Portfolio Theory

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Lecture 7

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Mean-variance utility maximization

We can model an investor's behavior as an attempt to maximize a utility function.

A very simple one is the **linear utility function**:

$$U(V) = a + bV, \qquad b > 0$$

where $U(V)$ is the utility that the investor gets depending on the value V of the portfolio. Only the mean is considered.

Markowitz (1952) adds risk by assuming that the investor cares both about the mean and the variance of the portfolio returns, i.e. the investor has a **mean-variance utility**.

Mean-variance utility maximization

- Given a certain mean return, the utility increases as the variance (which quantifies risk) gets lower.
- Equivalently, given a certain level of variance, the utility increases with a higher mean return.

The decision in the trade-off between return and risk is quantified by a **risk aversion parameter** γ: a higher γ means that the investor is more risk averse and will therefore require a higher compensation for an increased risk.

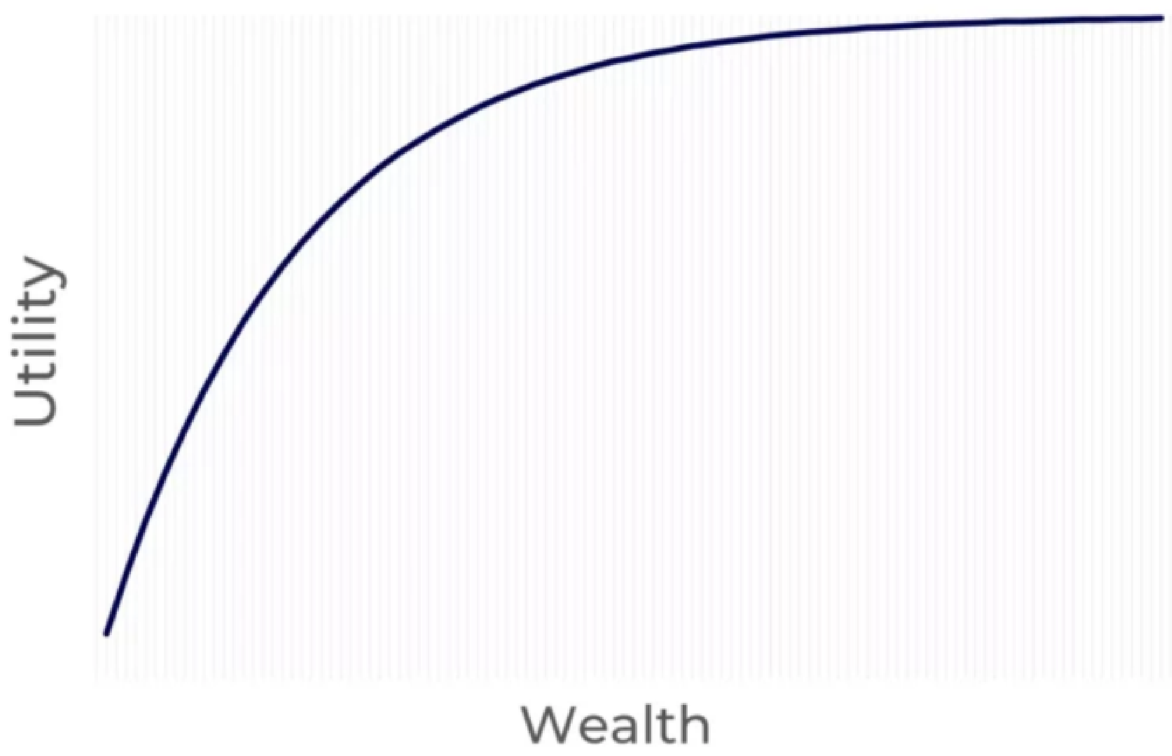
All else equal, the lower γ is, the more the investor will create a portfolio with a higher mean return but also a higher variance.

Mean-variance utility maximization

We model such preferences with a **quadratic utility function**:

$$U(V) = V - \frac{\gamma}{2}V^2, \qquad \gamma > 0$$

γ is assumed to be positive to have a concave utility function:



Mean-variance utility maximization

- $\gamma > 0$ The investor is risk-averse
- $\gamma = 0$ The investor is indifferent to risk
- $\gamma < 0$ The investor is risk-taker (i.e., prefers more risk with the same amount of wealth)

For a portfolio we have $\mu_P = \boldsymbol{w}'\boldsymbol{\mu} = V$, where $\boldsymbol{w}$ is the vector of portfolio weights and $\boldsymbol{\mu}$ is the vector of mean return of the single assets. Moreover, the variance is the expected value of the squared deviation from the mean.

