will be invaded—the probability of a digger being invaded is 1. In general, a digger can invade T_d/T_i burrows during a time period of length T_d. For every digger there are $(1-p)/p$ invaders, so the probability that a digger is invaded is $q = [(1-p)/p]T_d/T_i$, or $q = (1-p)T_d/(pT_i)$, assuming that this number is at most 1.

A wasp that digs its own burrow thus faces the following lottery: with probability $1 - q$ it is not invaded, with probability $q\pi$ it is invaded and wins the fight, and with probability $q(1-\pi)$ it is invaded and loses the fight (in which case assume that the whole time T_d is wasted). Thus the payoff—the expected number of eggs laid per unit time—of such a wasp is

$$(1 - q + q\pi)/T_d.$$

Similarly the expected number of eggs laid per unit time by an invader is $(1-\pi)/T_i$.

If $1/T_d \geq (1-\pi)/T_i$ there is an equilibrium in which every wasp digs its own burrow: the expected payoff to digging is at least the expected payoff to invading, given that $q = 0$. Clearly there is no equilibrium in which all wasps invade—for then there are no nests to invade! The remaining possibility is that there is an equilibrium in which diggers and invaders coexist in the population. In such an equilibrium the expected payoffs to the two activities must be equal, or $(1-q+q\pi)/T_d = (1-\pi)/T_i$. Substituting $(1-p)T_d/(pT_i)$ for q we find that $p = (1-\pi)T_d/T_i$. Looking back at the definition of q, we find that if the parameters π, T_i, and T_d satisfy $\pi T_i \leq (1-\pi)T_d$ then $q \leq 1$ for this value of p, so that we do indeed have an equilibrium.

Are these equilibria evolutionarily stable? First consider the equilibrium in which every wasp digs its own burrow. If $1/T_d > (1-\pi)/T_i$—that is, if the condition for the equilibrium to exist is satisfied *strictly*—then mutants that invade obtain a smaller payoff than the normal wasps that dig, and hence die out. Thus in this case the equilibrium is stable. (I do not consider the unlikely case that $1/T_d = (1-\pi)/T_i$.)

Now consider the equilibrium in which diggers and invaders coexist in the population. Suppose that there is a small mutation that increases slightly the fraction of diggers in the population. That is, p rises slightly. Then q, the probability of being invaded, falls, and the expected payoff to digging increases; the expected payoff to invading does not change. Thus a slight increase in p leads to an increase in the relative attractiveness of digging; diggers prosper relative to invaders, further increasing the value of p. We conclude that the equilibrium is not evolutionarily stable.

The polymorphic equilibrium I have analyzed can alternatively be interpreted as a mixed strategy equilibrium, in which each individual wasp randomizes between digging and invading, choosing to dig with probability p. In the populations that Brockmann et al. (1979) observe, digging and invading do coexist, and in fact individual wasps pursue mixed strategies—sometimes they dig and sometimes they invade. This evidence raises the question of how the model could be modified so that the mixed strategy equilibrium is evolutionarily stable. Brockman et al. suggest two such variants. In one case, for example, they assume that a wasp who digs a nest is better off if she is invaded and wins the fight than she is if she is not invaded (the invader may have helped to stock the nest with katydids before it got into a fight with the digger). The data Brockman et al. collected in one site generates a value of p that fits their observations very well; the data from another site does not fit well.

The following exercise illustrates another application of the main ideas of evolutionary game theory.

? EXERCISE 303.1 (*Darwin's theory of the sex ratio*) A population of males and females mate pairwise to produce offspring. Suppose that each offspring is male with probability p and female with probability $1 - p$. Then there is a steady state in which the fraction p of the population is male and the fraction $1 - p$ is female. If $p \neq \frac{1}{2}$ then males and females have different numbers of offspring (on average). Is such an equilibrium evolutionarily stable? Denote the number of children born to each female by n, so that the number of children born to each male is $(p/(1-p))n$. Suppose a mutation occurs that produces boys and girls each with probability $\frac{1}{2}$. Assume for simplicity that the mutant trait is dominant: if one partner in a couple has it, then all the offspring of the couple have it. Assume also that the number of children produced by a female with the trait is n, the same as for "normal" members of the population. Since both normal and mutant females produce the same number of children, it might seem that the fitness of a mutant is the same as that of a normal organism. But compare the number of *grand*children of mutants and normal organisms. How many female offspring does a normal organism produce? How many male offspring? Use your answers to find the number of grandchildren born to each mutant and to each normal organism. Does the mutant invade the population? Which value (values?) of p is evolutionarily stable?

Notes

[Incomplete.]

The main ideas in this chapter are due to Maynard Smith.

The chapter draws on the expositions of Hammerstein and Selten (1994) and van Damme (1987, Chapter 9).

Darwin's theory of sex ratio evolution (see the box on page 285) was independently discovered by Ronald A. Fisher (1930, 141–143), and is often referred to as "Fisher's theory". In the second edition of Darwin's book (1874, 256), he retracted his theory for reasons that are not apparent, and Fisher appears to have been aware only of the retraction, not of the original theory. Bulmer (1994, 207–208) appears to have been the first to notice that "Fisher's theory" was given by Darwin.

Hawk–Dove (Example 284.2) is due to Maynard Smith and Price (1973).

The discussion in Section 13.4 is based on van Damme (1987, Section 9.5).

Exercise 295.2 is a slightly less general version of Lemma 9.2.4 of van Damme (1987).

The material in Section 13.5 is taken from Bergstrom (1995).

The model in Section 13.6 is taken from Brockmann, Grafen, and Dawkins (1979), simplified along the lines of Bergstrom and Varian (1987, 324–327).

14 Repeated games: The Prisoner's Dilemma

Prerequisite: Chapters 5 and 7.

14.1 The main idea

MANY of the strategic interactions in which we are involved are ongoing: we repeatedly interact with the same people. In many such interactions we have the opportunity to "take advantage" of our co-players, but do not. We look after our neighbors' house while they're away, even if it is time-consuming for us to do so; we may give money to friends who are temporarily in need. The theory of repeated games provides a framework that we can use to study such behavior.

The basic idea in the theory is that a player may be deterred from exploiting her short-term advantage by the "threat" of "punishment" that reduces her long-term payoff. Suppose, for example, that two people are involved repeatedly in an interaction for which the short-term incentives are captured by the *Prisoner's Dilemma* (see Section 2.2), with payoffs as in Figure 389.1. Think of C as "cooperation" and D as "defection".

	C	D
C	$2, 2$	$0, 3$
D	$3, 0$	$1, 1$

Figure 389.1 The *Prisoner's Dilemma*.

As we know, the *Prisoner's Dilemma* has a unique Nash equilibrium, in which each player chooses D. Now suppose that a player adopts the following long-term strategy: choose C so long as the other player chooses C; if in any period the other player chooses D, then choose D in *every subsequent period*. What should the other player do? If she chooses C in every period then the outcome is (C, C) in every period and she obtains a payoff of 2 in every period. If she switches to D in some period then she obtains a payoff of 3 in that period and a payoff of 1 in every subsequent period. She may value the present more highly than the future—she may be *impatient*—but as long as the value she attaches to future payoffs is not too small compared with the value she attaches to her current payoff, the stream of payoffs $(3, 1, 1, \ldots)$ is worse for her than the stream $(2, 2, 2, \ldots)$, so that she is better off choosing C in every period.

This argument shows that if a player is sufficiently patient, the strategy that chooses C after every history is a best response to the strategy that starts off choosing C and "punishes" any defection by switching to D. Clearly another best response is this same punishment strategy: if your opponent is using this punishment strategy then the outcome is the same if you use the strategy that chooses C after every history, or the same punishment strategy as your opponent is using. In both cases, the outcome in every period is (C, C) (the other player never defects, so if you use the punishment strategy you are never induced to switch to punishment). Thus the strategy pair in which both players use the punishment strategy is a Nash equilibrium of the game: neither player can do better by adopting another long-term strategy.

The conclusion that the repeated *Prisoner's Dilemma* has a Nash equilibrium in which the outcome is (C, C) in every period accords with our intuition that in long-term relationships there is scope for mutually supportive strategies that do not relentlessly exploit short-term gain. However, this strategy pair is not the only Nash equilibrium of the game. Another Nash equilibrium is the strategy pair in which each player chooses D after every history: if one player adopts this strategy then the other player can do no better than to adopt the strategy herself, regardless of how she values the future, since whatever she does has no effect on the other player's behavior.

This analysis leaves open many questions.

- We have seen that the outcome in which (C, C) occurs in every period is supported as a Nash equilibrium if the players are sufficiently patient. Exactly how patient do they have to be?

- We have seen also that the outcome in which (D, D) occurs in every period is supported as a Nash equilibrium. What other outcomes are supported?

- We saw in Chapter 5 that Nash equilibria of extensive games are not always intuitively appealing, since the actions they prescribe after histories that result from deviations may not be optimal. The notion of subgame perfect equilibrium, which requires actions to be optimal after every possible history, not only those that are reached if the players adhere to their strategies, may be more appealing. Is the strategy pair in which each player uses the punishment strategy I have described a subgame perfect equilibrium? That is, is it optimal for each player to punish the other player for deviating? If not, is there any other strategy pair that supports desirable outcomes and is a subgame perfect equilibrium?

- The punishment strategy studied above is rather severe; in switching permanently to D in response to a deviation it leaves no room for error. Are there any Nash equilibria or subgame perfect equilibria in which the players' strategies punish deviations less severely?

- The arguments above are restricted to the *Prisoner's Dilemma*. To what other games do they apply?

I now formulate the model of a repeated game more precisely in order to answer these questions.

14.2 Preferences

14.2.1 Discounting

The outcome of a repeated game is a sequence of outcomes of a strategic game. How does each player evaluate such sequences? I assume that she associates a payoff with each outcome of the strategic game, and evaluates each sequence of outcomes by the **discounted sum** of the associated sequence of payoffs. More precisely, each player i has a payoff function u_i for the strategic game and a discount factor δ between 0 and 1 such that she evaluates the sequence $(a^1, a^2, \ldots, a^T)$ of outcomes of the strategic game by the sum

$$u_i(a^1) + \delta u_i(a^2) + \delta^2 u_i(a^3) + \cdots + \delta^{T-1} u_i(a^T) = \sum_{t=1}^{T} \delta^{t-1} u_i(a^t).$$

(Note that in this expression superscripts are used for two purposes: a^t is the action profile in period t, while δ^t is the discount factor δ raised to the power t.) I assume throughout that all players have the same discount factor δ. A player whose discount factor is close to zero cares very little about the future—she is very impatient; a player whose discount factor is close to one is very patient.

Why should a person value future payoffs less than current ones? Possibly she is simply impatient. Or, possibly, her underlying preferences do not display impatience, but in comparing streams of outcomes she takes into account the positive probability with which she may die in any given period.[1] Or, if the outcome in each period involves the payment to her of some amount of money, possibly impatience is induced by the fact that she can borrow and lend at a positive interest rate. For example, suppose her underlying preferences over streams of monetary payoffs do not display impatience. Then if she can borrow and lend at the interest rate r she is indifferent between the sequence $(\$100, \$100, 0, 0, \ldots)$ of amounts of money

[1] Alternatively, the hazard of death may have favored those who reproduce early, leading to the evolution of people who are "impatient".

and the sequence $(\$100 + \$100/(1 + r), 0, 0, \ldots)$, since by lending $\$100/(1 + r)$ of the amount she obtains in the first period she obtains \$100 in the second period. In fact, under these assumptions her preferences are represented precisely by the discounted sum of her payoffs with a discount factor of $1/(1 + r)$: any stream can be obtained from any other stream with the same discounted sum by borrowing and lending. (If you win one of the North American lotteries that promises \$1m you will quickly learn about discounted values: you will receive a stream of 20 yearly payments each of \$50,000, which at an interest rate of 7% is equivalent to receiving about \$567,000 as a lump sum.)

Obviously the assumption that everyone's preferences over sequences of outcomes are represented by a discounted sum of payoffs is restrictive: people's preferences do not *necessarily* take this form. However, a discounted sum captures simply the idea that people may value the present more highly than the future and appears not to obscure any other feature of preferences significant to the problem we are considering.

14.2.2 Equivalent payoff functions

When we considered preferences over atemporal outcomes and atemporal lotteries, we found that many payoff functions represent the same preferences. Specifically, if u is a payoff function that represents a person's preferences over deterministic outcomes, then any increasing function of u also represents her preferences. If u is a Bernoulli payoff function whose expected value represents a person's preferences over lotteries, then the expected value of any increasing affine function of u also represents her preferences.

Consider the same question for preferences over sequences of outcomes. Suppose that a person's preferences are represented by the discounted sum of payoffs with payoff function u and discount factor δ. Then if the two sequences of outcomes $(x^1, x^2, \ldots)$ and $(y^1, y^2, \ldots)$ are indifferent, we have

$$\sum_{t=0}^{\infty} \delta^{t-1} u(x^t) = \sum_{t=0}^{\infty} \delta^{t-1} u(y^t).$$

Now let v be an increasing affine function of u: $v(x) = \alpha + \beta u(x)$ with $\beta > 0$. Then

$$\sum_{t=0}^{\infty} \delta^{t-1} v(x^t) = \sum_{t=0}^{\infty} \delta^{t-1} [\alpha + \beta u(x^t)] = \sum_{t=0}^{\infty} \delta^{t-1} \alpha + \beta \sum_{t=0}^{\infty} \delta^{t-1} u(x^t)$$

and similarly

$$\sum_{t=0}^{\infty} \delta^{t-1} v(y^t) = \sum_{t=0}^{\infty} \delta^{t-1} [\alpha + \beta u(y^t)] = \sum_{t=0}^{\infty} \delta^{t-1} \alpha + \beta \sum_{t=0}^{\infty} \delta^{t-1} u(y^t),$$

so that

$$\sum_{t=0}^{\infty} \delta^{t-1} v(x^t) = \sum_{t=0}^{\infty} \delta^{t-1} v(y^t).$$

Thus the person's preferences are represented also by the discounted sum of payoffs with payoff function v and discount factor δ. That is, if a person's preferences are represented by the discounted sum of payoffs with payoff function u and discount factor δ then they are also represented by the discounted sum of payoffs with payoff function $\alpha + \beta u$ and discount factor δ, for any α and any $\beta > 0$.

In fact, as in the case of payoff representations of preferences over lotteries (see Lemma 145.1), the converse is also true: if preferences over a stream of outcomes are represented by the discounted sum of payoffs with payoff function u and discount factor δ, and also by the discounted sum of payoffs with payoff function v and discount factor δ, then v must be an increasing affine function of u.

■ LEMMA 393.1 (Equivalence of payoff functions under discounting) *Suppose there are at least three possible outcomes. The discounted sum of payoffs with the payoff function u and discount factor δ represents the same preferences over streams of payoffs as the discounted sum of payoffs with the payoff function v and discount factor δ if and only if there exist α and $\beta > 0$ such that $u(x) = \alpha + \beta v(x)$ for all x.*

The significance of this result is that the payoffs in the strategic games that generate the repeated games we now study are no longer simply ordinal, even if we restrict attention to deterministic outcomes. For example, the players' preferences in the repeated game based on a *Prisoner's Dilemma* with the payoffs given in Figure 389.1 are different from the players' preferences in the repeated game based on the variant of this game in which the payoff pairs $(0, 3)$ and $(3, 0)$ are replaced by $(0, 5)$ and $(5, 0)$. (When the discount factor is close enough to 1, for instance, each player prefers the sequence of outcomes $((C, C), (C, C))$ to the sequence of outcomes $((D, C), (C, D))$ in the first case, but not in the second case.) Thus I refer to *a* repeated *Prisoner's Dilemma*, rather than *the* repeated *Prisoner's Dilemma*. More generally, throughout the remainder of this chapter I define strategic games in terms of payoff functions rather than preferences: a **strategic game** consists of a set of players, and, for each player, a set of actions and a payoff function.

If a player's preferences over streams $(w^1, w^2, \ldots)$ of payoffs are represented by the discounted sum $\sum_{t=1}^{\infty} \delta^{t-1} w^t$ of these payoffs, where $\delta < 1$, then they are also represented by the **discounted average** $(1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} w^t$ of these payoffs (since this discounted average is simply a constant times the discounted sum). The discounted average has the advantage that its values are directly comparable to the payoffs in a single period. Specifically, for any discount factor δ between 0 and 1 the constant stream of payoffs $(c, c, \ldots)$ has discounted average $(1 - \delta)(c + \delta c + \delta^2 c + \cdots) = c$ (see (449.2)). For this reason I subsequently work with the discounted average rather than the discounted sum.

14.3 Infinitely repeated games

I start by studying a model of a repeated interaction in which play may continue indefinitely—there is no fixed final period. In many situations play cannot continue indefinitely. But the assumption that it can may nevertheless capture well the players' perceptions. The players may be aware that play cannot go on forever, but, especially if the termination date is very far in the future, may ignore this fact in their strategic reasoning. (I consider a model in which there is a definite final period in Section 15.3.)

A repeated game is an extensive game with perfect information and simultaneous moves. A history is a sequence of action profiles in the strategic game. After every nonterminal history, *every* player i chooses an action from the set of actions available to her in the strategic game.

▶ DEFINITION 394.1 Let G be a strategic game. Denote the set of players by N and the set of actions and payoff function of each player i by A_i and u_i respectively. The **infinitely repeated game of** G for the discount factor δ is the extensive game with perfect information and simultaneous moves in which

- the set of players is N
- the set of terminal histories is the set of infinite sequences $(a^1, a^2, \ldots)$ of action profiles in G
- the player function assigns the set of all players to every proper subhistory of every terminal history
- the set of actions available to player i after any history is A_i
- each player i evaluates each terminal history $(a^1, a^2, \ldots)$ according to its discounted average $(1-\delta)\sum_{t=1}^{\infty} \delta^{t-1} u_i(a^t)$.

14.4 Strategies

A player's strategy in an extensive game specifies her action after all possible histories after which it is her turn to move, including histories that are inconsistent with her strategy (Definition 203.2). Thus a strategy of player i in an infinitely repeated game of the strategic game G specifies an action of player i (a member of A_i) for every sequence $(a^1, \ldots, a^T)$ of outcomes of G.

For example, if player i's strategy s_i is the one discussed at the beginning of this chapter, it is defined as follows: $s_i(\varnothing) = C$ and

$$s_i(a^1, \ldots, a^t) = \begin{cases} C & \text{if } a_j^\tau = C \text{ for } \tau = 1, \ldots, t \\ D & \text{otherwise.} \end{cases} \qquad (394.2)$$

That is, player i chooses C at the start of the game (after the initial history $\varnothing$) and after any history in which every previous action of player j was C; she chooses D after every other history. We refer to this strategy as a *grim trigger strategy*, since it is a mode of behavior in which a defection by the other player triggers relentless ("grim") punishment.

We can think of the strategy as having two *states*: one, call it $\mathcal{C}$, in which C is chosen, and another, call it $\mathcal{D}$, in which D is chosen. Initially the state is $\mathcal{C}$; if the other player chooses D in any period then the state changes to $\mathcal{D}$, where it stays

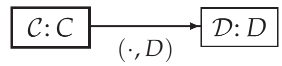


Figure 395.1 A grim trigger strategy for an infinitely repeated *Prisoner's Dilemma*.

forever. Figure 395.1 gives a natural representation of the strategy when we think of it in these terms. The box with a bold outline is the initial state, $\mathcal{C}$, in which the player chooses the action C. If the other player chooses D (indicated by the $(\cdot, D)$ under the arrow) then the state changes to $\mathcal{D}$, in which the player chooses D. If the other player does not choose D (i.e. chooses C) then the state remains $\mathcal{C}$. (The convention in the diagrams is that the state remains the same unless an event occurs that is a label for one of the arrows emanating from the state.) Once $\mathcal{D}$ is reached it is never left: there is no arrow leaving the box for state $\mathcal{D}$.

Any strategy can be represented in a diagram like Figure 395.1. In many cases, such a diagram is easier to interpret than a symbolic specification of the action taken after each history like (394.2). Note that since a player's strategy must specify her action after all histories, including those that do not occur if she follows her strategy, the diagram that represents a strategy must include, for every state, a transition for each of the possible outcomes in the game. In particular, if in some state the strategy calls for the player to choose the action B, then there must be one transition from the state for each of the cases in which the player chooses an action *different* from B. Figure 395.1 obscures this fact, since the event that triggers a change in the player's action is an action of her opponent; none of her own actions trigger a change in the state, so that the (null) transitions that her own actions induce are not indicated explicitly in the diagram.

A strategy that entails less draconian punishment is shown in Figure 395.2. This strategy punishes deviations for only three periods: it responds to a deviation by choosing the action D for three periods, then reverting to C, no matter how the other player behaved during her punishment.

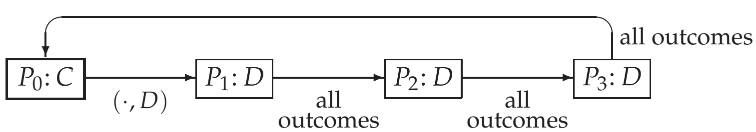


Figure 395.2 A strategy in an infinitely repeated *Prisoner's Dilemma* that punishes deviations for three periods.

In the strategy *tit-for-tat* the length of the punishment depends on the behavior of the player being punished. If she continues to choose D then *tit-for-tat* continues to do so; if she reverts to C then *tit-for-tat* reverts to C also. The strategy can be given a very compact description: do whatever the other player did in the previous period. It is illustrated in Figure 396.1.

? EXERCISE 395.1 (Strategies in the infinitely repeated *Prisoner's dilemma*) Represent each of the following strategies s in an infinitely repeated *Prisoner's Dilemma* in a

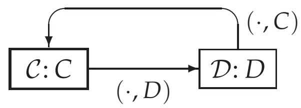


Figure 396.1 The strategy *tit-for-tat* in an infinitely repeated *Prisoner's Dilemma*.

diagram like Figure 395.1.

a. Choose C in period 1, and after any history in which the other player chose C in every period except, possibly, the previous period; choose D after any other history. (That is, punishment is grim, but its initiation is delayed by one period.)

b. Choose C in period 1 and after any history in which the other player chose D in at most one period; choose D after any other history. (That is, punishment is grim, but a single lapse is forgiven.)

c. (*Pavlov*, or *win-stay, lost-shift*) Choose C in period 1 and after any history in which the outcome in the last period is either (C, C) or (D, D); choose D after any other history. (That is, choose the same action again if the outcome was relatively good for you, and switch actions if it was not.)

14.5 Some Nash equilibria of the infinitely repeated Prisoner's Dilemma

If one player chooses D after every history in an infinitely repeated *Prisoner's Dilemma* then it is clearly optimal for the other player to do the same (since (D, D) is a Nash equilibrium of the *Prisoner's Dilemma*). The argument at the start of the chapter suggests that an infinitely repeated *Prisoner's Dilemma* has other, less dismal, equilibria, so long as the players are sufficiently patient—for example, the strategy pair in which each player uses the grim trigger strategy defined in Figure 395.1. I now make this argument precise. Throughout I consider the infinitely repeated *Prisoner's Dilemma* in which each player's discount factor is δ and the one-shot payoffs are given in Figure 389.1.

14.5.1 Grim trigger strategies

Suppose that player 1 adopts the grim trigger strategy. If player 2 does so then the outcome is (C, C) in every period and she obtains the stream of payoffs $(2, 2, \ldots)$, whose discounted average is 2. If she adopts a strategy that generates a different sequence of outcomes then there is one period (at least) in which she chooses D. In all subsequent periods player 1 chooses D (player 2's choice of D triggers the grim punishment), so the best deviation for player 2 chooses D in every subsequent period (since D is her unique best response to D). Further, if she can increase her payoff by deviating then she can do so by deviating to D in the first period. If she does so she obtains the stream of payoffs $(3, 1, 1, \ldots)$ (she gains one unit of payoff in the first period, then loses one unit in every subsequent period), whose discounted average is

$$(1-\delta)[3+\delta+\delta^2+\delta^3+\cdots] = 3(1-\delta)+\delta.$$

Thus she cannot increase her payoff by deviating if and only if

$$2 \geq 3(1-\delta)+\delta,$$

or $\delta \geq \frac{1}{2}$. We conclude that if $\delta \geq \frac{1}{2}$ then the strategy pair in which each player's strategy is the grim trigger strategy defined in Figure 395.1 is a Nash equilibrium of the infinitely repeated *Prisoner's Dilemma* with one-shot payoffs as in Figure 389.1.

14.5.2 Limited punishment

Now consider a generalization of the limited punishment strategy in Figure 395.2 in which a player who chooses D is punished for k periods. (The strategy in Figure 395.2 has $k = 3$; the grim punishment strategy corresponds to $k = \infty$.) If one player adopts this strategy, is it optimal for the other to do so? Suppose that player 1 does so. As in the argument for the grim trigger strategy, if player 2 can increase her payoff by deviating then she can increase her payoff by deviating in the first period. So suppose she chooses D in the first period. Then player 1 chooses D in each of the next k periods, regardless of player 2's choices, so player 2 also should choose D in these periods. In the $(k+1)$st period after the deviation player 1 switches back to C (regardless of player 2's behavior in the previous period), and player 2 faces precisely the same situation that she faced at the beginning of the game. Thus if her deviation increases her payoff, it increases her payoff during the first $k+1$ periods. If she adheres to her strategy then her discounted average payoff during these periods is

$$(1-\delta)[2+2\delta+2\delta^2+\cdots+2\delta^k] = 2(1-\delta^{k+1})$$

(see (449.1)), whereas if she deviates as described above then her payoff during these periods is

$$(1-\delta)[3+\delta+\delta^2+\cdots+\delta^k] = 3(1-\delta)+\delta(1-\delta^k).$$

Thus she cannot increase her payoff by deviating if and only if

$$2(1-\delta^{k+1}) \geq 3(1-\delta)+\delta(1-\delta^k),$$

or $\delta^{k+1} - 2\delta + 1 \leq 0$. If $k = 1$ then no value of δ less than 1 satisfies the inequality: one period of punishment is not severe enough to discourage a deviation, however patient the players are. If $k = 2$ then the inequality is satisfied for $\delta \geq 0.62$, and if $k = 3$ it is satisfied for $\delta \geq 0.55$. As k increases the lower bound on δ approaches $\frac{1}{2}$, the lower bound for the grim strategy.

We conclude that the strategy pair in which each player punishes the other for k periods in the event of a deviation is a Nash equilibrium of the infinitely repeated game so long as $k \geq 2$ and δ is large enough; the larger is k, the smaller is the lower bound on δ. Thus short punishment is effective in sustaining the mutually desirable outcome (C, C) only if the players are very patient.

14.5.3 Tit-for-tat

Now consider the conditions under which the strategy pair in which each player uses the strategy *tit-for-tat* is a Nash equilibrium. Suppose that player 1 adheres to this strategy. Then, as above, if player 2 can gain by deviating then she can gain by choosing D in the first period. If she does so, then player 1 chooses D in the second period, and continues to choose D until player 2 reverts to C. Thus player 2 has two options: she can revert to C, in which case in the next period she faces the same situation as she did at the start of the game, or she can continue to choose D, in which case player 1 will continue to do so too. We conclude that if player 2 can increase her payoff by deviating then she can do so either by alternating between D and C or by choosing D in every period. If she alternates between D and C then her stream of payoffs is $(3, 0, 3, 0, \ldots)$, with a discounted average of $(1-\delta) \cdot 3/(1-\delta^2) = 3/(1+\delta)$, while if she chooses D in every period her stream of payoffs is $(3, 1, 1, \ldots)$, with a discounted average of $3(1-\delta) + \delta = 3 - 2\delta$. Since her discounted average payoff to adhering to the strategy *tit-for-tat* is 2, we conclude that *tit-for-tat* is a best response to *tit-for-tat* if and only if

$$2 \geq \frac{3}{1+\delta} \text{ and } 2 \geq 3 - 2\delta.$$

Both of these conditions are equivalent to $\delta \geq \frac{1}{2}$.

Thus if $\delta \geq \frac{1}{2}$ then the strategy pair in which the strategy of each player is *tit-for-tat* is a Nash equilibrium of the infinitely repeated *Prisoner's Dilemma* with payoffs as in Figure 389.1.

? EXERCISE 398.1 (Nash equilibria of the infinitely repeated *Prisoner's Dilemma*) For each of the three strategies s in Exercise 395.1 determine the values of δ, if any, for which the strategy pair (s, s) is a Nash equilibrium of an infinitely repeated *Prisoner's Dilemma* with discount factor δ and the one-shot payoffs given in Figure 389.1. For each strategy s for which there is no value of δ such that (s, s) is a Nash equilibrium of this game, determine whether there are any payoffs for the *Prisoner's Dilemma* such that for some δ the strategy pair (s, s) is a Nash equilibrium of the infinitely repeated game with discount factor δ.

14.6 Nash equilibrium payoffs of the infinitely repeated Prisoner's Dilemma when the players are patient

All the Nash equilibria of the infinitely repeated *Prisoner's Dilemma* that I have discussed so far generate either the outcome (C, C) in every period or the outcome

(D, D) in every period. The first outcome path yields the discounted average payoff of 2 to each player, while the second outcome path yields the discounted average payoff of 1 to each player. What other discounted average payoffs are consistent with Nash equilibrium? It turns out that this question is hard to answer for an arbitrary discount factor. The question is relatively straightforward to answer, however, in the case that the discount factor is close to 1 (the players are very patient). Before tackling it, we need to determine the set of discounted average pairs of payoffs that are *feasible*—i.e. can be achieved by outcome paths.

14.6.1 Feasible discounted average payoffs

If the outcome is (X, Y) in every period then the discounted average payoff is $(u_1(X, Y), u_2(X, Y))$, for any X and Y. Thus $(2, 2)$, $(3, 0)$, $(0, 3)$, and $(1, 1)$ can all be achieved as pairs of discounted average payoffs.

Now consider the path in which the outcome alternates between (C, C) and (C, D). Along this path player 1's payoff alternates between 2 and 0 while player 2's alternates between 2 and 3. Thus the players' average payoffs along the path are 1 and $\frac{5}{2}$ respectively. Since player 1 receives more of her payoff in the first period of each two-period cycle than in the second period (in fact, she obtains nothing in the second period), her *discounted* average payoff exceeds 1, whatever the discount factor. But if the discount factor is close to 1 then her discounted average payoff is close to 1: the fact that more payoff is obtained in the first period of each two-period cycle is insignificant if the discount factor is close to 1. Similarly, since player 2 receives most of her payoff in the second period of each two-period cycle, her discounted average payoff is less than $\frac{5}{2}$, whatever the discount factor, but is close to $\frac{5}{2}$ when the discount factor is close to 1. Thus $(1, \frac{5}{2})$ can approximately be achieved as a pair of discounted average payoffs when the discount factor is close to 1.

This argument can be extended to any outcome path in which a sequence of outcomes is repeated. If the discount factor is close to 1 then a player's discounted average payoff on such a path is close to her average payoff in the sequence. For example, the outcome path that consists of repetitions of the sequence $((C, C), (D, C), (D, C))$ yields player 1 a discounted average payoff close to $\frac{1}{3}(2 + 3 + 3) = \frac{8}{3}$ and player 2 a discounted average payoff close to $\frac{1}{3}(2 + 0 + 0) = \frac{2}{3}$.

We conclude that the average of the payoffs to any sequence of outcomes can approximately be achieved as the discounted average payoff if the discount factor is close to 1. Further, if the discount factor is close to 1 then only such discounted average payoffs can be achieved. Thus if the discount factor is close to 1, the set of *feasible* discounted average payoff pairs in the infinitely repeated game is approximately the set of all pairs of weighted averages of payoffs in the component game. The same argument applies to any strategic game, and for convenience I make the following definition.

▶ DEFINITION 399.1 The set of **feasible** payoff profiles of a strategic game is the set of all weighted averages of payoff profiles in the game.

This definition is standard. Note, however, that the name "feasible" is a little misleading, in the sense that a feasible payoff profile is *not* in general achievable in the game, but only (approximately) as a discounted average payoff profile in the infinitely repeated game.

It is useful to represent the set of feasible payoff pairs in the *Prisoner's Dilemma* geometrically. Suppose that (x_1, x_2) and (y_1, y_2) are in the set. Now fix integers k and m with $m > k$ and consider the outcome path that consists of k repetitions of the cycle of outcomes that yields (x_1, x_2) followed by $m - k$ repetitions of the cycle that yields (y_1, y_2), and continues indefinitely with repetitions of this whole cycle. The average payoff pair on this outcome path is $(k/m)(x_1, x_2) + (1 - k/m)(y_1, y_2)$. This point lies on the straight line joining (x_1, x_2) and (y_1, y_2). As we vary k and m essentially all points on this straight line are achieved. (Precisely, every point that is a weighted average of (x_1, x_2) and (y_1, y_2) with rational weights are achieved.) We conclude that the set of feasible discounted average payoffs is the parallelogram in Figure 400.1.

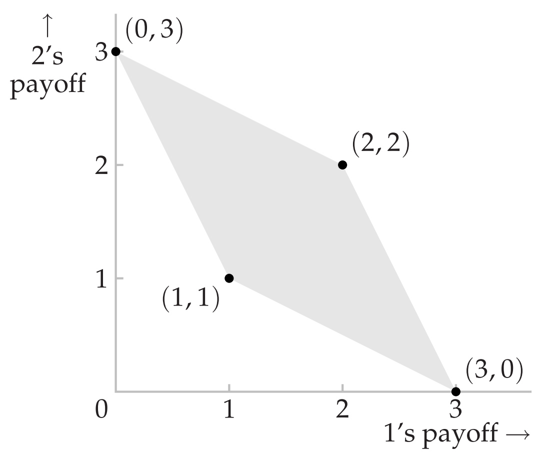


Figure 400.1 The set of feasible payoffs in the *Prisoner's Dilemma* with payoffs as in Figure 389.1. Any pair of payoffs in this set can approximately be achieved as a pair of discounted average payoffs in the infinitely repeated game when the discount factor is close to 1.

14.6.2 Nash equilibrium discounted average payoffs

We have seen that the feasible payoff pairs $(2, 2)$ and $(1, 1)$ can be achieved as discounted average payoffs pairs in Nash equilibria. Which other feasible payoff pairs can be achieved in Nash equilibria? By choosing D in every period, each player can obtain a payoff of at least 1 in each period, and hence a discounted average payoff of at least 1. Thus no pair of payoffs in which either player's payoff is less than 1 is the discounted average payoff pair of a Nash equilibrium.

I claim further that every feasible pair of payoffs in which each player's payoff is greater than 1 is close to a pair of payoffs that is the discounted average payoff pair of a Nash equilibrium when the discount factor is close enough to 1. For any feasible pair (x_1, x_2) of payoffs there is a finite sequence $(a^1, \ldots, a^k)$ of outcomes for which each player i's average payoff is x_i, so that her *discounted* average payoff can be made as close as we want to x_i by taking the discount factor close enough to 1.

Now consider the outcome path of the infinitely repeated games that consists of repetitions of the sequence $(a^1, \ldots, a^k)$; denote this outcome path by $(b^1, b^2, \ldots)$. (That is, $b^1 = b^{k+1} = b^{2k+1} = \ldots = a^1$, $b^2 = b^{k+2} = b^{2k+2} = \ldots = a^2$, and so on.) I now construct a strategy profile that yields this outcome path and, for a large enough discount factor, is a Nash equilibrium. In each period, each player's strategy chooses the action specified for her by the path so long as the other player did so in every previous period, and otherwise chooses the "punishment" action D. Precisely, player i's strategy s_i chooses the action b_i^1 in the first period and the action

$$s_i(h^1, \ldots, h^{t-1}) = \begin{cases} b_i^t & \text{if } h_j^r = b_j^r \text{ for } r = 1, \ldots, t-1 \\ D & \text{otherwise,} \end{cases}$$

after any other history $(h^1, \ldots, h^{t-1})$, where j is the other player. If every player adheres to this strategy then the outcome in each period t is b^t, so that the average payoff of each player i is x_i. Thus the discounted average payoff of each player i can be made arbitrarily close to x_i by choosing the discount factor to be close enough to 1.

If $x_i > 1$ for each player i then the strategy profile is a Nash equilibrium by the following argument. First note that since for each player i we have $x_i > 1$, for each player i there is an integer, say t_i, for which $u_i(a^{t_i}) > 1$. Now suppose that one of the players, say i, deviates from the path $(b^1, b^2, \ldots)$ in some period. In every subsequent period player j chooses D, so that player i's payoff is at most 1. In particular, in every period in which the outcome was supposed to be a^{t_i}, player i obtains the payoff 1 rather than $u_i(a^{t_i}) > 1$. If the discount factor is close enough to 1 then the discounted value of these future losses more than outweigh any gain that player i may have pocketed in the period in which she deviated. Hence for a discount factor close enough to 1, each player i is better off adhering to the strategy s_i than she is deviating, so that (s, s) is a Nash equilibrium. Further, by taking the discount factor close enough to 1 we can ensure that the discounted average payoff pair of the outcome path that (s, s) generates is arbitrarily close to (x_1, x_2).

In summary, we have proved the following result for the infinitely repeated *Prisoner's Dilemma* generated by the one-shot game with payoffs given in Figure 389.1:

- for any discount factor, each player's payoff in every discounted average payoff pair generated by a Nash equilibrium of the infinitely repeated game is at least 1

- for every feasible pair (x_1, x_2) of payoffs in the game for which $x_i > 1$ for each player i, there is a pair (y_1, y_2) close to (x_1, x_2) such that for a discount factor close enough to 1 there is a Nash equilibrium of the infinitely repeated game in which the pair of discounted average payoffs is (y_1, y_2).

(This result is a special case of a result I state, precisely, later; see Proposition 413.1.)

You may wonder why the second part of this statement is not simpler: why do I not claim that any outcome path in which every player's discounted average payoff exceeds 1 can be generated by a Nash equilibrium? The reason is simple: this claim is not true! Consider, for example, the outcome path $((C, C), (D, D), (D, D), \ldots)$ in which the outcome in every period but the first is (D, D). For any discount factor less than 1 each player's discounted average payoff exceeds 1 on this path, but no Nash equilibrium generates the path: a player who deviates to D in the first period obtains a higher payoff in the first period and at least the *same* payoff in every subsequent period, however her opponent behaves.

The set in Figure 402.1 illustrates the set of discounted average payoffs generated by Nash equilibria. For every point (x_1, x_2) in the set, by choosing the discount factor close enough to 1 we can ensure that there is a point (y_1, y_2) as close as we want to (x_1, x_2) that is the pair of discounted average payoffs of the infinitely repeated game. The diagram makes it clear how large the set of Nash equilibrium payoffs of the repeated game is: even though the one-shot game has a unique Nash equilibrium, and hence a unique pair of Nash equilibrium payoffs, the repeated game has a large set of Nash equilibria, with payoffs that vary from dismal to jointly maximal.

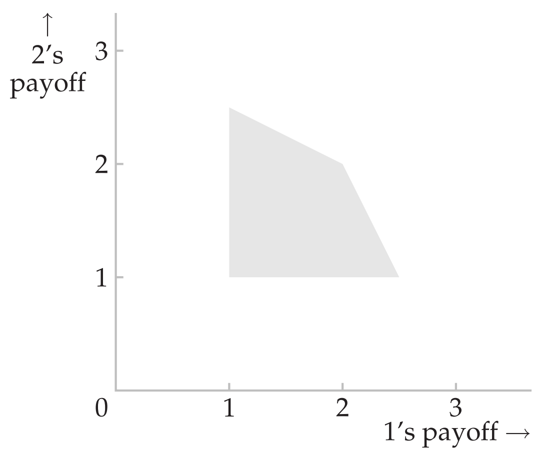


Figure 402.1 The approximate set of Nash equilibrium discounted average payoffs for the infinitely repeated *Prisoner's Dilemma* with one-shot payoffs as in Figure 389.1 when the discount factor is close to 1.

14.7 Subgame perfect equilibria and the one-deviation property

We saw in Section ????? that a strategy profile in a finite horizon extensive game is a subgame perfect equilibrium if and only if it satisfies the *one-deviation property*: no player can increase her payoff by changing her action at the start of any subgame in which she is the first mover, *given* the other player's strategies *and* the rest of her own strategy. I now argue that the same is true in an infinitely repeated game, a fact that can greatly simplify the process of determining whether or not a strategy profile is a subgame perfect equilibrium.

As in the case of a finite horizon game, if a strategy profile is a subgame perfect equilibrium then certainly it satisfies the one-deviation property, since no player must be able to increase her payoff by *any* change in her strategy. What we need to show is the converse: if a strategy profile is not a subgame perfect equilibrium then there is some subgame in which the first-mover can increase her payoff by changing *only* her initial action.

Let s be a strategy profile that is not a subgame perfect equilibrium. Specifically, suppose that in the subgame following the nonterminal history h, player i can increase her payoff by using the strategy s_i' rather than s_i. Now, since payoffs in the distant future are worth very little, there is some period T such that any strategy that coincides with s_i' through period T is better than any strategy that coincides with s_i through period T: T can be chosen to be sufficiently large that the first strategy yields a higher discounted average payoff than the second one even if the first strategy induces the best possible outcome for player i in every period after T, and the second strategy induces the worst possible outcome in every such period. In particular, the strategy s_i'' that coincides with s_i' through period T and with s_i after period T is better for player i than s_i.

But now by the same argument as for finite horizon games (Proposition ????), we can find a strategy for player i and a subgame such that in the subgame the strategy differs from s_i only in its first action and yields a payoff higher than that yielded by s_i (given that the other players adhere to s_{-i}). A more precise statement of the result and proof follows.

■ PROPOSITION 403.1 (One-deviation property of subgame perfect equilibria of infinitely repeated games) *A strategy profile in an infinitely repeated game is a subgame perfect equilibrium if and only if no player can gain by changing her action after any history, given both the strategies of the other players and the remainder of her own strategy.*

Proof. If the strategy profile s is a subgame perfect equilibrium then no player can gain by any deviation, so that if some player can gain by a one-period deviation then s is definitely not a subgame perfect equilibrium.

I now need to show that if s is not a subgame perfect equilibrium then in the subgame that follows some history h, some player, say i, can gain by a one-period deviation from s_i. Without loss of generality, assume that h is the initial history.

Now, since payoffs in the sufficiently distant future have an arbitrarily small value from today's point of view, there is some period T such that the payoff to any strategy that follows s_i' through period T exceeds the payoff to any strategy that follows by s_i through period T (given that the other players adhere to s_{-i}). (The integer T can be chosen to be sufficiently large that the first strategy yields a higher discounted average payoff than the second one even if the first strategy induces the best possible outcome for player i in every period after T, and the second strategy induces the worst possible outcome in every such period.) In particular, the strategy s_i'' that coincides with s_i' through period T and with s_i subsequently is better for player i than the strategy s_i.

Now, s_i and s_i'' differ only in the actions they prescribe after finitely many histories, so we can apply the argument in the proof of Proposition ??? to find a strategy of player i and a history such that in the subgame that follows the history, the strategy differs from s_i only in the action it prescribes initially, and player i is better off following the strategy than following s_i.

Thus we have shown that if s is not a subgame perfect equilibrium then some player can increase her payoff by making a one-period deviation after some history. □

14.8 Some subgame perfect equilibria of the infinitely repeated Prisoner's Dilemma

The notion of Nash equilibrium requires only that each player's strategy be optimal in the whole game, given the other players' strategies; after histories that do not occur if the players follow their strategies, the actions specified by a player's Nash equilibrium strategy may not be optimal. In some cases we can think of the actions prescribed by a strategy for histories that will not occur if the players follow their strategies as "threats"; the notion of Nash equilibrium does not require that it be optimal for a player to carry out these threats if called upon to do so. In the previous chapter we studied the notion of subgame perfect equilibrium, which does impose such a requirement: a strategy profile is a subgame perfect equilibrium if every player's strategy is optimal not only in the whole game, but after *every* history (including histories that do not occur if the players adhere to their strategies).

Are the Nash equilibria we considered in the previous section subgame perfect equilibria of the infinitely repeated *Prisoner's Dilemma* with payoffs as in Figure 389.1? Clearly the Nash equilibrium in which each player chooses D after every history is a subgame perfect equilibrium: whatever happens, each player chooses D, so it is optimal for the other player to do likewise. Now consider the other Nash equilibria we studied.

14.8.1 Grim trigger strategies

Suppose that the outcome in the first period is (C, D). Is it optimal for each player to subsequently adhere to the grim trigger strategy, given that the other player does so? In particular, is it optimal for player 1 to carry out the punishment that the grim trigger strategy prescribes? If both players adhere to the strategy then player 1 chooses D in every subsequent period while player 2 chooses C in period 2 and then D subsequently, so that the sequence of outcomes in the subgame following the history (C, D) is $((D, C), (D, D), (D, D), \ldots)$, yielding player 1 a discounted average payoff of

$$3(1 - \delta) + \delta = 3 - 2\delta.$$

If player 1 refrains from punishing player 2 for her lapse, and simply chooses C in every subsequent period, then the outcome in period 2 and subsequently is (C, C), so that the sequence of outcomes in the game yields player 1 a discounted average payoff of 2. If $\delta > \frac{1}{2}$ then $2 > 3 - 2\delta$, so that player 1 prefers not to punish player 2 for a deviation, and hence the strategy pair in which each player uses the grim trigger strategy is not a subgame perfect equilibrium.

In fact, the strategy pair in which each player uses the grim trigger strategy is not a subgame perfect equilibrium for *any* value of δ, for the following reason. If player 1 adheres to the grim trigger strategy, then in the subgame following the outcome (C, D), player 2 prefers to choose D in period 2 and subsequently, regardless of the value of δ (since the outcome is then (D, D) in every period, rather than (D, C) in the first period of the subgame and (D, D) subsequently).

In summary, the strategy pair in which both players use the grim trigger strategy defined in Figure 395.1 is not a subgame perfect equilibrium of the infinitely repeated game for any value of the discount factor: after the history (C, D) player 1 has no incentive to punish player 2, and player 2 prefers to choose D in every subsequent period if she is going to be punished, rather than choosing C in the second period of the game and then D subsequently.

However, a small modification of the grim trigger strategy fixes both of these problems. Consider the variant of the grim trigger strategy in which a player chooses D after any history in which *either* player chose D in some period. This strategy is illustrated in Figure 405.1. If both players adopt this strategy then in the subgame following a deviation, the miscreant chooses D in every period, so that her opponent is better off "punishing" her by choosing D than she is by choosing C. Further, a player's behavior during her punishment is optimal—she chooses D in every period. The point is that (D, D) is a Nash equilibrium of a *Prisoner's Dilemma*, so that neither player has any quarrel with the prescription of the modified grim trigger strategy that she choose D after any history in which some player chose D. The fact that the strategy specifies that a player choose D after any history in which she deviated means that it is optimal for the other player to punish her, and since she is punished it is optimal for her to choose D. Effectively, a player's strategy "punishes" her opponent—by choosing D—if her opponent does not "punish" her for deviating.

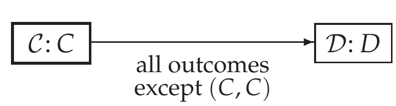


Figure 405.1 A variant of the grim strategy in an infinitely repeated *Prisoner's Dilemma*.

14.8.2 Limited punishment

The pair of strategies (s, s) in which s is the limited punishment strategy studied in Section 14.5.2 is not a subgame perfect equilibrium of the infinitely repeated *Prisoner's Dilemma* for the same reason that a pair of grim trigger strategies is not a subgame perfect equilibrium. However, as in the case of grim trigger strategies, we can modify the limited punishment strategy in order to obtain a subgame perfect equilibrium. Specifically, we need the transition from state P_0 to state P_1 in Figure 395.2 to occur whenever *either* player chooses D (not just if the *other* player chooses D). A player using this modified strategy chooses D during her punishment, which both is optimal for her and makes the other player's choice to punish optimal. When the punishment ends she, like her punisher, reverts to C.

? EXERCISE 406.1 (Lengths of punishment in subgame perfect equilibrium) Is there any subgame perfect equilibrium of an infinitely repeated *Prisoner's Dilemma* (with payoffs as in Figure 389.1), for any value of δ, in which each player's strategy involves limited punishment, but the lengths of the punishment are different for each player? If so, describe such a subgame perfect equilibrium; if not, argue why not.

14.8.3 Tit-for-tat

The behavior in a subgame of a player who uses the strategy *tit-for-tat* depends only on the last outcome in the history that preceded the subgame. Thus to examine whether the strategy pair in which both players use the strategy *tit-for-tat* is a subgame perfect equilibrium we need to consider four types of subgame, following histories in which the last outcome is (C, C), (C, D), (D, C), and (D, D).

The optimality of *tit-for-tat* in a subgame following a history ending in (C, C), given that the other player uses *tit-for-tat*, is covered by our analysis of Nash equilibrium: if $\delta \geq \frac{1}{2}$ then *tit-for-tat* is a best response to *tit-for-tat* in such a subgame.

In studying subgames following histories ending in other outcomes, I appeal to the fact that a strategy profile is a subgame perfect equilibrium if and only if it satisfies the one-deviation property (Propostion ???).

Consider the subgame following a history ending in the outcome (C, D). Suppose that player 2 adheres to *tit-for-tat*. If player 1 also adheres to *tit-for-tat* then the outcome alternates between (D, C) and (C, D), and player 1's discounted average payoff in the subgame is

$$(1-\delta)(3 + 3\delta^2 + \cdots) = \frac{3}{1+\delta}.$$

If player 1 instead chooses C in the first period of the subgame, and subsequently adheres to *tit-for-tat*, then the outcome is (C, C) in every period of the subgame, so that player 1's discounted average payoff is 2. Thus in order that *tit-for-tat* be optimal in such a subgame we need

$$\frac{3}{1+\delta} \geq 2, \text{ or } \delta \leq \tfrac{1}{2}.$$

In the subgame following a history ending with the outcome (D, C), the outcome alternates between (C, D) and (D, C) if both players adhere to *tit-for-tat*, yielding player 1 a discounted average payoff of $3\delta/(1+\delta)$ (the first outcome is (C, D), rather than (D, C) as in the previous case). If player 1 deviates to D in the first period, and then adheres to *tit-for-tat* then the outcome is (D, D) in every period, yielding player 1 a discounted average payoff of 1. Thus for *tit-for-tat* to be optimal for player 1 we need

$$\frac{3\delta}{1+\delta} \geq 1, \text{ or } \delta \geq \tfrac{1}{2}.$$

Finally, in a subgame following a history ending with the outcome (D, D), the outcome is (D, D) in every period if both players adhere to *tit-for-tat*, yielding player 1 a discounted average payoff of 1. If player 1 deviates to C in the first period of the subgame, then adheres to *tit-for-tat*, the outcome alternates between (C, D) and (D, C), yielding player 1 a discounted average payoff of $3\delta/(1+\delta)$. Thus *tit-for-tat* is optimal for player 1 only if $\delta \leq \frac{1}{2}$.

We conclude that (*tit-for-tat*,*tit-for-tat*) is a subgame perfect equilibrium of the infinitely repeated *Prisoner's Dilemma* with payoffs as in Figure 389.1 if and only if $\delta = \frac{1}{2}$. In fact, the existence of any value of the discount factor for which (*tit-for-tat*,*tit-for-tat*) is a subgame perfect equilibrium depends on the specific payoffs I have assumed for the component game: this strategy pair is a subgame perfect equilibrium of an infinitely repeated *Prisoner's Dilemma* only if the payoffs of the component game are rather special, as you are asked to show in the following exercise.

? EXERCISE 407.1 (*Tit-for-tat* as a subgame perfect equilibrium in the infinitely repeated *Prisoner's Dilemma*) Consider the infinitely repeated *Prisoner's Dilemma* in which the payoffs of the component game are those given in Figure 407.1. Show that (*tit-for-tat*,*tit-for-tat*) is a subgame perfect equilibrium if and only if $y - x = 1$ and $\delta = 1/x$. (Use the fact that subgame perfect equilibria have the one-deviation property.)

	C	D
C	x, x	$0, y$
D	$y, 0$	$1, 1$

Figure 407.1 The component game for the infinitely repeated *Prisoner's Dilemma* considered in Exercise 407.1.

14.8.4 Subgame perfect equilibrium payoffs of the infinitely repeated Prisoner's Dilemma *when the players are patient*

In Section 14.6 we saw that every pair (x_1, x_2) in which $x_i > 1$ is close to a pair of discounted average payoffs to some Nash equilibrium of the infinitely repeated

Prisoner's Dilemma with payoffs as in Figure 389.1 when the players are sufficiently patient. Since every subgame perfect equilibrium is a Nash equilibrium, the set of subgame perfect equilibrium payoff pairs is a subset of the set of Nash equilibrium payoff pairs. I now argue that, in fact, the two sets are the same. The strategy pair that I used in the argument of Section 14.6 is not a subgame perfect equilibrium, but can be modified, along the lines we considered in the previous section, to turn it into such an equilibrium.

Let (x_1, x_2) be a pair of feasible payoffs in the *Prisoner's Dilemma* for which $x_i > 1$ for each player i. Let $(a^1, \ldots, a^k)$ be a sequence of outcomes of the game for which each player i's average payoff is x_i, and let $(b^1, b^2, \ldots)$ be the outcome path of the infinitely repeated game that consists of repetitions of the sequence $(a^1, \ldots, a^k)$. I claim that the strategy pair in which each player follows the path $(b^1, b^2, \ldots)$ so long as *both* she and the other player have done so in the past, and otherwise chooses D, is a subgame perfect equilibrium. If one player deviates then subsequent to her deviation she continues to choose D, making it optimal for her opponent to "punish" her by choosing D. Precisely, the strategy s_i of player i chooses the action b_i^1 in the first period and the action

$$s_i(h^1, \ldots, h^{t-1}) = \begin{cases} b_i^t & \text{if } h^r = b^r \text{ for } r = 1, \ldots, t-1 \\ D & \text{otherwise,} \end{cases}$$

after any other history $(h^1, \ldots, h^{t-1})$.

I claim that (s, s) is a subgame perfect equilibrium of the infinitely repeated game. There are two types of subgame to consider. First, consider a history in which the outcome was b^r in every period r. The argument that if one player acts according to s in the subgame that follows such a history then it is optimal for the other to do so is the same as the argument that the strategy pair defined in Section 14.6 is a Nash equilibrium. Briefly, if both players adhere to the strategy s in the subgame, the outcome is b^t in every period t, yielding each player i a discounted average payoff close to x_i when the discount factor is close to 1. If one player deviates from s, then she may gain in the period in which she deviates, but her deviation will trigger her opponent to choose D in every subsequent period, so that given $x_i > 1$ for each i, her deviation makes her worse off if her discount factor is close enough to 1.

Now consider a history in which the outcome was different from b^r in some period r. If, in the subgame following this history, the players both use the strategy s, then they both choose D regardless of the outcomes in the subgame. Since the strategy pair in which both players always choose D regardless of history is a Nash equilibrium of the infinitely repeated game, the strategy pair that (s, s) induces in such a subgame is a Nash equilibrium.

We conclude that the strategy pair (s, s) is a subgame perfect equilibrium. The point is that after any deviation the players' strategies lead them to choose Nash equilibrium actions of the component game in every subsequent period, so that neither player has any incentive to deviate.

Since no player's discounted average payoff can be less than 1 in any Nash equilibrium of the infinitely repeated game, we conclude that the set of discounted average payoffs possible in subgame perfect equilibria is exactly the same as the set of discounted average payoffs possible in Nash equilibria:

- for any discount factor, each player's payoff in every discounted average payoff pair generated by a subgame perfect equilibrium of the infinitely repeated game is at least 1
- for every pair (x_1, x_2) of feasible payoffs in the game for which $x_i > 1$ for each player i, there is a pair (y_1, y_2) close to (x_1, x_2) such that for a discount factor close enough to 1 there is a subgame perfect equilibrium of the infinitely repeated game in which the pair of discounted average payoffs is (y_1, y_2).

Notes

15 Repeated games: General Results

Prerequisite: Chapter 14

15.1 Nash equilibria of general infinitely repeated games

THE IDEA behind the analysis of an infinitely repeated *Prisoner's Dilemma* applies to any infinitely repeated game: every feasible payoff profile in the one shot game in which each player's payoff exceeds some minimum is close (at least) to the discounted average payoff profile of a Nash equilibrium in which a deviation triggers each player to begin an indefinite "punishment" of the deviant.

For the *Prisoner's Dilemma* the minimum payoff of player i that is supported by a Nash equilibrium is $u_i(D, D)$. The significance of this payoff is that player j can ensure (by choosing D) that player i's payoff does not exceed $u_i(D, D)$, and there is no lower payoff with this property. That is, $u_i(D, D)$ is the lowest payoff that player j can force upon player i.

How can we find this minimum payoff in an arbitrary strategic game? Suppose that the deviant is player i. For any collection a_{-i} of the other players' actions, player i's highest possible payoff is her payoff when she chooses a best response to a_{-i}, namely

$$\max_{a_i \in A_i} u_i(a_i, a_{-i}).$$

As a_{-i} varies, this maximal payoff varies. We seek a collection a_{-i} of "punishment" actions that make this maximum as small as possible. That is, we seek a solution to the problem

$$\min_{a_{-i} \in A_{-i}} \left(\max_{a_i \in A_i} u_i(a_i, a_{-i}) \right).$$

This payoff is known, not surprisingly, as player i's *minmax* payoff.

▶ DEFINITION 411.1 Player i's **minmax** payoff in a strategic game in which the action set and payoff function of each player i are A_i and u_i respectively is

$$\min_{a_{-i} \in A_{-i}} \left(\max_{a_i \in A_i} u_i(a_i, a_{-i}) \right). \tag{411.2}$$

(Note that I am restricting attention to pure strategies in the strategic game; a player's minmax payoff is different if we consider mixed strategies.)

For example, in the *Prisoner's Dilemma* with the payoffs in Figure 389.1, each player's minmax payoff is 1; in *BoS* (Example 16.2) each player's minmax payoff is also 1.

? EXERCISE 412.1 (Minmax payoffs) Find each player's minmax payoff in each of the following games.

a. The game of dividing money in Exercise 36.2.

b. Cournot's oligopoly game (Section 3.1) when $C_i(0) = 0$ for each firm i and $P(Q) = 0$ for some sufficiently large value of Q.

c. Hotelling's model of electoral competition (Section 3.3) when (*i*) there are two candidates and (*ii*) there are three candidates, under the assumptions that the set of possible positions is the interval $[0, 1]$, the distribution of the candidates' ideal positions has a unique median, a tie results in each candidate's winning with probability $\frac{1}{2}$, and each candidate's payoff is her probability of winning.

Whatever the other players' strategies, any player can obtain at least her minmax payoff in every period, and hence a discounted average payoff at least equal to her minmax payoff, by choosing in each period a best response to the other players' actions. More precisely, player i can ensure that her payoff in every period is at least her minmax payoff by using a strategy that, after every history h, chooses a best response to $s_{-i}(h)$, the collection of actions prescribed for the other players' strategies after the history h. Thus in no Nash equilibrium of the infinitely repeated game is player i's discounted average payoff less than her minmax payoff.

We saw that in the *Prisoner's Dilemma*, a converse of this result holds: for every feasible payoff profile x in the game in which x_i exceeds player i's minmax payoff for $i = 1, 2$, for a discount factor sufficiently close to 1 there is a Nash equilibrium of the infinitely repeated game in which the discounted average payoff of player i is close to x_i for $i = 1, 2$.

An analogous result holds in general. The simplest case to consider is that in which x is a payoff profile of the game. Let x be the payoff profile generated by the action profile a; assume that each x_i exceeds player i's minmax payoff. For each player i, let p_{-i} be a collection of actions for the players other than i that holds player i down to her minmax payoff. (That is, p_{-i} is a solution of the minimization problem (411.2).) Define a strategy for each player as follows. In each period, the strategy of each player i chooses a_i as long as every other player j chose a_j in every previous period, and otherwise chooses the action $(p_{-j})_i$, where j is the player who deviated in the first period in which exactly one player deviated. Precisely, let H^* be the set of histories in which there is at least one period in which exactly one player j chose an action different from a_j. Refer to such a player as a *lone deviant*. The strategy of player i is defined by $s_i(\varnothing) = a_i$ (her action at the start of the game is a_i) and

$$s_i(h) = \begin{cases} a_i & \text{if } h \text{ is not in } H^* \\ (p_{-j})_i & \text{if } h \in H^* \text{ and } j \text{ is the first lone deviant in } h. \end{cases}$$

The strategy profile s is a Nash equilibrium by the following argument. If player i adheres to s_i then, given that every other player j adheres to s_j, her payoff is x_i in every period. If player i deviates from s_i, while every other player j adheres to s_j, then she may gain in the period in which she deviates, but she loses in every subsequent period, obtaining at most her minmax payoff, rather than x_i. Thus for a discount factor close enough to 1, s_i is a best response to s_{-i} for every player i, so that s is a Nash equilibrium.

(Note that the strategies I have defined do not react when more than one player deviates in any one period. They do not need to, since the notion of Nash equilibrium requires only that no *single* player has an incentive to deviate.)

This argument can be extended to deal with the case in which x is a feasible payoff profile that is not the payoff profile of a single action profile in the component game, along the same lines as the argument in the case of the *Prisoner's Dilemma* in the previous section. The result we obtain is known as a "folk theorem", since the basic form of the result was known long before it was written down precisely.[1]

■ PROPOSITION 413.1 (Nash folk theorem) *Let G be a strategic game.*

- *For any discount factor δ with $0 < \delta < 1$, the discounted average payoff of every player in any Nash equilibrium of the infinitely repeated game of G is at least her minmax payoff.*
- *Let w be a feasible payoff profile of G for which each player's payoff exceeds her minmax payoff. Then for all $\epsilon > 0$ there exists $\overline{\delta} < 1$ such that if the discount factor exceeds $\overline{\delta}$ then the infinitely repeated game of G has a Nash equilibrium whose discounted average payoff profile w' satisfies $|w - w'| < \epsilon$.*

? EXERCISE 413.2 (Nash equilibrium payoffs in infinitely repeated games) For the infinitely repeated games for which each of the following strategic games is the component game, find the set of discounted average payoffs to Nash equilibria of these infinitely repeated games when the discount factor is close to 1. (Parts *b* and *c* of Exercise 412.1 are relevant to parts *b* and *c*.)

a. *BoS* (Example 16.2).

b. Cournot's oligopoly game (Section 3.1) when there are two firms, $C_i(q_i) = q_i$ for all q_i for each firm i, and $P(Q) = \max\{0, \alpha - \beta Q\}$.

c. Hotelling's model of electoral competition (Section 3.3) when there are two candidates, under the assumptions that the set of possible positions is the interval $[0, 1]$, the distribution of the citizens' ideal positions has a unique median, a tie results in each candidate's winning with probability $\frac{1}{2}$, and each candidate's payoff is her probability of winning.

[1] If $x = (x_1, \ldots, x_n)$ is a vector then $|x|$ is the norm of x, namely $(x_1^2 + \cdots + x_n^2)^{1/2}$. If x and y are vectors and $|x - y|$ is small then the components of x and y are close to each other.

The strategies in the Nash equilibrium used to prove Proposition 413.1 are grim trigger strategies: any transgression leads to interminable punishment. As in the case of the *Prisoner's Dilemma*, less draconian punishment is sufficient to deter deviations; grim trigger strategies are simply easy to work with. The punishment embedded in a strategy has only to be severe enough that any deviation ultimately results in a net loss for its perpetrator.

? EXERCISE 414.1 (Repeated Bertrand duopoly) Consider Bertrand's model of duopoly (Section 3.2) in the case that each firm's unit cost is constant, equal to c. Let $\Pi(p) = (p - c)D(p)$ for any price p, and assume that Π is continuous and is uniquely maximized at the price p^m (the "monopoly price").

a. Let s be the strategy for the infinitely repeated game that charges p^m in the first period and subsequently as long as the other firm continues to charge p^m, and punishes any deviation from p^m by the other firm by choosing the price c for k periods, then reverting to p^m. Given any value of δ, for what values of k is the strategy pair (s, s) a Nash equilibrium of the infinitely repeated game?

b. Let s be the strategy for the infinitely repeated game defined as follows:

- in the first period charge the price p^m
- in every subsequent period charge the lowest of all the prices charged by the other firm in all previous periods.

Is the strategy pair (s, s) a Nash equilibrium of the infinitely repeated game for any discount factor less than 1?

15.2 Subgame perfect equilibria of general infinitely repeated games

The *Prisoner's Dilemma* has a feature that makes it easy to construct a subgame perfect equilibrium of the infinitely repeated game to prove the result in the previous section: it has a Nash equilibrium in which each player's payoff is her minmax payoff. In any game, each player's payoff is at least her minmax payoff, but in general there is no Nash equilibrium in which the payoffs are exactly the minmax payoffs. It may be clear how to generalize the arguments above to define a subgame perfect equilibrium of any infinitely repeated game in which both players' discounted average payoffs exceed their payoffs in some Nash equilibrium of the component game. However, it is not clear whether there are subgame perfect equilibrium payoff pairs in which the players' payoffs are between their minmax payoffs and their payoffs in the worst Nash equilibrium of the component game.

Consider the game in Figure 415.1. Each player's minmax payoff is 1: by choosing C, each player can ensure that the other player's payoff does not exceed 1, and there is no action that ensures that the other player's payoff is less than 1. In the unique Nash equilibrium (A, A), on the other hand, each player's payoff is 4. Payoffs between 1 and 4 cannot be achieved by strategies that react to deviations by choosing A, since one player's choosing A allows the other to obtain a payoff of 4 (by choosing A also), which exceeds her payoff if she does not deviate.

	A	B	C
A	$4,4$	$3,0$	$1,0$
B	$0,3$	$2,2$	$1,0$
C	$0,1$	$0,1$	$0,0$

Figure 415.1 A strategic game with a unique Nash equilibrium in which each player's payoff exceeds her minmax payoff.

Nevertheless, such payoffs can be achieved in subgame perfect equilibria. The punishments built into the players' strategies in these equilibria need to be carefully designed. A deviation cannot lead to the indefinite play of (C, C), since each player has an incentive to deviate from this action pair. In order to make it worthwhile for a player to punish her opponent for deviating, she must be made worse off if she fails to punish than if she does so. We can achieve this effect by designing strategies that punish deviations for a limited amount of time—enough to wipe out the gain from a deviation—so long as both players act as they are supposed to during the punishment, but are extended whenever one of the players misbehaves.

Specifically, consider the strategy s shown in Figure 415.2 for a player in the game in Figure 415.1. This strategy starts a two-period punishment after a de-

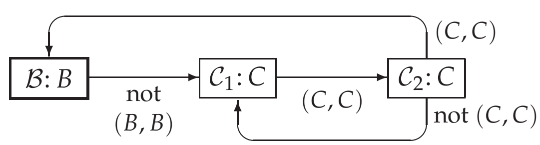


Figure 415.2 A subgame perfect equilibrium strategy for a player in the infinitely repeated game for which the component game is that given in Figure 415.1.

viation from the outcome (B, B). If both players choose the action C during the punishment phase then after two periods they both revert to choosing B. If, however, one of them does not choose C in the first period of the punishment then the punishment starts again: the transition from the first punishment state $\mathcal{C}_1$ to the second punishment state $\mathcal{C}_2$ does not occur unless both players choose C after a deviation from (B, B). Further, if there is a deviation from C in the second period of the punishment then there is a transition back to $\mathcal{C}_1$: the punishment starts again. Thus built into the strategy is punishment for a player who does not carry out a punishment.

I claim that if the discount factor is close enough to 1 then the strategy pair in which both players use this strategy is a subgame perfect equilibrium of the infinitely repeated game. The players' behavior in period t is determined only by the current state, so we need to consider only three cases. Suppose that player 2 adheres to the strategy, and in each case consider whether player 1 can increase her payoff by deviating at the start of the subgame, holding the rest of her strategy fixed.

State $\mathcal{B}$: If player 1 adheres to the strategy her payoffs in the next three periods are $(2,2,2)$, while if she deviates they are at most $(3,0,0)$; in both cases her payoff is subsequently 2. Thus adhering to the strategy is optimal if $2+2\delta+2\delta^2 \geq 3$, or $\delta \geq \frac{1}{2}(\sqrt{3}-1)$.

State $\mathcal{C}_1$: If player 1 adheres to the strategy her payoffs in the next three periods are $(0,0,2)$, while if she deviates they are at most $(1,0,0)$; in both cases, her payoff is subsequently 2. Thus adhering to the strategy is optimal if $2\delta^2 \geq 1$, or $\delta \geq \frac{1}{2}\sqrt{2}$.

State $\mathcal{C}_2$: If player 1 adheres to the strategy her payoffs in the next three periods are $(0,2,2)$, while if she deviates they are at most $(1,0,0)$; in both cases, her payoff is subsequently 2. Thus adhering to the strategy is optimal if $2\delta+2\delta^2 \geq 1$, or certainly if $2\delta^2 \geq 1$, as required by the previous case.

We conclude, using the fact that a strategy profile is a subgame perfect equilibrium if and only if it satisfies the one-deviation property, that if $\delta \geq \frac{1}{2}\sqrt{2}$ then (s,s) is a subgame perfect equilibrium.

The idea behind this example can be extended to any two-player game. Consider an outcome a of such a game for which both players' payoffs exceed their minmax payoffs. I construct a subgame perfect equilibrium in which the outcome is a in every period. Let p_j be an action of player i that holds player j down to her minmax payoff (a "punishment" for player j), and let $p=(p_2,p_1)$ (each player punishes the other). Let s_i be a strategy of player i of the form shown in Figure 416.1, for some value of k. This strategy starts off choosing a_i, and continues to choose a_i so long as the outcome is a; otherwise, it chooses the action p_j that holds player j to her minmax payoff. Once punishment begins, it continues for k periods as long as both players choose their punishment actions. If any player deviates from her assigned punishment action then the punishments are re-started (from each state $\mathcal{P}_\ell$ there is a transition to state $\mathcal{P}_1$ if the outcome in the previous period is not p).

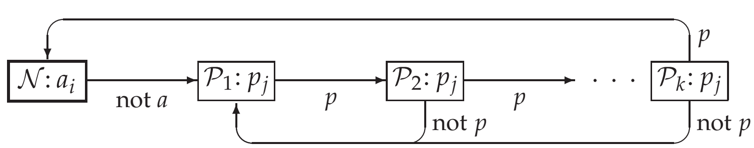


Figure 416.1 A subgame perfect equilibrium strategy for player i in a two-player infinitely repeated game. The outcome p is that in which each player's action is one that holds the other player down to her minmax payoff.

I claim that we can find $\underline{\delta}$ and $k(\delta)$ such that if $\delta > \underline{\delta}$ then the strategy pair (s_1,s_2) is a subgame perfect equilibrium of the infinitely repeated game. Suppose that player j adheres to s_j. If player i adheres to s_i in state $\mathcal{N}$ then her discounted average payoff is $u_i(a)$. If she deviates, she obtains at most her maximal payoff in the game, say $\overline{u}_i$, in the period of her deviation, then $u_i(p)$ for k periods, and subsequently $u_i(a)$ in the future. Thus her discounted average payoff from the deviation is at most

$$(1-\delta)[\overline{u}_i + \delta u_i(p) + \cdots + \delta^k u_i(p)] + \delta^{k+1} u_i(a) = \\ (1-\delta)\overline{u}_i + \delta(1-\delta^k)u_i(p) + \delta^{k+1}u_i(a).$$

In order for her not to want to deviate it is thus sufficient that

$$u_i(a) \geq (1-\delta)\overline{u}_i + \delta(1-\delta^k)u_i(p) + \delta^{k+1}u_i(a). \tag{417.1}$$

If player i adheres to s_i in any state $\mathcal{P}_\ell$ then she obtains $u_i(p)$ for at most k periods, then $u_i(a)$ in every subsequent period, which yields a discounted average payoff of at least

$$(1-\delta^k)u_i(p) + \delta^k u_i(a)$$

(since $u_i(p)$ is at most player i's minmax payoff and $u_i(a)$ exceeds this minmax payoff). If she deviates from s_i, she obtains at most her minmax payoff in the period of her deviation, then $u_i(p)$ for k periods, then $u_i(a)$ in the future, which yields a discounted average payoff of at most

$$(1-\delta)m_i + \delta(1-\delta^k)u_i(p) + \delta^{k+1}u_i(a),$$

where m_i is her minmax payoff. Thus in order that she not want to deviate it is sufficient that

$$(1-\delta^k)u_i(p) + \delta^k u_i(a) \geq (1-\delta)m_i + \delta(1-\delta^k)u_i(p) + \delta^{k+1}u_i(a)$$

or

$$(1-\delta^k)u_i(p) + \delta^k u_i(a) \geq m_i. \tag{417.2}$$

Thus if for each value of δ sufficiently close to 1 we can find $k(\delta)$ such that $(\delta, k(\delta))$ satisfies (417.1) and (417.2) then the strategy pair (s_1, s_2) is a subgame perfect equilibrium. [Need to make this argument.]

This argument shows that for any outcome of the component game in which each player's payoff exceeds her minmax payoff there is a subgame perfect equilibrium that yields this outcome path. More generally, for any two-player strategic game and any feasible payoff pair (x_1, x_2) in which each player's payoff exceeds her minmax payoff, we can construct a Nash equilibrium strategy pair that generates an outcome path for which the discounted average payoff of each player i is x_i. A precise statement of this result follows.

■ PROPOSITION 417.3 (Subgame perfect folk theorem for two-player games) *Let G be a two-player strategic game.*

- *For any discount factor δ with $0 < \delta < 1$, the discounted average payoff of every player in any subgame perfect equilibrium of the infinitely repeated game of G is at least her minmax payoff.*

- *Let w be a feasible payoff profile of G for which each player's payoff exceeds her minmax payoff. Then for all $\epsilon > 0$ there exists $\overline{\delta} < 1$ such that if the discount factor exceeds $\overline{\delta}$ then the infinitely repeated game of G has a subgame perfect equilibrium whose discounted average payoff profile w' satisfies $|w - w'| < \epsilon$.*

The conclusion of this result does not hold for all multi-player games.

AXELROD'S EXPERIMENTS

In the late 1970s, Robert Axelrod (a political scientist at the University of Michigan) invited some economists, psychologists, mathematicians, and sociologists familiar with the repeated *Prisoner's Dilemma* to submit strategies (written in computer code) for a finitely repeated *Prisoner's Dilemma* with payoffs of $(3,3)$ for (C,C), $(5,0)$ for (D,C), $(0,5)$ for (C,D), and $(1,1)$ for (D,D). He received 14 entries, which he pitted against each other, and against a strategy that randomly chooses C and D each with probability $\frac{1}{2}$, in 200-fold repetitions of the game. Each strategy was paired against each other five times. (Strategies could involve random choices, so a pair of strategies could generate different outcomes when paired repeatedly.) The strategy with the highest payoff was *tit-for-tat* (submitted by Anatol Rapoport, then a member of the Psychology Department of the University of Toronto). (See Axelrod (1980a, 1984).)

Axelrod, intrigued by the result, subsequently ran a second tournament. He invited the participants in the first tournament to compete again, and also recruited entrants by advertising in journals read by microcomputer users (a relatively small crowd in the early 1980s); contestants were informed of the results of the first round. Sixty-two strategies were submitted. The contest was run slightly differently from the previous one: the length of each game was determined probabilistically. Again *tit-for-tat* (again submitted by Anatol Rapoport) won. (See Axelrod (1980b, 1984).

Using the strategies submitted in his second tournament, Axelrod simulated an environment in which strategies that do well reproduce faster than other strategies. He repeatedly matched the strategies against each other, increasing the number of representatives of strategies that achieved high payoffs. A strategy that obtained a high payoff initially might, under these conditions, obtain a low one later on if the opponents against which it did well become much less numerous relative to the others. Axelrod found that after a large number of "generations" *tit-for-tat* had the most representatives in the population.

However, *tit-for-tat*'s supremacy has been subsequently shown to be fragile. [Discussion to be added.]

Axelrod's simulations are limited by the set of strategies that were submitted to him. Other simulations have included all strategies of a particular type. One type of strategy that has been examined is the class of "reactive strategies", in which a player's action in any period depends only on the other player's action in the previous period (Nowak and Sigmund (1992)). In evolutionary simulations in which the initial population consists of randomly selected reactive strategies, the strategy that chooses D in every period, regardless of the history, is found to come to dominate. However, if *tit-for-tat* is included in the set of strategies initially in the population, a strategy known as *generous tit-for-tat*, which differs from *tit-for-tat* only in that after its opponent chooses D it chooses D with probability $\frac{1}{3}$ (given the payoffs for the *Prisoner's Dilemma* used by Axelrod), XXXXXXXXXXXXX.

The results are different when the larger class of strategies in which the action chosen in any period depends on *both* actions chosen in the previous period is studied. In this case the strategy *Pavlov* (also known as *win–stay, lose–shift*; see Exercise 398.1), which chooses C when the outcome in the previous period was either (C, C) or (D, D) and otherwise chooses D, tends to come to dominate the population.

In summary, simulations show that a variety of strategies may emerge as "winners" in the repeated *Prisoner's Dilemma*; Axelrod's conclusions about the robustness of *tit-for-tat* appear to have been premature.

Given these results, it is natural to ask if the theory of evolutionary games (Chapter 13) can offer insights into the strategies that might be expected to survive. Unfortunately, the existing results are negative: depending on how one defines an evolutionarily stable strategy (ESS) in an extensive game, an infinitely repeated *Prisoner's Dilemma* either has no ESS, or the only ESS is the strategy that chooses D in every period regardless of history, or every feasible pair of payoffs can be sustained by some pair of ESSs (Kim (1994)).

Reciprocal altruism among sticklebacks

The idea that a population of animals repeatedly involved in a conflict with the structure of a *Prisoner's Dilemma* might evolve a mode of behavior involving reciprocal altruism (as in the strategy *tit-for-tat*), was suggested by Trivers (1971) and led biologists to look for examples of such behavior.

One much-discussed example involves predator inspection by sticklebacks. Sticklebacks often approach a predator in pairs, the members of a pair taking turns to be the first to move forward a few centimeters. (It is advantageous for them to approach the predator closely, since they thereby obtain more information about it.) The process can be modeled as a repeated *Prisoner's Dilemma*, in which moving forward is analogous to cooperating and holding back is like defecting. Milinski (1987) reports an experiment in which he put a stickleback into one compartment of a tank and a cichlid, which resembles a perch, a common predator of sticklebacks, in another compartment, separated by glass. In one condition he placed a mirror along one side of the tank (a "cooperating mirror"), so that as the stickleback approached the predator is had the impression that there was another stickleback mimicking its actions, as if following the strategy *tit-for-tat*. In a second condition he placed the mirror at an angle (a "defecting mirror"), so that a stickleback that approached the cichlid had the impression that there was another stickleback that was increasingly holding back. He found that the stickleback approached the cichlid much more closely with a cooperating mirror than with a defecting mirror. With a defecting mirror, the apparent second stickleback held back when the real one moved forward, and disappeared entirely when the real stickleback moved into the front half of the tank—that is, it tended to defect. Milinski interpreted the behavior of the real stickleback as consistent with its following the strategy *tit-for-tat*. (The same behavior was subsequently observed in guppies approaching a pumpkinseed sunfish (Dugatkin (1988, 1991)).)

Other explanations have been offered for the observed behavior of the fish. For example, one stickleback might simply be attracted to another, since sticklebacks shoal, or a stickleback might be bolder if in the company of another one, since its chances of being captured by the predator are lower (Lazarus and Metcalfe (1990)). Milinski (1990) argues that neither of these alternative theories fits the evidence; in Milinski (1993) he suggests that further evidence indicates that the strategy that his sticklebacks follow may not be *tit-for-tat* but rather *Pavlov* (see Exercise 398.1).

15.3 Finitely repeated games

To be written.

Notes

Early discussions of the notion of a repeated game and the ideas behind the Nash folk theorem (Proposition 413.1) appear in Luce and Raiffa (1957, pp. 97–105 (especially p. 102) and Appendix 8), Shubik (1959b, Ch. 10 (especially p. 226)), and Friedman (1971). Proposition 417.3 (a perfect folk theorem) is due to Fudenberg and Maskin (1986); related results were established earlier (see Aumann and Shapley (1994), Rubinstein (1994), and Rubinstein (1979)).

17 Appendix: Mathematics

17.1 Introduction

THIS CHAPTER presents informal definitions and discussions of the mathematical concepts used in the text. Much of the material should be familiar to you, though a few concepts may be new.

17.2 Numbers

I take the concept of a **number** as basic; 3, -7.4, $\frac{1}{2}$, and $\sqrt{2}$ are all numbers. The whole numbers $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$ are called **integers**. Let x be a number. If $x > 0$ then x is **positive**; if $x \geq 0$ then x is **nonnegative**; if $x < 0$ then x is **negative**; and if $x \leq 0$ then x is **nonpositive**. Note that 0 is both nonnegative and nonpositive, but neither positive nor negative.

When working with sums of numbers, a shorthand that uses the symbol $\sum$ (a large uppercase Greek sigma) is handy. Instead of writing $x_1 + x_2 + x_3 + x_4$, for example, where x_1, x_2, x_3, and x_4 are numbers, we can write

$$\sum_{i=1}^{4} x_i.$$

This expression is read as "the sum from $i = 1$ to $i = 4$ of x_i". The name we give the indexing variable is arbitrary; we frequently use i or j, but can alternatively use any other letter. If the number of items in the sum is a variable, say n, the notation is even more useful. Instead of writing $x_k + \cdots + x_n$, which leaves in doubt the variables indicated by the ellipsis, we can write

$$\sum_{i=k}^{n} x_i,$$

which has a precise meaning: first set $i = k$ and take x_i; then increase i by one and add the new x_i; continue increasing i by one at a time and adding x_i to the sum at each step, until $i = n$.

