PORTFOLIO THEORY – EXERCISES 4

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EXERCISE 1

Given the following series of returns

$R_1 = 0.05,\ R_2 = -0.02,\ R_3 = -0.01,\ R_4 = 0.1$

and the following series of risk-free rates

$R_{f,1} = 0.005,\ R_{f,2} = 0.005,\ R_{f,3} = 0,\ R_{f,4} = 0$

compute the Sharpe ratio

The Sharpe ratio is given by:

$$SR = \frac{E[R_p - R_f]}{SD(R_p - R_f)} = \frac{\text{mean excess return}}{\text{standard deviation of excess returns}}$$

First, we compute the excess returns:

$$R_{ex,1} = 0.05 - 0.005 = 0.045$$

$$R_{ex,2} = -0.02 - 0.005 = -0.025$$

$$R_{ex,3} = -0.01 - 0 = -0.01$$

$$R_{ex,4} = 0.1 - 0 = 0.1$$

Then we compute the mean excess return and the standard deviation of excess returns:

$$\overline{R_{ex}} = \frac{0.045 - 0.025 - 0.01 + 0.1}{4} = 0.0275$$

$$\sigma^2 = \frac{1}{T-1}\sum_{t=1}^{T}\left(R_{ex,t} - \overline{R_{ex}}\right)^2$$

$$= \frac{(0.045 - 0.0275)^2 + (-0.025 - 0.0275)^2 + (-0.01 - 0.0275)^2 + (0.1 - 0.0275)^2}{3}$$

$$\approx \frac{0.0003 + 0.0028 + 0.0014 + 0.0053}{3} \approx 0.0032$$

$$\sigma = \sqrt{0.0032} \approx 0.0566$$

So, the Sharpe ratio is:

$$SR = \frac{0.0275}{0.0566} \approx 0.48$$

EXERCISE 2

Given the following series of returns, compute the downside deviation with benchmark $B = 0.01$

$R_1 = 0.04,\ R_2 = -0.02,\ R_3 = -0.01,\ R_4 = 0.1,\ R_5 = 0.005$

The semivariance is given by:

$$\sigma_B^2 = \frac{1}{T}\sum_{t=1}^{T}[\mathrm{Min}(R_t - B, 0)]^2$$

which means that every return above the benchmark must be replaced with 0 in the computations. Therefore, we subtract the benchmark from each return:

$$R_1 - B = 0.04 - 0.01 = 0.03$$

$$R_2 - B = -0.02 - 0.01 = -0.03$$

$$R_3 - B = -0.01 - 0.01 = -0.02$$

$$R_4 - B = 0.1 - 0.01 = 0.09$$

$$R_5 - B = 0.005 - 0.01 = -0.005$$

Now we apply the formula:

$$\sigma_B^2 = \frac{0 + (-0.03)^2 + (-0.02)^2 + 0 + (-0.005)^2}{5} = 0.000265$$

From this we get the downside deviation, which is simply the square root of the semivariance:

$$\sigma_B = \sqrt{0.000265} \approx 0.0163$$

EXERCISE 3

Consider the set of weights

$$w_{t-1} = \begin{bmatrix} 0.2 \\ 0.4 \\ 0.1 \end{bmatrix} \quad w_t = \begin{bmatrix} 0.2 \\ 0.2 \\ 0.1 \end{bmatrix}$$

Compute the turnover taking into account the effect of the realized returns $R_{t-1} = \begin{bmatrix} -0.1 \\ 0.05 \\ 0.2 \end{bmatrix}$

We have to use the formula

$$TO_t = \sum_{i=1}^{N}\left|w_{i,t} - w_{i,t-1}^{+}\right|$$

where

$$w_{i,t-1}^{+} = \frac{(w_{i,t-1} + w_{i,t-1} \times R_{i,t-1}) \times \sum_{i=1}^{N} w_{i,t-1}}{\sum_{i=1}^{N}\left(w_{i,t-1} + w_{i,t-1} \times R_{i,t-1}\right)}$$

First, we compute the second term in the numerator, and the term in the denominator:

$$\sum_{i=1}^{N} w_{i,t-1} = 0.2 + 0.4 + 0.1 = 0.7$$

$$\sum_{i=1}^{N} \left(w_{i,t-1} + w_{i,t-1} \times R_{i,t-1}\right) = [0.2 + 0.2 \times (-0.1)] + [0.4 + 0.4 \times 0.05] + [0.1 + 0.1 \times 0.2] = 0.72$$

