PORTFOLIO THEORY – EXAM 14/6/2024

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EXERCISE 1

A three-year 1000-euro coupon bond pays a 200-euro coupon each year.

Knowing that a bond that pays 100 euro after one-year costs 97 euro, one that pays 100 euro after two years costs 95 euro, and one that pays 100 euro after three years costs 90 euro, what is the price at which the coupon bond should trade in a market in which the law of one price holds?

Due to the law of one price, equal cash flows should have the same price. The coupon bond pays 200 euro after one year, 200 euro after two years and 1200 (the 1000 euro plus the 200 coupon) after three years. Therefore, we can replicate its cash flows with two one-year zero-coupon bonds, two two-year zero-coupon bonds and twelve three-year zero-coupon bonds. The price of this portfolio of zero-coupon bonds, which must be equal to the price of the coupon bond, is:

$$97 * 2 + 95 * 2 + 90 * 12 = 1464$$

EXERCISE 2

The security S pays 990 euro after one year and it costs 900 euro.

If there is a 20% tax on financial profits and the inflation rate is 2% per year, what is the real return paid by security S after taxes?

After one year we earn $990 - 900 = 90$ euro.

On this we have to pay a 20% tax, so we actually earn $90 - 90 * 0.2 = 72$ euro

The nominal return net of taxes is therefore:

$$R = \frac{972 - 900}{900} = 8\%$$

We now need to account for the inflation. The real return is:

$$R_r = \frac{1 + R}{1 + i} - 1 = \frac{1 + 0.08}{1 + 0.02} - 1 \approx 0.0588$$

EXERCISE 3

Consider a portfolio of three assets. The vector of weights and log-returns are:

$$\boldsymbol{w} = \begin{bmatrix} 0.6 \\ 0.5 \\ -0.1 \end{bmatrix} \qquad \boldsymbol{r} = \begin{bmatrix} 0.1 \\ -0.02 \\ 0.04 \end{bmatrix}$$

What is the return of the portfolio?

Log-returns are not asset additive. We first need to convert them to simple returns:

$$R_1 = \exp(r_1) - 1 = \exp(0.1) - 1 \approx 0.1052$$

$$R_2 = \exp(r_2) - 1 = \exp(-0.02) - 1 \approx -0.0198$$

$$R_3 = \exp(r_3) - 1 = \exp(0.04) - 1 \approx 0.0408$$

We can now compute the return of the portfolio:

$$R_p = w_1 R_1 + w_2 R_2 + w_3 R_3 = 0.6 \times 0.1052 + 0.5 \times (-0.0198) - 0.1 \times 0.0408$$

$$= 0.06312 - 0.0099 - 0.00408 = 0.04914 \approx 4.9\%$$

EXERCISE 4

Given a risk-free asset and two risky assets with mean return, covariance matrix and inverse covariance matrix equal to

$$\mu = \begin{pmatrix} 0.01 \\ 0.008 \end{pmatrix} \quad \Sigma = \begin{bmatrix} 0.004 & 0.002 \\ 0.002 & 0.003 \end{bmatrix} \quad \Sigma^{-\mathbf{1}} = \begin{bmatrix} 375 & -250 \\ -250 & 500 \end{bmatrix}$$

compute the weights for the optimal mean-variance portfolio with target return $Re = 0.01$.

The weights for the risky assets are:

$$w_v = \frac{Re}{\boldsymbol{\mu}'\Sigma^{-\mathbf{1}}\boldsymbol{\mu}}\Sigma^{-\mathbf{1}}\boldsymbol{\mu} = \frac{0.01}{(0.01 \quad 0.008)\begin{bmatrix} 375 & -250 \\ -250 & 500 \end{bmatrix}\begin{pmatrix} 0.01 \\ 0.008 \end{pmatrix}}\begin{bmatrix} 375 & -250 \\ -250 & 500 \end{bmatrix}\begin{pmatrix} 0.01 \\ 0.008 \end{pmatrix} =$$

$$\frac{0.01}{[0.01 * 375 + 0.008 * (-250) \quad 0.01 * (-250) + 0.008 * 500]\begin{pmatrix} 0.01 \\ 0.008 \end{pmatrix}}\begin{bmatrix} 375 * 0.01 + (-250) * 0.008 \\ -250 * 0.01 + 500 * 0.008 \end{bmatrix} =$$

$$\frac{0.01}{[1.75 \quad 1.5]\begin{pmatrix} 0.01 \\ 0.012 \end{pmatrix}}\begin{bmatrix} 1.75 \\ 1.5 \end{bmatrix} = \frac{0.01}{0.0175 + 0.018}\begin{bmatrix} 1.75 \\ 1.5 \end{bmatrix} = \frac{0.01}{0.0355}\begin{bmatrix} 1.75 \\ 1.5 \end{bmatrix} \approx \begin{bmatrix} 0.49 \\ 0.42 \end{bmatrix}$$

The weight for the risk-free asset is $1 - (0.49 + 0.42) = 0.09$

EXERCISE 5

Consider the set of weights

$$w_{t-1} = \begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix} \quad w_t = \begin{bmatrix} -0.1 \\ 1.1 \end{bmatrix}$$

Compute the turnover taking into account the effect of the realized returns $R_t = \begin{bmatrix} 0.05 \\ 0.1 \end{bmatrix}$

We have to use the formula

$$TO_t = \sum_{i=1}^{N} \left| w_{i,t} - w_{i,t-1}^{+} \right|$$

To apply it correctly we first need to compute $w_{i,t-1}^{+}$.

The first asset experienced a +5% returns, therefore we have $0.2 + 0.2 \times 0.05 = 0.21$

The second asset experienced a +10% returns, therefore we have $0.8 + 0.8 \times 0.1 = 0.88$

The weights now sum to 1.09. We need to normalize them so that they sum up to 1 again:

$$w_{i,t-1}^{+} = \begin{bmatrix} \frac{0.21}{1.09} \\ \\ \frac{0.88}{1.09} \end{bmatrix} =\approx \begin{bmatrix} 0.19 \\ 0.81 \end{bmatrix}$$

Now we can apply the formula and compute the turnover:

$$TO_t = |-0.1 - 0.19| + |1.1 - 0.81| = 0.29 + 0.29 = 0.58$$

This means we need to trade 58% of our wealth in order to update the weights.

EXERCISE 6

1. *What are the four properties of a coherent risk measure?*
2. *Assuming no change in the risk and in the interest rate, how does the price of a zero-coupon bond change over time on the secondary market?*
3. *The effective annual rate of an investment is 10%. What is the monthly rate?*
4. *What is the efficient portfolio in a situation in which all the assumptions of the CAPM perfectly hold?*
5. *What is the bid-ask spread?*

1. Monotonicity, translation invariance, positive homogeneity, subadditivity.
2. The price gradually rises until it reaches the face value.
3. The monthly rate is $(1 + 0.1)^{1/12} - 1 \approx 0.008 = 0.8\%$
4. The efficient portfolio is equal to the market portfolio.
5. It is the difference between the price at which you can buy a security on the market and the price at which you can sell it.