17.3 Sets

A **set** is a collection of objects. If we can count the members of a set, and, when we do so, we eventually exhaust the members of the set, then the set is **finite**. We can specify a finite set by listing the names of its members within braces: {Paris, Venice, Havana} is a set of (beautiful) cities, for example. Neither the order in which the members of the set are listed nor the number of times each one appears has any significance: {Paris, Venice, Havana} is the same set as {Venice, Paris, Havana}, which is the same set as {Paris, Venice, Paris, Havana} (and has three members).

The symbol $\in$ is used to denote set membership: for example, Havana $\in$ {Paris, Venice, Havana}. We read the statement "$a \in A$" as "a is in A".

If every member of the set B is a member of the set A, we say that B is a **subset** of A. For example, the set {Paris} consisting of the single city Paris is a subset of the set {Paris, Venice, Havana}, since Paris is a member of this set. The set {Paris, Havana} is also a subset of {Paris, Venice, Havana}, since both Paris and Havana are members of the set. Further, the set {Paris, Venice, Havana} is a subset of itself: saying that A is a subset of B does *not* rule out the possibility that A and B are equal.

A **partition** of a set A is a collection $\{A_1, \ldots, A_k\}$ of subsets of A such that every member of A is in exactly one of the sets A_j. The set {Paris, Venice, Havana}, for example, has five partitions: {{Paris}, {Venice}, {Havana}}, {{Paris, Venice}, {Havana}}, {{Paris, Havana}, {Venice}}, {{Paris}, {Venice, Havana}, and {{Paris, Venice, Havana}}.

Some sets are not finite. We can divide such sets into two groups. The members of some sets can be counted, but if we count them then we go on counting forever. The set of positive integers is a set of this type. The members of other sets cannot be counted. For example, the set of all numbers between 0 and 1 cannot be counted. (Of course, one can arbitrarily choose one number in this set, then arbitrarily choose another number, and so on. But there is no systematic way of counting all the numbers.) We say that both types of sets have **infinitely many** members.

A set with infinitely many members obviously cannot be described by listing all its members! One way to describe such a set is to state a property that characterizes its members. For example, if a person's set of actions is a set of numbers A then we can describe the subset of her actions that exceed 1 as

$$\{a \in A\colon a > 1\}.$$

We read this as "the set of a in A such that a exceeds 1". If the set from which the objects come—in this case, the set A—is the set of all numbers, I do not include it explicitly. Thus

$$\{p\colon 0 \leq p \leq 1\}$$

is the set of all nonnegative numbers that are at most 1.

Sometimes we wish to calculate the sum of the numbers x_i for every i in some set S. If S is a set of consecutive numbers of the form $\{1, \ldots, k\}$ then we can write this sum as

$$\sum_{i=1}^{k} x_i,$$

as described at the end of the previous section. If S is not a set of consecutive numbers then we can use a variant of the previous notation to denote the sum

$$\sum_{i \in S} x_i,$$

which means "the sum of all values of x_i for i in the set S".

For example, if S is the set of cities {Paris, Venice, Havana} and the population of city i is x_i then the total population of the cities in S is

$$\sum_{i \in S} x_i.$$

17.4 Functions

A **function** is a rule defining a relationship between two variables. We usually specify a function by giving the formula that defines it. For example, the function, say f, that associates with every number twice that number is defined by $f(x) = 2x$ for each number x; the function, say g, that associates with every number its square is defined by $g(x) = x^2$.

If the variables that a function relates are both numbers then the function can be represented in a graph, like the one in Figure 446.1. We usually put the independent variable (denoted x in the examples above) on the horizontal axis, and the value of the function, $f(x)$, on the vertical axis. To read the graph, find a value of x on the horizontal axis, go vertically up to the graph, then horizontally to the vertical axis; the number on this axis is the value $f(x)$ of the function at x.

Two classes of functions figure prominently in the examples in this book. A function f defining a relationship between two numbers is **affine** if it takes the form $f(x) = ax + b$, where a and b are constants. For example, the functions $-3x + 1$ and $4x$ are both affine. (Sometimes such functions are called "linear", rather than "affine"; I follow the convention that a linear function is an affine function for which $b = 0$.) The graph of a general affine function $ax + b$ is a straight line with slope a that goes through the points $(0, b)$ and $(-b/a, 0)$ (since $a \cdot 0 + b = b$ and $a \cdot (-b/a) + b = 0$). In particular, if $a > 0$ then the slope is positive and if $a < 0$ then the slope is negative. An example is given in Figure 446.2.

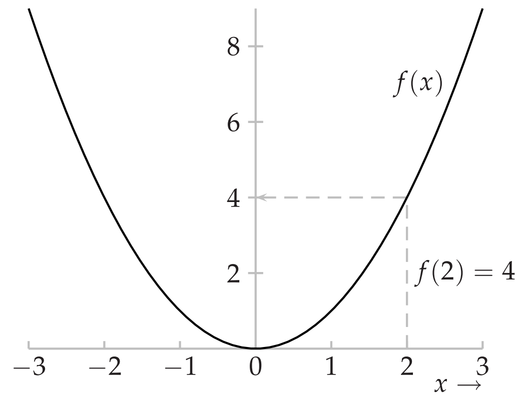


Figure 446.1 The graph of the function f defined by $f(x) = x^2$, for $-3 \leq x \leq 3$.

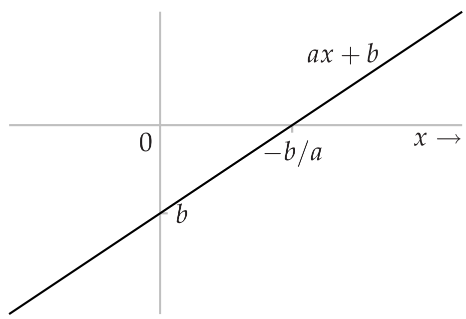


Figure 446.2 The graph of the affine function $ax + b$ (with $a > 0$).

A function f defining a relationship between two numbers is **quadratic** if it takes the form $f(x) = ax^2 + bx + c$, where a, b, and c are constants. If $a > 0$ then the graph of a quadratic function is U-shaped, as in the left-hand panel of Figure 447.1; if $a < 0$ then the shape of the graph is an inverted U, as in the right-hand panel of Figure 447.1.

In both cases the graph is symmetric about a vertical line through the extremum of the function (the minimum when the graph of the function is U-shaped and the maximum when it is an inverted U). Thus if we know the points x_0 and x_1 at which the graph of the function intersects some horizontal line (e.g. the horizontal axis) then we know that its extremum occurs at the midpoint of x_0 and x_1, namely $\frac{1}{2}(x_0 + x_1)$.

We can write the quadratic function $ax^2 + bx + c$ as $x(ax + b) + c$. Doing so allows us to see that the value of the function is c when $x = 0$ and when $x = -b/a$. That is, the function crosses the horizontal line of height c when $x = 0$ and when $x = -b/a$, so that its maximum (if $a < 0$) or minimum (if $a > 0$) occurs at $-\frac{1}{2}b/a$ (the midpoint of 0 and $-b/a$).

? EXERCISE 446.1 (Maximizer of quadratic function) Find the maximizer of the func-

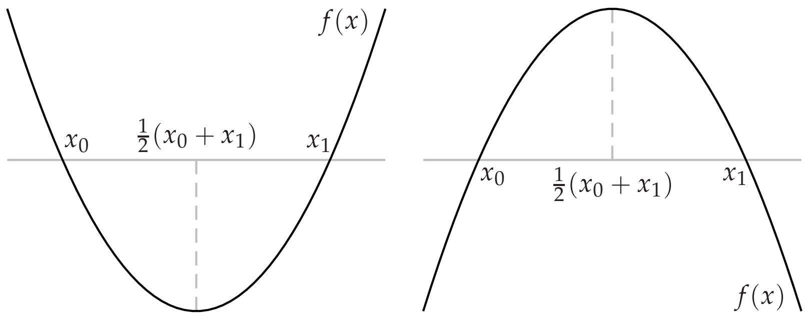


Figure 447.1 The graphs of two quadratic functions. In both cases the function takes the form $ax^2 + bx + c$; in the left panel $a > 0$, while in the right panel $a < 0$.

tion $x(\alpha - x)$, where α is a constant.

The graphs of the functions in Figures 446.1 and 447.1 do not have any jumps in them: for every point x, by choosing x' close enough to x we can ensure that the values $f(x)$ and $f(x')$ of the function at x and x' are as close as we wish. A function that has this property is **continuous**. The graph of a continuous function may be very steep, but does not have any holes in it. For example, the function whose graph is shown in the left panel of Figure 447.2 is continuous, while the function whose graph is shown in the right panel is not continuous. In graphs of discontinuous functions I use the convention that a small disk indicates a point that is included and a small circle indicates a point that is excluded.

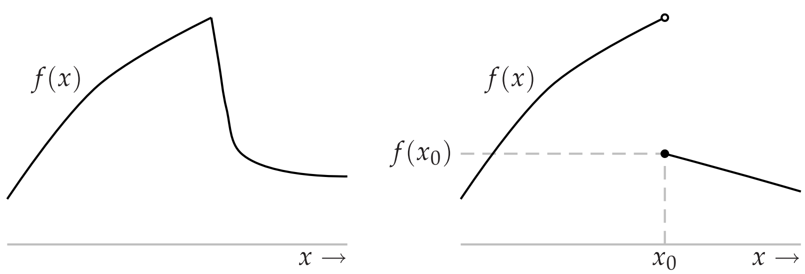


Figure 447.2 The function in the left panel is continuous, while the function in the right panel is not. The small disk indicates a point that is included in the graph, while the small circle indicates a point that is excluded.

For all the functions I have described so far, for each value of x the value $f(x)$ of the function is a single number. In this book we sometimes need to work with functions whose values are sets rather than points. Suppose, for example, that we need a function that assigns to each starting point x in some city the best route from x to city hall. For some values of x there may be a single best route, but for other values of x there are quite possibly several routes that are equally good. At these latter points, the value of our function would be the *set* of all the optimal routes. Since we should like our function to assign the same "type" of object to every value of x, we would take all the values to be sets; if the single route A is optimal from the starting point x then we take the value of the function to be the set $\{A\}$ consisting of the single route A.

We can specify a set-valued function, like a point-valued function, by giving its graph. I indicate values of the function that are sets of points by shading in gray; I indicate boundaries that are included by drawing lines along them. For example, for the function in Figure 448.1, $f(x_1) = \{y : y_2 < y \leq y_3\}$ and $f(x_2) = \{y \colon y = y_0 \text{ or } y_1 < y \leq y_4\}$.

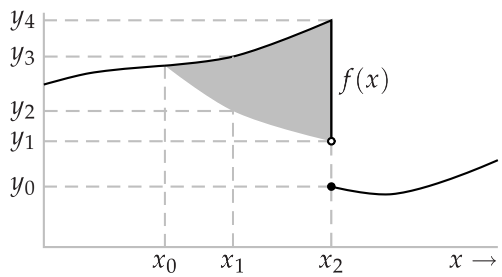


Figure 448.1 The graph of a set-valued function. For $x_0 < x \leq x_2$ the set $f(x)$ consists of more than one point. We have $f(x_1) = \{y : y_2 < y \leq y_3\}$ and $f(x_2) = \{y \colon y = y_0 \text{ or } y_1 < y \leq y_4\}$.

17.5 Profiles

Frequently in this book we wish to associate an object with each member of a set of players. For example, we often need to refer to the action chosen by each player. We can describe the correspondence between players and actions by specifying the function that associates each player with the action she takes. For example, if the players are Ernesto, whose action is R, and Hilda, whose action is S, then the correspondence between players and actions is described by the function a defined by $a(\text{Ernesto}) = R$ and $a(\text{Hilda}) = S$. We can alternatively present the function a by writing $(a_{\text{Ernesto}}, a_{\text{Hilda}}) = (R, S)$. We call such a function a a **profile**. The order in which we write the elements is irrelevant: we can alternatively write the profile above as $(a_{\text{Hilda}}, a_{\text{Ernesto}}) = (S, R)$.

In most of the book I sacrifice color for convenience and name the players 1, 2, 3, and so on. Doing so allows me to write a profile of actions as a list like (R, S), without saying explicitly which action belongs to which player: the convention is that the first action is that of player 1, the second is that of player 2, and so on. When the number of players is arbitrary, equal to say n, I follow convention and write an action profile as $(a_1, \ldots, a_n)$, where the ellipsis stands for the actions of players 2 through $n - 1$.

I frequently need to refer to the action profile that differs from $(a_1, \ldots, a_n)$ only in that the action of player i is b_i (say) rather than a_i. I denote this variant of $(a_1, \ldots, a_n)$ by (b_i, a_{-i}). The $-i$ subscript on a stands for "except i": every player except i chooses her component of a. If $(a_1, a_2, a_3) = (T, L, M)$ and $b_2 = R$, for example, then $(b_2, a_{-2}) = (T, R, M)$.

17.6 Sequences

A **sequence** is an ordered list. In this book the sequences consist of events that unfold over time; the first element of a sequence is an event that occurs before the second element of the sequence, and so on. A sequence that continues indefinitely is **infinite**; one that ends eventually is **finite**.

In Chapters 14 and 15 the formula for the sum of a sequence of numbers of the form $a, ar, ar^2, ar^3, \ldots$ is useful. For a finite sequence we have

$$a + ar + ar^2 + \cdots + ar^T = \frac{a(1 - r^{T+1})}{1 - r} \tag{449.1}$$

if $r \neq 1$ and $r \neq -1$. (Note that the exponent of r in the numerator of the formula is the number of terms in the sequence.) For an infinite sequence we have

$$a + ar + ar^2 + \cdots = \frac{a}{1 - r} \tag{449.2}$$

if $-1 < r < 1$.

? EXERCISE 449.3 (Sums of sequences) Find the sums $1 + \delta^2 + \delta^4 + \cdots$ and $1 + 2\delta + \delta^2 + 2\delta^3 + \cdots$, where δ is a constant with $0 < \delta < 1$. (Split the second sum into two parts.)

17.7 Probability

17.7.1 Basic concepts

We may sometimes conveniently model events as "random". Rather than modeling the causes of such an event, we assume that if the event occurs many times then sometimes it takes one value, sometimes another value, in no regular pattern. We refer to the proportion of times it takes any given value as the **probability** of its taking that value.

A simple example is the outcome of a coin toss. We could model this outcome as depending on the initial position of the coin, the speed and direction in which it is tossed, the nature of the air currents, and so on. But it is simpler, and for many purposes satisfactory, to model the outcome as being a head with probability $\frac{1}{2}$ and a tail with probability $\frac{1}{2}$. Given the sensitivity of the outcome to tiny changes in the initial position of the coin and the speed and direction in which it is tossed, and the inability of a person to precisely control these factors, the probabilistic theory is likely to work very well over many tosses: if the coin is tossed a large number n of times, then the number of heads is likely to be close to $n/2$.

We refer to an assignment of probabilities to events as a **probability distribution**. If, for example, there are three possible events, A, B, and C, then one probability distribution assigns probability $\frac{1}{3}$ to A, probability $\frac{1}{2}$ to B, and probability $\frac{1}{6}$ to C. In any probability distribution the sum of the probabilities of all possible events is 1 (on any given occasion, *one* of the events must occur), and each probability is nonnegative and at most 1. Saying that an event occurs with positive probability is equivalent to saying that there is some chance that it may occur; saying that an event occurs with probability zero is equivalent to saying that it will never occur. Similarly, saying that an event occurs with probability less than one is equivalent to saying that there is some chance that it may not occur; saying that an event occurs with probability one is equivalent to saying that it is certain to occur. We sometimes denote the probability of an event E by $\Pr(E)$.

If the events E and F cannot both occur, then the probability that *either* E *or* F occurs is the sum $\Pr(E) + \Pr(F)$. For example, suppose we model the outcome of the toss of a die as random, with the probability of each side equal to $\frac{1}{6}$. Then the probability that the side is either 3 or 4 is $\Pr(3) + \Pr(4) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$.

17.7.2 Independence

Two events E and F are **independent** if the probability $\Pr(E \text{ and } F)$ that they both occur is the product $\Pr(E)\Pr(F)$ of the probabilities that each occurs. Events may sensibly be modeled as independent if the occurrence of one has no bearing on the occurrence of the other. For example, the outcome of an election may sensibly be modeled as independent of the outcome of a coin toss, but not independent of the weather on the polling day (which may affect the candidates' supporters differently). In a strategic game, we model the players' choices of actions as independent: the probability that player 1 chooses action a_1 and player 2 chooses action a_2 is assumed to be the product of the probability that player 1 chooses a_1 and the probability that player 2 chooses a_2.

17.7.3 Lotteries and expected values

The material in this section is used only in Chapter 4 (Mixed strategy equilibrium), Section 7.6 (Extensive games with perfect information, simultaneous moves, and chance moves), Chapter 9 (Bayesian Games), Chapter 10 (Extensive games with imperfect information), Chapter 11 (Strictly competitive games and maxminimization), and Chapter 12 (Rationalizability).

Consider a decision-maker who faces a situation in which there are probabilistic elements. Each action that she chooses induces a probability distribution over outcomes. If you make an offer for an item in a classified advertisement, for example, then given the behavior of other potential buyers, your offer may be accepted with probability $\frac{1}{3}$ and rejected with probability $\frac{2}{3}$. We refer to a probability distribution over outcomes as a **lottery** over outcomes.

If the outcomes of a lottery are numerical (for example, amounts of money), we may be interested in their average value—the value we should expect to get if we found the total of the values on a large number n of trials and divided by n. For the lottery that yields the amount x_i with probability p_i, for $i = 1, \ldots, n$, this average value is

$$p_1 x_1 + \cdots + p_n x_n$$

or, more compactly, $\sum_{i=1}^{n} p_n x_n$. It is called the **expected value** of the lottery. A lottery that yields \$12 with probability $\frac{1}{3}$, \$4 with probability $\frac{1}{2}$, and \$6 with probability $\frac{1}{6}$, for example, has an expected value of $\frac{1}{3} \cdot 12 + \frac{1}{2} \cdot 4 + \frac{1}{6} \cdot 6 = 7$. On no single occasion does the lottery yield \$7, but over a large number of occasions the average amount that it yields is likely to be close to \$7 (the more likely, the larger the number of occasions).

17.7.4 Cumulative probability distributions

The material in this section is used only in Section 4.11 (Mixed strategy equilibrium in games in which each player has a continuum of actions) and Chapter 9 (Bayesian Games).

If the events in our model are associated with numbers, we can describe the probabilities assigned to them by giving the **cumulative probability distribution**, which assigns to each number x the total of the probabilities of all numbers at most equal to x. The cumulative probability distribution of the number of dots on the exposed side of a die, for example, is the function F for which $F(1) = \frac{1}{6}$, $F(2) = \frac{1}{3}$, $F(3) = \frac{1}{2}$, and so on. Given a cumulative probability distribution we can recover the probabilities of the events by calculating the differences between values of F: the probability of x is $F(x) - F(x')$, where x' is the next smaller event.

When the number of events is finite, we can represent the assignment of probabilities to events either by a probability distribution or by a cumulative probability distribution. When the number of events is infinite, we can usefully represent the probabilities only by a cumulative probability distribution, because the probability of any single event is typically zero. If the set of events is the set of numbers from $\underline{a}$ to $\overline{a}$ then a cumulative probability distribution is a nondecreasing function, say F, for which $F(x) = 0$ if $x < \underline{a}$ (the probability of a number less than $\underline{a}$ is 0) and $F(\overline{a}) = 1$ (the probability of a number at most equal to $\overline{a}$ is 1). The number $F(x)$ is the probability of an event at most equal to x.

For example, if $\underline{a} = 0$ and $\overline{a} = 1$ then the function $F(x) = x$ is a cumulative probability distribution. This distribution represents uniform randomization over the interval (sets of the same size have the same probability). Another cumulative probability distribution is given by the function $F(x) = x^2$. In this distribution the probabilities of sets of numbers close to 0 are lower than the probabilities of sets of numbers close to 1.

17.7.5 Conditional probability and Bayes' rule

The material in this section is used only in Section 9.8 (Juries) and Chapter 10 (Extensive Games with Imperfect Information).

We sometimes use the notion of probability to refer to the character of a person's belief, in a situation in which there is no possibility of an event's being repeated. For example, a jury in a civil case is asked to determine whether the probability of a person's being guilty is greater than or less than one half; you may form a belief about the probability of your carrying a particular gene or of your getting into graduate school. In some cases these beliefs may be tightly linked to numerical evidence. If, for example, the only information you have about the prevalence of a particular gene is that it is carried by 10% of the population, then it is reasonable for you to believe that your probability of carrying the gene is 0.1. In other cases beliefs may be at most loosely linked to numerical evidence. The evidence presented to a jury, for example, is likely to be qualitative, and open to alternative interpretations.

Whatever the basis for probabilistic beliefs, however, the theory of probability gives a specific rule for how they should be modified in the light of new probabilistic evidence. In this context in which a belief is changed by evidence, the initial belief is called the **prior belief** and the belief modified by the evidence is called the **posterior belief**.

Suppose that 10% of the population carries the gene X, so that in the absence of any other information your prior belief is that you carry the gene with probability 0.1. An imperfect test for the presence of X is available. The test is positive in 90% of subjects who carry X and in 20% of subjects who do not carry X. The test on you is positive. What should be your posterior belief about your carrying X? The probabilities are illustrated in Figure 452.1.

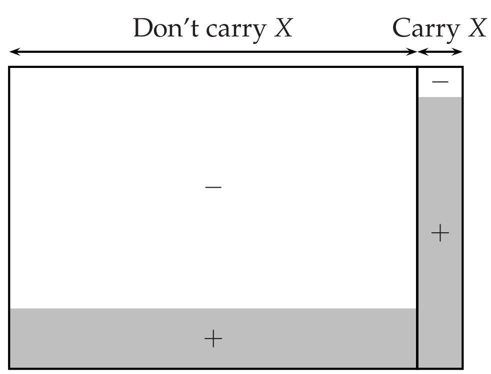


Figure 452.1 The outer box represents the set of people. People to the right of the vertical line carry gene X, while people to the left of this line do not. People in the shaded areas test positive for the gene.

Consider a random group of 100 people from the population. Of these, on average 10 carry X and 90 do not. If all these 100 people were tested, then, on average, 9 of the 10 (90%) who carry X and 18 of the 90 (20%) who do not carry X would test positive. These sets are represented by the shaded areas in Figure 452.1. Of all the people who test positive, what fraction of them carry the gene? That is, what fraction of the total shaded area in Figure 452.1 is the shaded area to the right of the vertical line? Of the 100 people, a total of $9 + 18 = 27$ test positive, and one-third of these (9/27) carry the gene. Thus after testing positive, your posterior belief that you carry the gene is $\frac{1}{3}$: the positive test raises the probability you assign to your carrying X from $\frac{1}{10}$ to $\frac{1}{3}$.

To generalize the analysis in this example, we introduce the concept of conditional probability. Let E and F be two events that may be related; assume that $\Pr(F) > 0$. Suppose that F is true. Define the **probability** $\Pr(E \mid F)$ **of** E **conditional on** F by

$$\Pr(E \mid F) = \frac{\Pr(E \text{ and } F)}{\Pr(F)}. \tag{453.1}$$

This number makes sense as the probability that E is true *given that F is true*. One way to see that it makes sense is to consider Figure 452.1 again. Let E be the event that you carry X and let F be the event that you test positive. If you test positive then we know you lie in the shaded area. Given you lie in this area, what is the probability $\Pr(E \mid F)$ that you lie to the right of the vertical line? This probability is the ratio of the shaded area to the right of the vertical line—the probability $\Pr(E \text{ and } F)$ that you carry the gene and test positive—to the total shaded area—the probability $\Pr(F)$ that you test positive.

If the events E and F are independent then

$$\Pr(E \mid F) = \Pr(E) \text{ and } \Pr(F) > 0$$

or, alternatively,

$$\Pr(F \mid E) = \Pr(F) \text{ and } \Pr(E) > 0.$$

These conditions express directly the idea that the occurrence of one event has no bearing on the occurrence of the other event.

In using the expression for conditional probability to find the posterior belief in this case, we needed to calculate $\Pr(E \text{ and } F)$ and $\Pr(F)$, which were not given directly as data in the problem. The data we were given were the prior belief $\Pr(E)$, the probability $\Pr(F \mid E)$ of a person who carries the gene testing positive, and the probability $\Pr(F \mid \text{not } E)$ of a person who does not carry the gene testing positive.

Bayes' rule expresses the conditional probability $\Pr(E \mid F)$ directly in terms of $\Pr(E)$, $\Pr(F \mid E)$, and $\Pr(F \mid \text{ not } E)$:

$$\Pr(E \mid F) = \frac{\Pr(E)\Pr(F \mid E)}{\Pr(E)\Pr(F \mid E) + \Pr(\text{not } E)\Pr(F \mid \text{not } E)}. \tag{453.2}$$

(The probability $\Pr(\text{not } E)$ is of course equal to $1 - \Pr(E)$; recall that I have assumed that $\Pr(F) > 0$.) This formula follows from the definition of conditional probability (453.1) and the properties of probabilities. First, interchanging E and F in (453.1) we deduce $\Pr(E)\Pr(F \mid E) = \Pr(E \text{ and } F)$. Thus the numerator of (453.2) is equal to $\Pr(E \text{ and } F)$. Second, again using (453.1) we see that the denominator of (453.2) is equal to $\Pr(E \text{ and } F) + \Pr((\text{not } E) \text{ and } F)$. Now, either the event E or the event not E occurs, but not both. Thus $\Pr(E \text{ and } F) + \Pr((\text{not } E) \text{ and } F) = \Pr(F)$. (The probability that either "it rains and you carry an umbrella" or "it rains and you do not carry an umbrella" is equal to the probability that "it rains"!)

? EXERCISE 454.1 (Bayes' rule) Consider a generalization of the example of testing positive for a gene in which the fraction p of the population carry the gene. Verify that as p decreases, the posterior probability that you carry X given that you test positive decreases. What value does this posterior probability take when p is 0.001? What value does the posterior probability take when p is 0.001 and the test is positive for 99% of those who carry X and is negative for 99% of those who do not carry X? (Are you surprised?)

In the cases I have described so far, the event about which we form a belief takes two possible values (E, or not E). In a more general setting, this event may take many values. For example, we may form a belief about the quality of an item—a variable that may take many values—on its price. In general, let F be an event and let $E_1, \ldots, E_n$ be a collection of events, exactly one of which must occur. (In the example above, F is the event that you test positive, $n = 2$, E_1 is the event you carry the gene, and E_2 is the event you do not carry the gene.) Then the probability of E_k conditional on F is

$$\Pr(E_k \mid F) = \frac{\Pr(F \mid E_k)\Pr(E_k)}{\sum_{j=1}^{n} \Pr(F \mid E_j)\Pr(E_j)}. \tag{454.2}$$

This general formula is known as **Bayes' rule**, after Thomas Bayes (1702–61). In the context in which we use this rule in a Bayesian game to find the probability of a state given the observed signal, the events $E_1, \ldots, E_n$ are the states and the event F is a signal. Thus every probability $\Pr(F \mid E_k)$ is either one or zero, depending on whether the state E_k generates the signal F or not.

17.8 Proofs

This book focuses on concepts, but contains precise arguments, and, in some cases, proofs of results. The results are given three names: Lemma, Proposition, and Corollary. These names have no formal significance—they do not have any implications for the type of logic used—but are intended to convey the role of the result in the analysis. Lemmas are results whose importance lies mainly in their being steps on the way to proving further results. Propositions are the main results. Corollaries are more or less direct implications of the main results.

A result consists of a series of statements of the form "if A is true then B is true". Frequently the series contains only one such statement, which may not be explicitly rendered as "if A then B". For example, "all prime numbers are odd" is a result; it can be transformed into the "if ... then" form: "if a number is prime then it is odd". A result that makes the two claims "if A is true then B is true" and "if B is true then A is true" is sometimes stated compactly as "A is true if and only if B is true".

A proof of the result "if A then B" is a series of arguments that lead from A to B, each of which follows from a known fact (including an earlier member of the series). Except for the proofs of very simple results, most proofs are not, and should not sensibly be, “complete”. To spell out how each step follows from the basic principles of mathematics would make a proof extremely long and very difficult to read. Some facts must be taken for granted; judging which to put in and which to leave out is an art. A good proof convinces readers that the result is true and gives them some understanding of *why* it is true (the features of A that are significant, and those that are not significant).

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1 Introduction

5.3 Altruistic preferences

Person 1 is indifferent between $(1,4)$ and $(3,0)$, and prefers both of these to $(2,1)$. Any function that assigns the same number to $(1,4)$ and to $(3,0)$, and a lower number to $(2,1)$ is a payoff function that represents her preferences.

6.1 Alternative representations of preferences

The function v represents the same preferences as does u (since $u(a) < u(b) < u(c)$ and $v(a) < v(b) < v(c)$), but the function w does not represent the same preferences, since $w(a) = w(b)$ while $u(a) < u(b)$.

2 Nash Equilibrium

14.1 Working on a joint project

The game in Figure 3.1 models this situation (as does any other game with the same players and actions in which the ordering of the payoffs is the same as the ordering in Figure 3.1).

	Work hard	*Goof off*
Work hard	$3,3$	$0,2$
Goof off	$2,0$	$1,1$

Figure 3.1 Working on a joint project (alternative version).

16.1 Hermaphroditic fish

A strategic game that models the situation is shown in Figure 3.2.

	Either role	*Preferred role*
Either role	$\frac{1}{2}(H+L), \frac{1}{2}(H+L)$	L, H
Preferred role	H, L	S, S

Figure 3.2 A model of encounters between pairs of hermaphroditic fish whose preferred roles differ.

In order for this game to differ from the *Prisoner's Dilemma* only in the names of the players' actions, there must be a way to associate each action with an action in the *Prisoner's Dilemma* so that each player's preferences over the four outcomes are the same as they are in the *Prisoner's Dilemma*. Thus we need $L < S < \frac{1}{2}(H+L)$. That is, the probability of a fish's encountering a potential partner must be large enough that $S > L$, but small enough that $S < \frac{1}{2}(H+L)$.

17.2 Games without conflict

Any two-player game in which each player has two actions and the players have the same preferences may be represented by a table of the form given in Figure 4.1, where a, b, c, and d are any numbers.

	L	R
T	a, a	b, b
B	c, c	d, d

Figure 4.1 A strategic game in which conflict is absent.

25.1 Altruistic players in the Prisoner's Dilemma

a. A game that model the situation is given in Figure 4.2.

	Quiet	*Fink*
Quiet	$4, 4$	$3, 3$
Fink	$3, 3$	$2, 2$

Figure 4.2 The payoffs in a variant of the *Prisoner's Dilemma* in which the players are altruistic.

This game is not the *Prisoner's Dilemma* because one (in fact both) of the players' preferences are not the same as they are in the *Prisoner's Dilemma*. Specifically, player 1 prefers (*Quiet*, *Quiet*) to (*Fink*, *Quiet*), while in the *Prisoner's Dilemma* she prefers (*Fink*, *Quiet*) to (*Quiet*, *Quiet*). (Alternatively, you may note that player 1 prefers (*Quiet*, *Fink*) to (*Fink*, *Fink*), while in the *Prisoner's Dilemma* she prefers (*Fink*, *Fink*) to (*Quiet*, *Fink*), or that player 2's preferences are similarly not the same as they are in the *Prisoner's Dilemma*.)

b. For an arbitrary value of α the payoffs are given in Figure 4.3. In order that the game be the *Prisoner's Dilemma* we need $3 > 2(1 + \alpha)$ (each player prefers *Fink* to *Quiet* when the other player chooses *Quiet*), $1 + \alpha > 3\alpha$ (each player prefers *Fink* to *Quiet* when the other player choose *Fink*), and $2(1 + \alpha) > 1 + \alpha$ (each player prefers (*Quiet*, *Quiet*) to (*Fink*, *Fink*)). The last condition is satisfied for all nonnegative values of α. The first two conditions are both equivalent to $\alpha < \frac{1}{2}$. Thus the game is the *Prisoner's Dilemma* if and only if $\alpha < \frac{1}{2}$.

If $\alpha = \frac{1}{2}$ then all four outcomes (*Quiet*, *Quiet*), (*Quiet*, *Fink*), (*Fink*, *Quiet*), and (*Fink*, *Fink*) are Nash equilibria; if $\alpha > \frac{1}{2}$ then only (*Quiet*, *Quiet*) is a Nash equilibrium.

	Quiet	*Fink*
Quiet	$2(1+\alpha), 2(1+\alpha)$	$3\alpha, 3$
Fink	$3, 3\alpha$	$1+\alpha, 1+\alpha$

Figure 4.3 The payoffs in a variant of the *Prisoner's Dilemma* in which the players are altruistic.

25.2 Selfish and altruistic social behavior

a. A game that model the situation is shown in Figure 5.1.

	Sit	*Stand*
Sit	1, 1	2, 0
Stand	0, 2	0, 0

Figure 5.1 Behavior on a bus when the players' preferences are selfish (Exercise 25.2).

This game is not the *Prisoner's Dilemma*. If we identify *Sit* with *Quiet* and *Stand* with *Fink* then, for example, (*Stand*, *Sit*) is worse for player 1 than (*Sit*, *Sit*), rather than better. If we identify *Sit* with *Fink* and *Stand* with *Quiet* then, for example, (*Stand*, *Stand*) is worse for player 1 than (*Sit*, *Sit*), rather than better. The game has a unique Nash equilibrium, (*Sit*, *Sit*).

b. A game that models the situation is shown in Figure 5.2, where α is some positive number.

	Sit	*Stand*
Sit	1, 1	0, 2
Stand	2, 0	α, α

Figure 5.2 Behavior on a bus when the players' preferences are selfish (Exercise 25.2).

If $\alpha < 1$ then this game is the *Prisoner's Dilemma*. It has a unique Nash equilibrium, (*Stand*, *Stand*) (regardless of the value of α).

c. Both people are more comfortable in the equilibrium that results when they act according to their selfish preferences.

28.1 Variants of the Stag Hunt

a. The equilibria of the game are the same as those of the original game: (*Stag*, ..., *Stag*) and (*Hare*, ..., *Hare*). Any player that deviates from the first profile obtains a hare rather than the fraction $1/n$ of the stag. Any player that deviates from the second profile obtains nothing, rather than a hare.

An action profile in which at least 1 and at most $m - 1$ hunters pursue the stag is not a Nash equilibrium, since any one of them is better off catching a hare. An action profile in which at least m and at most $n - 1$ hunters pursue the stag is not a Nash equilibrium, since any one of the remaining hunters is better off joining the pursuit of the stag (thereby earning herself the right to a share of the stag).

b. The set of Nash equilibria consists of the action profile $(Hare, \ldots, Hare)$ in which all hunters catch hares, and any action profile in which exactly k hunters pursue the stag and the remaining hunters catch hares. Any player that deviates from the first profile obtains nothing, rather than a hare. A player who switches from the pursuit of the stag to catching a hare in the second type of profile is worse off, since she obtains a hare rather than the fraction $1/k$ of the stag; a player who switches from catching a hare to pursuing the stag is also worse off since she obtains the fraction $1/(k+1)$ of the stag rather than a hare, and $1/(k+1) < 1/k$.

No other action profile is a Nash equilibrium, by the following argument.

- If some hunters, but fewer than m, pursue the stag then each of them obtains nothing, and is better off catching a hare.
- If at least m and fewer than k hunters pursue the stag then each one that pursues a hare is better off switching to the pursuit of the stag.
- If more than k hunters pursue the stag then the fraction of the stag that each of them obtains is less than $1/k$, so each of them is better off catching a hare.

28.2 Extension of the Stag Hunt

Every profile $(e, \ldots, e)$, where e is an integer from 0 to K, is a Nash equilibrium. In the equilibrium $(e, \ldots, e)$, each player's payoff is e. The profile $(e, \ldots, e)$ is a Nash equilibrium since if player i chooses $e_i < e$ then her payoff is $2e_i - e_i = e_i < e$, and if she chooses $e_i > e$ then her payoff is $2e - e_i < e$.

Consider an action profile $(e_1, \ldots, e_n)$ in which not all effort levels are the same. Suppose that e_i is the minimum. Consider some player j whose effort level exceeds e_i. Her payoff is $2e_i - e_j < e_i$, while if she deviates to the effort level e_i her payoff is $2e_i - e_i = e_i$. Thus she can increase her payoff by deviating, so that $(e_1, \ldots, e_n)$ is not a Nash equilibrium.

(This game is studied experimentally by van Huyck, Battalio, and Beil (1990). See also Ochs (1995, 209–233).)

29.1 Hawk–Dove

A strategic game that models the situation is shown in Figure 6.1. The game has two Nash equilibria, $(Aggressive, Passive)$ and $(Passive, Aggressive)$.

	Aggressive	*Passive*
Aggressive	$0, 0$	$3, 1$
Passive	$1, 3$	$2, 2$

Figure 6.1 *Hawk–Dove.*

31.1 Contributing to a public good

The following game models the situation.

Players The n people.

Actions Each person's set of actions is $\{Contribute, Don't\ contribute\}$.

Preferences Each person's preferences are those given in the problem.

An action profile in which more than k people contribute is not a Nash equilibrium: any contributor can induce an outcome she prefers by deviating to not contributing.

An action profile in which k people contribute is a Nash equilibrium: if any contributor stops contributing then the good is not provided; if any noncontributor switches to contributing then she is worse off.

An action profile in which fewer than k people contribute is a Nash equilibrium only if no one contributes: if someone contributes, she can increase her payoff by switching to noncontribution.

In summary, the set of Nash equilibria is the set of action profiles in which k people contribute together with the action profile in which no one contributes.

32.1 Guessing two-thirds of the average

If all three players announce the same integer $k \geq 2$ then any one of them can deviate to $k - 1$ and obtain \$1 (since her number is now closer to $\frac{2}{3}$ of the average than the other two) rather than $\$\frac{1}{3}$. Thus no such action profile is a Nash equilibrium. If all three players announce 1, then no player can deviate and increase her payoff; thus $(1, 1, 1)$ is a Nash equilibrium.

Now consider an action profile in which not all three integers are the same; denote the highest by k^*.

- Suppose only one player names k^*; denote the other integers named by k_1 and k_2, with $k_1 \geq k_2$. The average of the three integers is $\frac{1}{3}(k^* + k_1 + k_2)$, so that $\frac{2}{3}$ of the average is $\frac{2}{9}(k^* + k_1 + k_2)$. If $k_1 \geq \frac{2}{9}(k^* + k_1 + k_2)$ then k^* is further from $\frac{2}{3}$ of the average than is k_1, and hence does not win. If $k_1 < \frac{2}{9}(k^* + k_1 + k_2)$ then the difference between k^* and $\frac{2}{3}$ of the average is $k^* - \frac{2}{9}(k^* + k_1 + k_2) = \frac{7}{9}k^* - \frac{2}{9}k_1 - \frac{2}{9}k_2$, while the difference between k_1 and $\frac{2}{3}$ of the average is $\frac{2}{9}(k^* + k_1 + k_2) - k_1 = \frac{2}{9}k^* - \frac{7}{9}k_1 + \frac{2}{9}k_2$. The difference between the former and the latter is $\frac{5}{9}k^* + \frac{5}{9}k_1 - \frac{4}{9}k_2 > 0$, so k_1 is closer to $\frac{2}{3}$ of the average than is k^*. Hence the player who names k^* does not win, and is better off naming k_2, in which case she obtains a share of the prize. Thus no such action profile is a Nash equilibrium.

- Suppose two players name k^*, and the third player names $k < k^*$. The average of the three integers is then $\frac{1}{3}(2k^* + k)$, so that $\frac{2}{3}$ of the average is

$\frac{4}{9}k^* + \frac{2}{9}k$. We have $\frac{4}{9}k^* + \frac{2}{9}k < \frac{1}{2}(k^* + k)$ (since $\frac{4}{9} < \frac{1}{2}$ and $\frac{2}{9} < \frac{1}{2}$), so that the player who names k is the sole winner. Thus either of the other players can switch to naming k and obtain a share of the prize rather obtaining nothing. Thus no such action profile is a Nash equilibrium.

We conclude that there is only one Nash equilibrium of this game, in which all three players announce the number 1.

(This game is studied experimentally by Nagel (1995).)

32.2 Voter participation

a. For $k = m = 1$ the game is shown in Figure 8.1. It is the same, except for the names of the actions, as the *Prisoner's Dilemma*.

		B supporter	
		abstain	vote
A supporter	abstain	$1, 1$	$0, 2 - c$
	vote	$2 - c, 0$	$1 - c, 1 - c$

Figure 8.1 The game of voter participation in Exercise 32.2.

b. For $k = m$, denote the number of citizens voting for A by n_A and the number voting for B by n_B. The cases in which $n_A \leq n_B$ are symmetric with those in which $n_A \geq n_B$; I restrict attention to the latter.

$n_A = n_B = k$ (all citizens vote): A citizen who switches from voting to abstaining causes the candidate she supports to lose rather than tie, reducing her payoff from $1 - c$ to 0. Since $c < 1$, this situation is a Nash equilibrium.

$n_A = n_B < k$ (not all citizens vote; the candidates tie): A citizen who switches from abstaining to voting causes the candidate she supports to win rather than tie, increasing her payoff from 1 to $2 - c$. Thus this situation is not a Nash equilibrium.

$n_A = n_B + 1$ or $n_B = n_A + 1$ (a candidate wins by one vote): A supporter of the losing candidate who switches from abstaining to voting causes the candidate she supports to tie rather than lose, increasing her payoff from 0 to $1 - c$. Thus this situation is not a Nash equilibrium.

$n_A \geq n_B + 2$ or $n_B \geq n_A + 2$ (a candidate wins by two or more votes): A supporter of the winning candidate who switches from voting to abstaining does not affect the outcome, so such a situation is not a Nash equilibrium.

We conclude that the game has a unique Nash equilibrium, in which all citizens vote.

c. If $k < m$ then a similar logic shows that there is no Nash equilibrium.

$n_A = n_B \leq k$: A supporter of B who switches from abstaining to voting causes B to win rather than tie, increasing her payoff from 1 to $2 - c$. Thus this situation is not a Nash equilibrium.

$n_A = n_B + 1$ or $n_B = n_A + 1$: A supporter of the losing candidate who switches from abstaining to voting causes the candidates to tie, increasing her payoff from 0 to $1 - c$. Thus this situation is not a Nash equilibrium.

$n_A \geq n_B + 2$ or $n_B \geq n_A + 2$: A supporter of the winning candidate who switches from voting to abstaining does not affect the outcome, so such a situation is not a Nash equilibrium.

32.3 Choosing a route

A strategic game that models this situation is:

Players The four people.

Actions The set of actions of each person is {X, Y} (the route via X and the route via Y).

Preferences Each player's payoff is the negative of her travel time.

In every Nash equilibrium, two people take each route. (In any other case, a person taking the more popular route is better off switching to the other route.) For any such action profile, each person's travel time is either 29.9 or 30 minutes (depending on the route they take). If a person taking the route via X switches to the route via Y her travel time becomes $12 + 21.8 = 33.8$ minutes; if a person taking the route via Y switches to the route via X her travel time becomes $22 + 12 = 34$ minutes. For any other allocation of people to routes, at least one person can decrease her travel time by switching routes. Thus the set of Nash equilibria is the set of action profiles in which two people take the route via X and two people take the route via Y.

Now consider the situation after the road from X to Y is built. There is no equilibrium in which the new road is not used, by the following argument. Because the only equilibrium before the new road is built has two people taking each route, the only possibility for an equilibrium in which no one uses the new road is for two people to take the route A–X–B and two to take A–Y–B, resulting in a total travel time for each person of either 29.9 or 30 minutes. However, if a person taking A–X–B switches to the new road at X and then takes Y–B her total travel time becomes $9 + 7 + 12 = 28$ minutes.

I claim that in any Nash equilibrium, one person takes A–X–B, two people take A–X–Y–B, and one person takes A–Y–B. For this assignment, each person's travel time is 32 minutes. No person can change her route and decrease her travel time, by the following argument.

- If the person taking A–X–B switches to A–X–Y–B, her travel time increases to $12 + 9 + 15 = 36$ minutes; if she switches to A–Y–B her travel time increases to $21 + 15 = 36$ minutes.
- If one of the people taking A–X–Y–B switches to A–X–B, her travel time increases to $12 + 20.9 = 32.9$ minutes; if she switches to A–Y–B her travel time increases to $21 + 12 = 33$ minutes.
- If the person taking A–Y–B switches to A–X–B, her travel time increases to $15 + 20.9 = 35.9$ minutes; if she switches to A–X–Y–B, her travel time increases to $15 + 9 + 12 = 36$ minutes.

For every other allocation of people to routes at least one person can switch routes and reduce her travel time. For example, if one person takes A–X–B, one person takes A–X–Y–B, and two people take A–Y–B, then the travel time of those taking A–Y–B is $21 + 12 = 33$ minutes; if one of them switches to A–X–B then her travel time falls to $12 + 20.9 = 32.9$ minutes. Or if one person takes A–Y–B, one person takes A–X–Y–B, and two people take A–X–B, then the travel time of those taking A–X–B is $12 + 20.9 = 32.9$ minutes; if one of them switches to A–X–Y–B then her travel time falls to $12 + 8 + 12 = 32$ minutes.

Thus in the equilibrium with the new road every person's travel time *increases*, from either 29.9 or 30 minutes to 32 minutes.

35.1 Finding Nash equilibria using best response functions

a. The *Prisoner's Dilemma* and *BoS* are shown in Figure 10.1; *Matching Pennies* and the two-player *Stag Hunt* are shown in Figure 10.2.

	Quiet	*Fink*
Quiet	$2\ ,2$	$0\ ,3^*$
Fink	$3^*,0$	$1^*,1^*$

Prisoner's Dilemma

	Bach	*Stravinsky*
Bach	$2^*,1^*$	$0\ ,0$
Stravinsky	$0\ ,0$	$1^*,2^*$

BoS

Figure 10.1 The best response functions in the *Prisoner's Dilemma* (left) and in *BoS* (right).

	Head	*Tail*
Head	$1^*,-1$	$-1\ ,\ \ 1^*$
Tail	$-1\ ,\ \ 1^*$	$1^*,-1$

Matching Pennies

	Stag	*Hare*
Stag	$2^*,2^*$	$0\ ,1$
Hare	$1\ ,0$	$1^*,1^*$

Stag Hunt

Figure 10.2 The best response functions in *Matching Pennies* (left) and the *Stag Hunt* (right).

b. The best response functions are indicated in Figure 11.1. The Nash equilibria are (T, C), (M, L), and (B, R).

	L	C	R
T	2 ,2	1*,3*	0*,1
M	3*,1*	0 ,0	0*,0
B	1 ,0*	0 ,0*	0*,0*

Figure 11.1 The game in Exercise 35.1.

36.1 Constructing best response functions

The analogue of Figure 36.2 is given in Figure 11.2.

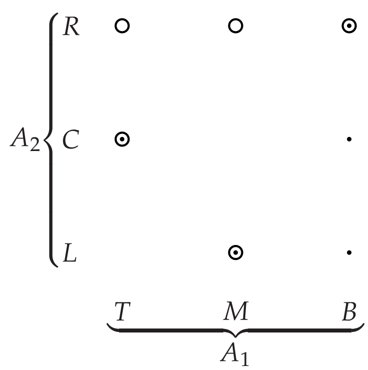


Figure 11.2 The players' best response functions for the game in Exercise 36.1b. Player 1's best responses are indicated by circles, and player 2's by dots. The action pairs for which there is both a circle and a dot are the Nash equilibria.

36.2 Dividing money

For each amount named by one of the players, the other player's best responses are given in the following table.

Other player's action	Sets of best responses
0	$\{10\}$
1	$\{9, 10\}$
2	$\{8, 9, 10\}$
3	$\{7, 8, 9, 10\}$
4	$\{6, 7, 8, 9, 10\}$
5	$\{5, 6, 7, 8, 9, 10\}$
6	$\{5, 6\}$
7	$\{6\}$
8	$\{7\}$
9	$\{8\}$
10	$\{9\}$

The best response functions are illustrated in Figure 12.1 (circles for player 1, dots for player 2). From this figure we see that the game has four Nash equilibria: $(5,5)$, $(5,6)$, $(6,5)$, and $(6,6)$.

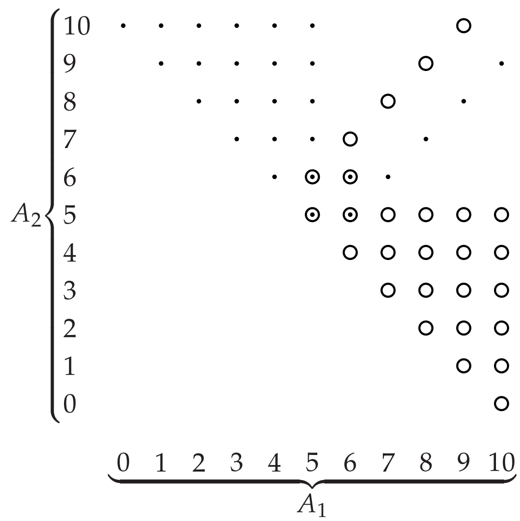


Figure 12.1 The players' best response functions for the game in Exercise 36.2.

39.1 Strict and nonstrict Nash equilibria

Only the Nash equilibrium (a_1^*, a_2^*) is strict. For each of the other equilibria, player 2's action a_2 satisfies $a_2^{***} \leq a_2 \leq a_2^{**}$, and for each such action player 1 has multiple best responses, so that her payoff is the same for a range of actions, only one of which is such that (a_1, a_2) is a Nash equilibrium.

40.1 Finding Nash equilibria using best response functions

First find the best response function of player 1. For any fixed value of a_2, player 1's payoff function $a_1(a_2 - a_1)$ is a quadratic in a_1. The coefficient of a_1^2 is negative and the function is zero at $a_1 = 0$ and at $a_1 = a_2$. Thus, using the symmetry of quadratic functions, $b_1(a_2) = \frac{1}{2}a_2$.

Now find the best response function of player 2. For any fixed value of a_1, player 2's payoff function $a_2(1 - a_1 - a_2)$ is a quadratic in a_2. The coefficient on a_2^2 is negative and the function is zero at $a_2 = 0$ and at $a_2 = 1 - a_1$. Thus if $a_1 \leq 1$ we have $b_2(a_1) = \frac{1}{2}(1 - a_1)$ and if $a_1 > 1$ we have $b_2(a_1) = 0$.

The best response functions are shown in Figure 13.1.

A Nash equilibrium is a pair (a_1^*, a_2^*) such that $a_1^* = b_1(a_2^*)$ and $a_2^* = b_2(a_1^*)$. From the figure we see that there is a unique Nash equilibrium, with $a_1^* < 1$. Thus

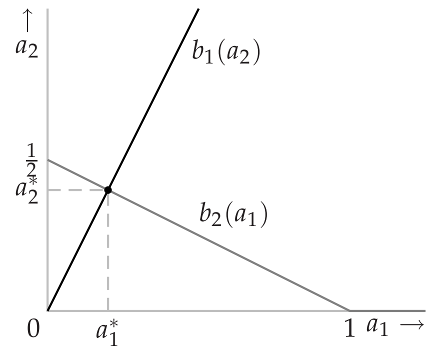


Figure 13.1 The best response functions for the game in Exercise 40.1.

in this equilibrium $a_1^* = \frac{1}{2}a_2^*$ and $a_2^* = \frac{1}{2}(1 - a_1^*)$. Hence $a_1^* = \frac{1}{4}(1 - a_1^*)$, or $5a_1^* = 1$, or $a_1^* = \frac{1}{5}$. Hence $a_2^* = \frac{2}{5}$. Thus the game has a unique Nash equilibrium, $(\frac{1}{5}, \frac{2}{5})$.

40.2 A joint project

A strategic game that models this situation is:

Players The two people.

Actions The set of actions of each person i is the set of effort levels (the set of numbers x_i with $0 \leq x_i \leq 1$).

Preferences Person i's payoff to the action pair (x_1, x_2) is $\frac{1}{2}f(x_1, x_2) - c(x_i)$.

a. Assume that $f(x_1, x_2) = 3x_1x_2$ and $c(x_i) = x_i^2$. To find the Nash equilibria of the game, first find the players' best response functions. Player 1's best response to x_2 is the action x_1 that maximizes $\frac{3}{2}x_1x_2 - x_1^2$, or $x_1(\frac{3}{2}x_2 - x_1)$. This function is a quadratic that is zero when $x_1 = 0$ and when $x_1 = \frac{3}{2}x_2$. The coefficient of x_1^2 is negative, so the maximum of the function occurs at $x_1 = \frac{3}{4}x_2$. Thus player 1's best response function is

$$b_1(x_2) = \tfrac{3}{4}x_2.$$

Similarly, player 2's best response function is

$$b_2(x_1) = \tfrac{3}{4}x_1.$$

The best response functions are shown in Figure 14.1.

In a Nash equilibrium (x_1^*, x_2^*) we have $x_1^* = b_1(x_2^*)$ and $x_2^* = b_2(x_1^*)$, or $x_1^* = \frac{3}{4}x_2^*$ and $x_2^* = \frac{3}{4}x_1^*$. Substituting x_2^* in the first equation we obtain $x_1^* = \frac{9}{16}x_1^*$, so that $x_1^* = 0$. Thus $x_2^* = 0$.

We conclude that the game has a unique Nash equilibrium, $(x_1^*, x_2^*) = (0, 0)$. In this equilibrium, both players' payoffs are zero.

If each player i chooses $x_i = 1$ then the total output is 3, and each player's payoff is $\frac{3}{2} - 1 = \frac{1}{2}$, rather than 0 as in the Nash equilibrium.

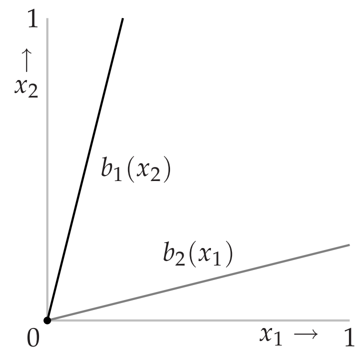


Figure 14.1 The best response functions for the game in Exercise 40.2a.

b. When $f(x_1, x_2) = 4x_1x_2$ and $c(x_i) = x_i$, player 1's payoff function is

$$2x_1x_2 - x_1 = x_1(2x_2 - 1).$$

Thus if $x_2 < \frac{1}{2}$ her best response is $x_1 = 0$, if $x_2 = \frac{1}{2}$ then all values of x_1 are best responses, and if $x_2 > \frac{1}{2}$ her best response is $x_1 = 1$. That is, player 1's best response function is

$$b_1(x_2) = \begin{cases} 0 & \text{if } x_2 < \frac{1}{2} \\ \{x_1 : 0 \leq x_1 \leq 1\} & \text{if } x_2 = \frac{1}{2} \\ 1 & \text{if } x_2 > \frac{1}{2}. \end{cases}$$

Player 2's best response function is the same. (That is, $b_2(x) = b_1(x)$ for all x.) The best response functions are shown in Figure 14.2.

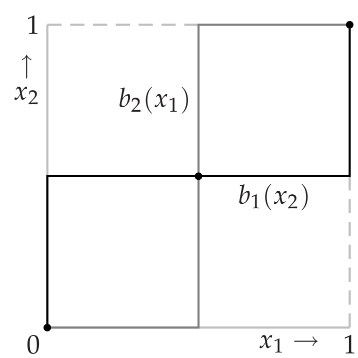


Figure 14.2 The best response functions for the game in Exercise 40.2b.

We see that the game has three Nash equilibria, $(0, 0)$, $(\frac{1}{2}, \frac{1}{2})$, and $(1, 1)$.

The players' payoffs at these equilibria are $(0, 0)$, $(0, 0)$, and $(1, 1)$. There is no pair of effort levels that yields both players payoffs higher than 1, but there are pairs of effort levels that yield both players payoffs higher than 0, for example $(1, 1)$, which yields the payoffs $(1, 1)$.

42.1 Contributing to a public good

The best response of player 1 to the contribution c_2 of player 2 is the value of c_1 that maximizes player 1's payoff $w + c_2 + (w - c_1)(c_1 + c_2)$. This function is a quadratic in c_1 (remember that $w + c_2$ is a constant). The coefficient of c_1^2 is negative, and the value of the function is equal to $w + c_2$ when $c_1 = w$ and when $c_1 = -c_2$. Thus the function attains a maximum at $c_1 = \frac{1}{2}(w - c_2)$. We conclude that player 1's best response function is

$$b_1(c_2) = \tfrac{1}{2}(w - c_2).$$

Player 2's best response function is similarly

$$b_2(c_1) = \tfrac{1}{2}(w - c_1).$$

A Nash equilibrium is a pair (c_1^*, c_2^*) such that $c_1^* = b_1(c_2^*)$ and $c_2^* = b_2(c_1^*)$, so that

$$c_1^* = \tfrac{1}{2}(w - c_2^*) = \tfrac{1}{2}(w - \tfrac{1}{2}(w - c_1^*)) = \tfrac{1}{4}w + \tfrac{1}{4}c_1^*$$

and hence $c_1^* = \frac{1}{3}w$. Substituting this value into player 2's best response function we get $c_2^* = \frac{1}{3}w$.

We conclude that the game has a unique Nash equilibrium $(c_1^*, c_2^*) = (\frac{1}{3}w, \frac{1}{3}w)$, in which each person contributes one third of her wealth to the public good.

In this equilibrium each player's payoff is $\frac{4}{3}w + \frac{4}{9}w^2$. If each player contributes $\frac{1}{2}w$ to the public good then her payoff is $\frac{3}{2}w + \frac{1}{2}w^2$, which exceeds $\frac{4}{3}w + \frac{4}{9}w^2$ for all w (since $\frac{3}{2} > \frac{4}{3}$ and $\frac{1}{2} > \frac{4}{9}$).

When there are n players the payoff function of player 1 is

$$\begin{aligned} w - c_1 + c_1 + c_2 + \cdots + c_n + (w - c_1)(c_1 + c_2 + \cdots + c_n) &= \\ w + c_2 + \cdots + c_n + (w - c_1)(c_1 + c_2 + \cdots + c_n). \end{aligned}$$

This function is a quadratic in c_1. The coefficient of c_1^2 is negative, and the value of the function is equal to $w + c_2 + \cdots + c_n$ when $c_1 = w$ and when $c_1 = -c_2 - c_3 - \cdots - c_n$. Thus the function attains a maximum at $c_1 = \frac{1}{2}(w - c_2 - c_3 - \cdots - c_n)$. We conclude that player 1's best response function is

$$b_1(c_{-1}) = \tfrac{1}{2}(w - c_2 - c_3 - \cdots - c_n)$$

where c_{-1} is the list of the contributions of the players other than 1. Similarly, any player i's best response function is

$$b_i(c_{-i}) = \tfrac{1}{2}(w - (c_1 + c_2 + \cdots + c_n) + c_i).$$

A Nash equilibrium is an action profile $(c_1^*, \ldots, c_n^*)$ such that $c_i^* = b_i(c_{-i}^*)$ for all i. We can write the condition $c_1^* = b_1(c_{-1}^*)$ as

$$2c_1^* = w - c_2^* - c_3^* - \cdots - c_n^*,$$

or

$$w = 2c_1^* + c_2^* + c_3^* + \cdots + c_n^*.$$

Writing the other conditions $c_i^* = b_i(c_{-i}^*)$ similarly, we obtain the system of equations

$$\begin{aligned} w &= 2c_1^* + c_2^* + c_3^* + \cdots + c_n^* \\ w &= c_1^* + 2c_2^* + c_3^* + \cdots + c_n^* \\ &\vdots \\ w &= c_1^* + c_2^* + c_3^* + \cdots + 2c_n^* \end{aligned}$$

Subtracting the second equation from the first we conclude that $c_1^* = c_2^*$. Similarly subtracting each equation from the next we deduce that c_i^* is the same for all i. Denote the common value by c^*. From any of the equations we deduce that $w = (n+1)c^*$. Hence $c^* = w/(n+1)$.

In conclusion, when there are n players the game has a unique Nash equilibrium $(c_1^*, \ldots, c_n^*) = (w/(n+1), \ldots, w/(n+1))$. The total amount contributed in this equilibrium is $nw/(n+1)$, which increases as n increases, approaching w as n increases without bound.

Player 1's payoff in the equilibrium is $w + (n-1)w/(n+1) + (nw/(n+1))^2$. As n increases without bound, this payoff increases, approaching $2w + w^2$. If each player contributes $\frac{1}{2}w$ to the public good, each player's payoff is $w + \frac{1}{2}(n-1)w + n(w/2)^2$, which increases without bound as n increases without bound.

45.2 Strict equilibria and dominated actions

For player 1, T is weakly dominated by M, and strictly dominated by B. For player 2, no action is weakly or strictly dominated. The game has a unique Nash equilibrium, (M, L). This equilibrium is not strict. (When player 2 choose L, B yields player 1 the same payoff as does M.)

46.1 Nash equilibrium and weakly dominated actions

The only Nash equilibrium of the game in Figure 16.1 is (T, L). The action T is weakly dominated by M and the action L is weakly dominated by C. (There are of course many other games that satisfy the conditions.)

	L	C	R
T	1,1	0,1	0,0
M	1,0	2,1	1,2
B	0,0	1,1	2,0

Figure 16.1 A game with a unique Nash equilibrium, in which both players' equilibrium actions are weakly dominated. (The unique Nash equilibrium is (T, L).)

47.1 Voting

First consider an action profile in which the winner receives one more vote than the loser and at least one citizen who votes for the winner prefers the loser to the winner. Any citizen who votes for the winner and prefers the loser to the winner can, by switching her vote, cause her favorite candidate to win rather than lose. Thus no such action profile is a Nash equilibrium.

Next consider an action profile in which the winner receives one more vote than the loser and all citizens who vote for the winner prefer the winner to the loser. Because a majority of citizens prefer A to B, the winner in any such case must be A. No citizen who prefers A to B can induce a better outcome by changing her vote, since her favorite candidate wins. Now consider a citizen who prefers B to A. By assumption, every such citizen votes for B; a change in her vote has no effect on the outcome (A still wins). Thus every such action profile is a Nash equilibrium.

Finally consider an action profile in which the winner receives at least three more votes than the loser. In this case no change in any citizen's vote has any effect on the outcome. Thus every such profile is a Nash equilibrium.

In summary, the Nash equilibria are: any action profile in which A receives one more vote than B and all the citizens who vote for A prefer A to B, and any action profile in which the winner receives at least three more votes than the loser.

The only equilibrium in which no citizen uses a weakly dominated action is that in which every citizen votes for her favorite candidate.

47.2 Voting between three candidates

Fix some citizen, say i; suppose she prefers A to B to C. By the argument in the text, citizen i's voting for C is weakly dominated by her voting for A (and by her voting for B). Her voting for B is clearly not weakly dominated by her voting for C. I now argue that her voting for B is not weakly dominated by her voting for A. Suppose that the other citizens' votes are equally divided between B and C; no one votes for A. Then if citizen i votes for A the outcome is a tie between B and C, while if she votes for B the outcome is that B wins. Thus for this configuration of the other citizens' votes, citizen i is better off voting for B than she is voting for A. Thus her voting for B is not weakly dominated by her voting for A.

Now fix some citizen, say i, and consider the candidate she ranks in the middle, say candidate B. The action profile in which all citizens vote for B is a Nash equilibrium. (No citizen's changing her vote affects the outcome.) In this equilibrium, citizen i does not vote for her favorite candidate, but the action she takes is not weakly dominated. (Other Nash equilibria also satisfy the conditions in the exercise.)

47.3 Approval voting

First I argue that any action a_i of player i that includes a vote for i's least preferred candidate, say candidate k, is weakly dominated by the action a_i' that differs from a_i only in that candidate k does not receive a vote in a_i'. For any list a_{-i} of the other players' actions, the outcome of (a_i', a_{-i}) differs from that of (a_i, a_{-i}) only in that the total number of votes received by candidate k is one less in (a_i', a_{-i}) than it is in (a_i, a_{-i}). There are two possible implications for the winners of the election, depending on a_{-i}: either the set of winners is the same in (a_i, a_{-i}) as it is in (a_i', a_{-i}), or candidate k is a winner in (a_i, a_{-i}) but not in (a_i', a_{-i}). Because candidate k is player i's least preferred candidate, a_i' thus weakly dominates a_i.

I now argue that any action a_i of player i that excludes a vote for i's most preferred candidate, say candidate 1, is weakly dominated by the action a_i' that differs from a_i only in that candidate 1 receives a vote in a_i'. The argument is symmetric with the one in the previous paragraph. For any list a_{-i} of the other players' actions, the outcome of (a_i', a_{-i}) differs from that of (a_i, a_{-i}) only in that the total number of votes received by candidate 1 is one more in (a_i', a_{-i}) than it is in (a_i, a_{-i}). There are two possible implications for the winners of the election, depending on a_{-i}: either the set of winners is the same in (a_i, a_{-i}) as it is in (a_i', a_{-i}), or candidate 1 is a winner in (a_i', a_{-i}) but not in (a_i, a_{-i}). Because candidate 1 is player i's most preferred candidate, a_i' thus weakly dominates a_i.

Finally I argue that if citizen i prefers candidate 1 to candidate 2 to $\ldots$ to candidate k then the action a_i that consists of votes for candidates 1 and $k-1$ is not weakly dominated.

- The action a_i is not weakly dominated by any action that excludes votes for either candidate 1 or candidate $k-1$ (or both). Suppose a_i' excludes a vote for candidate 1. Then if the total votes by the other citizens for candidates 1 and 2 are equal, and the total votes for all other candidates are less, then citizen i's taking the action a_i leads candidate 1 to win, while the action a_i' leads to at best (from the point of view of citizen i) a tie between candidates 1 and 2. Thus a_i' does not weakly dominate a_i. Similarly, suppose that a_i' excludes a vote for candidate $k-1$. Then if the total votes by the other citizens for candidates $k-1$ and k are equal, while the total votes for all other candidates are less, then citizen i's taking the action a_i leads candidate $k-1$ to win, while the action a_i' leads to at best (from the point of view of citizen i) a tie between candidates $k-1$ and k.

- Now let a_i' be an action that includes votes for both candidate 1 and candidate $k-1$, and also for at least one other candidate, say candidate j. Suppose that the total votes by the other citizens for candidates 1 and j are equal, and the total votes for all other candidates are less. Then citizen i's taking the action a_i leads candidate 1 to win, while the action a_i' leads to at best (from the point of view of citizen i) a tie between candidates 1 and j. Thus a_i' does not weakly dominate a_i.

49.1 Other Nash equilibria of the game modeling collective decision-making

Denote by i the player whose favorite policy is the median favorite policy. The set of Nash equilibria includes every action profile in which (*i*) i's action is her favorite policy x_i^*, (*ii*) every player whose favorite policy is less than x_i^* names a policy equal to at most x_i^*, and (*iii*) every player whose favorite policy is greater than x_i^* names a policy equal to at least x_i^*.

To show this, first note that the outcome is x_i^*, so player i cannot induce a better outcome for herself by changing her action. Now, if a player whose favorite position is less than x_i^* changes her action to some $x < x_i^*$, the outcome does not change; if such a player changes her action to some $x > x_i^*$ then the outcome either remains the same (if some player whose favorite position exceeds x_i^* names x_i^*) or increases, so that the player is not better off. A similar argument applies to a player whose favorite position is greater than x_i^*.

The set of Nash equilibria also includes, for any positive integer $k \leq n$, every action profile in which k players name the median favorite policy x_i^*, at most $\frac{1}{2}(n-3)$ players name policies less than x_i^*, and at most $\frac{1}{2}(n-3)$ players name policies greater than x_i^*. (In these equilibria, the favorite policy of a player who names a policy less than x_i^* may be greater than x_i^*, and vice versa. The conditions on the numbers of players who name policies less than x_i^* and greater than x_i^* ensure that no such player can, by naming instead her favorite policy, move the median policy closer to her favorite policy.)

Any action profile in which all players name the same, arbitrary, policy is also a Nash equilibrium; the outcome is the common policy named.

More generally, any profile in which at least three players name the same, arbitrary, policy x, at most $(n-3)/2$ players name a policy less than x, and at most $(n-3)/2$ players name a policy greater than x is a Nash equilibrium. (In both cases, no change in any player's action has any effect on the outcome.)

49.2 Another mechanism for collective decision-making

When the policy chosen is the mean of the announced policies, player i's announcing her favorite policy does not weakly dominate all her other actions. For example, if there are three players, the favorite policy of player 1 is 0.3, and the other players both announce the policy 0, then player 1 should announce the policy 0.9, which leads to the policy 0.3 ($= (0+0+0.9)/3$) being chosen, rather than 0.3, which leads to the policy 0.1.

50.1 Symmetric strategic game

The games in Exercise 29.1, Example 37.1, and Figure 46.1 (both games) are symmetric. The game in Exercise 40.1 is not symmetric. The game in Section 2.8.4 is symmetric if and only if $u_1 = u_2$.

51.1 Equilibrium for pairwise interactions in a single population

The Nash equilibria are (A, A), (A, C), and (C, A). Only the equilibrium (A, A) is relevant if the game is played between the members of a single population—this equilibrium is the only *symmetric* equilibrium.

3 Nash Equilibrium: Illustrations

57.1 Cournot's duopoly game with linear inverse demand and different unit costs

Following the analysis in the text, the best response function of firm 1 is

$$b_1(q_2) = \begin{cases} \frac{1}{2}(\alpha - c_1 - q_2) & \text{if } q_2 \leq \alpha - c_1 \\ 0 & \text{otherwise} \end{cases}$$

while that of firm 2 is

$$b_2(q_1) = \begin{cases} \frac{1}{2}(\alpha - c_2 - q_1) & \text{if } q_1 \leq \alpha - c_2 \\ 0 & \text{otherwise.} \end{cases}$$

To find the Nash equilibrium, first plot these two functions. Each function has the same general form as the best response function of either firm in the case studied in the text. However, the fact that $c_1 \neq c_2$ leads to two qualitatively different cases when we combine the two functions to find a Nash equilibrium. If c_1 and c_2 do not differ very much then the functions in the analogue of Figure 56.2 intersect at a pair of outputs that are both positive. If c_1 and c_2 differ a lot, however, the functions intersect at a pair of outputs in which $q_1 = 0$.

Precisely, if $c_1 \leq \frac{1}{2}(\alpha + c_2)$ then the downward-sloping parts of the best response functions intersect (as in Figure 56.2), and the game has a unique Nash equilibrium, given by the solution of the two equations

$$\begin{aligned} q_1 &= \tfrac{1}{2}(\alpha - c_1 - q_2) \\ q_2 &= \tfrac{1}{2}(\alpha - c_2 - q_1). \end{aligned}$$

This solution is

$$(q_1^*, q_2^*) = \left(\tfrac{1}{3}(\alpha - 2c_1 + c_2), \tfrac{1}{3}(\alpha - 2c_2 + c_1)\right).$$

If $c_1 > \frac{1}{2}(\alpha + c_2)$ then the downward-sloping part of firm 1's best response function lies below the downward-sloping part of firm 2's best response function (as in Figure 22.1), and the game has a unique Nash equilibrium, $(q_1^*, q_2^*) = (0, \frac{1}{2}(\alpha - c_2))$.

In summary, the game always has a unique Nash equilibrium, defined as follows:

$$\begin{cases} \left(\frac{1}{3}(\alpha - 2c_1 + c_2), \frac{1}{3}(\alpha - 2c_2 + c_1)\right) & \text{if } c_1 \leq \frac{1}{2}(\alpha + c_2) \\ \left(0, \frac{1}{2}(\alpha - c_2)\right) & \text{if } c_1 > \frac{1}{2}(\alpha + c_2). \end{cases}$$

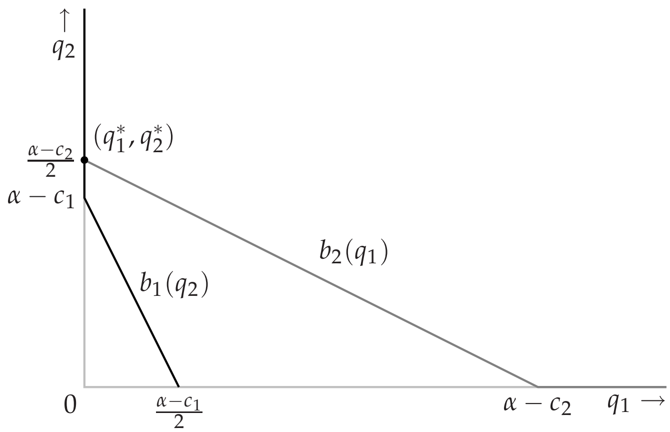


Figure 22.1 The best response functions in Cournot's duopoly game under the assumptions of Exercise 57.1 when $\alpha - c_1 < \frac{1}{2}(\alpha - c_2)$. The unique Nash equilibrium in this case is $(q_1^*, q_2^*) = (0, \frac{1}{2}(\alpha - c_2))$.

The output of firm 2 exceeds that of firm 1 in every equilibrium.

If c_2 decreases then firm 2's output increases and firm 1's output either falls, if $c_1 \leq \frac{1}{2}(\alpha + c_2)$, or remains equal to 0, if $c_1 > \frac{1}{2}(\alpha + c_2)$. The total output increases and the price falls.

57.2 Cournot's duopoly game with linear inverse demand and a quadratic cost function

Firm 1's profit is

$$\pi_1(q_1, q_2) = \begin{cases} q_1(\alpha - q_1 - q_2) - q_1^2 & \text{if } q_1 + q_2 \leq \alpha \\ -q_1^2 & \text{if } q_1 + q_2 > \alpha \end{cases}$$

or

$$\pi_1(q_1, q_2) = \begin{cases} q_1(\alpha - 2q_1 - q_2) & \text{if } q_1 + q_2 \leq \alpha \\ -q_1^2 & \text{if } q_1 + q_2 > \alpha. \end{cases}$$

When it is positive, this function is a quadratic in q_1 that is zero at $q_1 = 0$ and at $q_1 = (\alpha - q_2)/2$. Thus firm 1's best response function is

$$b_1(q_2) = \begin{cases} \frac{1}{4}(\alpha - q_2) & \text{if } q_2 \leq \alpha \\ 0 & \text{if } q_2 > \alpha. \end{cases}$$

Since the firms' cost functions are the same, firm 2's best response function is the same as firm 1's: $b_2(q) = b_1(q)$ for all q. The firms' best response functions are shown in Figure 23.1.

Solving the two equations $q_1^* = b_1(q_2^*)$ and $q_2^* = b_2(q_1^*)$ we find that there is a unique Nash equilibrium, in which the output of firm i ($i = 1, 2$) is $q_i^* = \frac{1}{5}\alpha$.

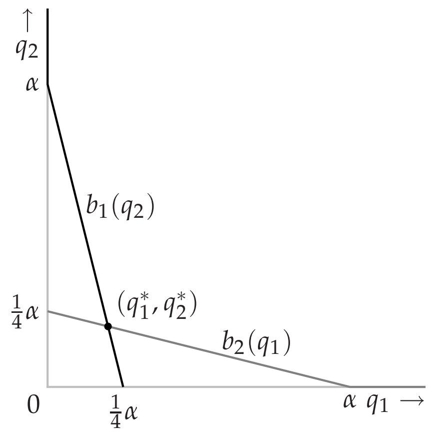


Figure 23.1 The best response functions in Cournot's duopoly game with linear inverse demand and a quadratic cost function, as in Exercise 57.2. The unique Nash equilibrium is $(q_1^*, q_2^*) = (\frac{1}{5}\alpha, \frac{1}{5}\alpha)$.

57.3 Cournot's duopoly game with linear inverse demand and a fixed cost

Firm i's payoff function is

$$\begin{cases} 0 & \text{if } q_i = 0 \\ q_i(P(q_1 + q_2) - c) - f & \text{if } q_i > 0. \end{cases}$$

As before firm 1's best response to q_2 is $(\alpha - c - q_2)/2$ if firm 1's profit is nonnegative for this output; otherwise its best response is the output of zero. Firm 1's profit when it produces $(\alpha - c - q_2)/2$ and firm 2 produces q_2 is

$$\frac{\alpha - c - q_2}{2}\left(\alpha - c - \frac{\alpha - c - q_2}{2} - q_2\right) - f = \left(\frac{\alpha - c - q_2}{2}\right)^2 - f,$$

which is nonnegative if

$$\left(\frac{\alpha - c - q_2}{2}\right)^2 > f,$$

or if $q_2 \leq \alpha - c - 2\sqrt{f}$. Let $\overline{q} = \alpha - c - 2\sqrt{f}$. Then firm 1's best response function is

$$b_1(q_2) = \begin{cases} \frac{1}{2}(\alpha - c - q_2) & \text{if } q_2 < \overline{q} \\ \{0, \frac{1}{2}(\alpha - c - q_2)\} & \text{if } q_2 = \overline{q} \\ 0 & \text{if } q_2 > \overline{q}. \end{cases}$$

(If $q_2 = \overline{q}$ then firm 1's profit is zero whether it produces the output $\frac{1}{2}(\alpha - c - q_2)$ or the output 0; both outputs are optimal.)

Thus firm 1's best response function has a "jump": for outputs of firm 2 slightly less than $\overline{q}$ firm 1 wants to produce a positive output (and earn a small profit), while for outputs of firm 2 slightly greater than $\overline{q}$ it wants to produce an output of zero.

Firm 2's cost function is the same as firm 1's, so its best response function is the same.

Because of the jumps in the best response functions, there are four qualitatively different cases, depending on the value of f. If f is small enough that $\overline{q} > \frac{1}{2}(\alpha - c)$ (or, equivalently, $f < (\alpha - c)^2/16$) then the best response functions take the form given in Figure 24.1. In this case the existence of the fixed cost has no impact on the equilibrium, which remains $(q_1^*, q_2^*) = (\frac{1}{3}(\alpha - c), \frac{1}{3}(\alpha - c))$.

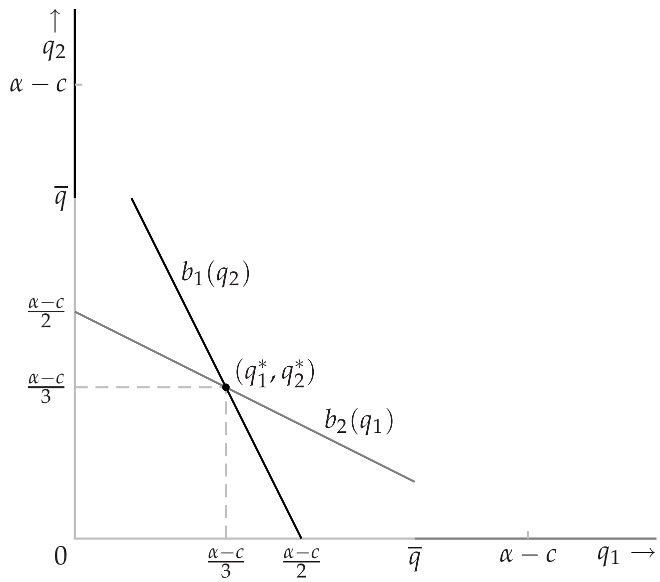


Figure 24.1 The best response functions in Cournot's duopoly game when the inverse demand function is $P(Q) = \alpha - Q$ (where this is positive) and the cost function of each firm is $f + cq$, with $f < (\alpha - c)^2/16$. The unique Nash equilibrium is $(q_1^*, q_2^*) = (\frac{1}{3}(\alpha - c), \frac{1}{3}(\alpha - c))$ (as in the case in which $f = 0$).

As f increases, the point at which the best response functions jump moves closer to the origin. Eventually $\overline{q}$ enters the range from $\frac{1}{3}(\alpha - c)$ to $\frac{1}{2}(\alpha - c)$ (which implies that $(\alpha - c)^2/16 < f < (\alpha - c)^2/9$), in which case the best response functions take the forms shown in the left panel of Figure 25.1. In this case there are three Nash equilibria: $(0, \frac{1}{2}(\alpha - c))$, $((\alpha - c)/3, (\alpha - c)/3)$, and $(\frac{1}{2}(\alpha - c), 0)$.

As f increases further, there is a point at which $\overline{q}$ becomes less than $\frac{1}{3}(\alpha - c)$ but is still positive (implying that $(\alpha - c)^2/9 < f < (\alpha - c)^2/4$), so that the best response functions take the forms shown in the right panel of Figure 25.1. In this case there are two Nash equilibria: $(0, \frac{1}{2}(\alpha - c))$ and $(\frac{1}{2}(\alpha - c), 0)$.

Finally, if f is extremely large then a firm does not want to produce any output even if the other firm produces no output. This occurs when $f > (\alpha - c)^2/4$; the unique Nash equilibrium in this case is $(0, 0)$.