So, the **mean-variance utility function** is:

$$U(\boldsymbol{w}) = \boldsymbol{w}'\boldsymbol{\mu} - \frac{\gamma}{2}\boldsymbol{w}'\Sigma\boldsymbol{w}$$

Mean-variance utility maximization

Given a risk-free asset and N risky assets with mean returns $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$, and a certain risk aversion coefficient γ, the investor wants to select the weights $\boldsymbol{w}$ in a way that maximizes the utility function:

$$\max_{\boldsymbol{w}} \boldsymbol{w}'\boldsymbol{\mu} - \frac{\gamma}{2}\boldsymbol{w}'\Sigma\boldsymbol{w}$$

We set the first-order condition, i.e., take the partial derivative with respect to $\boldsymbol{w}$ and set it equal to zero:

$$\frac{\partial U(\boldsymbol{w})}{\partial \boldsymbol{w}} = \boldsymbol{\mu} - \frac{2\gamma}{2}\Sigma\boldsymbol{w} = \boldsymbol{\mu} - \gamma\Sigma\boldsymbol{w} = \mathbf{0}$$

Mean-variance utility maximization

Solving for $\boldsymbol{w}$, we obtain the optimal weights for the risky assets:

$$\Sigma \boldsymbol{w} = \frac{1}{\gamma}\boldsymbol{\mu}$$

$$\boldsymbol{w_{mv}} = \frac{1}{\gamma}\Sigma^{-\mathbf{1}}\boldsymbol{\mu}$$

The weight for the risk-free asset is equal to $1 - \boldsymbol{w_{mv}}'\mathbf{1}$, where $\mathbf{1}$ is a vector of 1 with length equal to the number of risky assets.

The resulting optimal expected utility is:

$$U(\boldsymbol{w_{mv}}) = \frac{1}{2\gamma}\boldsymbol{\mu}'\Sigma^{-\mathbf{1}}\boldsymbol{\mu}$$

Mean-variance portfolio optimization

- Choosing a value for γ might be difficult for real investors
- The value of the utility U is not very informative

More intuitive approach: specify a desired mean return and minimize the variance.

This is a constrained optimization problem:

$$\min_{\boldsymbol{w}} \boldsymbol{w}'\Sigma\boldsymbol{w}$$

subject to:

$$\boldsymbol{w}'\boldsymbol{\mu} + (1 - \boldsymbol{w}'\mathbf{1})R_f = Re$$

$\boldsymbol{w}'\boldsymbol{\mu}$ is the return of the risky assets, and $(1 - \boldsymbol{w}'\mathbf{1})R_f$ is the return of the risk-free asset.

Mean-variance portfolio optimization

We solve it with the method of Lagrange multipliers.

First we define the Lagrangian function, i.e., a modified version of the objective function that incorporates the constraint in this way:

$$L(\boldsymbol{w},\lambda) = \boldsymbol{w}'\sum\boldsymbol{w} + \lambda\left[Re - \boldsymbol{w}'\boldsymbol{\mu} - (1-\boldsymbol{w}'\mathbf{1})R_f\right]$$

where λ is the Lagrange multiplier.

We can now solve an unconstrained problem instead of a constrained one by setting the first order conditions for the Lagrangian.

Mean-variance portfolio optimization

The conditions involve two simultaneous equations, as we have to compute the partial derivative both with respect to $\boldsymbol{w}$ and to λ:

$$\frac{\partial L}{\partial \boldsymbol{w}} = 2\Sigma\boldsymbol{w} - \lambda\boldsymbol{\mu} + \lambda R_f\mathbf{1} = \mathbf{0}$$

$$\frac{\partial L}{\partial \lambda} = Re - \boldsymbol{w}'\boldsymbol{\mu} - (1 - \boldsymbol{w}'\mathbf{1})R_f = 0$$

Solving this system for $\boldsymbol{w}$ gives the following closed-form solution:

$$\boldsymbol{w}_{\boldsymbol{mv}} = \frac{Re}{\boldsymbol{\mu}'\Sigma^{-\mathbf{1}}\boldsymbol{\mu}}\Sigma^{-\mathbf{1}}\boldsymbol{\mu}$$

Mean-variance portfolio optimization

It is also possible (and mathematically equivalent) to specify a given level of variance and maximize the mean return. However, it is more intuitive and more common to specify the desired mean and minimize the variance.

In practice, we do not need to explicitly include the risk-free asset in the asset menu, as it is equivalent and simpler to work with the excess returns of the risky assets.

