Jean-Philippe Bouchaud, Lecture 3: Models with Time-Varying Volatility

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In this lecture we study several different financial time series models which include time-varying volatility. We study the resulting effects on the autocorrelations and moments of the distribution.

1 Time-Varying Volatility with Volatility Correlation

In the first model we define the increments δx_i by

$$\delta x_i = \epsilon_i \sigma_i \tag{1}$$

The cumulative increment Δx_N is given by

$$\Delta x_N = \sum_{i=1}^{N} \delta x_i \tag{2}$$

where

$$< \sigma_i^2 \sigma_j^2 > - < \sigma_i^2 >^2 \quad \sim \quad \frac{1}{|i-j|^\nu} \tag{3}$$

for $\nu < 1$. In this model, we have for the second and fourth moments

$$< \Delta x_N^2 > \quad \sim \quad N \tag{4}$$

$$< \Delta x_N^4 > \quad \sim \quad N^2 + N^{2-\nu} \tag{5}$$

which implies for the kurtosis

$$\kappa = \frac{< \Delta x_N^4 > - < \Delta x_N^2 >^2}{< \Delta x_N^2 >^2} \tag{6}$$

(7) $$\kappa \sim N^{-\nu}$$

Note that for Gaussian iid increments, we have

(8) $$< \Delta x_N^q > \sim N^{q/2}$$

and more generally,

(9) $$< \Delta x_N^q > \sim N^{\zeta(q)}$$

where $\zeta(q)$ is a characteristic of the specific distribution.

2 Multi-Fracticality

We next study the multi-fractal model of Bacry-Delour-Muzy, which may be found online at http://xxx.lanl.gov/archive/cond-mat. In this model, we have

(10) $$\delta x_i = \epsilon_i e^{w_i}$$

where the w_i's are Gaussian, and

(11) $$< w_i w_j >= -\lambda^2 \log \frac{T}{|i-j|+1}$$

for $|i-j| < T$, where T is a parameter to be specified for the series. For $|i-j| > T$, we have $< w_i w_j >= 0$. The moments of this time series have the form

(12) $$< \Delta x^q > \sim A_q N^{\zeta(q)}$$

for $1 \ll N \ll T$. That is, a large number of steps have been taken, but still many less steps than the correlation time T. The exponent $\zeta(q)$ is given by

(13) $$\zeta(q) = \frac{q}{2} - \lambda^2 q(q-2)$$

for $q < q^*$, where q^* is a parameter not specified in the lecture. For $q > q^*$, the moments diverge.

3 Relative vs. Absolute Increments, "Retarded Volatility" Model

Time-dependent volatility models may describe the absolute or relative price increments. We may have either one of

(14) $$\delta x_i = \sigma_i \epsilon_i$$

(15) $$\frac{\delta x_i}{x_i} = \sigma_i \epsilon_i$$

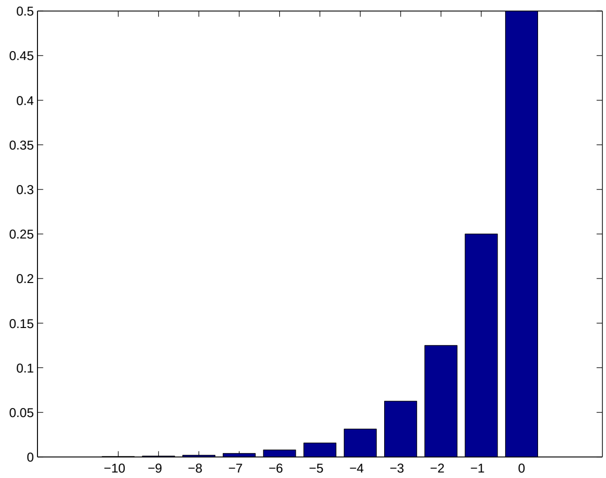

Figure 1: The example $\kappa(j-i) = (1/2)\alpha^{j-i}, \alpha = 0.5$. The x-axis plots the lagged time $j-i$, and the y-axis shows the corresponding values of κ. Note that κ is normalized to sum to 1.

The absolute model (14) is found to be more valid over short time scales, while the relative model (15) is better for longer time scales. A simple example of a model of the type (15) is Geometric Brownian motion

$$\frac{\delta x_i}{x_i} = \epsilon_i \sigma_0 \tag{16}$$

whose relative increments have constant volatility through time.

A generalization of the time-dependent volatility is given by the "Retarded Volatility" model,

$$\delta x_i = x_i^R \epsilon_i \sigma_i$$

$$x_i^R = \sum_{j=1}^{\infty} \kappa(j) x_{i-j} \tag{17}$$

where the κ's are chosen so that $\sum_{i=1}^{\infty} \kappa(i) = 1$. An example $\kappa(i)$ is graphed in Fig. (1). The Retarded Volatility model may be used to "interpolate" between the models for time-varying volatility of absolute vs. relative price increments.