# LECTURE 5

## Introduction to Econometrics

## Nonlinear specifications and dummy variables

March 17, 2023

# TESTING MULTIPLE HYPOTHESES REVISITED

- Suppose we have a model

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3} + \varepsilon_i$$

- Suppose we want to test multiple linear hypotheses in this model
- For example, we want to see if the following restrictions on coefficients hold jointly:

$$\beta_1 + \beta_2 = 1 \quad \text{and} \quad \beta_3 = 0$$

- We cannot use a $t$-test in this case ($t$-test can be used only for one hypothesis at a time)
- We will use an $F$-test

# Restricted vs. unrestricted model

- We can reformulate the model by plugging the restrictions as if they were true (model under $H_0$)
- We call this model *restricted model* as opposed to the *unrestricted model*
- The unrestricted model is

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3} + \varepsilon_i$$

- Restricted model can be derived to have the following form:

$$y^*_i = \beta_0 + \beta_1 x^*_i + \varepsilon_i \,,$$

where $y^*_i = y_i - x_{i2}$ and $x^*_i = x_{i1} - x_{i2}$

# IDEA OF THE $F$-TEST

- If the restrictions are true, then the restricted model fits the data in the same way as the unrestricted model

  residuals are nearly the same

- If the restrictions are false, then the restricted model fits the data poorly

  residuals from the restricted model are much larger than those from the unrestricted model

- The idea is thus to compare the residuals from the two models

# IDEA OF THE $F$-TEST

- How to compare residuals in the two models?
  - Calculate the sum of squared residuals in the two models
  - Test if the difference between the two sums is equal to zero (statistically)
  - $H_0$: the difference is zero (residuals in the two models are the same, restrictions hold)
  - $H_A$: the difference is positive (residuals in the restricted model are bigger, restrictions do not hold)

- Sum of squared residuals

$$SSR = \sum_{i=1}^{n} (y_i - \widehat{y}_i)^2 = \sum_{i=1}^{n} e_i^2$$

# $F$-TEST

- The test statistic is defined as

$$F = \frac{(SSR_r - SSR_{ur})/q}{SSR_{ur}/(n-k-1)} \sim F_{q,n-k-1} \ ,$$

where:

| | | |
|---|---|---|
| $SSR_r$ | … | sum of squared residuals from the restricted model |
| $SSR_{ur}$ | … | sum of squared residuals from the unrestricted model |
| $q$ | … | number of restrictions |
| $n$ | … | number of observations |
| $k$ | … | number of estimated coefficients |

## GOODNESS OF FIT MEASURE

- We know that education and experience have a significant influence on wages
- But how important are they in determining wages?
- How much of difference in wages between people is explained by differences in education and in experience?
- How well variation in the independent variable(s) explains variation in the dependent variable?
- This are the questions answered by the goodness of fit measure - $R^2$

## Total and explained variation

- **Total variation** in the dependent variable:

$$\sum_{i=1}^{n} (y_i - \overline{y}_n)^2$$

- Predicted value of the dependent variable = part that is explained by independent variables:

$$\widehat{y}_i = \widehat{\beta}_0 + \widehat{\beta}_1 x_i$$

(case of regression line - for simplicity of notation)

- **Explained variation** in the dependent variable:

$$\sum_{i=1}^{n} (\widehat{y}_i - \overline{y}_n)^2$$

# Goodness of fit - $R^2$

- Denote:
  - $SST = \sum_{i=1}^{n} \left(y_i - \overline{y}_n\right)^2$ … *Total Sum of Squares*
  - $SSE = \sum_{i=1}^{n} (\widehat{y}_i - \overline{y}_n)^2$ … *Regression (Explained) Sum of Squares*
- Define the measure of the goodness of fit:

$$R^2 = \frac{SSE}{SST} = \frac{\text{Explained variation in } y}{\text{Total variation in } y}$$

# GOODNESS OF FIT - $R^2$

- In all models: $0 \leq R^2 \leq 1$
- $R^2$ tells us what percentage of the total variation in the dependent variable is explained by the variation in the independent variable(s)
  - $R^2 = 0.3$ means that the independent variables can explain 30% of the variation in the dependent variable
- Higher $R^2$ means better fit of the regression model (not necessarily a better model!)

