PORTFOLIO THEORY – EXERCISES 07/05/2024

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EXERCISE 1

The vector of weights and the covariance matrix of a portfolio with two assets are:

$$\boldsymbol{w} = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} \quad \boldsymbol{\Sigma} = \begin{bmatrix} 0.002 & 0.001 \\ 0.001 & 0.003 \end{bmatrix}$$

Compute, using matrix form, the variance of the portfolio.

$$Var(R_P) = [0.6 \quad 0.4] \begin{bmatrix} 0.002 & 0.001 \\ 0.001 & 0.003 \end{bmatrix} \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} =$$

$$[0.6 * 0.002 + 0.4 * 0.001 \quad 0.6 * 0.001 + 0.4 * 0.003] \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} =$$

$$[0.0016 \quad 0.0018] \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} = 0.0016 * 0.6 + 0.0018 * 0.4 = 0.00168$$

EXERCISE 2

An investor with risk-aversion $\gamma = 4$ invests in a portfolio of one risk-free asset and two risky assets with mean excess return, covariance matrix and inverse covariance matrix equal to:

$$\mu = \begin{pmatrix} 0.006 \\ 0.004 \end{pmatrix} \quad \Sigma = \begin{bmatrix} 0.003 & 0.002 \\ 0.002 & 0.0015 \end{bmatrix} \quad \Sigma^{-1} \approx \begin{bmatrix} 3000 & -4000 \\ -4000 & 6000 \end{bmatrix}$$

Compute the portfolio weights.

The weights for the risky assets are:

$$\boldsymbol{w_{mv}} = \frac{1}{\gamma} \Sigma^{-1} \boldsymbol{\mu} = \frac{1}{4} \begin{bmatrix} 3000 & -4000 \\ -4000 & 6000 \end{bmatrix} \begin{pmatrix} 0.006 \\ 0.004 \end{pmatrix}$$

$$= \begin{bmatrix} 750 & -1000 \\ -1000 & 1500 \end{bmatrix} \begin{pmatrix} 0.006 \\ 0.004 \end{pmatrix} =$$

$$\begin{bmatrix} 750 * 0.006 + (-1000) * 0.004 \\ -1000 * 0.006 + 1500 * 0.004 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}$$

The weight for the risk-free asset is: 1 - 0.5 = 0.5

EXERCISE 3

Given two risky assets with mean return, covariance matrix and inverse covariance matrix equal to

$$\mu = \begin{pmatrix} 0.007 \\ 0.004 \end{pmatrix} \quad \Sigma = \begin{bmatrix} 0.004 & 0.002 \\ 0.002 & 0.003 \end{bmatrix} \quad \Sigma^{-\mathbf{1}} = \begin{bmatrix} 375 & -250 \\ -250 & 500 \end{bmatrix}$$

compute the weights for the minimum variance portfolio.

The weights for the risky assets are:

$$w_v = \frac{1}{\mathbf{1}'\Sigma^{-\mathbf{1}}\mathbf{1}}\Sigma^{-\mathbf{1}}\mathbf{1} = \frac{1}{(1 \quad 1)\begin{bmatrix} 375 & -250 \\ -250 & 500 \end{bmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}}\begin{bmatrix} 375 & -250 \\ -250 & 500 \end{bmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \frac{1}{[1 * 375 + 1 * (-250) \quad 1 * (-250) + 1 * 500]\begin{pmatrix} 1 \\ 1 \end{pmatrix}}\begin{bmatrix} 375 * 1 + (-250) * 1 \\ -250 * 1 + 500 * 1 \end{bmatrix}$$

$$= \frac{1}{[125 \quad 250]\begin{pmatrix} 1 \\ 1 \end{pmatrix}}\begin{bmatrix} 125 \\ 250 \end{bmatrix} = \frac{1}{125 + 250}\begin{bmatrix} 125 \\ 250 \end{bmatrix} = \frac{1}{375}\begin{bmatrix} 125 \\ 250 \end{bmatrix} \approx \begin{bmatrix} 0.33 \\ 0.67 \end{bmatrix}$$

EXERCISE 4

Write the first order conditions for the problem of computing portfolio weights that minimize the variance given a target return of 1% and a risk-free rate of 0.1%:

$$\min_{\boldsymbol{w}} \boldsymbol{w}'\Sigma\boldsymbol{w}$$

subject to:

$$\boldsymbol{w}'\boldsymbol{\mu} + (1 - \boldsymbol{w}'\mathbf{1})0.001 = 0.01$$

