Portfolio Theory

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Lecture 7

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- Equally weighted portfolio

Global minimum variance portfolio

In real applications $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ need to be estimated.

Plug-in approach: the sample estimates of the inputs are computed from past data and are then plugged into the optimization problem as if they were the true values.

Sample estimates can be very imprecise, especially that of the mean. Therefore, ignoring $\boldsymbol{\mu}$ and only minimizing the variance usually gives better results.

Therefore, the investor might want to compute the **global minimum variance portfolio (GMV)**, also simply called **minimum variance portfolio**.

Global minimum variance portfolio

The weights of the risky assets must sum up to 1, otherwise everything would be invested in the risk-free asset:

$$\min_{\boldsymbol{w}} \boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w}$$

$$\text{subject to: } \boldsymbol{w}'\mathbf{1} = 1$$

The Lagrangian and the first order conditions are:

$$L(\boldsymbol{w}, \lambda) = \frac{1}{2}\boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w} + \lambda[1 - \boldsymbol{w}'\mathbf{1}]$$

$$\frac{\partial L}{\partial \boldsymbol{w}} = \boldsymbol{\Sigma}\boldsymbol{w} - \lambda\mathbf{1} = \mathbf{0}$$

$$\frac{\partial L}{\partial \lambda} = 1 - \boldsymbol{w}'\mathbf{1} = 0$$

Global minimum variance portfolio

Through some simple rearrangement we get:

$$\boldsymbol{w} = \lambda \boldsymbol{\Sigma}^{-1}\mathbf{1}$$

$$\boldsymbol{w}'\mathbf{1} = 1$$

In the first equation we multiply both sides by $\mathbf{1}'$:

$$\mathbf{1}'\boldsymbol{w} = \lambda \mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}$$

From the second equation we know that:

$$\boldsymbol{w}'\mathbf{1} = \mathbf{1}'\boldsymbol{w} = 1$$

Hence, the first equation becomes:

$$1 = \lambda \mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}$$

Global minimum variance portfolio

$$\lambda = \frac{1}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}}$$

We take this last result and replace λ in $\boldsymbol{w} = \lambda\boldsymbol{\Sigma}^{-1}\mathbf{1}$, obtaining the solution:

$$\boldsymbol{w}_v = \frac{1}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}}\boldsymbol{\Sigma}^{-1}\mathbf{1}$$

- As nothing is invested in the risk-free rate, it is equivalent to work with returns or excess returns.
- However, working with excess returns makes it easier to compare the results with those obtained by the mean-variance portfolio.

Global minimum variance portfolio

To further reduce the impact of estimation errors, we can compute a **long-only minimum variance portfolio**:

$$\min_{\boldsymbol{w}} \boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w}$$

subject to:

$$\boldsymbol{w}'\mathbf{1} = 1$$

$$\boldsymbol{w} \geq \mathbf{0}$$

This problem does not have a closed form solution, but it can easily be solved in R using available packages.

This portfolio is theoretically sub-optimal, but in reality it typically improves the results out-of-sample.

Equally weighted portfolio

When a value for γ is specified, another strategy that mitigates the impact of estimation error is the **1/N rule**.

$\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are used to optimally allocate the wealth between the risk-free asset and the equally weighted risky assets.

The mean-variance utility optimization problem is

$$\max_{\boldsymbol{w}} \boldsymbol{w}'\boldsymbol{\mu} - \frac{\gamma}{2}\boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w}$$

But if all the risky assets must have the same weight, it means we are imposing $\boldsymbol{w} = c\mathbf{1}$, and we need to find the value of the scalar c that determines the weights.

Equally weighted portfolio

Hence, we substitute $\boldsymbol{w} = c\mathbf{1}$ in the original problem:

$$\max_{c} \; (c\mathbf{1})'\boldsymbol{\mu} - \frac{\gamma}{2}(c\mathbf{1})'\boldsymbol{\Sigma}(c\mathbf{1})$$

$$\max_{c} c\mathbf{1}'\boldsymbol{\mu} - \frac{\gamma}{2}c^2\mathbf{1}'\boldsymbol{\Sigma}\mathbf{1}$$

The first order condition of this unconstrained problem is:

$$\frac{\partial U(c)}{\partial c} = \mathbf{1}'\boldsymbol{\mu} - \gamma c\mathbf{1}'\boldsymbol{\Sigma}\mathbf{1} = \mathbf{0}$$

from which we easily get

$$c = \frac{1}{\gamma}\frac{\mathbf{1}'\boldsymbol{\mu}}{\mathbf{1}'\boldsymbol{\Sigma}\mathbf{1}}$$

Equally weighted portfolio

We then get the optimal weights for the risky assets by simply plugging this expression into $\boldsymbol{w} = c\mathbf{1}$:

$$\boldsymbol{w}_{\mathbf{1}/N} = \frac{1}{\gamma}\frac{\mathbf{1}'\boldsymbol{\mu}}{\mathbf{1}'\boldsymbol{\Sigma}\mathbf{1}}\mathbf{1}$$

while the weight for the riskless asset is $1 - \boldsymbol{w}'_{\mathbf{1}/N}\mathbf{1}$.

For example, if $N = 5$ and this rule returns a weight of 0.15 for each risky asset, we equally divide 75% of our wealth among the risky assets (i.e., 15% on each risky asset), and then place the remaining 25% on the risk-free asset.