# DECOMPOSING THE VARIANCE

- For models with intercept, $R^2$ can be rewritten using the decomposition of variance.
- Variance decomposition:

$$\sum_{i=1}^{n} (y_i - \overline{y}_n)^2 = \sum_{i=1}^{n} (\widehat{y}_i - \overline{y}_n)^2 + \sum_{i=1}^{n} e_i^2$$

- $SST = \sum_{i=1}^{n} (y_i - \overline{y}_n)^2$ ... *Total Sum of Squares*
- $SSE = \sum_{i=1}^{n} (\widehat{y}_i - \overline{y}_n)^2$ ... *Regression (Explained) Sum of Squares*
- $SSR = \sum_{i=1}^{n} e_i^2$ ... *Sum of Squared Residuals*

# Variance decomposition and $R^2$

- Variance decomposition: $SST = SSE + SSR$

- Intuition: total variation can be divided between the explained variation and the unexplained variation

  the true value $y$ is a sum of estimated (explained) $\hat{y}$ and the

  residual $e_i$ (unexplained part)

  $y_i = \widehat{y}_i + e_i$

- We can rewrite $R^2$:

$$R^2 = \frac{SSE}{SST} = \frac{SST - SSR}{SST} = 1 - \frac{SSR}{SST}$$

# Adjusted $R^2$

- The sum of squared residuals ($SSR$) decreases when additional explanatory variables are introduced in the model, whereas total sum of squares ($SST$) remains the same

$$R^2 = 1 - \frac{SSR}{SST} \text{ increases if we add explanatory variables}$$

Models with more variables automatically have better fit.

- To deal with this problem, we define the *adjusted* $R^2$:

$$R^2_{adj} = 1 - \frac{\frac{SSR}{n-k-1}}{\frac{SST}{n-1}} \leq R^2$$

($k$ is the number of coefficients)

- This measure introduces a "punishment" for including more explanatory variables

# FOUR IMPORTANT SPECIFICATION CRITERIA

Does a variable belong to the equation?

1. *Theory:* Is the variable's place in the equation unambiguous and theoretically sound? Does intuition tells you it should be included?

2. *t-test:* Is the variable's estimated coefficient significant in the expected direction?

3. *$R^2$:* Does the overall fit of the equation improve (enough) when the variable is added to the equation?

4. *Bias:* Do other variables' coefficients change significantly when the variable is added to the equation?

# Four important specification criteria

- If all conditions hold, the variable belongs in the equation
- If none of them holds, the variable is irrelevant and can be safely excluded
- If the criteria give contradictory answers, most importance should be attributed to theoretical justification

  Therefore, if theory (intuition) says that variable belongs to the equation, we include it (even though its coefficients might be insignificant!).

# NONLINEAR SPECIFICATION

- We will discuss different specifications nonlinear in dependent and independent variables and their interpretation

- We will define the notion of a dummy variable and we will show its different uses in linear regression models

# Nonlinear specification

- There is not always a linear relationship between dependent variable and explanatory variables
  - The use of OLS requires that the equation be linear in coefficients
  - However, there is a wide variety of functional forms that are linear in coefficients while being nonlinear in variables!
- We have to choose carefully the functional form of the relationship between the dependent variable and each explanatory variable
  - The choice of a functional form should be based on the underlying economic theory and/or intuition
  - Do we expect a curve instead of a straight line? Does the effect of a variable peak at some point and then start to decline?

# LINEAR FORM

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \varepsilon$$

- Assumes that the effect of the explanatory variable on the dependent variable is constant:

$$\frac{\partial y}{\partial x_k} = \beta_k \qquad k = 1, 2$$

- Interpretation: if $x_k$ increases by 1 **unit** (in which $x_k$ is measured), then $y$ will change by $\beta_k$ **units** (in which $y$ is measured)

- Linear form is used as default functional form until strong evidence that it is inappropriate is found

## LOG-LOG FORM

$$\ln y = \beta_0 + \beta_1 \ln x_1 + \beta_2 \ln x_2 + \varepsilon$$

- Assumes that the elasticity of the dependent variable with respect to the explanatory variable is constant:

$$\frac{\partial \ln y}{\partial \ln x_k} = \frac{\partial y / y}{\partial x_k / x_k} = \beta_k \qquad k = 1, 2$$

- Interpretation: if $x_k$ increases by 1 **percent**, then $y$ will change by $\beta_k$ **percents**

- Before using a double-log model, make sure that there are no negative or zero observations in the data set

# EXAMPLE

- Estimating the production function of Indian sugar industry:

$$\widehat{\ln Q} = \underset{}{2.70} + \underset{(0.14)}{0.59} \ln L + \underset{(0.17)}{0.33} \ln K$$

$Q$ . . . output $L$
. . . labor $K$
. . . capital employed

Interpretation: if we increase the amount of labor by 1%, the production of sugar will increase by 0.59%, ceteris paribus.

