APPLIED OPTIMIZATION

Jitka Dupačová, Jan Hurt and Josef Štěpán

STOCHASTIC MODELING IN ECONOMICS AND FINANCE

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by

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To my husband Václav

Jitka Dupačová

To Jarmila, Eva, and in memory of my parents

Jan Hurt

To my wife Iva

Josef Štěpán

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PREFACE

The three authors of this book are my colleagues (moreover, one of them is my wife). I followed their work on the book from initial discussions about its concept, through disputes over notation, terminology and technicalities, till bringing the manuscript to its present form. I am honored by having been asked to write the preface.

The book consists of three Parts. Though they may seem disparate at first glance, they are purposively tied together. Many topics are discussed in all three Parts, always from a different point of view or on a different level.

Part I presents basics of financial mathematics including some supporting topics, such as utility or index numbers. It is very concise, covering a surprisingly broad range of concepts and statements about them on not more than 100 pages. The mathematics of this Part is undemanding but precise within the limits of the chosen level. Being primarily an introductory text for a beginner, Part I will be useful to the enlightened reader as well, as a manual of notions and formulas used later on.

The more extensive Part II deals with stochastic decision models. Multistage stochastic programming is the main methodology here. The scenario-based approach is adopted with special attention to scenarios generation and via scenarios approximation. The output analysis is discussed, i.e. the question how to draw inference about the true problem from the approximating one. Numerous applications of the presented theory vary from portfolio optimal control to planning electric power generation systems or to managing technological processes. A case study on a bond investment problem is reported in detail. A survey of numerical techniques and available software is added. Mathematics of Part II is still of standard level but the application of the presented methods may be laborious.

The final Part III requires from the reader higher mathematical education including measure-theoretical probability theory. In fact, Part III is a brief textbook on stochastic analysis oriented to what is called diffusion financial mathematics. The apparatus built up in chapters on martingales and on stochastic integration leads to a precise formulation and to rigorous proving of many results talked about already in Part I. The author calls his proofs honest; indeed, he does not facilitate his task by unnecessarily simplifying assumptions or by skipping laborious algebra.

The audience of the book may be diverse. Students in mathematics interested in applications to economics and finance may read with benefit all Parts I,II,III and then study deeper those topics they find most attractive. Students and researchers in economics and finance may learn from the book of using stochastic methods in their fields. Specialists in optimization methods or in numerical mathematics will get acquainted with important optimization problems in finance and economics and with their numerical solution, mainly through Part II of the book. The probabilistic Part III can be appreciated especially by professional mathematicians; otherwise, this Part will be a challenge to the reader to raise his/her mathematical culture. After all, a challenge is present in all three Parts of the book through numerous unsolved exercises and through suggestions for further reading given in bibliographical notes.

I wish the book many readers with deep interest.

V. Dupač

ACKNOWLEDGMENTS

This volume could not come into being without support of several institutions and a number of individuals. We wish to express our sincere gratitude to every one of them.

First of all we thank to Ministry of Education of Czech Republic[1], Grant Agency of Czech Republic2 and Directorate General III (Industry) of the European Commission3 who supported the scientific and applied projects listed below that substantially influenced the contents and form of the text. We gratefully acknowledge the financial support from the companies NEWTON Investment Ltd and ALAX Ltd and appreciate the particularly helpful technical assistance provided by the Czech Statistical Office.

The authors are very indebted to Pavel Popela from the Brno University of Technology who, using his extensive experience with the numerical solutions to the problems in the field of Stochastic Programing, wrote Chapter II.8. Horrand I. Gassmann from the Dalhousie University read very carefully this Chapter and offered some valuable proposals for improvements. We thank also Marida Bertocchi from the University of Bergamo whose effective cooperation within the project (3) influenced the presentation of results in Chapter II.6.

We have to say many thanks, indeed, to our colleagues and friends Václav Dupač and Josef Machek who agreed to read the text. They expended a great effort using their extensive knowledge both of Mathematics and English to make many invaluable suggestions, pressing for higher clarity and consistency of our presentation. Further, we are particularly grateful to Jaromír Antoch for his invaluable help in the process of technical preparation of the book. The authors are also indebted to their present and former PhD students at the Charles University of Prague: Alena Henclová and Petr Dobiáš deserve credits for their efficient and swift technical assistance. Part III owes much to Petr Dostál, Daniel Hlubinka, Karel Janeček and Petr Ševčík who, cruelly tried out as the first readers, have then become enthusiastic and respected critics.

Finally, we thank our publisher *Kluwer Academic Publishers* and, above all, the senior editor John R. Martindale for publishing the book.

J. Dupačová, J. Hurt, and J. Štěpán

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Part I

FUNDAMENTALS

I.1 MONEY, CAPITAL, AND SECURITIES

money, capital, investment, interest, cash flows, financing business, securities, financial market, financial institutions, financial system

1.1 Money and Capital

Money is the means which facilitates the exchange of goods and services. Commonly, money appears in forms like banknotes, coins, and bank deposits. There are three functions ascribed to money: (i) a medium of exchange, (ii) a unit of value, expressing the value of goods and services in terms of a single unit of measure (Czech Krones, e.g.), (iii) a store of wealth. Money is, no doubts, better means for trade than *barter* (direct exchange of goods or services without monetary consideration), but still insufficient for more complicated and/or sophisticated financial operations like investment.

Capital is wealth (usually unspent money) or better to say accumulated money which is used to produce or generate more wealth via an economic activity.

1.2 Investment

Individuals or companies face the problem how to handle their income. They can either spend it immediately, or save it, or partly spend and partly save. In either of the mentioned possibilities, they must decide how to spend and how to save. In the latter case (saving), they postpone their immediate consumption in favour of investment. In that case, they become *investors* and *investment* may therefore be defined as postponed consumption. Usually, the consumption–investment decision is made so as to maximize the expected utility (level of satisfaction) of the investor. While the immediate consumption is sure (up to certain limits), the result of an investment is almost always uncertain. Investments (or assets) can be classified into two classes; *real* and *financial.* A *real asset* is a physical commodity like land, a building, a car. A (financial) *security* or a *financial asset* represents a claim (expressed in money terms) on some other economic unit. (see [143], e.g.).

1.3 Interest

The reward for both postponed consumption and the uncertainty of investment is usually paid in the form of interest. Interest is a time dependent quantity depending on, roughly speaking, time to the postponed consumption. *Interest* in wider sense is either a charge for borrowed money that is generally a percentage of the amount borrowed or the return received by capital on its investment. Simply, interest is

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the price of deferred consumption paid to ultimate savers. Note that the actual allocation of savings in a reasonably functioning economy is accomplished through interest rates, see next Section. In other words, capital in a free economy is allocated through a certain price system and the interest rate expresses the cost of money.

1.4 Cash Flows

A *cash flow* is a stream of payments at some time instances generated by the investment or business involved. The *inflows* to the investor have plus sign while the *outflows* have minus sign. In accounts, the inflows are called *black figures* while outflows are called *red* or *bracket figures* since they appear either in red color or in brackets. As a rule, *net cash flows* are considered; it means that at any time instant all inflows and outflows are summed up and only the resulting sum is displayed. See I.3 for a more detailed analysis of cash flows.

1.4.1 Cash Flows Example. An investor buys an equipment for USD 90000 today. After one year he or she still is not in black figures and the loss is USD 15200. In the successive years 2, 3, 4, 5, 6 the profits (in USD) are 45000, 60000, 25000, 22000, 12000, respectively. At the end of the sixth year the investor sells the equipment for the salvage value USD 15000. The net cash flow for years $0, \ldots, 6$ is (-90000, -15200, 45000, 60000, 25000, 22000, 27000=12000+15000). Graphical illustration is given in Figure 1.

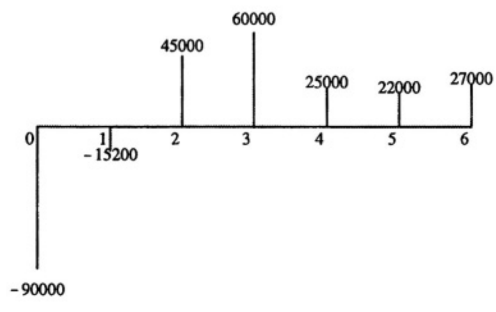

Figure 1: A cash flow

1.5 Financial and Real Estate Investment

Since handling money and capital itself is a rather complicated task, there are *financial intermediaries* and other financial institutions which should, in principle, handle money and capital efficiently. *Financial institutions* are business firms with assets in the form of either financial assets or claims like stocks, bonds, and loans. Financial institutions make loans and offer a variety of financial services (investment, life and general insurance, savings, pensions, credits, mortgages, leasing, real estates, etc.).

1.5.1 Financing the Business – Description

Almost every economic activity (of an individual, firm, bank, city, government) must be financed. In principle, there are two possibilities how to realize it; either from own funds or from outside sources (creditors, debt financing). Own funds of a company may be increased by issuing stocks resulting in the increase of equity while the debt financing usually takes form of either bank credit or issuing the debt instruments like corporate bonds. The better the expected performance of the firm is, the cheaper funds (money) are available. The financial public look on the performance of a firm through the *ratings* and the *prices* of financial instruments already issued by the firm on the market (mainly *Stock Exchange*). The most important corporations providing rating are Moody's Investor Service (shortly *Moody's*) and Standard & Poor's Corporation (shortly *Standard & Poor's*). Both the rating and price are important signals to the investors.

1.5.2 Financing the Business – Summary

We have seen that there are three main possible ways of financing; by equity (issuing stocks), and two ways of debt financing, i.e., by issuing the debt instruments like corporate bonds or just by acquiring a bank credit. A modern firm uses all the above possibilities and it is the task of financial managers to balance them. It is not so surprising that some very prospective American companies have debt to equity ratio about 70 per cent. The idea is simple; if you borrow at some 7 per cent and gain 11 per cent from the business, you are better off.