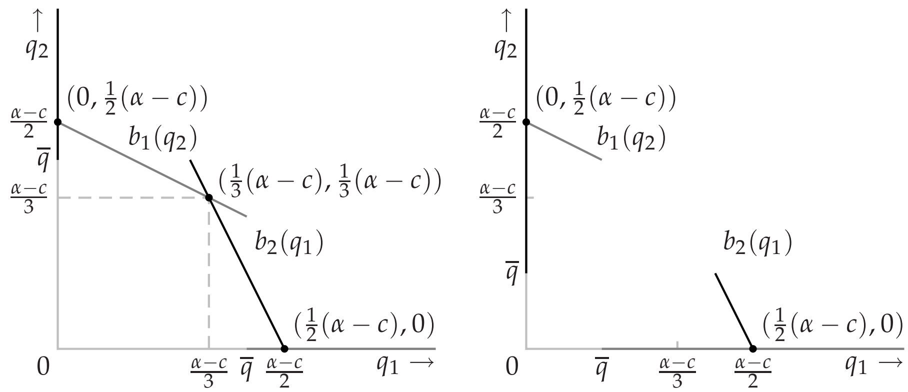


Figure 25.1 The best response functions in Cournot's duopoly game when the inverse demand function is $P(Q) = \alpha - Q$ (where this is positive) and the cost function of each firm is $f + cq$, with $(\alpha - c)^2/16 < f < (\alpha - c)^2/9$ (left panel) and $f > (\alpha - c)^2/9$ (right panel). In the first case the game has three Nash equilibria: $(0, \frac{1}{2}(\alpha - c))$, $(\frac{1}{3}(\alpha - c), \frac{1}{3}(\alpha - c))$, and $(\frac{1}{2}(\alpha - c), 0)$. In the second case it has two Nash equilibria: $(0, \frac{1}{2}(\alpha - c))$ and $(\frac{1}{2}(\alpha - c), 0)$.

58.2 Nash equilibrium of Cournot's duopoly game and the collusive outcome

The firms' total profit is $(q_1 + q_2)(\alpha - c - q_1 - q_2)$, or $Q(\alpha - c - Q)$, where Q denotes total output. This function is a quadratic in Q that is zero when $Q = 0$ and when $Q = \alpha - c$, so that its maximizer is $Q^* = \frac{1}{2}(\alpha - c)$.

If each firm produces $\frac{1}{4}(\alpha - c)$ then its profit is $\frac{1}{8}(\alpha - c)^2$. This profit exceeds its Nash equilibrium profit of $\frac{1}{9}(\alpha - c)^2$.

If one firm produces $Q^*/2$, the other firm's best response is $b_i(Q^*/2) = \frac{1}{2}(\alpha - c - \frac{1}{4}(\alpha - c)) = \frac{3}{8}(\alpha - c)$. That is, if one firm produces $Q^*/2$, the other firm wants to produce *more* than $Q^*/2$.

58.1 Variant of Cournot's game, with market-share maximizing firms

Let firm 1 be the market-share maximizing firm. If $q_2 > \alpha - c$, there is no output of firm 1 for which its profit is nonnegative. Thus its best response to such an output of firm 2 is $q_1 = 0$. If $q_2 \leq \alpha - c$ then firm 1 wants to choose its output q_1 large enough that the price is c (and hence its profit is zero). Thus firm 1's best response to such a value of q_2 is $q_1 = \alpha - c - q_2$. We conclude that firm 1's best response function is

$$b_1(q_2) = \begin{cases} \alpha - c - q_2 & \text{if } q_2 \leq \alpha - c \\ 0 & \text{if } q_2 > \alpha - c. \end{cases}$$

Firm 2's best response function is the same as in Section 3.1.3, namely

$$b_2(q_1) = \begin{cases} (\alpha - c - q_2)/2 & \text{if } q_2 \leq \alpha - c \\ 0 & \text{if } q_2 > \alpha - c. \end{cases}$$

These best response functions are shown in Figure 26.1. The game has a unique

Nash equilibrium, $(q_1^*, q_2^*) = (\alpha - c, 0)$, in which firm 2 does not operate. (The price is c, and firm 1's profit is zero.)

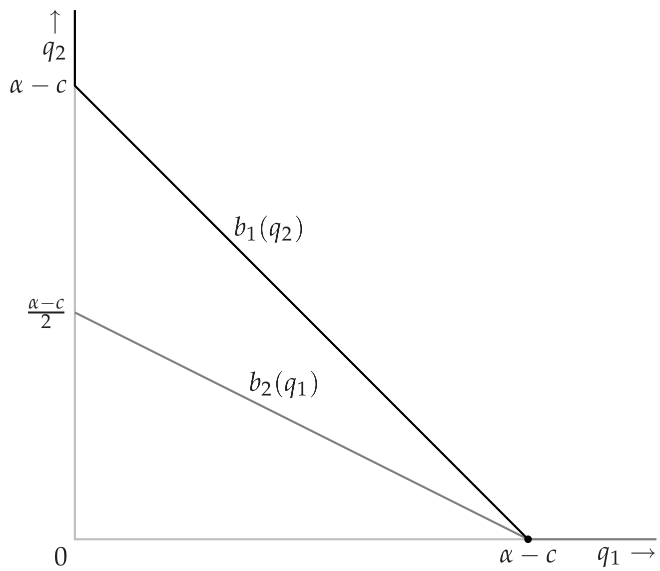


Figure 26.1 The best response functions in a variant of Cournot's duopoly game in which in which the inverse demand function is $P(Q) = \alpha - Q$ (where this is positive) and the cost function of each firm is cq, and firm 1 maximizes its market share, rather than its profit. The unique Nash equilibrium is $(q_1^*, q_2^*) = (\alpha - c, 0)$.

If both firms maximize their market shares, then the downward-sloping parts of their best response functions coincide in the analogue of Figure 26.1. Thus every pair (q_1, q_2) with $q_1 + q_2 = \alpha - c$ is a Nash equilibrium.

59.1 Cournot's game with many firms

Firm 1's payoff function is

$$\begin{cases} q_1(\alpha - c - q_1 - q_2 - \cdots - q_n) & \text{if } q_1 + q_2 + \cdots + q_n \leq \alpha \\ -cq_1 & \text{if } q_1 + q_2 + \cdots + q_n > \alpha. \end{cases}$$

As in the case of two firms, this function is a quadratic in q_1 where it is positive, and is zero when $q_1 = 0$ and when $q_1 = \alpha - c - q_2 - \cdots - q_n$. Thus firm 1's best response function is

$$b_1(q_{-1}) = \begin{cases} (\alpha - c - q_2 - \cdots - q_n)/2 & \text{if } q_2 + \cdots + q_n \leq \alpha - c \\ 0 & \text{if } q_2 + \cdots + q_n > \alpha - c. \end{cases}$$

(Recall that q_{-1} stands for the list of the outputs of all the firms except firm 1.)

The best response functions of every other firm is the same.

The conditions for $(q_1^*, \ldots, q_n^*)$ to be a Nash equilibrium are

$$\begin{aligned} q_1^* &= b_1(q_{-1}^*) \\ q_2^* &= b_2(q_{-2}^*) \\ &\vdots \\ q_n^* &= b_2(q_{-n}^*) \end{aligned}$$

or, in an equilibrium in which all the firms' outputs are positive,

$$\begin{aligned} q_1^* &= \tfrac{1}{2}(\alpha - c - q_2^* - q_3^* - \cdots - q_n^*) \\ q_2^* &= \tfrac{1}{2}(\alpha - c - q_1^* - q_3^* - \cdots - q_n^*) \\ &\vdots \\ q_n^* &= \tfrac{1}{2}(\alpha - c - q_1^* - q_2^* - \cdots - q_{n-1}^*). \end{aligned}$$

We can write these equations as

$$\begin{aligned} 0 &= \alpha - c - 2q_1^* - q_2^* - \cdots - q_{n-1}^* - q_n^* \\ 0 &= \alpha - c - q_1^* - 2q_2^* - \cdots - q_{n-1}^* - q_n^* \\ &\vdots \\ 0 &= \alpha - c - q_1^* - q_2^* - \cdots - q_{n-1}^* - 2q_n^*. \end{aligned}$$

If we subtract the second equation from the first we obtain $0 = -q_1^* + q_2^*$, or $q_1^* = q_2^*$. Similarly subtracting the third equation from the second we conclude that $q_2^* = q_3^*$, and continuing with all pairs of equations we deduce that $q_1^* = q_2^* = \cdots = q_n^*$. Let the common value of the firms' outputs be q^*. Then each equation is $0 = \alpha - c - (n+1)q^*$, so that $q^* = (\alpha - c)/(n+1)$.

In summary, the game has a unique Nash equilibrium, in which the output of every firm i is $(\alpha - c)/(n+1)$.

The price at this equilibrium is $\alpha - n(\alpha - c)/(n+1)$, or $(\alpha + nc)/(n+1)$. As n increases this price decreases, approaching c as n increases without bound: $\alpha/(n+1)$ decreases to 0 and $nc/(n+1)$ decreases to c.

60.1 Nash equilibrium of Cournot's game with small firms

- If $P(Q^*) < \underline{p}$ then every firm producing a positive output makes a negative profit, and can increase its profit (to 0) by deviating and producing zero.
- If $P(Q^* + \underline{q}) > \underline{p}$, take a firm that is either producing no output, or an arbitrarily small output. (Such a firm exists, since demand is finite.) Such a firm earns a profit of either zero or arbitrarily close to zero. If it deviates and chooses the output $\underline{q}$ then total output changes to at most $Q^* + \underline{q}$, so that the price still exceeds $\underline{p}$ (since $P(Q^* + \underline{q}) > \underline{p}$). Hence the deviant makes a positive profit.

61.1 Interaction among resource-users

The game is given as follows.

Players The firms.

Actions Each firm's set of actions is the set of all nonnegative numbers (representing the amount of input it uses).

Preferences The payoff of each firm i is

$$\begin{cases} x_i(1-(x_1+\cdots+x_n)) & \text{if } x_1+\cdots+x_n \leq 1 \\ 0 & \text{if } x_1+\cdots+x_n > 1. \end{cases}$$

This game is the same as that in Exercise 59.1 for $c = 0$ and $\alpha = 1$. Thus it has a unique Nash equilibrium, $(x_1, \ldots, x_n) = (1/(n+1), \ldots, 1/(n+1))$.

In this Nash equilibrium, each firm's output is $(1/(n+1))(1-n/(n+1)) = 1/(n+1)^2$. If $x_i = 1/(2n)$ for $i = 1, \ldots, n$ then each firm's output is $1/(4n)$, which exceeds $1/(n+1)^2$ for $n \geq 2$. (We have $1/(4n) - 1/(n+1)^2 = (n-1)^2/(4n(n+1)^2) > 0$ for $n \geq 2$.)

65.1 Bertrand's duopoly game with constant unit cost

The pair (c, c) of prices remains a Nash equilibrium; the argument is the same as before. Further, as before, there is no other Nash equilibrium. The argument needs only very minor modification. For an arbitrary function D there may exist no monopoly price p^m; in this case, if $p_i > c$, $p_j > c$, $p_i \geq p_j$, and $D(p_j) = 0$ then firm i can increase its profit by reducing its price slightly below $\overline{p}$ (for example).

65.2 Bertrand's duopoly game with discrete prices

Yes, (c, c) is still a Nash equilibrium, by the same argument as before.

In addition, $(c+1, c+1)$ is a Nash equilibrium (where c is given in cents). In this equilibrium both firms' profits are positive. If either firm raises its price or lowers it to c, its profit becomes zero. If either firm lowers its price below c, its profit becomes negative.

No other pair of prices is a Nash equilibrium, by the following argument, similar to the argument in the text for the case in which a price can be any nonnegative number.

- If $p_i < c$ then the firm whose price is lowest (or either firm, if the prices are the same) can increase its profit (to zero) by raising its price to c.
- If $p_i = c$ and $p_j \geq c + 1$ then firm i can increase its profit from zero to a positive amount by increasing its price to $c + 1$.

- If $p_i > p_j \geq c + 1$ then firm i can increase its profit (from zero) by lowering its price to $c + 1$.

- If $p_i = p_j \geq c + 2$ and $p_j < \alpha$ then either firm can increase its profit by lowering its price by one cent. (If firm i does so, its profit changes from $\frac{1}{2}(p_i - c)(\alpha - p_i)$ to $(p_i - 1 - c)(\alpha - p_i + 1) = (p_i - 1 - c)(\alpha - p_i) + p_i - 1 - c$. We have $p_i - 1 - c \geq \frac{1}{2}(p_i - c)$ and $p_i - 1 - c > 0$, since $p_i \geq c + 2$.)

- If $p_i = p_j \geq c + 2$ and $p_j \geq \alpha$ then either firm can increase its profit by lowering its price to p^m.

66.1 Bertrand's oligopoly game

Consider a profile $(p_1, \ldots, p_n)$ of prices in which $p_i \geq c$ for all i and at least two prices are equal to c. Every firm's profit is zero. If any firm raises its price its profit remains zero. If a firm charging more than c lowers its price, but not below c, its profit also remains zero. If a firm lowers its price below c then its profit is negative. Thus any such profile is a Nash equilibrium.

To show that no other profile is a Nash equilibrium, we can argue as follows.

- If some price is less than c then the firm charging the lowest price can increase its profit (to zero) by increasing its price to c.

- If exactly one firm's price is equal to c then that firm can increase its profit by raising its price a little (keeping it less than the next highest price).

- If all firms' prices exceed c then the firm charging the highest price can increase its profit by lowering its price to some price between c and the lowest price being charged.

66.2 Bertrand's duopoly game with different unit costs

a. If all consumers buy from firm 1 when both firms charge the price c_2, then $(p_1, p_2) = (c_2, c_2)$ is a Nash equilibrium by the following argument. Firm 1's profit is positive, while firm 2's profit is zero (since it serves no customers).

- If firm 1 increases its price, its profit falls to zero.

- If firm 1 reduces its price, say to p, then its profit changes from $(c_2 - c_1)(\alpha - c_2)$ to $(p - c_1)(\alpha - p)$. Since c_2 is less than the maximizer of $(p - c_1)(\alpha - p)$, firm 1's profit falls.

- If firm 2 increases its price, its profit remains zero.

- If firm 2 decreases its price, its profit becomes negative (since its price is less than its unit cost).

Under this rule no other pair of prices is a Nash equilibrium, by the following argument.

- If $p_i < c_1$ for $i = 1, 2$ then the firm with the lower price (or either firm, if the prices are the same) can increase its profit (to zero) by raising its price above that of the other firm.
- If $p_1 > p_2 \geq c_2$ then firm 2 can increase its profit by raising its price a little.
- If $p_2 > p_1 \geq c_1$ then firm 1 can increase its profit by raising its price a little.
- If $p_2 \leq p_1$ and $p_2 < c_2$ then firm 2's profit is negative, so that it can increase its profit by raising its price.
- If $p_1 = p_2 > c_2$ then at least one of the firms is not receiving all of the demand, and that firm can increase its profit by lowering its price a little.

b. Now suppose that the rule for splitting up the customers when the prices are equal specifies that firm 2 receives some customers when both prices are c_2. By the argument for part *a*, the only possible Nash equilibrium is $(p_1, p_2) = (c_2, c_2)$. (The argument in part *a* that every other pair of prices is not a Nash equilibrium does not use the fact that customers are split equally when $(p_1, p_2) = (c_2, c_2)$.) But if $(p_1, p_2) = (c_2, c_2)$ and firm 2 receives some customers, firm 1 can increase its profit by reducing its price a little and capturing the entire market.

67.1 Bertrand's duopoly game with fixed costs

At the pair of prices $(\overline{p}, \overline{p})$, both firms' profits are zero. (Firm 1 receives all the demand and obtains the profit $(\overline{p} - c)(\alpha - \overline{p}) - f = 0$, and firm 2 receives no demand.) This pair of prices is a Nash equilibrium by the following argument.

- If either firm raises its price its profit remains zero (it receives no customers).
- If either firm lowers its price then it receives all the demand and earns a negative profit (since f is less than the maximum of $(p - c)(\alpha - p)$).

No other pair of prices (p_1, p_2) is a Nash equilibrium, by the following argument.

- If $p_1 = p_2 < \overline{p}$ then firm 1's profit is negative; firm 1 can increase its profit by raising its price.
- If $p_1 = p_2 > \overline{p}$ then firm 2's profit is zero; firm 2 can obtain a positive profit by lowering its price a little.
- If $p_i < p_j$ and firm i's profit is positive then firm j can increase its profit from zero to almost the current level of i's profit by changing its price to be slightly less than p_i.

- If $p_i < p_j$ and firm i's profit is zero then firm i can earn a positive profit by raising its price a little.

- If $p_i < p_j$ and firm i's profit is negative then firm i can increase its profit to zero by raising its price above p_j.

72.1 Electoral competition with asymmetric voters' preferences

The unique Nash equilibrium remains (m, m); the direct argument is exactly the same as before. (The dividing line between the supporters of two candidates with different positions changes. If $x_i < x_j$, for example, the dividing line is $\frac{1}{3}x_i + \frac{2}{3}x_j$ rather than $\frac{1}{2}(x_i + x_j)$. The resulting change in the best response functions does not affect the Nash equilibrium.)

72.2 Electoral competition with three candidates

If a single candidate enters, then either of the remaining candidates can enter and either win outright or tie for first place. Thus there is no Nash equilibrium in which a single candidate enters.

In any Nash equilibrium in which more than one candidate enters, all the candidates that enter tie for first place, since if they do not then some candidate loses, and hence can do better by staying out of the race.

If two candidates enter, then by the argument in the text for the case in which there are two candidates, each takes the position m. But then the third candidate can enter and win outright. Thus there is no Nash equilibrium in which two candidates enter.

If all three candidates enter and choose the same position, each candidate receives one third of the votes. If the common position is equal to m then any candidate can win outright (obtaining close to one-half of the votes) by moving slightly to one side of m. If the common position is different from m then any candidate can win outright (obtaining more than one-half of the votes) by moving to m. Thus there is no Nash equilibrium in which all three candidates enter and choose the same position.

If all three candidates enter and do not all choose the same position then they all tie for first place, by the second argument. At least one candidate (*i*) does not share her position with any other candidate and (*ii*) is an extremist (her position is not between the positions of the other candidates). This candidate can move slightly closer to the other candidates and win outright. Thus there is no Nash equilibrium in which all three candidates enter and not all of them choose the same position.

We conclude that the game has no Nash equilibrium.

72.3 Electoral competition in two districts

The game has a unique equilibrium, in which the both candidates choose the position m_1 (the median favorite position in the district with the most electoral college votes). The outcome is a tie.

The following argument shows that this pair of positions is a Nash equilibrium. If a candidate deviates to a position less than m_1, she loses in district 1 and wins in district 2, and thus loses overall. If a candidate deviates to a position greater than m_1, she loses in both districts.

To see that there is no other Nash equilibrium, first consider a pair of positions for which candidate 1 loses in district 1, and hence loses overall. By deviating to m_1, she either wins in district 1, and hence wins overall, or, if candidate 2's position is m_1, ties in district 1, and ties overall. Thus her deviation induces an outcome she prefers. The same argument applies to candidate 2, so that in any equilibrium the candidates tie in district 1. Now, if the candidates' positions are either different, or the same and different from m_1, either candidate can win outright rather than tying for first place by moving to m_1. Thus there is a single equilibrium, in which both candidates' positions are m_1.

73.1 Electoral competition between candidates who care only about the winning position

First consider a pair (x_1, x_2) of positions for which either $x_1 < m$ and $x_2 < m$, or $x_1 > m$ and $x_2 > m$.

- If $x_1 \neq x_2$ and the winner's position is different from her favorite position then the winner can move slightly closer to her favorite position and still win.

- If $x_1 \neq x_2$ and the winner's position is equal to her favorite position then the other candidate can move to m, which is closer to her favorite position than the winner's position, and win.

- If $x_1 = x_2 < m$ then the candidate whose favorite position exceeds m can move to m and cause the winning position to be m rather than $x_1 = x_2$.

- If $x_1 = x_2 > m$ then the candidate whose favorite position is less than m can move to m and cause the winning position to be m rather than $x_1 = x_2$.

Now suppose the candidates' positions are on opposite sides of m: either $x_1 < m < x_2$, or $x_2 < m < x_1$.

- If each candidate's position is on the same side of m as her favorite position and one candidate wins outright, then the loser can win outright by moving to m, which she prefers to the position of the other candidate.

- If each candidate's position is on the same side of m as her favorite position and the candidates tie for first place, then by moving slightly closer to m either candidate can win. If her movement is small enough she prefers her new position to the previous compromise $\frac{1}{2}(x_1 + x_2)$ $(= m)$.

- If each candidate's position is on the opposite side of m to her favorite position then the winner, or either player in the case of a tie, can move to her favorite position and either win outright or cause the winning position to be the other candidate's position, in both cases improving the outcome from her point of view.

Now suppose that $x_1 = m$ and $x_2 < m$. If $x_1^* < m$ then candidate 1 is better off choosing a slightly smaller value of x_1 (in which case she still wins). If $x_1^* > m$ then candidate 1 is better off choosing a slightly larger value of x_1 (in which case she still wins). Thus (x_1, x_2) is not a Nash equilibrium. A similar argument applies to pairs (x_1, x_2) for which $x_1 = m$ and $x_2 > m$, and for which $x_1 \neq m$ and $x_2 = m$.

Finally, if $(x_1, x_2) = (m, m)$, then the candidates tie. If either candidate changes her position then she loses, and the winning position does not change. Thus this pair of positions is a Nash equilibrium.

73.2 Citizen-candidates

If $b \leq 2c$ then the game has a Nash equilibrium in which a single citizen, with favorite position m, stands as a candidate. Another citizen with the same favorite position who stands obtains the payoff $\frac{1}{2}b - c$, as opposed to the payoff of 0 if she does not stand. Given $b \leq 2c$, it is optimal for any such citizen not to stand. A citizen with any other favorite position who stands loses, and hence is worse off than if she does not stand.

If two citizens with favorite position m become candidates, each candidate's payoff is $\frac{1}{2}b - c$; if one withdraws then she obtains the payoff of 0, so for equilibrium we require $b \geq 2c$. Now consider a citizen whose favorite position is close to m. If she enters she wins outright, obtaining the payoff $b - c$. Since $b \geq 2c$, this payoff is positive, and hence exceeds her payoff if she does not stand (which is negative, since the winner's position is then different from her favorite position). Thus there is no equilibrium in which two citizens with favorite position m stand as candidates.

Now consider the possibility of an equilibrium in which two citizens with favorite positions different from m stand as candidates. For an equilibrium the candidates must tie, otherwise one loses, and can do better by withdrawing. Thus the positions, say x_1 and x_2, must satisfy $\frac{1}{2}(x_1 + x_2) = m$. If x_1 and x_2 are close enough to m then any other citizen loses if she becomes a candidate. Thus there are equilibria in which two citizens with positions symmetric about m, and sufficiently close to m, become candidates.

74.1 Electoral competition for more general preferences

a. If x^* is a Condorcet winner then for any $y \neq x^*$ a majority of voters prefer x^* to y, so y is not a Condorcet winner. Thus there is no more than one Condorcet winner.

b. Suppose that one of the remaining voters prefers y to z to x, and the other prefers z to x to y. For each position there is another position preferred by a majority of voters, so no position is a Condorcet winner.

c. Now suppose that x^* is a Condorcet winner. Then the strategic game described the exercise has a unique Nash equilibrium in which both candidates choose x^*. This pair of actions is a Nash equilibrium because if either candidate chooses a different position she loses. For any other pair of actions either one candidate loses, in which case that candidate can deviate to the position x^* and at least tie, or the candidates tie at a position different from x^*, in which case either of them can deviate to x^* and win.

   If there is no Condorcet winner then for every position there is another position preferred by a majority of voters. Thus for every pair of distinct positions the loser can deviate and win, and for every pair of identical positions either candidate can deviate and win. Thus there is no Nash equilibrium.

75.1 Competition in product characteristics

Suppose there are two firms. If the products are different, then either firm increases its market share by making its product more similar to that of its rival. Thus in every possible equilibrium the products are the same. But if $x_1 = x_2 \neq m$ then each firm's market share is 50%, while if it changes its product to be closer to m then its market share rises above 50%. Thus the only possible equilibrium is $(x_1, x_2) = (m, m)$. This pair of positions is an equilibrium, since each firm's market share is 50%, and if either firm changes its product its market share falls below 50%.

Now suppose there are three firms. If all firms' products are the same, each obtains one-third of the market. If $x_1 = x_2 = x_3 = m$ then any firm, by changing its product a little, can obtain close to one-half of the market. If $x_1 = x_2 = x_3 \neq m$ then any firm, by changing its product a little, can obtain more than one-half of the market. If the firms' products are not all the same, then at least one of the extreme products is different from the other two products, and the firm that produces it can increase its market share by making it more similar to the other products. Thus when there are three firms there is no Nash equilibrium.

76.1 Direct argument for Nash equilibria of War of Attrition

- If $t_1 = t_2$ then either player can increase her payoff by conceding slightly later (in which case she obtains the object for sure, rather than getting it with probability $\frac{1}{2}$).

- If $0 < t_i < t_j$ then player i can increase her payoff by conceding at 0.
- If $0 = t_i < t_j < v_i$ then player i can increase her payoff (from 0 to almost $v_i - t_j > 0$) by conceding slightly after t_j.

Thus there is no Nash equilibrium in which $t_1 = t_2$, $0 < t_i < t_j$, or $0 = t_i < t_j < v_i$ (for $i = 1$ and $j = 2$, or $i = 2$ and $j = 1$). The remaining possibility is that $0 = t_i < t_j$ and $t_j \geq v_i$ for $i = 1$ and $j = 2$, or $i = 2$ and $j = 1$. In this case player i's payoff is 0, while if she concedes later her payoff is negative; player j's payoff is v_j, her highest possible payoff in the game.

77.1 Variant of War of Attrition

The game is

Players The two parties to the dispute.

Actions Each player's set of actions is the set of possible concession times (nonnegative numbers).

Preferences Player i's preferences are represented by the payoff function

$$u_i(t_1, t_2) = \begin{cases} 0 & \text{if } t_i < t_j \\ \frac{1}{2}(v_i - t_i) & \text{if } t_i = t_j \\ v_i - t_j & \text{if } t_i > t_j. \end{cases}$$

where j is the other player.

Three representative cross-sections of player i's payoff function are shown in Figure 35.1.

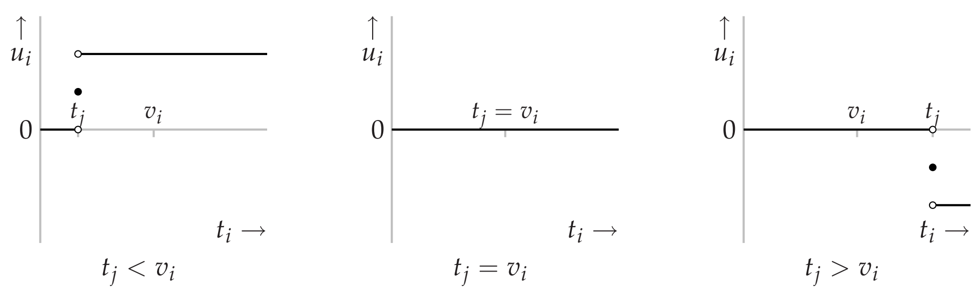


Figure 35.1 Three cross-sections of player i's payoff function in the variant of the *War of Attrition* in Exercise 77.1.

From this figure we deduce that the best response function of player i is

$$B_i(t_j) = \begin{cases} \{t_i : t_i > t_j\} & \text{if } t_j < v_i \\ \{t_i : t_i \geq 0\} & \text{if } t_j = v_i \\ \{t_i : 0 \leq t_i < t_j\} & \text{if } t_j > v_i. \end{cases}$$

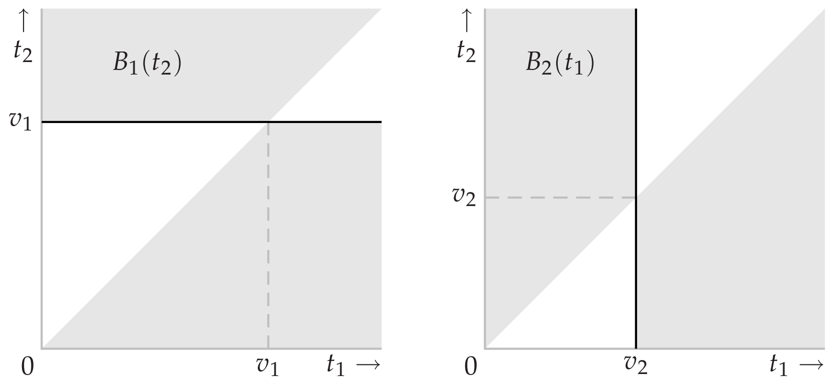


Figure 36.1 The players' best response functions in the variant of the *War of Attrition* in Exercise 77.1 for $v_1 > v_2$. Player 1's best response function is in the left panel; player 2's is in the right panel. (The sloping edges are excluded.)

The best response functions are shown in Figure 36.1 for a case in which $v_1 > v_2$.

Superimposing the two best response functions, we see that if $v_1 > v_2$ then the set of Nash equilibrium action pairs is the union of the shaded regions in Figure 36.2, namely the set of all pairs (t_1, t_2) such that either

$$t_1 \leq v_2 \text{ and } t_2 \geq v_1,$$

or

$$t_1 \geq v_2,\ t_1 > t_2, \text{ and } t_2 \leq v_1.$$

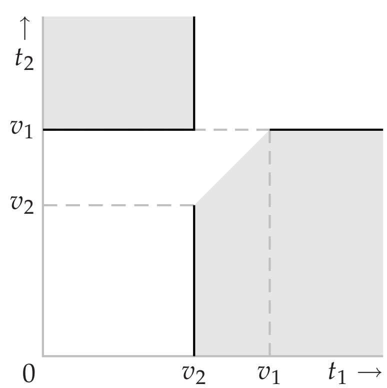


Figure 36.2 The set of Nash equilibria of the variant of the *War of Attrition* in Exercise 77.1 when $v_1 > v_2$.

78.1 Timing product release

A strategic game that models this situation is:

Players The two firms

Actions The set of actions of each player is the set of possible release times, which we can take to be the set of numbers t for which $0 \leq t \leq T$.

Preferences Each firm's preferences are represented by its market share; the market share of firm i when it releases its product at time t_i and its rival releases its product at time t_j is

$$\begin{cases} h(t_i) & \text{if } t_i < t_j \\ \frac{1}{2} & \text{if } t_i = t_j \\ 1 - h(t_j) & \text{if } t_i > t_j. \end{cases}$$

Three representative cross-sections of firm i's payoff function are shown in Figure 37.1.

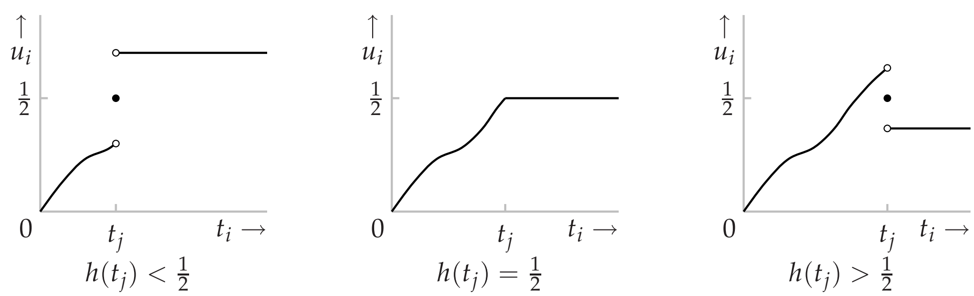


Figure 37.1 Three cross-sections of firm i's payoff function in the game in Exercise 78.1.

From the payoff function we see that if t_j is such that $h(t_j) < \frac{1}{2}$ then the set of firm i's best responses is the set of release times after t_j. If t_j is such that $h(t_j) = \frac{1}{2}$ then the set of firm i's best responses is the set of release times greater than or equal to t_j. If t_j is such that $h(t_j) > \frac{1}{2}$ then firm i wants to release its product just before t_j. Since there is no latest time before t_j, firm i has no *best* response in this case. (It has good responses, but none is optimal.) Denoting the time t for which $h(t) = \frac{1}{2}$ by t^*, the firms' best response functions are shown in Figure 38.1.

Combining the best response functions we see that the game has a unique Nash equilibrium, in which both firms release their products at the time t^* (where $h(t^*) = \frac{1}{2}$).

78.2 A fight

The game is defined as follows.

Players The two people.

Actions The set of actions of each player i is the set of amounts of the resource that player i can devote to fighting (the set of numbers y_i with $0 \leq y_i \leq 1$).

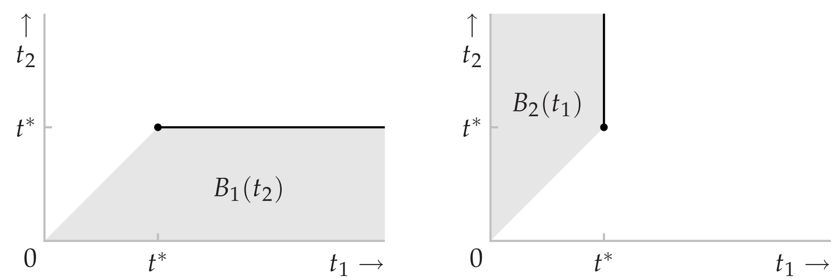


Figure 38.1 The firms' best response functions in the game in Exercise 78.1. Firm 1's best response function is in the left panel; firm 2's is in the right panel.

Preferences The preferences of player i are represented by the payoff function

$$u_i(y_1, y_2) = \begin{cases} f(y_1, y_2) & \text{if } y_i > y_j \\ \frac{1}{2} f(y_1, y_2) & \text{if } y_1 = y_2 \\ 0 & \text{if } y_i < y_j. \end{cases}$$

If $y_i < y_j$ then player j can increase her payoff by reducing y_j a little, keeping it greater than y_i (output increases, and she still wins). So no action profile in which $y_1 \neq y_2$ is a Nash equilibrium.

If $y_1 = y_2 < 1$ then either player i can increase her payoff by increasing y_i to slightly above y_j (output falls a little, but i's share of it increases from $\frac{1}{2}$ to 1). So no action profile in which $y_1 = y_2 < 1$ is a Nash equilibrium.

The only action profile that remains is $(y_1, y_2) = (1, 1)$. This profile is a Nash equilibrium: each player's payoff is 0, and remains 0 if she reduces the amount of the resource she devotes to fighting (given the other player's action).

82.1 Nash equilibrium of second-price sealed-bid auction

The action profile $(v_n, 0, \ldots, 0, v_1)$ is a Nash equilibrium of a second-price sealed-bid auction, by the following argument.

- If player 1 increases her bid she wins and obtains the payoff 0, equal to her current payoff. If she reduces her bid her payoff also remains 0.
- If player n increases her bid or reduces it to a level greater than v_n then the outcome does not change. If she reduces her bid to v_n or less then she loses, and her payoff remains 0.
- If any other player increases her bid, either the outcome remains the same or the player wins and pays the price v_1, thus obtaining a negative payoff.

83.1 Second-price sealed-bid auction with two bidders

If player 2's bid b_2 is less than v_1 then any bid of b_2 or more is a best response of player 1 (she wins and pays the price b_2). If player 2's bid is equal to v_1 then every bid of player 1 yields her the payoff zero (either she wins and pays v_1, or she loses), so every bid is a best response. If player 2's bid b_2 exceeds v_1 then any bid of less than b_2 is a best response of player 1. (If she bids b_2 or more she wins, but pays the price $b_2 > v_1$, and hence obtains a negative payoff.) In summary, player 1's best response function is

$$B_1(b_2) = \begin{cases} \{b_1 : b_1 \geq b_2\} & \text{if } b_2 < v_1 \\ \{b_1 : b_1 \geq 0\} & \text{if } b_2 = v_1 \\ \{b_1 : 0 \leq b_1 < b_2\} & \text{if } b_2 > v_1. \end{cases}$$

By similar arguments, player 2's best response function is

$$B_2(b_1) = \begin{cases} \{b_2 : b_2 > b_1\} & \text{if } b_1 < v_2 \\ \{b_2 : b_2 \geq 0\} & \text{if } b_1 = v_2. \\ \{b_2 : 0 \leq b_2 \leq b_1\} & \text{if } b_1 > v_2. \end{cases}$$

These best response functions are shown in Figure 39.1.

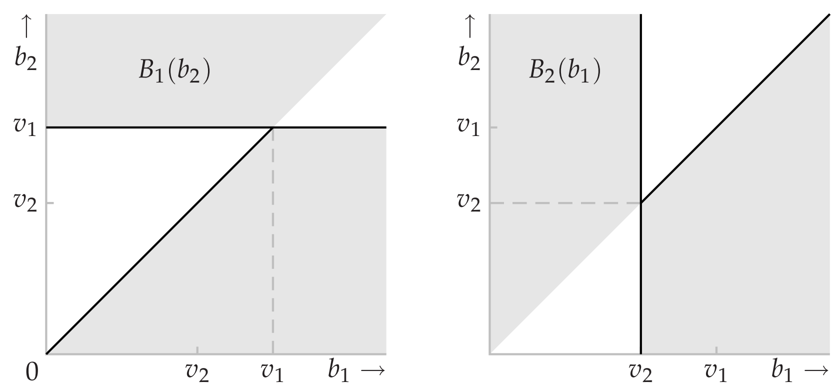


Figure 39.1 The players' best response functions in a two-player second-price sealed-bid auction (Exercise 83.1). Player 1's best response function is in the left panel; player 2's is in the right panel. (Only the edges marked by a black line are included.)

Superimposing the best response functions, we see that the set of Nash equilibria is the shaded set in Figure 40.1, namely the set of pairs (b_1, b_2) such that either

$$b_1 \leq v_2 \text{ and } b_2 \geq v_1$$

or

$$b_1 \geq v_2,\ b_1 \geq b_2, \text{ and } b_2 \leq v_1.$$

84.1 Nash equilibrium of first-price sealed-bid auction

The profile $(b_1, \ldots, b_n) = (v_2, v_2, v_3, \ldots, v_n)$ is a Nash equilibrium by the following argument.

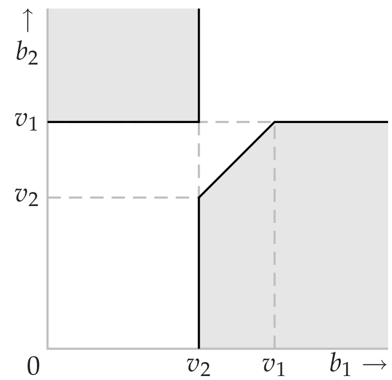


Figure 40.1 The set of Nash equilibria of a two-player second-price sealed-bid auction (Exercise 83.1).

- If player 1 raises her bid she still wins, but pays a higher price and hence obtains a lower payoff. If player 1 lowers her bid then she loses, and obtains the payoff of 0.
- If any other player changes her bid to any price at most equal to v_2 the outcome does not change. If she raises her bid above v_2 she wins, but obtains a negative payoff.

85.1 First-price sealed-bid auction

A profile of bids in which the two highest bids are not the same is not a Nash equilibrium because the player naming the highest bid can reduce her bid slightly, continue to win, and pay a lower price.

By the argument in the text, in any equilibrium player 1 wins the object. Thus she submits one of the highest bids.

If the highest bid is less than v_2, then player 2 can increase her bid to a value between the highest bid and v_2, win, and obtain a positive payoff. Thus in an equilibrium the highest bid is at least v_2.

If the highest bid exceeds v_1, player 1's payoff is negative, and she can increase this payoff by reducing her bid. Thus in an equilibrium the highest bid is at most v_1.

Finally, any profile $(b_1, \ldots, b_n)$ of bids that satisfies the conditions in the exercise is a Nash equilibrium by the following argument.

- If player 1 increases her bid she continues to win, and reduces her payoff. If player 1 decreases her bid she loses and obtains the payoff 0, which is at most her payoff at $(b_1, \ldots, b_n)$.
- If any other player increases her bid she either does not affect the outcome, or wins and obtains a negative payoff. If any other player decreases her bid she does not affect the outcome.

86.1 Third-price auction

a. The argument that a bid of v_i weakly dominates any lower bid is the same as for a second-price auction.

Now compare bids of $b_i > v_i$ and v_i. Suppose that one of the other players' bids is between v_i and b_i and all the remaining bids are less than v_i. If player i bids v_i she loses, and obtains the payoff of 0. If she bids b_i she wins, and pays the third highest bid, which is less than v_i. Thus she is better off bidding b_i than she is bidding v_i.

b. Each player's bidding her valuation is not a Nash equilibrium because player 2 can deviate and bid more than v_1 and obtain the object at the price v_3 instead of not obtaining the object.

c. Any action profile in which every player bids b, where $v_2 \leq b \leq v_1$ is a Nash equilibrium. (Player 1's changing her bid has no effect on her payoff. If any other player raises her bid then she wins and pays b, obtaining a nonpositive payoff; if any other player lowers her bid the outcome does not change.)

Any action profile in which player 1's bid b_1 satisfies $v_2 \leq b_1 \leq v_1$, every other player's bid is at most b_1, and at least two other players' bids are at least v_2 is also a Nash equilibrium.

88.3 Lobbying as an auction

First-price auction In the action pair, each interest group's payoff is -100. Consider group A. If it raises the price it will pay for y, then the government still chooses y, and A is worse off. If it lowers the price it will pay for y, then the government chooses z and A's payoff remains -100. Now suppose it changes its bid from y to x and bids p. If $p < 103$, then the government chooses z and A's payoff remains -100. If $p \geq 103$, then the government chooses x and A's payoff is at most -103. Group A cannot increase its payoff by changing its bid from y to z, for similar reasons. A similar argument applies to group B's bid.

Menu auction In the action pair, each group's payoff is -3. Consider group A. If it changes its bids then either the outcome remains x and it pays at least 3, so that its payoff is at most -3, or the outcome becomes y and it pays at least 6, in which case its payoff is at most -3, or the outcome becomes z and it pays at least 0, in which case its payoff is at most -100. (Note that if it reduces its bids for both x and y then z is chosen.) Thus no change in its bids increases its payoff. Similar considerations apply to group B's bid.

87.1 Multi-unit auctions

Discriminatory auction To show that the action of bidding v_i and w_i is not dominant for player i, we need only find actions for the other players and alternative bids for player i such that player i's payoff is higher under the alternative bids than it is under the v_i and w_i, given the other players' actions. Suppose that each of the other players submits two bids of 0. Then if player i submits one bid between 0 and v_i and one bid between 0 and w_i she still wins two units, and pays less than when she bids v_i and w_i.

Uniform-price auction Suppose that some bidder other than i submits one bid between w_i and v_i and one bid of 0, and all the remaining bidders submit two bids of 0. Then bidder i wins one unit, and pays the price w_i. If she replaces her bid of w_i with a bid between 0 and w_i then she pays a lower price, and hence is better off.

Vickrey auction Suppose that player i bids v_i and w_i. Consider separately the cases in which the bids of the players other than i are such that player i wins 0, 1, and 2 units.

Player i wins 0 units: In this case the second highest of the other players' bids is at least v_i, so that if player i changes her bids so that she wins one or more units, for any unit she wins she pays at least v_i. Thus no change in her bids increases her payoff from its current value of 0 (and some changes lower her payoff).

Player i wins 1 unit: If player i raises her bid of v_i then she still wins one unit and the price remains the same. If she lowers this bid then either she still wins and pays the same price, or she does not win any units. If she raises her bid of w_i then either the outcome does not change, or she wins a second unit. In the latter case the price she pays is the previously-winning bid she beat, which is at least w_i, so that her payoff either remains zero or becomes negative.

Player i wins 2 units: Player i's raising either of her bids has no effect on the outcome; her lowering a bid either has no effect on the outcome or leads her to lose rather than to win, leading her to obtain the payoff of zero.

88.1 Waiting in line

The situation is modeled by a variant of a discriminatory multi-unit auction in which 100 units are available, and each person attaches a positive value only to one unit and submits a bid for only one unit.

We can argue along the lines of Exercise 85.1.

- The first 100 people to arrive must do so at the same time. If not, at least one of them could arrive a little later and still be in the first 100.

- The first 100 people to arrive must be persons 1 through 100. Suppose, to the contrary, that one of these people is person i with $i \geq 101$, and person j with $j \leq 100$ is not in the group that arrives first. Then the common waiting time of the first 100 must be at most v_{101}, otherwise person i obtains a negative payoff. But then person j can deviate and arrive slightly earlier than the group of 100, and obtain a positive payoff.

- The common waiting time of the first 100 people must be at least v_{101}. If not, then person 101 could arrive slightly before the first 100 and obtain a positive payoff.

- The common waiting time of the first 100 people must be at most v_{100}. If not, then person 100 obtains a negative payoff, while by arriving later her payoff is zero.

- At least one person i with $i \geq 101$ arrives at the same time as the first 100 people. If not, then any person i with $i \leq 100$ can arrive slightly later and still be one of the first 100 to arrive.

This argument shows that in a Nash equilibrium persons 1 through 100 choose the same waiting time t^* with $v_{101} \leq t^* \leq v_{100}$, all the remaining people choose waiting times of at most t^*, and at least one of the remaining people chooses a waiting time equal to t^*. Any such action profile is a Nash equilibrium: any person i with $i \leq 100$ obtains a smaller payoff if she arrives earlier and a payoff of zero if she arrives later. Any person i with $i \geq 101$ obtains a negative payoff if she arrives before the first 100 people and a payoff of zero if she arrives at or after the first 100 people.

Thus the set of Nash equilibria is the set of action profiles $(t_1, \ldots, t_{200})$ in which $t_1 = \cdots = t_{100}$, this common waiting time, say t^*, satisfies $v_{101} \leq t^* \leq v_{100}$, $t_i \geq t^*$ for all $i \geq 101$, and $t_j = t^*$ for some $j \geq 101$.

When goods are rationed by line-ups in the world, people in general do not all arrive at the same time. The feature missing from the model that seems to explain the dispersion in arrival times is uncertainty on the part of each player about the other players' valuations.

88.2 Internet pricing

The situation may be modeled as a multi-unit auction in which k units are available, and each player attaches a positive value to only one unit and submits a bid for only one unit. The k highest bids win, and each winner pays the $(k+1)$st highest bid.

By a variant of the argument for a second-price auction, in which "highest of the other players' bids" is replaced by "highest rejected bid", each player's action of bidding her value is weakly dominates all her other actions.

94.3 Alternative standards of care under negligence with contributory negligence

First consider the case in which $X_1 = \hat{a}_1$ and $X_2 \le \hat{a}_2$. The pair $(\hat{a}_1, \hat{a}_2)$ is a Nash equilibrium by the following argument.

If $a_2 = \hat{a}_2$ then the victim's level of care is sufficient (at least X_2), so that the injurer's payoff is given by (91.1) in the text. Thus the argument that the injurer's action $\hat{a}_1$ is a best response to $\hat{a}_2$ is exactly the same as the argument for the case $X_2 = \hat{a}_2$ in the text.

Since X_1 is the same as before, the victim's payoff is the same also, so that by the argument in the text the victim's best response to $\hat{a}_1$ is $\hat{a}_2$. Thus $(\hat{a}_1, \hat{a}_2)$ is a Nash equilibrium.

To show that $(\hat{a}_1, \hat{a}_2)$ is the only Nash equilibrium of the game, we study the players' best response functions. First consider the injurer's best response function. As in the text, we split the analysis into three cases.

$a_2 < X_2$: In this case the injurer does not have to pay any compensation, regardless of her level of care; her payoff is $-a_1$, so that her best response is $a_1 = 0$.

$a_2 = X_2$: In this case the injurer's best response is $\hat{a}_1$, as argued when showing that $(\hat{a}_1, \hat{a}_2)$ is a Nash equilibrium.

$a_2 > X_2$: In this case the injurer's best response is at most $\hat{a}_1$, since her payoff is equal to $-a_1$ for larger values of a_1.

Thus the injurer's best response takes a form like that shown in the left panel of Figure 44.1. (In fact, $b_1(a_2) = \hat{a}_1$ for $X_2 \le a_2 \le \hat{a}_2$, but the analysis depends only on the fact that $b_1(a_2) \le \hat{a}_1$ for $a_2 > X_2$.)

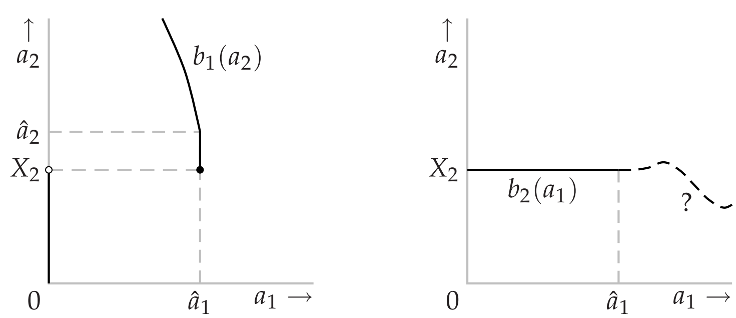


Figure 44.1 The players' best response functions under the rule of negligence with contributory negligence when $X_1 = \hat{a}_1$ and $X_2 = \hat{a}_2$. Left panel: the injurer's best response function b_1. Right panel: the victim's best response function b_2. (The position of the victim's best response function for $a_1 > \hat{a}_1$ is not significant, and is not determined in the solution.)

Now consider the victim's best response function. The victim's payoff function is

$$u_2(a_1, a_2) = \begin{cases} -a_2 & \text{if } a_1 < \hat{a}_1 \text{ and } a_2 \ge X_2 \\ -a_2 - L(a_1, a_2) & \text{if } a_1 \ge \hat{a}_1 \text{ or } a_2 < X_2. \end{cases}$$

As before, for $a_1 < \hat{a}_1$ we have $-a_2 - L(a_1, a_2) < -\hat{a}_2$ for all a_2, so that the victim's best response is X_2. As in the text, the nature of the victim's best responses to levels of care a_1 for which $a_1 > \hat{a}_1$ are not significant.

Combining the two best response functions we see that $(\hat{a}_1, \hat{a}_2)$ is the unique Nash equilibrium of the game.

Now consider the case in which $X_1 = M$ and $a_2 = \hat{a}_2$, where $M \geq \hat{a}_1$. The injurer's payoff is

$$u_1(a_1, a_2) = \begin{cases} -a_1 - L(a_1, a_2) & \text{if } a_1 < M \text{ and } a_2 \geq \hat{a}_2 \\ -a_1 & \text{if } a_1 \geq M \text{ or } a_2 < \hat{a}_2. \end{cases}$$

Now, the maximizer of $-a_1 - L(a_1, \hat{a}_2)$ is $\hat{a}_1$ (see the argument following (91.1) in the text), so that if M is large enough then the injurer's best response to $\hat{a}_2$ is $\hat{a}_1$. As before, if $a_2 < \hat{a}_2$ then the injurer's best response is 0, and if $a_2 > \hat{a}_2$ then the injurer's payoff decreases for $a_1 > M$, so that her best response is less than M. The injurer's best response function is shown in the left panel of Figure 45.1.

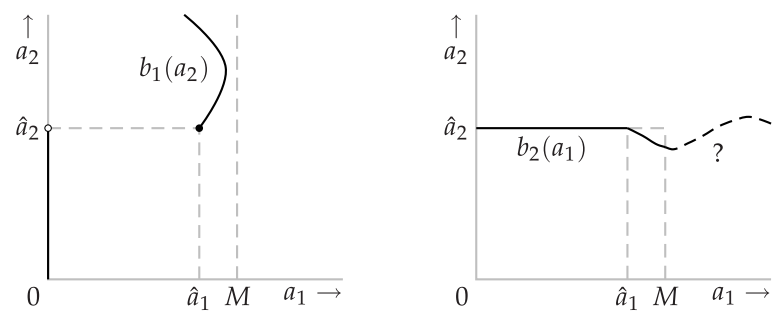


Figure 45.1 The players' best response functions under the rule of negligence with contributory negligence when $(X_1, X_2) = (M, \hat{a}_2)$, with $M \geq \hat{a}_1$. Left panel: the injurer's best response function b_1. Right panel: the victim's best response function b_2. (The position of the victim's best response function for $a_1 > M$ is not significant, and is not determined in the text.)

The victim's payoff is

$$u_2(a_1, a_2) = \begin{cases} -a_2 & \text{if } a_1 < M \text{ and } a_2 \geq \hat{a}_2 \\ -a_2 - L(a_1, a_2) & \text{if } a_1 \geq M \text{ or } a_2 < \hat{a}_2. \end{cases}$$

If $a_1 \leq \hat{a}_1$ then the victim's best response is $\hat{a}_2$ by the same argument as the one in the text. If a_1 is such that $\hat{a}_1 < a_1 < M$ then the victim's best response is at most $\hat{a}_2$ (since her payoff is decreasing for larger values of a_2). This information about the victim's best response function is recorded in the right panel of Figure 45.1; it is sufficient to deduce that $(\hat{a}_1, \hat{a}_2)$ is the unique Nash equilibrium of the game.

94.4 Equilibrium under strict liability

In this case the injurer's payoff is $-a_1 - L(a_1, a_2)$ and the victim's is $-a_2$ for all (a_1, a_2). Thus the victim's optimal action is 0, regardless of the injurer's action.

(The victim takes no care, given that, regardless of her level of care, the injurer is obliged to compensate her for any loss.) Thus in a Nash equilibrium the injurer chooses the level of care that maximizes $-a_1 - L(a_1, 0)$ and the victim chooses $a_2 = 0$.

If the function $-a_1 - L(a_1, 0)$ has a unique maximizer then the game has a unique Nash equilibrium; if there are multiple maximizers then the game has many Nash equilibria, though the players' payoffs are the same in all the equilibria. The relation between $\hat{a}_1$ and the equilibrium value of a_1 depends on the character of $L(a_1, a_2)$. If, for example, L decreases more sharply as a_1 increases when $a_2 = 0$ than when a_2 is positive, the equilibrium value of a_1 exceeds $\hat{a}_1$.

4 Mixed strategy equilibrium

99.1 Variant of Matching Pennies

The analysis is the same as for *Matching Pennies*. There is a unique steady state, in which each player chooses each action with probability $\frac{1}{2}$.

104.1 Extensions of BoS with vNM preferences

In the first case, when player 1 is indifferent between going to her less preferred concert in the company of player 2 and the lottery in which with probability $\frac{1}{2}$ she and player 2 go to different concerts and with probability $\frac{1}{2}$ they both go to her more preferred concert, the Bernoulli payoffs that represent her preferences satisfy the condition

$$u_1(S,S) = \tfrac{1}{2}u_1(S,B) + \tfrac{1}{2}u_1(B,B).$$

If we choose $u_1(S,B) = 0$ and $u_1(B,B) = 2$, then $u_1(S,S) = 1$. Similarly, for player 2 we can set $u_2(B,S) = 0$, $u_2(S,S) = 2$, and $u_2(B,B) = 1$. Thus the Bernoulli payoffs in the left panel of Figure 47.1 are consistent with the players' preferences.

In the second case, when player 1 is indifferent between going to her less preferred concert in the company of player 2 and the lottery in which with probability $\frac{3}{4}$ she and player 2 go to different concerts and with probability $\frac{1}{4}$ they both go to her more preferred concert, the Bernoulli payoffs that represent her preferences satisfy the condition

$$u_1(S,S) = \tfrac{3}{4}u_1(S,B) + \tfrac{1}{4}u_1(B,B).$$

If we choose $u_1(S,B) = 0$ and $u_1(B,B) = 2$ (as before), then $u_1(S,S) = \frac{1}{2}$. Similarly, for player 2 we can set $u_2(B,S) = 0$, $u_2(S,S) = 2$, and $u_2(B,B) = \frac{1}{2}$. Thus the Bernoulli payoffs in the right panel of Figure 47.1 are consistent with the players' preferences.

	Bach	*Stravinsky*
Bach	$2, 1$	$0, 0$
Stravinsky	$0, 0$	$1, 2$

	Bach	*Stravinsky*
Bach	$2, \frac{1}{2}$	$0, 0$
Stravinsky	$0, 0$	$\frac{1}{2}, 2$

Figure 47.1 The Bernoulli payoffs for two extensions of *BoS*.

107.1 Expected payoffs

For *BoS*, player 1's expected payoff is shown in Figure 48.1.

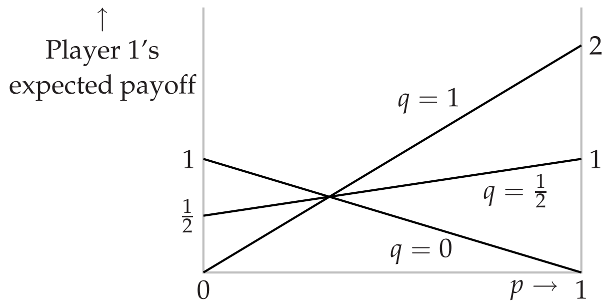


Figure 48.1 Player 1's expected payoff as a function of the probability p that she assigns to B in *BoS*, when the probability q that player 2 assigns to B is 0, $\frac{1}{2}$, and 1.

For the game in Figure 19.1 in the book, player 1's expected payoff is shown in Figure 48.2.

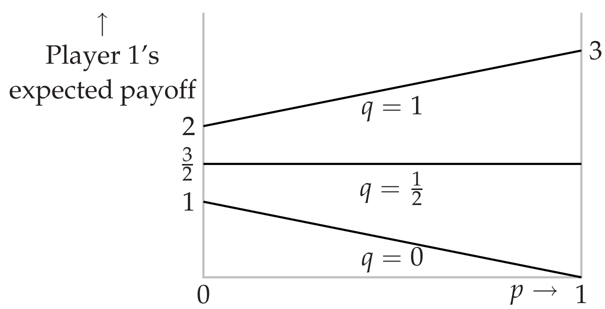


Figure 48.2 Player 1's expected payoff as a function of the probability p that she assigns to *Refrain* in the game in Figure 19.1 in the book, when the probability q that player 2 assigns to *Refrain* is 0, $\frac{1}{2}$, and 1.

108.1 Examples of best responses

For *BoS*: for $q = 0$ player 1's unique best response is $p = 0$ and for $q = \frac{1}{2}$ and $q = 1$ her unique best response is $p = 1$. For the game in Figure 19.1: for $q = 0$ player 1's unique best response is $p = 0$, for $q = \frac{1}{2}$ her set of best responses is the set of all her mixed strategies (all values of p), and for $q = 1$ her unique best response is $p = 1$.

111.1 Mixed strategy equilibrium of Hawk–Dove

Denote by u_i a payoff function whose expected value represents player i's preferences. The conditions in the problem imply that for player 1 we have

$$u_1(Passive, Passive) = \tfrac{1}{2}u_1(Aggressive, Aggressive) + \tfrac{1}{2}u_1(Aggressive, Passive)$$

and

$$u_1(Passive, Aggressive) = \tfrac{2}{3}u_1(Aggressive, Aggressive) + \tfrac{1}{3}u_1(Passive, Passive).$$

Given $u_1(Aggressive, Aggressive) = 0$ and $u_1(Passive, Aggressive = 1$, we have

$$u_1(Passive, Passive) = \tfrac{1}{2}u_1(Aggressive, Passive)$$

and

$$1 = \tfrac{1}{3}u_1(Passive, Passive),$$

so that

$$u_1(Passive, Passive) = 3 \text{ and } u_1(Aggressive, Passive) = 6.$$

Similarly,

$$u_2(Passive, Passive) = 3 \text{ and } u_2(Passive, Aggressive) = 6.$$

Thus the game is given in the left panel of Figure 49.1. The players' best response functions are shown in the right panel. The game has three mixed strategy Nash equilibria: $((0,1),(1,0))$, $((\frac{3}{4},\frac{1}{4}),(\frac{3}{4},\frac{1}{4}))$, and $((1,0),(0,1))$.

	Aggressive	*Passive*
Aggressive	0, 0	6, 1
Passive	1, 6	3, 3

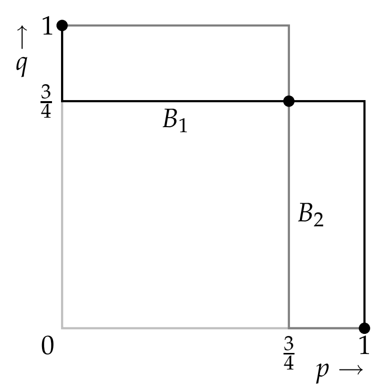


Figure 49.1 An extension of *Hawk–Dove* (left panel) and the players' best response functions when randomization is allowed in this game (right panel). The probability that player 1 assigns to *Aggressive* is p and the probability that player 2 assigns to *Aggressive* is q. The disks indicate the Nash equilibria (two pure, one mixed).

111.2 Games with mixed strategy equilibria

The best response functions for the left game are shown in the left panel of Figure 50.1. We see that the game has a unique mixed strategy Nash equilibrium $((\frac{1}{4}, \frac{3}{4}), (\frac{2}{3}, \frac{1}{3}))$. The best response functions for the right game are shown in the right panel of Figure 50.1. We see that the mixed strategy Nash equilibria are $((0, 1), (1, 0))$ and any $((p, 1 - p), (0, 1))$ with $\frac{1}{2} \leq p \leq 1$.

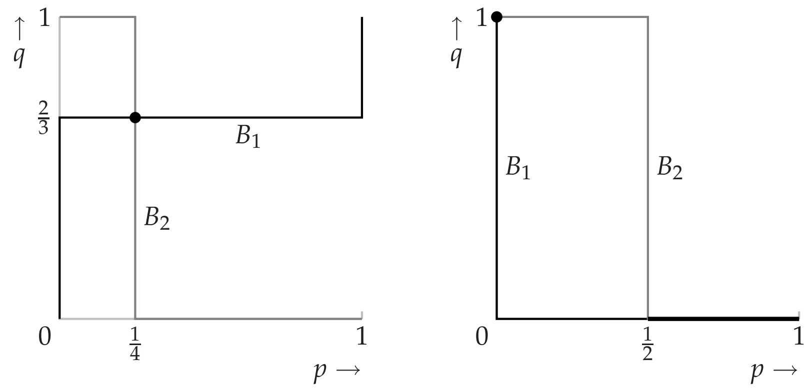


Figure 50.1 The players' best response functions in the left game (left panel) and right game (right panel) in Exercise 111.2. The probability that player 1 assigns to T is p and the probability that player 2 assigns to L is q. The disks and the heavy line indicate Nash equilibria.

112.1 A coordination game

The best response functions are shown in Figure 51.1. From the figure we see that the game has three mixed strategy Nash equilibria, $((1, 0), (1, 0))$ (the pure strategy equilibrium (*No effort*, *No effort*)), $((0, 1), (0, 1))$ (the pure strategy equilibrium (*Effort*, *Effort*)), and $((1 - c, c), (1 - c, c))$.

An increase in c has no effect on the pure strategy equilibria, and *increases* the probability that each player chooses to exert effort in the mixed strategy equilibrium (because this probability is precisely c).

The pure Nash equilibria are not affected by the cost of effort because a change in c has no effect on the players' rankings of the four outcomes. An increase in c reduces a player's payoff to the action *Effort*, given the other player's mixed strategy; the probability the other player assigns to *Effort* must increase in order to keep the player indifferent between *No effort* and *Effort*, as required in an equilibrium.

112.2 Swimming with sharks

As argued in the question, if you swim today, your expected payoff is $-\pi c + 2(1 - \pi)$, regardless of your friend's action. If you do not swim today and your friend

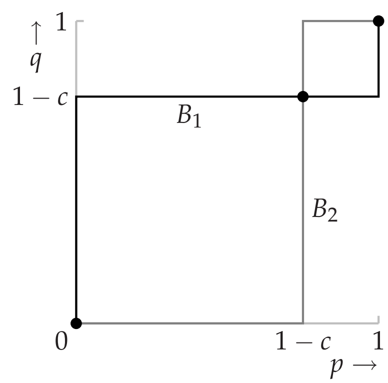


Figure 51.1 The players' best response functions in the coordination game in Exercise 112.1. The probability that player 1 assigns to *No effort* is p and the probability that player 2 assigns to *No effort* is q. The disks indicate the Nash equilibria (two pure, one mixed).

does, then with probability π your friend is attacked and you do not swim tomorrow, and with probability $1 - \pi$ your friend is not attacked and you do swim tomorrow. Thus your expected payoff in this case is $\pi \cdot 0 + (1 - \pi) \cdot 1 = 1 - \pi$. If neither of you swims today then your expected payoff if you swim tomorrow is $\pi(-c) + (1 - \pi) \cdot 1 = -\pi c + 1 - \pi$; if this is negative you prefer to stay on the beach tomorrow, getting a payoff of 0, and if it is positive you prefer to swim tomorrow, getting a payoff of $-\pi c + 1 - \pi$. The game is given in Figure 51.2.

	Swim today	*Wait*
Swim today	$-\pi c + 2(1 - \pi), -\pi c + 2(1 - \pi)$	$-\pi c + 2(1 - \pi), 1 - \pi$
Wait	$1 - \pi, -\pi c + 2(1 - \pi)$	$\max\{0, -\pi c + 1 - \pi\}, \max\{0, -\pi c + 1 - \pi\}$

Figure 51.2 Swimming with sharks.

To find the mixed strategy Nash equilibria, first note that if $-\pi c + 1 - \pi > 0$, or $c < (1 - \pi)/\pi$, then *Swim today* is the best response to both *Swim today* and *Wait*. Thus in this case there is a unique mixed strategy Nash equilibrium, in which both players choose *Swim today*.

At the other extreme, if $-\pi c + 2(1 - \pi) < 0$, or $c > 2(1 - \pi)/\pi$, then *Wait* is the best response to both *Swim today* and *Wait*. Thus in this case there is a unique mixed strategy Nash equilibrium, in which neither of you swims today, and consequently neither of you swims tomorrow.

In the intermediate case in which $0 < -\pi c + 2(1 - \pi) < 1 - \pi$, or $(1 - \pi)/\pi < c < 2(1 - \pi)/\pi$, the best response to *Swim today* is *Wait* and the best response to *Wait* is *Swim today*. Denoting by q the probability that player 2 chooses *Swim today*, player 1's expected payoff to *Swim today* is $-\pi c + 2(1 - \pi)$ and her expected payoff to *Wait* is $q(1 - \pi)$. (Because $-\pi c + 2(1 - \pi) < 1 - \pi$, we have $-\pi c + 1 - \pi < 0$, so that each player's payoff if both players *Wait* is 0.) Thus player 1's expected payoffs to her two actions are equal if and only if

$$-\pi c + 2(1-\pi) = q(1-\pi),$$

or $q = [-\pi c + 2(1-\pi)]/(1-\pi)$. The same calculation implies that player 2's expected payoffs to her two actions are equal if and only if the probability that player 1 assigns to *Swim today* is $[-\pi c + 2(1-\pi)]/(1-\pi) = 2 - \pi c/(1-\pi)$.

We conclude that if $(1-\pi)/\pi < c < 2(1-\pi)/\pi$ then the game has a unique mixed strategy Nash equilibrium, in which each person swims today with probability $2 - \pi c/(1-\pi)$.

If $c = (1-\pi)/\pi$ the payoffs simplify to those given in the left panel of Figure 52.1. The set of mixed strategy Nash equilibria in this case is the set of all mixed strategy pairs $((p, 1-p), (q, 1-q))$ for which either $p = 1$ or $q = 1$. If $c = 2(1-\pi)/\pi$ the payoffs simplify to those given in the right panel of Figure 52.1. The set of mixed strategy Nash equilibria in this case is the set of all mixed strategy pairs $((p, 1-p), (q, 1-q))$ for which either $p = 0$ or $q = 0$.

	Swim	*Wait*
Swim	$1-\pi, 1-\pi$	$1-\pi, 1-\pi$
Wait	$1-\pi, 1-\pi$	$0, 0$

	Swim	*Wait*
Swim	$0, 0$	$0, 1-\pi$
Wait	$1-\pi, 0$	$0, 0$

Figure 52.1 The game if Figure 51.2 for $c = (1-\pi)/\pi$ (left panel) and $c = 2(1-\pi)/\pi$ (right panel).

If you were alone your expected payoff to swimming on the first day would be $-\pi c + 2(1-\pi)$; your expected payoff to staying out of the water on the first day and acting optimally on the second day would be $\max\{0, -\pi c + 1 - \pi\}$. Thus if $-\pi c + 2(1-\pi) > 0$, or $c < 2(1-\pi)/\pi$, you swim on the first day (and stay out of the water on the second day if you get attacked on the first day), and if $c > 2(1-\pi)/\pi$ you stay out of the water on both days. In the presence of your friend, you also swim on the first day only if $c < (1-\pi)/\pi$. If $(1-\pi)/\pi < c < 2(1-\pi)/\pi$ you do not swim for sure on the first day as you would if you were alone, but rather swim with probability less than one. That is, the presence of your friend decreases the probability of your swimming on the first day when c lies in this range. (For other values of c your decision is the same whether or not you are alone.)

115.1 Choosing numbers

a. To show that the pair of mixed strategies in the question is a mixed strategy equilibrium, it suffices to verify the conditions in Proposition 113.2. Thus, given that each player's strategy specifies a positive probability for every action, it suffices to show that each action of each player yields the same expected payoff. Player 1's expected payoff to each pure strategy is $1/K$, because with probability $1/K$ player 2 chooses the same number, and with probability $1 - 1/K$ player 2 chooses a different number. Similarly, player 2's expected payoff to each pure strategy is $-1/K$, because with probability $1/K$ player 1 chooses the same number, and with probability $1 - 1/K$ player 2 chooses a different number. Thus the pair of strategies is a mixed strategy Nash equilibrium.

b. Let (p^*, q^*) be a mixed strategy equilibrium, where p^* and q^* are vectors, the jth components of which are the probabilities assigned to the integer j by each player. Given that player 2 uses the mixed strategy q^*, player 1's expected payoff if she chooses the number k is q_k^*. Hence if $p_k^* > 0$ then (by the first condition in Proposition 113.2) we need $q_k^* \geq q_j^*$ for all j, so that, in particular, $q_k^* > 0$ (q_j^* cannot be zero for all j!). But player 2's expected payoff if she chooses the number k is $-p_k$, so given $q_k^* > 0$ we need $p_k^* \leq p_j^*$ for all j (again by the first condition in Proposition 113.2), and, in particular, $p_k^* \leq 1/K$ (p_j^* cannot exceed $1/K$ for all j!). We conclude that any probability p_k^* that is positive must be at most $1/K$. The only possibility is that $p_k^* = 1/K$ for all k. A similar argument implies that $q_k^* = 1/K$ for all k.

115.2 Silverman's game

The game has no pure strategy Nash equilibrium in which the players' integers are the same because either player can increase her payoff from 0 to 1 by naming the next higher integer. It has no Nash equilibrium in which the players' integers are different because the losing player (the player whose payoff is -1) can increase her payoff to 1 by changing her integer to be one more than the other player's integer. Thus the game has no pure strategy Nash equilibrium.

To show that the pair of mixed strategies in the question is a mixed strategy equilibrium, it suffices to verify the conditions in Proposition 113.2. That is, it suffices to show that for each player, each action to which the player's mixed strategy assigns positive probability yields the player the same expected payoff, and every other action yields her a payoff at most as large. The game is symmetric and the players' strategies are the same, so we need to make an argument only for one player.

Suppose player 2 uses the mixed strategy in the question. Player 1's expected payoffs to her actions are as follows:

1: $\frac{1}{3} \cdot 0 + \frac{1}{3} \cdot (-1) + \frac{1}{3} \cdot 1 = 0$.

2: $\frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot (-1) = 0$.

3 or 4: $\frac{1}{3} \cdot (-1) + \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot (-1) = -\frac{1}{3}$.

5: $\frac{1}{3} \cdot (-1) + \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 0 = 0$.

6–14: $\frac{1}{3} \cdot (-1) + \frac{1}{3} \cdot (-1) + \frac{1}{3} \cdot 1 = -\frac{1}{3}$.

15 or more: $\frac{1}{3} \cdot (-1) + \frac{1}{3} \cdot (-1) + \frac{1}{3} \cdot (-1) = -1$.

Thus the pair of strategies is a mixed strategy equilibrium.

115.3 Voter participation

I verify that the conditions in Proposition 113.2 are satisfied.

First consider a supporter of candidate A. If she votes then candidate A ties if all $k-1$ of her comrades vote, an event with probability p^{k-1}, and otherwise candidate A loses. Thus her expected payoff is

$$p^{k-1} - c.$$

If she abstains, then candidate A surely loses, so her payoff is 0. Thus in an equilibrium in which $0 < p < 1$ the first condition in Proposition 113.2 implies that $p^{k-1} = c$, or

$$p = c^{1/(k-1)}.$$

Now consider a supporter of candidate B who votes. With probability p^k all of the supporters of candidate A vote, in which case the election is a tie; with probability $1 - p^k$ at least one of the supporters of candidate A does not vote, in which case candidate B wins. Thus the expected payoff of a supporter of candidate B who votes is

$$p^k + 2(1 - p^k) - c.$$

If the supporter of candidate B switches to abstaining, then

- candidate B loses if all supporters of candidate A vote, an event with probability p^k
- candidate B ties if exactly $k-1$ supporters of candidate A vote, an event with probability $kp^{k-1}(1-p)$
- candidate B wins if fewer than $k-1$ supporters of candidate A vote, an event with probability $1 - p^k - kp^{k-1}(1-p)$.

Thus a supporter of candidate B who switches from voting to abstaining obtains an expected payoff of

$$kp^{k-1}(1-p) + 2(1 - p^k - kp^{k-1}(1-p)) = 2 - (2-k)p^k - kp^{k-1}.$$

Hence in order for it to be optimal for such a citizen to vote (i.e. in order for the second condition in Proposition 113.2 to be satisfied), we need

$$p^k + 2(1 - p^k) - c \geq 2 - (2-k)p^k - kp^{k-1},$$

or

$$kp^{k-1}(1-p) + p^k \geq c.$$

Finally, consider a supporter of candidate B who abstains. With probability p^k all the supporters of candidate A vote, in which case the candidates tie; with probability $1 - p^k$ at least one of the supporters of candidate A does not vote, in which case candidate B wins. Thus the expected payoff of a supporter of candidate B who abstains is

$$p^k + 2(1 - p^k).$$

If this citizen instead votes, candidate B surely wins (she gets $k + 1$ votes, while candidate A gets at most k). Thus the citizen's expected payoff is

$$2 - c.$$

Hence in order for the citizen to wish to abstain, we need

$$p^k + 2(1 - p^k) \geq 2 - c$$

or

$$c \geq p^k.$$

In summary, for equilibrium we need $p = c^{1/(k-1)}$ and

$$p^k \leq c \leq kp^{k-1}(1 - p) + p^k.$$

Given $p = c^{1/(k-1)}$, $c = p^{k-1}$, so that the two inequalities are satisfied. Thus $p = c^{1/(k-1)}$ defines an equilibrium.

As c increases, the probability p, and hence the expected number of voters, *increases*.

115.4 Defending territory

(The solution to this problem, which corrects an error in Shubik (1982, 226), is due to Nick Vriend.) The game is shown in Figure 55.1, where each action (x, y) gives the number x of divisions allocated to the first pass and the number y allocated to the second pass.

General A \ General B	$(2,0)$	$(1,1)$	$(0,2)$
$(3,0)$	$1,-1$	$-1,\ 1$	$-1,\ 1$
$(2,1)$	$1,-1$	$1,-1$	$-1,\ 1$
$(1,2)$	$-1,\ 1$	$1,-1$	$1,-1$
$(0,3)$	$-1,\ 1$	$-1,\ 1$	$1,-1$

Figure 55.1 The game in Exercise 115.4.

Denote a mixed strategy of A by (p_1, p_2, p_3, p_4) and a mixed strategy of B by (q_1, q_2, q_3).

First I argue that in every equilibrium $q_2 = 0$. If $q_2 > 0$ then A's expected payoff to $(3,0)$ is less than her expected payoff to $(2,1)$, and her expected payoff to $(0,3)$ is less than her expected payoff to $(1,2)$, so that $p_1 = p_4 = 0$. But then B's expected payoff to at least one of her actions $(2,0)$ and $(0,2)$ exceeds her expected payoff to $(1,1)$, contradicting $q_2 > 0$.

Now I argue that in every equilibrium $q_1 = q_3 = 0$. Given $q_2 = 0$ we have $q_3 = 1 - q_1$, and A's payoffs are $2q_1 - 1$ to $(3,0)$ and to $(2,1)$, and $1 - 2q_1$ to $(1,2)$ and $(0,3)$. Thus if $q_1 < \frac{1}{2}$ then in any equilibrium we have $p_1 = p_2 = 0$. Then B's action $(2,0)$ yields her a higher payoff than does $(0,2)$, so that in any equilibrium $q_1 = 1$. But then A's actions $(3,0)$ and $(2,1)$ both yield higher payoffs than do $(1,2)$ and $(0,3)$, contradicting $p_1 = p_2 = 0$. Similarly, $q_1 > \frac{1}{2}$ is inconsistent with equilibrium. Hence in any equilibrium $q_1 = q_3 = \frac{1}{2}$.

Now, given $q_1 = q_3 = \frac{1}{2}$, A's payoffs to her four actions are all equal. Thus $((p_1, p_2, p_3, p_4), (q_1, q_2, q_3))$ is a Nash equilibrium if and only if B's payoff to $(2,0)$ is the same as her payoff to $(0,2)$, and this payoff is at least her payoff to $(1,1)$. The first condition is $-p_1 - p_2 + p_3 + p_4 = p_1 + p_2 - p_3 - p_4$, or $p_1 + p_2 = p_3 + p_4 = \frac{1}{2}$. Thus B's payoff to $(2,0)$ and to $(0,2)$ is zero, and the second condition is $p_1 - p_2 - p_3 + p_4 \leq 0$, or $p_1 + p_4 \leq \frac{1}{2}$ (using $p_1 + p_2 + p_3 + p_4 = 1$).

We conclude that the set of mixed strategy Nash equilibria of the game is the set of strategy pairs $((p_1, \frac{1}{2} - p_1, \frac{1}{2} - p_4, p_4), (\frac{1}{2}, 0, \frac{1}{2}))$ with $p_1 + p_4 \leq \frac{1}{2}$.

In this equilibrium general A splits her resources between the two passes with probability at least $\frac{1}{2}$ ($p_2 + p_3 = \frac{1}{2} - p_1 + \frac{1}{2} - p_4 = 1 - (p_1 + p_4) \geq \frac{1}{2}$) while general B concentrates all of her resources in one or other of the passes (with equal probability).

118.1 Strictly dominated actions

Denote the probability that player 1 assigns to T by p and the probability she assigns to M by r (so that the probability she assigns to B is $1 - p - r$). A mixed strategy of player 1 strictly dominates T if and only if

$$p + 4r > 1 \quad \text{and} \quad p + 3(1 - p - r) > 1,$$

or if and only if $1 - 4r < p < 1 - \frac{3}{2}r$. For example, the mixed strategies $(\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$ and $(0, \frac{1}{4}, \frac{3}{4})$ both strictly dominate T.

119.1 Eliminating dominated actions when finding equilibria

Player 2's action L is strictly dominated by the mixed strategy that assigns probability $\frac{1}{4}$ to M and probability $\frac{3}{4}$ to R (for example), so that we can ignore the action L. The players' best response functions in the reduced game in which player 2's actions are M and R are shown in Figure 57.1. We see that the game has a single mixed strategy Nash equilibrium, namely $((\frac{2}{3}, \frac{1}{3}), (0, \frac{1}{2}, \frac{1}{2}))$.

124.1 Equilibrium in the expert diagnosis game

When $E = rE' + (1 - r)I'$ the consumer is indifferent between her two actions when $p = 0$, so that her best response function has a vertical segment at $p = 0$.

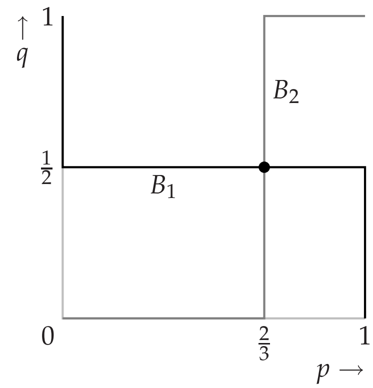


Figure 57.1 The players' best response functions in the game in Figure 119.1 after player 2's action L has been eliminated. The probability assigned by player 1 to T is p and the probability assigned by player 2 to M is q. The best response function of player 1 is black and that of player 2 is gray. The disk indicates the unique Nash equilibrium.

Referring to Figure 123.1 in the text, we see that the set of mixed strategy Nash equilibria correspond to $p = 0$ and $\pi/\pi' \leq q \leq 1$.

125.1 Incompetent experts

The payoffs are given in Figure 57.2. (The actions are the same as those in the game in which every expert is fully competent.)

	A	R
H	$\pi, -rE - (1-r)[sI + (1-s)E]$	$(1-r)s\pi, -rE' - (1-r)[sI + (1-s)I']$
D	$r\pi + (1-r)[s\pi' + (1-s)\pi], -E$	$0, -rE' - (1-r)I'$

Figure 57.2 A game between a consumer with a problem and a not-fully-competent expert.

Following the method in the text for the case $s = 1$, we find that in the case $E > rE' + (1-r)I'$ there is a unique mixed strategy equilibrium, in which the probability the expert's strategy assigns to H is

$$p^* = \frac{E - [rE' + (1-r)I']}{(1-r)s(E - I')}$$

and the probability the consumer's strategy assigns to A is

$$q^* = \frac{\pi}{\pi'}.$$

We see that q^* is independent of s. That is, the degree of competence has no effect on consumer behavior: consumers do not become more, or less, wary. The fraction of experts who are honest is a decreasing function of s, so that greater incompetence (smaller s) leads to a *higher* fraction of honest experts: incompetence breeds honesty! The intuition is that when experts become less competent, the potential gain from ignoring their advice increases (since $I' < E$), so that they need to be more honest to attract business.

125.2 Choosing a seller

The game is given in Figure 58.1.

		Buyer 2: Seller 1	Buyer 2: Seller 2
Buyer 1	Seller 1	$\frac{1}{2}(1-p_1), \frac{1}{2}(1-p_1)$	$1-p_1, 1-p_2$
	Seller 2	$1-p_2, 1-p_1$	$\frac{1}{2}(1-p_2), \frac{1}{2}(1-p_2)$

Figure 58.1 The game in Exercise 125.2.