First we write the Lagrangian function:

$$L(\boldsymbol{w},\lambda) = \boldsymbol{w}'\Sigma\boldsymbol{w} + \lambda[0.01 - \boldsymbol{w}'\boldsymbol{\mu} - (1 - \boldsymbol{w}'\mathbf{1})0.001]$$
$$= \boldsymbol{w}'\Sigma\boldsymbol{w} + \lambda[0.009 - \boldsymbol{w}'\boldsymbol{\mu} + \boldsymbol{w}'\mathbf{0.001}]$$

The first order conditions are the two partial derivatives of the Lagrangian with respect to $\boldsymbol{w}$ and λ set equal to zero:

$$\frac{\partial L}{\partial \boldsymbol{w}} = 2\Sigma\boldsymbol{w} - \lambda\boldsymbol{\mu} + \lambda\mathbf{0.001} = \mathbf{0}$$
$$\frac{\partial L}{\partial \lambda} = 0.009 - \boldsymbol{w}'\boldsymbol{\mu} - \boldsymbol{w}'\mathbf{0.001} = 0$$

EXERCISE 5

Given a vector of theoretically optimal weights

$$\boldsymbol{w} = \begin{bmatrix} -0.2 \\ 0.4 \\ 0.5 \\ 0.3 \end{bmatrix}$$

compute the weights $\boldsymbol{w}^*$ of the shrinkage portfolio obtained as

$$\boldsymbol{w}^* = \delta\boldsymbol{w}_{\boldsymbol{NAIVE}} + (1-\delta)\boldsymbol{w}$$

with shrinkage parameter $\delta = 0.4$

The weights of the shrinkage portfolio are:

$$\boldsymbol{w}^* = 0.4\begin{bmatrix} 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \end{bmatrix} + 0.6\begin{bmatrix} -0.2 \\ 0.4 \\ 0.5 \\ 0.3 \end{bmatrix} = \begin{bmatrix} 0.1 \\ 0.1 \\ 0.1 \\ 0.1 \end{bmatrix} + \begin{bmatrix} -0.12 \\ 0.24 \\ 0.3 \\ 0.18 \end{bmatrix} = \begin{bmatrix} -0.02 \\ 0.34 \\ 0.4 \\ 0.28 \end{bmatrix}$$

EXERCISE 6

Given the following series of returns:

0.05 , -0.01 , 0.02 , 0.03 , 0 , 0.005

Compute the downside deviation for an investor that sets the benchmark B=0.01

We compute the semivariance:

$$\sigma_B^2 = \frac{1}{6}[(-0.01 - 0.01)^2 + (0 - 0.01)^2 + (0.005 - 0.01)^2]$$
$$= \frac{1}{6}0.000525 = 0.0000875$$

The downside deviation is simply the square root of the semivariance:

$$\sigma_B = \sqrt{0.0000875} = 0.009354143$$

EXERCISE 7

Consider the set of weights

$$w_{t-1} = \begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix} \quad w_t = \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix}$$

Compute the turnover taking into account the effect of the realized returns $R_t = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}$

We have to use the formula

$$TO_t = \sum_{i=1}^{N} \left| w_{i,t} - w_{i,t-1}^{+} \right|$$

To apply it correctly we first need to compute $w_{i,t-1}^{+}$.

The first asset experienced a +10% returns, therefore we have $0.3 + 0.3 \times 0.1 = 0.33$

The second asset experienced a +20% returns, therefore we have $0.7 + 0.7 \times 0.2 = 0.84$

The weights now sum to 1.17. We need to normalize them so that they sum up to 1 again:

$$w_{i,t-1}^{+} = \begin{bmatrix} \frac{0.33}{1.17} \\ \frac{0.84}{1.17} \end{bmatrix} \approx \begin{bmatrix} 0.28 \\ 0.72 \end{bmatrix}$$

Now we can apply the formula and compute the turnover:

$$TO_t = |0.4 - 0.28| + |0.6 - 0.72| = 0.12 + 0.12 = 0.24$$

This means we need to trade 14% of our wealth in order to update the weights.