Ceteris paribus is a Latin phrase meaning 'other things being equal'.

# LOG-LINEAR FORMS

- Linear-log form:

$$y = \beta_0 + \beta_1 \ln x_1 + \beta_2 \ln x_2 + \varepsilon$$

Interpretation: if $x_k$ increases by 1 **percent**, then $y$ will change by $(\beta_k/100)$ **units** $(k = 1, 2)$

- Log-linear form:

$$\ln y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \varepsilon$$

Interpretation: if $x_k$ increases by 1 **unit**, then $y$ will change by $(\beta_k * 100)$ **percent** $(k = 1, 2)$

# Examples of log linear forms

- Estimating demand for chicken meat:

$$\widehat{Y} = -6.94 - \underset{(0.19)}{0.57}\ PC + \underset{(0.11)}{0.25}\ PB + \underset{(2.81)}{12.2}\ \ln YD$$

$Y$ . . . annual chicken consumption (kg.)
$PC$ . . . price of chicken
$PB$ . . . price of beef
$YD$ . . . annual disposable income

- Interpretation: An increase in the annual disposable income by 1% increases chicken consumption by 0.12 kg per year, ceteris paribus.

# Examples of log linear forms

- Estimating the influence of education and experience on wages:

$$\widehat{\ln wage} = 0.217 + \underset{(0.008)}{0.098}\ educ + \underset{(0.002)}{0.010}\ exper$$

| | |
|---|---|
| *wage* | . . . annual wage (USD) |
| *educ* | . . . years of education |
| *exper* | . . . years of experience |

- Interpretation: An increase in education by one year increases annual wage by 9.8%, ceteris paribus. An increase in experience by one year increases annual wage by 1%, ceteris paribus.

# Polynomial form

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_1^2 + \varepsilon$$

- To determine the effect of $x_1$ on $y$, we need to calculate the derivative:

$$\frac{\partial y}{\partial x_1} = \beta_1 + 2 \cdot \beta_2 \cdot x_1$$

- Clearly, the effect of $x_1$ on $y$ is not constant, but changes with the level of $x_1$

- We might also have higher order polynomials, e.g.:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_1^2 + \beta_3 x_1^3 + \beta_4 x_1^4 + \varepsilon$$

## EXAMPLE OF POLYNOMIAL FORM

- The impact of the number of hours of studying on the grade from Introductory Econometrics:

$$\widehat{grade} = 30 + 1.4 \cdot hours - 0.009 \cdot hours^2$$

- To determine the effect of hours on grade, calculate the derivative:

$$\frac{\partial y}{\partial x} = \frac{\partial grade}{\partial hours} = 1.4 - 2 \cdot 0.009 \cdot hours = 1.4 - 0.018 \cdot hours$$

Decreasing returns to hours of studying: more hours implies higher grade, but the positive effect of additional hour of studying decreases with more hours

# Choice of correct functional form

- The functional form has to be correctly specified in order to avoid biased and inconsistent estimates

  Remember that one of the OLS assumptions is that the model is correctly specified

- Ideally: the specification is given by underlying theory of the equation

- In reality: underlying theory does not give precise functional form

- In most cases, either linear form is adequate, or common sense will point out an easy choice from among the alternatives

# Choice of correct functional form

- Nonlinearity of explanatory variables
  - often approximated by polynomial form
  - missing higher powers of a variable can be detected as omitted variables (see next lecture)

- Nonlinearity of dependent variable
  - harder to detect based on statistical fit of the regression
  - $R^2$ is incomparable across models where the $y$ is transformed
  - dependent variables are often transformed to log-form in order to make their distribution closer to the normal distribution

# DUMMY VARIABLES

- Dummy variable - takes on the values of 0 or 1, depending on a qualitative attribute

- Examples of dummy variables:

$$Male = \begin{cases} 1 & \text{if the person is male} \\ 0 & \text{if the person is female} \end{cases}$$

$$Weekend = \begin{cases} 1 & \text{if the day is on weekend} \\ 0 & \text{if the day is a work day} \end{cases}$$

$$NewStadium = \begin{cases} 1 & \text{if the team plays on new stadium} \\ 0 & \text{if the team plays on old stadium} \end{cases}$$