The fully self-financed company seems to be rather old-fashioned now. The tradition of the European family business may serve as an example. There are rare exceptions still surviving in these days, even among big firms in Europe. Nevertheless, the prosperous debt financed firm makes usually more profit than a comparable self-financed company.

1.6 Securities

Security (in what follows here we mean *a financial security*) is a medium of investment in the money market or capital market like shares (English) or stocks (American), bonds, options, mortgages, etc. Security is a kind of *financial asset* (everything which has a value or earning power). Speaking in accounting terms, the holder (purchaser) of it has an *asset* while the issuer or borrower (seller) has a *liability.* Security usually takes the form of an agreement (contract) between the seller and the purchaser providing an evidence of debt or of property. The holder of a security is called to be in a *long position* while the issuer is in a *short position.* Security usually gives the holder some of the following rights:

(1) returning back money or property
(2) warranted reward
(3) share on the profit generated by money provided
(4) share on the property
(5) right on decision making concerning the use of money provided.

But a security may also be an agreement between two parties (often called *Party* and *Counterparty*) on a financial or real transaction between the two. This is the case of swaps, partly the case of forwards and futures. It is difficult to say who is the issuer and who is the holder, in this case.

The basic types of securities and their forms are listed below. See [143], [138], [105], [172] for more details.

1.6.1 Fixed-Income Securities

Fixed-income securities are debt instruments characterized by a specified maturity date (the date of payoff the debt) and a known schedule of repaying the principal and interest.

1.6.1.2 Demand Deposits

Commercial banks and saving societies offer to their clients *checking accounts* or *demand deposits* which are interest bearing but the interest is usually very small. A better situation is with *savings accounts*, a type of *time deposit.* Here money is saved for a prescribed period of time and any early withdrawal is subject to penalty which usually does not exceed the interest for the period involved. The interest is higher than that of applied to demand deposits and sometimes may vary.

1.6.1.3 Certificates of Deposit

Very popular, particularly for the institutional investors, are the *Certificates of Deposit,* shortly *CD's*, mainly issued by commercial banks in large denominations. They also take the form of time deposits with fixed interest but the early withdrawal is severely penalized. CD's are usually issued on the *discounted base,* at a discount from their face value. Roughly spoken, if you want to buy a CD of the face CZK 1000000, say, payable after one year, you buy it for some CZK 920000. Remember that the return in this case is not 8 per cent.

1.6.1.4 Treasury Bills

A typical money market securities issued by the central bank are *Treasury bills*, *T-bills.* Their main purpose is to finance the government or their fiancées. They have maturities typically varying from weeks to one year and are also issued on the discount base.

There is one interesting point in issuing securities of the above type. A careful government (even the Czech one, now) issues T-bills through the *auction.* Prior to each auction, the central bank (representing the government, in many countries behaving independently of the government) announces the par (face) value of the security and the upper limit of the bid expressed in terms of the interest rate. Also the intended volume (*total face value*) is announced.

For example, the issuer (the bank in this case), announces that the accepted offers are up to 8 per cent p.a. It means that the issuer will only accept the offers below this rate. The submitted bids are collected and ranked according to the offers with respect to the volume and interest rate. Since the offer of the issuer is competitive, the investors who wish to catch the offer must carefully choose both the offered interest and the volume. The strategy of the issuer is the question of allocation, the problem which will be discussed later.

Note that similar policy or technique (auction) is also often used by commercial banks as well as by highly rated firms (rated as blue chips, AAA, in Standard & Poor's rating scale).

For a detailed analysis including a discussion of auctions see [143].

1.6.1.5 Coupon Bonds

A *coupon bond* is the long-term (usually from 5 to 30 years) financial instrument issued by either central or local governments (municipals), banks, and corporations. It is a debt security in which the issuer promises the holder to repay the *principal*, *par value, face value, redemption value,* or *nominal value* F at the *maturity date* and to pay (periodically, at equally spaced dates up to and including the maturity date) a fixed amount of *interest* C called *coupon* for historical reasons. The ratio $c = C/F$ is called *coupon rate,* sometimes simply *interest.* A typical period for the coupon payment is semiannual, rarely annual, but both the coupon and coupon rate are expressed on the annual base. The number of periods in a year is called *frequency.* In case of semiannually paid coupon, the frequency is 2. The bond is usually valued at a time instant between the issuing date and the maturity date. So that more important for the valuation purposes is the length of time to the maturity date called *maturity of the bond.* Maturity differs from the whole life of the bond in that only remaining payments of coupons and principal are considered. A cash flow coming from a coupon bond is illustrated in Figure 2.

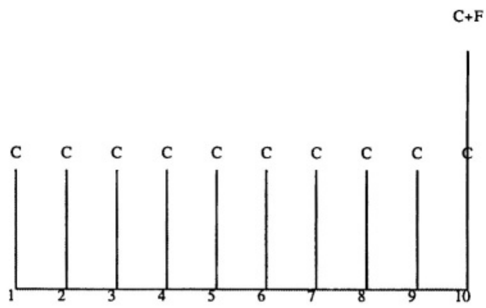

Figure 2: Cash flow of a bond

1.6.1.6 Callable Bonds

The simple coupon bond described above has an obvious disadvantage for the issuer; if the interest rates fall during the bond life, it is often possible for the issuer to get cheaper funds, for instance by issuing bonds with lower coupon. The security which partly gets rid of this feature is *callable bond.* The situation is the same as with the usual coupon bonds but in this case, the issuer has the right to buy some or all issued bonds prior to the original maturity date or *to call* them, in other words. Since the earlier repayment of the face value may cause an inconvenience to the bondholder (particularly with the reinvestment at lower interest than the coupon), the issuer should pay a reward to the bondholder in the form of *call premium.* The *call dates* and *call premiums* are stated in the offering statement. For example, if the bond is called one year before the maturity date, the payment is 101 per cent of the par value, if two years before, the payment is 102 per cent, etc. The call premium generally decreases with the date of call closer to the maturity date. Strictly speaking, the callable bond is not a fixed-income security since the payments coming from it are uncertain and depend both on the issuer policy and market interest rates.

1.6.1.7 Zero Coupon Bonds

A *zero coupon bond,* shortly *zero*, or *discount bond* pays only the face value at maturity. It is issued at discount to par value (like CD) and it pays par value at maturity. One reason for issuing such a type of bonds is that in some countries (like USA) the issuer may deduct the yearly accrued interest from taxes even though the payment is not made in cash. The bondholder (purchaser) must calculate interest income in the same way as the issuer calculates the tax deduction and should pay either corporate or personal tax even though no cash has been received. However, if the purchaser is a tax-exempt entity, like a pension fund or an individual who buys the bond for its individual retirement account, it pays no tax from the accrued interest. See [25] p. 578 for more details.

A coupon bond may be considered as a series of zero coupon bonds, all but last with face value equal to the coupon payment, and the last with the face value equal to the coupon payment plus the face value of the underlying coupon bond. This is not only a theoretical construction; the coupon components and face value of US Treasury bonds may be traded separately and such securities are called *STRIPS* – Separate Trading of Registered Interest and Principal of Securities. There are also *derivative zero coupon bonds*; a brokerage house buys usual coupon bonds, strips the coupons, and resells the stripped securities as zero coupon bonds.

1.6.1.8 Mortgage-Backed Securities

A lending institution that loans money for mortgages combines a large group of mortgages and thus creates a *pool.* The *mortgage-backed (pass-through)* security is then a long term (15 to 30 years) instrument that is collateralized by the pool of mortgages. As the homeowners make their (usually monthly) payments of the principal and interest to the lending institution, these payments are then "passed through" to the security holders in the form of coupon payments and the principal. The coupon is naturally less than the interest paid by homeowners, but the level of default is low. First, there is a warranty in real estate, second, there is a large pool of loans which diversifies the default risk.

See [143] for more details.

1.6.2 Floating-Rate Securities

Floating-rate securities' payments are not fixed in advance and rather depend on some underlying asset. The reason for issuing such securities is to reduce the interest rate risk for both the seller and the buyer. Typical examples are *floating-rate bonds* and *notes* with a coupon or interest periodically adjusted according on the underlying instrument (base rate) like LIBOR, PRIBOR, discount rate of the central bank etc. or they are simply tied to some interest rate like prime rate of a commercial bank (the interest rate for highly rated clients of the bank).

Note that LIBOR (*London InterBank Offered Rate*) is the daily published interest rate for leading currencies (GBP, EUR, USD, JPY, …) with a variety of maturities (one day or overnight, 7 days, 14 days, 1 month, 3 months, 6 months, 1 year). LIBOR is calculated as the trimmed average (two smallest and two largest values are not considered) of the interest rates on large deposits among 8 leading banks in Great Britain. Similarly PRIBOR is an abbreviation for *Prague Interbank*

Offered Rate and is calculated in a similar way like LIBOR. Usually the calendar Actual/360 applies to all transactions.

Typically, the actual coupon rate is the interest rate of the underlying asset plus *margin* (*spread*). If the underlying instrument is LIBOR, e.g., the actual coupon rate may be actual LIBOR plus 100 basis points or actual LIBOR plus 3 per cent. The floating rates may be reset more than once a year leading to *short-term floating rates* while in the opposite case we speak of *long-term floating rates.* We also speak about *adjustable-rate securities* or *variable-rate securities*, see [60], [61].

1.6.2.1 Example (I bonds). *I bonds* are U.S. Treasury *inflation-indexed saving bonds* introduced in September 1998 with maturity on September 2028 in denominations varying from USD 50 to USD 10000. The rate – currently 6.49% p.a. – consists of two components; a *fixed rate* 3.6% which applies for the life of the bond, and *inflation rate* measured by the Consumer Price Index which can change every six months. I bonds earnings are added every month (coupon is added to the principal) and the interest is compounded semiannually. Only Federal income tax applies to the earnings. Investors cashing before 5 years are subject to a 3-month earnings penalty.