Global minimum variance portfolio

In real applications $\boldsymbol{\mu}$ and $\sum$ need to be estimated.

Plug-in approach: the sample estimates of the inputs are computed from past data, and are then plugged into the optimization problem as if they were the true values.

Sample estimates can be very imprecise, especially that of the mean. Therefore, ignoring $\boldsymbol{\mu}$ and only minimizing the variance usually gives better results.

Therefore, the investor might want to compute the **global minimum variance portfolio (GMV)**, also simply called **minimum variance portfolio**.

Global minimum variance portfolio

This is a constrained optimization problem, as we need the weights of the risky assets to sum up to one, which means that nothing is invested in the risk-free asset (otherwise, everything would be invested in the risk-free asset):

$$\min_{\boldsymbol{w}} \boldsymbol{w}'\sum\boldsymbol{w}$$

subject to:

$$\boldsymbol{w}'\mathbf{1} = 1$$

We write the Lagrangian function:

$$L(\boldsymbol{w},\lambda) = \boldsymbol{w}'\sum\boldsymbol{w} + \lambda[1 - \boldsymbol{w}'\mathbf{1}]$$

Global minimum variance portfolio

It is convenient to multiply the first term by 0.5, which does not alter the result. So the Lagrangian becomes:

$$L(\boldsymbol{w}, \lambda) = \frac{1}{2}\boldsymbol{w}'\sum\boldsymbol{w} + \lambda[1 - \boldsymbol{w}'\mathbf{1}]$$

The first order conditions are:

$$\frac{\partial L}{\partial \boldsymbol{w}} = \sum\boldsymbol{w} - \lambda\mathbf{1} = \mathbf{0}$$

$$\frac{\partial L}{\partial \lambda} = 1 - \boldsymbol{w}'\mathbf{1} = 0$$

Global minimum variance portfolio

Through some simple rearrangement we get:

$$\boldsymbol{w} = \lambda \Sigma^{-1} \mathbf{1}$$

$$\boldsymbol{w}' \mathbf{1} = 1$$

In the first equation we multiply both sides by $\mathbf{1}'$:

$$\mathbf{1}' \boldsymbol{w} = \lambda \mathbf{1}' \Sigma^{-1} \mathbf{1}$$

From the second equation we know that:

$$\boldsymbol{w}' \mathbf{1} = \mathbf{1}' \boldsymbol{w} = 1$$

Hence, the first equation becomes:

$$1 = \lambda \mathbf{1}' \Sigma^{-1} \mathbf{1}$$

$$\lambda = \frac{1}{\mathbf{1}' \Sigma^{-1} \mathbf{1}}$$

Global minimum variance portfolio

We take this last result and replace λ in $\boldsymbol{w} = \lambda \Sigma^{-\mathbf{1}} \mathbf{1}$, obtaining the closed form-solution that gives the minimum variance portfolio weights:

$$\boldsymbol{w}_v = \frac{1}{\mathbf{1}' \Sigma^{-\mathbf{1}} \mathbf{1}} \Sigma^{-\mathbf{1}} \mathbf{1}$$

- As nothing is invested in the risk-free rate, it is equivalent to work with returns or excess returns.
- However, working with excess returns makes it easier to compare the results with those obtained by the mean-variance portfolio.

Portfolios that mitigate parameter uncertainty

To further reduce the impact of estimation errors, we can compute a **long-only minimum variance portfolio**.

In other words, we add a second constraint that requires all the weights be positive. This typically improves the results out-of-sample.

This problem does not have a closed form solution, but it can easily be solved in R using available packages.

Portfolios that mitigate parameter uncertainty

When a value for γ is specified, another strategy that mitigates the impact of the estimation error is the **1/N rule**.

The (sample) estimates of $\boldsymbol{\mu}$ and $\sum$ are used to optimally allocate the wealth between the risk-free asset and the equally weighted risky assets.

The weights for the risky assets using this strategy are:

$$\boldsymbol{w}_{\mathbf{1}/N} = \frac{1}{\gamma} \frac{\mathbf{1}'\boldsymbol{\mu}}{\mathbf{1}'\sum\mathbf{1}} \mathbf{1}$$

and the weight for the riskless asset is equal to $1 - \boldsymbol{w}'_{\mathbf{1}/N}\mathbf{1}$.

Portfolios that mitigate parameter uncertainty

For example, if $N = 5$ and this rule returns a weight of 0.15 for each risky asset, we equally divide 75% of our wealth among the risky assets (i.e., 15% on each risky asset), and then place the remaining 25% on the risk-free asset.

This rule typically performs very well in terms of mean-variance utility.