# Intercept dummy

- Dummy variable included in a regression alone (not interacted with other variables) is an intercept dummy
- It changes the intercept for the subset of data defined by a dummy variable condition:

$$y_i = \beta_0 + \beta_1 D_i + \beta_2 x_i + \varepsilon_i$$

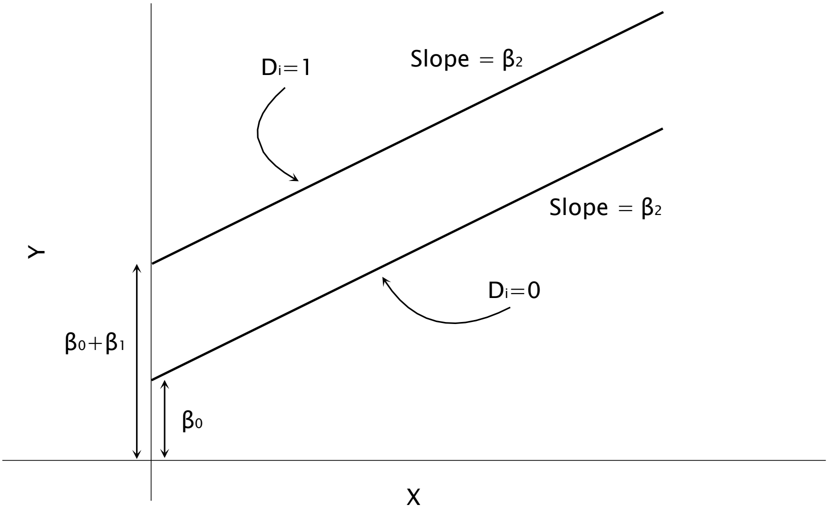
where

$$D_i = \begin{cases} 1 & \text{if the } i\text{-th observation meets a particular condition} \\ 0 & \text{otherwise} \end{cases}$$

- Wehave

$$\begin{aligned} y_i &= (\beta_0 + \beta_1) + \beta_2 x_i + \varepsilon_i \quad \text{if } D_i = 1 \\ y_i &= \beta_0 + \beta_2 x_i + \varepsilon_i \quad \text{if } D_i = 0 \end{aligned}$$

# Intercept dummy



# EXAMPLE

- Estimating the determinants of wages:

$$\widehat{wage}_i = \underset{}{-3.890} + \underset{(0.270)}{2.156}\ M_i + \underset{(0.051)}{0.603}\ educ_i + \underset{(0.064)}{0.010}\ exper_i$$

where $M_i = \begin{cases} 1 & \text{if the } i\text{-th person is male} \\ 0 & \text{if the } i\text{-th person is female} \end{cases}$

$wage$ ... average hourly wage in USD

- Interpretation of the dummy variable $M$: men earn on average \$2.156 per hour more than women, ceteris paribus

## SLOPE DUMMY

- If a dummy variable is interacted with another variable ($x$), it is a slope dummy.
- It changes the relationship between $x$ and $y$ for a subset of data defined by a dummy variable condition:

$$y_i = \beta_0 + \beta_1 x_i + \beta_2 (x_i \cdot D_i) + \varepsilon_i$$

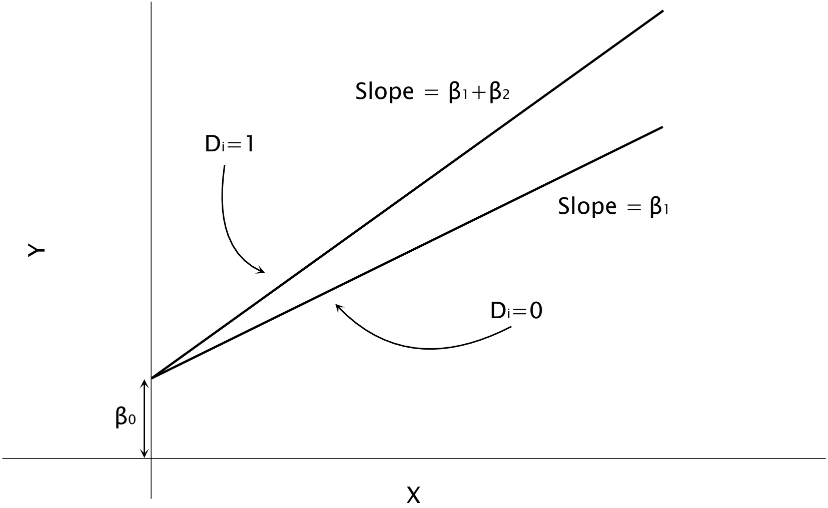
where

$$D_i = \begin{cases} 1 & \text{if the } i\text{-th observation meets a particular condition} \\ 0 & \text{otherwise} \end{cases}$$