1.6.3 Corporate Stocks

Issuing stocks is a very popular method of financing business and further development of a company (corporation, firm). The most important types of stocks are common and preferred stocks. A *common stock* (US), *ordinary share* (UK) is the security that represents an ownership in a company. The equity of a company is the property of the common stock holders, hence these stocks are often called *equities.* For the investors, the stock is a piece of paper or a record in the computer giving him or her the right to engage in the decision processes concerning the company policy according to the share on common stock (voting right). Also it entitles the owner to *dividends* which consist of the amount of company's profit distributed to stockholders. This amount equals earnings less *retained earnings* (the part of earnings intended for reserves and reinvestment).

A *preferred stock* gives the holder priority over common stockholders. Preferred stockholders receive their dividend prior to common stockholders. Usually the dividend does not depend on the company's earnings and often is constant, thus resembling a coupon bond. In case of bankruptcy, the preferred stockholders have higher chance to see their claims to be satisfied. On the other hand, often they do not have voting right.

Stocks have another feature which is called *limited liability* that means that their value cannot be negative in any case.

1.6.4 Financial Derivatives

Financial derivative securities or *contingent claims* are the instruments where the payment of either party depends on the value of an *underlying asset* or assets. The underlying assets in question may be of a rather general form, e.g. stocks, bonds, commodities, currencies, stock exchange indexes, interbank offer rates, and even derivatives themselves. The underlying assets thus fall into two main groups;

commodity assets and *financial assets.* The derivatives are now traded in enormous volumes all over the world. Estimated figure for options only at 1996 was about $35 trillion. The most common derivatives are forwards, futures, options, and swaps.

1.6.4.1 Forwards and Futures

A *forward contract* is an agreement between two parties, a buyer and a seller, such that the seller undertakes to provide the buyer with a fixed amount of the currency or commodity at a fixed future date called *delivery date* for a fixed price called *delivery price* agreed today, at the beginning of the contract. For both parties this agreement is an obligation. By fixing the price today the buyer is protected against price increase while the seller is protected against price decrease. Forward is typically a privately negotiable agreement and it is not traded on exchanges. The forward contract is a risky investment from two reasons, at least. First reason is obvious; since the spot price of the underlying asset generally differs from the delivery price, the loss of one party equals the profit of the counterparty and vice versa. The second reason is the default risk in which case the seller is not willing to provide the buyer with delivery. There are also nonnegligible costs in finding a partner for this contract and fair delivery price. Therefore, the forward contracts are usually realized between reliable, highly rated parties. No money changes hands prior to delivery.

A simple example is a forward contract between a miller and a farmer producing corn. Today, April 11, 2001, they agree that the farmer will deliver 1000 bushels of corn for the delivery price USD 2.5 per bushel on September 30, 2001, the delivery date. Both parties consider these conditions of the contract as good. Assume that the spot price of corn on the delivery date would increase to USD 3 per bushel. Without the forward contract, the miller would have to buy for this price which might cause problems to him. On the other hand, with the spot price decrease to USD 2 per bushel on delivery date, the farmer who would have to sell for this price might have to go to the bankruptcy.

A *futures contract* shortly *futures*, is of a similar form as the forward but it has additional features. The futures is *standardized* (specified quality and quantity, prescribed delivery dates dependent on the type of the underlying asset). The futures are traded (they are marketable instruments) on exchanges. One of the most popular is Chicago Board of Trade (CBT). To reduce the default risk to minimum, both parties in a futures must pay so called *margins*. These margins serve as reserves and the account of any party in the contract is daily recalculated according to the actual price of the futures, the *futures price.* Such a procedure is called *marking to market.* The *initial margin* must be paid by both parties at the initiation of the contract and usually takes values between 5 to 10 per cent of the contract volume. The *maintenance margin* is a prescribed amount below the initial margin. If the account falls below this margin, it must be recovered to the initial margin by an additional payment called a *variation margin.* The contractors' accounts bring interest. The futures exchange also imposes a *daily price limit* which restricts price movements within one business day, ±10 per cent, say. The responsibility for default is transferred to a clearing house that is also responsible for the clients' accounts, see [25] and [143].

The reports on futures prices in financial press provide the daily opening, highest, lowest, and closing price, the percentage change, the highest and lowest price during the lifetime of the contract, and the total number of currently outstanding contracts called *open interest.*

1.6.4.2 Options

An *option* is a contract giving its owner (*holder, buyer*) the right to buy or sell a specified underlying asset at a price fixed at the beginning of the contract (today) at any time before or just on a fixed date. The seller of an option is also called *writer.* It must be emphasized that an option contract gives the holder a **right** and not an **obligation** as it was the case of futures. For the writer, the contract has a potential obligation. He **must** sell or buy the underlying asset accordingly to the holder's decision. We distinguish between a *call option* (CALL) which is the right to buy and a *put option* (PUT) which is the right to sell. The fixed date of a possible delivery is called *expiry* or *maturity* date. The price fixed in the contract is called *exercise* or *strike price.* If the right is imposed we say that the option is *exercised.* If the option may be exercised at any time up to expiry date, we speak of an *American option* and if the option may be exercised **only** on expiry date, we speak of a *European option.* These are the simplest forms of options contracts and in literature such options are called *vanilla options.*

The right to buy/sell has a value called an *option premium* or *option price* which must be paid to the seller of the contract. It must be stressed that the option price is **different** from the exercise price!

Like futures, options are mostly *standardized* contracts and are traded on exchanges since 1973. The first such exchange was the Chicago Board Options Exchange (CBOE). Most common underlying assets are common stocks, stock market indexes, fixed-income securities, and foreign currencies. Options are usually short-term securities with typical maturities 3, 6, and 9 months. At any time there are options with different maturities and different strike prices available on the market. An example (taken from [143]) shows how the long term options are quoted in financial press on January 15, 1992, is in the following table:

Option	*Expiry*	*Strike*	*Last*
ATT	*Jan* 93	25	$16\frac{1}{2}$
ATT	*Jan* 93	35	$7\frac{1}{2}$
ATT	*Jan* 93	35p	$1\frac{1}{4}$
ATT	*Jan* 93	40	$4\frac{1}{4}$
ATT	*Jan* 93	40p	$2\frac{3}{4}$
ATT	*Jan* 93	50	$1\frac{3}{16}$

This is an example of American options with different strike prices with the underlying asset AT & T common stock and with the same expiry date, the third Friday January 1993. "p" standing at strike price means a PUT option, the others are CALLs. "Last" means the closing price.

Another type of options are *exotic* or *path-dependent* options. These options (if exercised) pay the holder the amount dependent on the history of the underlying asset. Despite their "exotic" features, they are successfully used for hedging purposes. Since the creativity and fantasy of the developers of such products is practically unbounded, we only give some examples. Note that most of the mentioned options may be either of European or American type. For more details see [172] and [105], e.g.

A *binary* or *digital* option pays the holder a fixed amount of money if the value of the underlying asset rises above or falls below the exercise price. The payoff is independent of how far from the exercise price the asset value was at the exercise time.

A *barrier option* is a usual vanilla option but it may only be exercised if either the asset value does not cross a certain value – an *out-barrier*, or if the asset price crosses a certain value – an *in-barrier* during the life of the option contract. There are four possible cases:

(1) *up-and-in*; the option pays only if the barrier is reached from below,
(2) *down-and-in*; the option pays only if the barrier is reached from above,
(3) *up-and-out*; the option pays only if the barrier is not reached from below,
(4) *down-and-out*; the option pays only if the barrier is not reached from above.

A *compound option* is simply an option where the underlying asset is another option. If we consider only plain vanilla options, we have four possibilities again. For brevity, we describe the mechanism of a *call-on-a-call* European type compound option. Such an option gives the holder the right to buy a call option for the price K_1 at the expiry T_1. The second call option is on an underlying asset with the exercise K_2 and the expiry $T_2 > T_1$.

A *chooser option* or *as-you-like-it option* is an option which gives the holder the right to buy or sell either a call or a put option. We give an example of a *call-on-a-call-or-put.* Such a chooser option gives the holder the right to purchase for the exercise price K_1 at expiry time T_1 either a call or a put with exercise price K_2 at time T_2.

An *Asian option* is a path-dependent option with payoffs dependent on the average price of the underlying asset during the life time of the option. Such an average plays the role of the exercise price. Thus, the *average strike call* pays the holder the difference between the asset price at expiry and the average of the asset prices over some period of time, if positive, and zero otherwise. The problems arise from the proper definition of the average involved, continuous or discrete sampling, if discrete, then from prices sampled hourly or from closing prices, etc.

A *lookback option* has a payoff which also depends on maximum or minimum reached by the underlying asset over some period prior to expiry. Such a maximum or minimum plays the role of the exercise price.

1.6.4.3 Swaps

Swaps, like forwards, are mostly individual contracts between two highly rated, reliable parties which well fit the needs of both. Although the swaps are individual contracts, in practise they often follow the recommendations of the *International Swaps and Derivatives Association* (ISDA). A *swap* may be briefly characterized as an agreement on exchange of cash flows in future times with a prescribed schedule. There are two main categories of swaps; interest rate swaps and currency swaps.

In practise, the two are often combined. Swaps are used to manage interest rate exposure or uncertainty concerning the future exchange rates.

An *interest rate swap* is a contract between two parties to exchange interest streams with different characteristics based on a principal, notional amount, sometimes called the volume of a swap. The interest rates may be either fixed or floating in the same or different currencies.

A pure *currency swap* is a forward contract on the exchange of different currencies on some future date (maturity) in amounts fixed today. Another type of a currency swap is a *cross-currency swap* that consists of the initial exchange of fixed amounts of currencies and reverse final exchange of the same amounts at maturity. One or both parties may pay interest during the lifetime of the swap.

