PORTFOLIO THEORY – EXERCISES 3

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EXERCISE 1

An investor with risk-aversion $\gamma = 4$ invests in a portfolio of one risk-free asset and two risky assets whose excess returns have mean vector, covariance matrix and inverse covariance matrix equal to:

$$\mu = \begin{bmatrix} 0.006 \\ 0.004 \end{bmatrix} \quad \boldsymbol{\Sigma} = \begin{bmatrix} 0.003 & 0.002 \\ 0.002 & 0.0015 \end{bmatrix} \quad \boldsymbol{\Sigma^{-1}} \approx \begin{bmatrix} 3000 & -4000 \\ -4000 & 6000 \end{bmatrix}$$

Compute the portfolio weights that maximize mean-variance utility, and the corresponding utility.

The weights for the risky assets are:

$$\boldsymbol{w_U} = \frac{1}{\gamma}\boldsymbol{\Sigma^{-1}\mu} = \frac{1}{4}\begin{bmatrix} 3000 & -4000 \\ -4000 & 6000 \end{bmatrix}\begin{bmatrix} 0.006 \\ 0.004 \end{bmatrix} = \begin{bmatrix} 750 & -1000 \\ -1000 & 1500 \end{bmatrix}\begin{bmatrix} 0.006 \\ 0.004 \end{bmatrix}$$
$$= \begin{bmatrix} 750 * 0.006 + (-1000) * 0.004 \\ -1000 * 0.006 + 1500 * 0.004 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}$$

The weight for the risk-free asset is: $1 - 0.5 = 0.5$

The formula we used maximizes the utility given by $U(\boldsymbol{w}) = \boldsymbol{w'\mu} - \frac{\gamma}{2}\boldsymbol{w'}\Sigma\boldsymbol{w}$. Therefore, the utility is:

$$U(\boldsymbol{w}) = \boldsymbol{w'\mu} - \frac{\gamma}{2}\boldsymbol{w'}\Sigma\boldsymbol{w} = [0.5 \quad 0]\begin{bmatrix} 0.006 \\ 0.004 \end{bmatrix} - \frac{4}{2}[0.5 \quad 0]\begin{bmatrix} 0.003 & 0.002 \\ 0.002 & 0.0015 \end{bmatrix}\begin{bmatrix} 0.5 \\ 0 \end{bmatrix}$$

$$= 0.5 * 0.006 + 0 * 0.004 - 2 * [0.5 * 0.003 + 0 * 0.002 \quad 0.5 * 0.002 + 0 * 0.0015]\begin{bmatrix} 0.5 \\ 0 \end{bmatrix}$$

$$= 0.003 - 2 * [0.0015 \quad 0.001]\begin{bmatrix} 0.5 \\ 0 \end{bmatrix} = 0.003 - 2 * 0.0015 * 0.5 = 0.003 - 0.0015 = 0.0015$$

Equivalently, we can use the formula for the utility of an optimal mean-variance portfolio, <u>which is only valid for a portfolio with optimal weights</u>:

$$U(\boldsymbol{w_U}) = \frac{1}{2\gamma}\boldsymbol{\mu'}\Sigma^{-1}\boldsymbol{\mu} = \frac{1}{2 * 4}[0.006 \quad 0.004]\begin{bmatrix} 3000 & -4000 \\ -4000 & 6000 \end{bmatrix}\begin{bmatrix} 0.006 \\ 0.004 \end{bmatrix}$$
$$= \frac{1}{8}[0.006 * 3000 + 0.004 * (-4000) \quad 0.006 * (-4000) + 0.004 * 6000]\begin{bmatrix} 0.006 \\ 0.004 \end{bmatrix}$$

$$= \frac{1}{8}[2 \quad 0]\begin{bmatrix} 0.006 \\ 0.004 \end{bmatrix} = \frac{1}{8} * 2 * 0.006 = 0.0015$$

EXERCISE 2

An investor with risk-aversion $\gamma = 2$ invests in a portfolio of one risk-free asset and two risky assets whose excess returns have mean vector, covariance matrix and inverse covariance matrix equal to:

$$\mu = \begin{bmatrix} 0.005 \\ 0.004 \end{bmatrix} \quad \boldsymbol{\Sigma} = \begin{bmatrix} 0.003 & 0.002 \\ 0.002 & 0.0015 \end{bmatrix} \quad \boldsymbol{\Sigma^{-1}} \approx \begin{bmatrix} 3000 & -4000 \\ -4000 & 6000 \end{bmatrix}$$

Compute the portfolio weights that maximize mean-variance utility given a full-investment constraint.