The character of its equilibria depend on the value of (p_1, p_2). If $p_1 = p_2 = 1$ every pair $((\pi_1, 1-\pi_1), ((\pi_2, 1-\pi_2))$ is a mixed strategy equilibrium (where π_i is the probability of buyer i's choosing seller 1) is a equilibrium. Now suppose that at least one price is less than 1.

- If $\frac{1}{2}(1-p_2) > 1-p_1$ (i.e. $p_2 < 2p_1 - 1$), each buyer's action of approaching seller 2 strictly dominates her action of approaching seller 1. Thus the game has a unique mixed strategy equilibrium, in which both buyers use a pure strategy: each approaches seller 2.

- If $\frac{1}{2}(1-p_2) = 1-p_1$ (i.e. $p_2 = 2p_1 - 1$), every mixed strategy is a best response of a buyer to the other buyer's approaching seller 2, and the pure strategy of approaching seller 2 is the unique best response to the other buyer's using any other strategy. Thus $((\pi_1, 1-\pi_1), ((\pi_2, 1-\pi_2))$ is a mixed strategy equilibrium if and only if either $\pi_1 = 0$ or $\pi_2 = 0$.

- If $\frac{1}{2}(1-p_1) > 1-p_2$ (i.e. $p_2 > \frac{1}{2}(1+p_1)$), each buyer's action of approaching seller 1 strictly dominates her action of approaching seller 2. Thus the game has a unique mixed strategy equilibrium, in which both buyers use a pure strategy: each approaches seller 1.

- If $\frac{1}{2}(1-p_1) = 1-p_2$ (i.e. $p_2 = \frac{1}{2}(1+p_1)$), every mixed strategy is a best response of a buyer to the other buyer's strategy of approaching seller 1, and the pure strategy of approaching seller 1 is the unique best response to any other strategy of the other buyer. Thus $((\pi_1, 1-\pi_1), ((\pi_2, 1-\pi_2))$ is a mixed strategy equilibrium if and only if either $\pi_1 = 1$ or $\pi_2 = 1$.

- For the case $2p_1 - 1 < p_2 < \frac{1}{2}(1+p_1)$, a buyer's expected payoff to choosing each seller is the same when

$$\tfrac{1}{2}(1-p_1)\pi + (1-p_1)(1-\pi) = (1-p_2)\pi + \tfrac{1}{2}(1-p_2)(1-\pi),$$

where π is the probability that the other buyer chooses seller 1, or when

$$\pi = \frac{1 - 2p_1 + p_2}{2 - p_1 - p_2}.$$

The players' best response functions are shown in Figure 59.1. We see that the game has three mixed strategy equilibria: two pure equilibria in which the buyers approach different sellers, and one mixed strategy equilibrium in which each buyer approaches seller 1 with probability $(1 - 2p_1 + p_2)/(2 - p_1 - p_2)$.

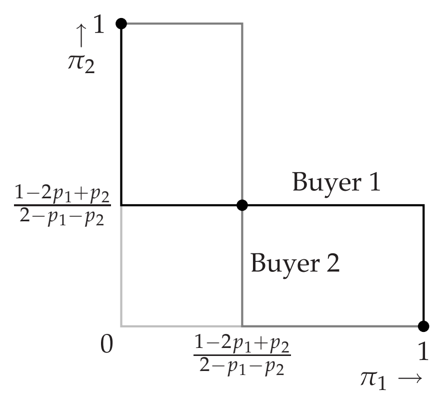


Figure 59.1 The players' best response functions in the game in Exercise 125.2. The probability with which buyer i approaches seller 1 is π_i.

The three main cases are illustrated in Figure 60.1. If the prices are relatively close, there are two pure strategy equilibria, in which the buyers choose different sellers, and a symmetric mixed strategy equilibrium in which both buyers approach seller 1 with the same probability. If seller 2's price is high relative to seller 1's, there is a unique equilibrium, in which both buyers approach seller 1. If seller 1's price is high relative to seller 2's, there is a unique equilibrium, in which both buyers approach seller 2.

127.2 Approaching cars

The game has three Nash equilibria: (*Stop*, *Continue*), (*Continue*, *Stop*), and a mixed strategy equilibrium in which each player chooses *Stop* with probability

$$\frac{1 - \epsilon}{2 - \epsilon}.$$

Only the mixed strategy equilibrium is symmetric; the expected payoff of each player in this equilibrium is $2(1 - \epsilon)/(2 - \epsilon)$.

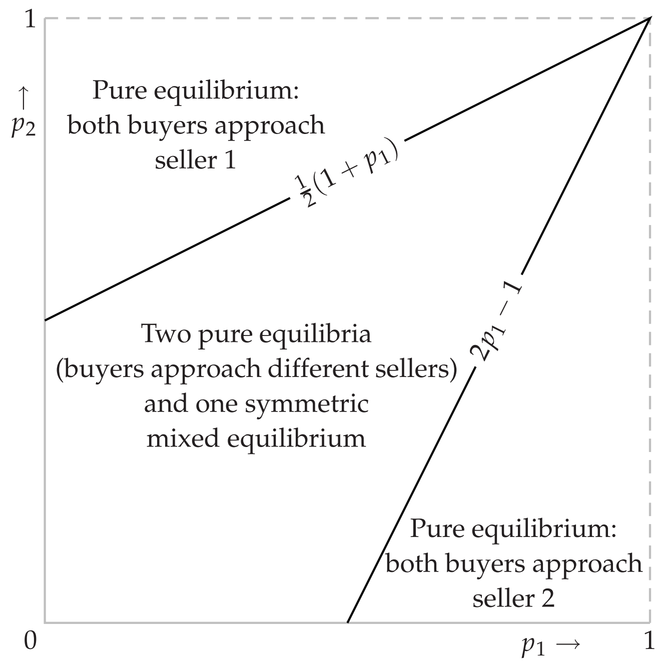


Figure 60.1 Equilibria of the game in Exercise 125.2 as a function of the sellers' prices.

The modified game also has a unique symmetric equilibrium. In this equilibrium each player chooses *Stop* with probability

$$\frac{1 - \epsilon + \delta}{2 - \epsilon}$$

if $\delta \leq 1$ and chooses *Stop* with probability 1 if $\delta \geq 1$. The expected payoff of each player in this equilibrium is $(2(1 - \epsilon) + \epsilon\delta)/(2 - \epsilon)$ if $\delta \leq 1$ and 1 if $\delta \geq 1$, both of which are larger than her payoff in the original game (given $\delta > 0$).

After reeducation, each driver's payoffs to stopping stay the same, while those to continuing fall. Thus if the behavioral norm (the probability of stopping) were to remain the same, every driver would find it beneficial to stop. Equilibrium is restored only if enough drivers switch to *Stop*, raising everyone's expected payoff. (Each player's expected payoff in a mixed strategy Nash equilibrium is her expected payoff to choosing *Stop*, which is $p + (1 - \epsilon)(1 - p)$, where p is the probability of a player's choosing *Stop*.)

128.1 Bargaining

The game is given in Figure 61.1.

By inspection it has a single symmetric pure strategy Nash equilibrium, $(10, 10)$.

Now consider situations in which the common mixed strategy assigns positive probability to two actions. Suppose that player 2 assigns positive probability only

	0	2	4	6	8	10
0	5,5	4,6	3,7	2,8	1,9	0,10
2	6,4	5,5	4,6	3,7	2,8	0,0
4	7,3	6,4	5,5	4,6	0,0	0,0
6	8,2	7,3	6,4	0,0	0,0	0,0
8	9,1	8,2	0,0	0,0	0,0	0,0
10	10,0	0,0	0,0	0,0	0,0	0,0

Figure 61.1 A bargaining game.

to 0 and 2. Then player 1's payoff to her action 4 exceeds her payoff to either 0 or 2. Thus there is no symmetric equilibrium in which the actions assigned positive probability are 0 and 2. By a similar argument we can rule out equilibria in which the actions assigned positive probability are any pair except 2 and 8, or 4 and 6.

If the actions to which player 2 assigns positive probability are 2 and 8 then player 1's expected payoffs to 2 and 8 are the same if the probability player 2 assigns to 2 is $\frac{2}{5}$ (and the probability she assigns to 8 is $\frac{3}{5}$). Given these probabilities, player 1's expected payoff to her actions 2 and 8 is $\frac{16}{5}$, and her expected payoff to every other action is less than $\frac{16}{5}$. Thus the pair of mixed strategies in which every player assigns probability $\frac{2}{5}$ to 2 and $\frac{3}{5}$ to 8 is a symmetric mixed strategy Nash equilibrium.

Similarly, the game has a symmetric mixed strategy equilibrium (α^*, α^*) in which α^* assigns probability $\frac{4}{5}$ to the demand of 4 and probability $\frac{1}{5}$ to the demand of 6.

In summary, the game has three symmetric mixed strategy Nash equilibria in which each player's strategy assigns positive probability to at most two actions: one in which probability 1 is assigned to 10, one in which probability $\frac{2}{5}$ is assigned to 2 and probability $\frac{3}{5}$ is assigned to 8, and one in which probability $\frac{4}{5}$ is assigned to 4 and probability $\frac{1}{5}$ is assigned to 6.

130.1 Contributing to a public good

In a mixed strategy equilibrium each player obtains the same expected payoff whether or not she contributes. A player's contribution makes a difference to the outcome only if exactly $k-1$ of the other players contribute. Thus the difference between the expected benefit of contributing and that of not contributing is

$$vQ_{n-1,k-1}(p) - c,$$

which must be 0 in a mixed strategy equilibrium.

For $v = 1$, $n = 4$, $k = 2$, and $c = \frac{3}{8}$ this equilibrium condition is

$$Q_{3,1}(p) = \tfrac{3}{8}.$$

Now, $Q_{3,1}(p) = 3p(1-p)^2$, so an equilibrium value of p satisfies

$$3p(1-p)^2 = \tfrac{3}{8},$$

or

$$p^3 - 2p^2 + p - \tfrac{1}{8} = 0,$$

or

$$(p - \tfrac{1}{2})(p^2 - \tfrac{3}{2}p + \tfrac{1}{4}) = 0.$$

Thus $p = \frac{1}{2}$ or $p = \frac{3}{4} - \frac{1}{2}\sqrt{\frac{5}{4}} \approx 0.19$. (The other root of the quadratic is greater than one, and thus not meaningful as a solution of the problem.)

We conclude that the game has two symmetric mixed strategy Nash equilibria: one in which the common probability is $\frac{1}{2}$ and one in which this probability is $\frac{3}{4} - \frac{1}{2}\sqrt{\frac{5}{4}}$.

133.1 Best response dynamics in Cournot's duopoly game

The best response functions of both firms are the same, so if the firms' outputs are initially the same, they are the same in every period: $q_1^t = q_2^t$ for every t. For each period t, we thus have

$$q_i^t = \tfrac{1}{2}(\alpha - c - q_i^t).$$

Given that $q_i^1 = 0$ for $i = 1, 2$, solving this first-order difference equation we have

$$q_i^t = \tfrac{1}{3}(\alpha - c)[1 - (-\tfrac{1}{2})^{t-1}]$$

for each period t. When t is large, q_i^t is close to $\frac{1}{3}(\alpha - c)$, a firm's equilibrium output.

In the first few periods, these outputs are 0, $\frac{1}{2}(\alpha - c)$, $\frac{1}{4}(\alpha - c)$, $\frac{3}{8}(\alpha - c)$, $\frac{5}{16}(\alpha - c)$.

133.2 Best response dynamics in Bertrand's duopoly game

If $p_i > c + 1$ then firm j has a unique best response, equal to the lesser of $p_i - 1$ and the monopoly price. Thus if both prices initially exceed $c + 1$ then for every period t in which at least one price exceeds $c + 1$ the maximal price in period $t + 1$ is (*i*) less than the maximal price in period t and (*ii*) at least $c + 1$. Thus the process converges to the Nash equilibrium $(c + 1, c + 1)$.

If $p_i = c$ then all prices $p_j \geq c$ are best responses. Thus if the pair of prices is initially (c, c), many subsequent sequences of prices are consistent with best response dynamics. We can divide the sequences into three cases.

- Both prices are equal to c in every subsequent period.
- In some period both prices are at least $c + 1$, in which case eventually the Nash equilibrium $(c + 1, c + 1)$ is reached (by the analysis for the first part of the exercise).

- In every period one of the prices is equal to c, while the other price is greater than c; the identity of the firm charging c changes from period to period. The pairs of prices eventually alternate between $(c, c+1)$ and $(c+1, c)$ (neither of which are Nash equilibria).

136.1 Finding all mixed strategy equilibria of two-player games

Left game:

- There is no equilibrium in which each player's mixed strategy assigns positive probability to a single action (i.e. there is no pure equilibrium).
- Consider the possibility of an equilibrium in which one player assigns probability 1 to a single action while the other player assigns positive probability to both her actions. For neither action of player 1 is player 2's payoff the same for both her actions, and for neither action of player 2 is player 1's payoff the same for both her actions, so there is no mixed strategy equilibrium of this type.
- Consider the possibility of a mixed strategy equilibrium in which each player assigns positive probability to both her actions. Denote by p the probability player 1 assigns to T and by q the probability player 2 assigns to L. For player 1's expected payoff to her two actions to be the same we need

$$6q = 3q + 6(1-q),$$

or $q = \frac{2}{3}$. For player 2's expected payoff to her two actions to be the same we need

$$2(1-p) = 6p,$$

or $p = \frac{1}{4}$. We conclude that the game has a unique mixed strategy equilibrium, $((\frac{1}{4}, \frac{3}{4}), (\frac{2}{3}, \frac{1}{3}))$.

Right game:

- By inspection, (T, R) and (B, L) are the pure strategy equilibria.
- Consider the possibility of a mixed strategy equilibrium in which one player assigns probability 1 to a single action while the other player assigns positive probability to both her actions.
  - $\{T\}$ for player 1, $\{L, R\}$ for player 2: no equilibrium, because player 2's payoffs to (T, L) and (T, R) are not the same.
  - $\{B\}$ for player 1, $\{L, R\}$ for player 2: no equilibrium, because player 2's payoffs to (B, L) and (B, R) are not the same.
  - $\{T, B\}$ for player 1, $\{L\}$ for player 2: no equilibrium, because player 1's payoffs to (T, L) and (B, L) are not the same.

   - $\{T, B\}$ for player 1, $\{R\}$ for player 2: player 1's payoffs to (T, R) and (B, R) are the same, so there is an equilibrium in which player 1 uses T with probability p if player 2's expected payoff to R, which is $2p + 1 - p$, is at least her expected payoff to L, which is $p + 2(1 - p)$. That is, the game has equilibria in which player 1's mixed strategy is $(p, 1 - p)$, with $p \geq \frac{1}{2}$, and player 2 uses R with probability 1.

- Consider the possibility of an equilibrium in which both players assign positive probability to both their actions. Denote by q the probability that player 2 assigns to L. For player 1's expected payoffs to T and B to be the same we need $0 = 2q$, or $q = 0$, so there is no equilibrium in which both players assign positive probability to both their actions.

In summary, the mixed strategy equilibria of the game are $((0, 1), (1, 0))$ (i.e. the pure equilibrium (B, L)) and $((p, 1 - p), (0, 1))$ for $\frac{1}{2} \leq p \leq 1$ (of which one equilibrium is the pure equilibrium (T, R)).

138.1 Finding all mixed strategy equilibria of a two-player game

By inspection, (T, R) and (B, L) are pure strategy equilibria.

Now consider the possibility of an equilibrium in which player 1's strategy is pure while player 2's strategy assigns positive probability to two or more actions.

- If player 1's strategy is T then player 2's payoffs to M and R are the same, and her payoff to L is less, so an equilibrium in which player 2 randomizes between M and R is possible. In order that T be optimal we need $1 - q \geq q$, or $q \leq \frac{1}{2}$, where q is the probability player 2's strategy assigns to M. Thus every mixed strategy pair $((1, 0), (0, q, 1 - q))$ in which $q \leq \frac{1}{2}$ is a mixed strategy equilibrium.

- If player 1's strategy is B then player 2's payoffs to L and R are the same, and her payoff to M is less, so an equilibrium in which player 2 randomizes between L and R is possible. In order that B be optimal we need $2q + 1 - q \leq 3q$, or $q \geq \frac{1}{2}$, where q is the probability player 2's strategy assigns to L. Thus every mixed strategy pair $((0, 1), (q, 0, 1 - q))$ in which $q \geq \frac{1}{2}$ is a mixed strategy equilibrium.

Now consider the possibility of an equilibrium in which player 2's strategy is pure while player 1's strategy assigns positive probability to both her actions. For each action of player 2, player 1's two actions yield her different payoffs, so there is no equilibrium of this sort.

Next consider the possibility of an equilibrium in which both player 1's and player 2's strategies assign positive probability to two actions. Denote by p the probability player 1's strategy assigns to T. There are three possibilities for the pair of player 2's actions that have positive probability.

L and *M*: For an equilibrium we need player 2's expected payoff to *L* to be equal to her expected payoff to *M* and at least her expected payoff to *R*. That is, we need

$$2 = 3p + 1 - p \geq 3p + 2(1 - p).$$

The inequality implies that $p = 1$, so that player 1's strategy assigns probability zero to *B*. Thus there is no equilibrium of this type.

L and *R*: For an equilibrium we need player 2's expected payoff to *L* to be equal to her expected payoff to *R* and at least her expected payoff to *M*. That is, we need

$$2 = 3p + 2(1 - p) \geq 3p + 1 - p.$$

The equation implies that $p = 0$, so there is no equilibrium of this type.

M and *R*: For an equilibrium we need player 2's expected payoff to *M* to be equal to her expected payoff to *R* and at least her expected payoff to *L*. That is, we need

$$3p + 1 - p = 3p + 2(1 - p) \geq 2.$$

The equation implies that $p = 1$, so there is no equilibrium of this type.

The final possibility is that there is an equilibrium in which player 1's strategy assigns positive probability to both her actions and player 2's strategy assigns positive probability to all three of her actions. Let p be the probability player 1's strategy assigns to T. Then for player 2's expected payoffs to her three actions to be equal we need

$$2 = 3p + 1 - p = 3p + 2(1 - p).$$

For the first equality we need $p = \frac{1}{2}$, violating the second equality. That is, there is no value of p for which player 2's expected payoffs to her three actions are equal, and thus no equilibrium in which she chooses each action with positive probability.

We conclude that the mixed strategy equilibria of the game are the strategy pairs of the forms $((1, 0), (0, q, 1 - q))$ for $0 \leq q \leq \frac{1}{2}$ ($q = 0$ is the pure equilibrium (T, R)) and $((0, 1), (q, 0, 1 - q))$ for $\frac{1}{2} \leq q \leq 1$ ($q = 1$ is the pure equilibrium (B, L)).

138.2 Rock, paper, scissors

The game is shown in Figure 65.1.

	Rock	*Paper*	*Scissors*
Rock	0, 0	−1, 1	1, −1
Paper	1, −1	0, 0	−1, 1
Scissors	−1, 1	1, −1	0, 0

Figure 65.1 *Rock, paper, scissors*

By inspection the game has no pure strategy equilibrium, and no mixed strategy equilibrium in which one player's strategy is pure and the other's is strictly mixed.

In the remaining possibilities both players use at least two actions with positive probability. Suppose that player 1's mixed strategy assigns positive probability to *Rock* and to *Paper*. Then player 2's expected payoff to *Paper* exceeds her expected payoff to *Rock*, so in any such equilibrium player 2 must assign positive probability only to *Paper* and *Scissors*. Player 1's expected payoffs to *Rock* and *Paper* are equal only if player 2 assigns probability $\frac{2}{3}$ to *Paper* and probability $\frac{1}{3}$ to *Scissors*. But then player 1's expected payoff to *Scissors* exceeds her expected payoffs to *Rock* and *Paper*. So there is no mixed strategy equilibrium in which player 1 assigns positive probability only to *Rock* and to *Paper*.

Given the symmetry of the game, the same argument implies that there is no equilibrium in which player 1 assigns positive probability to only two actions, nor any equilibrium in which player 2 assigns positive probability to only two actions.

The remaining possibility is that each player assigns positive probability to all three of her actions. Denote the probabilities player 1 assigns to her three actions by (p_1, p_2, p_3) and the probabilities player 2 assigns to her three actions by (q_1, q_2, q_3). Player 1's actions all yield her the same expected payoff if and only if there is a value of c for which

$$\begin{aligned} -q_2 + q_3 &= c \\ q_1 \qquad - q_3 &= c \\ -q_1 + q_2 \qquad &= c. \end{aligned}$$

Adding the three equations we deduce $c = 0$, and hence $q_1 = q_2 = q_3 = \frac{1}{3}$. A similar calculation for player 2 yields $p_1 = p_2 = p_3 = \frac{1}{3}$.

In conclusion, the game has a unique mixed strategy equilibrium, in which each player uses the strategy $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Each player's equilibrium payoff is 0.

In the modified game in which player 1 is prohibited from using the action *Scissors*, player 2's action *Rock* is strictly dominated. The remaining game has a unique mixed strategy equilibrium, in which player 1 chooses *Rock* with probability $\frac{1}{3}$ and *Paper* with probability $\frac{2}{3}$, and player 2 chooses *Paper* with probability $\frac{2}{3}$ and *Scissors* with probability $\frac{1}{3}$. The equilibrium payoff of player 1 is $-\frac{1}{3}$ and that of player 2 is $\frac{1}{3}$.

139.1 Election campaigns

A strategic game that models the situation is shown in Figure 67.1, where action k means devote resources to locality k.

By inspection the game has no pure strategy equilibrium and no equilibrium in which one player's strategy is pure and the other is strictly mixed. (For each action of each player, the other player has a single best action.)

		Party B 1	2	3
	1	$0, 0$	$a_1, -a_1$	$a_1, -a_1$
Party A	2	$a_2, -a_2$	$0, 0$	$a_2, -a_2$
	3	$a_3, -a_3$	$a_3, -a_3$	$0, 0$

Figure 67.1 The game in Exercise 139.1.

Now consider the possibility of an equilibrium in which party A assigns positive probability to exactly two actions. There are three possible pairs of actions. Throughout the argument I denote the probability party A's strategy assigns to her action i by p_i, and the probability party B's strategy assigns to her action i by q_i.

1 and 2: Party B's action 3 is strictly dominated by her mixed strategy that assigns probability $\frac{1}{2}$ to each of her actions 1 and 2, so that we can eliminate it from consideration. For party A's actions 1 and 2 to yield the same expected payoff we need $q_2 a_1 = q_1 a_2$, or, given $q_2 = 1 - q_1$, $q_1 = a_1/(a_1 + a_2)$. For party B's actions 1 and 2 to yield the same expected payoff we similarly need $p_1 = a_2/(a_1 + a_2)$. Finally, for party A's expected payoff to her action 3 to be no more than her expected payoff to her other two actions, we need

$$a_3 \leq \frac{a_1 a_2}{a_1 + a_2}.$$

We conclude that if $a_3 \leq a_1 a_2/(a_1 + a_2)$ (or equivalently $a_1 a_3 + a_2 a_3 \leq a_1 a_2$) then the game has a mixed strategy equilibrium

$$\left(\left(\frac{a_2}{a_1 + a_2}, \frac{a_1}{a_1 + a_2}, 0\right), \left(\frac{a_1}{a_1 + a_2}, \frac{a_2}{a_1 + a_2}, 0\right)\right). \tag{67.1}$$

1 and 3: Party B's action 2 is strictly dominated her mixed strategy that assigns probability $\frac{1}{2}$ to each of her actions 1 and 3, so that we can eliminate it from consideration. But then party A's action 2 strictly dominates her action 3, so there is no equilibrium in which she assigns positive probability to action 3. Thus there is no equilibrium of this type.

2 and 3: For similar reasons, there is no equilibrium of this type.

The remaining possibility is that there is an equilibrium in which each player assigns positive probability to all three of her actions. In order that party A's actions yield the same expected payoff we need

$$a_1(q_2 + q_3) = a_2(q_1 + q_3) = a_3(q_1 + q_2),$$

or, using $q_1 + q_2 + q_3 = 1$,

$$q_1 = \frac{a_1 a_2 + a_1 a_3 - a_2 a_3}{a_1 a_2 + a_1 a_3 + a_2 a_3}, \quad q_2 = \frac{a_1 a_2 - a_1 a_3 + a_2 a_3}{a_1 a_2 + a_1 a_3 + a_2 a_3}, \quad q_3 = \frac{-a_1 a_2 + a_1 a_3 + a_2 a_3}{a_1 a_2 + a_1 a_3 + a_2 a_3}. \tag{67.2}$$

For these three numbers to be positive we need

$$a_1a_2 + a_1a_3 - a_2a_3 > 0, \quad a_1a_2 - a_1a_3 + a_2a_3 > 0, \quad -a_1a_2 + a_1a_3 + a_2a_3 > 0.$$

Since $a_1 > a_2 > a_3$, these inequalities are satisfied if and only if $a_1a_3 + a_2a_3 > a_1a_2$.

Similarly, in order that party B's actions yield the same expected payoff we need

$$p_1 = \frac{a_2a_3}{a_1a_2 + a_1a_3 + a_2a_3}, \quad p_2 = \frac{a_1a_3}{a_1a_2 + a_1a_3 + a_2a_3}, \quad p_3 = \frac{a_1a_2}{a_1a_2 + a_1a_3 + a_2a_3}. \tag{68.1}$$

These three numbers are positive, given $a_i > 0$ for all i.

Thus if $a_1a_3 + a_2a_3 > a_1a_2$ there is an equilibrium in which player 1's mixed strategy is (p_1, p_2, p_3) and player 2's mixed strategy is (q_1, q_2, q_3).

In summary,

- if $(a_1 + a_2)a_3 \leq a_1a_2$ then the game has a unique mixed strategy equilibrium given by (67.1)
- if $(a_1 + a_2)a_3 > a_1a_2$ then the game has a unique mixed strategy equilibrium given by (67.2) and (68.1).

That is, if the first two localities are sufficiently more valuable than the third then both parties concentrate all their efforts on these two localities, while otherwise they both randomize between all three localities.

139.2 A three-player game

By inspection the game has two pure strategy equilibria, namely (A, A, A) and (B, B, B).

Now consider the possibility of an equilibrium in which one or more of the players' strategies is pure, and at least one is strictly mixed. If player 1 uses the action A and player 2 uses a strictly mixed strategy then player 3's uniquely best action is A, in which case player 2's uniquely best action is A. Thus there is no equilibrium in which player 1 uses the action A and at least one of the other players randomizes. By similar arguments, there is no equilibrium in which player 1 uses the action B and at least one of the other players randomizes, or indeed any equilibrium in which some player's strategy is pure while some other player's strategy is mixed.

The remaining possibility is that there is an equilibrium in which each player's strategy assigns positive probability to each of her actions. Denote the probabilities that players 1, 2, and 3 assign to A by p, q, and r respectively. In order that player 1's expected payoffs to her two actions be the same we need

$$qr = 4(1 - q)(1 - r).$$

Similarly, for player 2's and player 3's expected payoffs to their two actions to be the same we need

$$pr = 4(1-p)(1-r) \quad \text{and} \quad pq = 4(1-p)(1-q).$$

The unique solution of these three equations is $p = q = r = \frac{2}{3}$ (isolate r in the second equation and q in the third equation, and substitute into the first equation).

We conclude that the game has three mixed strategy equilibria: $((1,0),(1,0),(1,0))$ (i.e. the pure strategy equilibrium (A, A, A)), $((0,1),(0,1),(0,1))$ (i.e. the pure strategy equilibrium (B, B, B)), and $((\frac{2}{3},\frac{1}{3}),(\frac{2}{3},\frac{1}{3}),(\frac{2}{3},\frac{1}{3}))$.

143.1 All-pay auction with many bidders

Denote the common mixed strategy by F. Look for an equilibrium in which the largest value of z for which $F(z) = 0$ is 0 and the smallest value of z for which $F(z) = 1$ is $z = K$.

A player who bids a_i wins if and only if the other $n-1$ players all bid less than she does, an event with probability $(F(a_i))^{n-1}$. Thus, given that the probability that she ties for the highest bid is zero, her expected payoff is

$$(K - a_i)(F(a_i))^{n-1} + (-a_i)(1 - (F(a_i))^{n-1}).$$

Given the form of F, for an equilibrium this expected payoff must be constant for all values of a_i with $0 \le a_i \le K$. That is, for some value of c we have

$$K(F(a_i))^{n-1} - a_i = c \text{ for all } 0 \le a_i \le K.$$

For $F(0) = 0$ we need $c = 0$, so that $F(a_i) = (a_i/K)^{1/(n-1)}$ is the only candidate for an equilibrium strategy.

The function F is a cumulative probability distribution on the interval from 0 to K because $F(0) = 0$, $F(K) = 1$, and F is increasing. Thus F is indeed an equilibrium strategy.

We conclude that the game has a mixed strategy Nash equilibrium in which each player randomizes over all her actions according to the probability distribution $F(a_i) = (a_i/K)^{1/(n-1)}$; each player's equilibrium expected payoff is 0.

Each player's mean bid is K/n.

143.2 Bertrand's duopoly game

Denote the common mixed strategy by F. If firm 1 charges p it earns a profit only if the price charged by firm 2 exceeds p, an event with probability $1 - F(p)$. Thus firm 1's expected profit is

$$(1 - F(p))(p - c)D(p).$$

This profit is constant, equal to B, over some range of prices, if $F(p) = 1 - B/((p - c)D(p))$ over this range of prices. Because $(p-c)D(p)$ increases without bound as

p increases without bound, for any value of B the number $F(p)$ approaches 1 as p increases without bound. Further, for any $B > 0$, there exists some $\underline{p} > c$ such that $(\underline{p} - c)D(\underline{p}) = B$, so that $F(\underline{p}) = 0$. Finally, because $(p - c)D(p)$ is an increasing function, so is F. Thus F is a cumulative probability distribution function.

We conclude that for any $\underline{p} > c$, the game has a mixed strategy equilibrium in which each firm's mixed strategy is given by

$$F(p) = \begin{cases} 0 & \text{if } p < \underline{p} \\ 1 - \dfrac{(\underline{p} - c)D(\underline{p})}{(p - c)D(p)} & \text{if } p \geq \underline{p}. \end{cases}$$

144.2 Preferences over lotteries

The first piece of information about the decision-maker's preferences among lotteries is consistent with her preferences being represented by the expected value of a payoff function. For example, set $u(a_1) = 0$, $u(a_2) = 1$, and $u(a_3) = \frac{1}{3}$ (or any number between $\frac{1}{2}$ and $\frac{1}{4}$).

The second piece of information about the decision-maker's preferences is not consistent with these preferences being represented by the expected value of a payoff function, by the following argument. For consistency with the information about the decision-maker's preferences among the four lotteries, we need

$$0.4u(a_1) + 0.6u(a_3) > 0.5u(a_2) + 0.5u(a_3) >$$
$$0.3u(a_1) + 0.2u(a_2) + 0.5u(a_3) > 0.45u(a_1) + 0.55u(a_3).$$

The first inequality implies $u(a_2) < 0.8u(a_1) + 0.2u(a_3)$ and the last inequality implies $u(a_2) > 0.75u(a_1) + 0.25u(a_3)$. Because $u(a_1) < u(a_3)$, we have $0.75u(a_1) + 0.25u(a_3) > 0.8u(a_1) + 0.2u(a_3)$, so that the two inequalities are incompatible.

146.2 Normalized vNM payoff functions

Let $\overline{a}$ be the best outcome according to her preferences and let $\underline{a}$ be the worse outcome. Let $\eta = -u(\underline{a})/(u(\overline{a}) - u(\underline{a}))$ and $\theta = 1/(u(\overline{a}) - u(\underline{a})) > 0$. Lemma 145.1 implies that the function v defined by $v(x) = \eta + \theta u(x)$ represents the same preferences as does u; we have $v(\underline{a}) = 0$ and $v(\overline{a}) = 1$.

147.1 Games equivalent to the Prisoner's Dilemma

The left-hand game is not equivalent, by the following argument. Using either player's payoffs, for equivalence we need η and $\theta > 0$ such that

$$0 = \eta + \theta \cdot 0, 2 = \eta + \theta \cdot 1, 3 = \eta + \theta \cdot 2, \text{and } 4 = \eta + \theta \cdot 3.$$

From the first equation we have $\eta = 0$ and hence from the second we have $\theta = 2$. But these values do not satisfy the last two equations. (Alternatively, note that in the game in the left panel of Figure 104.1, player 1 is indifferent between (D, D) and the lottery in which (C, D) occurs with probability $\frac{1}{2}$ and (D, C) occurs with probability $\frac{1}{2}$, while in the left-hand game in Figure 148.1 she is not.)

The right-hand game is equivalent, by the following argument. For the equivalence of player 1's payoffs, we need η and $\theta > 0$ such that

$$0 = \eta + \theta \cdot 0, 3 = \eta + \theta \cdot 1, 6 = \eta + \theta \cdot 2, \text{ and } 9 = \eta + \theta \cdot 3.$$

The first two equations yield $\eta = 0$ and $\theta = 3$; these values satisfy the second two equations. A similar argument for player 2's payoffs yields $\eta = -4$ and $\theta = 2$.

5 Extensive games with perfect information: Theory

154.2 Examples of extensive games with perfect information

a. The game is given in Figure 73.1.

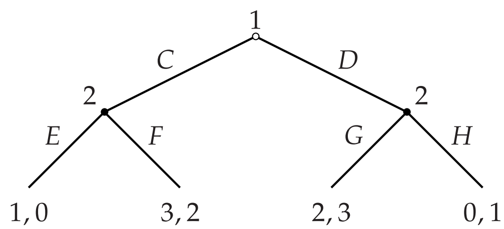


Figure 73.1 The game in Exercise 154.2*a*.

b. The game is specified as follows.

Players 1 and 2.

Terminal histories (C, E, G), (C, E, H), (C, F), D.

Player function $P(\varnothing) = 1$, $P(C) = 2$, $P(C, E) = 1$.

Preferences Player 1 prefers (C, F) to D to (C, E, G) to (C, E, H); player 2 prefers (C, E, G) to (C, F) to (C, E, H), and is indifferent between this outcome and D.

c. The game in shown in Figure 73.2, where the order of the payoffs is Karl, Rosa, Ernesto.

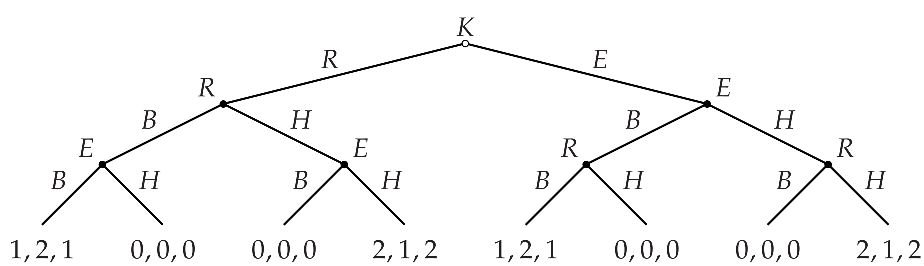


Figure 73.2 The game in Exercise 154.2*c*.

159.1 Strategies in extensive games

In the entry game, the challenger moves only at the start of the game, where it has two actions, *In* and *Out*. Thus it has two strategies, *In* and *Out*. The incumbent moves only after the history *In*, when it has two actions, *Acquiesce* and *Fight*. Thus it also has two strategies, *Acquiesce* and *Fight*.

In the game in Exercise 154.2c, Rosa moves after the histories R (Karl chooses her to move first), (E, B) (Karl chooses Ernesto to move first, and Ernesto chooses B), and (E, H) (Karl chooses Ernesto to move first, and Ernesto chooses H). In each case Rosa has two actions, B and H. Thus she has eight strategies. Each strategy takes the form (x, y, z), where each of x, y, and z are either B or H; the strategy (x, y, z) means that she chooses x after the history R, y after the history (E, B), and z after the history (E, H).

161.1 Nash equilibria of extensive games

The strategic form of the game in Exercise 154.2*a* is given in Figure 74.1.

	EG	*EH*	*FG*	*FH*
C	1, 0	1, 0	3, 2	3, 2
D	2, 3	0, 1	2, 3	0, 1

Figure 74.1 The strategic form of the game in Exercise 154.2*a*.

The Nash equilibria of the game are (C, FG), (C, FH), and (D, EG).

The strategic form of the game in Figure 158.1 is given in Figure 74.2.

	E	*F*
CG	1, 2	3, 1
CH	0, 0	3, 1
DG	2, 0	2, 0
DH	2, 0	2, 0

Figure 74.2 The strategic form of the game in Figure 158.1.

The Nash equilibria of the game are (CH, F), (DG, E), and (DH, E).

161.2 Voting by alternating veto

The following extensive game models the situation.

Players The two people.

Terminal histories $(\not{X}, \not{Y})$, $(\not{X}, \not{Z})$, $(\not{Y}, \not{X})$, $(\not{Y}, \not{Z})$, $(\not{Z}, \not{X})$, and $(\not{Z}, \not{Y})$ (where $\not{A}$ means veto A).

Player function $P(\varnothing) = 1$ and $P(\not X) = P(\not Y) = P(\not Z) = 2$.

Preferences Person 1's preferences are represented by the payoff function u_1 for which $u_1(\not Y, \not Z) = u_1(\not Z, \not Y) = 2$ (both of these terminal histories result in X's being chosen), $u_1(\not X, \not Z) = u_1(\not Z, \not X) = 1$, and $u_1(\not X, \not Y) = u_1(\not Y, \not X) = 0$. Person 2's preferences are represented by the payoff function u_2 for which $u_2(\not X, \not Y) = u_2(\not Y, \not X) = 2$, $u_2(\not X, \not Z) = u_2(\not Z, \not X) = 1$, and $u_2(\not Y, \not Z) = u_2(\not Z, \not Y) = 0$.

This game is shown in Figure 75.1.

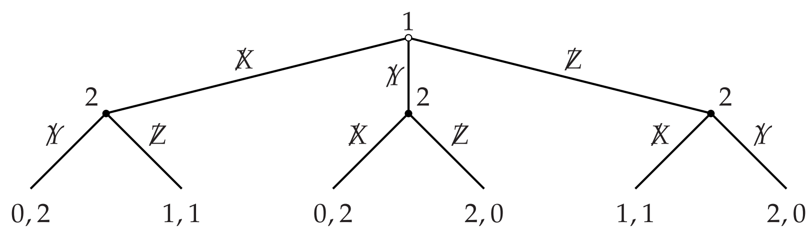


Figure 75.1 An extensive game that models the alternate strikeoff method of selecting an arbitrator, as specified in Exercise 161.2.

The strategic form of the game is given in Figure 75.2 (where $\not A \not B \not C$ is person 2's strategy in which it vetoes A if person 1 vetoes X, B if person 1 vetoes Y, and C if person 1 vetoes Z). Its Nash equilibria are $(\not Z, \not Y \not X \not X)$ and $(\not Z, \not Z \not X \not X)$.

	$\not Y \not X \not X$	$\not Y \not X \not Y$	$\not Y \not Z \not X$	$\not Y \not Z \not Y$	$\not Z \not X \not X$	$\not Z \not X \not Y$	$\not Z \not Z \not X$	$\not Z \not Z \not Y$
$\not X$	0, 2	0, 2	0, 2	0, 2	1, 1	1, 1	1, 1	1, 1
$\not Y$	0, 2	0, 2	2, 0	2, 0	0, 2	0, 2	2, 0	2, 0
$\not Z$	1, 1	2, 0	1, 1	2, 0	1, 1	2, 0	1, 1	2, 0

Figure 75.2 The strategic form of the game in Figure 75.1.

163.1 Subgames

The subgames of the game in Exercise 154.2*c* are the whole game and the six games in Figure 76.1.

166.2 Checking for subgame perfect equilibria

The Nash equilibria (CH, F) and (DH, E) are not subgame perfect equilibria: in the subgame following the history (C, E), player 1's strategies CH and DH induce the strategy H, which is not optimal.

The Nash equilibrium (DG, E) is a subgame perfect equilibrium: (*a*) it is a Nash equilibrium, so player 1's strategy is optimal at the start of the game, given

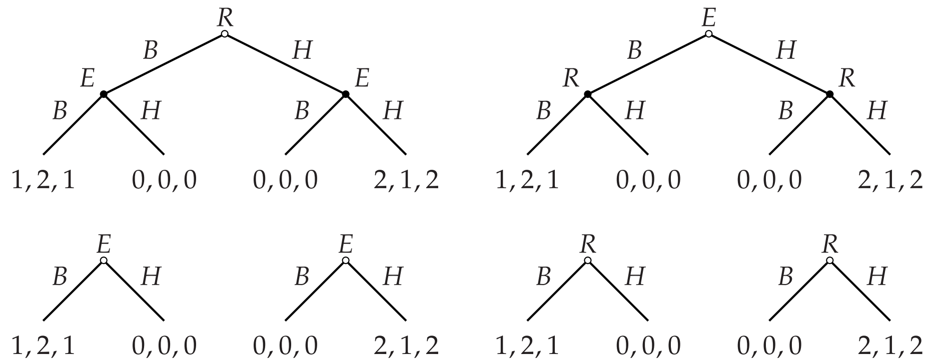


Figure 76.1 The proper subgames of the game in Exercise 154.2*c*.

player 2's strategy, (*b*) in the subgame following the history C, player 2's strategy E induces the strategy E, which is optimal given player 1's strategy, and (*c*) in the subgame following the history (C, E), player 1's strategy DG induces the strategy G, which is optimal.

171.2 Finding subgame perfect equilibria

The game in Exercise 154.2*a* has a unique subgame perfect equilibrium, (C, FG).

The game in Exercise 154.2*c* has a unique subgame perfect equilibrium in which Karl's strategy is E, Rosa's strategy is to choose B after the history R, B after the history (E, B), and H after the history (E, H), and Ernesto's strategy is to chooses B after the history (R, B), H after the history (R, H), and H after the history E. (The outcome is that Karl chooses Ernesto to move first, he chooses H, and then Rosa chooses H.)

The game in Figure 171.1 has six subgame perfect equilibria: (C, EG), (D, EG), (C, EH), (D, FG), (C, FH), (D, FH).

171.3 Voting by alternating veto

The game has a unique subgame perfect equilibrium $(\not{Z}, \not{Y}\not{X}\not{X})$. The outcome is that action Y is taken.

Thus the Nash equilibrium $(\not{Z}, \not{Z}\not{X}\not{X})$ (see Exercise 161.2) is not a subgame perfect equilibrium. However, this equilibrium generates the same outcome as the unique subgame perfect equilibrium.

If player 2 prefers Y to X to Z then in the unique subgame perfect equilibrium of the game in which player 1 moves first the outcome is that X is chosen, while in the unique subgame perfect equilibrium of the game in which player 2 moves first the outcome is that Y is chosen. (For all other strict preferences of player 2 (i.e.

preferences in which player 2 is not indifferent between any pair of policies) the outcome of the subgame perfect equilibria of the two games are the same.)

171.4 Burning a bridge

An extensive game that models the situation has the same structure as the entry game in Figure 154.1 in the book. The challenger is army 1, the incumbent army 2. The action *In* corresponds to attacking; *Acquiesce* corresponds to retreating. The game has a single subgame perfect equilibrium, in which army 1 attacks, and army 2 retreats.

If army 2 burns the bridge, the game has a single subgame perfect equilibrium in which army 1 does not attack.

172.1 Sharing heterogeneous objects

Let $n = 2$ and $k = 3$, and call the objects a, b, and c. Suppose that the values person 1 attaches to the objects are 3, 2, and 1 respectively, while the values player 2 attaches are 1, 3, 2. If player 1 chooses a on the first round, then in any subgame perfect equilibrium player 2 chooses b, leaving player 1 with c on the second round. If instead player 1 chooses b on the first round, in any subgame perfect equilibrium player 2 chooses c, leaving player 1 with a on the second round. Thus in every subgame perfect equilibrium player 1 chooses b on the first round (though she values a more highly.)

Now I argue that for any preferences of the players, $G(2,3)$ has a subgame perfect equilibrium of the type described in the exercise. For any object chosen by player 1 in round 1, in any subgame perfect equilibrium player 2 chooses her favorite among the two objects remaining in round 2. Thus player 2 never obtains the object she least prefers; in any subgame perfect equilibrium, player 1 obtains that object. Player 1 can ensure she obtains her more preferred object of the two remaining by choosing that object on the first round. That is, there is a subgame perfect equilibrium in which on the first round player 1 chooses her more preferred object out of the set of objects excluding the object player 2 least prefers, and on the last round she obtains x_3. In this equilibrium, player 2 obtains the object less preferred by player 1 out of the set of objects excluding the object player 2 least prefers. That is, player 2 obtains x_2. (Depending on the players' preferences, the game also may have a subgame perfect equilibrium in which player 1 chooses x_3 on the first round.)

172.2 An entry game with a financially-constrained firm

a. Consider the last period, after any history. If the incumbent chooses to fight, the challenger's best action is to exit, in which case both firms obtain the profit zero. If the incumbent chooses to cooperate, the challenger's best action is to stay in, in which case both firms obtain the profit $C > 0$. Thus the incumbent's best action at the start of the period is to cooperate.

Now consider period $T - 1$. Regardless of the outcome in this period, the incumbent will cooperate in the last period, and the challenger will stay in (as we have just argued). Thus each player's action in the period affects its payoff only because it affects its profit in the period. Thus by the same argument as for the last period, in period $T - 1$ the incumbent optimally cooperates, and the challenger optimally stays in if the incumbent cooperates. If, in period $T - 1$, the incumbent fights, then the challenger also optimally stays in, because in the last period it obtains $C > F$.

Working back to the start of the game, using the same argument in each period, we conclude that in every period before the last the incumbent cooperates and the challenger stays in regardless of the incumbent's action. Given $C > f$, the challenger optimally enters at the start of the game.

That is, the game has a unique subgame perfect equilibrium, in which

- the challenger enters at the start of the game, exits in the last period if the challenger fights in that period, and stays in after every other history after which it moves
- the incumbent cooperates after every history after which it moves.

The incumbent's payoff in this equilibrium is TC and the challenger's payoff is $TC - f$.

b. First consider the incumbent's action after the history in which the challenger enters, the incumbent fights in the first $T - 2$ periods, and in each of these periods the challenger stays in. Denote this history h_{T-2}. If the incumbent fights after h_{T-2}, the challenger exits (it has no alternative), and the incumbent's profit in the last period is M. If the incumbent cooperates after h_{T-2} then by the argument for the game in part *a*, the challenger stays in, and in the last period the incumbent also cooperates and the challenger stays in. Thus the incumbent's payoff if it cooperates after the history h_{T-2} is $2C$. Because $M > 2C$, we conclude that the incumbent fights after the history h_{T-2}.

Now consider the incumbent's action after the history in which the challenger enters, the incumbent fights in the first $T - 3$ periods, and in each period the challenger stays in. Denote this history h_{T-3}. If the incumbent fights after h_{T-3}, we know, by the previous paragraph, that if the challenger stays in then the incumbent will fight in the next period, driving the challenger out. Thus the challenger will obtain an additional profit of $-F$ if it stays in and 0 if it exits. Consequently the challenger exits if the incumbent fights after h_{T-3}, making a fight by the incumbent optimal (it yields the incumbent the additional profit $2M$).

Working back to the first period we conclude that the incumbent fights and the challenger exits. Thus the challenger's optimal action at the start of the game is to stay out.

In summary, the game has a unique subgame perfect equilibrium, in which

- the challenger stays out at the start of the game, exits after any history in which the incumbent fought in every period, exits in the last period if the incumbent fights in that period, and stays in after every other history.
- the incumbent fights after the challenger enters and after any history in which it has fought in every period, and cooperates after every other history.

The incumbent's payoff in this equilibrium is TM and the challenger's payoff is 0.

173.2 Dollar auction

The game is shown in Figure 80.1. It has four subgame perfect equilibria. In all the equilibria player 2 passes after player 1 bids $2. After other histories the actions in the equilibria are as follows.

- Player 1 bids $3 after the history ($1, $2), player 2 passes after the history $1, and player 1 bids $1 at the start of the game.
- Player 1 passes after the history ($1, $2), player 2 passes after the history $1, and player 1 bids $1 at the start of the game.
- Player 1 passes after the history ($1, $2), player 2 bids $2 after the history $1, and player 1 passes at the start of the game.
- Player 1 passes after the history ($1, $2), player 2 bids $2 after the history $1, and player 1 bids $2 at the start of the game.

There are three subgame perfect equilibrium outcomes: player 1 passes at the start of the game (player 2 gets the object without making any payment), player 1 bids $1 and then player 2 passes (player 1 gets the object for $1), and player 1 bids $2 and then player 2 passes (player 1 gets the object for $2).

174.2 Firm–union bargaining

a. The following extensive game models the situation.

Players The firm and the union.

Terminal histories All sequences of the form (w, Y, L) and (w, N) for nonnegative numbers w and L (where w is a wage, Y means accept, N means reject, and L is the number of workers hired).

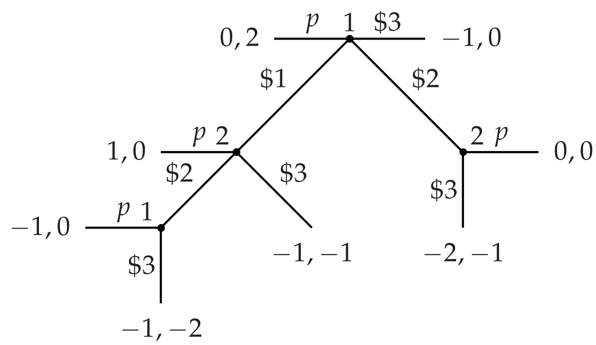


Figure 80.1 The extensive form of the dollar auction for $w = 3$ and $v = 2$. A pass is denoted p.

Player function $P(\varnothing)$ is the union, and, for any nonnegative number w, $P(w)$ and $P(w, Y)$ are the firm.

Preferences The firm's preferences are represented by its profit, and the union's preferences are represented by the value of wL (which is zero after any history (w, N)).

b. First consider the subgame following a history (w, Y), in which the firm accepts the wage demand w. In a subgame perfect equilibrium, the firm chooses L to maximize its profit, given w. For $L \leq 50$ this profit is $L(100 - L) - wL$, or $L(100 - w - L)$. This function is a quadratic in L that is zero when $L = 0$ and when $L = 100 - w$ and reaches a maximum in between. Thus the value of L that maximizes the firm's profit is $\frac{1}{2}(100 - w)$ if $w \leq 100$, and 0 if $w > 100$.

Given the firm's optimal action in such a subgame, consider the subgame following a history w, in which the firm has to decide whether to accept or reject w. For any w the firm's profit, given its subsequent optimal choice of L, is nonnegative; if $w < 100$ this profit is positive, while if $w \geq 100$ it is 0. Thus in a subgame perfect equilibrium, the firm accepts any demand $w < 100$ and either accepts or rejects any demand $w \geq 100$.

Finally consider the union's choice at the beginning of the game. If it chooses $w < 100$ then the firm accepts and chooses $L = (100 - w)/2$, yielding the union a payoff of $w(100 - w)/2$. If it chooses $w > 100$ then the firm either accepts and chooses $L = 0$ or rejects; in both cases the union's payoff is 0. Thus the best value of w for the union is the number that maximizes $w(100 - w)/2$. This function is a quadratic that is zero when $w = 0$ and when $w = 100$ and reaches a maximum in between; thus its maximizer is $w = 50$.

In summary, in a subgame perfect equilibrium the union's strategy is $w = 50$, and the firm's strategy accepts any demand $w < 100$ and chooses $L = (100 - w)/2$, and either rejects a demand $w \geq 100$ or accepts such a demand and chooses $L = 0$. The outcome of any equilibrium is that the union demands

$w = 50$ and the firm chooses $L = 25$.

c. Yes. In any subgame perfect equilibrium the union's payoff is $(50)(25) = 1250$ and the firm's payoff is $(25)(75) - (50)(25) = 625$. Thus both parties are better off at the outcome (w, L) than they are in the unique subgame perfect equilibrium if and only if $L \leq 50$ and

$$\begin{aligned} wL &> 1250 \\ L(100 - L) - wL &> 625 \end{aligned}$$

or $L \geq 50$ and

$$\begin{aligned} wL &> 1250 \\ 2500 - wL &> 625. \end{aligned}$$

These conditions are satisfied for a nonempty set of pairs (w, L). For example, if $L = 50$ the conditions are satisfied by $25 < w < 37.5$; if $L = 100$ they are satisfied by $12.5 < w < 18.75$.

d. There are many Nash equilibria in which the firm "threatens" to reject high wage demands. In one such Nash equilibrium the firm threatens to reject any positive wage demand. In this equilibrium the union's strategy is $w = 0$, and the firm's strategy rejects any demand $w > 0$, and accepts the demand $w = 0$ and chooses $L = 50$. (The union's payoff is 0 no matter what demand it makes; given $w = 0$, the firm's optimal action is $L = 50$.)

175.1 The "rotten kid theorem"

The situation is modeled by the following extensive game.

Players The parent and the child.

Terminal histories The set of sequences (a, t), where a (an action of the child) and t (a transfer from the parent to the child) are numbers.

Player function $P(\varnothing)$ is the child, $P(a)$ is the parent for every value of a.

Preferences The child's preferences are represented by the payoff function $c(a) + t$ and the parent's preferences are represented by the payoff function $\min\{p(a) - t, c(a) + t\}$.

To find the subgame perfect equilibria of this game, first consider the parent's optimal actions in the subgames of length 1. Consider the subgame following the choice of a by the child. We have $p(a) > c(a)$ (by assumption), so if the parent makes no transfer her payoff is $c(a)$. If she transfers \$1 to the child then her payoff increases to $c(a) + 1$. As she increases the transfer her payoff increases until $p(a) - t = c(a) + t$; that is, until $t = \frac{1}{2}(p(a) - c(a))$. (If she increases the transfer any more, she has less money than her child.) Thus the parent's optimal action in the subgame following the choice of a by the child is $t = \frac{1}{2}(p(a) - c(a))$.

Now consider the whole game. Given the parent's optimal action in each subgame, a child who chooses a receives the payoff $c(a) + \frac{1}{2}(p(a) - c(a)) = \frac{1}{2}(p(a) + c(a))$. Thus in a subgame perfect equilibrium the child chooses the action that maximizes $p(a) + c(a)$, the sum of her own private income and her parent's income.

175.2 Comparing simultaneous and sequential games

a. Denote by (a_1^*, a_2^*) a Nash equilibrium of the strategic game in which player 1's payoff is maximal in the set of Nash equilibria. Because (a_1^*, a_2^*) is a Nash equilibrium, a_2^* is a best response to a_1^*. By assumption, it is the only best response to a_1^*. Thus if player 1 chooses a_1^* in the extensive game, player 2 must choose a_2^* in any subgame perfect equilibrium of the extensive game. That is, by choosing a_1^*, player 1 is assured of a payoff of at least $u_1(a_1^*, a_2^*)$. Thus in any subgame perfect equilibrium player 1's payoff must be at least $u_1(a_1^*, a_2^*)$.

b. Suppose that $A_1 = \{T, B\}$, $A_2 = \{L, R\}$, and the payoffs are those given in Figure 82.1. The strategic game has a unique Nash equilibrium, (T, L), in which player 2's payoff is 1. The extensive game has a unique subgame perfect equilibrium, (B, LR) (where the first component of player 2's strategy is her action after the history T and the second component is her action after the history B). In this subgame perfect equilibrium player 2's payoff is 2.

	L	R
T	$1,1$	$3,0$
B	$0,0$	$2,2$

Figure 82.1 The payoffs for the example in Exercise 175.2*a*.

c. Suppose that $A_1 = \{T, B\}$, $A_2 = \{L, R\}$, and the payoffs are those given in Figure 83.1. The strategic game has a unique Nash equilibrium, (T, L), in which player 2's payoff is 2. A subgame perfect equilibrium of the extensive game is (B, RL) (where the first component of player 2's strategy is her action after the history T and the second component is her action after the history B). In this subgame perfect equilibrium player 1's payoff is 1. (If you read Chapter 4, you can find the mixed strategy Nash equilibria of the strategic game; in all these equilibria, as in the pure strategy Nash equilibrium, player 1's expected payoff exceeds 1.)

	L	R
T	$2,2$	$0,2$
B	$1,1$	$3,0$

Figure 83.1 The payoffs for the example in Exercise 175.2*b*.

176.1 Subgame perfect equilibria of ticktacktoe

Player 2 puts her O in the center. If she does so, each player has a strategy that guarantees at least a draw in the subgame. Player 1 guarantees at least a draw by next marking one of the two squares adjacent to her first X and then subsequently completing a line of X's, if possible, or, if not possible, blocking a line of O's, if necessary, or, if not necessary, moving arbitrarily. Player 2 guarantees at least a draw as follows.

- If player 1's second X is adjacent to her first X or is in a corner not diagonally opposite player 1's first X, player 2 should, on each move, either complete a line of O's, if possible, or, if not possible, block a line of X's, if necessary, or, if not necessary, move arbitrarily.
- If player 1's second X is in some other square then player 2 should, on her second move, mark one of the corners not diagonally opposite player 1's first X, and then, on each move, either complete a line of O's, if possible, or, if not possible, block a line of X's, if necessary, or, if not necessary, move arbitrarily.

For each of player 2's other opening moves, player 1 has a strategy in the subgame that wins, as follows.

- Suppose player 2 marks the corner diagonally opposite player 1's first X. If player 1 next marks another corner, player 2 must next mark the square between player 1's two X's; by marking the remaining corner, player 1 wins on her next move.
- Suppose player 2 marks one of the other corners. If player 1 next marks the corner diagonally opposite her first X, player 2 must mark the center, then player 1 must mark the remaining corner, leading her to win on her next move.
- Suppose player 2 marks one of the two squares adjacent to player 1's X. If player 1 next marks the center, player 2 must mark the corner opposite player 1's first X, in which case player 1 can mark the other square adjacent to her first X, leading her to win on her next move.
- Suppose player 2 marks one of the other squares, other than the center. If player 1 next marks the center, player 2 must mark the corner opposite player 1's first X, in which case player 1 can mark the corner that blocks a row of O's, leading her to win on her next move.

176.2 Toetacktick

The following strategy leads to either a draw or a win for player 1: mark the central square initially, and on each subsequent move mark the square symmetrically opposite the one just marked by the second player.

177.1 Three Men's Morris, or Mill

Number the squares 1 through 9, starting at the top left, working across each row. The following strategy of player 1 guarantees she wins, so that the subgame perfect equilibrium outcome is that she wins. First player 1 chooses the central square (5).

- Suppose player 2 then chooses a corner; take it to be square 1. Then player 1 chooses square 6. Now player 2 must choose square 4 to avoid defeat; player 1 must choose square 7 to avoid defeat; and then player 2 must choose square 3 to avoid defeat (otherwise player 1 can move from square 6 to square 3 on her next turn). If player 1 now moves from square 6 to square 9, then whatever player 2 does she can subsequently move her counter from square 5 to square 8 and win.

- Suppose player 2 then chooses a noncorner; take it to be square 2. Then player 1 chooses square 7. Now player 2 must choose square 3 to avoid defeat; player 1 must choose square 1 to avoid defeat; and then player 2 must choose square 4 to avoid defeat (otherwise player 1 can move from square 5 to square 4 on her next turn). If player 1 now moves from square 7 to square 8, then whatever player 2 does she can subsequently move from square 8 to square 9 and win.

6 Extensive Games with Perfect Information: Illustrations

180.1 Nash equilibria of the ultimatum game

For *every* amount x there are Nash equilibria in which person 1 offers x. For example, for any value of x there is a Nash equilibrium in which person 1's strategy is to offer x and person 2's strategy is to accept x and any offer more favorable, and reject every other offer. (Given person 2's strategy, person 1 can do no better than offer x. Given person 1's strategy, person 2 should accept x; whether person 2 accepts or rejects any other offer makes no difference to her payoff, so that rejecting all less favorable offers is, in particular, optimal.)

180.2 Subgame perfect equilibria of the ultimatum game with indivisible units

In this case each player has finitely many actions, and for both possible subgame perfect equilibrium strategies of player 2 there is an optimal strategy for player 1.

If player 2 accepts all offers then player 1's best strategy is to offer 0, as before.

If player 2 accepts all offers except 0 then player 1's best strategy is to offer one cent (which player 2 accepts).

Thus the game has two subgame perfect equilibria: one in which player 1 offers 0 and player 2 accepts all offers, and one in which player 1 offers one cent and player 2 accepts all offers except 0.

180.3 Dictator game and impunity game

Dictator game Person 2 has no choice; person 1 optimally chooses the offer 0.
Impunity game The analysis of the subgames of length one is the same as it is in the ultimatum game. That is, in any subgame perfect equilibrium person 2 either accepts all offers, or accepts all positive offers and rejects 0. Now consider the whole game. Regardless of person 2's behavior in the subgames, person 1's best action is to offer 0.

Thus the game has two subgame perfect equilibria. In both equilibria person 1 offers 0. In one equilibrium person 2 accepts all offers, and in the other equilibrium she accepts all positive offers and rejects 0. The outcome of the first equilibrium is that person 1 offers 0, which person 2 accepts; the outcome of the second equilibrium is that person 1 offers 0, which person 2 rejects. In both equilibria person 1's payoff is c and person 2's payoff is 0.

181.1 Variant of ultimatum game and impunity game with equity-conscious players

Ultimatum game First consider the optimal response of person 2 to each possible offer. If person 2 accepts an offer x her payoff is $x - \beta_2|(1-x) - x|$, while if she rejects an offer her payoff is 0. Thus she accepts an offer x if $x - \beta_2|(1-x) - x| > 0$, or

$$x - \beta_2|1 - 2x| > 0, \tag{86.1}$$

rejects an offer x if $x - \beta_2|1-2x| < 0$, and is indifferent between accepting and rejecting if $x - \beta_2|1-2x| = 0$.

Which values of x satisfy (86.1)? Because of the absolute value in the expression, we can conveniently consider the cases $x \leq \frac{1}{2}$ and $x > \frac{1}{2}$ separately.

- For $x \leq \frac{1}{2}$ the condition is $x - \beta_2(1-2x) > 0$, or $x > \beta_2/(1+2\beta_2)$.
- For $x \geq \frac{1}{2}$ the condition is $x + \beta_2(1-2x) > 0$, or $x(1-2\beta_2) + \beta_2 > 0$. The values of x that satisfy this inequality depend on whether β_2 is greater than or less than $\frac{1}{2}$.

  $\beta_2 \leq \frac{1}{2}$: All values of x satisfy the inequality.

  $\beta_2 > \frac{1}{2}$: The inequality is $x < \beta_2/(2\beta_2 - 1)$ (the right-hand side of which is less than 1 only if $\beta_2 > 1$).

In summary, person 2 accepts any offer x with $\beta_2/(1+2\beta_2) < x < \beta_2/(2\beta_2 - 1)$, may accept or reject the offers $\beta_2/(1+2\beta_2)$ and $\beta_2/(2\beta_2-1)$, and rejects any offer x with $x < \beta_2/(1+2\beta_2)$ or $x > \beta_2/(2\beta_2-1)$. The shaded region of Figure 86.1 shows, for each value of β_2, the set of offers that person 2 accepts. Note, in particular, that, for every value of β_2, person 2 accepts the offer $\frac{1}{2}$.

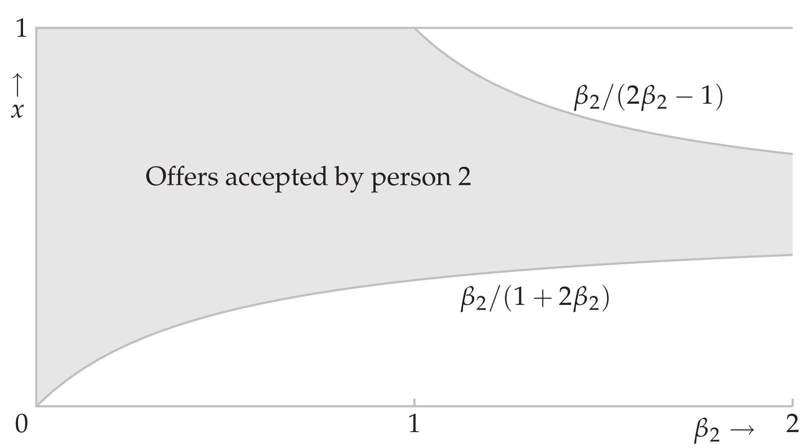


Figure 86.1 The set of offers x that person 2 accepts for each value of $\beta_2 \leq 2$ in the variant of the ultimatum game with equity-conscious players studied in Exercise 181.1.

Now consider person 1's decision. Her payoff is 0 if her offer is rejected and $1 - x - \beta_1|(1 - x) - x| = 1 - x - \beta_1|1 - 2x|$ if it is accepted. We can conveniently separate the analysis into three cases.

$\beta_1 < \frac{1}{2}$: Person 1's payoff when her offer x is accepted is positive for $0 \leq x < 1$ and is decreasing in x. Thus person 1's optimal offer is the smallest one that person 2 accepts. If person 2's strategy rejects the offer $\beta_2/(1 + 2\beta_2)$, then as in the analysis of the original game when person 2's strategy rejects 0, person 1 has no optimal response. Thus in any subgame perfect equilibrium person 2 accepts $\beta_2/(1 + 2\beta_2)$, and person 1 offers this amount.

$\beta_1 = \frac{1}{2}$: Person 1's payoff to an offer that is accepted is positive and constant from $x = 0$ to $x = \frac{1}{2}$, then decreasing. Thus if person 2 accepts the offer $\beta_2/(1+2\beta_2)$ then every offer x with $\beta_2/(1+2\beta_2) \leq x \leq \frac{1}{2}$ is optimal, while if person 2 rejects the offer $\beta_2/(1+2\beta_2)$ then every offer x with $\beta_2/(1+2\beta_2) < x \leq \frac{1}{2}$ is optimal.

$\beta_1 > \frac{1}{2}$: Person 1's payoff to an offer that is accepted is increasing up to $x = \frac{1}{2}$ and then decreasing, and is positive at $x = \frac{1}{2}$, so that her optimal offer is $\frac{1}{2}$ (which person 2 accepts).

We conclude that the set of subgame perfect equilibria depends on the values of β_1 and β_2, as follows.

$\beta_1 < \frac{1}{2}$: the set of subgame perfect equilibria is the set of all strategy pairs for which

- person 1 offers $\beta_2/(1 + 2\beta_2)$
- person 2 accepts all offers x with $\beta_2/(1 + 2\beta_2) \leq x < \beta_2/(2\beta_2 - 1)$, rejects all offers x with $x < \beta_2/(1 + 2\beta_2)$ or $x > \beta_2/(2\beta_2 - 1)$, and either accepts or rejects the offer $\beta_2/(2\beta_2 - 1)$.

$\beta_1 = \frac{1}{2}$: the set of subgame perfect equilibria is the set of all strategy pairs for which

- person 1's offer x satisfies $\beta_2/(1 + 2\beta_2) \leq x \leq \frac{1}{2}$
- person 2 accepts all offers x with $\beta_2/(1 + 2\beta_2) < x < \beta_2/(2\beta_2 - 1)$, rejects all offers x with $x < \beta_2/(1 + 2\beta_2)$ or $x > \beta_2/(2\beta_2 - 1)$, either accepts or rejects the offer $\beta_2/(2\beta_2 - 1)$, and either accepts or rejects the offer $\beta_2/(1 + 2\beta_2)$ unless person 1 makes this offer, in which case person 2 definitely accepts it.

$\beta_1 > \frac{1}{2}$: the set of subgame perfect equilibria is the set of all strategy pairs for which

- person 1 offers $\frac{1}{2}$

- person 2 accepts all offers x with $\beta_2/(1+2\beta_2) < x < \beta_2/(2\beta_2-1)$, rejects all offers x with $x < \beta_2/(1+2\beta_2)$ or $x > \beta_2/(2\beta_2-1)$, and either accepts or rejects the offer $\beta_2/(2\beta_2-1)$ and the offer $\beta_2/(1+2\beta_2)$.

The subgame perfect equilibrium outcomes are:

$\beta_1 < \frac{1}{2}$: person 1 offers $\beta_2/(1+2\beta_2)$, which person 2 accepts

$\beta_1 = \frac{1}{2}$: person 1 makes an offer x that satisfies $\beta_2/(1+2\beta_2) \leq x \leq \frac{1}{2}$, and person 2 accepts this offer

$\beta_1 > \frac{1}{2}$: person 1 offers $\frac{1}{2}$, which person 2 accepts.

In particular, in all cases the offer made by person 1 in equilibrium is accepted by person 2.

Impunity game First consider the optimal response of person 2 to each possible offer. If person 2 accepts an offer x her payoff is $x - \beta_2|(1-x)-x|$, while if she rejects an offer her payoff is $-\beta_2(1-x)$. Thus she accepts an offer x if $x - \beta_2|(1-x)-x| > -\beta_2(1-x)$, or

$$x(1-\beta_2) + \beta_2(1-|1-2x|) > 0, \tag{88.1}$$

rejects an offer x if $x(1-\beta_2) + \beta_2(1-|1-2x|) < 0$, and is indifferent between accepting and rejecting if $x(1-\beta_2) + \beta_2(1-|1-2x|) = 0$.

As before, we can conveniently consider the cases $x \leq \frac{1}{2}$ and $x > \frac{1}{2}$ separately.

- For $x \leq \frac{1}{2}$ the condition is $x(1+\beta_2) > 0$, or $x > 0$.
- For $x \geq \frac{1}{2}$ the condition is $x(1-3\beta_2) + 2\beta_2 > 0$, which is satisfied by all values of x if $\beta_2 \leq \frac{1}{3}$, and for all x with $x < 2\beta_2/(3\beta_2-1)$ if $\beta_2 > \frac{1}{3}$.

In summary, person 2 accepts any offer x with $0 < x < 2\beta_2/(3\beta_2-1)$, may accept or reject the offers 0 and $2\beta_2/(3\beta_2-1)$, and rejects any offer x with $x > 2\beta_2/(3\beta_2-1)$.

Now consider person 1. If she offers x, her payoff is

$$\begin{cases} 1-x-\beta_1|1-2x| & \text{if person 1 accepts } x \\ 1-x-\beta_1(1-x) & \text{if person 1 rejects } x. \end{cases}$$

If $\beta_1 < \frac{1}{2}$ then in both cases person 1's payoff is decreasing in x; for $x = 0$ the payoffs are equal. Thus, given person 2's optimal strategy, in any subgame perfect equilibrium person 1's optimal offer is 0, which person 2 may accept or reject.

If $\beta_1 = \frac{1}{2}$ then person 1's payoff when person 2 accepts x is constant from 0 to $\frac{1}{2}$, then decreases. Her payoff when person 2 rejects x is decreasing in x, and the two payoffs are equal when $x = 0$. Thus the optimal offers of person 1 are 0, which person 2 may accept or reject, and any x with $0 < x \leq \frac{1}{2}$, which person 2 accepts.

If $\beta_1 > \frac{1}{2}$ then person 1's highest payoff is obtained when $x = \frac{1}{2}$, which person 2 accepts. Thus $x = \frac{1}{2}$ is her optimal offer.

In summary, in all subgame perfect equilibria the strategy of person 2 accepts all offers x with $0 < x < 2\beta_2/(3\beta_2 - 1)$, rejects all offers x with $x > 2\beta_2/(3\beta_2 - 1)$, and either accepts or rejects the offer 0 and the offer $2\beta_2/(3\beta_2 - 1)$. Person 1's offer depends on the value of β_1 and β_2, as follows.

$\beta_1 < \frac{1}{2}$: person 1 offers 0

$\beta_1 = \frac{1}{2}$: person 1's offer x satisfies $0 \le x \le \frac{1}{2}$

$\beta_1 > \frac{1}{2}$: person 1 offers $x = \frac{1}{2}$.

The subgame perfect equilibrium outcomes are:

$\beta_1 < \frac{1}{2}$: person 1 offers 0, which person 2 may accept or reject

$\beta_1 = \frac{1}{2}$: person 1 either offers 0, which person 2 either accepts or rejects, or makes an offer x that satisfies $0 < x \le \frac{1}{2}$, which person 2 accepts

$\beta_1 > \frac{1}{2}$: person 1 offers $\frac{1}{2}$, which person 2 accepts.

In particular, if $\beta_1 \le \frac{1}{2}$ there are equilibria in which person 1 offers 0, and person 2 rejects this offer.

Comparison of subgame perfect equilibria of ultimatum and impunity games The equilibrium outcomes of the two games are the same unless $0 < \beta_1 \le \frac{1}{2}$, or $\beta_1 = 0$ and $\beta_2 > 0$, in which case person 1's offer in the ultimatum game is higher than her offer in the impunity game.

183.1 Bargaining over two indivisible objects

An extensive game that models the situation is shown in Figure 89.1, where the action $(x, 2 - x)$ of player 1 means that she keeps x objects and offers $2 - x$ objects to player 2.

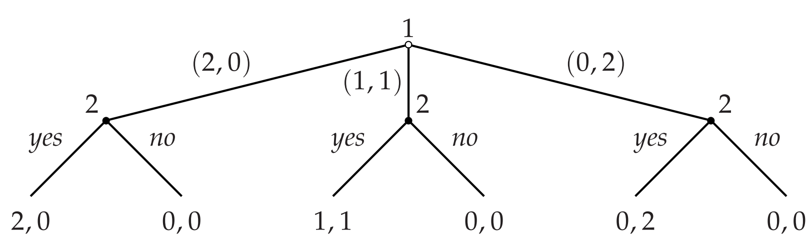


Figure 89.1 An extensive game that models the procedure described in Exercise 183.1 for allocating two identical indivisible objects between two people.

Denote a strategy of player 2 by a triple *abc*, where *a* is the action (*y* or *n*, for *yes* or *no*) taken after the offer $(2, 0)$, *b* is the action taken after the offer $(1, 1)$, and *c* is the action taken after the offer $(0, 2)$.

The subgame perfect equilibria of the game are $((2, 0), yyy)$ (resulting in the division $(2, 0)$), and $((1, 1), nyy)$ (resulting in the division $(1, 1)$).

The strategic form of the game is given in Figure 90.1. Its Nash equilibria are $((2,0), yyy)$, $((2,0), yyn)$, $((2,0), yny)$, $((2,0), ynn)$, $((2,0), nny)$, $((1,1), nyy)$, $((1,1), nyn)$, $((0,2), nny)$, and $((2,0), nnn)$. The first four equilibria result in the division $(2,0)$, the next two result in the division $(1,1)$, and the last two result in the divisions $(0,2)$ and $(0,0)$ respectively.

	yyy	yyn	yny	ynn	nyy	nyn	nny	nnn
$(2,0)$	$2,0$	$2,0$	$2,0$	$2,0$	$0,0$	$0,0$	$0,0$	$0,0$
$(1,1)$	$1,1$	$1,1$	$0,0$	$0,0$	$1,1$	$1,1$	$0,0$	$0,0$
$(0,2)$	$0,2$	$0,0$	$0,2$	$0,0$	$0,2$	$0,0$	$0,2$	$0,0$

Figure 90.1 The strategic form of the game in Figure 89.1

The outcomes $(0,2)$ and $(0,0)$ are generated by Nash equilibria but not by any subgame perfect equilibria.

183.2 Dividing a cake fairly

a. If player 1 divides the cake unequally then player 2 chooses the larger piece. Thus in any subgame perfect equilibrium player 1 divides the cake into two pieces of equal size.

b. In a subgame perfect equilibrium player 2 chooses P_2 over P_1, so she likes P_2 at least as much as P_1. To show that in fact she is indifferent between P_1 and P_2, suppose to the contrary that she prefers P_2 to P_1. I argue that in this case player 1 can slightly increase the size of P_1 in such a way that player 2 still prefers the now-slightly-smaller P_2. Precisely, by the continuity of player 2's preferences, there is a subset P of P_2, not equal to P_2, that player 2 prefers to its complement $C \setminus P$ (the remainder of the cake). Thus if player 1 makes the division $(C \setminus P, P)$, player 2 chooses P. The piece P_1 is a subset of $C \setminus P$ not equal to $C \setminus P$, so player 1 prefers $C \setminus P$ to P_1. Thus player 1 is better off making the division $(C \setminus P, P)$ than she is making the division (P_1, P_2), contradicting the fact that (P_1, P_2) is a subgame perfect equilibrium division. We conclude that in any subgame perfect equilibrium player 2 is indifferent between the two pieces into which player 1 divides the cake.

I now argue that player 1 likes P_1 as least as much as P_2. Suppose that, to the contrary, she prefers P_2 to P_1. If she deviates and makes a division $(P, C \setminus P)$ in which P is slightly bigger than P_1 but still such that she prefers $C \setminus P$ to P, then player 2, who is indifferent between P_1 and P_2, chooses P, leaving $C \setminus P$ for player 1, who prefers it to P and hence to P_1. Thus in any subgame perfect equilibrium player 1 likes P_1 at least as much as P_2.

To show that player 1 may strictly prefer P_1 to P_2, consider a cake that is perfectly homogeneous except for the presence of a single cherry. Assume that player 2 values a piece of the cherry in exactly the same way that she values a piece of the cake of the same size, while player 1 prefers a piece of the cherry to a piece of the cake of the same size. Then there is a subgame perfect equilibrium in which player 1 divides the cake equally, with one piece containing all of the cherry, and player 2 chooses the piece without the cherry. (In this equilibrium, as in all equilibria, player 2 is indifferent between the two pieces—but note that there is no subgame perfect equilibrium in which she chooses the piece with the cherry in it. A strategy pair in which she acts in this way is not an equilibrium, because player 1 can deviate and increase slightly the size of the cherryless piece of cake, inducing player 2 to choose that piece.)

183.3 Holdup game

The game is defined as follows.

Players Two people, person 1 and person 2.

Terminal histories The set of all sequences (low, x, Z), where x is a number with $0 \leq x \leq c_L$ (the amount of money that person 1 offers to person 2 when the pie is small), and $(high, x, Z)$, where x is a number with $0 \leq x \leq c_H$ (the amount of money that person 1 offers to person 2 when the pie is large) and Z is either Y ("yes, I accept") or N ("no, I reject").

Player function $P(\varnothing) = 2$, $P(low) = P(high) = 1$, and $P(low, x) = P(high, x) = 2$ for all x.

Preferences Person 1's preferences are represented by payoffs equal to the amounts of money she receives, equal to $c_L - x$ for any terminal history (low, x, Y) with $0 \leq x \leq c_L$, equal to $c_H - x$ for any terminal history $(high, x, Y)$ with $0 \leq x \leq c_H$, and equal to 0 for any terminal history (low, x, N) with $0 \leq x \leq c_L$ and for any terminal history $(high, x, N)$ with $0 \leq x \leq c_H$. Person 2's preferences are represented by payoffs equal to $x - L$ for the terminal history (low, x, Y), $x - H$ for the terminal history $(high, x, Y)$, $-L$ for the terminal history (low, x, N), and $-H$ for the terminal history $(high, x, N)$.

186.1 Stackelberg's duopoly game with quadratic costs

From Exercise 57.2, the best response function of firm 2 is the function b_2 defined by

$$b_2(q_1) = \begin{cases} \frac{1}{4}(\alpha - q_1) & \text{if } q_1 \leq \alpha \\ 0 & \text{if } q_1 > \alpha. \end{cases}$$

Firm 1's subgame perfect equilibrium strategy is the value of q_1 that maximizes $q_1(\alpha - q_1 - b_2(q_1)) - q_1^2$, or $q_1(\alpha - q_1 - \frac{1}{4}(\alpha - q_1)) - q_1^2$, or $\frac{1}{4}q_1(3\alpha - 7q_1)$. The maximizer is $q_1 = \frac{3}{14}\alpha$.

We conclude that the game has a unique subgame perfect equilibrium, in which firm 1's strategy is the output $\frac{3}{14}\alpha$ and firm 2's strategy is its best response function b_2.

The outcome of the subgame perfect equilibrium is that firm 1 produces $q_1^* = \frac{3}{14}\alpha$ units of output and firm 2 produces $q_2^* = b_2(\frac{3}{14}\alpha) = \frac{11}{56}\alpha$ units. In a Nash equilibrium of Cournot's (simultaneous-move) game each firm produces $\frac{1}{5}\alpha$ (see Exercise 57.2). Thus firm 1 produces more in the subgame perfect equilibrium of the sequential game than it does in the Nash equilibrium of Cournot's game, and firm 2 produces less.

188.1 Stackelberg's duopoly game with fixed costs

We have $f < (\alpha - c)^2/16$ ($f = 4$; $(\alpha - c)^2/16 = 9$), so the best response function of firm 2 takes the form shown in Figure 24.1 (in the solution to Exercise 57.3). To determine the subgame perfect equilibrium we need to compare firm 1's profit when it produces $\overline{q} = 8$ units of output, so that firm 2 produces 0, with its profit when it produces the output that maximizes its profit on the positive part of firm 2's best response function.

If firm 1 produces 8 units of output and firm 2 produces 0, firm 1's profit is $8(12 - 8) = 32$. Firm 1's best output on the positive part of firm 2's best response function is $\frac{1}{2}(\alpha - c) = 6$. If it produces this output then firm 2 produces $\frac{1}{2}(\alpha - c - q_1) = \frac{1}{2}(12 - 6) = 3$, and firm 1's profit is $6(12 - 9) = 18$. Thus firm 1's profit is higher when it produces enough to induce firm 2 to produce zero. We conclude that the game has a unique subgame perfect equilibrium, in which firm 1's strategy is to produce 8 units, and firm 2's strategy is to produce $\frac{1}{2}(\alpha - c - q_1) = \frac{1}{2}(12 - q_1)$ units if firm 1 produces $q_1 < 8$ and 0 if firm 1 produces $q_1 \geq 8$ units.