1.6.4.4 Example (Combined swap). Notional amount: CZK 34,500,000

Fixed amounts:

Initial exchange: Party A pays EUR 1,000,000 to party B, party B pays CZK 34,500,000 to party A. Maturity 10 years.

Final exchange (after 10 years): Party B pays EUR 1,000,000 to party A, party A pays CZK 34,500,000 to party B.

Floating amounts:

Party A pays to party B semiannually E6M - 3.5 per cent (spread or margin) from notional amount based on the floating rate day count fraction Actual/360, i.e., CZK $((\mathrm{E6M}-3.5)/100)\cdot(182/360)\cdot 34{,}500{,}000$. Here E6M stands for LIBOR interest rate on EUR with maturity 6 months.

1.6.5 Miscellaneous Securities

Here we briefly mention a sample of other types of derivatives met in financial practise.

A *warrant* is a derivative security which gives the holder the right to buy a specified number of common stocks for a fixed price called *exercise price* at any time during the lifetime of the warrant. Such a security resembles a CALL option but there are two differences. First, warrant is a long-term security, 10 years say, while options have maturities up to two years. Second, perhaps a more important feature of the warrant is, that it is issued by the same company which issues the underlying stock while options are traded among investors.

Another type of security with an option is a *convertible bond.* Such a bond gives the bondholder the right to exchange the bond for another security, typically the common stock issued by the same company or just to sell back the bond to the issuing company. This is an example of a convertible bond with *put option*. Firms usually add the conversion option to lower the coupon rate. On the other hand, the issuer may reserve the right to *call back* the bonds and upon call, the bondholder either converts the bond into stocks or redeems it at the call price (convertible bond with *call option*). In this case, the coupon rate must be higher than that of usual coupon bond. In both cases we speak of *conversion premiums*.

Let us turn to floating-rate bonds (see 1.6.2). Most issuers cap their obligations to ensure that the floating coupon rate does not rise above a prespecified rate called *cap*. Thus if the face value of a bond is F, the floating rate r (say LIBOR on EUR with maturity 6 month + 3 percent) and the cap r_c, then the payment is $F \cdot \min(r, r_c)$. On the other hand, some issuers offer buyers an interest rate below which the coupon rate will not decline; such a rate is called *floor.* If the floor is r_f, then the payment is

$$F \cdot \max(\min(r, r_c), r_f) = F \cdot \min(\max(r, r_f), r_c).$$

Usually caps and floors take the form of consequent payments called caplets and floorlets, respectively.

1.7 Financial Market

Financial market consists of money market and capital market. *Money market* is a market with short-term assets or funds, up to one year say, like bills of exchange, Treasury bills (*T-bills*), and Certificates of Deposit (*CD*'s). *Capital market* is a market which deals with longer-term loanable funds mainly used by industry and commerce for investment and acquisition. Usually capital markets handle securities which are related to the time horizon longer than one year.

1.8 Financial Institutions

The role of financial institutions is simple. Financial intermediaries (commercial banks, insurance companies, pension funds, e.g.) acquire debts issued by borrowers (IOU – the abbreviation for "I Owe You") and at the same time sell their own IOUs to savers. Every bank (with rare exceptions in the Czech Republic) is happy to accept your savings and handle them. It is a debt which is used by the bank in the form of loans and investments. Examples of other financial institutions are security brokers (bringing buyers and sellers of securities together), dealers, who – like brokers – intermediate but moreover purchase securities for their own accounts. There are investment bankers, mortgage bankers, and other miscellaneous financial institutions in this category, as well.

1.9 Financial System

In a civilized country, all the activities mentioned above go through the *financial system* which can be simply illustrated by the following scheme:

Ultimate borrowers, savings-deficit units $\longleftrightarrow$

$\longleftrightarrow$ Financial Intermediaries $\longleftrightarrow$

$\longleftrightarrow$ Ultimate savers, savings-surplus units.

The needs or wishes of borrowers and savers are different, of course. The borrowers need long-term loans, acceptance of significant risk by the lenders, and larger amounts of credit. Perhaps the highest priority of the lenders is *liquidity*, which means the availability of the funds (money) at the moment when these are requested. The natural needs of the savers are *safety of funds* and, particularly for small investors, accessibility of the securities in small denominations.

I.2 INTEREST RATE

interest rate, compounding, present value, future value, calendar convention, determinants of the interest rate, term structure, continuous compounding

2.1 Simple and Compound Interest

Interest rate (also *rate of interest*) is a quantitative measure of interest expressed as a proportion of a sum of money in question that is paid over a specified time period. So if the initial amount of money is *PV* (also called *principal* or *present value*) and the interest rate is i for the given time period, then the interest paid at the end of the period is $PV \cdot i$ and the accumulated amount of money at the end of the period (called *future value* or *terminal value*) is

$$FV = PV(1+i). \tag{1}$$

Alternatively, the interest rate is quoted *per cent.* It will be clear from the context where $i = 0.13$ means $i = 13\%$ and vice versa. Note that r is another frequently used symbol for the rate of interest, particularly if speaking of the rate of return.

Let us consider more than one time period, say T periods, with T not necessarily integer, and the same interest rate i for one period. There are two approaches how to handle interests after each period. Under *simple interest* model, only interest from principal is received at any period. Thus the future value after T periods is

$$FV_T = PV(1+iT). \tag{2}$$

Under *compound interest* model, the interest after each period is added to the previous principal and the interest for the next period is calculated from this increased value of the principal. The corresponding future value is

$$FV_T = PV(1+i)^T. \tag{3}$$

In the context of the compound interest model, the process of going from present values to future values is called *compounding.*

2.1.1 Remark (Mixed Simple and Compound Interest)

Some banks or saving companies use a combination of simple and compound interest if T is not an integer. Let $T = \lfloor T \rfloor + \{T\}$ where $\lfloor\rfloor$ denotes the entire part and $\{\}$ denotes the fractional part of the argument. Then the future value is calculated as

$$FV_T = PV(1+i)^{\lfloor T \rfloor}(1+i\{T\}). \tag{4}$$

2.1.2 Exercise. Decide what is better for the saver: future value of the savings calculated from (3) or (4).

Speaking of interest rates, it is important to state clearly the corresponding *unit of time*. In most cases, the interest rate is given as the *annual interest rate*, often stressed by the abbreviation *p.a.* (per annum). The usual notation is $i = 0.13$ p.a. or equivalently $i = 13\%$ p.a. Rarely, interest rates are given semiannually (*p.s.*, per semestre), quarterly (*p.q.*, per quartale), monthly (*p.m.*, per mensem), daily (*p.d.*, per diem). The *period of compounding* is similarly one year, six months, three months, one month, or one day. If the unit of time for the given interest rate differs from the period of compounding (which is often the case), it is very important to emphasize that we consider $i\%$ p.a. interest rate compounded semiannually, say. In this case it means that the interest rate i is so called *nominal interest rate,* and for every six month's period the actual interest rate is $i/2$. Generally, let $i^{(m)}$ be the nominal rate of interest per unit time compounded $m$times within the unit time so that there are m periods, each of length $1/m$, and the interest rate is $i^{(m)}/m$ per period. We also say that the nominal interest rate $i^{(m)}$ is *payable* $m$thly. Thus the future value of *PV* after *T* periods is

$$FV_T = PV\left(1+\frac{i^{(m)}}{m}\right)^T. \tag{5}$$

Of course, the actual interest rate per unit time i_{eff}, called *effective rate of interest* is *not* equal to the nominal rate of interest. Obviously,

$$1 + i_{\text{eff}} = \left(1+\frac{i^{(m)}}{m}\right)^m. \tag{6}$$

2.1.3 Exercise. Compare the effective rates of interests if $i^{(m)} = 0.13$ p.a. for $m = 1, 2, 12, 365$ and comment the result.

2.2 Calendar Conventions

Assume the unit time is one year. If the number of periods n is not an integer, there are different methods to count the difference between two dates. Consider two dates, $DATE_1$, $DATE_2$, say, expressed in the form $DATE_j = YYYY_jMM_jDD_j$, $j = 1, 2$. January 13, 2013, is therefore expressed as 20130113. The most frequent conventions:

Calendar 30/360 or Euro-30/360. Under this convention all months have 30 days and every year has 360 days. The number of periods *T* is calculated as

$$T = \frac{1}{360}(360(YYYY_2 - YYYY_1) + 30(MM_2 - MM_1) + \min(DD_2, 30) - \min(DD_1, 30)).$$

Calendar US-30/360. In this case, all dates ending on the 31st are changed to the 30th with the following exception: if $DD_1 < 30$ and $DD_2 = 31$ then $DATE_2$ is changed to the first of the next month.

Calendar Actual/Actual. This convention assumes the actual number of days between two dates with the actual number of days in the year.

Calendar Actual/360. The actual number of days in each month but 360 days in the year are considered. As a result, the number of periods within one year can exceed one.

Calendar Actual/365. The actual number of days in each month and 365 days in each year are considered. The leap year assumes 365 days.

Most computer systems are equipped with calendar functions, particularly with the function which returns the number of days between two dates. For example, *Mathematica* offers the function `DaysBetween` [*date2, date1*] which returns the actual number of days between two dates. The arguments date takes the form `{year, month, day}` so that March 14, 2001 is `{2001,3,14}`in this notation. In financial packages, the same *Mathematica* function has option `DayCountBasis` either `"Actual/Actual"`or `"30/360"` .

2.2.1 Exercise. Analyze the effect of the calendar conventions on savings from the point of view of a saver or a borrower.

2.3 Determinants of the Interest Rate

In a free economy, interest rates, as a price of money, are mainly determined by market supply and demand, and partly mastered by the government or central bank via money supply policy. Interest rates vary with economic environment, market position, used financial instrument, and time. The economic units which are willing to pay higher interest rates for the *funds* (=borrowed money in this case) expect higher returns on their investments. The returns are usually measured by the *rate of return* defined by:

$$\text{Rate of return} = \frac{\text{Ending price} + \text{Cash income} - \text{Beginning price}}{\text{Beginning price}},$$

sometimes quoted in per cent.