The weights are given by:

$$\boldsymbol{w}_{U^*} = \frac{\Sigma^{-1}}{\gamma}\left(\boldsymbol{\mu} + \frac{\gamma - \boldsymbol{\mu}'\Sigma^{-1}\mathbf{1}}{\mathbf{1}'\Sigma^{-1}\mathbf{1}}\mathbf{1}\right)$$

$$= \frac{1}{2}\begin{bmatrix} 3000 & -4000 \\ -4000 & 6000 \end{bmatrix}\left\{\begin{bmatrix} 0.005 \\ 0.004 \end{bmatrix} + \frac{2 - [0.005 \quad 0.004]\begin{bmatrix} 3000 & -4000 \\ -4000 & 6000 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}{[1 \quad 1]\begin{bmatrix} 3000 & -4000 \\ -4000 & 6000 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$$

$$= \begin{bmatrix} 1500 & -2000 \\ -2000 & 3000 \end{bmatrix}\left\{\begin{bmatrix} 0.005 \\ 0.004 \end{bmatrix} + \frac{2 - [0.005 \quad 0.004]\begin{bmatrix} 3000 * 1 - 4000 * 1 \\ -4000 * 1 + 6000 * 1 \end{bmatrix}}{[1 \quad 1]\begin{bmatrix} 3000 * 1 - 4000 * 1 \\ -4000 * 1 + 6000 * 1 \end{bmatrix}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$$

$$= \begin{bmatrix} 1500 & -2000 \\ -2000 & 3000 \end{bmatrix}\left\{\begin{bmatrix} 0.005 \\ 0.004 \end{bmatrix} + \frac{2 - [0.005 \quad 0.004]\begin{bmatrix} -1000 \\ 2000 \end{bmatrix}}{[1 \quad 1]\begin{bmatrix} -1000 \\ 2000 \end{bmatrix}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$$

$$= \begin{bmatrix} 1500 & -2000 \\ -2000 & 3000 \end{bmatrix}\left\{\begin{bmatrix} 0.005 \\ 0.004 \end{bmatrix} + \frac{2 - [0.005 * (-1000) + 0.004 * 2000]}{-1000 + 2000}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$$

$$= \begin{bmatrix} 1500 & -2000 \\ -2000 & 3000 \end{bmatrix}\left\{\begin{bmatrix} 0.005 \\ 0.004 \end{bmatrix} + \frac{-1}{1000}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\} = \begin{bmatrix} 1500 & -2000 \\ -2000 & 3000 \end{bmatrix}\left\{\begin{bmatrix} 0.005 \\ 0.004 \end{bmatrix} + \begin{bmatrix} -0.001 \\ -0.001 \end{bmatrix}\right\}$$

$$= \begin{bmatrix} 1500 & -2000 \\ -2000 & 3000 \end{bmatrix}\begin{bmatrix} 0.004 \\ 0.003 \end{bmatrix} = \begin{bmatrix} 1500 * 0.004 - 2000 * 0.003 \\ -2000 * 0.004 + 3000 * 0.003 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

EXERCISE 3

Given a risk-free asset and two risky assets whose excess returns have mean vector, covariance matrix and inverse covariance matrix equal to

$$\mu = \begin{bmatrix} 0.01 \\ 0.008 \end{bmatrix} \qquad \boldsymbol{\Sigma} = \begin{bmatrix} 0.004 & 0.002 \\ 0.002 & 0.003 \end{bmatrix} \qquad \boldsymbol{\Sigma}^{-1} = \begin{bmatrix} 375 & -250 \\ -250 & 500 \end{bmatrix}$$

compute the weights for the optimal mean-variance portfolio with target return $Re = 0.01$.