Now we compute the weights in the previous periods accounting for the realized returns:

$$w_{1,t-1}^{+} = \frac{0.2 + 0.2 \times (-0.1)}{0.72} \times 0.7 = \frac{0.18}{0.72} \times 0.7 = 0.175$$

$$w_{2,t-1}^{+} = \frac{0.4 + 0.4 \times 0.05}{0.72} \times 0.7 = \frac{0.42}{0.72} \times 0.7 \approx 0.408$$

$$w_{3,t-1}^{+} = \frac{0.1 + 0.1 \times 0.2}{0.72} \times 0.7 = \frac{0.12}{0.72} \times 0.7 \approx 0.117$$

We can finally apply the formula for the turnover:

$$TO_t = \sum_{i=1}^{N} \left|w_{i,t} - w_{i,t-1}^{+}\right| = |0.2 - 0.175| + |0.2 - 0.408| + |0.1 - 0.117| = 0.25$$

This means we need to trade 25% of our wealth in order to update the weights.

EXERCISE 4

Consider an equally weighted portfolio of two assets, A and B, which experience the following monthly returns over three periods:

$R_{A,1} = 0.1,\ R_{A,2} = -0.05,\ R_{A,3} = 0.15$

$R_{B,1} = 0,\ R_{B,2} = 0.05,\ R_{B,3} = 0.1$

There are proportional transaction costs equal to 10 basis points.

If we invested 10000 euro in such portfolio (i.e., 5000 in A and 5000 in B) at $t = 0$, how much money would we have at period $t = 3$, net of transaction costs (ignore the initial transaction costs required to start investing at time $t = 0$)?

To compute the portfolio returns net of transaction costs we first need the turnover at each period:

$$TO_t = \sum_{i=1}^{N} \left|w_{i,t} - w_{i,t-1}^{+}\right|$$

where the formula for $w_{i,t-1}^{+}$ simplifies to

$$w_{i,t-1}^{+} = \frac{w_{i,t-1} + w_{i,t-1} \times R_{i,t-1}}{\sum_{i=1}^{N}\left(w_{i,t-1} + w_{i,t-1} \times R_{i,t-1}\right)}$$

because the weights always sum to 1.

At time $t = 1$ the turnover is zero. Therefore, we start from time $t = 2$:

$$w_{A,1}^{+} = \frac{0.5 + 0.5 \times 0.1}{(0.5 + 0.5 \times 0.1) + (0.5 + 0.5 \times 0)} = \frac{0.55}{0.55 + 0.5} \approx 0.524$$

$$w_{B,1}^{+} = \frac{0.5 + 0.5 \times 0}{(0.5 + 0.5 \times 0.1) + (0.5 + 0.5 \times 0)} = \frac{0.5}{0.55 + 0.5} \approx 0.476$$

So, the turnover at time $t = 2$ is

$$TO_2 = \sum_{i=1}^{2} \left| w_{i,2} - w_{i,1}^{+} \right| = |0.5 - 0.524| + |0.5 - 0.476| = 0.048$$

and the transaction costs at time $t = 2$ are

$$TC_2 = 0.048 \times 0.001 = 0.000048$$

We do the same computations for $t = 3$:

$$w_{A,2}^{+} = \frac{0.5 + 0.5 \times (-0.05)}{[0.5 + 0.5 \times (-0.05)] + (0.5 + 0.5 \times 0.05)} = \frac{0.475}{1} = 0.475$$

$$w_{B,2}^{+} = \frac{0.5 + 0.5 \times 0.05}{[0.5 + 0.5 \times (-0.05)] + (0.5 + 0.5 \times 0.05)} = \frac{0.525}{1} = 0.525$$

So, the turnover at time $t = 3$ is

$$TO_3 = \sum_{i=1}^{3} \left| w_{i,3} - w_{i,2}^{+} \right| = |0.5 - 0.475| + |0.5 - 0.525| = 0.05$$

and the transaction costs at time $t = 3$ are

$$TC_3 = 0.05 \times 0.001 = 0.00005$$

We can now compute the returns net of transaction costs in all three periods:

$$R_{1,net} = 0.5 \times 0.1 + 0.5 \times 0 - 0 = 0.05$$

$$R_{2,net} = 0.5 \times (-0.05) + 0.5 \times 0.05 - 0.000048 = -0.000048$$

$$R_{3,net} = 0.5 \times 0.15 + 0.5 \times 0.1 - 0.00005 = 0.12495$$

Finally, we compute the value of the investment at $t = 3$:

$$V_3 = V_0 + V_0 \left[\prod_{t=1}^{3} \left(1 + R_{t,net}\right) - 1 \right]$$

$$= 10000 + 10000[(1 + 0.05)(1 - 0.000048)(1 + 0.12495) - 1] = 11811.41$$