Every investment should be valued from the point of view of return, risk, inflation, and liquidity. The firm with higher return will pay higher interest for funds (money). With the rate of return 25 per cent the firm pays 20 per cent interest with pleasure. Another firm, with the rate of return 20 per cent, would not pay 20 per cent interest since then it would not have reason to develop any activity. More risky investment should be more expensive than an investment with (almost) certain return, in terms of interest rates. Inflation also makes funds more expensive. If the inflation is high, the funds may be not accessible. Short term funds (money borrowed for short time) are usually cheaper than long term funds (money borrowed for long time). Short term interest rates more or less reflect the actual state of the economy while long term interest rates reflect expectations, rational or less rational. Situation is more complicated, however, see the concept of yield curves next in this part. Denote r the rate of interest comprising all the factors mentioned above. In this context, r is also called *cost of capital.*

2.3.1 Remark (Taxation)

Almost all incomes coming from investment are subject to taxes. The few exemptions are returns on some government or municipal bonds, e.g. Thus the taxation reduces the returns. Moreover, the taxes are often different for various types of investment and sometimes are progressive, i.e., the higher the return, the higher the taxes. Thus any investment should be carefully valued with respect to tax consideration.

2.4 Decomposition of the Interest Rate

Taking into account all the factors which affect the so called *quoted* or *nominal interest rate* r, we can write

$$1 + r = (1 + r_0)(1 + r_{\text{infl}})(1 + r_{\text{default}})(1 + r_{\text{liquid}})(1 + r_{\text{mat}}) \tag{7}$$

where r_0 denotes the *risk free* interest rate if we do not consider inflation, r_{infl} (*inflation premium*) is the **expected** rate of inflation, r_{default} (*default risk premium*) is the premium charged for the default risk, that is the risk that the debtor will not pay either principal or interest or both. Sometimes it is called *credit risk.* The term r_{liquid} (*liquidity premium*) stands for the risk that an asset in question is not readily convertible into cash without considerable cost. Finally, r_{mat} (*maturity risk premium*) is the premium for the risk produced by possible changes of interest rates during the life of an asset. There are two types of the maturity risk. Consider bonds, e.g. For long-term bonds, it is the *interest rate risk*; if the market interest rate rises, the prices of bonds go down. This kind of premium rises when the interest rates are more volatile. For short-term bonds, it is the *reinvestment rate risk*; if these bills become due and the actual interest rates are low, the reinvestment will result in interest income loss.

Sometimes the decomposition is given in additive form (see [25], e.g.)

$$r = r_0 + r_{\text{infl}} + r_{\text{default}} + r_{\text{liquid}} + r_{\text{mat}} \tag{8}$$

which is a good approximation of (7) if the components of r are sufficiently small since the cross-factors of type $r_0 r_{\text{infl}}$ are small of twice higher order than the original components.

In real world, there is practically no riskless investment. For simplicity, however, the government bonds are usually considered riskless. In this case, the offered return also includes the expected rate of inflation, so that the risk free rate with a premium for expected inflation is

$$(1 + r_0)(1 + r_{\text{infl}}) - 1. \tag{9}$$

In what follows, without further notice we will consider the riskless rate with the inflation premium.

It is also necessary to note that the above decomposition depends on the time period involved. So if we consider the one-year quoted interest rate, the corresponding expected inflation is a one-year inflation, and the risk free-rate is derived from one-year T-bills rates and the maturity risk premium has a negligible influence on the nominal rate in a stable economy. For a ten-years' quoted interest rate we should take ten-years yields of the government bonds for the riskless rate and carefully consider the other factors affecting the nominal interest rate; default, liquidity, and maturity premium in this case.

2.4.1 Remark (Rating)

Useful guides to credit risk evaluation for corporate bonds are conducted by recognized agencies like *Standard and Poor's* and *Moody's.* Based on an analysis of the firms they provide a classification into rating categories. According to Standard and Poor's, AAA is the highest rating reflecting extremely strong capacity to pay interest and to repay principal, AA means very strong capacity, A may be effected by economic conditions, etc. Further categories are BBB, BB, B, CCC, CC, C, D. Categories below BBB are sometimes considered as speculative or junk bonds. Refinement may be made by adding + or – signs. Similar categories provided by Moody's are Aaa, Aa, A, Baa, Ba, B, Caa, Ca, C, D.

2.4.2 Example. In January 1991 the quoted interest rates for U.S. T-bonds, AAA, AA, and A were 8.0, 8.9, 9.1, and 9.4 per cent, respectively. See [25], p. 109. All these bonds had similar maturity, liquidity, and other features. So the only difference is in the default risk premium. Using formula $r_{\text{default}} = \frac{1+r}{1+r_0} - 1$ for the default premium risk we get $r_{\text{default}}(AAA) = 0.83$, $r_{\text{default}}(AA) = 1.02$, and $r_{\text{default}}(A) = 1.30$, respectively.

2.4.3 Real Return

If r is the nominal rate of interest on deposits and r_{infl} is the rate of inflation, then the *real return* on deposits is sometimes expressed in terms of the *real rate of interest* r_{real} which can be calculated from the obvious relation

$$1 + r = (1 + r_{\text{infl}})(1 + r_{\text{real}})$$

or

$$r_{\text{real}} = \frac{r - r_{\text{infl}}}{1 + r_{\text{infl}}}. \tag{10}$$

For small values of the components appearing in the last formula, we can use the approximation $r_{\text{real}} = r - r_{\text{infl}}$. Moreover, let r_{tax} be the tax rate imposed on the earned interest from deposits. Then we get

$$1 + r(1 - r_{\text{tax}}) = (1 + r_{\text{infl}})(1 + r_{\text{real}}) \tag{11}$$

for the real return.

2.4.4 Example. In the Czech Republic, year 1997, the inflation rate was 0.10 (official source), r could have been taken as 0.11 (an over-optimistic value at some banks), and tax on the return on deposits was 0.15 (by law). Then we obtain the negative real return –0.6 per cent. In April 2001, the yearly inflation has been estimated as 4.1 per cent and one year term deposits net yield was about 3 per cent. So again we get the negative real return at about –1.1 per cent.

2.4.5 Exercise. Derive the corresponding relation for the *real percentage increase* in purchasing power if the percentage increase in salaries is r, the inflation rate is r_{infl}, and r_{tax} is the tax rate.

2.4.6 Example. Let us consider two investments, A and B, say, with gross returns r_A and r_B, subject to taxes i_A and i_B, respectively. The two investments provide the same net yield if

$$r_A(1 - i_A) = r_B(1 - i_B)$$

holds.

2.5 Term Structure of Interest Rates

All the interest rates in this Section relate to the equal time periods. Suppose ${}_tR_n$ is the actual rate of interest at time t on an n-period investment called *spot interest rate* and ${}_{t+1}r_{1t}$, ${}_{t+2}r_{1t}$, ... , ${}_{t+n-1}r_{1t}$ are the one-period interest rates on an investment beginning at times $t+1$, $t+2$, ... , $t+n-1$, respectively, called *forward rates for one period implied in the term structure at time* t. At time t we know spot rates ${}_tR_1$, ${}_tR_2$, ... , ${}_tR_n$. Obviously, we can put ${}_tr_{1t} = {}_tR_1$. We have

$$(1 + {}_tR_k)^k = (1 + {}_tR_1)\prod_{j=1}^{k-1}(1 + {}_{t+j}r_{1t}), \qquad k = 1, \dots, n. \tag{12}$$

From this formula we can simply obtain the forward rates

$$1 + {}_{t+j}r_{1t} = \frac{(1 + {}_tR_{j+1})^{j+1}}{(1 + {}_tR_j)^j}, \qquad j = 1, \dots, n. \tag{13}$$

The one-period forward rates may simply span any desired length of time. Thus, j-period *forward rate beginning at time* $t+k$ *implied in the term structure at time* t is

$${}_{t+k}r_{jt} = \left(\frac{(1 + {}_tR_{k+j})^{k+j}}{(1 + {}_tR_k)^k}\right)^{1/j} - 1. \tag{14}$$

Due to the liquidity premium the relations between spot and forward rates are rarely fulfilled exactly in practise. Instead of ${}_{t+k}r_{1t}$ we should consider ${}_{t+k}r_{1t} + {}_{t+k}L_{1t}$ where the L's are the liquidity premiums embodied in the forward rates. Usually the liquidity premiums are increasing:

$$0 < {}_{t+1}L_{1t} < \cdots < {}_{t+n-1}L_{1t}.$$

2.6 Continuous Compounding

In theory, continuous compounding plays a crucial role. The idea of continuous compounding comes from the usual concept of compounding for the number of compounding periods approaching to infinity. In this case, we consider the nominal interest rate $i^{(\infty)} =: \delta$ (δ called the *force of interest* or often *interest rate* in the continuous financial mathematics) per unit time so that the future value FV of the initial investment PV (at time $t = 0$) after time T becomes

$$FV_T = PV \lim_{m\to\infty} \left(1 + \frac{i^{(\infty)}}{m}\right)^{Tm} = PVe^{\delta T}. \tag{15}$$

In other words, the future value grows exponentially with time according to

$$\frac{1}{FV_T}\frac{\partial FV_T}{\partial T} = \frac{\partial \ln FV_T}{\partial T} = \delta. \tag{16}$$

This formula is often presented in the form

$$\frac{dFV_T}{FV_T} = \delta dT. \tag{17}$$

If the investment is taken at time t instead of $t = 0$ (usually $t < T$), and is represented by the present value PV_t then

$$FV_T = PV_t\, e^{\delta(T-t)}. \tag{18}$$

So far, we have considered the force of interest to be a constant. But, the above formulation allows us to simply extend it to the case of variable force of interest δ_t depending on time t. The accumulation factor then becomes $\int_t^T \delta_s ds$ instead of $\delta(T-t)$ and the future value at time T of the unit investment at time t therefore is

$$FV_T = PV_t\, e^{\int_t^T \delta_s ds}. \tag{19}$$

Analogously, the expression of the present value in terms of the future value and the time dependent force of interest reads:

$$PV_t = FV_T\, e^{-\int_t^T \delta_s ds}. \tag{20}$$

The function

$$v(t,T) = e^{-\int_t^T \delta_s ds} \tag{21}$$

is called *discount function* and for $t = 0$ it is abbreviated to $v(T)$ so that $v(T) = v(0,T)$.