The weights for the risky assets are:

$$\boldsymbol{w}_{mv} = \frac{Re}{\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} = \frac{0.01}{[0.01 \quad 0.008]\begin{bmatrix} 375 & -250 \\ -250 & 500 \end{bmatrix}\begin{bmatrix} 0.01 \\ 0.008 \end{bmatrix}}\begin{bmatrix} 375 & -250 \\ -250 & 500 \end{bmatrix}\begin{bmatrix} 0.01 \\ 0.008 \end{bmatrix}$$

$$= \frac{0.01}{[0.01 * 375 + 0.008 * (-250) \quad 0.01 * (-250) + 0.008 * 500]\begin{bmatrix} 0.01 \\ 0.008 \end{bmatrix}}\begin{bmatrix} 375 * 0.01 + (-250) * 0.008 \\ -250 * 0.01 + 500 * 0.008 \end{bmatrix}$$

$$= \frac{0.01}{[1.75 \quad 1.5]\begin{bmatrix} 0.01 \\ 0.008 \end{bmatrix}}\begin{bmatrix} 1.75 \\ 1.5 \end{bmatrix} = \frac{0.01}{0.0175 + 0.012}\begin{bmatrix} 1.75 \\ 1.5 \end{bmatrix} = \frac{0.01}{0.0295}\begin{bmatrix} 1.75 \\ 1.5 \end{bmatrix} \approx \begin{bmatrix} 0.59 \\ 0.51 \end{bmatrix}$$

The weight for the risk-free asset is $1 - (0.59 + 0.51) = -0.1$

EXERCISE 4

Given a risk-free asset and two risky assets whose excess returns have mean vector, covariance matrix and inverse covariance matrix equal to

$$\mu = \begin{bmatrix} 0.01 \\ 0.008 \end{bmatrix} \qquad \boldsymbol{\Sigma} = \begin{bmatrix} 0.004 & 0.002 \\ 0.002 & 0.003 \end{bmatrix} \qquad \boldsymbol{\Sigma^{-1}} = \begin{bmatrix} 375 & -250 \\ -250 & 500 \end{bmatrix}$$

compute the weights for the optimal mean-variance portfolio with target return $Re = 0.01$ *given a full-investment constraint.*

The weights are:

$$\boldsymbol{w_{mv^*}} = \Sigma^{-\mathbf{1}} \left[\frac{CRe - B}{AC - B^2} \boldsymbol{\mu} + \frac{A - BRe}{AC - B^2} \mathbf{1} \right]$$

where $A = \boldsymbol{\mu}' \Sigma^{-\mathbf{1}} \boldsymbol{\mu}$, $B = \mathbf{1}' \Sigma^{-\mathbf{1}} \boldsymbol{\mu}$ and $C = \mathbf{1}' \Sigma^{-\mathbf{1}} \mathbf{1}$.

First we compute the three scalars:

$$A = \boldsymbol{\mu}' \Sigma^{-\mathbf{1}} \boldsymbol{\mu} = [0.01 \quad 0.008] \begin{bmatrix} 375 & -250 \\ -250 & 500 \end{bmatrix} \begin{bmatrix} 0.01 \\ 0.008 \end{bmatrix} =$$

$$= [0.01 * 375 + 0.008 * (-250) \quad 0.01 * (-250) + 0.008 * 500] \begin{bmatrix} 0.01 \\ 0.008 \end{bmatrix} =$$

$$= [1.75 \quad 1.5] \begin{bmatrix} 0.01 \\ 0.008 \end{bmatrix} = 1.75 * 0.01 + 1.5 * 0.008 = 0.0295$$