189.1 Sequential variant of Bertrand's duopoly game

a. *Players* The two firms.

Terminal histories The set of all sequences (p_1, p_2) of prices (where each p_i is a nonnegative number).

Player function $P(\varnothing) = 1$ and $P(p_1) = 2$ for all p_1.

Preferences The payoff of each firm i to the terminal history (p_1, p_2) is its profit

$$\begin{cases} (p_i - c)D(p_i) & \text{if } p_i < p_j \\ \frac{1}{2}(p_i - c)D(p_i) & \text{if } p_i = p_j \\ 0 & \text{if } p_i > p_j, \end{cases}$$

where j is the other firm.

b. A strategy of firm 1 is a price (e.g. the price c). A strategy of firm 2 is a function that associates a price with every price chosen by firm 1 (e.g.

$s_2(p_1) = p_1 - 1$, the strategy in which firm 2 always charges 1 cent less than firm 1).

c. First consider firm 2's best responses to each price p_1 chosen by firm 1.

- If $p_1 < c$, any price greater than p_1 is a best response for firm 2.
- If $p_1 = c$, any price at least equal to c is a best response for firm 2.
- If $p_1 = c + 1$, firm 2's unique best response is to set the same price.
- If $p_1 > c + 1$, firm 2's unique best response is to set the price $\min\{p^m, p_1 - 1\}$ (where p^m is the monopoly price).

Now consider the optimal action of firm 1. Given firm 2's best responses,

- if $p_1 < c$, firm 1's profit is positive
- if $p_1 = c$, firm 1's profit is zero
- if $p_1 = c + 1$, firm 1's profit is positive
- if $p_1 > c + 1$, firm 1's profit is zero.

Thus the only price p_1 for which there is a best response of firm 2 that leads to a positive profit for firm 1 is $c + 1$.

We conclude that in every subgame perfect equilibrium firm 1's strategy is $p_1 = c + 1$, and firm 2's strategy assigns to each price chosen by firm 1 one of its best responses, so that firm 2's strategy takes the form

$$s_2(p_1) = \begin{cases} k(p_1) & \text{if } p_1 < c \\ k' & \text{if } p_1 = c \\ c+1 & \text{if } p_1 = c+1 \\ \min\{p^m, p_1 - 1\} & \text{if } p_1 > c+1 \end{cases}$$

where $k(p_1) > p_1$ for all p_1 and $k' \geq c$.

The outcome of every subgame perfect equilibrium is that both firms choose the price $c + 1$.

193.1 Three interest groups buying votes

a. Consider the possibility of a subgame perfect equilibrium in which bill X passes. In any such equilibrium, groups Y and Z make no payments. But now given that Y makes no payments and that $V_X = V_Z$, group Z can match X's payments to the two legislators to whom X's payments are smallest, and gain the passage of bill Z. Thus there is no subgame perfect equilibrium in which bill X passes. Similarly there is no subgame perfect equilibrium in which bill Y passes. Thus in every subgame perfect equilibrium bill Z passes.

b. By making payments of more than 50 to each legislator, group X ensures that neither group Y nor group Z can profitably buy the passage of its favorite bill. (In any subgame perfect equilibrium, group X's payments to each legislator are exactly 50.) Thus in every subgame perfect equilibrium the outcome is that bill X is passed.

c. For any payments of group X that sum to at most 300, group Y can make payments that are (*i*) at least as high to at least two legislators and (*ii*) high enough that group Z cannot buy off more than one legislator. (Take the two legislators to whom group X pays the least. Let them be legislators 1 and 2, and denote group X's payments x_1 and x_2; suppose that $x_1 \geq x_2$. Group Y pays $x_1 + 1$ to legislator 1 and $200 - x_1$ to legislator 2.) Thus in every subgame perfect equilibrium the outcome is that bill Y is passed.

193.2 Interest groups buying votes under supermajority rule

a. However group X allocates payments summing to 700, group Y can buy off five legislators for at most 500. Thus in any subgame perfect equilibrium neither group makes any payment, and bill Y is passed.

b. If group X pays each legislator 80 then group Y is indifferent between buying off five legislators, in which case bill Y is passed, and in making no payments, in which case bill X is passed. If group Y makes no payments then X is selected, and group X is better off than it is if it makes no payments. There is no subgame perfect equilibrium in which group Y buys off five legislators, because if it were to do so group X could pay each legislator slightly more than 80 to ensure the passage of bill X. Thus in every subgame perfect equilibrium group X pays each legislator 80, group Y makes no payments, and bill X is passed.

c. If only a simple majority is required to pass a bill, in case *a* the outcome under majority rule is the same as it is when five votes are required.

In case *b*, group X needs to pay each legislator 100 in order to prevent group Y from winning. If it does so, its total payments are less than V_X, so doing so is optimal. Thus in this case the payment to each legislator is *higher* under majority rule.

193.3 Sequential positioning by two political candidates

The following extensive game models the situation.

Players The candidates.

Terminal histories The set of all sequences $(x_1, \ldots, x_n)$, where x_i is a position of candidate i (a number) for $i = 1, \ldots, n$.

Player function $P(\varnothing) = 1$, $P(x_1) = 2$ for all x_1, $P(x_1, x_2) = 3$ for all (x_1, x_2), $\ldots$, $P(x_1, \ldots, x_{n-1}) = n$ for all $(x_1, \ldots, x_{n-1})$.

Preferences Each candidate's preferences are represented by a payoff function that assigns n to every terminal history in which she wins outright, k to every terminal history in which she ties for first place with $n - k$ other candidates, for $1 \leq k \leq n - 1$, and 0 to every terminal history in which she loses, where positions attract votes as in Hotelling's model of electoral competition (Section 3.3).

This game has a finite horizon, so we may use backward induction to find its subgame perfect equilibria. Suppose there are two candidates. First consider candidate 2's best response to each strategy of candidate 1. Suppose candidate 1's strategy is m. Then candidate 2 loses if she chooses any position different from m and ties with candidate 1 if she chooses m. Thus candidate 2's best response to m is m. Now suppose candidate 1's strategy is $x_1 \neq m$. Then candidate 2 wins if she chooses any position between x_1 and $2m - x_1$; thus every such position is a best response.

Given candidate 2's best responses, the best strategy for candidate 1 is m, leading to a tie. (Every other strategy of candidate 1 leads her to lose.)

We conclude that in every subgame perfect equilibrium candidate 1's strategy is m; candidate 2's strategy chooses m after the history m and some position between x_1 and $2m - x_1$ after any other history x_1.

193.4 Sequential positioning by three political candidates

The following extensive game models the situation.

Players The candidates.

Terminal histories The set of all sequences $(x_1, \ldots, x_n)$, where x_i is either *Out* or a position of candidate i (a number) for $i = 1, \ldots, n$.

Player function $P(\varnothing) = 1$, $P(x_1) = 2$ for all x_1, $P(x_1, x_2) = 3$ for all (x_1, x_2), $\ldots$, $P(x_1, \ldots, x_{n-1}) = n$ for all $(x_1, \ldots, x_{n-1})$.

Preferences Each candidate's preferences are represented by a payoff function that assigns n to every terminal history in which she wins, k to every terminal history in which she ties for first place with $n - k$ other candidates, for $1 \leq k \leq n - 1$, 0 to every terminal history in which she stays out, and -1 to every terminal history in which she loses, where positions attract votes as in Hotelling's model of electoral competition (Section 3.3).

When there are two candidates the analysis of the subgame perfect equilibria is similar to that in the previous exercise. In every subgame perfect equilibrium candidate 1's strategy is m; candidate 2's strategy chooses m after the history m, some position between x_1 and $2m - x_1$ after the history x_1 for any position x_1, and any position after the history *Out*.

Now consider the case of three candidates when the voters' favorite positions are distributed uniformly from 0 to 1. I claim that every subgame perfect equilibrium results in the first candidate's entering at $\frac{1}{2}$, the second candidate's staying out, and the third candidate's entering at $\frac{1}{2}$.

To show this, first consider the best response of candidate 3 to each possible pair of actions of candidates 1 and 2. Figure 96.1 illustrates these optimal actions in every case that candidate 1 enters. (If candidate 1 does not enter then the subgame is exactly the two-candidate game.)

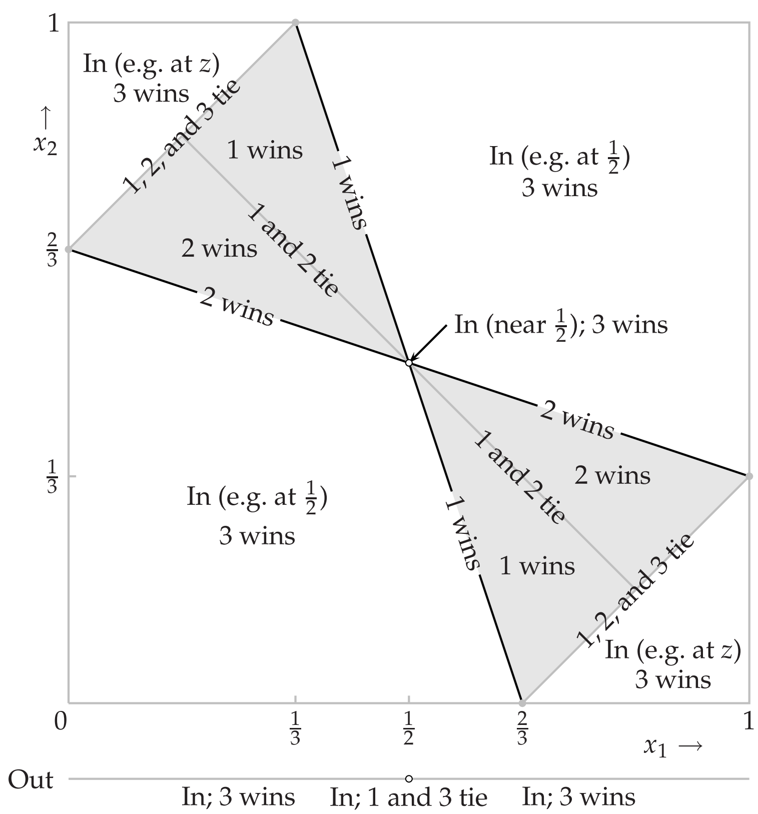


Figure 96.1 The outcome of a best response of candidate 3 to each pair of actions by candidates 1 and 2. The best response for any point in the gray shaded area (including the black boundaries of this area, but excluding the other boundaries) is *Out*. The outcome at each of the four small disks at the outer corners of the shaded area is that all three candidates tie. The value of z is $1 - \frac{1}{2}(x_1 + x_2)$.

Now consider the optimal action of candidate 2, given x_1 and the outcome of candidate 3's best response, as given in Figure 96.1. In the figure, take a value of x_1 and look at the outcomes as x_2 varies; find the value of x_2 that induces the best outcome for candidate 2. For example, for $x_1 = 0$ the only value of x_2 for which candidate 2 does not lose is $\frac{2}{3}$, at which point she ties with the other two candidates. Thus when candidate 1's strategy is $x_1 = 0$, candidate 2's best action, given candidate 3's best response, is $x_2 = \frac{2}{3}$, which leads to a three-way tie. We find that the outcome of the optimal value of x_2, for each value of x_1, is given as follows.

$$\begin{cases} 1, 2, \text{and 3 tie } (x_2 = \frac{2}{3}) & \text{if } x_1 = 0 \\ \text{2 wins} & \text{if } 0 < x_1 < \frac{1}{2} \\ \text{1 and 3 tie (2 stays out)} & \text{if } x_1 = \frac{1}{2} \\ \text{2 wins} & \text{if } \frac{1}{2} < x_1 < 1 \\ 1, 2, \text{and 3 tie } (x_2 = \frac{1}{3}) & \text{if } x_1 = 1. \end{cases}$$

Finally, consider candidate 1's best strategy, given the responses of candidates 2 and 3. If she stays out then candidates 2 and 3 enter at m and tie. If she enters then the best position at which to do so is $x_1 = \frac{1}{2}$, where she ties with candidate 3. (For every other position she either loses or ties with both of the other candidates.)

We conclude that in every subgame perfect equilibrium the outcome is that candidate 1 enters at $\frac{1}{2}$, candidate 2 stays out, and candidate 3 enters at $\frac{1}{2}$. (There are many subgame perfect equilibria, because after many histories candidate 3's optimal action is not unique.)

(If you're interested in what may happen when there are many potential candidates, look at `http://www.economics.utoronto.ca/osborne/research/CONJECT.HTM`.)

195.1 The race $G_1(2,2)$

The consequences of player 1's actions at the start of the game are as follows.

Take two steps: Player 1 wins.

Take one step: Go to the game $G_2(1,2)$, in which player 2 initially takes two steps and wins.

Do not move: If player 2 does not move, the game ends. If she takes one step we go to the game $G_1(2,1)$, in which player 1 takes two steps and wins. If she takes two steps, she wins. Thus in a subgame perfect equilibrium player 2 takes two steps, and wins.

We conclude that in a subgame perfect equilibrium of $G_1(2,2)$ player 1 initially takes two steps, and wins.

198.1 A race in which the players' valuations of the prize differ

By the arguments in the text for the case in which both players' valuations of the prize are between 6 and 7, the subgame perfect equilibrium outcomes of all games in which $k_1 \leq 2$ or $k_2 \leq 3$ are the same as they are when both players' valuations of the prize are between 6 and 7. If $k_2 \geq 5$ then player 1 is the winner in all subgame perfect equilibria, because even if player 2 reaches the finish line after taking one step at a time, her payoff is negative.

The games $G_i(3,4)$, $G_i(4,4)$, $G_i(5,4)$, and $G_i(6,4)$ remain. If, in the games $G_2(3,4)$ and $G_2(4,4)$, player 2 takes a single step then play moves to a game that player 1 wins. Thus player 2 is better off not moving; the subgame perfect equilibrium outcome is that player 1 takes one step at a time, and wins. In the game $G_i(5,4)$, the player who moves first can, by taking a single step, reach a game in which she wins regardless of the identity of the first-mover. Thus in this game the winner is the first-mover. Finally, in the game $G_1(6,4)$ it is not worth player 1's while taking two steps, to reach a game in which she wins, because her payoff would ultimately be negative. And if she takes one step, play moves to a game in which player 2 is the first-mover, and wins. Thus in this game player 2 wins. Figure 98.1 shows the subgame perfect equilibrium outcomes.

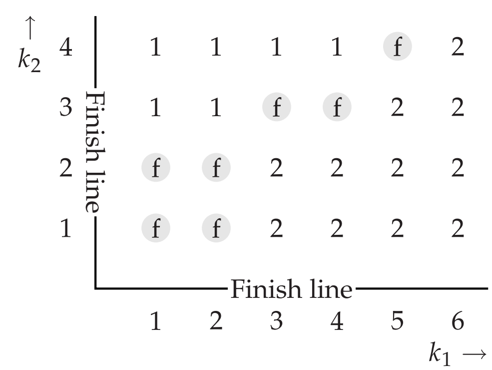


Figure 98.1 The subgame perfect equilibrium outcomes for the race in Exercise 198.1. Player 1 moves to the left, and player 2 moves down. The labels on the values of (k_1, k_2) indicate the subgame perfect equilibrium outcomes, as in the text.

198.2 Removing stones

For $n = 1$ the game has a unique subgame perfect equilibrium, in which player 1 takes one stone. The outcome is that player 1 wins.

For $n = 2$ the game has a unique subgame perfect equilibrium in which

- player 1 takes two stones
- after a history in which player 1 takes one stone, player 2 takes one stone.

The outcome is that player 1 wins.

For $n = 3$, the subgame following the history in which player 1 takes one stone is the game for $n = 2$ in which player 2 is the first mover, so player 2 wins. The subgame following the history in which player 1 takes two stones is the game for $n = 1$ in which player 2 is the first mover, so player 2 wins. Thus there is a subgame perfect equilibrium in which player 1 takes one stone initially, and one in which she takes two stones initially. In both subgame perfect equilibria player 2 wins.

For $n = 4$, the subgame following the history in which player 1 takes one stone is the game for $n = 3$ in which player 2 is the first-mover, so player 1 wins. The subgame following the history in which player 1 takes two stones is the game for $n = 2$ in which player 2 is the first-mover, so player 2 wins. Thus in every subgame perfect equilibrium player 1 takes one stone initially, and wins.

Continuing this argument for larger values of n, we see that if n is a multiple of 3 then in every subgame perfect equilibrium player 2 wins, while if n is not a multiple of 3 then in every subgame perfect equilibrium player 1 wins. We can prove this claim by induction on n. The claim is correct for $n = 1$, 2, and 3, by the arguments above. Now suppose it is correct for all integers through $n - 1$. I will argue that it is correct for n.

First suppose that n is divisible by 3. The subgames following player 1's removal of one or two stones are the games for $n - 1$ and $n - 2$ in which player 2 is the first-mover. Neither $n - 1$ nor $n - 2$ is divisible by 3, so by hypothesis player 2 is the winner in every subgame perfect equilibrium of both of these subgames. Thus player 2 is the winner in every subgame perfect equilibrium of the whole game.

Now suppose that n is not divisible by 3. As before, the subgames following player 1's removal of one or two stones are the games for $n - 1$ and $n - 2$ in which player 2 is the first-mover. Either $n - 1$ or $n - 2$ is divisible by 3, so in one of these subgames player 1 is the winner in every subgame perfect equilibrium. Thus player 1 is the winner in every subgame perfect equilibrium of the whole game.

199.1 Hungry lions

Denote by $G(n)$ the game in which there are n lions.

The game $G(1)$ has a unique subgame perfect equilibrium, in which the single lion eats the prey.

Consider the game $G(2)$. If lion 1 does not eat, it remains hungry. If it eats, we reach a subgame identical to $G(1)$, which we know has a unique subgame perfect equilibrium, in which lion 2 eats lion 1. Thus $G(2)$ has a unique subgame perfect equilibrium, in which lion 1 does not eat the prey.

In $G(3)$, lion 1's eating the prey leads to $G(2)$, in which we have just concluded that the first mover (lion 2) does not eat the prey (lion 1). Thus $G(3)$ has a unique subgame perfect equilibrium, in which lion 1 eats the prey.

For an arbitrary value of n, lion 1's eating the prey in $G(n)$ leads to $G(n - 1)$. If $G(n - 1)$ has a unique subgame perfect equilibrium, in which the prey is eaten, then $G(n)$ has a unique subgame perfect equilibrium, in which the prey is not eaten; if $G(n - 1)$ has a unique subgame perfect equilibrium, in which the prey is not eaten, then $G(n)$ has a unique subgame perfect equilibrium, in which the prey is eaten. Given that $G(1)$ has a unique subgame perfect equilibrium, in which the prey is eaten, we conclude that if n is odd then $G(n)$ has a unique subgame perfect equilibrium, in which lion 1 eats the prey, and if n is even it has a unique subgame perfect equilibrium, in which lion 1 does not eat the prey.

200.1 A race with a liquidity constraint

In the absence of the constraint, player 1 initially takes one step. Suppose she does so in the game with the constraint. Consider player 2's options after player 1's move.

Player 2 takes two steps: Because of the liquidity constraint, player 1 can take at most one step. If she takes one step, player 2's optimal action is to take one step, and win. Thus player 1's best action is not to move; player 2's payoff exceeds 1 (her steps cost 5, and the prize is worth more than 6).

Player 2 moves one step: Again because of the liquidity constraint, player 1 can take at most one step. If she takes one step, player 2 can take two steps and win, obtaining a payoff of more than 1 (as in the previous case).

Player 2 does not move: Player 1, as before, can take one step on each turn, and win; player 2's payoff is 0.

We conclude that after player 1 moves one step, player 2 should take either one or two steps, and ultimately win; player 1's payoff is -1. A better option for player 1 is not to move, in which case player 2 can move one step at a time, and win; player 1's payoff is zero.

Thus the subgame perfect equilibrium outcome is that player 1 does not move, and player 2 takes one step at a time and wins.

7 Extensive Games with Perfect Information: Extensions and Discussion

206.2 Extensive game with simultaneous moves

The game is shown in Figure 101.1.

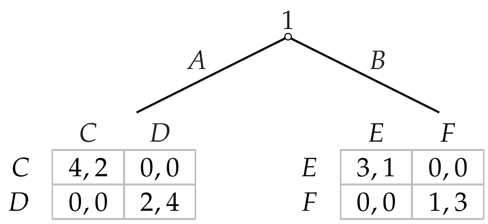


Figure 101.1 The game in Exercise 206.2.

The subgame following player 1's choice of A has two Nash equilibria, (C, C) and (D, D); the subgame following player 1's choice of B also has two Nash equilibria, (E, E) and (F, F). If the equilibrium reached after player 1 chooses A is (C, C), then regardless of the equilibrium reached after she chooses (E, E), she chooses A at the beginning of the game. If the equilibrium reached after player 1 chooses A is (D, D) and the equilibrium reached after she chooses B is (F, F), she chooses A at the beginning of the game. If the equilibrium reached after player 1 chooses A is (D, D) and the equilibrium reached after she chooses B is (E, E), she chooses B at the beginning of the game.

Thus the game has four subgame perfect equilibria: (ACE, CE), (ACF, CF), (ADF, DF), and (BDE, DE) (where the first component of player 1's strategy is her choice at the start of the game, the second component is her action after she chooses A, and the third component is her action after she chooses B, and the first component of player 2's strategy is her action after player 1 chooses A at the start of the game and the second component is her action after player 1 chooses B at the start of the game).

In the first two equilibria the outcome is that player 1 chooses A and then both players choose C, in the third equilibrium the outcome is that player 1 chooses A and then both players choose D, and in the last equilibrium the outcome is that player 1 chooses B and then both players choose E.

206.3 Two-period Prisoner's Dilemma

The extensive game is specified as follows.

Players The two people.

Terminal histories The set of pairs $((W, X), (Y, Z))$, where each component is either Q or F.

Player function $P(\varnothing) = \{1, 2\}$ and $P(W, X) = \{1, 2\}$ for any pair (W, X) in which both W and X are either Q or F.

Actions The set $A_i(\varnothing)$ of player i's actions at the initial history is $\{Q, F\}$, for $i = 1, 2$; the set $A_i(W, X)$ of player i's actions after any history (W, X) in which both W and X are either Q or F is $\{Q, F\}$, for $i = 1, 2$.

Preferences Each player's preferences are represented by the payoffs described in the problem.

Consider the subgame following some history (W, X) (where W and X are both either Q or F). In this subgame each player chooses either Q or F, and her payoff to each resulting terminal history is the sum of her payoff to (W, X) in the *Prisoner's Dilemma* given in Figure 13.1 and her payoff to the pair of actions chosen in the subgame, again as in the *Prisoner's Dilemma*. Thus the subgame differs from the *Prisoner's Dilemma* given in Figure 13.1 only in that every payoff to a given player is increased by her payoff to the pair of actions (W, X). Thus the subgame has a unique Nash equilibrium, in which both players choose F.

Now consider the whole game. Regardless of the actions chosen at the start of the game, the outcome in the second period is (F, F). Thus the payoffs to the pairs of actions chosen in the first period are the payoffs in the *Prisoner's Dilemma* plus the payoff to (F, F). We conclude that the game has a unique subgame perfect equilibrium, in which each player chooses F after every history.

207.1 Timing claims on an investment

The following extensive game models the situation.

Players The two people.

Terminal histories The sequences of the form $((N, N), (N, N), \ldots, (N, N), x_t)$, where $1 \leq t \leq T$, x_t is (C, C), (C, N), or (N, C) if $t \leq T - 1$ and (C, C), (C, N), (N, C), or (N, N) if $t = T$, C means "claim", and N means "do not claim".

Player function The set of players assigned to every nonterminal history is $\{1, 2\}$ (the two people).

Actions The set of actions of each player after every nonterminal history is $\{C, N\}$.

Preferences Each player's preferences are represented by a payoff equal to the amount of money she obtains.

The consequences of the players' actions in period T are given in Figure 103.1. We see that the subgame starting in period T has a unique Nash equilibrium, (C, C), in which each player's payoff is T.

	C	N
C	T, T	$2T, 0$
N	$0, 2T$	T, T

Figure 103.1 The consequences of the players' actions in period T of the game in Exercise 207.1.

Thus if $T = 1$ the game has a unique subgame perfect equilibrium, in which both players claim.

Now suppose that $T \geq 2$, and consider period $T - 1$. The consequences of the players' actions in this period, given the equilibrium in the subgame starting in period T, are shown in Figure 103.2. (The entry in the bottom right box, (T, T), is the pair of equilibrium payoffs in the subgame in period T.) If $T > 2$ then $2(T - 1) > T$, so that the subgame starting in period $T - 1$ has a unique subgame perfect equilibrium, (C, C), in which each player's payoff is $T - 1$. If $T = 2$ then the whole game has two subgame perfect equilibria, in one of which both players claim in both periods, and another in which neither claims in period 1 and both claim in period 2.

	C	N
C	$T-1, T-1$	$2(T-1), 0$
N	$0, 2(T-1)$	T, T

Figure 103.2 The consequences of the players' actions in period $T - 1$ of the game in Exercise 207.1, given the equilibrium actions in period T.

For $T > 2$, working back to period 1 we see that the game has two subgame perfect equilibria: one in which each player claims in every period, and one in which neither player claims in period 1 but both players claim in every subsequent period.

207.2 A market game

The following extensive game models the situation.

Players The seller and m buyers.

Terminal histories The set of sequences of the form $((p_1, \ldots, p_m), j)$, where each p_i is a price (nonnegative number) and j is either 0 or one of the sellers (an integer from 1 to m), with the interpretation that p_i is the offer of buyer i, $j = 0$ means that the seller accepts no offer, and $j \geq 1$ means that the seller accepts buyer j's offer.

Player function $P(\varnothing)$ is the set of buyers and $P(p_1, \ldots, p_m)$ is the seller for every history $(p_1, \ldots, p_m)$.

Actions The set $A_i(\varnothing)$ of actions of buyer i at the start of the game is the set of prices (nonnegative numbers). The set $A_s(p_1, \ldots, p_m)$ of actions of the seller after the buyers have made offers is the set of integers from 0 to m.

Preferences Each player's preferences are represented by the payoffs given in the question.

To find the subgame perfect equilibria of the game, first consider the subgame following a history $(p_1, \ldots, p_m)$ of offers. The seller's best action is to accept the highest price, or one of the highest prices in the case of a tie.

I claim that a strategy profile is a subgame perfect equilibrium of the whole game if and only if the seller's strategy is the one just described, and among the buyers' strategies $(p_1, \ldots, p_m)$, every offer p_i is at most v and at least two offers are equal to v.

Such a strategy profile is a subgame perfect equilibrium by the following argument. If the buyer with whom the seller trades raises her offer then her payoff becomes negative, while if she lowers her offer she no longer trades and her payoff remains zero. If any other buyer raises her offer then either she still does not trade, or she trades at a price greater than v and hence receives a negative payoff.

No other profile of actions for the buyers at the start of the game is part of a subgame perfect equilibrium by the following argument.

- If some offer exceeds v then the buyer who submits the highest offer can induce a better outcome by reducing her offer to a value below v, so that either the seller does not trade with her, or, if the seller does trade with her, she trades at a lower price.
- If all offers are at most v and only one is equal to v, the buyer who offers v can increase her payoff by reducing her offer a little.
- If all offers are less than v then one of the buyers whose offer is not accepted can increase her offer to some value between the winning offer and v, induce the seller to trade with her, and obtain a positive payoff.

In any equilibrium the buyer who trades with the seller does so at the price v. Thus her payoff is zero. The other buyers do not trade, and hence also obtain the payoff of zero.

208.1 Price competition

The following game models the situation.

Players The two sellers and the two buyers.

Terminal histories All sequences $((p_1, p_2), (x_1, x_2))$ where p_i (for $i = 1, 2$) is the price posted by seller i and x_i (for $i = 1$, 2) is the seller chosen by buyer i (either seller 1 or seller 2).

Player function $P(\varnothing)$ is the set consisting of the two sellers; $P(p_1, p_2)$ for any pair (p_1, p_2) of prices is the set consisting of the two buyers.

Actions The set of actions of each seller at the start of the game is the set of prices (nonnegative numbers), and the set of actions of each buyer after any history (p_1, p_2) is the set consisting of seller 1 and seller 2.

Preferences Each seller's preferences on lotteries over the terminal histories are represented by the expected value of a Bernoulli payoff function that assigns the payoff p to a sale at the price p. Each buyers' preferences on lotteries over the terminal histories are represented by the expected value of a Bernoulli payoff function that assigns the payoff $1 - p$ to a purchase at the price p. The payoff of a player who does not trade is 0.

In any subgame perfect equilibrium, the buyers' strategies in the subgame following any history (p_1, p_2) must be a Nash equilibrium of the game in Exercise 125.2. This game has a unique Nash equilibrium unless $\frac{1}{2}(1 + p_1) \leq p_2 \leq 2p_1 - 1$. If $\frac{1}{2}(1 + p_1) < p_2 < 2p_1 - 1$ the game has three Nash equilibria, two pure and one mixed.

I claim that for any price $p \geq \frac{1}{2}$ the extensive game in this exercise has a subgame perfect equilibrium in which if $\frac{1}{2}(1 + p_1) < p_2 < 2p_1 - 1$ then if either $p_1 \leq p$ or $p_2 \leq p$, the equilibrium in the subgame is the pure Nash equilibrium in which buyer 1 approaches seller 1 and buyer 2 approaches seller 2, while if $p_1 > p$ and $p_2 > p$, the equilibrium in the subgame is the mixed strategy equilibrium.

Precisely, I claim that for any $p \geq \frac{1}{2}$ the following strategy pair is a subgame perfect equilibrium of the game.

Sellers' strategies Each seller announces the price p.

Buyers' strategies

- After a history (p_1, p_2) in which $2p_1 - 1 < p_2 < \frac{1}{2}(1 + p_1)$ and either $p_1 \leq p$ or $p_2 \leq p$ (or both), buyer 1 approaches seller 1 and buyer 2 approaches seller 2.
- After a history (p_1, p_2) in which $2p_1 - 1 < p_2 < \frac{1}{2}(1 + p_1)$, $p_1 > p$, and $p_2 > p$, each buyer approaches seller 1 with probability $(1 - 2p_1 + p_2)/(2 - p_1 - p_2)$.

- After a history (p_1, p_2) in which $p_2 \leq 2p_1 - 1$, both buyers approach seller 2.
- After a history (p_1, p_2) in which $p_2 \geq \frac{1}{2}(1 + p_1)$, both buyers approach seller 1.

By Exercise 125.2, the buyers' strategy pair is a Nash equilibrium in every subgame. The sellers' payoffs in the pure equilibrium in which one buyer approaches each seller are (p_1, p_2); their payoffs in the pure equilibrium in which both buyers approach seller 1 is $(p_1, 0)$; and their payoffs in the pure equilibrium in which both buyers approach seller 1 is $(0, p_2)$. Their payoffs in the mixed strategy equilibrium are more difficult to calculate. They are $(\pi_1^*(p_1, p_2), \pi_2^*(p_1, p_2)) = ((1 - (1 - \pi)^2)p_1, (1 - \pi^2)p_2)$, where $\pi = (1 - 2p_1 + p_2)/(2 - p_1 - p_2)$. After some algebra we obtain

$$(\pi_1^*(p_1, p_2), \pi_2^*(p_1, p_2)) = \left(\frac{3p_1(1 - p_2)(1 - 2p_1 + p_2)}{(2 - p_1 - p_2)^2}, \frac{3p_2(1 - p_1)(1 - 2p_2 + p_1)}{(2 - p_1 - p_2)^2}\right).$$

These equilibrium payoffs are illustrated in Figure 106.1.

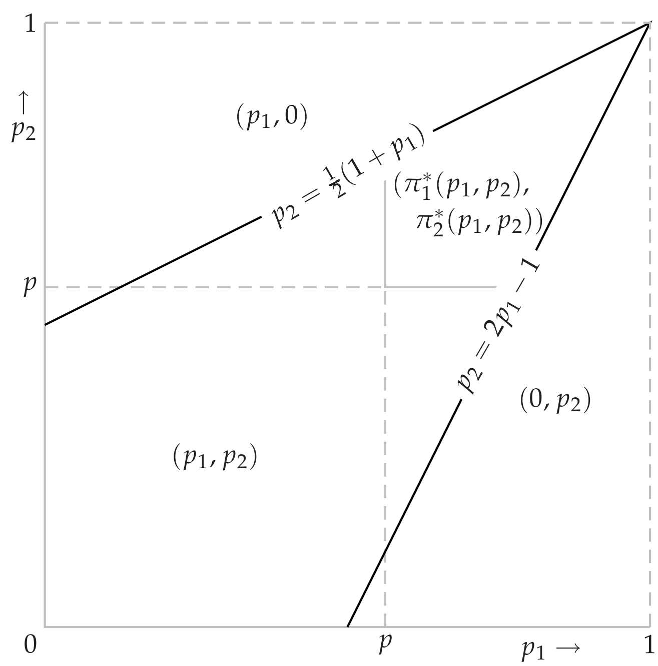


Figure 106.1 The sellers' payoffs in the game in Exercise 208.1 as a function of their prices, given the buyers' equilibrium strategies.

Now consider the sellers' choices of prices. Given that $p_2 = p \geq \frac{1}{2}$ and the buyers' strategies are those defined above, seller 1's payoff when she sets the price

p_1 is

$$\begin{cases} p_1 & \text{if } p_1 \le p \\ \pi_1^*(p_1, p) & \text{if } p < p_1 \le \frac{1}{2}(1+p) \\ 0 & \text{if } p > \frac{1}{2}(1+p). \end{cases}$$

By the claim in the question (verified at the end of this solution), $\pi_1^*(p_1, p_2)$ is decreasing in p_1 for $p_1 \ge p_2$, so that seller 1's best response to p is p. An analogous argument shows that seller 2's best response to p is p.

We conclude that the strategy pair defined above is a subgame perfect equilibrium.

The verification of the last claim of the question (not required as part of an answer) follows. We have

$$\pi_1^*(p_1, p_2) = \frac{3p_1(1-p_2)(1-2p_1+p_2)}{(2-p_1-p_2)^2}.$$

The derivative of this function with respect to p_1 is

$$\frac{3(1-p_2)\left[(2-p_1-p_2)^2(1-2p_1+p_2-2p_1)+2(2-p_1-p_2)p_1(1-2p_1+p_2)\right]}{(2-p_1-p_2)^4}$$

or

$$\frac{3(1-p_2)(2-p_1-p_2)\left[(2-p_1-p_2)(1-4p_1+p_2)+2p_1(1-2p_1+p_2)\right]}{(2-p_1-p_2)^4}.$$

This expression is negative if

$$(2-p_1-p_2)(1-4p_1+p_2)+2p_1(1-2p_1+p_2) < 0,$$

or

$$p_1 > \frac{(2-p_2)(1+p_2)}{7-5p_2}.$$

The right-hand side is less than p_2 if

$$(2p_2-1)(p_2-1) < 0,$$

which is true if $\frac{1}{2} < p_2 < 1$, so that seller 1's equilibrium payoff is decreasing in p_1 whenever $p_1 > p_2 > \frac{1}{2}$.

210.1 Bertrand's duopoly game with entry

The unique Nash equilibrium of the subgame that follows the challenger's entry is (c, c), as we found in Section 3.2.2. The challenger's profit is $-f < 0$ in this equilibrium. By choosing to stay out the challenger obtains the profit of 0, so in any subgame perfect equilibrium the challenger stays out. After the history in which the challenger stays out, the incumbent chooses its price p_1 to maximize its profit $(p_1 - c)(\alpha - p_1)$.

Thus for any value of $f > 0$ the whole game has a unique subgame perfect equilibrium, in which the strategies are:

Challenger

- at the start of the game: stay out
- after the history in which the challenger enters: choose the price c

Incumbent

- after the history in which the challenger enters: choose the price c
- after the history in which the challenger stays out: choose the price p_1 that maximizes $(p_1 - c)(\alpha - p_1)$.

212.1 Electoral competition with strategic voters

Consider the strategy profile in which each candidate chooses the median m of the citizens' favorite positions and the citizens' strategies are defined as follows.

- After a history in which every candidate chooses m, each citizen i votes for candidate j, where j is the smallest integer greater than or equal to in/q. (That is, the citizens split their votes equally among the n candidates. If there are 3 candidates and 15 citizens, for example, citizens 1 through 5 vote for candidate 1, citizens 6 through 10 vote for candidate 2, and citizens 11 through 15 vote for candidate 3.)
- After a history in which all candidates enter and every candidate but j chooses m, each citizen votes for candidate j if her favorite position is closer to j's position than it is to m, and for some candidate ℓ whose position is m otherwise. (All citizens who do not vote for j vote for the *same* candidate ℓ.)
- After any other history, the citizens' action profile is any Nash equilibrium of the voting subgame in which no citizen's action is weakly dominated.

The outcome induced by this strategy profile is that all candidates enter and choose the median of the citizens' favorite positions, and tie for first place. After every history of one of the first two types, every citizen votes for one of the candidates who is closest to her favorite position, so no citizen's strategy is weakly dominated. After a history of the third type, no citizen's strategy is weakly dominated by construction.

The strategy profile is a subgame perfect equilibrium by the following argument.

In each voting subgame the citizens' strategy profile is a Nash equilibrium:

- after the history in which the candidates' positions are the same, equal to m, no citizen's vote affects the outcome
- after a history in which all candidates enter and every candidate but j chooses m, a change in any citizen's vote either has no effect on the outcome or makes it worse for her

- after any other history the citizens' strategy profile is a Nash equilibrium by construction.

Now consider the candidates' choices at the start of the game. If any candidate deviates by choosing a position different from that of the other candidates, she loses, rather than tying for first place. If any candidate deviates by staying out of the race, the outcome is worse for her than adhering to the equilibrium, and tying for first place. Thus each candidate's strategy is optimal given the other players' strategies.

[The claim that every voting subgame has a (pure) Nash equilibrium in which no citizen's action is weakly dominated, which you are not asked to prove, may be demonstrated as follows. Given the candidates' positions, choose the candidate, say j, ranked last by the smallest number of citizens. Suppose that all citizens except those who rank j last vote for j; distribute the votes of the citizens who rank j last as equally as possible among the other candidates. Each citizen's action is not weakly dominated (no citizen votes for the candidate she ranks last) and, given $q \geq 2n$, no change in any citizen's vote affects the outcome, so that the list of citizens' actions is a Nash equilibrium of the voting subgame.]

213.1 Electoral competition with strategic voters

I first argue that in any equilibrium each candidate that enters is in the set of winners. If some candidate that enters is not a winner, she can increase her payoff by deviating to *Out*.

Now consider the voting subgame in which there are more than two candidates and not all candidates' positions are the same. Suppose that the citizens' votes are equally divided among the candidates. I argue that this list of citizens' strategies is not a Nash equilibrium of the voting subgame.

For either the citizen whose favorite position is 0 or the citizen whose favorite position is 1 (or both), at least two candidates' positions are better than the position of the candidate furthest from the citizen's favorite position. Denote a citizen for whom this condition holds by i. (The claim that citizen i exists is immediate if the candidates occupy at least three distinct positions, or they occupy two distinct positions and at least two candidates occupy each position. If the candidates occupy only two positions and one position is occupied by a single candidate, then take the citizen whose favorite position is 0 if the lone candidate's position exceeds the other candidates' position; otherwise take the citizen whose favorite position is 1.)

Now, given that each candidate obtains the same number of votes, if citizen i switches her vote to one of the candidates whose position is better for her than that of the candidate whose position is furthest from her favorite position, then this candidate wins outright. (If citizen i originally votes for one of these superior candidates, she can switch her vote to the other superior candidate; if she originally votes for neither of the superior candidates, she can switch her vote to either one of them.) Citizen i's payoff increases when she thus switches her vote, so that the list of citizens' strategies is not a Nash equilibrium of the voting subgame.

We conclude that in every Nash equilibrium of every voting subgame in which there are more than two candidates and not all candidates' positions are the same at least one candidate loses. Because no candidate loses in a subgame perfect equilibrium (by the first argument in the proof), in any subgame perfect equilibrium either only two candidates enter, or all candidates' positions are the same.

If only two candidates enter, then by the argument in the text for the case $n = 2$, each candidate's position is m (the median of the citizens' favorite positions).

Now suppose that more than two candidates enter, and their common position is not equal to m. If a candidate deviates to m then in the resulting voting subgame only two positions are occupied, so that for every citizen, any strategy that is not weakly dominated votes for a candidate at the position closest to her favorite position. Thus a candidate who deviates to m wins outright. We conclude that in any subgame perfect equilibrium in which more than two candidates enter, they all choose the position m.

216.1 Top cycle set

a. The top cycle set is the set $\{x, y, z\}$ of all three alternatives because x beats y beats z beats x.

b. The top cycle set is the set $\{w, x, y, z\}$ of all four alternatives. As in the previous case, x beats y beats z beats x; also y beats w.

217.1 Designing agendas

We have: x beats y beats z beats x; x, y, and z all beat v; v beats w; and w does not beat any alternative. Thus the top cycle set is $\{x, y, z\}$.

An agenda that yields x is shown in Figure 111.1. A similar agenda, with y and x interchanged, yields y, and one with x and z interchanged yields z.

No binary agenda yields w because for every other alternative a, a majority of committee members prefer a to w. No binary agenda yields v because the only alternative that v beats is w, which itself is beaten by every other alternative.

217.2 An agenda that yields an undesirable outcome

An agenda for which the outcome of sophisticated voting is z is given in Figure 111.2.

220.1 Exit from a declining industry

Period t_1 is the largest value of t for which $P_t(k_1) \geq c$, or $60 - t \geq 10$. Thus $t_1 = 50$. Similarly, $t_2 = 70$.

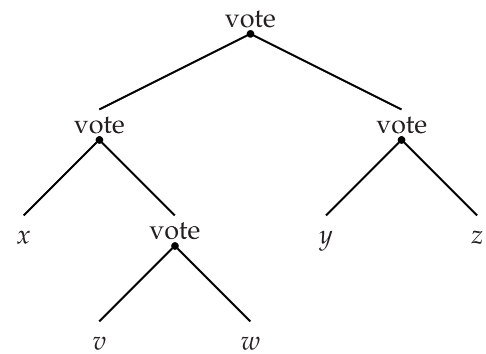


Figure 111.1 A binary agenda for which the alternative x is the outcome of sophisticated voting for the committee in Exercise 217.1.

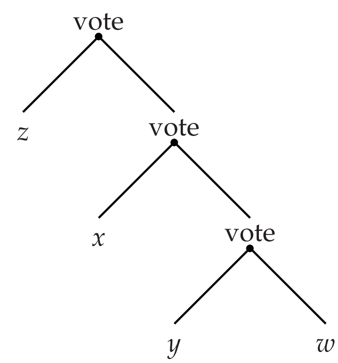


Figure 111.2 A binary agenda for which the alternative z is the outcome of sophisticated voting for the committee in Exercise 217.2.

If both firms are active in period t_1, then firm 2's profit in this period is $(100 - t_1 - c - k_1 - k_2)k_2 = (-20)(20) = -400$. Its profit in any period t in which it is alone in the market is $(100 - t - c - k_2)k_2 = (70 - t)(20)$. Thus its profit from period $t_1 + 1$ through period t_2 is

$$(19 + 18 + \ldots + 1)(20) = 3800.$$

Hence firm 2's loss in period t_1 when both firms are active is (much) less than the sum of its profits in periods $t_1 + 1$ through t_2 when it alone is active.

221.1 Effect of borrowing constraint in declining industry

Period t_0 is the largest value of t for which $P_t(k_1 + k_2) \geq c$, or $100 - t - 60 \geq 10$, or $t \leq 30$. Thus $t_0 = 30$. From Exercise 220.1 we have $t_1 = 50$ and $t_2 = 70$.

Suppose that firm 2 stays in the market for k periods after t_0, then exits in period $t_0 + k + 1$. Firm 1's total profit from period $t_0 + 1$ on if it stays until period t_1 is

$$(P_{t_0+1}(k_1 + k_2) - c)k_1 + \ldots + (P_{t_0+k}(k_1 + k_2) \;-\; c)k_1 +$$

$$(P_{t_0+k+1}(k_1) - c)k_1 + \ldots + (P_{t_1}(k_1) - c)k_1,$$

or

$$40[(100-30-1-60-10) + \ldots + (100-30-k-60-10) + (100-30-k-1-40-10) + \ldots + (100-50-40-10)],$$

or

$$40[-1-\ldots-k+(19-k)+\ldots+0],$$

or

$$40[-\tfrac{1}{2}k(k+1) + \tfrac{1}{2}(19-k)(20-k)]$$

(using the fact that the sum of the first n positive integers is $\frac{1}{2}n(n+1)$), or

$$20(380-40k).$$

In order that this profit be nonpositive we need $40k \geq 380$, or $k \geq 9.5$. Thus firm 2 needs to survive until at least period 40 ($30+10$) in order to make firm 1's exit in period t_0+1 optimal.

Firm 2's total loss from period 31 through period 40 when both firms are in the market is

$$(P_{31}(k_1+k_2) - c)k_2 + \ldots + (P_{40}(k_1+k_2) - c)k_2,$$

or

$$20[(100-31-60-10) + \ldots + (100-40-60-10)],$$

or

$$20(-1+\ldots+-10),$$

or 1100.

Thus firm 2 needs to be able to bear a debt of at least 1100 in order for there to be a subgame perfect equilibrium in which firm 1 exits in period t_0+1.

222.2 Variant of ultimatum game with equity-conscious players

The game is defined as follows.

Players The two people.

Terminal histories The set of sequences (x, β_2, Z), where x is a number with $0 \leq x \leq c$ (the amount of money that person 1 offers to person 2), β_2 is 0 or 1 (the value of β_2 selected by chance), and Z is either Y ("yes, I accept") or N ("no, I reject").

Player function $P(\varnothing) = 1$, $P(x) = c$ for all x, and $P(x, \beta_2) = 2$ for all x and all β_2.

Chance probabilities For every history x, chance chooses 0 with probability p and 1 with probability $1-p$.

Preferences Each person's preferences are represented by the expected value of a payoff equal to the amount of money she receives. For any terminal history (x, β_2, Y) person 1 receives $c - x$ and person 2 receives x; for any terminal history (x, β_2, N) each person receives 0.

Given the result from Exercise 181.1 given in the question, if person 1's offer x satisfies $0 < x < \frac{1}{3}$ then the offer is rejected with probability $1 - p$, so that person 1's expected payoff is $p(1 - x)$, while if $x > \frac{1}{3}$ the offer is certainly accepted, independent of the type of person 2. Thus person 1's optimal offer is

$$\begin{cases} \frac{1}{3} & \text{if } p < \frac{2}{3} \\ 0 & \text{if } p > \frac{2}{3}; \end{cases}$$

if $p = \frac{2}{3}$ then both offers are optimal.

If $p > \frac{2}{3}$ we see that in a subgame perfect equilibrium person 1's offers are rejected by every person 2 with whom she is matched for whom $\beta_2 = 1$ (that is, with probability $1 - p$).

223.1 Sequential duel

The following game models the situation.

Players The two people.

Terminal histories All sequences of the form $(X_1, X_2, \ldots, X_k, S, H)$, where each X_i is either N ("don't shoot") or (S, M) ("shoot", "miss"), and H means "hit", together with the infinite sequence $(S, M, S, M, S, M, \ldots)$.

Player function $P(h) = 1$ for any history h in which the total number of S's and N's is even and $P(h) = 2$ for any history h in which the total number of S's and N's is odd.

Chance probabilities Whenever chance moves after a move of player 1 it chooses H with probability p_1 and M with probability $1 - p_1$; whenever chance moves after a move of player 2 it chooses H with probability p_2 and M with probability $1 - p_2$;

Preferences Each player's preferences are represented by the expected value of a Bernoulli payoff function that assigns 1 to any history in which she survives and 0 to any history in which she is killed.

If neither player ever shoots, both players survive. No outcome is better for either player, so in particular neither player has a strategy that leads to a better outcome for her, given the other player's strategy.

Now suppose that player 2 shoots whenever it is her turn to move. I claim that a best response for player 1 is to shoot whenever it is her turn to move. Denote player 1's probability of survival when she follows this strategy by π_1.

Suppose that player 1 deviates to not shooting at the start of the game (but does not change the remainder of her strategy). If player 2 hits her in the next round, she does not survive. If player 2 misses her, an event with probability $1 - p_2$, then we reach a subgame identical to the whole game in which both players always shoot, so that in this subgame player 1's survival probability is π_1. Thus if player 1 deviates to not shooting at the start of the game her survival probability is $(1 - p_2)\pi_1$. We conclude that player 1 is not better off (and worse off if $p_2 > 0$) by deviating at the start of the game.

The same argument shows that, given player 2's strategy, player 1 is not better off deviating after any history that ends with player 2's shooting and missing, or after any collection of such histories. A change in player 1's strategy after a history that ends with player 2's not shooting has no effect on the outcome (because player 2's is to shoot whenever it is her turn to move). Thus no change in player 1's strategy increases her expected payoff.

A symmetric argument shows that player 2 is not better off changing her strategy. Thus the strategy pair in which each player always shoots is a subgame perfect equilibrium.

223.2 Sequential truel

The games are shown in Figure 115.1. (The action marked "0" is that of shooting into the air, which is available only in the second version of the game.)

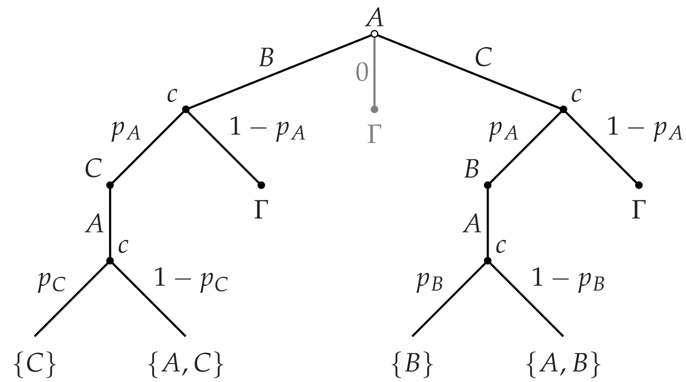
To find the subgame perfect equilibria, first consider the subgame Γ' in Figure 115.1. Whomever player C aims at, if she misses then she survives in the company of both A and B. If she aims at B and hits her, then she survives in the company of A; if she aims at A and hits her then she survives in the company of B. Thus C aims at B if $p_A < p_B$ and at A if $p_A > p_B$.

Now consider the subgame Γ. Whomever B aims at, the outcome is the same if she misses (because Γ' has a unique subgame perfect equilibrium). If B aims at A and hits her, then she survives with probability $1 - p_C$; if she aims at C and hits her, then she survives with probability 1. Thus (given $p_C > 0$), the subgame Γ thus has a unique subgame perfect equilibrium, in which B aims at C.

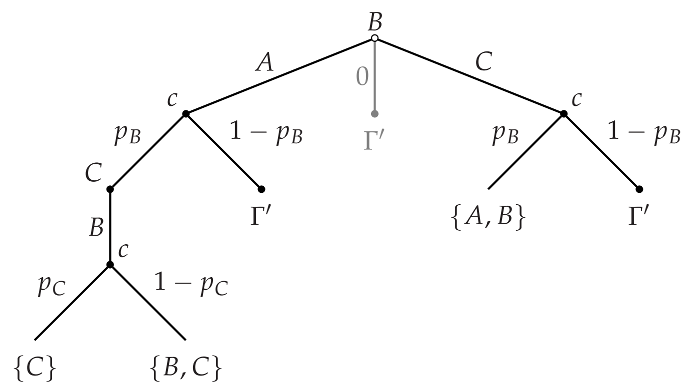
Finally, consider the whole game. Whomever A aims at, the outcome is the same if she misses (because Γ has a unique subgame perfect equilibrium). If she aims at B and hits her, then she survives with probability $1 - p_C$; if she aims at C and hits her, then she survives with probability $1 - p_B$. Thus A aims at C if $p_B < p_C$ and at B if $p_B > p_C$.

In summary, the game in which no player has the option of shooting into the air has the following unique subgame perfect equilibrium.

- At the start of the game, A aims at C if $p_B < p_C$ and at B if $p_B > p_C$.
- After a history in which A misses, B aims at C.



where the game Γ is



and the game Γ′ is

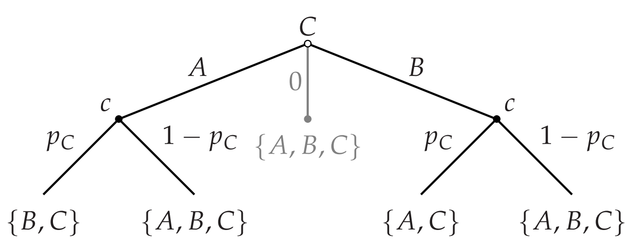


Figure 115.1 The games in Exercise 223.2. Only the actions indicated by black lines are available when players do not have the option of shooting into the air (the action "0"). The labels beside the actions of chance are the probabilities with which the actions are chosen; in each case the left action is "hit" and the right action is "miss".

- After a history in which both A and B miss, C aims at B if $p_A < p_B$ and at A if $p_A > p_B$.

Player A aims the player who is her more dangerous opponent; she is better off if she eliminates this opponent than if she eliminates her weaker opponent.

Player C's survival probability is $(1 - p_A)(1 - p_B) = 1 - p_A - p_B(1 - p_A)$ if

$p_C > p_B$, and $1 - p_B(1 - p_A)$ if $p_C < p_B$. Thus she is better off if $p_C < p_B$ than if $p_C > p_B$.

Now consider the game in which each player has the option of shooting into the air. In the subgame Γ', player C's best action is to aim at B (given $p_A < p_B$). (If she shoots into the air then the set of survivors is $\{A, B, C\}$; if she aims at B she has some chance of eliminating her.)

In the subgame Γ we know that if B shoots, her target should be C. If she does so her probability of survival is $1 - (1 - p_B)p_C$. If she shoots into the air her probability of survival is $1 - p_C$. The former exceeds the latter, so in the subgame Γ player B aims at C.

Finally, given the equilibrium actions in the subgames, at the start of the game we know that if A fires she aims at C if $p_B < p_C$ and at B if $p_B > p_C$. Given $p_A < p_B$, her shooting into the air results in her certain survival, while her aiming at B or C results in her surviving with probability less than 1. Thus she shoots into the air.

We conclude that if $p_A < p_B$ then the game in which each player has the option of shooting into the air has a unique subgame perfect equilibrium, which differs from the subgame perfect equilibrium in which this option is absent only in that A shoots into the air at the beginning of the game.

Player A fires into the air because when she does so B and C fight between themselves; if she shoots at one of them she may eliminate her from the game, giving the remaining player an incentive to shoot at her.

224.1 Cohesion in legislatures

Let the initial governing coalition consist of legislators 1 and 2. The US game is defined as follows.

Players The three legislators.

Terminal histories All sequences $(i, (x_1, x_2, x_3), (A, B, C), j, (y_1, y_2, y_3), (A', B', C'))$, where i and j are members of the governing coalition (possibly $i = j$), (x_1, x_2, x_3) and (y_1, y_2, y_3) are partitions of one unit of payoff ($x_1 + x_2 + x_3 = y_1 + y_2 + y_3 = 1$, $x_i \geq 0$, and $y_i \geq 0$ for $i = 1, 2, 3$), and A, B, C, A', B', and C' are either *yes* (vote for bill) or *no* (vote against bill).

Player function

- $P(\varnothing) = c$ (chance)
- $P(i) = i$
- $P(i, (x_1, x_2, x_3)) = \{1, 2, 3\}$
- $P(i, (x_1, x_2, x_3), (A, B, C)) = c$
- $P(i, (x_1, x_2, x_3), (A, B, C), j) = j$
- $P(i, (x_1, x_2, x_3), (A, B, C), j, (y_1, y_2, y_3)) = \{1, 2, 3\}$.

Chance probabilities Chance assigns probability $\frac{1}{2}$ to 1 and probability $\frac{1}{2}$ to 2 whenever it moves.

Actions

- $A(\varnothing) = \{1, 2\}$
- $A(i) = \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 = 1, x_i \geq 0 \textit{ for all } i\}$ for $i = 1, 2$
- $A_k(i, (x_1, x_2, x_3)) = \{\textit{yes}, \textit{no}\}$ for all k, $i = 1, 2$, and all (x_1, x_2, x_3)
- $A(i, (x_1, x_2, x_3), (A, B, C)) = \{1, 2\}$ for all i, all (x_1, x_2, x_3), and all triples (A, B, C) in which A, B, and C are all either *yes* or *no*
- $A(i, (x_1, x_2, x_3), (A, B, C), j) = \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 = 1, x_i \geq 0 \textit{ for all } i\}$ for $i = 1$, 2, all (x_1, x_2, x_3), all triples (A, B, C) in which A, B, and C are all either *yes* or *no*, and $j = 1, 2$
- $A_k(i, (x_1, x_2, x_3), (A, B, C), j, (y_1, y_2, y_3)) = \{\textit{yes}, \textit{no}\}$ for all k, $i = 1, 2$, all (x_1, x_2, x_3), all triples (A, B, C) in which A, B, and C are all either *yes* or *no*, $j = 1, 2$, and all (y_1, y_2, y_3).

Preferences Each legislator i ranks the terminal histories by the amount of money she receives: $x_i + y_i$ if both bills are passed, $x_i + d_i^2$ if only the first bill is passed, $d_i^1 + y_i$ if only the second bill is passed, and $d_i^1 + d_i^2$ if neither bill is passed.

We find a subgame perfect equilibrium as follows. Refer to d_i^t as legislator i's *reservation value* in period t. In the second period, denote by k the legislator whose reservation value is lower between the two who do not propose a bill. Each legislator i gets d_i^t if a bill does not pass, and hence votes for a bill only if it gives her a payoff of at least d_i^t. The proposer needs one vote in addition to her own to pass a bill, and can obtain it most cheaply by proposing a bill that gives k the payoff d_k^2 and gives herself the remaining payoff $1 - d_k^2$ (which exceeds her reservation value, because all reservation values are less than $\frac{1}{2}$). Legislator k and the proposer vote for the bill, which thus passes. (Legislator k is indifferent between voting for or against the bill, but there is no subgame perfect equilibrium in which she votes against the bill, because relative if she uses such a strategy the proposer can increase her offer to k a little, leading k to strictly prefer voting for the bill.) The third player may vote for or against the bill (her vote has no effect on the outcome).

In the first period, the pattern of behavior is the same: the bill proposed gives the non-proposer with the lower reservation value that value.

In summary, in every subgame perfect equilibrium of the US game the strategy of each member i of the governing coalition has the following properties:

- after the move of chance in either period, propose the bill that gives the legislator with the smallest reservation value in the that period her reservation value and gives i the remaining payoff

- after a bill is proposed in either period, vote for the bill if it assigns i a positive amount.

The equilibrium strategy of the other legislator j satisfies the condition:

- after a bill is proposed in either period, vote for the bill if it assigns j a positive amount.

(Each legislator's equilibrium strategy may either vote for or vote against a bill that gives her a payoff of zero.)

Thus in the US game there is no cohesion: the supporters of a bill may change from period to period, depending on the values of the reservation values.

The UK game is defined as follows.

Players The three legislators.

Terminal histories All sequences $(i, (x_1, x_2, x_3), (A, B, C), j, (y_1, y_2, y_3), (A', B', C'))$, where i is a member of the governing coalition and j is any legislator, (x_1, x_2, x_3) and (y_1, y_2, y_3) are partitions of one unit of payoff ($x_1 + x_2 + x_3 = y_1 + y_2 + y_3 = 1$, $x_i \geq 0$, and $y_i \geq 0$ for $i = 1, 2, 3$), and A, B, C, A', B', and C' are either *yes* (vote for bill) or *no* (vote against bill).

Player function

- $P(\varnothing) = c$ (chance)
- $P(i) = i$
- $P(i, (x_1, x_2, x_3)) = \{1, 2, 3\}$
- $P(i, (x_1, x_2, x_3), (A, B, C)) = c$
- $P(i, (x_1, x_2, x_3), (A, B, C), j) = j$
- $P(i, (x_1, x_2, x_3), (A, B, C), j, (y_1, y_2, y_3)) = \{1, 2, 3\}$.

Chance probabilities Chance assigns probability $\frac{1}{2}$ to 1 and probability $\frac{1}{2}$ to 2 at the start of the game and after a history $(i, (x_1, x_2, x_3), (A, B, C))$ in which at least two of the votes A, B, and C are *yes*. Chance assigns probability $\frac{1}{3}$ to each legislator after a history $(i, (x_1, x_2, x_3), (A, B, C))$ in which at least two of the votes A, B, and C are *no*.

Actions

- $A(\varnothing) = \{1, 2\}$
- $A(i) = \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 = 1, x_i \geq 0 \textit{ for all } i\}$ for $i = 1, 2$
- $A_k(i, (x_1, x_2, x_3)) = \{\textit{yes}, \textit{no}\}$ for all k, $i = 1, 2$, and all (x_1, x_2, x_3)
- $A(i, (x_1, x_2, x_3), (A, B, C)) = \{1, 2\}$ for all i, all (x_1, x_2, x_3), and all triples (A, B, C) in which A, B, and C are all either *yes* or *no* and at least two are *yes*, and $A(i, (x_1, x_2, x_3), (A, B, C)) = \{1, 2, 3\}$ for all i, all (x_1, x_2, x_3), and all triples (A, B, C) in which A, B, and C are all either *yes* or *no* and at most one is *yes*

- $A(i,(x_1,x_2,x_3),(A,B,C),j) = \{(x_1,x_2,x_3) : x_1+x_2+x_3 = 1, x_i \geq 0 \textit{ for all } i\}$ for $i = 1, 2$, all (x_1,x_2,x_3), all triples (A,B,C) in which A, B, and C are all either *yes* or *no*, and $j = 1,2,3$
- $A_k(i,(x_1,x_2,x_3),(A,B,C),j,(y_1,y_2,y_3)) = \{yes, no\}$ for all k, $i = 1,2$, all (x_1,x_2,x_3), all triples (A,B,C) in which A, B, and C are all either *yes* or *no*, $j = 1,2,3$, and all (y_1,y_2,y_3).

Preferences Each legislator i ranks the terminal histories by the amount of money she receives: $x_i + y_i$ if both bills are passed, x_i if only the first bill is passed, y_i if only the second bill is passed, and 0 if neither bill is passed.

To find the subgame perfect equilibria, start with the second period. The defeat of a bill leads each legislator to obtain the payoff of 0, so each legislator optimally votes for every bill. Thus in any subgame perfect equilibrium the proposer's bill gives the proposer all the pie, and at least one of the other legislators votes for the bill. (As before, each of the other legislators is indifferent between voting for and voting against the bill, but there is no subgame perfect equilibrium in which the bill is voted down.)

In the first period, the same argument shows that the proposer's bill gives the proposer all the pie and that this bill passes. Further, in this period the other member of the governing coalition definitely votes for the bill. The reason is that if she does so, then her chance of being the proposer in the next period is $\frac{1}{2}$, so that her expected payoff is $\frac{1}{2}$. If she votes against, then the bill fails, so that she obtains a payoff of 0 in the first period and has a probability of $\frac{2}{3}$ of being in the governing coalition in the second period, so that her expected payoff is $\frac{1}{3}$. Thus she is better off voting for her comrade's bill than against it.

In summary, in every subgame perfect equilibrium of the UK game the strategy of each legislator i has the following properties:

- after the move of chance in either period, propose the bill that gives legislator i the payoff 1
- after a bill is proposed in the first period, vote for the bill if i is a member of the governing coalition.

Thus in the UK game the governing coalition is entirely cohesive.

226.1 Nash equilibria when players may make mistakes

The players' best response functions are indicated in Figure 120.1. We see that the game has two Nash equilibria, (A,A,A) and (B,A,A).

The action A is not weakly dominated for any player. For player 1, A is better than B if players 2 and 3 both choose B; for players 2 and 3, A is better than B for all actions of the other players.

If players 2 and 3 choose A in the modified game, player 1's expected payoffs to A and B are

	A	B
A	$1^*, 1^*, 1^*$	$0, 0, 1^*$
B	$1^*, 1^*, 1^*$	$1^*, 0, 1^*$

A

	A	B
A	$0, 1^*, 0$	$1^*, 0, 0$
B	$1^*, 1^*, 0$	$0, 0, 0$

B

Figure 120.1 The player's best response functions in the game in Exercise 226.1.

$$A\colon\ (1-p_2)(1-p_3) + p_1p_2(1-p_3) + p_1(1-p_2)p_3 + (1-p_1)p_2p_3$$

$$B\colon\ (1-p_2)(1-p_3) + (1-p_1)p_2(1-p_3) + (1-p_1)(1-p_2)p_3 + p_1p_2p_3.$$

The difference between the expected payoff to B and the expected payoff to A is

$$(1-2p_1)[p_2 + p_3 - 3p_2p_3].$$

If $0 < p_i < \frac{1}{2}$ for $i = 1, 2, 3$, this difference is positive, so that (A, A, A) is not a Nash equilibrium of the modified game.

228.1 Nash equilibria of the chain-store game

Any terminal history in which the event in each period is either *Out* or (In, A) is the outcome of a Nash equilibrium. In any period in which challenger chooses *Out*, the strategy of the chain-store specifies that it choose F in the event that the challenger chooses *In*.

229.1 Subgame perfect equilibrium of the chain-store game

The outcome of the strategy pair is that the only the last 10 challengers enter, and the chain-store acquiesces to their entry. The payoff of each of the first 90 challengers is 1 and the payoff to the remaining 10 is 2. The chain-store's payoff is $90 \times 2 + 10 \times 1 = 190$.

No challenger can profitably deviate in any subgame (if one of the first 90 enters it is fought). However, I claim that the chain-store can increase its payoff by deviating after a history in which the first 89 challengers enter and are fought, and then challenger 90 enters. The chain-store's strategy calls for it to fight challenger 90 and then subsequently acquiesce to any entry, and the remaining challengers' strategies call for them to enter. But if instead the chain-store acquiesces to challenger 90, keeping the rest of its strategy the same, it increases its payoff by 1.

(Note that the chain-store cannot profitably deviate after a history in which fewer than 89 challengers enter and each of them is fought. Suppose, for example, that each of the first 88 challengers enters and is fought, and then challenger 89 enters. The chain-store's strategy calls for it to fight challenger 89, which induces challenger 90 to stay out; the remaining challengers enter, and the chain-store acquiesces. Its best deviation is to acquiesce to challenger 89's entry and that of all subsequent entrants, in which case all remaining challengers, including challenger 90, enter. The outcomes of the two strategies differ in periods 89 and 90. If the challenger sticks to its original strategy it obtains 0 in period 89 and 2 in period 90; if it deviates it obtains 1 in each period.)

229.3 Nash equilibria of the centipede game

Consider a strategy pair that results in an outcome in which player 1 stops the game in period $k \geq 2$. (That is, each player chooses C through period $k - 1$ and the player who moves in period k chooses S.) Such a pair is not a Nash equilibrium because the player who moves in period $k - 1$ can do better (in the whole game, not only the subgame) by choosing S rather than C, given the other player's strategy. Similarly the strategy pair in which each player always chooses C is not a Nash equilibrium. Thus in every Nash equilibrium player 1 chooses S at the start of the game.

8 Coalitional Games and the Core

241.1 Three-player majority game

Let (x_1, x_2, x_3) be an action of the grand coalition. Every coalition consisting of two players can obtain one unit of output, so for (x_1, x_2, x_3) to be in the core we need

$$\begin{aligned} x_1 + x_2 &\geq 1 \\ x_1 + x_3 &\geq 1 \\ x_2 + x_3 &\geq 1 \\ x_1 + x_2 + x_3 &= 1. \end{aligned}$$

Adding the first three conditions we conclude that

$$2x_1 + 2x_2 + 2x_3 \geq 3,$$

or $x_1 + x_2 + x_3 \geq \frac{3}{2}$, contradicting the last condition. Thus no action of the grand coalition satisfies all the conditions, so that the core of the game is empty.

In the variant in which player 1 has three votes, a coalition can obtain one unit of output if and only if it contains player 1. (Note that players 2 and 3 together do not have a majority of the votes.) Thus for (x_1, x_2, x_3) to be in the core we need

$$\begin{aligned} x_1 &\geq 1 \\ x_1 + x_2 &\geq 1 \\ x_1 + x_3 &\geq 1 \\ x_1 + x_2 + x_3 &= 1. \end{aligned}$$

The first and last conditions (and the restriction that amounts of output must be nonnegative) imply that $(x_1, x_2, x_3) = (1, 0, 0)$, which satisfies the other two conditions. Thus the core consists of the single action $(1, 0, 0)$ in which player 1 obtains all the output.

242.1 Market with one owner and two heterogeneous buyers

By the arguments in Example 241.2, in any action in the core the owner does not keep the good, the buyer who obtains the good pays at most her valuation, and the other buyer makes no payment. Let a_N be an action of the grand coalition in which buyer 2 obtains the good and pays the owner p, and buyer 1 makes no payment. Then $p \leq v < 1$, so that the coalition consisting of the owner and buyer 1 can improve upon a_N: if the owner transfers the good to buyer 1 in exchange for $\frac{1}{2}(1 + p)$ units of money, both the owner and buyer 1 are better off than they are in a_N. Thus in any action in the core, buyer 1 obtains the good. The price she pays is at least v (otherwise the coalition consisting of the owner and buyer 2 can improve upon the action). No coalition can improve upon any action in which buyer 1 obtains the good and pays the owner at least v and at most 1 (and buyer 2 makes no payment), so the core consists of all such actions.

242.2 Vote trading

a. The core consists of the single action in which all three bills pass, yielding each legislator a payoff of 2. This action cannot be improved upon by any coalition because no single bill or pair of bills gives every member of any majority coalition a payoff of more than 2.

No other action is in the core, by the following argument.

- The action in which no bill passes (so that each legislator's payoff is 0) can be improved upon by the coalition of all three legislators, which by passing all three bills raises the payoff of each legislator to 2.
- The action in which only A passes can be improved upon by the coalition of legislators 2 and 3, who by passing bills A and B raise both of their payoffs.
- Similarly the action in which only B passes can be improved upon by the coalition of legislators 1 and 3, and the action in which only C passes can be improved upon by the coalition of legislators 1 and 2.
- The action in which bills A and B pass can be improved upon by the coalition of legislators 1 and 3, who by passing all three bills raise both their payoffs.
- Similarly the action in which bills A and C pass can be improved upon by the coalition of legislators 2 and 3, and the action in which bills B and C pass can be improved upon by the coalition of legislators 1 and 2.

b. The core consists of two actions: all three bills pass, and bills A and B pass. As in part *a*, the action in which all three bills pass cannot be improved upon by any coalition. The action in which bills A and B cannot be improved upon either: for no other set of bills are at least two legislators better off.

No other action is in the core, by the following argument.

- The action in which A passes can be improved upon by the coalition consisting of legislators 2 and 3, who can pass B instead.

- The action in which B passes can be improved upon by the coalition consisting of legislators 1 and 2, who can pass A and B instead.
- The action in which C passes can be improved upon by the coalition consisting of legislators 2 and 3, who can pass B instead.
- The action in which A and C pass can be improved upon by the coalition consisting of legislators 2 and 3, who can pass A and B instead.
- The action in which B and C pass can be improved upon by the coalition consisting of legislators 1 and 2, who can pass A and B instead.

c. The core is empty.

- The action in which no bill passes can be improved upon by the coalition consisting of legislators 1 and 2, who can pass A and B instead.
- The action in which any single bill passes can be improved upon by the coalition consisting of the two legislators whose payoffs are -1 if this bill passes; this coalition can do better by passing the other two bills.
- The action in which bills A and B pass can be improved upon by the coalition consisting of legislators 2 and 3, who can pass B instead.
- Similarly the action in which A and C pass can be improved upon by the coalition consisting of legislators 1 and 2, who can pass A instead, and the action in which B and C pass can be improved upon by the coalition consisting of legislators 1 and 2, who can pass B instead.
- The action in which all three bills pass can be improved upon by the coalition consisting of legislators 1 and 2, who can pass A and B instead.

244.1 Core of landowner–worker game

Let a_N be an action of the grand coalition in which the output received by each worker is at most $f(n) - f(n-1)$. No coalition consisting solely of workers can obtain any output, so no such coalition can improve upon a_N. Let S be a coalition of the landowner and $k-1$ workers. The total output received by the members of S in a_N is at least

$$f(n) - (n-k)(f(n) - f(n-1))$$

(because the total output is $f(n)$, and every *other* worker receives at most $f(n) - f(n-1)$). Now, the output that S can obtain is $f(k)$, so for S to improve upon a_N we need

$$f(k) > f(n) - (n-k)(f(n) - f(n-1)),$$

which contradicts the inequality given in the exercise.

244.2 Unionized workers in landowner–worker game

The following game models the situation.

Players The landowner and the workers.

Actions The set of actions of the grand coalition is the set of all allocations of the output $f(n)$. Every other coalition has a single action, which yields the output 0.

Preferences Each player's preferences are represented by the amount of output she obtains.

The core of this game consists of every allocation of the output $f(n)$ among the players. The grand coalition cannot improve upon any allocation x because for every other allocation x' there is at least one player whose payoff is lower in x' than it is in x. No other coalition can improve upon any allocation because no other coalition can obtain any output.

245.1 Landowner–worker game with increasing marginal products

We need to show that no coalition can improve upon the action a_N of the grand coalition in which every player receives the output $f(n)/n$. No coalition of workers can obtain any output, so we need to consider only coalitions containing the landowner. Consider a coalition consisting of the landowner and k workers, which can obtain $f(k+1)$ units of output by itself. Under a_N this coalition obtains the output $(k+1)f(n)/n$, and we have $f(k+1)/(k+1) < f(n)/n$ because $k < n$. Thus no coalition can improve upon a_N.

250.1 Range of prices in horse market

The equality of the number of owners who sell their horses and the number of nonowners who buy horses implies that the common trading price p^*

- is not less than σ_{k^*}, otherwise at most $k^* - 1$ owners' valuations would be less than p^* and at least k^* nonowners' valuations would be greater than p^*, so that the number of buyers would exceed the number of sellers

- is not less than β_{k^*+1}, otherwise at most k^* owners' valuations would be less than p^* and at least $k^* + 1$ nonowners' valuations would be greater than p^*, so that the number of buyers would exceed the number of sellers

- is not greater than β_{k^*}, otherwise at least k^* owners' valuations would be less than p^* and at most $k^* - 1$ nonowners' valuations would be greater than p^*, so that the number of sellers would exceed the number of buyers

- is not greater than σ_{k^*+1}, otherwise at least $k^* + 1$ owners' valuations would be less than p^* and at most k^* nonowners' valuations would be greater than p^*, so that the number of sellers would exceed the number of buyers.

That is, $p^* \geq \max\{\sigma_{k^*}, \beta_{k^*+1}\}$ and $p^* \leq \min\{\beta_{k^*}, \sigma_{k^*+1}\}$.

251.1 Horse trading game with single seller

The core consists of the set of actions of the grand coalition in which the owner sells her horse to the nonowner with the highest valuation (nonowner 1) at a price p^* for which $\max\{\beta_2, \sigma_1\} \leq p^* \leq \beta_1$. (The coalition consisting of the owner and nonwoner 2 can improve any action in which the price is less than β_2, the owner alone can improve upon any action in which the price is less than σ_1, and nonowner 1 alone can improve upon any action in which the price is greater than β_1.)

251.2 Horse trading game with large seller

In every action in the core, the owner sells one horse to buyer 1 and one horse to buyer 2. The prices at which the trades occur are not necessarily the same. The price p_1 paid by buyer 1 satisfies $\max\{\beta_3, \sigma_1\} \leq p_1 \leq \beta_1$ and the price p_2 paid by buyer 2 satisfies $\max\{\beta_3, \sigma_1\} \leq p_1 \leq \beta_2$.

254.1 House assignment with identical preferences

Because the players rank the houses in the same way, we can refer to the "best house", the "second best house", and so on. In any assignment in the core, the player who owns the best house is assigned this house (because she has the option of keeping it). Among the remaining players, the one who owns the second best house must be assigned this house (again, because she has the option of keeping it). Continuing to argue in the same way, we see that there is a single assignment in the core, in which every player is assigned the house she owns initially.

255.1 Emptiness of the strong core when preferences are not strict

Of the six possible assignments, $h_1h_2h_3$ (i.e. every player keeps the house she owns) and $h_3h_2h_1$ can both be improved upon by $\{1,2\}$ (and by $\{2,3\}$). All four of the other assignments are in the core.

None of the assignments in the core is in the strong core. The assignments $h_1h_3h_2$ and $h_3h_1h_2$ can both be weakly improved upon by $\{1,2\}$, and $h_2h_1h_3$ and $h_2h_3h_1$ can both be weakly improved upon by $\{2,3\}$.

257.1 Median voter theorem

Denote the median favorite position by m. If $x < m$ then every player whose favorite position is m or greater—a majority of the players—prefers m to x. Similarly, if $x > m$ then every player whose favorite position is m or less—a majority of the players—prefers m to x.

258.1 Cores of q-rule games

a. Denote the favorite policy of player i by x_i^* and number the players so that $x_1^* \leq \cdots \leq x_n^*$. The q-core is the set of all policies x for which

$$x_{n-q+1}^* \leq x \leq x_q^*.$$

Any such policy x is in the core because every coalition of q players contains at least one player whose favorite position is less than x and at least one player whose favorite position is greater than x, so that there is no position $y \neq x$ that all members of the coalition prefer to x.

Any policy $x < x_{n-q+1}^*$ is not in the core because the coalition of players $n - q + 1$ through n can improve upon x: this coalition contains q players, all of whom prefer x_{n-q+1}^* to x. Similarly, no policy greater than x_q^* is in the core.

b. The core is the set of policies in the triangle defined by x_1^*, x_2^*, and x_3^*.

Every policy x in this set is in the core because for every other policy $y \neq x$ at least one player is worse off than she is at x.

No policy x outside the set is in the core because the policy $y \neq x$ closest to x in the set is preferred by all three players.

262.1 Deferred acceptance procedure with proposals by Y's

For the preferences given in Figure 260.1, the progress of the procedure when proposals are made by Y's is given in Figure 128.1. The matching produced is the same as that produced by the procedure when proposals are made by X's, namely (x_1, y_1), (x_2, y_2), x_3 (alone), and y_3 (alone).

	Stage 1		Stage 2		Stage 3	
y_1:	$\to x_1$					
y_2:	$\to x_2$					
y_3:	$\to x_1$	reject	$\to x_3$	reject	$\to x_2$	reject

Figure 128.1 The progress of the deferred acceptance procedure with proposals by Y's when the players' preferences are those given in Figure 260.1. Each row gives the proposals of one X.

262.2 Example of deferred acceptance procedure

For the preferences in Figure 262.1, the procedure when proposals are made by X's yields the matching (x_1, y_1), (x_2, y_2), (x_3, y_3); the procedure when proposals are made by Y's yields the matching (x_1, y_1), (x_2, y_3), (x_3, y_2).

In any matching in the core, x_1 and y_1 are matched, because each is the other's top-ranked partner. Thus the only two possible matchings are those generated by the two procedures. Player x_2 prefers y_2 to y_3 and player x_3 prefers y_3 to y_2, so the matching generated by the procedure when proposals are made by X's yields each X a better partner than does the matching generated by the procedure when proposals are made by Y's. Similarly, player y_2 prefers x_3 to x_2 and player y_3 prefers x_2 to x_3, so the matching generated by the procedure when proposals are made by Y's yields each Y a better partner than does the matching generated by the procedure when proposals are made by X's.

263.1 Strategic behavior under the deferred acceptance procedure

The matching produced by the deferred acceptance procedure with proposals by X's is (x_1, y_2), (x_2, y_3), (x_3, y_1). The matching produced by the deferred acceptance procedure with proposals by Y's is (x_1, y_1), (x_2, y_3), (x_3, y_2). Of the four other matchings, the coalition $\{x_3, y_2\}$ can improve upon (x_1, y_1), (x_2, y_2), (x_3, y_3) and (x_1, y_2), (x_2, y_1), (x_3, y_3), and the coalition $\{x_1, y_1\}$ can improve upon (x_1, y_3), (x_2, y_1), (x_3, y_2) and (x_1, y_3), (x_2, y_2), (x_3, y_1). Thus the core consists of the two matchings produced by the deferred acceptance procedures.

If y_1 names the ranking (x_1, x_2, x_3) and every other player names her true ranking, the deferred acceptance procedure with proposals by X's yields the matching (x_1, y_1), (x_2, y_3), (x_3, y_2), as illustrated in Figure 129.1. Players y_1 and y_2 are matched with their favorite partners, so cannot profitably deviate by submitting any other ranking. Player y_3's ranking does not affect the outcome of the procedure. Thus, given that submitting her true ranking is a dominant strategy for every X, the game thus has a Nash equilibrium in which player y_1 submits the ranking (x_1, x_2, x_3) and every other player submits her true ranking.

	Stage 1		Stage 2		Stage 3		Stage 4
x_1:	$\to y_2$			reject	$\to y_1$		
x_2:	$\to y_1$					reject	$\to y_3$
x_3:	$\to y_1$	reject	$\to y_2$				

Figure 129.1 The progress of the deferred acceptance procedure with proposals by X's when the players' preferences differ from those in Exercise 263.1 only in that y_1's ranking is (x_1, x_2, x_3). Each row gives the proposals of one X.

263.2 Empty core in roommate problem

Notice that ℓ is at the bottom of each of the other players' preferences. Suppose that she is matched with i. Then j and k are matched, and $\{i, k\}$ can improve upon the matching. Similarly, if ℓ is matched with j then $\{i, j\}$ can improve upon the matching, and if ℓ is matched with k then $\{j, k\}$ can improve upon the matching. Thus the core is empty (ℓ has to be matched with *someone*!).