2.6.1 Example (Stoodley's Formula). A flexible model has been suggested by Stoodley. In spite of the fact that this model is mainly of theoretical interest, it is useful for giving a sight of a possible behavior of the time development of the interest rate. The *Stoodley's formula* says that

$$\delta_t = p + \frac{s}{1 + rse^{st}} \tag{22}$$

where p, r, and s are properly chosen or estimated parameters.

2.6.2 Exercise. Study the behavior of the force of interest following the Stoodley's formula dependent on the parameters appearing in the formula.

2.6.3 Example (Discount Function of the Stoodley's Force of Interest). The calculation needs some algebra. Write t instead of T in the formula for $v(T)$. Then

$$\begin{aligned} v(t) &= \exp\left\{-\int_0^t \left(p + \frac{s}{1 + re^{sy}}\right) dy\right\} = \\ &\exp\left\{-\int_0^t \left(p + s - \frac{rse^{sy}}{1 + re^{sy}}\right) dy\right\} = \\ &\exp\{-(p+s)t + \ln(1 + re^{sy})|_0^t\} = \\ &\exp\{-(p+s)t\}\frac{1 + re^{st}}{1 + r} = \frac{1}{1+r}e^{-(p+s)t} + \frac{r}{1+r}e^{-pt}. \end{aligned} \tag{23}$$

If we put $v_1 := e^{-(p+s)}$, $v_2 := e^{-p}$, we get

$$v(t) = \frac{1}{1+r}v_1^t + \frac{r}{1+r}v_2^t. \tag{24}$$

From this formula it follows that the discount function can be expressed as the weighted average of the present values with **constant interest rates.**

I.3 MEASURES OF CASH FLOWS

present value, future value, annuities, equation of value, internal rate of return, duration, convexity, investment projects, payback method, yield curves

Consider first the sums (payments) $CF_0, \dots, CF_T$ related to the equally spaced time instants $0, \dots, T$. The interest rate for one period i will alternatively mean the *cost of capital,* the *opportunity cost rate,* i.e., the rate of return that can be earned on an alternative investment. Sometimes it is called *valuation interest rate.* The formulas below are formally valid for $i > -1$ but the case $i \geq 0$ is the only realistic one. The vector $\boldsymbol{CF} = (CF_0, \dots, CF_T)^{\mathsf{T}}$ represents a *cash flow.* Values $CF_t > 0$ are *inflows* (amounts received) and $CF_t < 0$ are *outflows* (amounts paid, deposits, costs, etc.) Define the *discount factor* v corresponding to the interest rate i by $v := 1/(1+i)$, the *discount* by $d = 1 - v$, and the *force of interest* δ by the relation $e^{\delta} = 1 + i$ or $\delta = \ln(1+i)$. **Beware** of the fact that here symbol d is different from the same symbol $\boldsymbol{d}$ used from notational reasons in Part III where $\boldsymbol{d}$ will mean the discount function, or more generally the discount process. Summary of the notation:

$$v = \frac{1}{1+i} \qquad d = 1 - v = \frac{i}{1+i} \qquad e^{\delta} = 1 + i \qquad \delta = \ln(1+i)$$

3.1 Present Value

One of the most important characteristics of a cash flow *CF* is its *present value, PV,* also called *net present value, NPV.* "Net" means that inflows and outflows at the same time t are added together and thus represented by a single number CF_t. If needed, the dependence of *PV* on $\boldsymbol{CF}$ and either i, v, or δ will be stressed:

$$(1) \quad PV(\boldsymbol{CF}, i) := PV(\boldsymbol{CF}, v) := PV(\boldsymbol{CF}, \delta) := \sum_{t=0}^{T} \frac{CF_t}{(1+i)^t} = \sum_{t=0}^{T} CF_t v^t = \sum_{t=0}^{T} CF_t e^{-\delta t}.$$

Note that the present value is expressed in *currency units* like USD or CZK.

Let $\mathbb{L}^{T+1}$ be the linear vector space of cash flows, i.e., the space of finite sequences of maximum length $T+1$. If the actual length of a cash flow is less than $T+1$, we complete it by zeros. The present value is a linear function on $\mathbb{L}^{T+1}$ in the following sense: if α, $\beta \in \mathbb{R}$, $\boldsymbol{CF_A}$, $\boldsymbol{CF_B} \in \mathbb{L}^{T+1}$ then

$$(2) \qquad PV(\alpha \boldsymbol{CF_A} + \beta \boldsymbol{CF_B}, i) = \alpha PV(\boldsymbol{CF_A}, i) + \beta PV(\boldsymbol{CF_B}, i).$$

Let us consider the payments $CF_0, \dots, CF_T$ at equally spaced time instants $0, \dots, T$, again, but with different interest rates in the compounding periods

$\boldsymbol{i} := (i_1, \ldots, i_T)$ where i_t is the interest rate applied in the period $(t-1, t)$, $t = 1, \ldots, T$. Then the present value of the given cash flow is

$$(3)\quad PV(\boldsymbol{CF}, \boldsymbol{i}) = CF_0 + \frac{CF_1}{1+i_1} + \cdots + \frac{CF_T}{(1+i_1)\ldots(1+i_T)} = \sum_{t=0}^{T} \frac{CF_t}{\prod_{j=1}^{t}(1+i_j)}$$

where $\prod_{j=1}^{0} := 1$, by definition.

Finally, let us assume that the payments $CF_{t_1}, \ldots, CF_{t_T}$ take place in some general time instants $0 < t_1 < \cdots < t_T$ and the corresponding discount factor is v. Then

$$(4)\qquad PV(\boldsymbol{CF}, v) = CF_{t_1} v^{t_1} + \ldots CF_{t_T} v^{t_T}.$$

This formula may be generalized to the case of an arbitrary starting (or valuating) date t_0. The present value related to this date is then

$$(5)\qquad PV(\boldsymbol{CF}, v) = CF_{t_1} v^{t_1 - t_0} + \ldots CF_{t_T} v^{t_T - t_0}.$$

One must be careful with proper interpretation of time in this case, however.

3.1.1 Example. Consider the calendar convention Actual/360 and a cash flow $CF_{t_1}, \ldots, CF_{t_T}$ where the t_j's now represent dates, the compounding is annual with the discount factor $v = 1/(1+i)$ and the starting date is t_0. Let $d(t_j, t_k)$ denote the number of days between the dates t_j, t_k. Then

$$(6)\qquad PV(\boldsymbol{CF}, v) = CF_{t_1} v^{d(t_1, t_0)/360} + \cdots + CF_{t_T} v^{d(t_T, t_0)/360}.$$

With daily compounding with the interest rate $i^{(360)} = i$ p.a., the formula for the present value reads

$$(7)\qquad PV(\boldsymbol{CF}, i^{(360)}) = \frac{CF_{t_1}}{(1 + i^{(360)}/360)^{d(t_1, t_0)}} + \cdots + \frac{CF_{t_T}}{(1 + i^{(360)}/360)^{d(t_T, t_0)}}.$$

A cash flow often represents an investment opportunity. The dependence of the net present value of such a cash flow is of vital importance for investment decision making. For the first insight, the graphical representation of the dependence of the present value on the cost of capital (valuation interest rate) is of interest.

3.1.2 Example. Let us consider the cash flow from 1.4.1

(–90000, –15200, 45000, 60000, 25000, 22000, 270000) at times $t = 0, \ldots, 6 (= T)$. The *PV* of this cash flow in dependence on the interest rate i is plotted in Figure 3. Such a type of graph is called the *present value profile.*

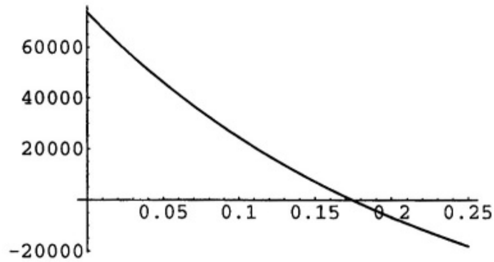

Figure 3: Present value of cash flow

3.1.3 Continuous Case

Speaking of interest rates, we were speaking of present values and future values with constant present values (investments) and a continuously varying force of interest. Here we deal with the case when even the respective cash flow changes continuously. For the sake of simplicity let us suppose that the starting point of time is set to 0 and the time at which the cash flow comes (received or paid, inflows or outflows) is t. Let us denote the cash flow coming for the period $(0, t)$ as $CF(t)$. It means that the net income for the corresponding period will be $CF(t)$, either with plus or minus sign. So the total payment made between (t_1, t_2) is $CF(t_2) - CF(t_1)$. Suppose that *CF* is differentiable so that the derivative $cf(t) = CF'(t)$ exists. Then the increment in income may be expressed as

(8) $$CF(t_2) - CF(t_1) = \int_{t_1}^{t_2} cf(s)ds.$$

Now we have to consider the time value of money. Between the time instants $t, t+dt$, dt being small enough, the total income is approximately $cf(t) \cdot dt$. Therefore, the present value of money received during the time interval $t, t + dt$ is $v(t)cf(t)dt$. So the present value of the cash flow over the whole period (t_1, t_2) is

(9) $$PV(CF, t_1, t_2) = \int_{t_1}^{t_2} v(t)cf(t)dt.$$

3.2 Annuities

Consider a series of T payments, each of amount 1 at times $1, \ldots, T$. Such a stream of payments is called *annuity immediate* (with payments at the end of the period). The present value of this cash flow for $i > 0$ is

(10) $$a_{\overline{T}|} := v + \cdots + v^T = \frac{1 - v^T}{i} = \frac{1 - (1+i)^{-T}}{i}$$

and often it is also called the *Present Value Interest Factor of an Annuity* abbreviated as $PVIFA_{i:T}$. For $i = 0$ we have $a_{\overline{T}|} = T$. Sometimes the interest rate is attached to symbol a: $a_{\overline{T}|i}$ or $a_{\overline{T}|i\%}$.

