Efficient frontier

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Content

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Shape of the set of admissible portfolios

- The model of H. Markowitz;
  - Wealth is defined

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  - Assets are divisible without limits

Indivisible assets

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- Variants of possible portfolios are determined/limited by *pseudo*-short sell

Two components portfolio - Basic concept

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A multi-component portfolio

Expansion to **three** assets . . .

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, by

$$\rho_{1,2} = -1, \rho_{1,3} = 0, \rho_{2,3} = 1$$

3-assets Portfolio with varying proportions

⋮

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With the help of R . . .

Combination of a risky and a risky-free assets

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- Properties of ICs:
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  - Axiom of **IN-SATURATION**
  - Axiom of **RISK AVERSION**
- All investors are **risk averse**, but the level of aversion is **individual**

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  - There is no portfolio in the set of **permissible** portfolios that has a higher return for the same or lower risk

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