264.1 Spatial preferences in roommate problem

The core consists of the single matching μ^* defined as follows. First match the pair of players whose characteristics are closest. Then match the pair of players in the remaining set whose characteristics are closest. Continue until all players are matched.

Number the matches in the order they are made according to this procedure. If a coalition can improve upon μ^*, then a coalition consisting of two players can do so. Now, neither member of match k is better off being matched with a member of match ℓ for any $\ell > k$, so no two-player coalition can improve upon the matching. Thus μ^* is in the core.

For any other matching μ', at least one of the members of some match k defined by the procedure is matched with a different partner. If she is matched with a member of some match $\ell < k$ then the coalition consisting of the two members of match ℓ can improve μ'; if she is matched with a member of some match $\ell > k$ then the coalition consisting of the two member of match k can improve upon μ'. Thus no matching $\mu' \neq \mu^*$ is in the core.

9 Bayesian games

274.1 Equilibria of a variant of BoS with imperfect information

If player 1 chooses S then type 1 of player 2 chooses S and type 2 chooses B. But if the two types of player 2 make these choices then player 1 is better off choosing B (which yields her an expected payoff of 1) than choosing S (which yields her an expected payoff of $\frac{1}{2}$). Thus there is no Nash equilibrium in which player 1 chooses S.

Now consider the mixed strategy Nash equilibria. If both types of player 2 use a pure strategy then player 1's two actions yield her different payoffs. Thus there is no equilibrium in which both types of player 2 use pure strategies and player 1 randomizes.

Now consider an equilibrium in which type 1 of player 2 randomizes. Denote by p the probability that player 1's mixed strategy assigns to B. In order for type 1 of player 2 to obtain the same expected payoff to B and S we need $p = \frac{2}{3}$. For this value of p the best action of type 2 of player 2 is S. Denote by q the probability that type 1 of player 2 assigns to B. Given these strategies for the two types of player 2, player 1's expected payoff if she chooses B is

$$\tfrac{1}{2} \cdot 2q = q$$

and her expected payoff if she chooses S is

$$\tfrac{1}{2} \cdot (1 - q) + \tfrac{1}{2} \cdot 1 = 1 - \tfrac{1}{2}q.$$

These expected payoffs are equal if and only if $q = \frac{2}{3}$. Thus the game has a mixed strategy equilibrium in which the mixed strategy of player 1 is $(\frac{2}{3}, \frac{1}{3})$, that of type 1 of player 2 is $(\frac{2}{3}, \frac{1}{3})$, and that of type 2 of player 2 is $(0, 1)$ (that is, type 2 of player 2 uses the pure strategy that assigns probability 1 to S).

Similarly the game has a mixed strategy equilibrium in which the strategy of player 1 is $(\frac{1}{3}, \frac{2}{3})$, that of type 1 of player 2 is $(0, 1)$, and that of type 2 of player 2 is $(\frac{2}{3}, \frac{1}{3})$.

For no mixed strategy of player 1 are both types of player 2 indifferent between their two actions, so there is no equilibrium in which both types randomize.

275.1 Expected payoffs in a variant of BoS with imperfect information

The expected payoffs are given in Figure 132.1.

	(B,B)	(B,S)	(S,B)	(S,S)
B	0	1	1	2
S	1	$\frac{1}{2}$	$\frac{1}{2}$	0

Type n_1 of player 1

	(B,B)	(B,S)	(S,B)	(S,S)
B	1	$\frac{2}{3}$	$\frac{1}{3}$	0
S	0	$\frac{2}{3}$	$\frac{4}{3}$	2

Type y_2 of player 2

	(B,B)	(B,S)	(S,B)	(S,S)
B	0	$\frac{1}{3}$	$\frac{2}{3}$	1
S	2	$\frac{4}{3}$	$\frac{2}{3}$	0

Type n_2 of player 2

Figure 132.1 The expected payoffs of type n_1 of player 1 and types y_2 and n_2 of player 2 in Example 274.2.

280.2 Fighting an opponent of unknown strength

The following Bayesian game models the situation.

Players The two people.

States The set of states is $\{strong, weak\}$.

Actions The set of actions of each player is $\{fight, yield\}$.

Signals Player 1 receives the same signal in each state, whereas player 2 receives different signals in the two states.

Beliefs The single type of player 1 assigns probability α to the state *strong* and probability $1 - \alpha$ to the state *weak*. Each type of player 2 assigns probability 1 to the single state consistent with her signal.

Payoffs The players' Bernoulli payoffs are shown in Figure 133.1.

The best responses of each type of player 2 are indicated by asterisks in Figure 133.1. Thus if $\alpha < \frac{1}{2}$ then player 1's best action is *fight*, whereas if $\alpha > \frac{1}{2}$ her best action is *yield*. Thus for $\alpha < \frac{1}{2}$ the game has a unique Nash equilibrium, in which player 1 chooses *fight* and player 2 chooses *fight* if she is strong and *yield* if she is weak, and if $\alpha > \frac{1}{2}$ the game has a unique Nash equilibrium, in which player 1 chooses *yield* and player 2 chooses *fight* whether she is strong or weak.

State: *strong*

	F	Y
F	$-1,\ 1^*$	$1,\ 0$
Y	$0,\ 1^*$	$0,\ 0$

State: *weak*

	F	Y
F	$1, -1$	$1,\ 0^*$
Y	$0,\ 1^*$	$0,\ 0$

Figure 133.1 The player's Bernoulli payoff functions in Exercise 280.2. The asterisks indicate the best responses of each type of player 2.

280.3 An exchange game

The following Bayesian game models the situation.

Players The two individuals.

States The set of all pairs (s_1, s_2), where s_i is the number on player i's ticket (an integer from 1 to m).

Actions The set of actions of each player is $\{Exchange, Don't\ exchange\}$.

Signals The signal function of each player i is defined by $\tau_i(s_1, s_2) = s_i$ (each player observes her own ticket, but not that of the other player)

Beliefs Type s_i of player i assigns the probability $\Pr_j(s_j)$ to the state (s_1, s_2), where j is the other player and $\Pr_j(s_j)$ is the probability with which player j receives a ticket with the prize s_j on it.

Payoffs Player i's Bernoulli payoff function is given by $u_i((X, Y), \omega) = \omega_j$ if $X = Y = Exchange$ and $u_i((X, Y), \omega) = \omega_i$ otherwise.

Let M_i be the highest type of player i that chooses *Exchange*. If $M_i > 1$ then type 1 of player j optimally chooses *Exchange*: by exchanging her ticket, she cannot obtain a smaller prize, and may receive a bigger one. Thus if $M_i \geq M_j$ and $M_i > 1$, type M_i of player i optimally chooses *Don't exchange*, because the expected value of the prizes of the types of player j that choose *Exchange* is less than M_i. Thus in any possible Nash equilibrium $M_i = M_j = 1$: the only prizes that may be exchanged are the smallest.

280.4 Adverse selection

The game is defined as follows.

Players Firms A and T.

States The set of possible values of firm T (the integers from 0 to 100).

Actions Firm A's set of actions is its set of possible bids (nonnegative numbers), and firm T's set of actions is the set of possible cutoffs (nonnegative numbers) above which it will accept A's offer.

Signals Firm A receives the same signal in every state; firm T receives a different signal in every state.

Beliefs The single type of firm A assigns an equal probability to each state; each type of firm T assigns probability 1 to the single state consistent with its signal.

Payoff functions If firm A bids y, firm T's cutoff is at most y, and the state is x, then A's payoff is $\frac{3}{2}x - y$ and T's payoff is y. If firm A bids y, firm T's cutoff is greater than y, and the state is x, then A's payoff is 0 and T's payoff is x.

To find the Nash equilibria of this game, first consider the behavior of each type of firm T. Type x is at least as well off accepting the offer y than it is rejecting it if and only if $y \geq x$. Thus type x's optimal cutoff for accepting offers is x, regardless of firm A's action.

Now consider firm A. If it bids y then each type x of T with $x < y$ accepts its offer, and each type x of T with $x > y$ rejects the offer. Thus the expected value of the type that accepts an offer $y \leq 100$ is $\frac{1}{2}y$, and the expected value of the type that accepts an offer $y > 100$ is 50. If the offer y is accepted then A's payoff is $\frac{3}{2}x - y$, so that its expected payoff is $\frac{3}{2}(\frac{1}{2}y) - y = -\frac{1}{4}y$ if $y \leq 100$ and $\frac{3}{2}(50) - y = 75 - y$ if $y > 100$. Thus firm A's optimal bid is 0!

We conclude that the game has a unique Nash equilibrium, in which firm A bids 0 and the cutoff for accepting an offer for each type x of firm T is x.

Even though firm A can increase firm T's value, it is not willing to make a positive bid in equilibrium because firm T's interest is in accepting only offers that exceed its value, so that the average type that accepts an offer has a value of only half the offer. As A decreases its offer, the value of the average firm that accepts the offer decreases: the selection of firms that accept the offer is adverse to A's interest.

282.1 Infection argument

In any Nash equilibrium, the action of player 1 when she receives the signal $\tau_1(\alpha)$ is R, because R strictly dominates L.

Now suppose that player 2's signal is $\tau_2(\alpha) = \tau_2(\beta)$. I claim that her best action is R, regardless of player 1's action in state β. If player 1 chooses L in state β then player 2's expected payoff to L is $\frac{3}{4} \cdot 0 + \frac{1}{4} \cdot 2 = \frac{1}{2}$, and her expected payoff to R is $\frac{3}{4} \cdot 1 + \frac{1}{4} \cdot 0 = \frac{3}{4}$. If player 1 chooses R in state β then player 2's expected payoff to L is 0, and her expected payoff to R is 1. Thus in any Nash equilibrium player 2's action when her signal is $\tau_2(\alpha) = \tau_2(\beta)$ is R.

Now suppose that player 1's signal is $\tau_1(\beta) = \tau_1(\gamma)$. By the same argument as in the previous paragraph, player 1's best action is R, regardless of player 2's action in state γ. Thus in any Nash equilibrium player 1's action in this case is R.

Finally, given that player 1's action in state γ is R, player 2's best action in this state is also R.

285.1 Cournot's duopoly game with imperfect information

We have

$$b_1(q_L, q_H) = \begin{cases} \frac{1}{2}(\alpha - c - (\theta q_L + (1-\theta)q_H)) & \text{if } \theta q_L + (1-\theta)q_H \leq \alpha - c \\ 0 & \text{otherwise.} \end{cases}$$

The best response function of each type of player 2 is similar:

$$b_I(q_1) = \begin{cases} \frac{1}{2}(\alpha - c_I - q_1) & \text{if } q_1 \leq \alpha - c_I \\ 0 & \text{otherwise} \end{cases}$$

for $I = L, H$.

The three equations that define a Nash equilibrium are

$$q_1^* = b_1(q_L^*, q_H^*),\ q_L^* = b_L(q_1^*), \text{ and } q_H^* = b_H(q_1^*).$$

Solving these equations under the assumption that they have a solution in which all three outputs are positive, we obtain

$$\begin{aligned} q_1^* &= \tfrac{1}{3}(\alpha - 2c + \theta c_L + (1-\theta)c_H) \\ q_L^* &= \tfrac{1}{3}(\alpha - 2c_L + c) - \tfrac{1}{6}(1-\theta)(c_H - c_L) \\ q_H^* &= \tfrac{1}{3}(\alpha - 2c_H + c) + \tfrac{1}{6}\theta(c_H - c_L) \end{aligned}$$

If both firms know that the unit costs of the two firms are c_1 and c_2 then in a Nash equilibrium the output of firm i is $\frac{1}{3}(\alpha - 2c_i + c_j)$ (see Exercise 57.1). In the case of imperfect information considered here, firm 2's output is less than $\frac{1}{3}(\alpha - 2c_L + c)$ if its cost is c_L and is greater than $\frac{1}{3}(\alpha - 2c_H + c)$ if its cost is c_H. Intuitively, the reason is as follows. If firm 1 knew that firm 2's cost were high then it would produce a relatively large output; if it knew this cost were low then it would produce a relatively small output. Given that it does not know whether the cost is high or low it produces a moderate output, less than it would if it knew firm 2's cost were high. Thus if firm 2's cost is in fact high, firm 2 benefits from firm 1's lack of knowledge and optimally produces more than it would if firm 1 knew its cost.

286.1 Cournot's duopoly game with imperfect information

The best response $b_0(q_L, q_H)$ of type 0 of firm 1 is the solution of

$$\max_{q_0} [\theta(P(q_0 + q_L) - c)q_0 + (1-\theta)(P(q_0 + q_H) - c)q_0].$$

The best response $b_\ell(q_L, q_H)$ of type ℓ of firm 1 is the solution of

$$\max_{q_\ell} (P(q_\ell + q_L) - c)q_\ell$$

and the best response $b_h(q_L, q_H)$ of type h of firm 1 is the solution of

$$\max_{q_h}(P(q_h + q_H) - c)q_h.$$

The best response $b_L(q_0, q_\ell, q_h)$ of type L of firm 2 is the solution of

$$\max_{q_L}[(1-\pi)(P(q_0 + q_L) - c_L)q_L + \pi(P(q_\ell + q_L) - c_L)q_L]$$

and the best response $b_H(q_0, q_\ell, q_h)$ of type H of firm 2 is the solution of

$$\max_{q_H}[(1-\pi)(P(q_0 + q_H) - c_H)q_H + \pi(P(q_h + q_H) - c_H)q_H].$$

A Nash equilibrium is a profile $(q_0^*, q_\ell^*, q_h^*, q_L^*, q_H^*)$ for which q_0^*, q_ℓ^*, and q_h^* are best responses to q_L^* and q_H^*, and q_L^* and q_H^* are best responses to q_0^*, q_ℓ^*, and q_h^*. When $P(Q) = \alpha - Q$ for $Q \leq \alpha$ and $P(Q) = 0$ for $Q > \alpha$ we find, after some exciting algebra, that

$$\begin{aligned}
q_0^* &= \frac{1}{3}\left(\alpha - 2c + c_H - \theta\left(c_H - c_L\right)\right) \\
q_\ell^* &= \frac{1}{3}\left(\alpha - 2c + c_L + \frac{(1-\theta)(1-\pi)(c_H - c_L)}{4-\pi}\right) \\
q_H^* &= \frac{1}{3}\left(\alpha - 2c + c_H - \frac{\theta(1-\pi)(c_H - c_L)}{4-\pi}\right) \\
q_L^* &= \frac{1}{3}\left(\alpha - 2c_L + c - \frac{2(1-\theta)(1-\pi)(c_H - c_L)}{4-\pi}\right) \\
q_H^* &= \frac{1}{3}\left(\alpha - 2c_H + c + \frac{2\theta(1-\pi)(c_H - c_L)}{4-\pi}\right).
\end{aligned}$$

When $\pi = 0$ we have

$$\begin{aligned}
q_0^* &= \frac{1}{3}\left(\alpha - 2c + c_H - \theta\left(c_H - c_L\right)\right) \\
q_\ell^* &= \frac{1}{3}\left(\alpha - 2c + c_L + \frac{(1-\theta)(c_H - c_L)}{4}\right) \\
q_H^* &= \frac{1}{3}\left(\alpha - 2c + c_H - \frac{\theta(c_H - c_L)}{4}\right) \\
q_L^* &= \frac{1}{3}\left(\alpha - 2c_L + c - \frac{(1-\theta)(c_H - c_L)}{2}\right) \\
q_H^* &= \frac{1}{3}\left(\alpha - 2c_H + c + \frac{\theta(c_H - c_L)}{2}\right),
\end{aligned}$$

so that q_0^* is equal to the equilibrium output of firm 1 in Exercise 285.1, and q_L^* and q_H^* are the same as the equilibrium outputs of the two types of firm 2 in that exercise.

When $\pi = 1$ we have

$$\begin{aligned} q_0^* &= \frac{1}{3}\left(\alpha - 2c + c_H - \theta\left(c_H - c_L\right)\right) \\ q_\ell^* &= \frac{1}{3}\left(\alpha - 2c + c_L\right) \\ q_H^* &= \frac{1}{3}\left(\alpha - 2c + c_H\right) \\ q_L^* &= \frac{1}{3}\left(\alpha - 2c_L + c\right) \\ q_H^* &= \frac{1}{3}\left(\alpha - 2c_H + c\right), \end{aligned}$$

so that q_ℓ^* and q_L^* are the same as the equilibrium outputs when there is perfect information and the costs are c and c_L (see Exercise 57.1), and q_h^* and q_H^* are the same as the equilibrium outputs when there is perfect information and the costs are c and c_H.

Now, for an arbitrary value of π we have

$$\begin{aligned} q_L^* &= \frac{1}{3}\left(\alpha - 2c_L + c - \frac{2(1-\theta)(1-\pi)(c_H - c_L)}{4-\pi}\right) \\ q_H^* &= \frac{1}{3}\left(\alpha - 2c_H + c + \frac{2\theta(1-\pi)(c_H - c_L)}{4-\pi}\right). \end{aligned}$$

To show that for $0 < \pi < 1$ the values of these variables lie between their values when $\pi = 0$ and when $\pi = 1$, we need to show that

$$0 \le \frac{2(1-\theta)(1-\pi)(c_H - c_L)}{4-\pi} \le \frac{(1-\theta)(c_L - c_H)}{2}$$

and

$$0 \le \frac{2\theta(1-\pi)(c_H - c_L)}{4-\pi} \le \frac{\theta(c_L - c_H)}{2}.$$

These inequalities follow from $c_H \ge c_L$, $\theta \ge 0$, and $0 \le \pi \le 1$.

288.1 Nash equilibria of game of contributing to a public good

Any type v_j of any player j with $v_j < c$ obtains a negative payoff if she contributes and 0 if she does not. Thus she optimally does not contribute.

Any type $v_i \ge c$ of player i obtains the payoff $v_i - c \ge 0$ if she contributes, and the payoff 0 if she does not, so she optimally contributes.

Any type $v_j \ge c$ of any player $j \ne i$ obtains the payoff $v_j - c$ if she contributes, and the payoff $(1 - F(c))v_j$ if she does not. (If she does not contribute, the probability that player i does so is $1 - F(c)$, the probability that player i's valuation is at least c.) Thus she optimally does not contribute if $(1 - F(c))v_j \ge v_j - c$, or $F(c) \le c/v_j$. This condition must hold for all types of every player $j \ne i$, so we need $F(c) \le c/\overline{v}$ for the strategy profile to be a Nash equilibrium.

290.1 Reporting a crime with an unknown number of witnesses

A Bayesian game that models the situation is given in Figure 138.1.

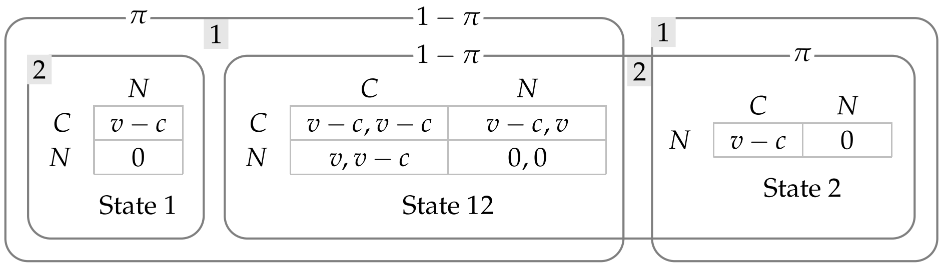


Figure 138.1 A Bayesian game that models the situation in Exercise 290.1. The action *Call* is denoted C, and the action *Don't call* is denoted N. In state 1, only player 1 is active, in state 2, only player 2 is active, and in state 12, both players are active. In states in which only one players is active, only that player's payoff is given.

A player obtains the payoff $v - c$ if she chooses C and the payoff $(1 - \pi)v$ if she chooses N. Thus the game has a pure strategy Nash equilibrium in which each player chooses C if and only if $v - c \geq (1 - \pi)v$, or $\pi \geq c/v$.

For a mixed strategy Nash equilibrium in which each player chooses C (if she is active) with probability p, where $0 < p < 1$, we need each player's expected payoffs to C and N to be the same, given that the other player chooses C with probability p. Thus we need $v - c = (1 - \pi)pv$, or

$$p = \frac{v - c}{(1 - \pi)v}.$$

If $\pi < c/v$, this number is less than 1, so that the game indeed has a mixed strategy Nash equilibrium in which each player calls with probability p.

When $\pi = 0$ we have $p = 1 - c/v$, as found in Section 4.8.

292.1 Weak domination in second-price sealed-bid action

Fix player i, and choose a bid for every type of every other player. Player i, who does not know the other players' types, is uncertain of the highest bid of the other players. Denote by $\overline{b}$ this highest bid. Consider a bid b_i of type v_i of player i for which $b_i < v_i$. The dependence of the payoff of type v_i of player i on $\overline{b}$ is shown in Figure 139.1.

Player i's expected payoffs to the bids b_i and v_i are weighted averages of the payoffs in the columns; each value of $\overline{b}$ gets the same weight when calculating the expected payoff to b_i as it does when calculating the expected payoff to v_i. The payoffs in the two rows are the same except when $b_i \leq \overline{b} < v_i$, in which case v_i yields a payoff higher than does b_i. Thus the expected payoff to v_i is at least as high as the expected payoff to b_i, and is greater than the expected payoff to b_i unless the other players' bids lead this range of values of $\overline{b}$ to get probability 0.

		Highest of other players' bids			
		$\overline{b} < b_i$	$b_i = \overline{b}$ (m-way tie)	$b_i < \overline{b} < v_i$	$\overline{b} \geq v_i$
i's bid	$b_i < v_i$	$v_i - \overline{b}$	$(v_i - \overline{b})/m$	0	0
	v_i	$v_i - \overline{b}$	$v_i - \overline{b}$	$v_i - \overline{b}$	0

Figure 139.1 Player i's payoffs to her bids $b_i < v_i$ and v_i in a second-price sealed-bid auction as a function of the highest of the other player's bids, denoted $\overline{b}$.

Now consider a bid b_i of type v_i of player i for which $b_i > v_i$. The dependence of the payoff of type v_i of player i on $\overline{b}$ is shown in Figure 139.2.

		Highest of other players' bids			
		$\overline{b} \leq v_i$	$v_i < \overline{b} < b_i$	$b_i = \overline{b}$ (m-way tie)	$\overline{b} > b_i$
i's bid	v_i	$v_i - \overline{b}$	0	0	0
	$b_i > v_i$	$v_i - \overline{b}$	$v_i - \overline{b}$	$(v_i - \overline{b})/m$	0

Figure 139.2 Player i's payoffs to her bids v_i and $b_i > v_i$ in a second-price sealed-bid auction as a function of the highest of the other player's bids, denoted $\overline{b}$.

As before, player i's expected payoffs to the bids b_i and v_i are weighted averages of the payoffs in the columns; each value of $\overline{b}$ gets the same weight when calculating the expected payoff to v_i as it does when calculating the expected payoff to b_i. The payoffs in the two rows are the same except when $v_i < \overline{b} \leq b_i$, in which case v_i yields a payoff higher than does b_i. (Note that $v_i - \overline{b} < 0$ for $\overline{b}$ in this range.) Thus the expected payoff to v_i is at least as high as the expected payoff to b_i, and is greater than the expected payoff to b_i unless the other players' bids lead this range of values of $\overline{b}$ to get probability 0.

We conclude that for type v_i of player i, every bid $b_i \neq v_i$ is weakly dominated by the bid v_i.

292.2 Nash equilibria of a second-price sealed-bid auction

For any player i, the game has a Nash equilibrium in which player i bids $\overline{v}$ (the highest possible valuation) regardless of her valuation and every other player bids $\underline{v}$ regardless of her valuation. The outcome is that player i wins and pays $\underline{v}$. Player i can do no better by bidding less; no other player can do better by bidding more, because unless she bids at least $\overline{v}$ she does not win, and if she makes such a bid her payoff is at best zero. (It is zero if her valuation is $\overline{v}$, negative otherwise.)

295.1 Auctions with risk-averse bidders

Consider player i. Suppose that the bid of each type v_j of player j is given by $\beta_j(v_j) = (1 - 1/[m(n-1)+1])v_j$. Then as far as player i is concerned, the bids of every other player are distributed uniformly between 0 and $1 - 1/[m(n-1)+1]$. Thus for $0 \leq x \leq 1 - 1/[m(n-1)+1]$, the probability that any given player's bid is less than x is $(1 + 1/[m(n+1)])x$ ($1 + 1/[m(n+1)]$ being the reciprocal of $1 - 1/[m(n-1)+1]$), and hence the probability that *all* the bids of the other $n-1$ players are less than x is $[(1+1/[m(n+1)])x]^{n-1}$. Consequently, if player i bids more than $1 - 1/[m(n-1)+1]$ then she surely wins, whereas if she bids $b_i \leq 1 - 1/[m(n-1)+1]$ she wins with probability $[(1+1/[m(n+1)])b_i]^{n-1}$. Thus player i's payoff as a function of her bid b_i is

$$\begin{cases} (v_i - b_i)^{1/m}\left\{\left(1 + \dfrac{1}{m(n+1)}\right)b_i\right\}^{n-1} & \text{if } 0 \leq b_i \leq 1 - \dfrac{1}{m(n-1)+1} \\ (v_i - b_i)^{1/m} & \text{if } b_i > 1 - \dfrac{1}{m(n-1)+1}. \end{cases} \tag{140.1}$$

Now, the value of b_i that maximizes the function

$$(v_i - b_i)^{1/m}\left\{\left(1 + \frac{1}{m(n+1)}\right)b_i\right\}^{n-1}$$

is the same as the value of b_i that maximizes the function

$$(v_i - b_i)^{1/m}(b_i)^{n-1},$$

which is $(n-1)v_i/(n-1+1/m)$ (by the mathematical fact stated in the exercise), or

$$\left(1 - \frac{1}{m(n-1)+1}\right)v_i.$$

We have

$$\left(1 - \frac{1}{m(n-1)+1}\right)v_i \leq 1 - \frac{1}{m(n-1)+1}$$

(because $v_i \leq 1$), and the function in (140.1) is decreasing in b_i for $b_i > 1 - 1/[m(n-1)+1]$, so $1 - 1/[m(n-1)+1]$ is the bid that maximizes player i's expected payoff, given that the bid of each type v_j of player j is $(1 - 1/[m(n-1)+1])v_j$.

We conclude that, as claimed, the game has a Nash equilibrium in which each type v_i of each player i bids $(1 - 1/[m(n-1)+1])v_i$.

In this equilibrium, the price paid by a bidder with valuation v who wins is $(1 - 1/[m(n-1)+1])v$ (the amount she bids). The expected price paid by a bidder in a second-price auction does not depend on the players' payoff functions. Thus this payoff is equal, by the revenue equivalence result, to the expected price paid by a bidder with valuation v who wins in a first-price auction in which each bidder is risk-neutral, namely $(1 - 1/n)v$. We have

$$\left(1 - \frac{1}{m(n-1)+1}\right) - \left(1 - \frac{1}{n}\right) = \frac{(m-1)(n-1)}{n(m(n-1)+1)},$$

which is positive because $m > 1$. Thus the expected price paid by a bidder with valuation v who wins is greater in a first-price auction than it is in a second-price auction. The probability that a bidder with any given valuation wins is the same in both auctions, so the auctioneer's expected revenue is greater in a first-price auction than it is in a second-price auction.

297.1 Asymmetric Nash equilibria of second-price sealed-bid common value auctions

Suppose that each type t_2 of player 2 bids $(1 + 1/\lambda)t_2$ and that type t_1 of player 1 bids b_1. Then by the calculations in the text, with $\alpha = 1$ and $\gamma = 1/\lambda$,

- a bid of b_1 by player 1 wins with probability $b_1/(1 + 1/\lambda)$
- the expected value of player 2's bid, *given that it is less than* b_1, is $\frac{1}{2}b_1$
- the expected value of signals that yield a bid of less than b_1 is $\frac{1}{2}b_1/(1 + 1/\lambda)$ (because of the uniformity of the distribution of t_2).

Thus player 1's expected payoff if she bids b_1 is $(t_1 + \frac{1}{2}b_1/(1 + 1/\lambda) - \frac{1}{2}b_1)b_1/(1 + 1/\lambda)$, or

$$\frac{\lambda}{2(1+\lambda)^2} \cdot (2(1+\lambda)t_1 - b_1)b_1.$$

This function is maximized at $b_1 = (1 + \lambda)t_1$. That is, if each type t_2 of player 2 bids $(1 + 1/\lambda)t_2$, any type t_1 of player 1 optimally bids $(1 + \lambda)t_1$. Symmetrically, if each type t_1 of player 1 bids $(1 + \lambda)t_1$, any type t_2 of player 2 optimally bids $(1 + 1/\lambda)t_2$. Hence the game has the claimed Nash equilibrium.

297.2 First-price sealed-bid auction with common valuations

Suppose that each type t_2 of player 2 bids $\frac{1}{2}(\alpha + \gamma)t_2$ and type t_1 of player 1 bids b_1. To determine the expected payoff of type t_1 of player 1, we need to find the probability with which she wins, and the expected value of player 2's signal if player 1 wins. (The price she pays is her bid, b_1.)

Probability of player 1's winning: Given that player 2's bidding function is $\frac{1}{2}(\alpha + \gamma)t_2$, player 1's bid of b_1 wins only if $b_1 \geq \frac{1}{2}(\alpha + \gamma)t_2$, or if $t_2 \leq 2b_1/(\alpha + \gamma)$. Now, t_2 is distributed uniformly from 0 to 1, so the probability that it is at most $2b_1/(\alpha + \gamma)$ is $2b_1/(\alpha + \gamma)$. Thus a bid of b_1 by player 1 wins with probability $2b_1/(\alpha + \gamma)$.

Expected value of player 2's signal if player 1 wins: Player 2's bid, given her signal t_2, is $\frac{1}{2}(\alpha + \gamma)t_2$, so that the expected value of signals that yield a bid of less than b_1 is $b_1/(\alpha + \gamma)$ (because of the uniformity of the distribution of t_2).

Thus player 1's expected payoff if she bids b_1 is $2(\alpha t_1 + \gamma b_1/(\alpha + \gamma) - b_1)b_1/(\alpha + \gamma)$, or

$$\frac{2\alpha}{(\alpha+\gamma)^2}((\alpha+\gamma)t_1 - b_1)b_1.$$

This function is maximized at $b_1 = \frac{1}{2}(\alpha + \gamma)t_1$. That is, if each type t_2 of player 2 bids $\frac{1}{2}(\alpha + \gamma)t_2$, any type t_1 of player 1 optimally bids $\frac{1}{2}(\alpha + \gamma)t_1$. Hence, as claimed, the game has a Nash equilibrium in which each type t_i of player i bids $\frac{1}{2}(\alpha + \gamma)t_i$.

304.1 Signal-independent equilibria in a model of a jury

If every juror votes for acquittal regardless of her signal then the action of any single juror has no effect on the outcome. Thus the strategy profile in which every juror votes for acquittal regardless of her signal is always a Nash equilibrium.

Now consider the possibility of a Nash equilibrium in which every juror votes for conviction regaredless of her signal. Suppose that every juror other than juror 1 votes for conviction independently of her signal. Then juror 1's vote determines the outcome, exactly as in the case in which there is a single juror. Thus from the calculations in Section 9.8.2, type b of juror 1 optimally votes for conviction if and only if

$$z \leq \frac{(1-p)\pi}{(1-p)\pi + q(1-\pi)}$$

and type g of juror 1 optimally votes for conviction if and only if

$$z \leq \frac{p\pi}{p\pi + (1-q)(1-\pi)}.$$

The assumption that $p > 1 - q$ implies that the term on the right side of the second inequality is greater than the term on the right side of the first inequality, so that we conclude that there is a Nash equilibrium in which every juror votes for conviction regardless of her signal if and only if

$$\frac{(1-p)\pi}{(1-p)\pi + q(1-\pi)} \leq z \leq \frac{p\pi}{p\pi + (1-q)(1-\pi)}.$$

305.1 Swing voter's curse

a. The Bayesian game is defined as follows.

Players Citizens 1 and 2.

States $\{A, B\}$.

Actions The set of actions of each player is $\{0, 1, 2\}$ (where 0 means do not vote).

Signals Citizen 1 receives different signals in states A and B, whereas citizen 2 receives the same signal in both states.

Beliefs Each type of citizen 1 assigns probability 1 to the single state consistent with her signal. The single type of citizen 2 assigns probability 0.9 to state A and probability 0.1 to state B.

Payoffs Both citizens' Bernoulli payoffs are 1 if either the state is A and candidate 1 receives the most votes or the state is B and candidate 2 receives the most votes; their payoffs are 0 if either the state is B and candidate 1 receives the most votes or the state is A and candidate 2 receives the most votes; and otherwise their payoffs are $\frac{1}{2}$. (These payoffs are shown in Figure 143.1.)

	0	1	2
0	$\frac{1}{2}, \frac{1}{2}$	$1, 1$	$0, 0$
1	$1, 1$	$1, 1$	$\frac{1}{2}, \frac{1}{2}$
2	$0, 0$	$\frac{1}{2}, \frac{1}{2}$	$0, 0$

State A

	0	1	2
0	$\frac{1}{2}, \frac{1}{2}$	$0, 0$	$1, 1$
1	$0, 0$	$0, 0$	$\frac{1}{2}, \frac{1}{2}$
2	$1, 1$	$\frac{1}{2}, \frac{1}{2}$	$1, 1$

State B

Figure 143.1 The payoffs in the Bayesian game for Exercise 305.1.

b. Type A of player 1's best action depends only on the action of player 2; it is to vote for 1 if player 2 votes for 2 or does not vote, and either to vote for 1 or not vote if player 2 votes for 1. Similarly, type B of player 1's best action is to vote for 2 if player 2 votes for 1 or does not vote, and either to vote for 2 or not vote if player 2 votes for 2.

Player 2's best action is to vote for 1 if type A of player 1 either does not vote or votes for 2 (regardless of how type B of player 1 votes), not to vote if type A of player 1 votes for 1 and type B of player 1 either votes for 2 or does not vote, and either to vote for 1 or not to vote if both types of player 1 vote for 1.

Given the best responses of the two types of player 1, their only possible equilibrium actions are $(0, 0)$ (i.e. both do not vote), $(0, 2)$, $(1, 0)$, and $(1, 2)$. Checking player 2's best responses we see that the only equilibria are

- $(0, 2, 1)$ (player 1 does not vote in state A and votes for 2 in state B; player 2 votes for 1)
- $(1, 2, 0)$ (player 1 votes for 1 in state A and for 2 in state B; player 2 does not vote).

c. In the equilibrium $(0, 2, 1)$, type A of player 1's action is weakly dominated by the action of voting for 1: voting for 1 instead of not voting never makes her worse off, and makes her better off in the event that player 2 does not vote.

d. In the equilibrium $(1, 2, 0)$, player 2 does not vote because if she does then in the only case in which her vote affects the outcome (i.e. the only case in which she is a "swing voter"), it affects it adversely: if she votes for 1 then her vote makes no difference in state A, whereas it causes a tie, instead of a win for candidate 2 in state B, and if she votes for 2 then her vote causes a tie, instead of a win for candidate 1 in state A, and makes no difference in state B.

307.2 Properties of the bidding function in a first-price sealed-bid auction

We have

$$\begin{aligned}
\beta^{*\prime}(v) &= 1 - \frac{(F(v))^{n-1}(F(v))^{n-1} - (n-1)(F(v))^{n-2}F'(v)\int_{\underline{v}}^{v}(F(x))^{n-1}\,dx}{(F(v))^{2n-2}} \\
&= 1 - \frac{(F(v))^{n} - (n-1)F'(v)\int_{\underline{v}}^{v}(F(x))^{n-1}\,dx}{(F(v))^{n}} \\
&= \frac{(n-1)F'(v)\int_{\underline{v}}^{v}(F(x))^{n-1}\,dx}{(F(v))^{n}} \\
&> 0 \quad \text{if } v > \underline{v}
\end{aligned}$$

because $F'(v) > 0$ (F is increasing). (The first line uses the quotient rule for derivatives and the fact that the derivative of $\int^{v} f(x)dx$ with respect to v is $f(v)$ for any function f.)

If $v > \underline{v}$ then the integral in (307.1) is positive, so that $\beta^*(v) < v$. If $v = \underline{v}$ then both the numerator and denominator of the quotient in (307.1) are zero, so we may use L'Hôpital's rule to find the value of the quotient as $v \to \underline{v}$. Taking the derivatives of the numerator and denominator we obtain

$$\frac{(F(v))^{n-1}}{(n-1)(F(v))^{n-2}F'(v)} = \frac{F(v)}{(n-1)F'(v)},$$

the numerator of which is zero and the denominator of which is positive. Thus the quotient in (307.1) is zero, and hence $\beta^*(\underline{v}) = \underline{v}$.

307.3 Example of Nash equilibrium in a first-price auction

From (307.1) we have

$$\begin{aligned}
\beta^*(v) &= v - \frac{\int_0^v x^{n-1}\,dx}{v^{n-1}} \\
&= v - \frac{\int_0^v x^{n-1}\,dx}{v^{n-1}} \\
&= v - v/n = (n-1)v/n.
\end{aligned}$$

11 Strictly competitive games and maxminimization

338.2 Nash equilibrium payoffs and maxminimized payoffs

In the game in Figure 147.1 each player's maxminimized payoff is 1, while her payoff in the unique Nash equilibrium is 2.

	L	R
T	$2,2$	$1,0$
B	$0,1$	$0,0$

Figure 147.1 A game in which each player's Nash equilibrium payoff exceeds her maxminimized payoff.

340.1 Strictly competitive games

Left-hand game: Strictly competitive both in pure and in mixed strategies. (Player 2's preferences are represented by the vNM payoff function $-u_1$ since $-u_1(a) = -\frac{1}{2} + \frac{1}{2}u_2(a)$ for every pure outcome a.)

Right-hand game: Strictly competitive in pure strategies (since player 1's ranking of the four outcomes is the reverse of player 2's ranking). Not strictly competitive in mixed strategies (there exist no values of α and $\beta > 0$ such that $-u_1(a) = \alpha + \beta u_2(a)$ for every outcome a; or, alternatively, player 1 is indifferent between (D, L) and the lottery that yields (U, L) with probability $\frac{1}{2}$ and (U, R) with probability $\frac{1}{2}$, while player 2 is not indifferent between these two outcomes).

343.2 Maxminimizing in BoS

The maxminimizer of player 1 is $(\frac{1}{3}, \frac{2}{3})$ while that of player 2 is $(\frac{2}{3}, \frac{1}{3})$.

It is clear that neither of the pure equilibrium strategies of either player guarantees her equilibrium payoff. In the mixed strategy equilibrium player 1's expected payoff is $\frac{2}{3}$; but if, for example, player 2 choose S instead of her equilibrium strategy, then player 1's expected payoff is $\frac{1}{3}$. Similarly for player 2.

343.3 Changing payoffs in strictly competitive game

a. Let u_i be player i's payoff function in the game G, let w_i be his payoff function in G', and let (x^*, y^*) be a Nash equilibrium of G'. Then, using part (a) of Proposition 341.1, we have $w_1(x^*, y^*) = \min_y \max_x w_1(x, y) \geq \min_y \max_x u_1(x, y)$, which is the value of G.

b. This follows from part (a) of Proposition 341.1 and the fact that for any function f we have $\max_{x \in X} f(x) \geq \max_{x \in Y} f(x)$ if $Y \subseteq X$.

c. In the unique equilibrium of the game on the left of Figure 148.1 player 1 receives a payoff of 3, while in the unique equilibrium of she receives a payoff of 2. If she is prohibited from using her second action in this second game then she obtains an equilibrium payoff of 3, however.

3, 3	1, 1
1, 0	0, 1

3, 3	1, 1
4, 0	2, 1

Figure 148.1 The games for part *c* of Exercise 343.3.

344.1 Equilibrium payoff in strictly competitive game

The claim is false. In the strictly competitive game in Figure 148.2 the action pair (T, L) is a Nash equilibrium, so that player 1's unique equilibrium payoff in the game is 0; but (B, R), which also yields player 1 a payoff of 0, is not a Nash equilibrium.

	L	R
T	0, 0	1, −1
B	−1, 1	0, 0

Figure 148.2 The game in Exercise 344.1.

344.2 Guessing Morra

In the strategic game there are two players, each of whom has four (relevant) actions, $S1G2$, $S1G3$, $S2G3$, and $S2G4$, where $SiGj$ denotes the strategy (Show i, Guess j). The payoffs in the game are shown in Figure 148.3.

	$S1G2$	$S1G3$	$S2G3$	$S2G4$
$S1G2$	0, 0	2, −2	−3, 3	0, 0
$S1G3$	−2, 2	0, 0	0, 0	3, −3
$S2G3$	3, −3	0, 0	0, 0	−4, 4
$S2G4$	0, 0	−3, 3	4, −4	0, 0

Figure 148.3 The game in Exercise 344.2.

Now, if there is a Nash equilibrium in which player 1's payoff is v then, given the symmetry of the game, there is a Nash equilibrium in which player 2's payoff is v, so that player 1's payoff is $-v$. Since the equilibrium payoff in a strictly competitive game is unique, we have $v = 0$.

Let (p_1, p_2, p_3, p_4) be the probabilities that player 1 assigns to her four actions. In order that she obtain a payoff of at least 0 if player 2 uses any of her pure strategies, we need

$$\begin{aligned} -2p_2 + 3p_3 \quad &\geq 0 \\ 2p_1 \quad - 3p_4 &\geq 0 \\ -3p_1 \quad + 4p_4 &\geq 0 \\ 3p_2 - 4p_3 \quad &\geq 0. \end{aligned}$$

The second and third inequalities imply that $p_1 \geq \frac{3}{2}p_4$ and $p_1 \leq \frac{4}{3}p_4$, so that $p_1 = p_4 = 0$, so that $p_3 = 1 - p_2$. The first and fourth inequalities imply that $p_2 \leq \frac{3}{2}p_3$ and $p_2 \geq \frac{4}{3}p_3$, or $p_2 \leq \frac{3}{5}$ and $p_2 \geq \frac{4}{7}$.

We conclude that any pair of mixed strategies $((0, p_2, 1 - p_2, 0), (0, q_2, 1 - q_2, 0))$ with $\frac{4}{7} \leq p_2 \leq \frac{3}{5}$ and $\frac{4}{7} \leq q_2 \leq \frac{3}{5}$ is an equilibrium.

344.3 Equilibria of a 4×4 game

a. Denote the probability with which player 1 chooses each of her actions 1, 2, and 3, by p and the probability with which player 2 chooses each of these actions by q. Then all four of player 1's actions yield the same expected payoff if and only if $4q - 1 = 1 - 6q$, or $q = \frac{1}{5}$, and similarly all four of player 2's actions yield the same expected payoff if and only if $p = \frac{1}{5}$. Thus $((\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{2}{5}), (\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{2}{5}))$ is a Nash equilibrium of the game. The players' payoffs in this equilibrium are $(-\frac{1}{5}, \frac{1}{5})$.

b. Let (p_1, p_2, p_3, p_4) be an equilibrium strategy of player 1. In order that it guarantee her the payoff of $-\frac{1}{5}$, we need

$$\begin{aligned} -p_1 + p_2 + p_3 - p_4 &\geq -\tfrac{1}{5} \\ p_1 - p_2 + p_3 - p_4 &\geq -\tfrac{1}{5} \\ p_1 + p_2 - p_3 - p_4 &\geq -\tfrac{1}{5} \\ -p_1 - p_2 - p_3 + p_4 &\geq -\tfrac{1}{5}. \end{aligned}$$

Adding these four inequalities, we deduce that $p_4 \leq \frac{2}{5}$. Adding each pair of the first three inequalities, we deduce that $p_1 \leq \frac{1}{5}$, $p_2 \leq \frac{1}{5}$, and $p_3 \leq \frac{1}{5}$. Since $p_1 + p_2 + p_3 + p_4 = 1$, we deduce that $(p_1, p_2, p_3, p_4) = (\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{2}{5})$. A similar analysis of the conditions for player 2's strategy to guarantee her the payoff of $\frac{1}{5}$ leads to the conclusion that $(q_1, q_2, q_3, q_4) = (\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{2}{5})$.

12 Rationalizability

354.3 Mixed strategy equilibria of game

There is no equilibrium in which player 2 assigns positive probability only to L and C, since if she does so then only M and B are possible best responses for player 1, but if player 1 assigns positive probability only to these actions then L is not optimal for player 2.

By a similar argument there is no equilibrium in which player 2 assigns positive probability only to C and R.

Assume that player 2 assigns positive probability only to L and R. There are no probabilities for L and R under which player 1 is indifferent between all three of her actions, so player 1 must assign positive probability to at most two actions. If these two actions are T and M then player 2 prefers L to R, while if the two actions are M and B then player 2 prefers R to L. The only possibility is thus that the two actions are T and B. In this case we need player 2 to assign probability $\frac{1}{2}$ to L and R (in order that player 1 be indifferent between T and B); but then M is better for player 1. Thus there is no equilibrium in which player 2 assigns positive probability only to L and R.

Finally, if player 2 assigns positive probability to all three of her actions then player 1's mixed strategy must be such that each of these three actions yields the same payoff. A calculation shows that there is no mixed strategy of player 1 with this property.

We conclude that the game has no mixed strategy equilibrium in which either player assigns positive probability to more than one action.

358.1 Example of rationalizable actions

I claim that the action R of player 2 is strictly dominated by some mixed strategies that assign positive probability to L and C. Consider such a mixed strategy that assigns probability p to L. In order for this mixed strategy to strictly dominate R we need $p + 4(1-p) > 3$ and $8p + 2(1-p) > 3$, or $\frac{1}{6} < p < \frac{1}{3}$. That is, any such value of p is associated with a mixed strategy that strictly dominated R. In the reduced game (i.e. after R is eliminated), B is dominated by T. Finally, L is dominated by C. Hence the only rationalizable action of player 1 is T and the only rationalizable action of player 2 is C.

358.2 Guessing Morra

Take Z_i to be all the actions of player i, for $i = 1, 2$. Then (Z_1, Z_2) satisfies Definition 354.1. (The action $S1G2$ is a best response to a belief that assigns probability 1 to $S1G3$, the action $S1G3$ is a best response to the belief that assigns probability one to $S2G4$, the action $S2G3$ is a best response to the belief that assigns probability one to $S1G2$, and the action $S2G4$ is a best response to the belief that assigns probability one to $S2G3$.)

358.3 Contributing to a public good

a. The derivative to player i's payoff with respect to c_i is

$$-2c_i - \sum_{j \neq i} c_j + w_i,$$

which, for every possible value of $\sum_{j \neq i} c_j$, is negative if $c_i > \frac{1}{2} w_i$. Thus the contribution $w_i/2$ yields player i a payoff higher than does any larger contribution, regardless of the other players' contributions. (Note that this result depends on the sum of the other players' contributions being nonnegative.)

b. The best response function of player i is given by

$$\max\{0, \tfrac{1}{2}(w - \sum_{j \neq i} c_j)\}.$$

Let $c \leq w/2$ and suppose that each of the other players contributes $\frac{1}{2}w - c$ (which is nonnegative). Then the other players' total contribution is $w - 2c$, so that player i's best response is to contribute c. That is, any contribution c of at most $w/2$ is a best response to the belief that assigns probability one to each of the other player's contributing $\frac{1}{2}w - c \leq \frac{1}{2}w$. Thus if we set $Z_i = [0, w/2]$ for all i in Definition 354.1 we see that any action of player i in $[0, w/2]$ is rationalizable for player i. [Note: This argument does not show that actions outside $[0, w/2]$ are not rationalizable.]

c. Denote $w_1 = w_2 = w$. First eliminate contributions of more than $w_i/2$ by each player i.

In the reduced game the most that players 1 and 2 together contribute is w (since each contributes at most $w/2$). Now consider player 3. Given the derivative of her payoff function found in part *a*, her payoff is increasing in her contribution for every remaining possible value of $c_1 + c_2$ so long as $c_3 < \frac{1}{2}(w_3 - (c_1 + c_2))$. Since $c_1 + c_2 \leq w$, player 3's payoff is thus definitely increasing for $c_3 < \frac{1}{2}(w_3 - w)$. But $w_3 \geq 3w$, so player 3's payoff is definitely increasing for $c_3 < w$. We conclude that in the reduced game every contribution of player 3 of less than w is strictly dominated. Eliminate all such actions of player 3.

In the newly reduced game every contribution of player 3 is in the interval $[w, w_3/2]$. Now consider player 1. Her payoff is decreasing in her contribution if $c_1 > \frac{1}{2}(w - (c_2 + c_3))$. We have $c_2 \geq 0$ and $c_3 \geq w$, so player 1's payoff is decreasing if $c_1 > 0$. Thus every action of player 1 is strictly dominated by a contribution of 0. The same analysis applies to player 2. Eliminate all such actions of player 1 and player 2.

Finally, in the game we now have, players 1 and 2 both contribute 0; it follows that all actions of player 3 are dominated except for a contribution of $w_3/2$, which is her best response to a total contribution of 0 by players 1 and 2.

We conclude that the unique action profile that survives iterated elimination of strictly dominated actions is $(0, 0, w_3/2)$.

358.4 Iterated elimination in location game

In the first round *Out* is strictly dominated by the position $\frac{1}{2}$ (since the position $\frac{1}{2}$ guarantees at least a draw, which each player prefers to staying out of the competition). In the next round the positions 0 and 1 are strictly dominated by the position $\frac{1}{2}$: a player who chooses $\frac{1}{2}$ rather than either 0 or 1 ties rather than loses if her opponent also chooses $\frac{1}{2}$, and wins outright rather than ties or loses if her opponent chooses any other position. In every subsequent round the two remaining extreme positions are strictly dominated by $\frac{1}{2}$. The only action that remains is $\frac{1}{2}$. [Note that under the procedure of iterated elimination of *weakly* dominated actions, discussed in the next section of the text, there is only one round of elimination: all actions other than $\frac{1}{2}$ are weakly dominated by $\frac{1}{2}$. (In particular, the game is dominance solvable.)]

361.1 Example of dominance solvability

The Nash equilibria of the game are (T, L), any $((0, 0, 1), (0, q, 1 - q))$ with $0 \leq q \leq 1$, and any $((0, p, 1 - p), (0, 0, 1))$ with $0 \leq p \leq 1$. The game is dominance solvable, because T and L are the only weakly dominated actions, and in they are eliminated the only weakly dominated actions are M and C, leaving (B, R), with payoffs $(0, 0)$.

If T is eliminated, then L and C, no remaining action is weakly dominated; (M, R) and (B, R) both remain.

361.2 Dominance solvability in demand game

In the first round the demands 0, 1, and 2 are eliminated for each player and in the second round the demand 4 is eliminated, leaving the outcome in which each player demands 3 (and receives 2).

361.3 Dominance solvability in Bertrand's duopoly game

In the first round every price in excess of the monopoly price is weakly dominated by the monopoly price and every price equal to at most c is weakly dominated by the price $c + 0.01$. At each subsequent round the highest remaining price is weakly dominated by the next highest price. (Note that for any $p \geq c + 0.01$ it is better to obtain all the demand at the price p than obtain half of the demand at the price $p + 0.01$.) The pair of prices that remains is $(c + 0.01, c + 0.01)$.

13 Evolutionary equilibrium

370.1 ESSs and weakly dominated actions

The ESS a^* does not necessarily weakly dominate every other action in the game. For example, in the game in Figure 155.1, a^* is an ESS but does not weakly dominate b.

	a^*	b
a^*	$1,1$	$0,0$
b	$0,0$	$2,2$

Figure 155.1 A game in which an ESS (a^*) does not weakly dominate another action.

No action can weakly dominate an ESS. To see why, let a^* be an ESS and let b be another action. Since a^* is an ESS, (a^*, a^*) is a Nash equilibrium, so that $u(b, a^*) \leq u(a^*, a^*)$. Now, if $u(b, a^*) < u(a^*, a^*)$, certainly b does not weakly dominate a^*, so suppose that $u(b, a^*) = u(a^*, a^*)$. Then by the second condition for an ESS we have $u(b, b) < u(a^*, b)$. We conclude that b does not weakly dominate a^*.

370.2 Pure ESSs

The payoff matrix of the game is given in Figure 155.2. The pure strategy symmet-

	1	2	3
1	$1,1$	$2,2\delta$	$3,3\delta$
2	$2\delta,2$	$2,2$	$3,3\delta$
3	$3\delta,3$	$3\delta,3$	$3,3$

Figure 155.2 The game in Exercise 370.2.

ric Nash equilibria are $(1,1)$, $(2,2)$, and $(3,3)$. The only pure evolutionarily stable strategy is 1, by the following argument. The action 1 is evolutionarily stable since $(1,1)$ is a strict Nash equilibrium. The action 2 is not evolutionarily stable, since 1 is a best response to 2 and

$$u(1,1) = 1 > 2\delta = u(2,1).$$

The action 3 is not evolutionarily stable, since 2 is a best response to 3 and

$$u(2,2) = 2 > 3\delta = u(3,2).$$

In the case that each player has n actions, every pair (i,i) is a Nash equilibrium; only the action 1 is an ESS.

375.1 Hawk–Dove–Retaliator

First suppose that $v \geq c$. In this case the game has two pure symmetric Nash equilibria, (A,A) and (R,R). However, A is not an ESS, since R is a best response to A and $u(R,R) > u(A,R)$. Since (R,R) is a strict equilibrium, R is an ESS. Now consider the possibility that the game has a mixed strategy equilibrium (α,α). If α assigns positive probability to either P or R (or both) then R yields a payoff higher than does P, so only A and R may be assigned positive probability in a mixed strategy equilibrium. But if a strategy α assigns positive probability to A and R and probability 0 to P, then R yields a payoff higher than does A against an opponent who uses α. Thus the game has no symmetric mixed strategy equilibrium in this case.

Now suppose that $v < c$. Then the only symmetric pure strategy equilibrium is (R,R). This equilibrium is strict, so that R is an ESS. Now consider the possibility that the game has a mixed strategy equilibrium (α,α). If α assigns probability 0 to A then R yields a payoff higher than does P against an opponent who uses α; if α assigns probability 0 to P then R yields a payoff higher than does A against an opponent who uses α. Thus in any mixed strategy equilibrium (α,α), the strategy α must assign positive probability to both A and P. If α assigns probability 0 to R then we need $\alpha = (v/c, 1 - v/c)$ (the calculation is the same as for *Hawk–Dove*). Since R yields a lower payoff against this strategy than do A and P, and since the strategy is an ESS in *Hawk–Dove*, it is an ESS in the present game. The remaining possibility is that the game has a mixed strategy equilibrium (α,α) in which α assigns positive probability to all three actions. If so, then the expected payoff to this strategy is less than $\frac{1}{2}v$, since the pure strategy P yields an expected payoff less than $\frac{1}{2}v$ against any such strategy. But then $U(R,R) = \frac{1}{2}v > U(\alpha,R)$, violating the second condition in the definition of an ESS.

In summary:

- If $v \geq c$ then R is the unique ESS of the game.
- If $v < c$ then both R and the mixed strategy that assigns probability v/c to A and $1 - v/c$ to P are ESSs.

375.2 Example of pure and mixed ESSs

Since (C,C) is a strict Nash equilibrium, C is an ESS.

The game also has a symmetric mixed strategy equilibrium in which each player's mixed strategy is $\alpha^* = (\frac{3}{4}, \frac{1}{4}, 0)$. Every mixed strategy $\beta = (p, 1-p, 0)$ is a best response to α^*, so in order that α^* is an ESS we need

$$U(\beta, \beta) < U(\alpha^*, \beta).$$

We have $U(\beta, \beta) = 4p(1-p)$ and $U(\alpha^*, \beta) = \frac{9}{4}(1-p) + \frac{1}{4}p$, so the inequality is equivalent to

$$(p - \tfrac{3}{4})^2 > 0,$$

which is true for all $p \neq \frac{3}{4}$. Thus α^* is an ESS.

The only other symmetric mixed strategy equilibrium is one in which each player's strategy is $\alpha^{**} = (\frac{3}{7}, \frac{1}{7}, \frac{3}{7})$. This strategy is not an ESS, since $u(C, C) = 1$ while $u(\alpha^{**}, C) = \frac{3}{7} < 1$.

375.3 Bargaining

The game is given in Figure 157.1. Let α be a mixed strategy that assigns positive

	0	2	4	6	8	10
0	5, 5	4, 6	3, 7	2, 8	1, 9	0, 10
2	6, 4	5, 5	4, 6	3, 7	2, 8	0, 0
4	7, 3	6, 4	5, 5	4, 6	0, 0	0, 0
6	8, 2	7, 3	6, 4	0, 0	0, 0	0, 0
8	9, 1	8, 2	0, 0	0, 0	0, 0	0, 0
10	10, 0	0, 0	0, 0	0, 0	0, 0	0, 0

Figure 157.1 A bargaining game.

probability only to the demands 2 and 8. For (α, α) to be a Nash equilibrium we need $\alpha = (\frac{2}{5}, \frac{3}{5})$. Each player's payoff at this strategy pair (α^*, α^*) is $\frac{16}{5}$. Thus the only actions a that are best responses to α^* are 2 and 8, so that the only mixed strategies that are best responses assign positive probability only to the actions 2 and 8. Let β be the mixed strategy that assigns probability p to 2 and probability $1-p$ to 8. We have

$$U(\beta, \beta) = 5p(2-p)$$

and

$$U(\alpha^*, \beta) = 6p + \tfrac{4}{5}.$$

We find that $U(\alpha^*, \beta) - U(\beta, \beta) = 5(p - \frac{2}{5})^2$, which is positive if $p \neq \frac{2}{5}$. Hence α^* is an ESS.

Now let α be a mixed strategy that assigns positive probability only to the demands 4 and 6. For (α, α) to be a Nash equilibrium we need $\alpha = \frac{4}{5}$. Each player's payoff at this strategy pair (α^*, α^*) is $\frac{24}{5}$. Thus the only actions a that are best responses to α^* are 4 and 6, so that the only mixed strategies that are best responses assign positive probability only to the actions 4 and 6. Let β be the mixed strategy that assigns probability p to 4 and probability $1-p$ to 6. We have

$$U(\beta,\beta) = 5p(2-p)$$

and

$$U(\alpha^*,\beta) = 2p + \tfrac{16}{5}.$$

We find that $U(\alpha^*,\beta) - U(\beta,\beta) = 5(p-\frac{4}{5})^2$, which is positive if $p \neq \frac{4}{5}$. Hence α^* is an ESS.

379.1 Mixed strategies in an asymmetric Hawk–Dove

Let p be the probability that β assigns to AA. In order that AA and DD yield a player the same expected payoff when her opponent uses β, we need

$$p(V+v-2c) + (1-p)(2V+2v) = (1-p)(V+v),$$

or

$$p = \frac{V+v}{2c}.$$

Now, if player 2 uses the strategy β then the difference between player 1's expected payoff to AA and her expected payoff to AP is

$$p(v-c) + (1-p)v = v - pc = \tfrac{1}{2}(v-V) < 0.$$

Thus the strategy pair (β,β) is not a Nash equilibrium.

379.2 Mixed strategy ESSs

Let β be an ESS that assigns positive probability to every action in A^*. Then (β,β) is a Nash equilibrium (since β is an ESS), so that every mixed strategy that assigns positive probability only to actions in A^* is a best response to β. In particular, α^* is a best response to β. Thus if $\beta \neq \alpha^*$ then the second condition in the definition of an ESS, when applied to β, requires that

$$U(\alpha^*,\alpha^*) < U(\beta,\alpha^*).$$

But this inequality contradicts the fact that (α^*,α^*) is a Nash equilibrium. Hence $\beta = \alpha^*$.

380.1 Asymmetric ESSs of BoS

The game is shown in Figure 159.1. The strategy pairs (LD,LD) and (DL,DL) are strict symmetric Nash equilibria. Thus both LD and DL are ESSs. By the same argument as in the analysis of *Hawk–Dove* in the text, the only possible mixed ESS

	LL	LD	DL	DD
LL	$0,0$	$1,\frac{1}{2}$	$1,\frac{1}{2}$	$2,1$
LD	$\frac{1}{2},1$	$\frac{3}{2},\frac{3}{2}$	$0,0$	$1,\frac{1}{2}$
DL	$\frac{1}{2},1$	$0,0$	$\frac{3}{2},\frac{3}{2}$	$1,\frac{1}{2}$
DD	$1,2$	$\frac{1}{2},1$	$\frac{1}{2},1$	$0,0$

Figure 159.1 The game *BoS* when the players' roles may differ.

assigns positive probability only to LL and DD. Let β be such a strategy; let p be the probability that it assigns to LL. Then for (β, β) to be a Nash equilibrium we need

$$2(1-p) = p,$$

or $p = \frac{2}{3}$. If one of the players uses such a strategy then the other player obtains the same expected payoff to all her four actions, namely $\frac{2}{3}$. Thus (β, β) is a Nash equilibrium. However, since

$$u(LD, LD) = \tfrac{3}{2} > \tfrac{5}{6} = u(\beta, LD),$$

the strategy β is not an ESS.

Thus the game has two ESSs, each of which is a pure strategy: LD and DL.

385.1 A coordination game between siblings

The games with payoff functions v and w are shown in Figure 159.2. If $x < 2$ then

	X	Y
X	x,x	$\frac{1}{2}x,\frac{1}{2}$
Y	$\frac{1}{2},\frac{1}{2}x$	$1,1$

v

	X	Y
X	x,x	$\frac{1}{5}x,\frac{1}{5}$
Y	$\frac{1}{5},\frac{1}{5}x$	$1,1$

w

Figure 159.2 The games with payoff functions v and w derived from the game in Exercise 385.1.

(Y, Y) is a strict Nash equilibrium of both games, so Y is an evolutionarily stable action in the game between siblings. If $x > 2$ then the only (pure) Nash equilibrium of the game is (X, X), and this equilibrium is strict. Thus the range of values of x for which the only evolutionarily stable action is X is $x > 2$.

387.1 Darwin's theory of the sex ratio

A normal organism produces pn female offspring and $(1-p)n$ male offspring (ignoring the small probability that the partner of a normal organism is a mutant). Thus it has $pn \cdot n + (1-p)n \cdot (p/(1-p))n = 2pn^2$ grandchildren.

A mutant has $\frac{1}{2}n$ female offspring and $\frac{1}{2}n$ male offspring, and hence has $\frac{1}{2}n \cdot n + \frac{1}{2}n \cdot (p/(1-p))n = \frac{1}{2}n^2/(1-p)$ grandchildren.

Thus the difference between the number of grandchildren produced by normal and mutant organisms is

$$\tfrac{1}{2}n^2/(1-p) - 2pn^2 = n^2\left(\frac{2}{1-p}\right)(p-\tfrac{1}{2})^2,$$

which is positive if $p \neq \frac{1}{2}$. (The point is that a higher fraction of the mutant's offspring are female, which each bear more offspring than each male.)

Thus the mutant invades the population; only $p = \frac{1}{2}$ is evolutionarily stable.

14 Repeated games: The Prisoner's Dilemma

395.1 Strategies for an infinitely repeated Prisoner's Dilemma

a. The strategy is shown in Figure 161.1.

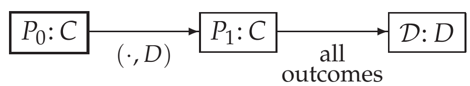


Figure 161.1 The strategy in Exercise 395.1a.

b. The strategy is shown in Figure 161.2.

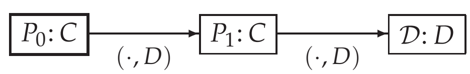


Figure 161.2 The strategy in Exercise 395.1b.

c. The strategy is shown in Figure 161.3.

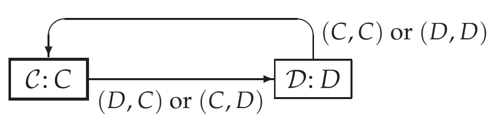


Figure 161.3 The strategy in Exercise 395.1c.

398.1 Nash equilibria of the infinitely repeated Prisoner's Dilemma

a. A player who adheres to the strategy obtains the discounted average payoff of 2. A player who deviates obtains the stream of payoffs $(3, 3, 1, 1, \ldots)$, with a discounted average of $(1-\delta)(3+3\delta)+\delta^2$. Thus for an equilibrium we require $(1-\delta)(3+3\delta)+\delta^2 \leq 2$, or $\delta \geq \frac{1}{2}\sqrt{2}$.

b. A player who adheres to the strategy obtains the payoff of 2 in every period. A player who chooses D in the first period and C in every subsequent period obtains the stream of payoffs $(3, 2, 2, \ldots)$. Thus for any value of δ a player can increase her payoff by deviating, so that the strategy pair is not a Nash equilibrium. Further, whatever the one-shot payoffs, a player can increase her payoff by deviating to D in a single period, so that for no payoffs is there any δ such that the strategy pair is a Nash equilibrium of the infinitely repeated game.

c. A player who adheres to the strategy obtains the discounted average payoff of 2 (the outcome is (C, C) in every period). If player 1 deviates to D in every period then she induces the outcome to alternate between (D, C) and (D, D), yielding her a discounted average payoff of $(1-\delta) \cdot (3 + 3\delta^2 + 3\delta^4 + \ldots) + (1-\delta)(\delta + \delta^3 + \delta^5 + \ldots) = (1-\delta)[3/(1-\delta^2) + \delta/(1-\delta^2)] = (3+\delta)/(1+\delta)$. For all $\delta < 1$ this payoff exceeds 2, so that the strategy pair is not a Nash equilibrium of the infinitely repeated game.

However, for different payoffs for the one-shot *Prisoner's Dilemma*, the strategy pair *is* a Nash equilibrium of the infinitely repeated game. The point is that the best deviation leads to the sequence of outcomes that alternates between (C, D) and (D, D). If the average payoff of player 2 in these two outcomes is less than her payoff to the outcome (C, C) then the strategy pair is a Nash equilibrium for some values of δ. (For the payoffs in Figure 389.1 the average payoff of the two outcomes (C, D) and (D, D) is exactly equal to the payoff to (C, C).) Consider the general payoffs in Figure 162.1. The discounted average payoff of the sequence of outcomes that alternates between (C, D) and (D, D) is $(y+\delta)/(1+\delta)$, while the discounted average of the constant sequence containing only (C, C) is x. Thus in order for the strategy pair to be a Nash equilibrium we need

	C	D
C	x, x	$0, y$
D	$y, 0$	$1, 1$

Figure 162.1 A *Prisoner's Dilemma.*

$$\frac{y+\delta}{1+\delta} \leq x,$$

or

$$\delta \geq \frac{y-x}{x-1},$$

an inequality that is compatible with $\delta < 1$ if $x > \frac{1}{2}(y+1)$—that is, if x exceeds the average of 1 and y.

406.1 Different punishment lengths in the infinitely repeated Prisoner's Dilemma

Yes, there are such subgame perfect equilibria. The only subtlety is that the number of periods for which a player chooses D after a history in which not all the outcomes were (C, C) depends on who first deviated. If, for example, player 1 punishes for two periods while player 2 punishes for three periods, then the outcome (C, D) induces player 1 to choose D for two periods (to punish player 2 for her deviation) while the outcome (D, C) induces her to choose D for three periods (while she is being punished by player 2). The strategy of each player in this case is shown in Figure 163.1.

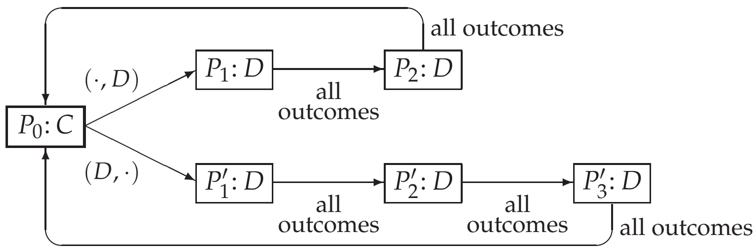


Figure 163.1 A strategy in an infinitely repeated *Prisoner's Dilemma* that punishes deviations for two periods and reacts to punishment by choosing D for three periods.

407.1 Tit-for-tat in the infinitely repeated Prisoner's Dilemma

Suppose that player 2 adheres to *tit-for-tat*. Consider player 1's behavior in subgames following histories that end in each of the following outcomes.

(C, C) If player 1 adheres to *tit-for-tat* the outcome is (C, C) in every period, so that her discounted average payoff in the subgame is x. If she chooses D, then adheres to *tit-for-tat*, the outcome alternates between (D, C) and (C, D), and player 1's discounted average payoff is $y/(1+\delta)$. Thus we need $x \geq y/(1+\delta)$, or $\delta \geq (y-x)/x$, in order that *tit-for-tat* be optimal for player 1.

(C, D) If player 1 adheres to *tit-for-tat* the outcome alternates between (D, C) and (C, D), so that her discounted average payoff is $y/(1+\delta)$. If she deviates to C, then adheres to *tit-for-tat*, the outcome is (C, C) in every period, and her discounted average payoff is x. Thus we need $y/(1+\delta) \geq x$, or $\delta \leq (y-x)/x$, in order that *tit-for-tat* be optimal for player 1.

(D, C) If player 1 adheres to *tit-for-tat* the outcome alternates between (C, D) and (D, C), so that her discounted average payoff is $\delta y/(1+\delta)$. If she deviates to D, then adheres to *tit-for-tat*, the outcome is (D, D) in every period, and her discounted average payoff is 1. Thus we need $\delta y/(1+\delta) \geq 1$, or $\delta \geq 1/(y-1)$, in order that *tit-for-tat* be optimal for player 1.

(D, D) If player 1 adheres to *tit-for-tat* the outcome is (D, D) in every period, so that her discounted average payoff is 1. If she deviates to C, then adheres to *tit-for-tat*, the outcome alternates between (C, D) and (D, C), and her discounted average payoff is $\delta y/(1+\delta)$. Thus we need $1 \geq \delta y/(1+\delta)$, or $\delta \leq 1/(y-1)$, in order that *tit-for-tat* be optimal for player 1.

We conclude that for (*tit-for-tat*,*tit-for-tat*) to be a subgame perfect equilibrium we need $\delta = (y - x)/x$ and $\delta = 1/(y - 1)$. Thus only if $(y - x)/x = 1/(y - 1)$, or $y - x = 1$, is the strategy pair a subgame perfect equilibrium. Given that a subgame perfect equilibrium satisfies the one-deviation property, the strategy pair is indeed a subgame perfect equilibrium in this case when $\delta = 1/x$.

17 Mathematical appendix

446.1 Maximizer of quadratic function

We can write the function as $-x(x-\alpha)$. Thus $r_1 = 0$ and $r_2 = \alpha$, and hence the maximizer is $\alpha/2$.

449.3 Sums of sequences

In the first case set $r = \delta^2$ to transform the sum into $1 + r + r^2 + \cdots$, which is equal to $1/(1-r) = 1/(1-\delta^2)$.

In the second case split the sum into $(1 + \delta^2 + \delta^4 + \cdots) + (2\delta + 2\delta^3 + 2\delta^5 + \cdots)$; the first part is equal to $1/(1-\delta^2)$ and the second part is equal to $2\delta(1 + \delta^2 + \delta^4 + \cdots)$, or $2\delta/(1-\delta^2)$. Thus the complete sum is

$$\frac{1+2\delta}{1-\delta^2}.$$

454.1 Bayes' law

Your posterior probability of carrying X given that you test positive is

$$\frac{\Pr(\text{positive test}|X)\Pr(X)}{\Pr(\text{positive test}|X)\Pr(X) + \Pr(\text{positive test}|\neg X)\Pr(\neg X)}$$

where $\neg X$ means "not X". This probability is equal to $0.9p/(0.9p + 0.2(1-p)) = 0.9p/(0.2 + 0.7p)$, which is increasing in p (i.e. a smaller value of p gives a smaller value of the probability). If $p = 0.001$ then the probability is approximately 0.004. (That is, if 1 in 1,000 people carry the gene then if you test positive on a test that is 90% accurate for people who carry the gene and 80% accurate for people who do not carry the gene, then you should assign probability 0.004 to your carrying the gene.) If the test is 99% accurate in both cases then the posterior probability is $(0.99 \cdot 0.001)/[0.99 \cdot 0.001 + 0.01 \cdot 0.999] \approx 0.09$.

Corrections and updates for first printing of Osborne's "An Introduction to Game Theory" (Oxford University Press, 2003)

2004/5/4

I thank the following people for pointing out errors and improvements: T. K. Ahn, Kyung Hwan Baik, Richard Boylan, Hao-Chen Liu, Nathan Nunn, David A. Malueg, Ahmer Tarar, Debraj Ray, Kaouthar Souki.

Corrections

Page, Line	*Correction*
4	The first letter of the text in Section 1.2 should be upper case.
6	Add a space after the period on the last line.
31	The "A" in the caption of Figure 31.1 should be upright, not italic.
78	In Figure 78.2, replace "$B_1(p_2)$" with "$B_1(t_2)$" and replace "$B_2(p_1)$" with "$B_2(t_1)$".
83	In the first line of the second paragraph, change "complete" to "perfect" (for consistency with other terminology).
83	The first sentence of the item "Preferences" just below the middle of the page is hard to follow. A better version is: "Denote by b_i the bid of player i and by $\overline{b}$ the highest bid submitted by a player other than i. If either (a) $b_i > \overline{b}$ or (b) $b_i = \overline{b}$ and the number of every other player who bids $\overline{b}$ is greater than i, then player i's payoff is $v_i - \overline{b}$."
85–87	In the third line of the text on page 85, in the third line of Section 3.5.3 on page 86, and in the fifth line from the bottom of page 87, change "complete" to "perfect" (for consistency with other terminology).
94	The fourth word of the caption of Figure 94.1 should be "shows".
110	Replace "an" at the end of line 16 with "a".
143	Replace $F(z)$ on line 21 with $F_i(z)$.
145	Replace "x_2 and y_2" on the line below the display with "x_1 and y_1".
145	Delete "a_1" at the end of line -8.
187	Replace "in" with "is" on line 2.
202–203	The term "equilibrium path" is used without explanation. It is synonymous with "equilibrium outcome". (That is, the equilibrium path is the terminal history generated by the equilibrium strategies.)
216	The word "that" on the fourth line from the bottom of the page should be "than".

289 In the description of the states above the figure, replace "$0 \leq v_i \leq \overline{v}$" with "$\underline{v} \leq v_i \leq \overline{v}$".

291 Replace "a decreasing" with "an increasing" on line 16 and "increases" with "decreases" on line 17.

295 The claim in the last sentence on the page is too strong: the appendix contains only suggestive arguments, not a proof.

303 In the description of the beliefs in the middle of the page, replace the π near the start of the second line with $\Pr(G \mid g)$, the $1 - \pi$ near the end of the second line with $\Pr(I \mid g)$, the π near the middle of the fifth line with $\Pr(G \mid b)$, and the expression involving $1 - \pi$ near the start of the sixth line with $\Pr(I \mid b)(1-q)^k q^{n-k-1}$.

307 In part *c* of Exercise 307.1, replace "one of the player's actions" with "an action of one of the players". In part *d*, replace "second" with "first".

308–309 To deduce the solution of the differential equation near the bottom of page 308, the initial condition $\beta(\underline{v}) = \underline{v}$ is needed. Given that this initial condition is needed to find the equilibrium bidding function, the part of Exercise 309.2 asking for a proof that the equilibrium bidding function satisfies the condition should be removed. See the website for the book for a version of Section 9.8.1 that corrects these two points, treats more carefully the boundary cases in which $v = \underline{v}$ and $v = \overline{v}$, and explains the argument more clearly.

310 Two lines below (310.1), replace $\Pr\{\mathbf{X} < v\}$ with $\Pr(\mathbf{X} < v)$. On the following line, delete "$= 0$".

319 Change the weak inequality on the next to last line to a strict inequality.

321 In the bottom row of the right-hand table in the bottom panel of Figure 321.1, interchange the entries in the columns headed XY and YX, so that $1/(2 - 4\epsilon)$ is in the column headed XY and 0 is in the column headed YX.

330 In the 7th line of Example 330.1, replace "the history is *Acquiesce*" with "the history is *Unready*".

331 Add a period to the end of the caption of Figure 331.2.

331 Replace Exercise 331.2 (which is incorrect) with the following exercise. EXERCISE 331.2 (Weak sequential equilibrium and Nash equilibrium in subgames) Consider the variant of the game in Figure 331.1 shown in Figure 332.1, in which the challenger's initial move is broken into two steps. Show that this game has a weak sequential equilibrium in which the players' actions in the subgame following the history *In* do not constitute a Nash equilibrium of the subgame.

332–333 Replace the last word on page 332 and the first word on page 333 with "a weak", and replace the penultimate word of the sentence with "strong".

344 Replace each of the seven occurrences of the string $t - b$ with $t + b$.

389 On line 11, the outcomes that survive are (T, L) and (T, C) (not (T, L) and (T, R)).

415 Add a period to the end of the caption of Figure 415.1.

457 Change $k - \ell$ to $k - \ell + 1$ on line -2.

Updates

Dhillon and Lockwood (2003) is now

Dhillon, Amrita, and Ben Lockwood (2004), "When are plurality rule voting games dominance-solvable?", *Games and Economic Behavior* **46**, 55–75.

Publicly-available solutions for

AN INTRODUCTION TO GAME THEORY

Publicly-available solutions for

AN INTRODUCTION TO GAME THEORY

Martin J. Osborne
University of Toronto

This manual was typeset by the author, who is greatly indebted to Donald Knuth (TEX), Leslie Lamport (LATEX), Diego Puga (mathpazo), Christian Schenk (MiKTEX), Ed Sznyter (ppctr), Timothy van Zandt (PSTricks), and others, for generously making superlative software freely available. The main font is 10pt Palatino.

Version 2: 2004-4-27

Contents

Preface

This manual contains all publicly-available solutions to exercises in my book *An Introduction to Game Theory* (Oxford University Press, 2004). The sources of the problems are given in the section entitled "Notes" at the end of each chapter of the book. Please alert me to errors.

MARTIN J. OSBORNE
Martin.Osborne@utoronto.ca
Department of Economics, 150 St. George Street,
University of Toronto, Toronto, Canada M5S 3G7

1 Introduction

5.3 Altruistic preferences

Person 1 is indifferent between $(1, 4)$ and $(3, 0)$, and prefers both of these to $(2, 1)$. The payoff function u defined by $u(x, y) = x + \frac{1}{2}y$, where x is person 1's income and y is person 2's, represents person 1's preferences. Any function that is an increasing function of u also represents her preferences. For example, the functions $k(x + \frac{1}{2}y)$ for any positive number k, and $(x + \frac{1}{2}y)^2$, do so.

6.1 Alternative representations of preferences

The function v represents the same preferences as does u (because $u(a) < u(b) < u(c)$ and $v(a) < v(b) < v(c)$), but the function w does not represent the same preferences, because $w(a) = w(b)$ while $u(a) < u(b)$.

2 Nash Equilibrium

16.1 Working on a joint project

The game in Figure 3.1 models this situation (as does any other game with the same players and actions in which the ordering of the payoffs is the same as the ordering in Figure 3.1).

	Work hard	*Goof off*
Work hard	3, 3	0, 2
Goof off	2, 0	1, 1

Figure 3.1 Working on a joint project (alternative version).

17.1 Games equivalent to the *Prisoner's Dilemma*

The game in the left panel differs from the *Prisoner's Dilemma* in both players' preferences. Player 1 prefers (Y, X) to (X, X) to (X, Y) to (Y, Y), for example, which differs from her preference in the *Prisoner's Dilemma*, which is (F, Q) to (Q, Q) to (F, F) to (Q, F), whether we let $X = F$ or $X = Q$.

The game in the right panel is equivalent to the *Prisoner's Dilemma*. If we let $X = Q$ and $Y = F$ then player 1 prefers (F, Q) to (Q, Q) to (F, F) to (Q, F) and player 2 prefers (Q, F) to (Q, Q) to (F, F) to (F, Q), as in the *Prisoner's Dilemma*.

20.1 Games without conflict

Any two-player game in which each player has two actions and the players have the same preferences may be represented by a table of the form given in Figure 3.2, where a, b, c, and d are any numbers.

	L	R
T	a, a	b, b
B	c, c	d, d

Figure 3.2 A strategic game in which conflict is absent.

31.1 Extension of the *Stag Hunt*

Every profile $(e, \ldots, e)$, where e is an integer from 0 to K, is a Nash equilibrium. In the equilibrium $(e, \ldots, e)$, each player's payoff is e. The profile $(e, \ldots, e)$ is a Nash equilibrium since if player i chooses $e_i < e$ then her payoff is $2e_i - e_i = e_i < e$, and if she chooses $e_i > e$ then her payoff is $2e - e_i < e$.

Consider an action profile $(e_1, \ldots, e_n)$ in which not all effort levels are the same. Suppose that e_i is the minimum. Consider some player j whose effort level exceeds e_i. Her payoff is $2e_i - e_j < e_i$, while if she deviates to the effort level e_i her payoff is $2e_i - e_i = e_i$. Thus she can increase her payoff by deviating, so that $(e_1, \ldots, e_n)$ is not a Nash equilibrium.

(This game is studied experimentally by van Huyck, Battalio, and Beil (1990). See also Ochs (1995, 209–233).)

34.1 Guessing two-thirds of the average

If all three players announce the same integer $k \geq 2$ then any one of them can deviate to $k - 1$ and obtain \$1 (since her number is now closer to $\frac{2}{3}$ of the average than the other two) rather than $\$\frac{1}{3}$. Thus no such action profile is a Nash equilibrium. If all three players announce 1, then no player can deviate and increase her payoff; thus $(1, 1, 1)$ is a Nash equilibrium.