Consider again a series of T payments, each of amount 1 but now at times $0, \ldots, T-1$. Such a stream of payments is called *annuity due* (with payments at the beginning of the period). The present value of this cash flow for $i > 0$ is

$$\ddot{a}_{\overline{T}|} = 1 + v + \cdots + v^{T-1} = \frac{1-v^T}{1-v} = \frac{(1+i)(1-(1+i)^{-T})}{i}. \tag{11}$$

Clearly, $\ddot{a}_{\overline{T}|} = T$ for $i = 0$. Further,

$$\ddot{a}_{\overline{T}|} = (1+i)a_{\overline{T}|}, \qquad \ddot{a}_{\overline{T}|} = 1 + a_{\overline{T-1}|} \quad \text{for } T \geq 2. \tag{12}$$

For an infinite stream of constant payments of amount 1, the annuity is called *perpetuity* and if it is immediate or due, its present value is

$$a_{\overline{\infty}|} = \frac{1}{i} \qquad \text{or} \quad \ddot{a}_{\overline{\infty}|} = \frac{1+i}{i}, \tag{13}$$

respectively.

3.3 Future Value

Let us consider the valuation date T, a cash flow $CF_0, \ldots, CF_T$, and the above interest rate characteristics i, v, δ. Then the future value is

$$FV(\boldsymbol{CF}, i) := CF_T + CF_{T-1}(1+i) + CF_{T-2}(1+i)^2 + \cdots + CF_0(1+i)^T = \sum_{t=0}^{T} CF_t(1+i)^{T-t}, \tag{14}$$

alternatively

$$FV(\boldsymbol{CF}, v) = \sum_{t=0}^{T} CF_t v^{t-T}, \qquad FV(\boldsymbol{CF}, \delta) = \sum_{t=0}^{T} CF_t\, e^{\delta(T-t)}. \tag{15}$$

Obviously, $FV(\boldsymbol{CF}, i) = (1+i)^T PV(\boldsymbol{CF}, i)$ in this case.

In the case of varying interest rates we have

$$FV(\boldsymbol{CF}, i) = CF_T + CF_{T-1}(1+i_T) + CF_{T-2}(1+i_T)(1+i_{T-1}) + \cdots + CF_0(1+i_T)(1+i_{T-1}) \cdot \ldots \cdot (1+i_1)$$

or

$$FV(\boldsymbol{CF}, i) = \sum_{t=0}^{T} CF_t \prod_{j=t+1}^{T} (1+i_j) \tag{16}$$

with $\prod_{j=T+1}^{T} := 1$.

In case of general time instants (see (4)) and a constant interest rate i we immediately get the obvious relationship

$$FV(\boldsymbol{CF}, i) = (1+i)^{t_T} PV(\boldsymbol{CF}, i). \tag{17}$$

3.3.1 Exercise. Modify the last result to the case of the calendar convention Actual/365.

Let us turn to the annuity immediate of an amount 1 and $i > 0$. The future value of this annuity is

$$s_{\overline{T}|} := 1 + (1+i) + \cdots + (1+i)^{T-1} = \frac{(1+i)^T - 1}{i} = (1+i)^T a_{\overline{T}|}. \tag{18}$$

Analogously, for an annuity due, the future value is

$$\ddot{s}_{\overline{T}|} = \frac{(1+i)^T - 1}{d} = (1+i)^T \ddot{a}_{\overline{T}|}. \tag{19}$$

Both $s_{\overline{T}|}$ and $\ddot{s}_{\overline{T}|}$ are equal to T for $i = 0$.

3.3.2 Exercise. Verify the following relations:

$$\ddot{s}_{\overline{T}|} = (1+i)s_{\overline{T}|} \qquad s_{\overline{T+1}|} = 1 + \ddot{s}_{\overline{T}|}. \tag{20}$$

3.3.3 Remark

Other useful and frequently used relations:

$$\boxed{1 = ia_{\overline{T}|} + v^T \quad 1 = d\ddot{a}_{\overline{T}|} + v^T \quad (1+i)^T = is_{\overline{T}|} + 1 \quad (1+i)^T = d\ddot{s}_{\overline{T}|} + 1} \tag{21}$$

3.3.4 Exercise. Verify and give the interpretation of the preceeding formulas. (Hint: the first formula may be explained as the present value of a loan of amount 1 over the period $0, 1, \ldots, T$).

3.3.5 Remark

If the regular payments are all equal to *PMT* (abbreviation for PayMenT), then the corresponding present and future values are simply multiples by *PMT* of the corresponding a's and s's.

3.3.6 Remark (Equation of Value)

Due to technical and accounting reasons, the strict convention on the signs (inflows plus, outflows minus) leads to the following relations between the five variables involved, i.e., the present value *PV*, the future value *FV*, the interest rate i, the annuity *PMT*, and the number of periods T:

Annuity of amount PMT immediate.

$$PV + PMT\, a_{\overline{T}|} + \frac{FV}{(1+i)^T} = 0 \tag{22}$$

Annuity of amount PMT due.

$$PV + PMT\,\ddot{a}_{\overline{T}|} + \frac{FV}{(1+i)^T} = 0 \tag{23}$$

In the introductory courses, such a type of formulas is known as the *equation of value.* This approach is often used on financial calculators or in spread sheets. The user should carefully input the data with proper plus or minus signs for inflows and outflows, respectively.

3.3.7 Example (Installment Savings). Consider the investment of CZK 5000 in installment savings for 3 years at 3.6 per cent p.a., compounded monthly, so that $i^{(12)} = 0.036$. What will be the total of principal and interest at the end? Reasonably, installment savings represent an annuity due (payments at the beginning of the period) so that the equation of value (23) applies with $PV = 0$, $PMT = -5000$, $i = i^{(12)}/12 = 0.003$, $T = 36$. We have $\ddot{a}_{\overline{36}|} = 38.069$ so that $FV = 190349$. Compare this result with the case of 3 installment savings CZK 60000 at the beginning of every year with yearly compounding at the interest rate $i = 3.6$ per cent p.a. This results in the total savings $FV = 193273$. Give an explanation as an exercise.

3.3.8 Example and Exercise (Loans). Suppose you are able to repay CZK 5000 monthly for a 3 years' loan at $i^{(12)} = 7.2$ per cent p.a., compounded monthly. The question is, how much you can borrow under these conditions. Reasonably, the payments represent an annuity immediate (payments at the end of period) so that (22) applies to loan borrowing power $PV = ?$, $PMT = -5000$, $i = i^{(12)}/12 = 0.006$, $FV = 0$, $T = 36$. Since $a_{\overline{36}|} = 32.29$, you can borrow $PV = 161454$. In case you are able to pay CZK 60000 at the end of each year at the same interest but compounded yearly, you will obtain from (22) with $PMT = -60000$, $i = 0.072$, $T = 3$ that your loan borrowing power will decrease to $PV = 156885$. As an exercise, calculate *PV* under the same conditions if your balance (= remaining debt) is compounded monthly.

3.4 Internal Rate of Return (IRR)

In a simple Example 3.1.2 we have seen that depending on the interest rate the present value of a cash flow takes either positive or negative values. So the critical point is the value of the interest rate that equates the present value to zero. Consequently, we are motivated to define an *internal rate of return* (shortly *IRR*) as a solution to the equation

$$PV(CF, IRR) = \sum_{t=0}^{T} \frac{CF_t}{(1+IRR)^t} = 0. \tag{24}$$

In other words, *IRR* is defined as the interest rate (or the cost of capital) which equates the present value of inflows (incomes) to the present value of outflows (costs):

$$\sum_{t:CF_t>0} \frac{CF_t}{(1+IRR)^t} = - \sum_{t:CF_t<0} \frac{CF_t}{(1+IRR)^t}. \tag{25}$$

The equivalent problem is to find a discount factor v such that

(26)
$$PV(\boldsymbol{CF}, v) = \sum_{t=0}^{T} CF_T v^t = 0.$$

If $CF_T \neq 0$ then the last equation is an algebraic equation of degree T and hence it has T roots. Therefore, by the above definition, we have T internal rates of return. All the solutions can be easily obtained by standard numerical methods. Only real roots greater than –1 may have an economic meaning, however. Some authors define *IRR* as a positive solution to (24). But it can be simply demonstrated that some (rather strange) cash flows possess only positive *IRR*'s with difficult economic interpretation. The cash flow (–1000, 3600, –4310, 1716) has *IRR*'s 0.1, 0.2, 0.3, e.g. Nevertheless for "well-behaved" cash flows we have the following theorem:

3.4.1 Theorem. *Let* $A_j = \sum_{t=0}^{j} CF_t$, $j = 0, 1 \ldots, T$, $A_0 \neq 0$, $A_T \neq 0$. *Suppose that in the sequence* $A_0, \ldots, A_T$ *with zeros excluded the sign changes just once. Then there is exactly one positive IRR.*

Proof. We have the equation

$$\sum_{t=0}^{T} CF_t e^{-\delta t} = 0$$

with $e^\delta = 1 + i$. Since $CF_t = A_t - A_{t-1}$, $(A_{-1} := 0)$, the equation reads