$$B = \mathbf{1}' \Sigma^{-\mathbf{1}} \boldsymbol{\mu} = [1 \quad 1] \begin{bmatrix} 375 & -250 \\ -250 & 500 \end{bmatrix} \begin{bmatrix} 0.01 \\ 0.008 \end{bmatrix} =$$

$$= [1 * 375 + 1 * (-250) \quad 1 * (-250) + 1 * 500] \begin{bmatrix} 0.01 \\ 0.008 \end{bmatrix} =$$

$$= [125 \quad 250] \begin{bmatrix} 0.01 \\ 0.008 \end{bmatrix} = 125 * 0.01 + 250 * 0.008 = 3.25$$

$$C = \mathbf{1}' \Sigma^{-\mathbf{1}} \mathbf{1} = [1 \quad 1] \begin{bmatrix} 375 & -250 \\ -250 & 500 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} =$$

$$= [1 * 375 + 1 * (-250) \quad 1 * (-250) + 1 * 500] \begin{bmatrix} 1 \\ 1 \end{bmatrix} =$$

$$= [125 \quad 250] \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 125 * 1 + 250 * 1 = 375$$

Now we can compute the weights:

$$\boldsymbol{w_{mv^*}} = \Sigma^{-\mathbf{1}} \left[\frac{CRe - B}{AC - B^2} \boldsymbol{\mu} + \frac{A - BRe}{AC - B^2} \mathbf{1} \right]$$

$$= \begin{bmatrix} 375 & -250 \\ -250 & 500 \end{bmatrix} \left\{ \frac{375 * 0.01 - 3.25}{0.0295 * 375 - 10.5625} \begin{bmatrix} 0.01 \\ 0.008 \end{bmatrix} + \frac{0.0295 - 3.25 * 0.01}{0.0295 * 375 - 10.5625} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

$$= \begin{bmatrix} 375 & -250 \\ -250 & 500 \end{bmatrix} \left\{ \frac{0.5}{0.5} \begin{bmatrix} 0.01 \\ 0.008 \end{bmatrix} + \frac{-0.003}{0.5} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

$$= \begin{bmatrix} 375 & -250 \\ -250 & 500 \end{bmatrix} \left\{ \begin{bmatrix} 0.01 \\ 0.008 \end{bmatrix} + \begin{bmatrix} -0.006 \\ -0.006 \end{bmatrix} \right\} = \begin{bmatrix} 375 & -250 \\ -250 & 500 \end{bmatrix} \begin{bmatrix} 0.004 \\ 0.002 \end{bmatrix}$$

$$= \begin{bmatrix} 375 * 0.004 - 250 * 0.002 \\ -250 * 0.004 + 500 * 0.002 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

EXERCISE 5

Given a risk-free asset and two risky assets whose excess returns have mean vector, covariance matrix and inverse covariance matrix equal to

$$\mu = \begin{bmatrix} 0.01 \\ 0.008 \end{bmatrix} \qquad \boldsymbol{\Sigma} = \begin{bmatrix} 0.004 & 0.002 \\ 0.002 & 0.003 \end{bmatrix} \qquad \boldsymbol{\Sigma}^{-\mathbf{1}} = \begin{bmatrix} 375 & -250 \\ -250 & 500 \end{bmatrix}$$

compute the tangency portfolio.

The weights are given by:

$$\boldsymbol{w_{tan}} = \frac{\Sigma^{-\mathbf{1}}\boldsymbol{\mu}}{\mathbf{1}'\Sigma^{-\mathbf{1}}\boldsymbol{\mu}} = \frac{\begin{bmatrix} 375 & -250 \\ -250 & 500 \end{bmatrix} \begin{bmatrix} 0.01 \\ 0.008 \end{bmatrix}}{\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 375 & -250 \\ -250 & 500 \end{bmatrix} \begin{bmatrix} 0.01 \\ 0.008 \end{bmatrix}} = \frac{\begin{bmatrix} 375 * 0.01 - 250 * 0.008 \\ -250 * 0.01 + 500 * 0.008 \end{bmatrix}}{\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 375 * 0.01 - 250 * 0.008 \\ -250 * 0.01 + 500 * 0.008 \end{bmatrix}}$$