Now consider an action profile in which not all three integers are the same; denote the highest by k^*.

- Suppose only one player names k^*; denote the other integers named by k_1 and k_2, with $k_1 \geq k_2$. The average of the three integers is $\frac{1}{3}(k^* + k_1 + k_2)$, so that $\frac{2}{3}$ of the average is $\frac{2}{9}(k^* + k_1 + k_2)$. If $k_1 \geq \frac{2}{9}(k^* + k_1 + k_2)$ then k^* is further from $\frac{2}{3}$ of the average than is k_1, and hence does not win. If $k_1 < \frac{2}{9}(k^* + k_1 + k_2)$ then the difference between k^* and $\frac{2}{3}$ of the average is $k^* - \frac{2}{9}(k^* + k_1 + k_2) = \frac{7}{9}k^* - \frac{2}{9}k_1 - \frac{2}{9}k_2$, while the difference between k_1 and $\frac{2}{3}$ of the average is $\frac{2}{9}(k^* + k_1 + k_2) - k_1 = \frac{2}{9}k^* - \frac{7}{9}k_1 + \frac{2}{9}k_2$. The difference between the former and the latter is $\frac{5}{9}k^* + \frac{5}{9}k_1 - \frac{4}{9}k_2 > 0$, so k_1 is closer to $\frac{2}{3}$ of the average than is k^*. Hence the player who names k^* does not win, and is better off naming k_2, in which case she obtains a share of the prize. Thus no such action profile is a Nash equilibrium.

- Suppose two players name k^*, and the third player names $k < k^*$. The average of the three integers is then $\frac{1}{3}(2k^* + k)$, so that $\frac{2}{3}$ of the average is $\frac{4}{9}k^* + \frac{2}{9}k$. We have $\frac{4}{9}k^* + \frac{2}{9}k < \frac{1}{2}(k^* + k)$ (since $\frac{4}{9} < \frac{1}{2}$ and $\frac{2}{9} < \frac{1}{2}$), so that the player who names k is the sole winner. Thus either of the other players can switch to naming k and obtain a share of the prize rather obtaining nothing. Thus no such action profile is a Nash equilibrium.

We conclude that there is only one Nash equilibrium of this game, in which all three players announce the number 1.

(This game is studied experimentally by Nagel (1995).)

34.3 Choosing a route

A strategic game that models this situation is:

Players The four people.

Actions The set of actions of each person is $\{X, Y\}$ (the route via X and the route via Y).

Preferences Each player's payoff is the negative of her travel time.

In every Nash equilibrium, two people take each route. (In any other case, a person taking the more popular route is better off switching to the other route.) For any such action profile, each person's travel time is either 29.9 or 30 minutes (depending on the route they take). If a person taking the route via X switches to the route via Y her travel time becomes $12 + 21.8 = 33.8$ minutes; if a person taking the route via Y switches to the route via X her travel time becomes $22 + 12 = 34$ minutes. For any other allocation of people to routes, at least one person can decrease her travel time by switching routes. Thus the set of Nash equilibria is the set of action profiles in which two people take the route via X and two people take the route via Y.

Now consider the situation after the road from X to Y is built. There is no equilibrium in which the new road is not used, by the following argument. Because the only equilibrium before the new road is built has two people taking each route, the only possibility for an equilibrium in which no one uses the new road is for two people to take the route A–X–B and two to take A–Y–B, resulting in a total travel time for each person of either 29.9 or 30 minutes. However, if a person taking A–X–B switches to the new road at X and then takes Y–B her total travel time becomes $9 + 7 + 12 = 28$ minutes.

I claim that in any Nash equilibrium, one person takes A–X–B, two people take A–X–Y–B, and one person takes A–Y–B. For this assignment, each person's travel time is 32 minutes. No person can change her route and decrease her travel time, by the following argument.

- If the person taking A–X–B switches to A–X–Y–B, her travel time increases to $12 + 9 + 15 = 36$ minutes; if she switches to A–Y–B her travel time increases to $21 + 15 = 36$ minutes.
- If one of the people taking A–X–Y–B switches to A–X–B, her travel time increases to $12 + 20.9 = 32.9$ minutes; if she switches to A–Y–B her travel time increases to $21 + 12 = 33$ minutes.
- If the person taking A–Y–B switches to A–X–B, her travel time increases to $15 + 20.9 = 35.9$ minutes; if she switches to A–X–Y–B, her travel time increases to $15 + 9 + 12 = 36$ minutes.

For every other allocation of people to routes at least one person can switch routes and reduce her travel time. For example, if one person takes A–X–B, one person takes A–X–Y–B, and two people take A–Y–B, then the travel time of those taking A–Y–B is $21 + 12 = 33$ minutes; if one of them switches to A–X–B then her travel time falls to $12 + 20.9 = 32.9$ minutes. Or if one person takes A–Y–B, one person takes A–X–Y–B, and two people take A–X–B, then the travel time of those taking A–X–B is $12 + 20.9 = 32.9$ minutes; if one of them switches to A–X–Y–B then her travel time falls to $12 + 8 + 12 = 32$ minutes.

Thus in the equilibrium with the new road every person's travel time *increases*, from either 29.9 or 30 minutes to 32 minutes.

37.1 Finding Nash equilibria using best response functions

a. The *Prisoner's Dilemma* and *BoS* are shown in Figure 6.1; *Matching Pennies* and the two-player *Stag Hunt* are shown in Figure 6.2.

	Quiet	*Fink*
Quiet	$2, 2$	$0, 3^*$
Fink	$3^*, 0$	$1^*, 1^*$

Prisoner's Dilemma

	Bach	*Stravinsky*
Bach	$2^*, 1^*$	$0, 0$
Stravinsky	$0, 0$	$1^*, 2^*$

BoS

Figure 6.1 The best response functions in the *Prisoner's Dilemma* (left) and in *BoS* (right).

	Head	*Tail*
Head	$1^*, -1$	$-1, 1^*$
Tail	$-1, 1^*$	$1^*, -1$

Matching Pennies

	Stag	*Hare*
Stag	$2^*, 2^*$	$0, 1$
Hare	$1, 0$	$1^*, 1^*$

Stag Hunt

Figure 6.2 The best response functions in *Matching Pennies* (left) and the *Stag Hunt* (right).

b. The best response functions are indicated in Figure 6.3. The Nash equilibria are (T, C), (M, L), and (B, R).

	L	C	R
T	$2, 2$	$1^*, 3^*$	$0^*, 1$
M	$3^*, 1^*$	$0, 0$	$0^*, 0$
B	$1, 0^*$	$0, 0^*$	$0^*, 0^*$

Figure 6.3 The game in Exercise 37.1.

38.1 Constructing best response functions

The analogue of Figure 38.2 in the book is given in Figure 7.1.

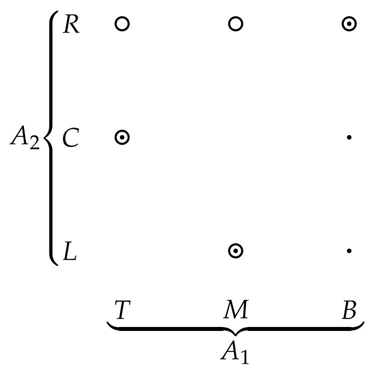


Figure 7.1 The players' best response functions for the game in Exercise 38.1b. Player 1's best responses are indicated by circles, and player 2's by dots. The action pairs for which there is both a circle and a dot are the Nash equilibria.

38.2 Dividing money

For each amount named by one of the players, the other player's best responses are given in the following table.

Other player's action	Sets of best responses
0	$\{10\}$
1	$\{9, 10\}$
2	$\{8, 9, 10\}$
3	$\{7, 8, 9, 10\}$
4	$\{6, 7, 8, 9, 10\}$
5	$\{5, 6, 7, 8, 9, 10\}$
6	$\{5, 6\}$
7	$\{6\}$
8	$\{7\}$
9	$\{8\}$
10	$\{9\}$

The best response functions are illustrated in Figure 8.1 (circles for player 1, dots for player 2). From this figure we see that the game has four Nash equilibria: $(5, 5)$, $(5, 6)$, $(6, 5)$, and $(6, 6)$.

41.1 Strict and nonstrict Nash equilibria

Only the Nash equilibrium (a_1^*, a_2^*) is strict. For each of the other equilibria, player 2's action a_2 satisfies $a_2^{***} \leq a_2 \leq a_2^{**}$, and for each such action player 1 has multiple best responses, so that her payoff is the same for a range of actions, only one of which is such that (a_1, a_2) is a Nash equilibrium.

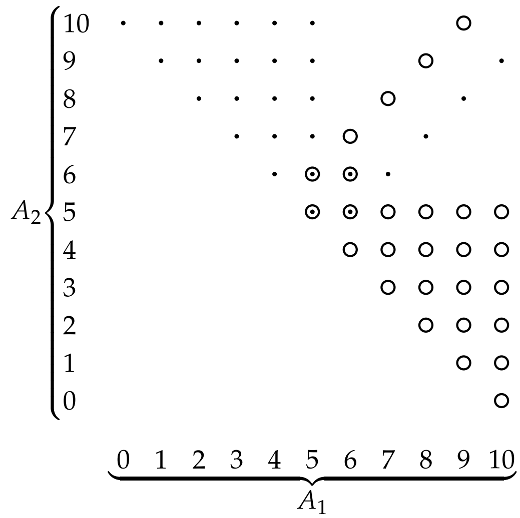


Figure 8.1 The players' best response functions for the game in Exercise 38.2.

47.1 Strict equilibria and dominated actions

For player 1, T is weakly dominated by M, and strictly dominated by B. For player 2, no action is weakly or strictly dominated. The game has a unique Nash equilibrium, (M, L). This equilibrium is not strict. (When player 2 choose L, B yields player 1 the same payoff as does M.)

47.2 Nash equilibrium and weakly dominated actions

The only Nash equilibrium of the game in Figure 8.2 is (T, L). The action T is weakly dominated by M and the action L is weakly dominated by C. (There are of course many other games that satisfy the conditions.)

	L	C	R
T	1, 1	0, 1	0, 0
M	1, 0	2, 1	1, 2
B	0, 0	1, 1	2, 0

Figure 8.2 A game with a unique Nash equilibrium, in which both players' equilibrium actions are weakly dominated. (The unique Nash equilibrium is (T, L).)

50.1 Other Nash equilibria of the game modeling collective decision-making

Denote by i the player whose favorite policy is the median favorite policy. The set of Nash equilibria includes every action profile in which (*i*) i's action is her favorite policy x_i^*, (*ii*) every player whose favorite policy is less than x_i^* names a policy equal to at most x_i^*, and (*iii*) every player whose favorite policy is greater than x_i^* names a policy equal to at least x_i^*.

To show this, first note that the outcome is x_i^*, so player i cannot induce a better outcome for herself by changing her action. Now, if a player whose favorite position is less than x_i^* changes her action to some $x < x_i^*$, the outcome does not change; if such a player changes her action to some $x > x_i^*$ then the outcome either remains the same (if some player whose favorite position exceeds x_i^* names x_i^*) or increases, so that the player is not better off. A similar argument applies to a player whose favorite position is greater than x_i^*.

The set of Nash equilibria also includes, for any positive integer $k \leq n$, every action profile in which k players name the median favorite policy x_i^*, at most $\frac{1}{2}(n - 3)$ players name policies less than x_i^*, and at most $\frac{1}{2}(n-3)$ players name policies greater than x_i^*. (In these equilibria, the favorite policy of a player who names a policy less than x_i^* may be greater than x_i^*, and vice versa. The conditions on the numbers of players who name policies less than x_i^* and greater than x_i^* ensure that no such player can, by naming instead her favorite policy, move the median policy closer to her favorite policy.)

Any action profile in which all players name the same, arbitrary, policy is also a Nash equilibrium; the outcome is the common policy named.

More generally, any profile in which at least three players name the same, arbitrary, policy x, at most $(n-3)/2$ players name a policy less than x, and at most $(n-3)/2$ players name a policy greater than x is a Nash equilibrium. (In both cases, no change in any player's action has any effect on the outcome.)

51.2 Symmetric strategic games

The games in Exercise 31.2, Example 39.1, and Figure 47.2 (both games) are symmetric. The game in Exercise 42.1 is not symmetric. The game in Section 2.8.4 is symmetric if and only if $u_1 = u_2$.

52.2 Equilibrium for pairwise interactions in a single population

The Nash equilibria are (A, A), (A, C), and (C, A). Only the equilibrium (A, A) is relevant if the game is played between the members of a single population—this equilibrium is the only *symmetric* equilibrium.

3 Nash Equilibrium: Illustrations

58.1 Cournot's duopoly game with linear inverse demand and different unit costs

Following the analysis in the text, the best response function of firm 1 is

$$b_1(q_2) = \begin{cases} \frac{1}{2}(\alpha - c_1 - q_2) & \text{if } q_2 \leq \alpha - c_1 \\ 0 & \text{otherwise} \end{cases}$$

while that of firm 2 is

$$b_2(q_1) = \begin{cases} \frac{1}{2}(\alpha - c_2 - q_1) & \text{if } q_1 \leq \alpha - c_2 \\ 0 & \text{otherwise.} \end{cases}$$

To find the Nash equilibrium, first plot these two functions. Each function has the same general form as the best response function of either firm in the case studied in the text. However, the fact that $c_1 \neq c_2$ leads to two qualitatively different cases when we combine the two functions to find a Nash equilibrium. If c_1 and c_2 do not differ very much then the functions in the analogue of Figure 59.1 intersect at a pair of outputs that are both positive. If c_1 and c_2 differ a lot, however, the functions intersect at a pair of outputs in which $q_1 = 0$.

Precisely, if $c_1 \leq \frac{1}{2}(\alpha + c_2)$ then the downward-sloping parts of the best response functions intersect (as in Figure 59.1), and the game has a unique Nash equilibrium, given by the solution of the two equations

$$\begin{aligned} q_1 &= \tfrac{1}{2}(\alpha - c_1 - q_2) \\ q_2 &= \tfrac{1}{2}(\alpha - c_2 - q_1). \end{aligned}$$

This solution is

$$(q_1^*, q_2^*) = \left(\tfrac{1}{3}(\alpha - 2c_1 + c_2), \tfrac{1}{3}(\alpha - 2c_2 + c_1)\right).$$

If $c_1 > \frac{1}{2}(\alpha + c_2)$ then the downward-sloping part of firm 1's best response function lies below the downward-sloping part of firm 2's best response function (as in Figure 12.1), and the game has a unique Nash equilibrium, $(q_1^*, q_2^*) = (0, \frac{1}{2}(\alpha - c_2))$.

In summary, the game always has a unique Nash equilibrium, defined as follows:

$$\begin{cases} \left(\tfrac{1}{3}(\alpha - 2c_1 + c_2), \tfrac{1}{3}(\alpha - 2c_2 + c_1)\right) & \text{if } c_1 \leq \tfrac{1}{2}(\alpha + c_2) \\ \left(0, \tfrac{1}{2}(\alpha - c_2)\right) & \text{if } c_1 > \tfrac{1}{2}(\alpha + c_2). \end{cases}$$

The output of firm 2 exceeds that of firm 1 in every equilibrium.

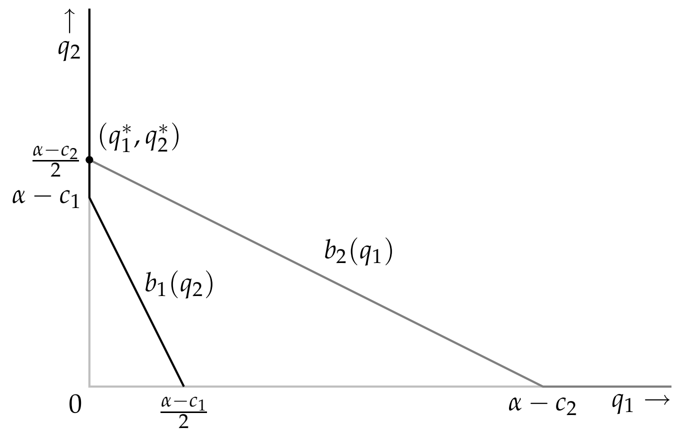


Figure 12.1 The best response functions in Cournot's duopoly game under the assumptions of Exercise 58.1 when $\alpha - c_1 < \frac{1}{2}(\alpha - c_2)$. The unique Nash equilibrium in this case is $(q_1^*, q_2^*) = (0, \frac{1}{2}(\alpha - c_2))$.

If c_2 decreases then firm 2's output increases and firm 1's output either falls, if $c_1 \leq \frac{1}{2}(\alpha + c_2)$, or remains equal to 0, if $c_1 > \frac{1}{2}(\alpha + c_2)$. The total output increases and the price falls.

60.2 Nash equilibrium of Cournot's duopoly game and the collusive outcome

The firms' total profit is $(q_1 + q_2)(\alpha - c - q_1 - q_2)$, or $Q(\alpha - c - Q)$, where Q denotes total output. This function is a quadratic in Q that is zero when $Q = 0$ and when $Q = \alpha - c$, so that its maximizer is $Q^* = \frac{1}{2}(\alpha - c)$.

If each firm produces $\frac{1}{4}(\alpha - c)$ then its profit is $\frac{1}{8}(\alpha - c)^2$. This profit exceeds its Nash equilibrium profit of $\frac{1}{9}(\alpha - c)^2$.

If one firm produces $Q^*/2$, the other firm's best response is $b_i(Q^*/2) = \frac{1}{2}(\alpha - c - \frac{1}{4}(\alpha - c)) = \frac{3}{8}(\alpha - c)$. That is, if one firm produces $Q^*/2$, the other firm wants to produce *more* than $Q^*/2$.

63.1 Interaction among resource-users

The game is given as follows.

Players The firms.

Actions Each firm's set of actions is the set of all nonnegative numbers (representing the amount of input it uses).

Preferences The payoff of each firm i is

$$\begin{cases} x_i(1 - (x_1 + \cdots + x_n)) & \text{if } x_1 + \cdots + x_n \leq 1 \\ 0 & \text{if } x_1 + \cdots + x_n > 1. \end{cases}$$

This game is the same as that in Exercise 61.1 for $c = 0$ and $\alpha = 1$. Thus it has a unique Nash equilibrium, $(x_1, \ldots, x_n) = (1/(n+1), \ldots, 1/(n+1))$.

In this Nash equilibrium, each firm's output is $(1/(n+1))(1 - n/(n+1)) = 1/(n+1)^2$. If $x_i = 1/(2n)$ for $i = 1, \ldots, n$ then each firm's output is $1/(4n)$, which exceeds $1/(n+1)^2$ for $n \geq 2$. (We have $1/(4n) - 1/(n+1)^2 = (n-1)^2/(4n(n+1)^2) > 0$ for $n \geq 2$.)

67.1 Bertrand's duopoly game with constant unit cost

The pair (c, c) of prices remains a Nash equilibrium; the argument is the same as before. Further, as before, there is no other Nash equilibrium. The argument needs only very minor modification. For an arbitrary function D there may exist no monopoly price p^m; in this case, if $p_i > c$, $p_j > c$, $p_i \geq p_j$, and $D(p_j) = 0$ then firm i can increase its profit by reducing its price slightly below $\overline{p}$ (for example).

68.1 Bertrand's oligopoly game

Consider a profile $(p_1, \ldots, p_n)$ of prices in which $p_i \geq c$ for all i and at least two prices are equal to c. Every firm's profit is zero. If any firm raises its price its profit remains zero. If a firm charging more than c lowers its price, but not below c, its profit also remains zero. If a firm lowers its price below c then its profit is negative. Thus any such profile is a Nash equilibrium.

To show that no other profile is a Nash equilibrium, we can argue as follows.

- If some price is less than c then the firm charging the lowest price can increase its profit (to zero) by increasing its price to c.
- If exactly one firm's price is equal to c then that firm can increase its profit by raising its price a little (keeping it less than the next highest price).
- If all firms' prices exceed c then the firm charging the highest price can increase its profit by lowering its price to some price between c and the lowest price being charged.

68.2 Bertrand's duopoly game with different unit costs

a. If all consumers buy from firm 1 when both firms charge the price c_2, then $(p_1, p_2) = (c_2, c_2)$ is a Nash equilibrium by the following argument. Firm 1's profit is positive, while firm 2's profit is zero (since it serves no customers).

- If firm 1 increases its price, its profit falls to zero.
- If firm 1 reduces its price, say to p, then its profit changes from $(c_2 - c_1)(\alpha - c_2)$ to $(p - c_1)(\alpha - p)$. Since c_2 is less than the maximizer of $(p - c_1)(\alpha - p)$, firm 1's profit falls.

- If firm 2 increases its price, its profit remains zero.
- If firm 2 decreases its price, its profit becomes negative (since its price is less than its unit cost).

Under this rule no other pair of prices is a Nash equilibrium, by the following argument.

- If $p_i < c_1$ for $i = 1, 2$ then the firm with the lower price (or either firm, if the prices are the same) can increase its profit (to zero) by raising its price above that of the other firm.
- If $p_1 > p_2 \geq c_2$ then firm 2 can increase its profit by raising its price a little.
- If $p_2 > p_1 \geq c_1$ then firm 1 can increase its profit by raising its price a little.
- If $p_2 \leq p_1$ and $p_2 < c_2$ then firm 2's profit is negative, so that it can increase its profit by raising its price.
- If $p_1 = p_2 > c_2$ then at least one of the firms is not receiving all of the demand, and that firm can increase its profit by lowering its price a little.

b. Now suppose that the rule for splitting up the customers when the prices are equal specifies that firm 2 receives some customers when both prices are c_2. By the argument for part *a*, the only possible Nash equilibrium is $(p_1, p_2) = (c_2, c_2)$. (The argument in part *a* that every other pair of prices is not a Nash equilibrium does not use the fact that customers are split equally when $(p_1, p_2) = (c_2, c_2)$.) But if $(p_1, p_2) = (c_2, c_2)$ and firm 2 receives some customers, firm 1 can increase its profit by reducing its price a little and capturing the entire market.

73.1 Electoral competition with asymmetric voters' preferences

The unique Nash equilibrium remains (m, m); the direct argument is exactly the same as before. (The dividing line between the supporters of two candidates with different positions changes. If $x_i < x_j$, for example, the dividing line is $\frac{1}{3}x_i + \frac{2}{3}x_j$ rather than $\frac{1}{2}(x_i + x_j)$. The resulting change in the best response functions does not affect the Nash equilibrium.)

75.3 Electoral competition for more general preferences

a. If x^* is a Condorcet winner then for any $y \neq x^*$ a majority of voters prefer x^* to y, so y is not a Condorcet winner. Thus there is no more than one Condorcet winner.

b. Suppose that one of the remaining voters prefers y to z to x, and the other prefers z to x to y. For each position there is another position preferred by a majority of voters, so no position is a Condorcet winner.

c. Now suppose that x^* is a Condorcet winner. Then the strategic game described the exercise has a unique Nash equilibrium in which both candidates choose x^*. This pair of actions is a Nash equilibrium because if either candidate chooses a different position she loses. For any other pair of actions either one candidate loses, in which case that candidate can deviate to the position x^* and at least tie, or the candidates tie at a position different from x^*, in which case either of them can deviate to x^* and win.

   If there is no Condorcet winner then for every position there is another position preferred by a majority of voters. Thus for every pair of distinct positions the loser can deviate and win, and for every pair of identical positions either candidate can deviate and win. Thus there is no Nash equilibrium.

76.1 Competition in product characteristics

Suppose there are two firms. If the products are different, then either firm increases its market share by making its product more similar to that of its rival. Thus in every possible equilibrium the products are the same. But if $x_1 = x_2 \neq m$ then each firm's market share is 50%, while if it changes its product to be closer to m then its market share rises above 50%. Thus the only possible equilibrium is $(x_1, x_2) = (m, m)$. This pair of positions is an equilibrium, since each firm's market share is 50%, and if either firm changes its product its market share falls below 50%.

Now suppose there are three firms. If all firms' products are the same, each obtains one-third of the market. If $x_1 = x_2 = x_3 = m$ then any firm, by changing its product a little, can obtain close to one-half of the market. If $x_1 = x_2 = x_3 \neq m$ then any firm, by changing its product a little, can obtain more than one-half of the market. If the firms' products are not all the same, then at least one of the extreme products is different from the other two products, and the firm that produces it can increase its market share by making it more similar to the other products. Thus when there are three firms there is no Nash equilibrium.

79.1 Direct argument for Nash equilibria of *War of Attrition*

- If $t_1 = t_2$ then either player can increase her payoff by conceding slightly later (in which case she obtains the object for sure, rather than getting it with probability $\frac{1}{2}$).
- If $0 < t_i < t_j$ then player i can increase her payoff by conceding at 0.
- If $0 = t_i < t_j < v_i$ then player i can increase her payoff (from 0 to almost $v_i - t_j > 0$) by conceding slightly after t_j.

Thus there is no Nash equilibrium in which $t_1 = t_2$, $0 < t_i < t_j$, or $0 = t_i < t_j < v_i$ (for $i = 1$ and $j = 2$, or $i = 2$ and $j = 1$). The remaining possibility is that $0 = t_i < t_j$ and $t_j \geq v_i$ for $i = 1$ and $j = 2$, or $i = 2$ and $j = 1$. In this case player i's payoff is 0, while if she concedes later her payoff is negative; player j's payoff is v_j, her highest possible payoff in the game.

85.1 Second-price sealed-bid auction with two bidders

If player 2's bid b_2 is less than v_1 then any bid of b_2 or more is a best response of player 1 (she wins and pays the price b_2). If player 2's bid is equal to v_1 then every bid of player 1 yields her the payoff zero (either she wins and pays v_1, or she loses), so every bid is a best response. If player 2's bid b_2 exceeds v_1 then any bid of less than b_2 is a best response of player 1. (If she bids b_2 or more she wins, but pays the price $b_2 > v_1$, and hence obtains a negative payoff.) In summary, player 1's best response function is

$$B_1(b_2) = \begin{cases} \{b_1 : b_1 \geq b_2\} & \text{if } b_2 < v_1 \\ \{b_1 : b_1 \geq 0\} & \text{if } b_2 = v_1 \\ \{b_1 : 0 \leq b_1 < b_2\} & \text{if } b_2 > v_1. \end{cases}$$

By similar arguments, player 2's best response function is

$$B_2(b_1) = \begin{cases} \{b_2 : b_2 > b_1\} & \text{if } b_1 < v_2 \\ \{b_2 : b_2 \geq 0\} & \text{if } b_1 = v_2. \\ \{b_2 : 0 \leq b_2 \leq b_1\} & \text{if } b_1 > v_2. \end{cases}$$

These best response functions are shown in Figure 16.1.

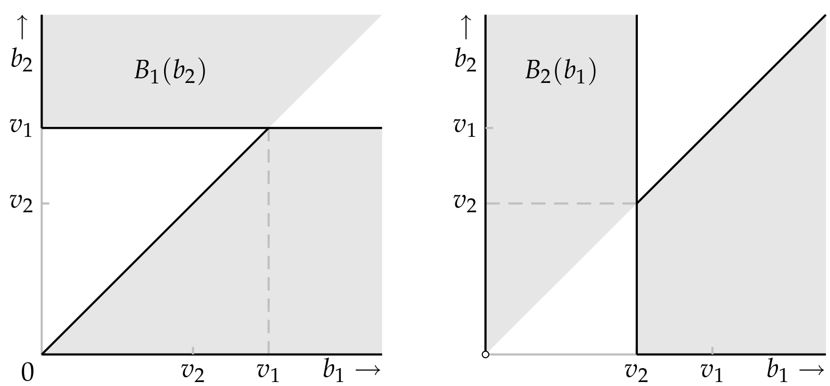


Figure 16.1 The players' best response functions in a two-player second-price sealed-bid auction (Exercise 85.1). Player 1's best response function is in the left panel; player 2's is in the right panel. (Only the edges marked by a black line are included.)

Superimposing the best response functions, we see that the set of Nash equilibria is the shaded set in Figure 17.1, namely the set of pairs (b_1, b_2) such that either

$$b_1 \leq v_2 \text{ and } b_2 \geq v_1$$

or

$$b_1 \geq v_2,\ b_1 \geq b_2, \text{ and } b_2 \leq v_1.$$

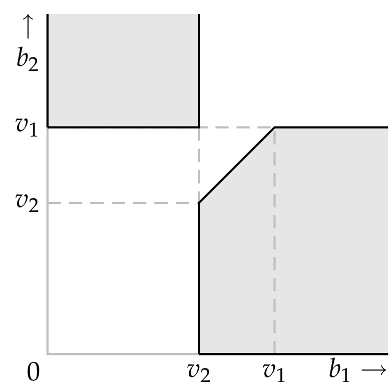


Figure 17.1 The set of Nash equilibria of a two-player second-price sealed-bid auction (Exercise 85.1).

86.2 Nash equilibrium of first-price sealed-bid auction

The profile $(b_1, \ldots, b_n) = (v_2, v_2, v_3, \ldots, v_n)$ is a Nash equilibrium by the following argument.

- If player 1 raises her bid she still wins, but pays a higher price and hence obtains a lower payoff. If player 1 lowers her bid then she loses, and obtains the payoff of 0.
- If any other player changes her bid to any price at most equal to v_2 the outcome does not change. If she raises her bid above v_2 she wins, but obtains a negative payoff.

87.1 First-price sealed-bid auction

A profile of bids in which the two highest bids are not the same is not a Nash equilibrium because the player naming the highest bid can reduce her bid slightly, continue to win, and pay a lower price.

By the argument in the text, in any equilibrium player 1 wins the object. Thus she submits one of the highest bids.

If the highest bid is less than v_2, then player 2 can increase her bid to a value between the highest bid and v_2, win, and obtain a positive payoff. Thus in an equilibrium the highest bid is at least v_2.

If the highest bid exceeds v_1, player 1's payoff is negative, and she can increase this payoff by reducing her bid. Thus in an equilibrium the highest bid is at most v_1.

Finally, any profile $(b_1, \ldots, b_n)$ of bids that satisfies the conditions in the exercise is a Nash equilibrium by the following argument.

- If player 1 increases her bid she continues to win, and reduces her payoff. If player 1 decreases her bid she loses and obtains the payoff 0, which is at most her payoff at $(b_1, \ldots, b_n)$.
- If any other player increases her bid she either does not affect the outcome, or wins and obtains a negative payoff. If any other player decreases her bid she does not affect the outcome.

89.1 All-pay auctions

Second-price all-pay auction with two bidders: The payoff function of bidder i is

$$u_i(b_1, b_2) = \begin{cases} -b_i & \text{if } b_i < b_j \\ v_i - b_j & \text{if } b_i > b_j, \end{cases}$$

with $u_1(b, b) = v_1 - b$ and $u_2(b, b) = -b$ for all b. This payoff function differs from that of player i in the *War of Attrition* only in the payoffs when the bids are equal. The set of Nash equilibria of the game is the same as that for the *War of Attrition*: the set of all pairs $(0, b_2)$ where $b_2 \geq v_1$ and $(b_1, 0)$ where $b_1 \geq v_2$. (The pair (b, b) of actions is not a Nash equilibrium for any value of b because player 2 can increase her payoff by either increasing her bid slightly or by reducing it to 0.)

First-price all-pay auction with two bidders: In any Nash equilibrium the two highest bids are equal, otherwise the player with the higher bid can increase her payoff by reducing her bid a little (keeping it larger than the other player's bid). But no profile of bids in which the two highest bids are equal is a Nash equilibrium, because the player with the higher index who submits this bid can increase her payoff by slightly increasing her bid, so that she wins rather than loses.

90.1 Multiunit auctions

Discriminatory auction To show that the action of bidding v_i and w_i is not dominant for player i, we need only find actions for the other players and alternative bids for player i such that player i's payoff is higher under the alternative bids than it is under the v_i and w_i, given the other players' actions. Suppose that each of the other players submits two bids of 0. Then if player i submits one bid between 0 and v_i and one bid between 0 and w_i she still wins two units, and pays less than when she bids v_i and w_i.

Uniform-price auction Suppose that some bidder other than i submits one bid between w_i and v_i and one bid of 0, and all the remaining bidders submit two bids of 0. Then bidder i wins one unit, and pays the price w_i. If she replaces her bid of w_i with a bid between 0 and w_i then she pays a lower price, and hence is better off.

Vickrey auction Suppose that player i bids v_i and w_i. Consider separately the cases in which the bids of the players other than i are such that player i wins 0, 1, and 2 units.

Player i wins 0 units: In this case the second highest of the other players' bids is at least v_i, so that if player i changes her bids so that she wins one or more units, for any unit she wins she pays at least v_i. Thus no change in her bids increases her payoff from its current value of 0 (and some changes lower her payoff).

Player i wins 1 unit: If player i raises her bid of v_i then she still wins one unit and the price remains the same. If she lowers this bid then either she still wins and pays the same price, or she does not win any units. If she raises her bid of w_i then either the outcome does not change, or she wins a second unit. In the latter case the price she pays is the previously-winning bid she beat, which is at least w_i, so that her payoff either remains zero or becomes negative.

Player i wins 2 units: Player i's raising either of her bids has no effect on the outcome; her lowering a bid either has no effect on the outcome or leads her to lose rather than to win, leading her to obtain the payoff of zero.

90.3 Internet pricing

The situation may be modeled as a multiunit auction in which k units are available, and each player attaches a positive value to only one unit and submits a bid for only one unit. The k highest bids win, and each winner pays the $(k+1)$st highest bid.

By a variant of the argument for a second-price auction, in which "highest of the other players' bids" is replaced by "highest rejected bid", each player's action of bidding her value is weakly dominates all her other actions.

96.2 Alternative standards of care under negligence with contributory negligence

First consider the case in which $X_1 = \hat{a}_1$ and $X_2 \leq \hat{a}_2$. The pair $(\hat{a}_1, \hat{a}_2)$ is a Nash equilibrium by the following argument.

If $a_2 = \hat{a}_2$ then the victim's level of care is sufficient (at least X_2), so that the injurer's payoff is given by (94.1) in the text. Thus the argument that the injurer's action $\hat{a}_1$ is a best response to $\hat{a}_2$ is exactly the same as the argument for the case $X_2 = \hat{a}_2$ in the text.

Since X_1 is the same as before, the victim's payoff is the same also, so that by the argument in the text the victim's best response to $\hat{a}_1$ is $\hat{a}_2$. Thus $(\hat{a}_1, \hat{a}_2)$ is a Nash equilibrium.

To show that $(\hat{a}_1, \hat{a}_2)$ is the only Nash equilibrium of the game, we study the players' best response functions. First consider the injurer's best response function. As in the text, we split the analysis into three cases.

$a_2 < X_2$: In this case the injurer does not have to pay any compensation, regardless of her level of care; her payoff is $-a_1$, so that her best response is $a_1 = 0$.

$a_2 = X_2$: In this case the injurer's best response is $\hat{a}_1$, as argued when showing that $(\hat{a}_1, \hat{a}_2)$ is a Nash equilibrium.

$a_2 > X_2$: In this case the injurer's best response is at most $\hat{a}_1$, since her payoff is equal to $-a_1$ for larger values of a_1.

Thus the injurer's best response takes a form like that shown in the left panel of Figure 20.1. (In fact, $b_1(a_2) = \hat{a}_1$ for $X_2 \leq a_2 \leq \hat{a}_2$, but the analysis depends only on the fact that $b_1(a_2) \leq \hat{a}_1$ for $a_2 > X_2$.)

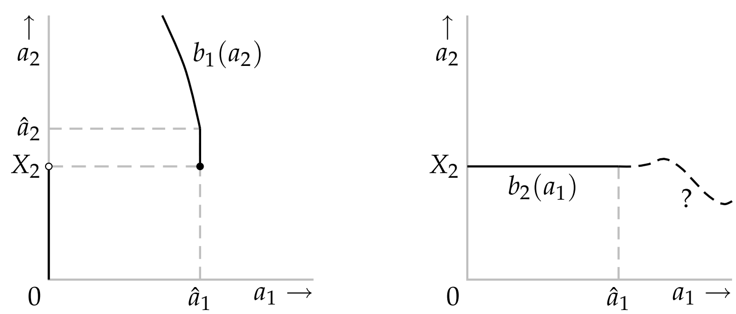


Figure 20.1 The players' best response functions under the rule of negligence with contributory negligence when $X_1 = \hat{a}_1$ and $X_2 = \hat{a}_2$. Left panel: the injurer's best response function b_1. Right panel: the victim's best response function b_2. (The position of the victim's best response function for $a_1 > \hat{a}_1$ is not significant, and is not determined in the solution.)

Now consider the victim's best response function. The victim's payoff function is

$$u_2(a_1, a_2) = \begin{cases} -a_2 & \text{if } a_1 < \hat{a}_1 \text{ and } a_2 \geq X_2 \\ -a_2 - L(a_1, a_2) & \text{if } a_1 \geq \hat{a}_1 \text{ or } a_2 < X_2. \end{cases}$$

As before, for $a_1 < \hat{a}_1$ we have $-a_2 - L(a_1, a_2) < -\hat{a}_2$ for all a_2, so that the victim's best response is X_2. As in the text, the nature of the victim's best responses to levels of care a_1 for which $a_1 > \hat{a}_1$ are not significant.

Combining the two best response functions we see that $(\hat{a}_1, \hat{a}_2)$ is the unique Nash equilibrium of the game.

Now consider the case in which $X_1 = M$ and $a_2 = \hat{a}_2$, where $M \geq \hat{a}_1$. The injurer's payoff is

$$u_1(a_1, a_2) = \begin{cases} -a_1 - L(a_1, a_2) & \text{if } a_1 < M \text{ and } a_2 \geq \hat{a}_2 \\ -a_1 & \text{if } a_1 \geq M \text{ or } a_2 < \hat{a}_2. \end{cases}$$

Now, the maximizer of $-a_1 - L(a_1, \hat{a}_2)$ is $\hat{a}_1$ (see the argument following (94.1) in the text), so that if M is large enough then the injurer's best response to $\hat{a}_2$ is $\hat{a}_1$. As before, if $a_2 < \hat{a}_2$ then the injurer's best response is 0, and if $a_2 > \hat{a}_2$ then the

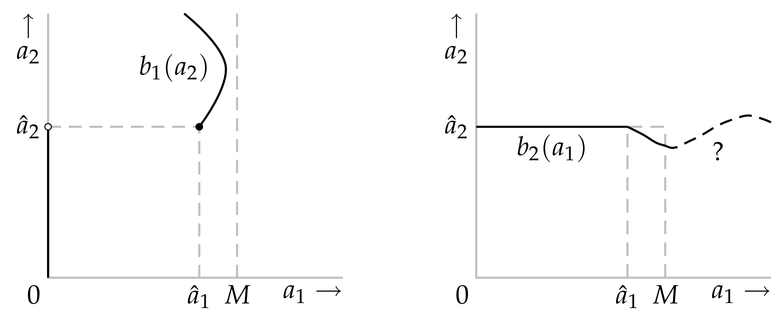


Figure 21.1 The players' best response functions under the rule of negligence with contributory negligence when $(X_1, X_2) = (M, \hat{a}_2)$, with $M \geq \hat{a}_1$. Left panel: the injurer's best response function b_1. Right panel: the victim's best response function b_2. (The position of the victim's best response function for $a_1 > M$ is not significant, and is not determined in the text.)

injurer's payoff decreases for $a_1 > M$, so that her best response is less than M. The injurer's best response function is shown in the left panel of Figure 21.1.

The victim's payoff is

$$u_2(a_1, a_2) = \begin{cases} -a_2 & \text{if } a_1 < M \text{ and } a_2 \geq \hat{a}_2 \\ -a_2 - L(a_1, a_2) & \text{if } a_1 \geq M \text{ or } a_2 < \hat{a}_2. \end{cases}$$

If $a_1 \leq \hat{a}_1$ then the victim's best response is $\hat{a}_2$ by the same argument as the one in the text. If a_1 is such that $\hat{a}_1 < a_1 < M$ then the victim's best response is at most $\hat{a}_2$ (since her payoff is decreasing for larger values of a_2). This information about the victim's best response function is recorded in the right panel of Figure 21.1; it is sufficient to deduce that $(\hat{a}_1, \hat{a}_2)$ is the unique Nash equilibrium of the game.

4 Mixed Strategy Equilibrium

101.1 Variant of *Matching Pennies*

The analysis is the same as for *Matching Pennies*. There is a unique steady state, in which each player chooses each action with probability $\frac{1}{2}$.

106.2 Extensions of *BoS* with vNM preferences

In the first case, when player 1 is indifferent between going to her less preferred concert in the company of player 2 and the lottery in which with probability $\frac{1}{2}$ she and player 2 go to different concerts and with probability $\frac{1}{2}$ they both go to her more preferred concert, the Bernoulli payoffs that represent her preferences satisfy the condition

$$u_1(S,S) = \tfrac{1}{2}u_1(S,B) + \tfrac{1}{2}u_1(B,B).$$

If we choose $u_1(S,B) = 0$ and $u_1(B,B) = 2$, then $u_1(S,S) = 1$. Similarly, for player 2 we can set $u_2(B,S) = 0$, $u_2(S,S) = 2$, and $u_2(B,B) = 1$. Thus the Bernoulli payoffs in the left panel of Figure 23.1 are consistent with the players' preferences.

In the second case, when player 1 is indifferent between going to her less preferred concert in the company of player 2 and the lottery in which with probability $\frac{3}{4}$ she and player 2 go to different concerts and with probability $\frac{1}{4}$ they both go to her more preferred concert, the Bernoulli payoffs that represent her preferences satisfy the condition

$$u_1(S,S) = \tfrac{3}{4}u_1(S,B) + \tfrac{1}{4}u_1(B,B).$$

If we choose $u_1(S,B) = 0$ and $u_1(B,B) = 2$ (as before), then $u_1(S,S) = \frac{1}{2}$. Similarly, for player 2 we can set $u_2(B,S) = 0$, $u_2(S,S) = 2$, and $u_2(B,B) = \frac{1}{2}$. Thus the Bernoulli payoffs in the right panel of Figure 23.1 are consistent with the players' preferences.

	Bach	*Stravinsky*
Bach	$2, 1$	$0, 0$
Stravinsky	$0, 0$	$1, 2$

	Bach	*Stravinsky*
Bach	$2, \frac{1}{2}$	$0, 0$
Stravinsky	$0, 0$	$\frac{1}{2}, 2$

Figure 23.1 The Bernoulli payoffs for two extensions of *BoS*.

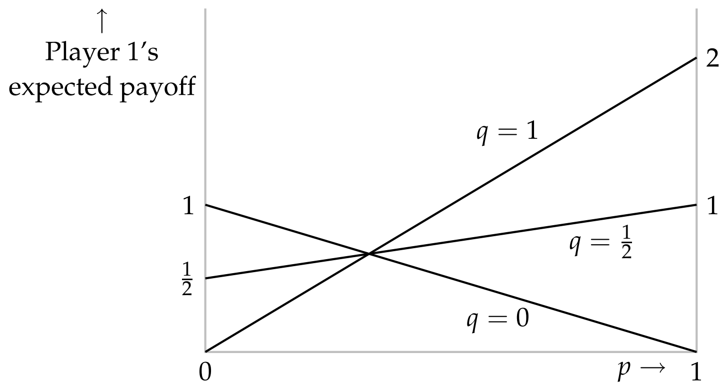


Figure 24.1 Player 1's expected payoff as a function of the probability p that she assigns to B in *BoS*, when the probability q that player 2 assigns to B is 0, $\frac{1}{2}$, and 1.

110.1 Expected payoffs

For *BoS*, player 1's expected payoff is shown in Figure 24.1.

For the game in the right panel of Figure 21.1 in the book, player 1's expected payoff is shown in Figure 24.2.

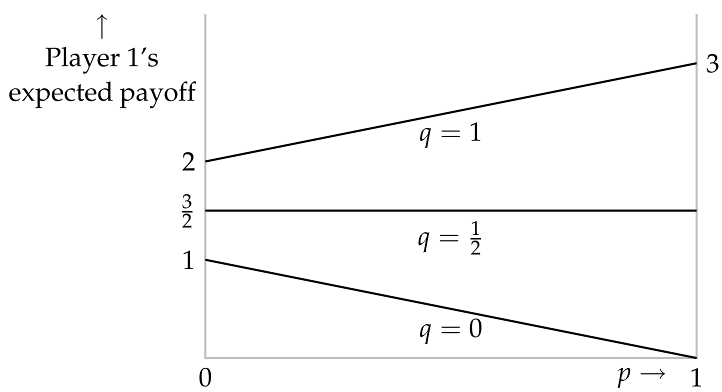


Figure 24.2 Player 1's expected payoff as a function of the probability p that she assigns to *Refrain* in the game in the right panel of Figure 21.1 in the book, when the probability q that player 2 assigns to *Refrain* is 0, $\frac{1}{2}$, and 1.

111.1 Examples of best responses

For *BoS*: for $q = 0$ player 1's unique best response is $p = 0$ and for $q = \frac{1}{2}$ and $q = 1$ her unique best response is $p = 1$. For the game in the right panel of Figure 21.1: for $q = 0$ player 1's unique best response is $p = 0$, for $q = \frac{1}{2}$ her set of best responses is the set of all her mixed strategies (all values of p), and for $q = 1$ her unique best response is $p = 1$.

114.1 Mixed strategy equilibrium of *Hawk–Dove*

Denote by u_i a payoff function whose expected value represents player i's preferences. The conditions in the problem imply that for player 1 we have

$$u_1(Passive, Passive) = \tfrac{1}{2}u_1(Aggressive, Aggressive) + \tfrac{1}{2}u_1(Aggressive, Passive)$$

and

$$u_1(Passive, Aggressive) = \tfrac{2}{3}u_1(Aggressive, Aggressive) + \tfrac{1}{3}u_1(Passive, Passive).$$

Given $u_1(Aggressive, Aggressive) = 0$ and $u_1(Passive, Aggressive = 1$, we have

$$u_1(Passive, Passive) = \tfrac{1}{2}u_1(Aggressive, Passive)$$

and

$$1 = \tfrac{1}{3}u_1(Passive, Passive),$$

so that

$$u_1(Passive, Passive) = 3 \text{ and } u_1(Aggressive, Passive) = 6.$$

Similarly,

$$u_2(Passive, Passive) = 3 \text{ and } u_2(Passive, Aggressive) = 6.$$

Thus the game is given in the left panel of Figure 25.1. The players' best response functions are shown in the right panel. The game has three mixed strategy Nash equilibria: $((0,1),(1,0))$, $((\frac{3}{4},\frac{1}{4}),(\frac{3}{4},\frac{1}{4}))$, and $((1,0),(0,1))$.

	Aggressive	*Passive*
Aggressive	0, 0	6, 1
Passive	1, 6	3, 3

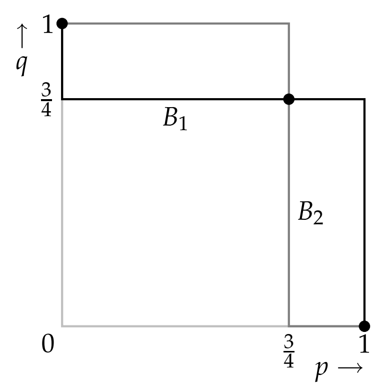


Figure 25.1 An extension of *Hawk–Dove* (left panel) and the players' best response functions when randomization is allowed in this game (right panel). The probability that player 1 assigns to *Aggressive* is p and the probability that player 2 assigns to *Aggressive* is q. The disks indicate the Nash equilibria (two pure, one mixed).

117.2 Choosing numbers

a. To show that the pair of mixed strategies in the question is a mixed strategy equilibrium, it suffices to verify the conditions in Proposition 116.2. Thus, given that each player's strategy specifies a positive probability for every action, it suffices to show that each action of each player yields the same expected payoff. Player 1's expected payoff to each pure strategy is $1/K$, because with probability $1/K$ player 2 chooses the same number, and with probability $1 - 1/K$ player 2 chooses a different number. Similarly, player 2's expected payoff to each pure strategy is $-1/K$, because with probability $1/K$ player 1 chooses the same number, and with probability $1 - 1/K$ player 2 chooses a different number. Thus the pair of strategies is a mixed strategy Nash equilibrium.

b. Let (p^*, q^*) be a mixed strategy equilibrium, where p^* and q^* are vectors, the jth components of which are the probabilities assigned to the integer j by each player. Given that player 2 uses the mixed strategy q^*, player 1's expected payoff if she chooses the number k is q_k^*. Hence if $p_k^* > 0$ then (by the first condition in Proposition 116.2) we need $q_k^* \geq q_j^*$ for all j, so that, in particular, $q_k^* > 0$ (q_j^* cannot be zero for all j!). But player 2's expected payoff if she chooses the number k is $-p_k^*$, so given $q_k^* > 0$ we need $p_k^* \leq p_j^*$ for all j (again by the first condition in Proposition 116.2), and, in particular, $p_k^* \leq 1/K$ (p_j^* cannot exceed $1/K$ for all j!). We conclude that any probability p_k^* that is positive must be at most $1/K$. The only possibility is that $p_k^* = 1/K$ for all k. A similar argument implies that $q_k^* = 1/K$ for all k.

120.2 Strictly dominating mixed strategies

Denote the probability that player 1 assigns to T by p and the probability she assigns to M by r (so that the probability she assigns to B is $1 - p - r$). A mixed strategy of player 1 strictly dominates T if and only if

$$p + 4r > 1 \quad \text{and} \quad p + 3(1 - p - r) > 1,$$

or if and only if $1 - 4r < p < 1 - \frac{3}{2}r$. For example, the mixed strategies $(\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$ and $(0, \frac{1}{4}, \frac{3}{4})$ both strictly dominate T.

120.3 Strict domination for mixed strategies

(*a*) True. Suppose that the mixed strategy α_i' assigns positive probability to the action a_i', which is strictly dominated by the action a_i. Then $u_i(a_i, a_{-i}) > u_i(a_i', a_{-i})$ for all a_{-i}. Let α_i be the mixed strategy that differs from α_i' only in the weight that α_i' assigns to a_i' is transferred to a_i. That is, α_i is defined by $\alpha_i(a_i') = 0$, $\alpha_i(a_i) = \alpha_i'(a_i') + \alpha_i'(a_i)$, and $\alpha_i(b_i) = \alpha_i'(b_i)$ for every other action b_i. Then α_i strictly dominates α_i': for any a_{-i} we have $U(\alpha_i, a_{-i}) - U(\alpha_i', a_{-i}) = \alpha_i'(a_i')(u(a_i, a_{-i}) - u_i(a_i', a_{-i})) > 0$.

(*b*) False. Consider a variant of the game in Figure 120.1 in the text in which player 1's payoffs to (T, L) and to (T, R) are both $\frac{5}{2}$ instead of 1. Then player 1's mixed strategy that assigns probability $\frac{1}{2}$ to M and probability $\frac{1}{2}$ to B is strictly dominated by T, even though neither M nor B is strictly dominated.

127.1 Equilibrium in the expert diagnosis game

When $E = rE' + (1 - r)I'$ the consumer is indifferent between her two actions when $p = 0$, so that her best response function has a vertical segment at $p = 0$. Referring to Figure 126.1 in the text, we see that the set of mixed strategy Nash equilibria correspond to $p = 0$ and $\pi/\pi' \leq q \leq 1$.

130.3 Bargaining

The game is given in Figure 27.1.

	0	2	4	6	8	10
0	5,5	4,6	3,7	2,8	1,9	0,10
2	6,4	5,5	4,6	3,7	2,8	0,0
4	7,3	6,4	5,5	4,6	0,0	0,0
6	8,2	7,3	6,4	0,0	0,0	0,0
8	9,1	8,2	0,0	0,0	0,0	0,0
10	10,0	0,0	0,0	0,0	0,0	0,0

Figure 27.1 A bargaining game.

By inspection it has a single symmetric pure strategy Nash equilibrium, $(10, 10)$.

Now consider situations in which the common mixed strategy assigns positive probability to two actions. Suppose that player 2 assigns positive probability only to 0 and 2. Then player 1's payoff to her action 4 exceeds her payoff to either 0 or 2. Thus there is no symmetric equilibrium in which the actions assigned positive probability are 0 and 2. By a similar argument we can rule out equilibria in which the actions assigned positive probability are any pair except 2 and 8, or 4 and 6.

If the actions to which player 2 assigns positive probability are 2 and 8 then player 1's expected payoffs to 2 and 8 are the same if the probability player 2 assigns to 2 is $\frac{2}{5}$ (and the probability she assigns to 8 is $\frac{3}{5}$). Given these probabilities, player 1's expected payoff to her actions 2 and 8 is $\frac{16}{5}$, and her expected payoff to every other action is less than $\frac{16}{5}$. Thus the pair of mixed strategies in which every player assigns probability $\frac{2}{5}$ to 2 and $\frac{3}{5}$ to 8 is a symmetric mixed strategy Nash equilibrium.

Similarly, the game has a symmetric mixed strategy equilibrium (α^*, α^*) in which α^* assigns probability $\frac{4}{5}$ to the demand of 4 and probability $\frac{1}{5}$ to the demand of 6.

In summary, the game has three symmetric mixed strategy Nash equilibria in which each player's strategy assigns positive probability to at most two actions: one in which probability 1 is assigned to 10, one in which probability $\frac{2}{5}$ is assigned to 2 and probability $\frac{3}{5}$ is assigned to 8, and one in which probability $\frac{4}{5}$ is assigned to 4 and probability $\frac{1}{5}$ is assigned to 6.

132.2 Reporting a crime when the witnesses are heterogeneous

Denote by p_i the probability with which each witness with cost c_i reports the crime, for $i = 1, 2$. For each witness with cost c_1 to report with positive probability less than one, we need

$$\begin{aligned} v - c_1 &= v \cdot \Pr\{\text{at least one other person calls}\} \\ &= v\left(1 - (1-p_1)^{n_1-1}(1-p_2)^{n_2}\right), \end{aligned}$$

or

$$c_1 = v(1-p_1)^{n_1-1}(1-p_2)^{n_2}. \tag{28.1}$$

Similarly, for each witness with cost c_2 to report with positive probability less than one, we need

$$\begin{aligned} v - c_2 &= v \cdot \Pr\{\text{at least one other person calls}\} \\ &= v\left(1 - (1-p_1)^{n_1}(1-p_2)^{n_2-1}\right), \end{aligned}$$

or

$$c_2 = v(1-p_1)^{n_1}(1-p_2)^{n_2-1}. \tag{28.2}$$

Dividing (28.1) by (28.2) we obtain

$$1 - p_2 = c_1(1-p_1)/c_2.$$

Substituting this expression for $1 - p_2$ into (28.1) we get

$$p_1 = 1 - \left(\frac{c_1}{v} \cdot \left(\frac{c_2}{c_1}\right)^{n_2}\right)^{1/(n-1)}.$$

Similarly,

$$p_2 = 1 - \left(\frac{c_2}{v} \cdot \left(\frac{c_1}{c_2}\right)^{n_1}\right)^{1/(n-1)}.$$

For these two numbers to be probabilities, we need each of them to be nonnegative and at most one, which requires

$$\left(\frac{c_2^{n_2}}{v}\right)^{1/(n_2-1)} < c_1 < \left(vc_2^{n_1-1}\right)^{1/n_1}.$$

136.1 Best response dynamics in Cournot's duopoly game

The best response functions of both firms are the same, so if the firms' outputs are initially the same, they are the same in every period: $q_1^t = q_2^t$ for every t. For each period t, we thus have

$$q_i^t = \tfrac{1}{2}(\alpha - c - q_i^t).$$

Given that $q_i^1 = 0$ for $i = 1, 2$, solving this first-order difference equation we have

$$q_i^t = \tfrac{1}{3}(\alpha - c)[1 - (-\tfrac{1}{2})^{t-1}]$$

for each period t. When t is large, q_i^t is close to $\frac{1}{3}(\alpha - c)$, a firm's equilibrium output.

In the first few periods, these outputs are 0, $\frac{1}{2}(\alpha - c)$, $\frac{1}{4}(\alpha - c)$, $\frac{3}{8}(\alpha - c)$, $\frac{5}{16}(\alpha - c)$.

139.1 Finding all mixed strategy equilibria of two-player games

Left game:

- There is no equilibrium in which each player's mixed strategy assigns positive probability to a single action (i.e. there is no pure equilibrium).
- Consider the possibility of an equilibrium in which one player assigns probability 1 to a single action while the other player assigns positive probability to both her actions. For neither action of player 1 is player 2's payoff the same for both her actions, and for neither action of player 2 is player 1's payoff the same for both her actions, so there is no mixed strategy equilibrium of this type.
- Consider the possibility of a mixed strategy equilibrium in which each player assigns positive probability to both her actions. Denote by p the probability player 1 assigns to T and by q the probability player 2 assigns to L. For player 1's expected payoff to her two actions to be the same we need

  $$6q = 3q + 6(1 - q),$$

  or $q = \frac{2}{3}$. For player 2's expected payoff to her two actions to be the same we need

  $$2(1 - p) = 6p,$$

  or $p = \frac{1}{4}$. We conclude that the game has a unique mixed strategy equilibrium, $((\frac{1}{4}, \frac{3}{4}), (\frac{2}{3}, \frac{1}{3}))$.

Right game:

- By inspection, (T, R) and (B, L) are the pure strategy equilibria.

- Consider the possibility of a mixed strategy equilibrium in which one player assigns probability 1 to a single action while the other player assigns positive probability to both her actions.
  - $\{T\}$ for player 1, $\{L, R\}$ for player 2: no equilibrium, because player 2's payoffs to (T, L) and (T, R) are not the same.
  - $\{B\}$ for player 1, $\{L, R\}$ for player 2: no equilibrium, because player 2's payoffs to (B, L) and (B, R) are not the same.
  - $\{T, B\}$ for player 1, $\{L\}$ for player 2: no equilibrium, because player 1's payoffs to (T, L) and (B, L) are not the same.
  - $\{T, B\}$ for player 1, $\{R\}$ for player 2: player 1's payoffs to (T, R) and (B, R) are the same, so there is an equilibrium in which player 1 uses T with probability p if player 2's expected payoff to R, which is $2p + 1 - p$, is at least her expected payoff to L, which is $p + 2(1 - p)$. That is, the game has equilibria in which player 1's mixed strategy is $(p, 1 - p)$, with $p \geq \frac{1}{2}$, and player 2 uses R with probability 1.
- Consider the possibility of an equilibrium in which both players assign positive probability to both their actions. Denote by q the probability that player 2 assigns to L. For player 1's expected payoffs to T and B to be the same we need $0 = 2q$, or $q = 0$, so there is no equilibrium in which both players assign positive probability to both their actions.

In summary, the mixed strategy equilibria of the game are $((0, 1), (1, 0))$ (i.e. the pure equilibrium (B, L)) and $((p, 1 - p), (0, 1))$ for $\frac{1}{2} \leq p \leq 1$ (of which one equilibrium is the pure equilibrium (T, R)).

145.1 All-pay auction with many bidders

Denote the common mixed strategy by F. Look for an equilibrium in which the largest value of z for which $F(z) = 0$ is 0 and the smallest value of z for which $F(z) = 1$ is $z = K$.

A player who bids a_i wins if and only if the other $n - 1$ players all bid less than she does, an event with probability $(F(a_i))^{n-1}$. Thus, given that the probability that she ties for the highest bid is zero, her expected payoff is

$$(K - a_i)(F(a_i))^{n-1} + (-a_i)(1 - (F(a_i))^{n-1}).$$

Given the form of F, for an equilibrium this expected payoff must be constant for all values of a_i with $0 \leq a_i \leq K$. That is, for some value of c we have

$$K(F(a_i))^{n-1} - a_i = c \text{ for all } 0 \leq a_i \leq K.$$

For $F(0) = 0$ we need $c = 0$, so that $F(a_i) = (a_i/K)^{1/(n-1)}$ is the only candidate for an equilibrium strategy.

The function F is a cumulative probability distribution on the interval from 0 to K because $F(0) = 0$, $F(K) = 1$, and F is increasing. Thus F is indeed an equilibrium strategy.

We conclude that the game has a mixed strategy Nash equilibrium in which each player randomizes over all her actions according to the probability distribution $F(a_i) = (a_i/K)^{1/(n-1)}$; each player's equilibrium expected payoff is 0.

Each player's mean bid is K/n.

147.2 Preferences over lotteries

The first piece of information about the decision-maker's preferences among lotteries is consistent with her preferences being represented by the expected value of a payoff function: set $u(a_1) = 0$, $u(a_2)$ equal to any number between $\frac{1}{2}$ and $\frac{1}{4}$, and $u(a_3) = 1$.

The second piece of information about the decision-maker's preferences is not consistent with these preferences being represented by the expected value of a payoff function, by the following argument. For consistency with the information about the decision-maker's preferences among the four lotteries, we need

$$0.4u(a_1) + 0.6u(a_3) > 0.5u(a_2) \;+\; 0.5u(a_3) > \\ 0.3u(a_1) + 0.2u(a_2) \;+\; 0.5u(a_3) > 0.45u(a_1) + 0.55u(a_3).$$

The first inequality implies $u(a_2) < 0.8u(a_1) + 0.2u(a_3)$ and the last inequality implies $u(a_2) > 0.75u(a_1) + 0.25u(a_3)$. Because $u(a_1) < u(a_3)$, we have $0.75u(a_1) + 0.25u(a_3) > 0.8u(a_1) + 0.2u(a_3)$, so that the two inequalities are incompatible.

149.2 Normalized vNM payoff functions

Let $\overline{a}$ be the best outcome according to her preferences and let $\underline{a}$ be the worse outcome. Let $\eta = -u(\underline{a})/(u(\overline{a}) - u(\underline{a}))$ and $\theta = 1/(u(\overline{a}) - u(\underline{a})) > 0$. Lemma 148.1 implies that the function v defined by $v(x) = \eta + \theta u(x)$ represents the same preferences as does u; we have $v(\underline{a}) = 0$ and $v(\overline{a}) = 1$.

5 Extensive Games with Perfect Information: Theory

163.1 Nash equilibria of extensive games

The strategic form of the game in Exercise 156.2*a* is given in Figure 33.1.

	EG	EH	FG	FH
C	$1,0$	$1,0$	$3,2$	$3,2$
D	$2,3$	$0,1$	$2,3$	$0,1$

Figure 33.1 The strategic form of the game in Exercise 156.2*a*.

The Nash equilibria of the game are (C, FG), (C, FH), and (D, EG).
The strategic form of the game in Figure 160.1 is given in Figure 33.2.

	E	F
CG	$1,2$	$3,1$
CH	$0,0$	$3,1$
DG	$2,0$	$2,0$
DH	$2,0$	$2,0$

Figure 33.2 The strategic form of the game in Figure 160.1.

The Nash equilibria of the game are (CH, F), (DG, E), and (DH, E).

164.2 Subgames

The subgames of the game in Exercise 156.2*c* are the whole game and the six games in Figure 34.1.

168.1 Checking for subgame perfect equilibria

The Nash equilibria (CH, F) and (DH, E) are not subgame perfect equilibria: in the subgame following the history (C, E), player 1's strategies CH and DH induce the strategy H, which is not optimal.

The Nash equilibrium (DG, E) is a subgame perfect equilibrium: (*a*) it is a Nash equilibrium, so player 1's strategy is optimal at the start of the game, given player 2's strategy, (*b*) in the subgame following the history C, player 2's strategy E induces the strategy E, which is optimal given player 1's strategy, and (*c*) in the subgame following the history (C, E), player 1's strategy DG induces the strategy G, which is optimal.

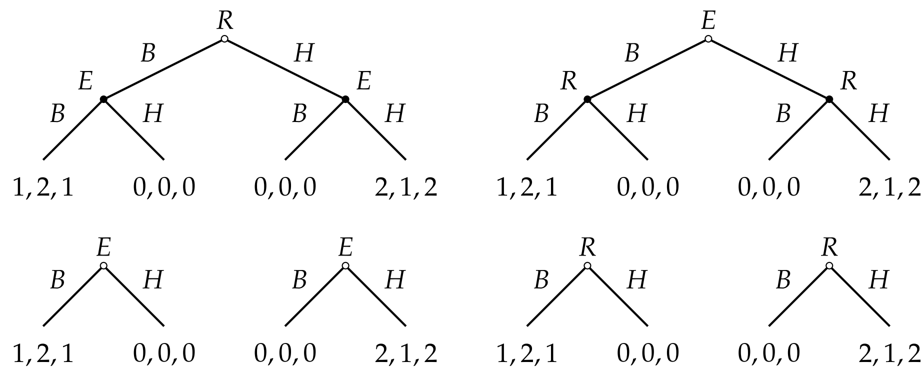


Figure 34.1 The proper subgames of the game in Exercise 156.2*c*.

174.1 Sharing heterogeneous objects

Let $n = 2$ and $k = 3$, and call the objects a, b, and c. Suppose that the values person 1 attaches to the objects are 3, 2, and 1 respectively, while the values player 2 attaches are 1, 3, 2. If player 1 chooses a on the first round, then in any subgame perfect equilibrium player 2 chooses b, leaving player 1 with c on the second round. If instead player 1 chooses b on the first round, in any subgame perfect equilibrium player 2 chooses c, leaving player 1 with a on the second round. Thus in every subgame perfect equilibrium player 1 chooses b on the first round (though she values a more highly.)

Now I argue that for any preferences of the players, $G(2,3)$ has a subgame perfect equilibrium of the type described in the exercise. For any object chosen by player 1 in round 1, in any subgame perfect equilibrium player 2 chooses her favorite among the two objects remaining in round 2. Thus player 2 never obtains the object she least prefers; in any subgame perfect equilibrium, player 1 obtains that object. Player 1 can ensure she obtains her more preferred object of the two remaining by choosing that object on the first round. That is, there is a subgame perfect equilibrium in which on the first round player 1 chooses her more preferred object out of the set of objects excluding the object player 2 least prefers, and on the last round she obtains x_3. In this equilibrium, player 2 obtains the object less preferred by player 1 out of the set of objects excluding the object player 2 least prefers. That is, player 2 obtains x_2. (Depending on the players' preferences, the game also may have a subgame perfect equilibrium in which player 1 chooses x_3 on the first round.)

177.3 Comparing simultaneous and sequential games

a. Denote by (a_1^*, a_2^*) a Nash equilibrium of the strategic game in which player 1's payoff is maximal in the set of Nash equilibria. Because (a_1^*, a_2^*) is a Nash equilibrium, a_2^* is a best response to a_1^*. By assumption, it is the only best response to a_1^*. Thus if player 1 chooses a_1^* in the extensive game, player 2 must choose a_2^* in any subgame perfect equilibrium of the extensive game. That is, by choosing a_1^*, player 1 is assured of a payoff of at least $u_1(a_1^*, a_2^*)$. Thus in any subgame perfect equilibrium player 1's payoff must be at least $u_1(a_1^*, a_2^*)$.

b. Suppose that $A_1 = \{T, B\}$, $A_2 = \{L, R\}$, and the payoffs are those given in Figure 35.1. The strategic game has a unique Nash equilibrium, (T, L), in which player 2's payoff is 1. The extensive game has a unique subgame perfect equilibrium, (B, LR) (where the first component of player 2's strategy is her action after the history T and the second component is her action after the history B). In this subgame perfect equilibrium player 2's payoff is 2.

	L	R
T	1,1	3,0
B	0,0	2,2

Figure 35.1 The payoffs for the example in Exercise 177.3*b*.

c. Suppose that $A_1 = \{T, B\}$, $A_2 = \{L, R\}$, and the payoffs are those given in Figure 35.2. The strategic game has a unique Nash equilibrium, (T, L), in which player 2's payoff is 2. A subgame perfect equilibrium of the extensive game is (B, RL) (where the first component of player 2's strategy is her action after the history T and the second component is her action after the history B). In this subgame perfect equilibrium player 1's payoff is 1. (If you read Chapter 4, you can find the mixed strategy Nash equilibria of the strategic game; in all these equilibria, as in the pure strategy Nash equilibrium, player 1's expected payoff exceeds 1.)

	L	R
T	2,2	0,2
B	1,1	3,0

Figure 35.2 The payoffs for the example in Exercise 177.3*c*.

179.3 Three Men's Morris, or Mill

Number the squares 1 through 9, starting at the top left, working across each row. The following strategy of player 1 guarantees she wins, so that the subgame perfect equilibrium outcome is that she wins. First player 1 chooses the central square (5).

- Suppose player 2 then chooses a corner; take it to be square 1. Then player 1 chooses square 6. Now player 2 must choose square 4 to avoid defeat; player 1 must choose square 7 to avoid defeat; and then player 2 must choose square

3 to avoid defeat (otherwise player 1 can move from square 6 to square 3 on her next turn). If player 1 now moves from square 6 to square 9, then whatever player 2 does she can subsequently move her counter from square 5 to square 8 and win.

- Suppose player 2 then chooses a noncorner; take it to be square 2. Then player 1 chooses square 7. Now player 2 must choose square 3 to avoid defeat; player 1 must choose square 1 to avoid defeat; and then player 2 must choose square 4 to avoid defeat (otherwise player 1 can move from square 5 to square 4 on her next turn). If player 1 now moves from square 7 to square 8, then whatever player 2 does she can subsequently move from square 8 to square 9 and win.

6 Extensive Games with Perfect Information: Illustrations

183.1 Nash equilibria of the ultimatum game

For *every* amount x there are Nash equilibria in which person 1 offers x. For example, for any value of x there is a Nash equilibrium in which person 1's strategy is to offer x and person 2's strategy is to accept x and any offer more favorable, and reject every other offer. (Given person 2's strategy, person 1 can do no better than offer x. Given person 1's strategy, person 2 should accept x; whether person 2 accepts or rejects any other offer makes no difference to her payoff, so that rejecting all less favorable offers is, in particular, optimal.)

183.2 Subgame perfect equilibria of the ultimatum game with indivisible units

In this case each player has finitely many actions, and for both possible subgame perfect equilibrium strategies of player 2 there is an optimal strategy for player 1.

If player 2 accepts all offers then player 1's best strategy is to offer 0, as before.

If player 2 accepts all offers except 0 then player 1's best strategy is to offer one cent (which player 2 accepts).

Thus the game has two subgame perfect equilibria: one in which player 1 offers 0 and player 2 accepts all offers, and one in which player 1 offers one cent and player 2 accepts all offers except 0.

186.1 Holdup game

The game is defined as follows.

Players Two people, person 1 and person 2.

Terminal histories The set of all sequences (low, x, Z), where x is a number with $0 \leq x \leq c_L$ (the amount of money that person 1 offers to person 2 when the pie is small), and $(high, x, Z)$, where x is a number with $0 \leq x \leq c_H$ (the amount of money that person 1 offers to person 2 when the pie is large) and Z is either Y ("yes, I accept") or N ("no, I reject").

Player function $P(\varnothing) = 2$, $P(low) = P(high) = 1$, and $P(low, x) = P(high, x) = 2$ for all x.

Preferences Person 1's preferences are represented by payoffs equal to the amounts of money she receives, equal to $c_L - x$ for any terminal history (low, x, Y) with $0 \leq x \leq c_L$, equal to $c_H - x$ for any terminal history

$(high, x, Y)$ with $0 \le x \le c_H$, and equal to 0 for any terminal history (low, x, N) with $0 \le x \le c_L$ and for any terminal history $(high, x, N)$ with $0 \le x \le c_H$. Person 2's preferences are represented by payoffs equal to $x - L$ for the terminal history (low, x, Y), $x - H$ for the terminal history $(high, x, Y)$, $-L$ for the terminal history (low, x, N), and $-H$ for the terminal history $(high, x, N)$.

189.1 Stackelberg's duopoly game with quadratic costs

From Exercise 59.1, the best response function of firm 2 is the function b_2 defined by

$$b_2(q_1) = \begin{cases} \frac{1}{4}(\alpha - q_1) & \text{if } q_1 \le \alpha \\ 0 & \text{if } q_1 > \alpha. \end{cases}$$

Firm 1's subgame perfect equilibrium strategy is the value of q_1 that maximizes $q_1(\alpha - q_1 - b_2(q_1)) - q_1^2$, or $q_1(\alpha - q_1 - \frac{1}{4}(\alpha - q_1)) - q_1^2$, or $\frac{1}{4}q_1(3\alpha - 7q_1)$. The maximizer is $q_1 = \frac{3}{14}\alpha$.

We conclude that the game has a unique subgame perfect equilibrium, in which firm 1's strategy is the output $\frac{3}{14}\alpha$ and firm 2's strategy is its best response function b_2.

The outcome of the subgame perfect equilibrium is that firm 1 produces $q_1^* = \frac{3}{14}\alpha$ units of output and firm 2 produces $q_2^* = b_2(\frac{3}{14}\alpha) = \frac{11}{56}\alpha$ units. In a Nash equilibrium of Cournot's (simultaneous-move) game each firm produces $\frac{1}{5}\alpha$ (see Exercise 59.1). Thus firm 1 produces more in the subgame perfect equilibrium of the sequential game than it does in the Nash equilibrium of Cournot's game, and firm 2 produces less.

196.4 Sequential positioning by three political candidates

The following extensive game models the situation.

Players The candidates.

Terminal histories The set of all sequences $(x_1, \ldots, x_n)$, where x_i is either *Out* or a position of candidate i (a number) for $i = 1, \ldots, n$.

Player function $P(\varnothing) = 1$, $P(x_1) = 2$ for all x_1, $P(x_1, x_2) = 3$ for all (x_1, x_2), ..., $P(x_1, \ldots, x_{n-1}) = n$ for all $(x_1, \ldots, x_{n-1})$.

Preferences Each candidate's preferences are represented by a payoff function that assigns n to every terminal history in which she wins, k to every terminal history in which she ties for first place with $n - k$ other candidates, for $1 \le k \le n - 1$, 0 to every terminal history in which she stays out, and -1 to every terminal history in which she loses, where positions attract votes as in Hotelling's model of electoral competition (Section 3.3).

When there are two candidates the analysis of the subgame perfect equilibria is similar to that in the previous exercise. In every subgame perfect equilibrium candidate 1's strategy is m; candidate 2's strategy chooses m after the history m, some position between x_1 and $2m - x_1$ after the history x_1 for any position x_1, and any position after the history *Out*.

Now consider the case of three candidates when the voters' favorite positions are distributed uniformly from 0 to 1. I claim that every subgame perfect equilibrium results in the first candidate's entering at $\frac{1}{2}$, the second candidate's staying out, and the third candidate's entering at $\frac{1}{2}$.

To show this, first consider the best response of candidate 3 to each possible pair of actions of candidates 1 and 2. Figure 39.1 illustrates these optimal actions in every case that candidate 1 enters. (If candidate 1 does not enter then the subgame is exactly the two-candidate game.)

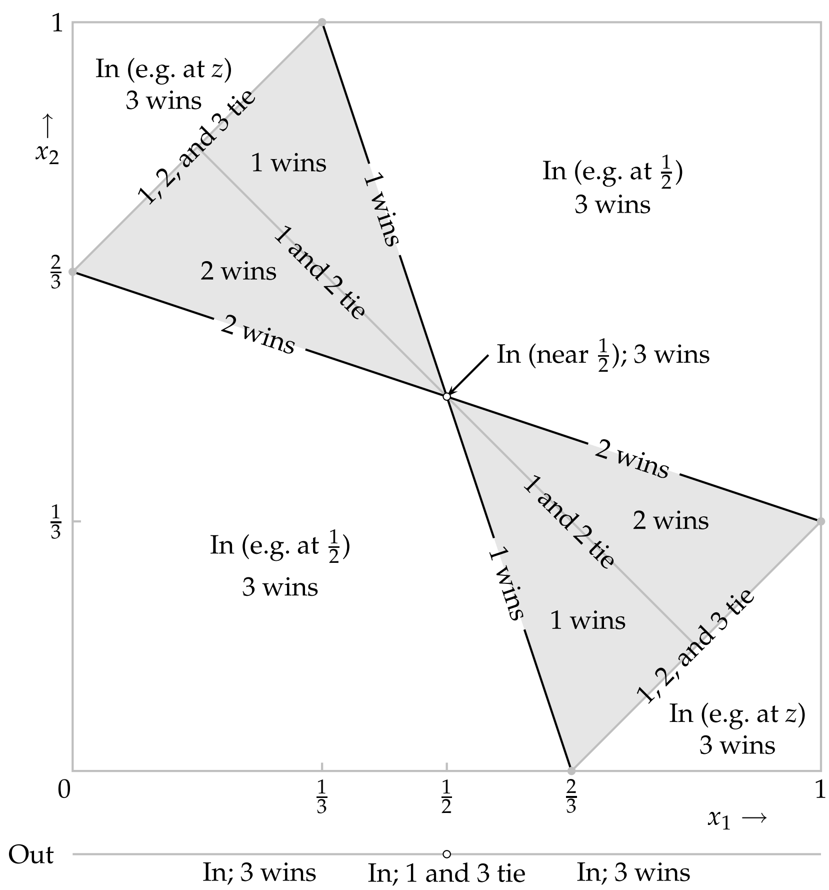


Figure 39.1 The outcome of a best response of candidate 3 to each pair of actions by candidates 1 and 2. The best response for any point in the gray shaded area (including the black boundaries of this area, but excluding the other boundaries) is *Out*. The outcome at each of the four small disks at the outer corners of the shaded area is that all three candidates tie. The value of z is $1 - \frac{1}{2}(x_1 + x_2)$.

Now consider the optimal action of candidate 2, given x_1 and the outcome of candidate 3's best response, as given in Figure 39.1. In the figure, take a value of x_1 and look at the outcomes as x_2 varies; find the value of x_2 that induces the best outcome for candidate 2. For example, for $x_1 = 0$ the only value of x_2 for which candidate 2 does not lose is $\frac{2}{3}$, at which point she ties with the other two candidates. Thus when candidate 1's strategy is $x_1 = 0$, candidate 2's best action, given candidate 3's best response, is $x_2 = \frac{2}{3}$, which leads to a three-way tie. We find that the outcome of the optimal value of x_2, for each value of x_1, is given as follows.

$$\begin{cases} 1, 2, \text{and } 3 \text{ tie } (x_2 = \frac{2}{3}) & \text{if } x_1 = 0 \\ 2 \text{ wins} & \text{if } 0 < x_1 < \frac{1}{2} \\ 1 \text{ and } 3 \text{ tie } (2 \text{ stays out}) & \text{if } x_1 = \frac{1}{2} \\ 2 \text{ wins} & \text{if } \frac{1}{2} < x_1 < 1 \\ 1, 2, \text{and } 3 \text{ tie } (x_2 = \frac{1}{3}) & \text{if } x_1 = 1. \end{cases}$$

Finally, consider candidate 1's best strategy, given the responses of candidates 2 and 3. If she stays out then candidates 2 and 3 enter at m and tie. If she enters then the best position at which to do so is $x_1 = \frac{1}{2}$, where she ties with candidate 3. (For every other position she either loses or ties with both of the other candidates.)

We conclude that in every subgame perfect equilibrium the outcome is that candidate 1 enters at $\frac{1}{2}$, candidate 2 stays out, and candidate 3 enters at $\frac{1}{2}$. (There are many subgame perfect equilibria, because after many histories candidate 3's optimal action is not unique.)

(The case in which there are many potential candidates, is discussed on the page `http://www.economics.utoronto.ca/osborne/research/CONJECT.HTM`.)

198.1 The race $G_1(2,2)$

The consequences of player 1's actions at the start of the game are as follows.

Take two steps: Player 1 wins.

Take one step: Go to the game $G_2(1,2)$, in which player 2 initially takes two steps and wins.

Do not move: If player 2 does not move, the game ends. If she takes one step we go to the game $G_1(2,1)$, in which player 1 takes two steps and wins. If she takes two steps, she wins. Thus in a subgame perfect equilibrium player 2 takes two steps, and wins.

We conclude that in a subgame perfect equilibrium of $G_1(2,2)$ player 1 initially takes two steps, and wins.

203.1 A race with a liquidity constraint

In the absence of the constraint, player 1 initially takes one step. Suppose she does so in the game with the constraint. Consider player 2's options after player 1's move.

Player 2 takes two steps: Because of the liquidity constraint, player 1 can take at most one step. If she takes one step, player 2's optimal action is to take one step, and win. Thus player 1's best action is not to move; player 2's payoff exceeds 1 (her steps cost 5, and the prize is worth more than 6).

Player 2 moves one step: Again because of the liquidity constraint, player 1 can take at most one step. If she takes one step, player 2 can take two steps and win, obtaining a payoff of more than 1 (as in the previous case).

Player 2 does not move: Player 1, as before, can take one step on each turn, and win; player 2's payoff is 0.

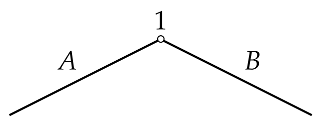
We conclude that after player 1 moves one step, player 2 should take either one or two steps, and ultimately win; player 1's payoff is -1. A better option for player 1 is not to move, in which case player 2 can move one step at a time, and win; player 1's payoff is zero.

Thus the subgame perfect equilibrium outcome is that player 1 does not move, and player 2 takes one step at a time and wins.

7 Extensive Games with Perfect Information: Extensions and Discussion

210.2 Extensive game with simultaneous moves

The game is shown in Figure 43.1.



	C	D
C	4, 2	0, 0
D	0, 0	2, 4

	E	F
E	3, 1	0, 0
F	0, 0	1, 3

Figure 43.1 The game in Exercise 210.2.

The subgame following player 1's choice of A has two Nash equilibria, (C, C) and (D, D); the subgame following player 1's choice of B also has two Nash equilibria, (E, E) and (F, F). If the equilibrium reached after player 1 chooses A is (C, C), then regardless of the equilibrium reached after she chooses (E, E), she chooses A at the beginning of the game. If the equilibrium reached after player 1 chooses A is (D, D) and the equilibrium reached after she chooses B is (F, F), she chooses A at the beginning of the game. If the equilibrium reached after player 1 chooses A is (D, D) and the equilibrium reached after she chooses B is (E, E), she chooses B at the beginning of the game.