- We have

$$\begin{aligned} y_i &= \beta_0 + (\beta_1 + \beta_2) x_i + \varepsilon_i \quad \text{if } D_i = 1 \\ y_i &= \beta_0 + \beta_1 x_i + \varepsilon_i \quad \text{if } D_i = 0 \end{aligned}$$

# SLOPE DUMMY



# EXAMPLE

- Estimating the determinants of wages:

$$\widehat{wage}_i = \underset{}{-2.620} + \underset{(0.054)}{0.450}\ educ_i + \underset{(0.021)}{0.170}\ M_i \cdot educ_i + \underset{(0.065)}{0.010}\ exper_i$$

$$\text{where} \quad M_i = \begin{cases} 1 & \text{if the } i\text{-th person is male} \\ 0 & \text{if the } i\text{-th person is female} \end{cases}$$

*wage* ... average hourly wage in USD

- Interpretation: men gain on average 17 cents per hour more than women for each additional year of education, ceteris paribus

# SLOPE AND INTERCEPT DUMMIES

- Allow both for different slope and intercept for two subsets of data distinguished by a qualitative condition:

$$y_i = \beta_0 + \beta_1 D_i + \beta_2 x_i + \beta_3 (x_i \cdot D_i) + \varepsilon_i$$

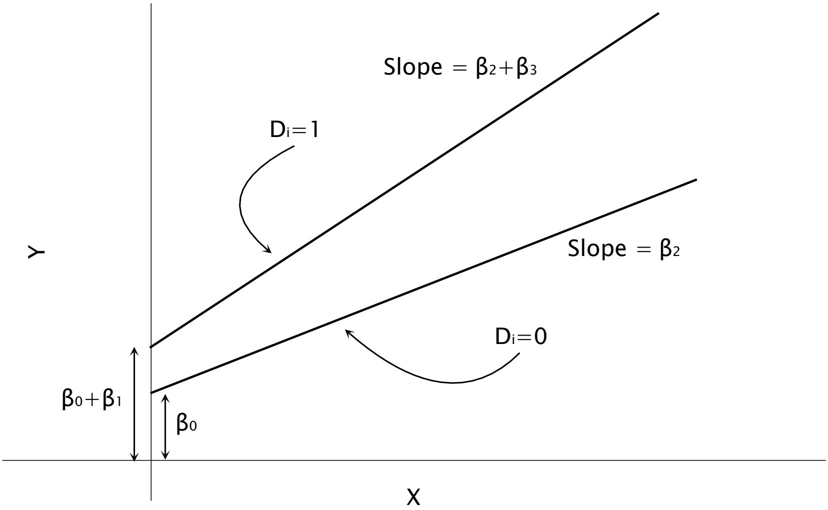
where

$$D_i = \begin{cases} 1 & \text{if the } i\text{-th observation meets a particular condition} \\ 0 & \text{otherwise} \end{cases}$$

- Wehave

$$\begin{aligned} y_i &= (\beta_0 + \beta_1) + (\beta_2 + \beta_3) x_i + \varepsilon_i \quad \text{if } D_i = 1 \\ y_i &= \beta_0 + \beta_2 x_i + \varepsilon_i \quad \text{if } D_i = 0 \end{aligned}$$

# SLOPE AND INTERCEPT DUMMIES



# Dummy variables - multiple categories

- What if a variable defines three or more qualitative attributes?
- Example: level of education - elementary school, high school, and college
- Define and use a set of dummy variables:

$$H = \begin{cases} 1 & \text{if high school} \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad C = \begin{cases} 1 & \text{if college} \\ 0 & \text{otherwise} \end{cases}$$

- Should we include also a third dummy in the regression, which is equal to 1 for people with elementary education?

  No, unless we exclude the intercept!

  Using full set of dummies leads to perfect multicollinearity (dummy variable trap)

# SUMMARY

- We discussed different nonlinear specifications of a regression equation and their interpretation
- We defined the concept of a dummy variable and we showed its use
- Further readings:
  Studenmund, Chapter 7
  Wooldridge, Chapters 6 & 7