$$= \frac{\begin{bmatrix} 1.75 \\ 1.5 \end{bmatrix}}{\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1.75 \\ 1.5 \end{bmatrix}} = \frac{\begin{bmatrix} 1.75 \\ 1.5 \end{bmatrix}}{1 * 1.75 + 1 * 1.5} = \frac{1}{3.25} \begin{bmatrix} 1.75 \\ 1.5 \end{bmatrix} \approx \begin{bmatrix} 0.54 \\ 0.46 \end{bmatrix}$$

EXERCISE 6

Given two risky assets with mean return, covariance matrix and inverse covariance matrix equal to

$$\mu = \begin{bmatrix} 0.007 \\ 0.004 \end{bmatrix} \qquad \boldsymbol{\Sigma} = \begin{bmatrix} 0.004 & 0.002 \\ 0.002 & 0.003 \end{bmatrix} \qquad \boldsymbol{\Sigma}^{-\mathbf{1}} = \begin{bmatrix} 375 & -250 \\ -250 & 500 \end{bmatrix}$$

compute the weights for the minimum variance portfolio.

The weights are:

$$\boldsymbol{w_v} = \frac{1}{\mathbf{1}'\boldsymbol{\Sigma}^{-\mathbf{1}}\mathbf{1}} \boldsymbol{\Sigma}^{-\mathbf{1}}\mathbf{1} = \frac{1}{\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 375 & -250 \\ -250 & 500 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}} \begin{bmatrix} 375 & -250 \\ -250 & 500 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{\begin{bmatrix} 1 * 375 + 1 * (-250) & 1 * (-250) + 1 * 500 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}} \begin{bmatrix} 375 * 1 + (-250) * 1 \\ -250 * 1 + 500 * 1 \end{bmatrix}$$

$$= \frac{1}{\begin{bmatrix} 125 & 250 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}} \begin{bmatrix} 125 \\ 250 \end{bmatrix} = \frac{1}{125 + 250} \begin{bmatrix} 125 \\ 250 \end{bmatrix} = \frac{1}{375} \begin{bmatrix} 125 \\ 250 \end{bmatrix} \approx \begin{bmatrix} 0.33 \\ 0.67 \end{bmatrix}$$

EXERCISE 7

A mean-variance utility investor with risk-aversion coefficient $\gamma = 5$ can invest in two risky assets and one risk-free asset. The mean and covariance matrix of the excess returns of the two risky assets are:

$$\mu = \begin{bmatrix} 0.01 \\ 0.008 \end{bmatrix} \quad \Sigma = \begin{bmatrix} 0.005 & 0.002 \\ 0.002 & 0.003 \end{bmatrix}$$

Compute the weights for the 1/N portfolio that optimally allocates the wealth between the risky assets and the risk-free asset.

We compute the weights for the risky assets:

$$\boldsymbol{w_{1/N}} = \frac{1}{\gamma}\frac{\mathbf{1}'\boldsymbol{\mu}}{\mathbf{1}'\Sigma\mathbf{1}}\mathbf{1} = \frac{1}{5}\frac{\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 0.01 \\ 0.008 \end{bmatrix}}{\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 0.005 & 0.002 \\ 0.002 & 0.003 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{5}\frac{1 * 0.01 + 1 * 0.008}{\begin{bmatrix} 1 * 0.005 + 1 * 0.002 & 1 * 0.002 + 1 * 0.003 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{5}\frac{0.018}{\begin{bmatrix} 0.007 & 0.005 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{5}\frac{0.018}{0.007 * 1 + 0.005 * 1}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{5}\frac{0.018}{0.007 * 1 + 0.005 * 1}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{5}\frac{0.018}{0.012}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{0.018}{0.06}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0.3\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.3 \\ 0.3 \end{bmatrix}$$

The two risky assets get a weight of 0.3 each, so the weight of the risk-free asset is:

$1 - (0.3 + 0.3) = 0.4$