Thus the game has four subgame perfect equilibria: (ACE, CE), (ACF, CF), (ADF, DF), and (BDE, DE) (where the first component of player 1's strategy is her choice at the start of the game, the second component is her action after she chooses A, and the third component is her action after she chooses B, and the first component of player 2's strategy is her action after player 1 chooses A at the start of the game and the second component is her action after player 1 chooses B at the start of the game).

In the first two equilibria the outcome is that player 1 chooses A and then both players choose C, in the third equilibrium the outcome is that player 1 chooses A and then both players choose D, and in the last equilibrium the outcome is that player 1 chooses B and then both players choose E.

217.1 Electoral competition with strategic voters

I first argue that in any equilibrium each candidate that enters is in the set of winners. If some candidate that enters is not a winner, she can increase her payoff by deviating to *Out*.

Now consider the voting subgame in which there are more than two candidates and not all candidates' positions are the same. Suppose that the citizens' votes are equally divided among the candidates. I argue that this list of citizens' strategies is not a Nash equilibrium of the voting subgame.

For either the citizen whose favorite position is 0 or the citizen whose favorite position is 1 (or both), at least two candidates' positions are better than the position of the candidate furthest from the citizen's favorite position. Denote a citizen for whom this condition holds by i. (The claim that citizen i exists is immediate if the candidates occupy at least three distinct positions, or they occupy two distinct positions and at least two candidates occupy each position. If the candidates occupy only two positions and one position is occupied by a single candidate, then take the citizen whose favorite position is 0 if the lone candidate's position exceeds the other candidates' position; otherwise take the citizen whose favorite position is 1.)

Now, given that each candidate obtains the same number of votes, if citizen i switches her vote to one of the candidates whose position is better for her than that of the candidate whose position is furthest from her favorite position, then this candidate wins outright. (If citizen i originally votes for one of these superior candidates, she can switch her vote to the other superior candidate; if she originally votes for neither of the superior candidates, she can switch her vote to either one of them.) Citizen i's payoff increases when she thus switches her vote, so that the list of citizens' strategies is not a Nash equilibrium of the voting subgame.

We conclude that in every Nash equilibrium of every voting subgame in which there are more than two candidates and not all candidates' positions are the same at least one candidate loses. Because no candidate loses in a subgame perfect equilibrium (by the first argument in the proof), in any subgame perfect equilibrium either only two candidates enter, or all candidates' positions are the same.

If only two candidates enter, then by the argument in the text for the case $n = 2$, each candidate's position is m (the median of the citizens' favorite positions).

Now suppose that more than two candidates enter, and their common position is not equal to m. If a candidate deviates to m then in the resulting voting subgame only two positions are occupied, so that for every citizen, any strategy that is not weakly dominated votes for a candidate at the position closest to her favorite position. Thus a candidate who deviates to m wins outright. We conclude that in any subgame perfect equilibrium in which more than two candidates enter, they all choose the position m.

220.1 Top cycle set

a. The top cycle set is the set $\{x, y, z\}$ of all three alternatives because x beats y beats z beats x.

b. The top cycle set is the set $\{w, x, y, z\}$ of all four alternatives. As in the previous case, x beats y beats z beats x; also y beats w.

224.1 Exit from a declining industry

Period t_1 is the largest value of t for which $P_t(k_1) \geq c$, or $60 - t \geq 10$. Thus $t_1 = 50$. Similarly, $t_2 = 70$.

If both firms are active in period t_1, then firm 2's profit in this period is $(100 - t_1 - c - k_1 - k_2)k_2 = (-20)(20) = -400$. Its profit in any period t in which it is alone in the market is $(100 - t - c - k_2)k_2 = (70 - t)(20)$. Thus its profit from period $t_1 + 1$ through period t_2 is

$$(19 + 18 + \ldots + 1)(20) = 3800.$$

Hence firm 2's loss in period t_1 when both firms are active is (much) less than the sum of its profits in periods $t_1 + 1$ through t_2 when it alone is active.

227.1 Variant of ultimatum game with equity-conscious players

The game is defined as follows.

Players The two people.

Terminal histories The set of sequences (x, β_2, Z), where x is a number with $0 \leq x \leq c$ (the amount of money that person 1 offers to person 2), β_2 is 0 or 1 (the value of β_2 selected by chance), and Z is either Y ("yes, I accept") or N ("no, I reject").

Player function $P(\varnothing) = 1$, $P(x) = c$ for all x, and $P(x, \beta_2) = 2$ for all x and all β_2.

Chance probabilities For every history x, chance chooses 0 with probability p and 1 with probability $1 - p$.

Preferences Each person's preferences are represented by the expected value of a payoff equal to the amount of money she receives. For any terminal history (x, β_2, Y) person 1 receives $c - x$ and person 2 receives x; for any terminal history (x, β_2, N) each person receives 0.

Given the result from Exercise 183.4 given in the question, if person 1's offer x satisfies $0 < x < \frac{1}{3}$ then the offer is rejected with probability $1 - p$, so that person 1's expected payoff is $p(1 - x)$, while if $x > \frac{1}{3}$ the offer is certainly accepted, independent of the type of person 2. Thus person 1's optimal offer is

$$\begin{cases} \frac{1}{3} & \text{if } p < \frac{2}{3} \\ 0 & \text{if } p > \frac{2}{3}; \end{cases}$$

if $p = \frac{2}{3}$ then both offers are optimal.

If $p > \frac{2}{3}$ we see that in a subgame perfect equilibrium person 1's offers are rejected by every person 2 with whom she is matched for whom $\beta_2 = 1$ (that is, with probability $1 - p$).

230.1 Nash equilibria when players may make mistakes

The players' best response functions are indicated in Figure 46.1. We see that the game has two Nash equilibria, (A, A, A) and (B, A, A).

	A	B
A	$1^*, 1^*, 1^*$	$0, 0, 1^*$
B	$1^*, 1^*, 1^*$	$1^*, 0, 1^*$

A

	A	B
A	$0, 1^*, 0$	$1^*, 0, 0$
B	$1^*, 1^*, 0$	$0, 0, 0$

B

Figure 46.1 The player's best response functions in the game in Exercise 230.1.

The action A is not weakly dominated for any player. For player 1, A is better than B if players 2 and 3 both choose B; for players 2 and 3, A is better than B for all actions of the other players.

If players 2 and 3 choose A in the modified game, player 1's expected payoffs to A and B are

$$A\colon\ (1-p_2)(1-p_3) + p_1 p_2 (1-p_3) + p_1(1-p_2)p_3 + (1-p_1)p_2 p_3$$

$$B\colon\ (1-p_2)(1-p_3) + (1-p_1)p_2(1-p_3) + (1-p_1)(1-p_2)p_3 + p_1 p_2 p_3.$$

The difference between the expected payoff to B and the expected payoff to A is

$$(1-2p_1)[p_2 + p_3 - 3p_2 p_3].$$

If $0 < p_i < \frac{1}{2}$ for $i = 1, 2, 3$, this difference is positive, so that (A, A, A) is not a Nash equilibrium of the modified game.

233.1 Nash equilibria of the chain-store game

Any terminal history in which the event in each period is either *Out* or (In, A) is the outcome of a Nash equilibrium. In any period in which challenger chooses *Out*, the strategy of the chain-store specifies that it choose F in the event that the challenger chooses *In*.

8 Coalitional Games and the Core

245.1 Three-player majority game

Let (x_1, x_2, x_3) be an action of the grand coalition. Every coalition consisting of two players can obtain one unit of output, so for (x_1, x_2, x_3) to be in the core we need

$$\begin{aligned} x_1 + x_2 &\geq 1 \\ x_1 + x_3 &\geq 1 \\ x_2 + x_3 &\geq 1 \\ x_1 + x_2 + x_3 &= 1. \end{aligned}$$

Adding the first three conditions we conclude that

$$2x_1 + 2x_2 + 2x_3 \geq 3,$$

or $x_1 + x_2 + x_3 \geq \frac{3}{2}$, contradicting the last condition. Thus no action of the grand coalition satisfies all the conditions, so that the core of the game is empty.

248.1 Core of landowner–worker game

Let a_N be an action of the grand coalition in which the output received by each worker is at most $f(n) - f(n-1)$. No coalition consisting solely of workers can obtain any output, so no such coalition can improve upon a_N. Let S be a coalition of the landowner and $k-1$ workers. The total output received by the members of S in a_N is at least

$$f(n) - (n-k)(f(n) - f(n-1))$$

(because the total output is $f(n)$, and every *other* worker receives at most $f(n) - f(n-1)$). Now, the output that S can obtain is $f(k)$, so for S to improve upon a_N we need

$$f(k) > f(n) - (n-k)(f(n) - f(n-1)),$$

which contradicts the inequality given in the exercise.

249.1 Unionized workers in landowner–worker game

The following game models the situation.

Players The landowner and the workers.

Actions The set of actions of the grand coalition is the set of all allocations of the output $f(n)$. Every other coalition has a single action, which yields the output 0.

Preferences Each player's preferences are represented by the amount of output she obtains.

The core of this game consists of every allocation of the output $f(n)$ among the players. The grand coalition cannot improve upon any allocation x because for every other allocation x' there is at least one player whose payoff is lower in x' than it is in x. No other coalition can improve upon any allocation because no other coalition can obtain any output.

249.2 Landowner–worker game with increasing marginal products

We need to show that no coalition can improve upon the action a_N of the grand coalition in which every player receives the output $f(n)/n$. No coalition of workers can obtain any output, so we need to consider only coalitions containing the landowner. Consider a coalition consisting of the landowner and k workers, which can obtain $f(k+1)$ units of output by itself. Under a_N this coalition obtains the output $(k+1)f(n)/n$, and we have $f(k+1)/(k+1) < f(n)/n$ because $k < n$. Thus no coalition can improve upon a_N.

254.1 Range of prices in horse market

The equality of the number of owners who sell their horses and the number of nonowners who buy horses implies that the common trading price p^*

- is not less than σ_{k^*}, otherwise at most $k^* - 1$ owners' valuations would be less than p^* and at least k^* nonowners' valuations would be greater than p^*, so that the number of buyers would exceed the number of sellers

- is not less than β_{k^*+1}, otherwise at most k^* owners' valuations would be less than p^* and at least $k^* + 1$ nonowners' valuations would be greater than p^*, so that the number of buyers would exceed the number of sellers

- is not greater than β_{k^*}, otherwise at least k^* owners' valuations would be less than p^* and at most $k^* - 1$ nonowners' valuations would be greater than p^*, so that the number of sellers would exceed the number of buyers

- is not greater than σ_{k^*+1}, otherwise at least $k^* + 1$ owners' valuations would be less than p^* and at most k^* nonowners' valuations would be greater than p^*, so that the number of sellers would exceed the number of buyers.

That is, $p^* \geq \max\{\sigma_{k^*}, \beta_{k^*+1}\}$ and $p^* \leq \min\{\beta_{k^*}, \sigma_{k^*+1}\}$.

258.1 House assignment with identical preferences

Because the players rank the houses in the same way, we can refer to the "best house", the "second best house", and so on. In any assignment in the core, the player who owns the best house is assigned this house (because she has the option of keeping it). Among the remaining players, the one who owns the second best house must be assigned this house (again, because she has the option of keeping it). Continuing to argue in the same way, we see that there is a single assignment in the core, in which every player is assigned the house she owns initially.

261.1 Median voter theorem

Denote the median favorite position by m. If $x < m$ then every player whose favorite position is m or greater—a majority of the players—prefers m to x. Similarly, if $x > m$ then every player whose favorite position is m or less—a majority of the players—prefers m to x.

267.2 Empty core in roommate problem

Notice that ℓ is at the bottom of each of the other players' preferences. Suppose that she is matched with i. Then j and k are matched, and $\{i,k\}$ can improve upon the matching. Similarly, if ℓ is matched with j then $\{i,j\}$ can improve upon the matching, and if ℓ is matched with k then $\{j,k\}$ can improve upon the matching. Thus the core is empty (ℓ has to be matched with *someone*!).

9 Bayesian Games

276.1 Equilibria of a variant of *BoS* with imperfect information

If player 1 chooses S then type 1 of player 2 chooses S and type 2 chooses B. But if the two types of player 2 make these choices then player 1 is better off choosing B (which yields her an expected payoff of 1) than choosing S (which yields her an expected payoff of $\frac{1}{2}$). Thus there is no Nash equilibrium in which player 1 chooses S.

Now consider the mixed strategy Nash equilibria. If both types of player 2 use a pure strategy then player 1's two actions yield her different payoffs. Thus there is no equilibrium in which both types of player 2 use pure strategies and player 1 randomizes.

Now consider an equilibrium in which type 1 of player 2 randomizes. Denote by p the probability that player 1's mixed strategy assigns to B. In order for type 1 of player 2 to obtain the same expected payoff to B and S we need $p = \frac{2}{3}$. For this value of p the best action of type 2 of player 2 is S. Denote by q the probability that type 1 of player 2 assigns to B. Given these strategies for the two types of player 2, player 1's expected payoff if she chooses B is

$$\tfrac{1}{2} \cdot 2q = q$$

and her expected payoff if she chooses S is

$$\tfrac{1}{2} \cdot (1 - q) + \tfrac{1}{2} \cdot 1 = 1 - \tfrac{1}{2}q.$$

These expected payoffs are equal if and only if $q = \frac{2}{3}$. Thus the game has a mixed strategy equilibrium in which the mixed strategy of player 1 is $(\frac{2}{3}, \frac{1}{3})$, that of type 1 of player 2 is $(\frac{2}{3}, \frac{1}{3})$, and that of type 2 of player 2 is $(0, 1)$ (that is, type 2 of player 2 uses the pure strategy that assigns probability 1 to S).

Similarly the game has a mixed strategy equilibrium in which the strategy of player 1 is $(\frac{1}{3}, \frac{2}{3})$, that of type 1 of player 2 is $(0, 1)$, and that of type 2 of player 2 is $(\frac{2}{3}, \frac{1}{3})$.

For no mixed strategy of player 1 are both types of player 2 indifferent between their two actions, so there is no equilibrium in which both types randomize.

277.1 Expected payoffs in a variant of *BoS* with imperfect information

The expected payoffs are given in Figure 52.1.

	(B,B)	(B,S)	(S,B)	(S,S)
B	0	1	1	2
S	1	$\frac{1}{2}$	$\frac{1}{2}$	0

Type n_1 of player 1

	(B,B)	(B,S)	(S,B)	(S,S)
B	1	$\frac{2}{3}$	$\frac{1}{3}$	0
S	0	$\frac{2}{3}$	$\frac{4}{3}$	2

Type y_2 of player 2

	(B,B)	(B,S)	(S,B)	(S,S)
B	0	$\frac{1}{3}$	$\frac{2}{3}$	1
S	2	$\frac{4}{3}$	$\frac{2}{3}$	0

Type n_2 of player 2

Figure 52.1 The expected payoffs of type n_1 of player 1 and types y_2 and n_2 of player 2 in Example 276.2.

282.2 An exchange game

The following Bayesian game models the situation.

Players The two individuals.

States The set of all pairs (s_1, s_2), where s_i is the number on player i's ticket (an integer from 1 to m).

Actions The set of actions of each player is $\{Exchange, Don't\ exchange\}$.

Signals The signal function of each player i is defined by $\tau_i(s_1, s_2) = s_i$ (each player observes her own ticket, but not that of the other player)

Beliefs Type s_i of player i assigns the probability $\Pr_j(s_j)$ to the state (s_1, s_2), where j is the other player and $\Pr_j(s_j)$ is the probability with which player j receives a ticket with the prize s_j on it.

Payoffs Player i's Bernoulli payoff function is given by $u_i((X, Y), \omega) = \omega_j$ if $X = Y = Exchange$ and $u_i((X, Y), \omega) = \omega_i$ otherwise.

Let M_i be the highest type of player i that chooses *Exchange*. If $M_i > 1$ then type 1 of player j optimally chooses *Exchange*: by exchanging her ticket, she cannot obtain a smaller prize, and may receive a bigger one. Thus if $M_i \geq M_j$ and $M_i > 1$, type M_i of player i optimally chooses *Don't exchange*, because the expected value of the prizes of the types of player j that choose *Exchange* is less than M_i. Thus in any possible Nash equilibrium $M_i = M_j = 1$: the only prizes that may be exchanged are the smallest.

287.1 Cournot's duopoly game with imperfect information

We have

$$b_1(q_L, q_H) = \begin{cases} \frac{1}{2}(\alpha - c - (\theta q_L + (1-\theta)q_H)) & \text{if } \theta q_L + (1-\theta)q_H \leq \alpha - c \\ 0 & \text{otherwise.} \end{cases}$$

The best response function of each type of player 2 is similar:

$$b_I(q_1) = \begin{cases} \frac{1}{2}(\alpha - c_I - q_1) & \text{if } q_1 \leq \alpha - c_I \\ 0 & \text{otherwise} \end{cases}$$

for $I = L, H$.

The three equations that define a Nash equilibrium are

$$q_1^* = b_1(q_L^*, q_H^*),\ q_L^* = b_L(q_1^*), \text{ and } q_H^* = b_H(q_1^*).$$

Solving these equations under the assumption that they have a solution in which all three outputs are positive, we obtain

$$\begin{aligned} q_1^* &= \tfrac{1}{3}(\alpha - 2c + \theta c_L + (1-\theta)c_H) \\ q_L^* &= \tfrac{1}{3}(\alpha - 2c_L + c) - \tfrac{1}{6}(1-\theta)(c_H - c_L) \\ q_H^* &= \tfrac{1}{3}(\alpha - 2c_H + c) + \tfrac{1}{6}\theta(c_H - c_L) \end{aligned}$$

If both firms know that the unit costs of the two firms are c_1 and c_2 then in a Nash equilibrium the output of firm i is $\frac{1}{3}(\alpha - 2c_i + c_j)$ (see Exercise 58.1). In the case of imperfect information considered here, firm 2's output is less than $\frac{1}{3}(\alpha - 2c_L + c)$ if its cost is c_L and is greater than $\frac{1}{3}(\alpha - 2c_H + c)$ if its cost is c_H. Intuitively, the reason is as follows. If firm 1 knew that firm 2's cost were high then it would produce a relatively large output; if it knew this cost were low then it would produce a relatively small output. Given that it does not know whether the cost is high or low it produces a moderate output, less than it would if it knew firm 2's cost were high. Thus if firm 2's cost is in fact high, firm 2 benefits from firm 1's lack of knowledge and optimally produces more than it would if firm 1 knew its cost.

288.1 Cournot's duopoly game with imperfect information

The best response $b_0(q_L, q_H)$ of type 0 of firm 1 is the solution of

$$\max_{q_0} [\theta(P(q_0 + q_L) - c)q_0 + (1-\theta)(P(q_0 + q_H) - c)q_0].$$

The best response $b_\ell(q_L, q_H)$ of type ℓ of firm 1 is the solution of

$$\max_{q_\ell} (P(q_\ell + q_L) - c)q_\ell$$

and the best response $b_h(q_L, q_H)$ of type h of firm 1 is the solution of

$$\max_{q_h}(P(q_h + q_H) - c)q_h.$$

The best response $b_L(q_0, q_\ell, q_h)$ of type L of firm 2 is the solution of

$$\max_{q_L}[(1-\pi)(P(q_0 + q_L) - c_L)q_L + \pi(P(q_\ell + q_L) - c_L)q_L]$$

and the best response $b_H(q_0, q_\ell, q_h)$ of type H of firm 2 is the solution of

$$\max_{q_H}[(1-\pi)(P(q_0 + q_H) - c_H)q_H + \pi(P(q_h + q_H) - c_H)q_H].$$

A Nash equilibrium is a profile $(q_0^*, q_\ell^*, q_h^*, q_L^*, q_H^*)$ for which q_0^*, q_ℓ^*, and q_h^* are best responses to q_L^* and q_H^*, and q_L^* and q_H^* are best responses to q_0^*, q_ℓ^*, and q_h^*. When $P(Q) = \alpha - Q$ for $Q \leq \alpha$ and $P(Q) = 0$ for $Q > \alpha$ we find, after some exciting algebra, that

$$\begin{aligned}
q_0^* &= \frac{1}{3}\left(\alpha - 2c + c_H - \theta\left(c_H - c_L\right)\right) \\
q_\ell^* &= \frac{1}{3}\left(\alpha - 2c + c_L + \frac{(1-\theta)(1-\pi)(c_H - c_L)}{4-\pi}\right) \\
q_H^* &= \frac{1}{3}\left(\alpha - 2c + c_H - \frac{\theta(1-\pi)(c_H - c_L)}{4-\pi}\right) \\
q_L^* &= \frac{1}{3}\left(\alpha - 2c_L + c - \frac{2(1-\theta)(1-\pi)(c_H - c_L)}{4-\pi}\right) \\
q_H^* &= \frac{1}{3}\left(\alpha - 2c_H + c + \frac{2\theta(1-\pi)(c_H - c_L)}{4-\pi}\right).
\end{aligned}$$

When $\pi = 0$ we have

$$\begin{aligned}
q_0^* &= \frac{1}{3}\left(\alpha - 2c + c_H - \theta\left(c_H - c_L\right)\right) \\
q_\ell^* &= \frac{1}{3}\left(\alpha - 2c + c_L + \frac{(1-\theta)(c_H - c_L)}{4}\right) \\
q_H^* &= \frac{1}{3}\left(\alpha - 2c + c_H - \frac{\theta(c_H - c_L)}{4}\right) \\
q_L^* &= \frac{1}{3}\left(\alpha - 2c_L + c - \frac{(1-\theta)(c_H - c_L)}{2}\right) \\
q_H^* &= \frac{1}{3}\left(\alpha - 2c_H + c + \frac{\theta(c_H - c_L)}{2}\right),
\end{aligned}$$

so that q_0^* is equal to the equilibrium output of firm 1 in Exercise 287.1, and q_L^* and q_H^* are the same as the equilibrium outputs of the two types of firm 2 in that exercise.

When $\pi = 1$ we have

$$\begin{aligned}
q_0^* &= \frac{1}{3}\left(\alpha - 2c + c_H - \theta\left(c_H - c_L\right)\right) \\
q_\ell^* &= \frac{1}{3}\left(\alpha - 2c + c_L\right) \\
q_H^* &= \frac{1}{3}\left(\alpha - 2c + c_H\right) \\
q_L^* &= \frac{1}{3}\left(\alpha - 2c_L + c\right) \\
q_H^* &= \frac{1}{3}\left(\alpha - 2c_H + c\right),
\end{aligned}$$

so that q_ℓ^* and q_L^* are the same as the equilibrium outputs when there is perfect information and the costs are c and c_L (see Exercise 58.1), and q_h^* and q_H^* are the same as the equilibrium outputs when there is perfect information and the costs are c and c_H.

Now, for an arbitrary value of π we have

$$\begin{aligned}
q_L^* &= \frac{1}{3}\left(\alpha - 2c_L + c - \frac{2(1-\theta)(1-\pi)(c_H - c_L)}{4-\pi}\right) \\
q_H^* &= \frac{1}{3}\left(\alpha - 2c_H + c + \frac{2\theta(1-\pi)(c_H - c_L)}{4-\pi}\right).
\end{aligned}$$

To show that for $0 < \pi < 1$ the values of these variables lie between their values when $\pi = 0$ and when $\pi = 1$, we need to show that

$$0 \le \frac{2(1-\theta)(1-\pi)(c_H - c_L)}{4-\pi} \le \frac{(1-\theta)(c_L - c_H)}{2}$$

and

$$0 \le \frac{2\theta(1-\pi)(c_H - c_L)}{4-\pi} \le \frac{\theta(c_L - c_H)}{2}.$$

These inequalities follow from $c_H \ge c_L$, $\theta \ge 0$, and $0 \le \pi \le 1$.

290.1 Nash equilibria of game of contributing to a public good

Any type v_j of any player j with $v_j < c$ obtains a negative payoff if she contributes and 0 if she does not. Thus she optimally does not contribute.

Any type $v_i \ge c$ of player i obtains the payoff $v_i - c \ge 0$ if she contributes, and the payoff 0 if she does not, so she optimally contributes.

Any type $v_j \ge c$ of any player $j \ne i$ obtains the payoff $v_j - c$ if she contributes, and the payoff $(1 - F(c))v_j$ if she does not. (If she does not contribute, the probability that player i does so is $1 - F(c)$, the probability that player i's valuation is at least c.) Thus she optimally does not contribute if $(1 - F(c))v_j \ge v_j - c$, or $F(c) \le c/v_j$. This condition must hold for all types of every player $j \ne i$, so we need $F(c) \le c/\overline{v}$ for the strategy profile to be a Nash equilibrium.

294.1 Weak domination in second-price sealed-bid action

Fix player i, and choose a bid for every type of every other player. Player i, who does not know the other players' types, is uncertain of the highest bid of the other players. Denote by $\overline{b}$ this highest bid. Consider a bid b_i of type v_i of player i for which $b_i < v_i$. The dependence of the payoff of type v_i of player i on $\overline{b}$ is shown in Figure 56.1.

		Highest of other players' bids			
		$\overline{b} < b_i$	$b_i = \overline{b}$ (m-way tie)	$b_i < \overline{b} < v_i$	$\overline{b} \geq v_i$
i's bid	$b_i < v_i$	$v_i - \overline{b}$	$(v_i - \overline{b})/m$	0	0
	v_i	$v_i - \overline{b}$	$v_i - \overline{b}$	$v_i - \overline{b}$	0

Figure 56.1 Player i's payoffs to her bids $b_i < v_i$ and v_i in a second-price sealed-bid auction as a function of the highest of the other player's bids, denoted $\overline{b}$.

Player i's expected payoffs to the bids b_i and v_i are weighted averages of the payoffs in the columns; each value of $\overline{b}$ gets the same weight when calculating the expected payoff to b_i as it does when calculating the expected payoff to v_i. The payoffs in the two rows are the same except when $b_i \leq \overline{b} < v_i$, in which case v_i yields a payoff higher than does b_i. Thus the expected payoff to v_i is at least as high as the expected payoff to b_i, and is greater than the expected payoff to b_i unless the other players' bids lead this range of values of $\overline{b}$ to get probability 0.

Now consider a bid b_i of type v_i of player i for which $b_i > v_i$. The dependence of the payoff of type v_i of player i on $\overline{b}$ is shown in Figure 56.2.

		Highest of other players' bids			
		$\overline{b} \leq v_i$	$v_i < \overline{b} < b_i$	$b_i = \overline{b}$ (m-way tie)	$\overline{b} > b_i$
i's bid	v_i	$v_i - \overline{b}$	0	0	0
	$b_i > v_i$	$v_i - \overline{b}$	$v_i - \overline{b}$	$(v_i - \overline{b})/m$	0

Figure 56.2 Player i's payoffs to her bids v_i and $b_i > v_i$ in a second-price sealed-bid auction as a function of the highest of the other player's bids, denoted $\overline{b}$.

As before, player i's expected payoffs to the bids b_i and v_i are weighted averages of the payoffs in the columns; each value of $\overline{b}$ gets the same weight when calculating the expected payoff to v_i as it does when calculating the expected payoff to b_i. The payoffs in the two rows are the same except when $v_i < \overline{b} \leq b_i$, in which case v_i yields a payoff higher than does b_i. (Note that $v_i - \overline{b} < 0$ for $\overline{b}$ in this range.) Thus the expected payoff to v_i is at least as high as the expected payoff to b_i, and is greater than the expected payoff to b_i unless the other players' bids lead this range of values of $\overline{b}$ to get probability 0.

We conclude that for type v_i of player i, every bid $b_i \neq v_i$ is weakly dominated by the bid v_i.

299.1 Asymmetric Nash equilibria of second-price sealed-bid common value auctions

Suppose that each type t_2 of player 2 bids $(1 + 1/\lambda)t_2$ and that type t_1 of player 1 bids b_1. Then by the calculations in the text, with $\alpha = 1$ and $\gamma = 1/\lambda$,

- a bid of b_1 by player 1 wins with probability $b_1/(1 + 1/\lambda)$
- the expected value of player 2's bid, *given that it is less than* b_1, is $\frac{1}{2}b_1$
- the expected value of signals that yield a bid of less than b_1 is $\frac{1}{2}b_1/(1 + 1/\lambda)$ (because of the uniformity of the distribution of t_2).

Thus player 1's expected payoff if she bids b_1 is

$$(t_1 + \tfrac{1}{2}b_1/(1 + 1/\lambda) - \tfrac{1}{2}b_1) \cdot \frac{b_1}{1 + 1/\lambda},$$

or

$$\frac{\lambda}{2(1 + \lambda)^2} \cdot (2(1 + \lambda)t_1 - b_1)b_1.$$

This function is maximized at $b_1 = (1 + \lambda)t_1$. That is, if each type t_2 of player 2 bids $(1 + 1/\lambda)t_2$, any type t_1 of player 1 optimally bids $(1 + \lambda)t_1$. Symmetrically, if each type t_1 of player 1 bids $(1 + \lambda)t_1$, any type t_2 of player 2 optimally bids $(1 + 1/\lambda)t_2$. Hence the game has the claimed Nash equilibrium.

299.2 First-price sealed-bid auction with common valuations

Suppose that each type t_2 of player 2 bids $\frac{1}{2}(\alpha + \gamma)t_2$ and type t_1 of player 1 bids b_1. To determine the expected payoff of type t_1 of player 1, we need to find the probability with which she wins, and the expected value of player 2's signal if player 1 wins. (The price she pays is her bid, b_1.)

Probability of player 1's winning: Given that player 2's bidding function is $\frac{1}{2}(\alpha + \gamma)t_2$, player 1's bid of b_1 wins only if $b_1 \geq \frac{1}{2}(\alpha + \gamma)t_2$, or if $t_2 \leq 2b_1/(\alpha + \gamma)$. Now, t_2 is distributed uniformly from 0 to 1, so the probability that it is at most $2b_1/(\alpha + \gamma)$ is $2b_1/(\alpha + \gamma)$. Thus a bid of b_1 by player 1 wins with probability $2b_1/(\alpha + \gamma)$.

Expected value of player 2's signal if player 1 wins: Player 2's bid, given her signal t_2, is $\frac{1}{2}(\alpha + \gamma)t_2$, so that the expected value of signals that yield a bid of less than b_1 is $b_1/(\alpha + \gamma)$ (because of the uniformity of the distribution of t_2).

Thus player 1's expected payoff if she bids b_1 is

$$2(\alpha t_1 + \gamma b_1/(\alpha + \gamma) - b_1) \cdot \frac{b_1}{\alpha + \gamma},$$

or

$$\frac{2\alpha}{(\alpha + \gamma)^2}((\alpha + \gamma)t_1 - b_1)b_1.$$

This function is maximized at $b_1 = \frac{1}{2}(\alpha+\gamma)t_1$. That is, if each type t_2 of player 2 bids $\frac{1}{2}(\alpha+\gamma)t_2$, any type t_1 of player 1 optimally bids $\frac{1}{2}(\alpha+\gamma)t_1$. Hence, as claimed, the game has a Nash equilibrium in which each type t_i of player i bids $\frac{1}{2}(\alpha+\gamma)t_i$.

309.2 Properties of the bidding function in a first-price sealed-bid auction

We have

$$\begin{aligned}
\beta^{*\prime}(v) &= 1 - \frac{(F(v))^{n-1}(F(v))^{n-1} - (n-1)(F(v))^{n-2}F'(v)\int_{\underline{v}}^{v}(F(x))^{n-1}\,dx}{(F(v))^{2n-2}} \\
&= 1 - \frac{(F(v))^{n} - (n-1)F'(v)\int_{\underline{v}}^{v}(F(x))^{n-1}\,dx}{(F(v))^{n}} \\
&= \frac{(n-1)F'(v)\int_{\underline{v}}^{v}(F(x))^{n-1}\,dx}{(F(v))^{n}} \\
&> 0 \quad \text{if } v > \underline{v}
\end{aligned}$$

because $F'(v) > 0$ (F is increasing). (The first line uses the quotient rule for derivatives and the fact that the derivative of $\int^{v} f(x)dx$ with respect to v is $f(v)$ for any function f.)

If $v > \underline{v}$ then the integral in (309.1) is positive, so that $\beta^*(v) < v$. If $v = \underline{v}$ then both the numerator and denominator of the quotient in (309.1) are zero, so we may use L'Hôpital's rule to find the value of the quotient as $v \to \underline{v}$. Taking the derivatives of the numerator and denominator we obtain

$$\frac{(F(v))^{n-1}}{(n-1)(F(v))^{n-2}F'(v)} = \frac{F(v)}{(n-1)F'(v)},$$

the numerator of which is zero and the denominator of which is positive. Thus the quotient in (309.1) is zero, and hence $\beta^*(\underline{v}) = \underline{v}$.

309.3 Example of Nash equilibrium in a first-price auction

From (309.1) we have

$$\begin{aligned}
\beta^*(v) &= v - \frac{\int_0^v x^{n-1}\,dx}{v^{n-1}} \\
&= v - \frac{\int_0^v x^{n-1}\,dx}{v^{n-1}} \\
&= v - v/n = (n-1)v/n.
\end{aligned}$$

10 Extensive Games with Imperfect Information

316.1 Variant of card game

An extensive game that models the game is shown in Figure 59.1.

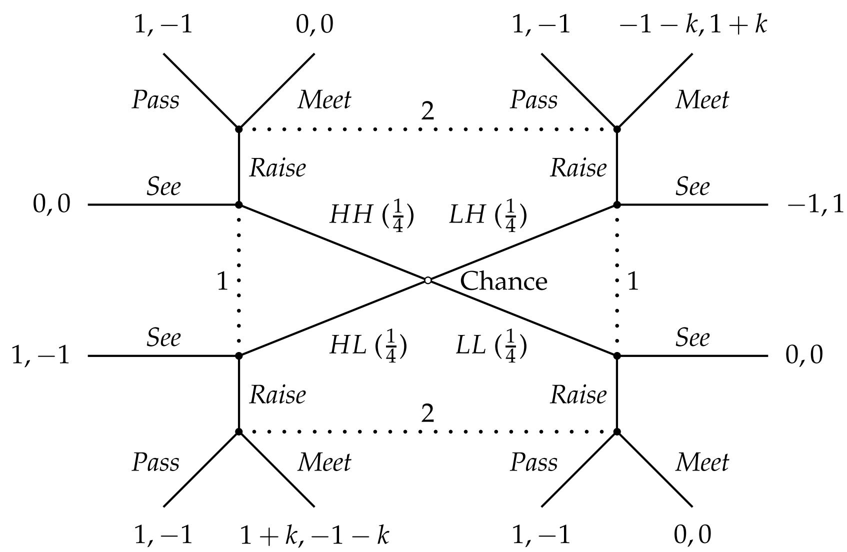


Figure 59.1 An extensive game that models the situation in Exercise 316.1.

318.2 Strategies in variants of card game and entry game

Card game: Each player has two information sets, and has two actions at each information set. Thus each player has four strategies: *SS*, *SR*, *RS*, and *RR* for player 1 (where *S* stands for *See* and *R* for *Raise*, the first letter of each strategy is player 1's action if her card is *High*, and the second letter if her action is her card is *Low*), and *PP*, *PM*, *MP*, and *MM* for player 2 (where *P* stands for *Pass* and *M* for *Meet*).

Entry game: The challenger has a single information set (the empty history) and has three actions after this history, so it has three strategies—*Ready*, *Unready*, and *Out*. The incumbent also has a single information set, at which two actions are available, so it has two strategies—*Acquiesce* and *Fight*.

331.2 Weak sequential equilibrium and Nash equilibrium in subgames

Consider the assessment in which the Challenger's strategy is (Out, R), the Incumbent's strategy is F, and the Incumbent's belief assigns probability 1 to the history (In, U) at her information set. Each player's strategy is sequentially rational. The Incumbent's belief satisfies the condition of weak consistency because her information set is not reached when the Challenger follows her strategy. Thus the assessment is a weak sequential equilibrium.

The players' actions in the subgame following the history *In* do not constitute a Nash equilibrium of the subgame because the Incumbent's action F is not optimal when the Challenger chooses R. (The Incumbent's action F *is* optimal given her belief that the history is (In, U), as it is in the weak sequential equilibrium. In a Nash equilibrium she acts as if she has a belief that coincides with the Challenger's action in the subgame.)

340.1 Pooling equilibria of game in which expenditure signals quality

We know that in the second period the high-quality firm charges the price H and the low-quality firm charges any nonnegative price, and the consumer buys the good from a high-quality firm, does not buy the good from a low-quality firm that charges a positive price, and may or may not buy from a low-quality firm that charges a price of 0.

Consider an assessment in which each type of firm chooses (p^*, E^*) in the first period, the consumer believes the firm is high-quality with probability π if it observes (p^*, E^*) and low quality if it observes any other (price, expenditure) pair, and buys the good if and only if it observes (p^*, E^*).

The payoff of a high-quality firm under this assessment is $p^* + H - E^* - 2c_H$, that of a low-quality firm is $p^* - E^*$, and that of the consumer is $\pi(H - p^*) + (1 - \pi)(-p^*) = \pi H - p^*$.

This assessment is consistent—the only first-period action of the firm observed in equilibrium is (p^*, E^*), and after observing this pair the consumer believes, correctly, that the firm is high-quality with probability π.

Under what conditions is the assessment sequentially rational?

Firm If the firm chooses a (price, expenditure) pair different from (p^*, E^*) then the consumer does not buy the good, and the firm's profit is 0. Thus for the assessment to be an equilibrium we need $p^* + H - E^* - 2c_H \geq 0$ (for the high-quality firm) and $p^* - E^* \geq 0$ (for the low-quality firm).

Consumer If the consumer does not buy the good after observing (p^*, E^*) then its payoff is 0, so for the assessment to be an equilibrium we need $\pi H - p^* \geq 0$.

In summary, the assessment is a weak sequential equilibrium if and only if

$$\max\{E^*, E^* - H + 2c_H\} \leq p^* \leq \pi H.$$

346.1 Comparing the receiver's expected payoff in two equilibria

The receiver's payoff as a function of the state t in each equilibrium is shown in Figure 61.1. The area above the black curve is smaller than the area above the gray curve: if you shift the black curve $\frac{1}{2}t_1$ to the left and move the section from 0 to $\frac{1}{2}t_1$ to the interval from $1 - \frac{1}{2}t_1$ to 1 then the area above the black curve is a subset of the area above the gray curve.

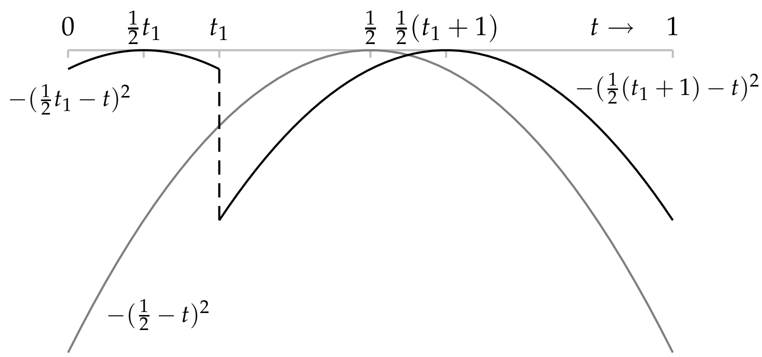


Figure 61.1 The gray curve gives the receiver's payoff in each state in the equilibrium in which no information is transferred. The black curve gives her payoff in each state in the two-report equilibrium.

350.1 Variant of model with piecewise linear payoff functions

The equilibria of the variant are exactly the same as the equilibria of the original model.

11 Strictly Competitive Games and Maxminimization

363.1 Maxminimizers in a bargaining game

If a player demands any amount x up to \$5 then her payoff is x regardless of the other player's action. If she demands \$6 then she may get as little as \$5 (if the other player demands \$5 or \$6). If she demands $x \geq \$7$ then she may get as little as $\$(11 - x)$ (if the other player demands $x - 1$). For each amount that a player demands, the smallest amount that you may get is given in Figure 63.1. We see that each player's maxminimizing pure strategies are \$5 and \$6 (for both of which the worst possible outcome is that the player receives \$5).

Amount demanded	0	1	2	3	4	5	6	7	8	9	10
Smallest amount obtained	0	1	2	3	4	5	5	4	3	2	1

Figure 63.1 The lowest payoffs that a player receives in the game in Exercise 38.2 for each of her possible actions, as the other player's action varies.

363.3 Finding a maxminimizer

The analog of Figure 364.1 in the text is Figure 63.2. From this figure we see that the maxminimizer for player 2 is the strategy that assigns probability $\frac{3}{5}$ to L. Player 2's maxminimized payoff is $-\frac{1}{5}$.

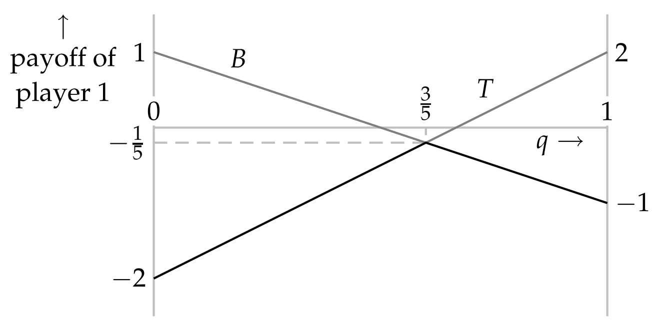


Figure 63.2 The expected payoff of player 2 in the game in Figure 363.1 for each of player 1's actions, as a function of the probability q that player 2 assigns to L.

366.2 Determining strictly competitiveness

Game in Exercise 365.1: Strictly competitive in pure strategies (because player 1's ranking of the four outcomes is the reverse of player 2's ranking). Not strictly competitive in mixed strategies (there exist no values of π and $\theta > 0$ such that $-u_1(a) = \pi + \theta u_2(a)$ for every outcome a; or, alternatively, player 1 is indifferent between (B, L) and the lottery that yields (T, L) with probability $\frac{1}{2}$ and (T, R) with probability $\frac{1}{2}$, whereas player 2 is not indifferent between these two outcomes).

Game in Figure 367.1: Strictly competitive both in pure and in mixed strategies. (Player 2's preferences are represented by the expected value of the Bernoulli payoff function $-u_1$ because $-u_1(a) = -\frac{1}{2} + \frac{1}{2}u_2(a)$ for every pure outcome a.)

370.2 Maxminimizing in BoS

Player 1's maxminimizer is $(\frac{1}{3}, \frac{2}{3})$ while player 2's is $(\frac{2}{3}, \frac{1}{3})$. Clearly neither pure equilibrium strategy of either player guarantees her equilibrium payoff. In the mixed strategy equilibrium, player 1's expected payoff is $\frac{2}{3}$. But if, for example, player 2 choose S instead of her equilibrium strategy, then player 1's expected payoff is $\frac{1}{3}$. Similarly for player 2.

372.2 Equilibrium in strictly competitive game

The claim is false. In the strictly competitive game in Figure 64.1 the action pair (T, L) is a Nash equilibrium, so that player 1's unique equilibrium payoff in the game is 0. But (B, R), which also yields player 1 a payoff of 0, is not a Nash equilibrium.

	L	R
T	$0, 0$	$1, -1$
B	$-1, 1$	$0, 0$

Figure 64.1 The game in Exercise 372.2.

372.4 O'Neill's game

a. Denote the probability with which player 1 chooses each of her actions 1, 2, and 3, by p, and the probability with which player 2 chooses each of these actions by q. Then all four of player 1's actions yield the same expected payoff if and only if $4q - 1 = 1 - 6q$, or $q = \frac{1}{5}$, and similarly all four of player 2's actions yield the same expected payoff if and only if $p = \frac{1}{5}$. Thus $((\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{2}{5}), (\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{2}{5}))$ is a Nash equilibrium of the game. The players' payoffs in this equilibrium are $(-\frac{1}{5}, \frac{1}{5})$.

b. Let (p_1, p_2, p_3, p_4) be an equilibrium strategy of player 1. In order that it guarantee her the payoff of $-\frac{1}{5}$, we need

$$\begin{aligned}
-p_1 + p_2 + p_3 - p_4 &\geq -\tfrac{1}{5} \\
p_1 - p_2 + p_3 - p_4 &\geq -\tfrac{1}{5} \\
p_1 + p_2 - p_3 - p_4 &\geq -\tfrac{1}{5} \\
-p_1 - p_2 - p_3 + p_4 &\geq -\tfrac{1}{5}.
\end{aligned}$$

Adding these four inequalities, we deduce that $p_4 \leq \frac{2}{5}$. Adding each pair of the first three inequalities, we deduce that $p_1 \leq \frac{1}{5}$, $p_2 \leq \frac{1}{5}$, and $p_3 \leq \frac{1}{5}$. We have $p_1 + p_2 + p_3 + p_4 = 1$, so we deduce that $(p_1, p_2, p_3, p_4) = (\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{2}{5})$. A similar analysis of the conditions for player 2's strategy to guarantee her the payoff of $\frac{1}{5}$ leads to the conclusion that $(q_1, q_2, q_3, q_4) = (\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{2}{5})$.

12 Rationalizability

379.2 Best responses to beliefs

Consider a two-player game in which player 1's payoffs are given in Figure 67.1. The action B of player 1 is a best response to the belief that assigns probability $\frac{1}{2}$ to both L and R, but is not a best response to any belief that assigns probability 1 to either action.

	L	R
T	3	0
M	0	3
B	2	2

Figure 67.1 The action B is a best response to a belief that assigns probability $\frac{1}{2}$ to L and to R, but is not a best response to any belief that assigns probability 1 to either L or R.

384.1 Mixed strategy equilibria of game in Figure 384.1

The game has no equilibrium in which player 2 assigns positive probability only to L and C, because if she does so then only M and B are possible best responses for player 1, but if player 1 assigns positive probability only to these actions then L is not optimal for player 2.

Similarly, the game has no equilibrium in which player 2 assigns positive probability only to C and R, because if she does so then only T and M are possible best responses for player 1, but if player 1 assigns positive probability only to these actions then R is not optimal for player 2.

Now assume that player 2 assigns positive probability only to L and R. There are no probabilities for L and R under which player 1 is indifferent between all three of her actions, so player 1 must assign positive probability to at most two actions. If these two actions are T and M then player 2 prefers L to R, while if the two actions are M and B then player 2 prefers R to L. The only possibility is thus that the two actions are T and B. In this case we need player 2 to assign probability $\frac{1}{2}$ to L and R (in order that player 1 be indifferent between T and B); but then M is better for player 1. Thus there is no equilibrium in which player 2 assigns positive probability only to L and R.

Finally, if player 2 assigns positive probability to all three of her actions then player 1's mixed strategy must be such that each of these three actions yields the same payoff. A calculation shows that there is no mixed strategy of player 1 with this property.

We conclude that the game has no mixed strategy equilibrium in which either player assigns positive probability to more than one action.

387.2 Finding rationalizable actions

I claim that the action R of player 2 is strictly dominated. Consider a mixed strategy of player 2 that assigns probability p to L and probability $1-p$ to C. Such a mixed strategy strictly dominates R if $p + 4(1-p) > 3$ and $8p + 2(1-p) > 3$, or if $\frac{1}{6} < p < \frac{1}{3}$. Now eliminate R from the game. In the reduced game, B is dominated by T. In the game obtained by eliminating B, L is dominated by C. Thus the only rationalizable action of player 1 is T and the only rationalizable action of player 2 is C.

387.5 Hotelling's model of electoral competition

The positions 0 and ℓ are strictly dominated by the position m:

- if her opponent chooses m, a player who chooses m ties whereas a player who chooses 0 loses
- if her opponent chooses 0 or ℓ, a player who chooses m wins whereas a player who chooses 0 or ℓ either loses or ties
- if her opponent chooses any other position, a player who chooses m wins whereas a player who chooses 0 or ℓ loses.

In the game obtained by eliminating the two positions 0 and ℓ, the positions 1 and $\ell - 1$ are similarly strictly dominated. Continuing in the same way, we are left with the position m.

388.2 Cournot's duopoly game

From Figure 58.1 we see that firm 1's payoff to any output greater than $\frac{1}{2}(\alpha - c)$ is less than its payoff to the output $\frac{1}{2}(\alpha - c)$ for any output q_2 of firm 2. Thus any output greater than $\frac{1}{2}(\alpha - c)$ is strictly dominated by the output $\frac{1}{2}(\alpha - c)$ for firm 1; the same argument applies to firm 2.

Now eliminate all outputs greater than $\frac{1}{2}(\alpha - c)$ for each firm. The maximizer of firm 1's payoff function for $q_2 = \frac{1}{2}(\alpha - c)$ is $\frac{1}{4}(\alpha - c)$, so from Figure 58.1 we see that firm 1's payoff to any output less than $\frac{1}{4}(\alpha - c)$ is less than its payoff to the output $\frac{1}{4}(\alpha - c)$ for any output $q_2 \leq \frac{1}{2}(\alpha - c)$ of firm 2. Thus any output less than $\frac{1}{4}(\alpha - c)$ is strictly dominated by the output $\frac{1}{4}(\alpha - c)$ for firm 1; the same argument applies to firm 2.

Now eliminate all outputs less than $\frac{1}{4}(\alpha - c)$ for each firm. Then by another similar argument, any output greater than $\frac{3}{8}(\alpha - c)$ is strictly dominated by $\frac{3}{8}(\alpha - c)$. Continuing in this way, we see from Figure 59.1 that in a finite number of rounds (given the finite number of possible outputs for each firm) we reach the Nash equilibrium output $\frac{1}{3}(\alpha - c)$.

391.1 Example of dominance-solvable game

The Nash equilibria of the game are (T, L), any $((0,0,1),(0,q,1-q))$ with $0 \leq q \leq 1$, and any $((0,p,1-p),(0,0,1))$ with $0 \leq p \leq 1$.

The game is dominance solvable, because T and L are the only weakly dominated actions, and when they are eliminated the only weakly dominated actions are M and C, leaving (B, R), with payoffs $(0, 0)$.

If T is eliminated, then L and C, no remaining action is weakly dominated; (M, R) and (B, R) both remain.

391.2 Dividing money

In the first round every action $a_i \leq 5$ of each player i is weakly dominated by 6. No other action is weakly dominated, because 100 is a strict best response to 0 and every other action $a_i \geq 6$ is a strict best response to $a_i + 1$. In the second round, 10 is weakly dominated by 6 for each player, and each other remaining action a_i of player i is a strict best response to $a_1 + 1$, so no other action is weakly dominated. Similarly, in the third round, 9 is weakly dominated by 6, and no other action is weakly dominated. In the fourth and fifth rounds 8 and 7 are eliminated, leaving the single action pair $(6, 6)$, with payoffs $(5, 5)$.

392.2 Strictly competitive extensive games with perfect information

Every finite extensive game with perfect information has a (pure strategy) subgame perfect equilibrium (Proposition 173.1). This equilibrium is a pure strategy Nash equilibrium of the strategic form of the game. Because the game has only two possible outcomes, one of the players prefers the Nash equilibrium outcome to the other possible outcome. By Proposition 368.1, this player's equilibrium strategy guarantees her equilibrium payoff, so this strategy weakly dominates all her nonequilibrium strategies. After all dominated strategies are eliminated, every remaining pair of strategies generates the same outcome.

13 Evolutionary Equilibrium

400.1 Evolutionary stability and weak domination

The ESS a^* does not necessarily weakly dominate every other action in the game. For example, in the game in Figure 395.1 of the text, X is an ESS but does not weakly dominate Y.

No action can weakly dominate an ESS. To see why, let a^* be an ESS and let b be another action. Because a^* is an ESS, (a^*, a^*) is a Nash equilibrium, so that $u(b, a^*) \leq u(a^*, a^*)$. Now, if $u(b, a^*) < u(a^*, a^*)$, certainly b does not weakly dominate a^*, so suppose that $u(b, a^*) = u(a^*, a^*)$. Then by the second condition for an ESS we have $u(b, b) < u(a^*, b)$. We conclude that b does not weakly dominate a^*.

405.1 *Hawk–Dove–Retaliator*

First suppose that $v \geq c$. In this case the game has two pure symmetric Nash equilibria, (A, A) and (R, R). However, A is not an ESS, because R is a best response to A and $u(R, R) > u(A, R)$. The action pair (R, R) is a strict equilibrium, so R is an ESS. Now consider the possibility that the game has a mixed strategy equilibrium (α, α). If α assigns positive probability to either P or R (or both) then R yields a payoff higher than does P, so only A and R may be assigned positive probability in a mixed strategy equilibrium. But if a strategy α assigns positive probability to A and R and probability 0 to P, then R yields a payoff higher than does A against an opponent who uses α. Thus the game has no symmetric mixed strategy equilibrium in this case.

Now suppose that $v < c$. Then the only symmetric pure strategy equilibrium is (R, R). This equilibrium is strict, so that R is an ESS. Now consider the possibility that the game has a mixed strategy equilibrium (α, α). If α assigns probability 0 to A then R yields a payoff higher than does P against an opponent who uses α; if α assigns probability 0 to P then R yields a payoff higher than does A against an opponent who uses α. Thus in any mixed strategy equilibrium (α, α), the strategy α must assign positive probability to both A and P. If α assigns probability 0 to R then we need $\alpha = (v/c, 1 - v/c)$ (the calculation is the same as for *Hawk–Dove*). Because R yields a lower payoff against this strategy than do A and P, and the strategy is an ESS in *Hawk–Dove*, it is an ESS in the present game. The remaining possibility is that the game has a mixed strategy equilibrium (α, α) in which α assigns positive probability to all three actions. If so, then the expected payoff to this strategy is less than $\frac{1}{2}v$, because the pure strategy P yields an expected payoff less than $\frac{1}{2}v$ against any such strategy. But then $U(R, R) = \frac{1}{2}v > U(\alpha, R)$, violating the second condition in the definition of an ESS.

In summary:

- If $v \geq c$ then R is the unique ESS of the game.
- If $v < c$ then both R and the mixed strategy that assigns probability v/c to A and $1 - v/c$ to P are ESSs.

405.3 Bargaining

The game is given in Figure 27.1.

The pure strategy of demanding 10 is not an ESS because 2 is a best response to 10 and $u(2, 2) > u(10, 2)$.

Now let α be the mixed strategy that assigns probability $\frac{2}{5}$ to 2 and $\frac{3}{5}$ to 8. Each player's payoff at the strategy pair (α, α) is $\frac{16}{5}$. Thus the only actions a that are best responses to α are 2 and 8, so that the only mixed strategies that are best responses to α assign positive probability only to the actions 2 and 8. Let β be the mixed strategy that assigns probability p to 2 and probability $1 - p$ to 8. We have

$$U(\beta, \beta) = 5p(2 - p)$$

and

$$U(\alpha, \beta) = 6p + \tfrac{4}{5}.$$

We find that $U(\alpha, \beta) - U(\beta, \beta) = 5(p - \frac{2}{5})^2$, which is positive if $p \neq \frac{2}{5}$. Hence α is an ESS.

Finally let α be the mixed strategy that assigns probability $\frac{4}{5}$ to 4 and $\frac{1}{5}$ to 6. Each player's payoff at the strategy pair (α, α) is $\frac{24}{5}$. Thus the only actions a that are best responses to α are 4 and 6, so that the only mixed strategies that are best responses assign positive probability only to the actions 4 and 6. Let β be the mixed strategy that assigns probability p to 4 and probability $1 - p$ to 6. We have

$$U(\beta, \beta) = 5p(2 - p)$$

and

$$U(\alpha^*, \beta) = 2p + \tfrac{16}{5}.$$

We find that $U(\alpha, \beta) - U(\beta, \beta) = 5(p - \frac{4}{5})^2$, which is positive if $p \neq \frac{4}{5}$. Hence α^* is an ESS.

408.1 Equilibria of C and of G

First suppose that (α_1, α_2) is a mixed strategy Nash equilibrium of C. Then for all mixed strategies β_1 of player 1 and all mixed strategies β_2 of player 2 we have

$$U_1(\alpha_1, \alpha_2) \geq U_1(\beta_1, \alpha_2) \text{ and } U_2(\alpha_1, \alpha_2) \geq U_2(\alpha_1, \beta_2).$$

Thus

$$\begin{aligned} u((\alpha_1,\alpha_2),(\alpha_1,\alpha_2)) &= \tfrac{1}{2}U_1(\alpha_1,\alpha_2) + \tfrac{1}{2}U_2(\alpha_1,\alpha_2) \\ &\geq \tfrac{1}{2}U_1(\beta_1,\alpha_2) + \tfrac{1}{2}U_2(\alpha_1,\beta_2) \\ &= u((\beta_1,\beta_2),(\alpha_1,\alpha_2)), \end{aligned}$$

so that $((\alpha_1,\alpha_2),(\alpha_1,\alpha_2))$ is a Nash equilibrium of G. If (α_1,α_2) is a strict Nash equilibrium of C then the inequalities are strict, and $((\alpha_1,\alpha_2),(\alpha_1,\alpha_2))$ is a strict Nash equilibrium of G.

Now assume that $((\alpha_1,\alpha_2),(\alpha_1,\alpha_2))$ is a Nash equilibrium of G. Then

$$u((\alpha_1,\alpha_2),(\alpha_1,\alpha_2)) \geq u((\beta_1,\beta_2),(\alpha_1,\alpha_2)),$$

or

$$\tfrac{1}{2}U_1(\alpha_1,\alpha_2) + \tfrac{1}{2}U_2(\alpha_1,\alpha_2) \geq \tfrac{1}{2}U_1(\beta_1,\alpha_2) + \tfrac{1}{2}U_2(\alpha_1,\beta_2),$$

for all conditional strategies (β_1,β_2). Taking $\beta_2 = \alpha_2$ we see that α_1 is a best response to α_2 in C, and taking $\beta_1 = \alpha_1$ we see that α_2 is a best response to α_1 in C. Thus (α_1,α_2) is a Nash equilibrium of G.

414.1 A coordination game between siblings

The game with payoff function v is shown in Figure 73.1. If $x < 2$ then (Y,Y) is a strict Nash equilibrium of the games, so Y is an evolutionarily stable action in the game between siblings. If $x > 2$ then the only Nash equilibrium of the game is (X,X), and this equilibrium is strict. Thus the range of values of x for which the only evolutionarily stable action is X is $x > 2$.

	X	Y
X	x, x	$\frac{1}{2}x, \frac{1}{2}$
Y	$\frac{1}{2}, \frac{1}{2}x$	$1, 1$

v

Figure 73.1 The game with payoff function v derived from the game in Exercise 414.1.

414.2 Assortative mating

Under assortative mating, all siblings take the same action, so the analysis is the same as that for asexual reproduction. (A difficulty with the assumption of assortative mating is that a rare mutant will have to go to great lengths to find a mate that is also a mutant.)

416.1 Darwin's theory of the sex ratio

A normal organism produces pn male offspring and $(1-p)n$ female offspring (ignoring the small probability that the partner of a normal organism is a mutant). Thus it has $pn \cdot ((1-p)/p)n + (1-p)n \cdot n = 2(1-p)n^2$ grandchildren.

A mutant has $\frac{1}{2}n$ male offspring and $\frac{1}{2}n$ female offspring, and hence $\frac{1}{2}n \cdot ((1-p)/p)n + \frac{1}{2}n \cdot n = \frac{1}{2}n^2/p$ grandchildren.

Thus the difference between the number of grandchildren produced by mutant and normal organisms is

$$\tfrac{1}{2}n^2/p - 2(1-p)n^2 = n^2\left(\frac{1}{2p}\right)(1-2p)^2,$$

which is positive if $p \neq \frac{1}{2}$. (The point is that if $p > \frac{1}{2}$ then the fraction of a mutant's offspring that are males is higher than the fraction of a normal organism's offspring that are males, and males each bear more offspring than females. Similarly, if $p < \frac{1}{2}$ then the fraction of a mutant's offspring that are females is higher than the fraction of a normal organism's offspring that are females, and females each bear more offspring than males.)

Thus any mutant with $p \neq \frac{1}{2}$ invades the population; only $p = \frac{1}{2}$ is evolutionarily stable.

14 Repeated Games: The *Prisoner's Dilemma*

423.1 Equivalence of payoff functions

Suppose that a person's preferences are represented by the discounted sum of payoffs with payoff function u and discount factor δ. Then if the two sequences of outcomes $(x^1, x^2, \ldots)$ and $(y^1, y^2, \ldots)$ are indifferent, we have

$$\sum_{t=0}^{\infty} \delta^{t-1} u(x^t) = \sum_{t=0}^{\infty} \delta^{t-1} u(y^t).$$

Now let $v(x) = \alpha + \beta u(x)$ for all x, with $\beta > 0$. Then

$$\sum_{t=0}^{\infty} \delta^{t-1} v(x^t) = \sum_{t=0}^{\infty} \delta^{t-1} [\alpha + \beta u(x^t)] = \sum_{t=0}^{\infty} \delta^{t-1} \alpha + \beta \sum_{t=0}^{\infty} \delta^{t-1} u(x^t)$$

and similarly

$$\sum_{t=0}^{\infty} \delta^{t-1} v(y^t) = \sum_{t=0}^{\infty} \delta^{t-1} [\alpha + \beta u(y^t)] = \sum_{t=0}^{\infty} \delta^{t-1} \alpha + \beta \sum_{t=0}^{\infty} \delta^{t-1} u(y^t),$$

so that

$$\sum_{t=0}^{\infty} \delta^{t-1} v(x^t) = \sum_{t=0}^{\infty} \delta^{t-1} v(y^t).$$

Thus the person's preferences are represented also by the discounted sum of payoffs with payoff function v and discount factor δ.

426.1 Subgame perfect equilibrium of finitely repeated *Prisoner's Dilemma*

Use backward induction. In the last period, the action C is strictly dominated for each player, so each player chooses D, regardless of history. Now consider period $T - 1$. Each player's action in this period affects only the outcome in this period—it has no effect on the outcome in period T, which is (D, D). Thus in choosing her action in period $T - 1$, a player considers only her payoff in that period. As in period T, her action D strictly dominates her action C, so that in any subgame perfect equilibrium she chooses D. A similar argument applies to all previous periods, leading to the conclusion that in every subgame perfect equilibrium each player chooses D in every period, regardless of history.

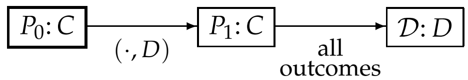


Figure 76.1 The strategy in Exercise 428.1a.

428.1 Strategies in an infinitely repeated *Prisoner's Dilemma*

a. The strategy is shown in Figure 76.1.

b. The strategy is shown in Figure 76.2.

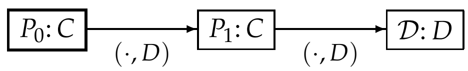


Figure 76.2 The strategy in Exercise 428.1b.

c. The strategy is shown in Figure 76.3.

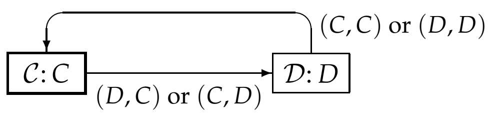


Figure 76.3 The strategy in Exercise 428.1c.

439.1 Finitely repeated *Prisoner's Dilemma* with switching cost

a. Consider deviations by player 1, given that player 2 adheres to her strategy, in the subgames following histories that end in each of the four outcomes of the game.

(C, C): If player 1 adheres to her strategy, her payoff is 3 in every period. If she deviates in the first period of the subgame, but otherwise follows her strategy, her payoff is $4 - \epsilon$ in the first period of the subgame, and 2 in every subsequent period. Given $\epsilon > 1$, player 1's deviation is not profitable, even if it occurs in the last period of the game.

(D, C) or (D, D): If player 1 adheres to her strategy, her payoff is 2 in every period. If she deviates in the first period of the subgame, but otherwise follows her strategy, her payoff is $-\epsilon$ in the first period of the subgame, $2 - \epsilon$ in the next period, and 2 subsequently. Thus adhering to her strategy is optimal for player 1.

(C, D): If player 1 adheres to her strategy, her payoff is $2 - \epsilon$ in the first period of the subgame, and 2 subsequently. If she deviates in the first period of the subgame, but otherwise follows her strategy, her payoff is 0 in the first period of the subgame, $2 - \epsilon$ in the next period, and 2 subsequently. Given $\epsilon < 2$, player 1's deviation is not optimal even if it occurs in the last period of the game.

b. Given $\epsilon > 2$, a player does not gain from deviating from (C, C) in the next-to-last or last periods, even if she is not punished, and does not optimally punish such a deviation by her opponent. Consider the strategy that chooses C at the start of the game and after any history that ends with (C, C), chooses D after any other history that has length at most $T - 2$, and chooses the action it chose in period $T - 1$ after any history of length $T - 1$ (where T is the length of the game). I claim that the strategy pair in which both players use this strategy is a subgame perfect equilibrium. Consider deviations by player 1, given that player 2 adheres to her strategy, in the subgames following the various possible histories.

History ending in (C, C), length $\leq T - 3$: If player 1 adheres to her strategy, her payoff is 3 in every period of the subgame. If she deviates in the first period of the subgame, but otherwise follows her strategy, her payoff is $4 - \epsilon$ in the first period of the subgame, and 2 in every subsequent period (her opponent switches to D). Given $\epsilon > 1$, player 1's deviation is not profitable.

History ending in (C, C), length $\geq T - 2$: If player 1 adheres to her strategy, her payoff is 3 in each period of the subgame. If she deviates to D in the first period of the subgame, her payoff is $4 - \epsilon$ in that period, and 4 subsequently (her deviation is not punished). The length of the subgame is at most 2, so given $\epsilon > 2$, her deviation is not profitable.

History ending in (D, C) or (D, D): If player 1 adheres to her strategy, her payoff is 2 in every period. If she deviates in the first period of the subgame, but otherwise follows her strategy, her payoff is $-\epsilon$ in the first period of the subgame, $2 - \epsilon$ in the next period, and 2 subsequently. Thus adhering to her strategy is optimal for player 1.

History ending in (C, D), length $\leq T - 2$: If player 1 adheres to her strategy, her payoff is $2 - \epsilon$ in the first period of the subgame (she switches to D), and 2 subsequently. If she deviates in the first period of the subgame, but otherwise follows her strategy, her payoff is 0 in the first period of the subgame, $2 - \epsilon$ in the next period, and 2 subsequently.

History ending in (C, D), length $T - 1$: If player 1 adheres to her strategy, her payoff is 0 in period T (the outcome is (C, D)). If she deviates to D, her payoff is $2 - \epsilon$ in period T. Given $\epsilon > 2$, adhering to her strategy is thus optimal.

442.1 Deviations from grim trigger strategy

- If player 1 adheres to the strategy, she subsequently chooses D (because player 2 chose D in the first period). Player 2 chooses C in the first period of the subgame (player 1 chose C in the first period of the game), and then chooses D (because player 1 chooses D in the first period of the subgame). Thus the sequence of outcomes in the subgame is $((D,C),(D,D),(D,D),\ldots)$, yielding player 1 a discounted average payoff in the subgame of

$$(1-\delta)(3+\delta+\delta^2+\delta^3+\cdots) = (1-\delta)\left(3+\frac{\delta}{1-\delta}\right) = 3-2\delta.$$

- If player 1 refrains from punishing player 2 for her lapse, and simply chooses C in every subsequent period, then the outcome in period 2 and subsequently is (C,C), so that the sequence of outcomes in the subgame yields player 1 a discounted average payoff of 2.

If $\delta > \frac{1}{2}$ then $2 > 3-2\delta$, so that player 1 prefers to ignore player 2's deviation rather than to adhere to her strategy and punish player 2 by choosing D. (Note that the theory does not consider the possibility that player 1 takes player 2's play of D as a signal that she is using a strategy different from the grim trigger strategy.)

443.2 Different punishment lengths in subgame perfect equilibrium

Yes, an infinitely repeated *Prisoner's Dilemma* has such subgame perfect equilibria. As for the modified grim trigger strategy, each player's strategy has to switch to D not only if the other player chooses D but also if the player herself chooses D. The only subtlety is that the number of periods for which a player chooses D after a history in which not all the outcomes were (C,C) must depend on the identity of the player who first deviated. If, for example, player 1 punishes for two periods while player 2 punishes for three periods, then the outcome (C,D) induces player 1 to choose D for two periods (to punish player 2 for her deviation) while the outcome (D,C) induces her to choose D for three periods (while she is being punished by player 2). The strategy of each player in this case is shown in Figure 79.1. Viewed as a strategy of player 1, the top part of the figure entails punishment of player 2 and the bottom part entails player 1's reaction to her own deviation. Similarly, viewed as a strategy of player 2, the bottom part of the figure entails punishment of player 1 and the top part entails player 2's reaction to her own deviation.

To find the values of δ for which the strategy pair in which each player uses the strategy in Figure 79.1 is a subgame perfect equilibrium, consider the result of each player's deviating at the start of a subgame.

First consider player 1. If she deviates when both players are in state P_0, she induces the outcome (D,C) followed by three periods of (D,D), and then (C,C) subsequently. This outcome path is worse for her than (C,C) in every period if

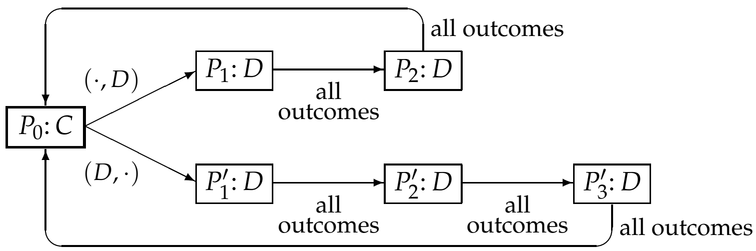


Figure 79.1 A strategy in an infinitely repeated *Prisoner's Dilemma* that punishes deviations for two periods and reacts to punishment by choosing D for three periods.

and only if $\delta^3 - 2\delta + 1 \leq 0$, or if and only if δ is at least around 0.62 (as we found in Section 14.7.2). If she deviates when both players are in one of the other states then she is worse off in the period of her deviation and her deviation does not affect the subsequent outcomes. Thus player 1 cannot profitably deviate in the first period of any subgame if δ is at least around 0.62.

The same argument applies to player 2, except that a deviation when both players are in state P_0 induces (C, D) followed by three, rather than two periods of (D, D). This outcome path is worse for player 2 than (C, C) in every period if and only if $\delta^4 - 2\delta + 1 \leq 0$, or if and only if δ is at least around 0.55 (as we found in Section 14.7.2).

We conclude that the strategy pair in which each player uses the strategy in Figure 79.1 is a subgame perfect equilibrium if and only if $\delta^3 - 2\delta + 1 \leq 0$, or if and only if δ is at least around 0.62.

445.1 *Tit-for-tat* as a subgame perfect equilibrium

Suppose that player 2 adheres to *tit-for-tat*. Consider player 1's behavior in subgames following histories that end in each of the following outcomes.

(C, C) If player 1 adheres to *tit-for-tat* the outcome is (C, C) in every period, so that her discounted average payoff in the subgame is x. If she chooses D in the first period of the subgame, then adheres to *tit-for-tat*, the outcome alternates between (D, C) and (C, D), and her discounted average payoff is $y/(1+\delta)$. Thus we need $x \geq y/(1+\delta)$, or $\delta \geq (y-x)/x$, for a one-period deviation from *tit-for-tat* not to be profitable for player 1.

(C, D) If player 1 adheres to *tit-for-tat* the outcome alternates between (D, C) and (C, D), so that her discounted average payoff is $y/(1+\delta)$. If she deviates to C in the first period of the subgame, then adheres to *tit-for-tat*, the outcome is (C, C) in every period, and her discounted average payoff is x. Thus we need $y/(1+\delta) \geq x$, or $\delta \leq (y-x)/x$, for a one-period deviation from *tit-for-tat* not to be profitable for player 1.

(D, C) If player 1 adheres to *tit-for-tat* the outcome alternates between (C, D) and (D, C), so that her discounted average payoff is $\delta y/(1+\delta)$. If she deviates to D in the first period of the subgame, then adheres to *tit-for-tat*, the outcome is (D, D) in every period, and her discounted average payoff is 1. Thus we need $\delta y/(1+\delta) \geq 1$, or $\delta \geq 1/(y-1)$, for a one-period deviation from *tit-for-tat* not to be profitable for player 1.

(D, D) If player 1 adheres to *tit-for-tat* the outcome is (D, D) in every period, so that her discounted average payoff is 1. If she deviates to C in the first period of the subgame, then adheres to *tit-for-tat*, the outcome alternates between (C, D) and (D, C), and her discounted average payoff is $\delta y/(1+\delta)$. Thus we need $1 \geq \delta y/(1+\delta)$, or $\delta \leq 1/(y-1)$, for a one-period deviation from *tit-for-tat* not to be profitable for player 1.

The same arguments apply to deviations by player 2, so we conclude that (*tit-for-tat*, *tit-for-tat*) is a subgame perfect equilibrium if and only if $\delta = (y-x)/x$ and $\delta = 1/(y-1)$, or $y - x = 1$ and $\delta = 1/x$.

15 Repeated Games: General Results

454.3 Repeated Bertrand duopoly

a. Suppose that firm i uses the strategy s_i. If the other firm, j, uses s_j, then its discounted average payoff is

$$(1-\delta)\left(\tfrac{1}{2}\pi(p^m) + \tfrac{1}{2}\delta\pi(p^m) + \cdots\right) = \tfrac{1}{2}\pi(p^m).$$

If, on the other hand, firm j deviates to a price p then the closer this price is to p^m, the higher is j's profit, because the punishment does not depend on p. Thus by choosing p close enough to p^m the firm can obtain a profit as close as it wishes to $\pi(p^m)$ in the period of its deviation. Its profit during its punishment in the following k periods is zero. Once its punishment is complete, it can either revert to p^m or deviate once again. If it can profit from deviating initially then it can profit by deviating once its punishment is complete, so its maximal profit from deviating is

$$(1-\delta)\left(\pi(p^m) + \delta^{k+1}\pi(p^m) + \delta^{2k+2}\pi(p^m) + \cdots\right) = \frac{(1-\delta)\pi(p^m)}{1-\delta^{k+1}}.$$

Thus for (s_1, s_2) to be a Nash equilibrium we need

$$\frac{1-\delta}{1-\delta^{k+1}} \leq \tfrac{1}{2},$$

or

$$\delta^{k+1} - 2\delta + 1 \leq 0.$$

(This condition is the same as the one we found for a pair of k-period punishment strategies to be a Nash equilibrium in the *Prisoner's Dilemma* (Section 14.7.2).)

b. Suppose that firm i uses the strategy s_i. If the other firm does so then its discounted average payoff is $\frac{1}{2}\pi(p^m)$, as in part *a*. If the other firm deviates to some price p with $c < p < p^m$ in the first period, and maintains this price subsequently, then it obtains $\pi(p)$ in the first period and shares $\pi(p)$ in each subsequent period, so that its discounted average payoff is

$$(1-\delta)\left(\pi(p) + \tfrac{1}{2}\delta\pi(p) + \tfrac{1}{2}\delta^2\pi(p) + \cdots\right) = \tfrac{1}{2}(2-\delta)\pi(p).$$

If p is close to p^m then $\pi(p)$ is close to $\pi(p^m)$ (because π is continuous). In fact, for any $\delta < 1$ we have $2-\delta > 1$, so that we can find $p < p^m$ such that $(2-\delta)\pi(p) > \pi(p^m)$. Hence the strategy pair is not a Nash equilibrium of the infinitely repeated game for any value of δ.

459.2 Detection lags

a. The best deviations involve prices slightly less than p^*. Such a deviation by firm i yields a discounted average payoff close to

$$(1-\delta)\left(\pi(p^*)+\delta\pi(p^*)+\cdots+\delta^{k_i-1}\pi(p^*)\right)=(1-\delta^{k_i})\pi(p^*),$$

whereas compliance with the strategy yields the discounted average payoff $\frac{1}{2}\pi(p^*)$. Thus the strategy pair is a subgame perfect equilibrium for any value of p^* if $\delta^{k_1}\geq\frac{1}{2}$ and $\delta^{k_2}\geq\frac{1}{2}$, and is not a subgame perfect equilibrium for any value of p^* if $\delta^{k_1}<\frac{1}{2}$ or $\delta^{k_2}<\frac{1}{2}$. That is, the most profitable price for which the strategy pair is a subgame perfect equilibrium is p^m if $\delta^{k_1}\geq\frac{1}{2}$ and $\delta^{k_2}\geq\frac{1}{2}$ and is c if $\delta^{k_1}<\frac{1}{2}$ or $\delta^{k_2}<\frac{1}{2}$.

b. Denote by k_i^* the critical value of k_i found in part *a*. (That is, $\delta^{k_i^*}\geq\frac{1}{2}$ and $\delta^{k_i^*+1}<\frac{1}{2}$.)

If $k_i>k_i^*$ then no change in k_j affects the outcome of the price-setting subgame, so j's best action at the start of the game is θ, in which case i's best action is the same. Thus in one subgame perfect equilibrium both firms choose θ at the start of the game, and c regardless of history in the rest of the game.

If $k_i\leq k_i^*$ then j's best action is k_j^* if the cost of choosing k_j^* is at most $\frac{1}{2}\pi(p^m)$. Thus if the cost of choosing k_i^* is at most $\frac{1}{2}\pi(p^m)$ for each firm then the game has another subgame perfect equilibrium, in which each firm i chooses k_i^* at the start of the game and the strategy s_i in the price-setting subgame.

A promise by firm i to beat another firm's price is an inducement for consumers to inform firm i of deviations by other firms, and thus reduce its detection time. To this extent, such a promise tends to promote collusion.

16 Bargaining

468.1 Two-period bargaining with constant cost of delay

In the second period, player 1 accepts any proposal that gives a positive amount of the pie. Thus in any subgame perfect equilibrium player 2 proposes $(0, 1)$ in period 2, which player 1 accepts, obtaining the payoff $-c_1$.

Now consider the first period. Given the second period outcome of any subgame perfect equilibrium, player 2 accepts any proposal that gives her more than $1 - c_2$ and rejects any proposal that gives her less than $1 - c_2$. Thus in any subgame perfect equilibrium player 1 proposes $(c_2, 1 - c_2)$, which player 2 accepts.

In summary, the game has a unique subgame perfect equilibrium, in which

- player 1 proposes $(c_2, 1 - c_2)$ in period 1, and accepts all proposals in period 2
- player 2 accepts a proposal in period 1 if and only if it gives her at least $1 - c_2$, and proposes $(0, 1)$ in period 2 after any history.

The outcome of the equilibrium is that the proposal $(c_2, 1 - c_2)$ is made by player 1 and immediately accepted by player 2.

468.2 Three-period bargaining with constant cost of delay

The subgame following a rejection by player 2 in period 1 is a two-period game in which player 2 makes the first proposal. Thus by the result of Exercise 468.1, the subgame has a unique subgame perfect equilibrium, in which player 2 proposes $(1 - c_1, c_1)$, which player 1 immediately accepts.

Now consider the first period.

- If $c_1 \geq c_2$, player 2 rejects any offer of less than $c_1 - c_2$ (which she obtains if she rejects an offer), and accepts any offer of more than $c_1 - c_2$. Thus in an equilibrium player 1 offers her $c_1 - c_2$, which she accepts.
- If $c_1 < c_2$, player 2 accepts all offers, so that player 1 proposes $(1, 0)$, which player 2 accepts.

In summary, the game has a unique subgame perfect equilibrium, in which

- player 1 proposes $(1 - (c_1 - c_2), c_1 - c_2)$ if $c_1 \geq c_2$ and $(1, 0)$ otherwise in period 1, accepts any proposal that gives her at least $1 - c_1$ in period 2, and proposes $(1, 0)$ in period 3

- player 2 accepts any proposal that gives her at least $c_1 - c_2$ if $c_1 \geq c_2$ and accepts all proposals otherwise in period 1, proposes $(1 - c_1, c_1)$ in period 2, and accepts all proposals in period 3.

17 Appendix: Mathematics

497.1 Maximizer of quadratic function

We can write the function as $-x(x-\alpha)$. Thus $r_1 = 0$ and $r_2 = \alpha$, and hence the maximizer is $\alpha/2$.

499.3 Sums of sequences

In the first case set $r = \delta^2$ to transform the sum into $1 + r + r^2 + \cdots$, which is equal to $1/(1-r) = 1/(1-\delta^2)$.

In the second case split the sum into $(1 + \delta^2 + \delta^4 + \cdots) + (2\delta + 2\delta^3 + 2\delta^5 + \cdots)$; the first part is equal to $1/(1-\delta^2)$ and the second part is equal to $2\delta(1 + \delta^2 + \delta^4 + \cdots)$, or $2\delta/(1-\delta^2)$. Thus the complete sum is

$$\frac{1+2\delta}{1-\delta^2}.$$

504.2 Bayes' law

Your posterior probability of carrying X given that you test positive is

$$\frac{\Pr(\text{positive test}|X)\Pr(X)}{\Pr(\text{positive test}|X)\Pr(X) + \Pr(\text{positive test}|\neg X)\Pr(\neg X)}$$

where $\neg X$ means "not X". This probability is equal to $0.9p/(0.9p + 0.2(1-p)) = 0.9p/(0.2 + 0.7p)$, which is increasing in p (i.e. a smaller value of p gives a smaller value of the probability). If $p = 0.001$ then the probability is approximately 0.004. (That is, if 1 in 1,000 people carry the gene then if you test positive on a test that is 90% accurate for people who carry the gene and 80% accurate for people who do not carry the gene, then you should assign probability 0.004 to your carrying the gene.) If the test is 99% accurate in both cases then the posterior probability is $(0.99 \cdot 0.001)/[0.99 \cdot 0.001 + 0.01 \cdot 0.999] \approx 0.09$.

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The page numbers on which the references are cited are given in brackets after each item.

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