(27)
$$A_0 + \sum_{t=1}^{T} (A_t - A_{t-1}) e^{-\delta t} = 0.$$

Further,

$$\sum_{t=1}^{T} (A_t - A_{t-1}) e^{-\delta t} = \sum_{t=1}^{T} A_t e^{-\delta t} - \sum_{t=1}^{T} A_{t-1} e^{-\delta t} =$$
$$\sum_{t=1}^{T} A_t e^{-\delta t} - \sum_{t=0}^{T-1} A_t e^{-\delta(t+1)} = \sum_{t=1}^{T} A_t e^{-\delta t} - e^{-\delta} \sum_{t=0}^{T-1} A_t e^{-\delta t} =$$
$$(1 - e^{-\delta}) \sum_{t=1}^{T-1} A_t e^{-\delta t} + A_T e^{-\delta T} - e^{-\delta} A_0.$$

Thus (27) may be written as

(28)
$$(1 - e^{-\delta}) \sum_{t=0}^{T-1} A_t e^{-\delta t} + A_T e^{-\delta T} = 0.$$

Without loss of generality suppose that $A_0 > 0$. Then there exists an index k such that $A_t \geq 0$, $t = 1, \ldots, k-1$, $A_k > 0$, $A_t \leq 0$, $t = k+1, \ldots T-1$, $A_T < 0$. Hence (28) becomes

$$(1 - e^{-\delta}) \left[\sum_{t=0}^{k} A_t e^{-\delta t} - \sum_{t=k+1}^{T-1} |A_t| e^{-\delta t} \right] - |A_T| e^{-\delta T} = 0$$

and after multiplication by $e^{\delta k}$ we get

$$(1 - e^{-\delta}) \left[\sum_{t=0}^{k} A_t e^{-\delta (t-k)} - \sum_{t=k+1}^{T-1} |A_t| e^{-\delta (t-k)} \right] - |A_T| e^{-\delta (T-k)} = 0$$

or

$$g_1(\delta) \left[g_2(\delta) - g_3(\delta) \right] - g_4(\delta) = 0,$$

say. All the g_i's are continuous, g_1, g_2 increasing, g_3, g_4 decreasing. Thus $g = g_1 [g_2 - g_3] - g_4$ is continuous and increasing. Moreover,

$$\lim_{\delta \to 0} g(\delta) = A_T < 0 \qquad \lim_{\delta \to +\infty} g(\delta) = +\infty$$

so that there is just one $\delta_0 > 0$ such that $g(\delta_0) = 0$ and $IRR = e^{\delta_0} - 1$ is the only positive *IRR*. □

3.4.2 Remark and Example (Leasing). Financial *leasing* is an alternative form of financing. It takes a form of an agreement between two parties, the *lessee* and the leasing company called *lessor.* The lessee obtains the right to use a (usually real) asset for a period of time while the ownership of that asset remains with the lessor. At the end of the lease the ownership still remains with the lessor. But the residual (or salvage) value is usually negligible. There are many reasons for leasing, let us mention some of them. First, a company or an individual may not have money available to purchase the asset. This is often the case if the asset is too expensive like tanker or airplane. Second, there is a risk that the asset will become obsolete. Third, in most countries there exists a tax deduction advantage to promote investment. See [141], p. 512 for details. The following numerical example presents an analysis of leasing a car. The SKODA car priced CZK 227900 is leased under the following conditions: the lessee pays the sum of CZK 34185 immediately. Then the lessee pays (i) CZK 6943 $=: PMT_{36}$ monthly for 36 months or (ii) CZK 6192 $=: PMT_{42}$ monthly for 42 months. In both cases the payments are at the end of the month and the salvage value of the car is CZK 122. The question arises, what is the effective interest rate counted by the lessor. The *IRR* methodology gives the answer. We have $PV = -34185 + 227900 = 193715$, annuities with the minus sign given above, $T = 36$ and $T = 42$ months, respectively. Using a financial calculator or a spreadsheet program, we find the respective *IRR*'s are $IRR_{36} = 1.446$ and $IRR_{42} = 1.449$ per cent monthly, so that $IRR_{36}^{12} = 17.35$ and $IRR_{42}^{12} = 17.39$ per cent p.a.

Investment projects represented by cash flows are called *normal* or *regular* if the payments change their sign just once, and are called *nonnormal* or *irregular* in the opposite case.

In the above definition of *IRR* we have implicitly supposed that the inflows from the project will be reinvested at the same interest rate, i.e., *IRR*. More often, the inflows are reinvested at the interest rate equal to the current cost of capital k, say. We can overcome this problem by a modification of the definition of *IRR* following the principle:

$$PV(\text{outflows}, k) = PV(FV(\text{inflows}, k), MIRR) \tag{29}$$

where MIRR is called *modified rate of return.* In symbols, MIRR is defined by the equation

$$-\sum_{t:CF_t<0} \frac{CF_t}{(1+k)^t} = \frac{1}{(1+MIRR)^T} \sum_{t:CF_t>0} CF_t(1+k)^{T-t}. \tag{30}$$

It is obvious that in this case (given k) MIRR can be expressed explicitly. Also note that for $k = IRR$ we have $MIRR = IRR$.

Another modification of IRR makes use of different interest rates for outflows and inflows, i.e., the different costs of investment and reinvestment capital, k_O and k_I, respectively. The *modified rate of return* MIRR (we use the same notation) is then defined by

$$-\sum_{t:CF_t<0} \frac{CF_t}{(1+k_O)^t} = \frac{1}{(1+MIRR)^T} \sum_{t:CF_t>0} CF_t(1+k_I)^{T-t}. \tag{31}$$

MIRR can be explicitly calculated again.

Note that sometimes this idea is also used for the valuation of cash flows if different valuation interest rates are used for outflows and inflows. Using the above notation, the present value is expressed as

$$PV(\boldsymbol{CF}, k_O, k_I) = \sum_{t:CF_t<0} \frac{CF_t}{(1+k_O)^t} + \sum_{t:CF_t>0} \frac{CF_t}{(1+k_I)^t}. \tag{32}$$

3.5 Duration

The *duration* is defined as the time-weighted average of the discounted payments:

$$D(\boldsymbol{CF}, v) = \frac{\sum_{t=0}^T tCF_t v^t}{\sum_{t=0}^T CF_t v^t} = \frac{1}{PV(\boldsymbol{CF}, v)} \sum_{t=0}^T tCF_t v^t. \tag{33}$$

Duration is expressed in time units. So if the payments are semiannual, for instance, the duration is expressed in halves of year. It is also called *discounted mean term of the cash flow.* We have

$$\frac{\partial PV(\boldsymbol{CF}, v)}{\partial v} = \frac{1}{v}\sum_{t=0}^T tCF_t v^t = \frac{1}{v} D(\boldsymbol{CF}, v) PV(\boldsymbol{CF}, v)$$

and thus the duration may be expressed as

$$D(\boldsymbol{CF}, v) = \frac{v}{PV(\boldsymbol{CF}, v)} \frac{\partial PV(\boldsymbol{CF}, v)}{\partial v}. \tag{34}$$

In economics the last expression is known as elasticity so that we may interpret the duration as an *elasticity of the net present value with respect to the discount factor.* An alternative formula for the duration expressed in terms of the interest rate reads

$$D(\boldsymbol{CF}, i) = -\frac{1+i}{PV(\boldsymbol{CF}, i)} \frac{\partial PV(\boldsymbol{CF}, i)}{\partial i}. \tag{35}$$

From the above expressions it follows that the duration may serve either as a measure of the sensitivity of the cash flow to the interest rate or as the duration of the corresponding investment project. The first interpretation will become clear if we write the first few terms of the Taylor expansion of the relative increment of the present value of the given cash flow as a function of the interest rate; the derivatives are taken with respect to the second argument:

$$\frac{PV(\boldsymbol{CF}, i+\Delta i) - PV(\boldsymbol{CF}, i)}{PV(\boldsymbol{CF}, i)} = \tag{36}$$
$$\frac{PV'(\boldsymbol{CF}, i)}{PV(\boldsymbol{CF}, i)}\Delta i + \frac{1}{2}\frac{PV''(\boldsymbol{CF}, i)}{PV(\boldsymbol{CF}, i)}(\Delta i)^2 + \dots \quad \approx -\frac{1}{1+i} D(\boldsymbol{CF}, i)\Delta i\,.$$

Note that duration, unlike the present value, **is not** a linear function of the ***CF*'s**. To overcome this disadvantage sometimes the *dollar duration* is used:

$$D_{\$}(\boldsymbol{CF}, v) = \sum_{t=0}^{T} tCF_t v^t. \tag{37}$$

In literature and applications we can also meet the *modified duration*:

$$D_{\text{mod}}(\boldsymbol{CF}, v) = v \cdot D(\boldsymbol{CF}, v)$$

or, in terms of i

$$D_{\text{mod}} = -\frac{1}{PV(\boldsymbol{CF}, i)} \frac{\partial PV(\boldsymbol{CF}, i)}{\partial i}.$$

3.6 Convexity

A finer measure of the sensitivity of a cash flow to the interest rate is the *convexity*:

$$C(\boldsymbol{CF}, v) = \frac{\sum_{t=0}^{T} t(t+1)CF_t v^t}{\sum_{t=0}^{T} CF_t v^t}. \tag{38}$$

Convexity is expressed in squared time units. If the payments are accomplished semiannually, convexity is expressed in [$\text{year}^2/4$], e.g. Taking into account that

$$PV''(\boldsymbol{CF}, i) = \frac{1}{(1+i)^2} C(\boldsymbol{CF}, i) PV(\boldsymbol{CF}, i)$$

we can substitute in (36) and get a more precise formula for the relative increment of the present value in the form

$$(39) \quad \frac{PV(\boldsymbol{CF}, i + \Delta i) - PV(\boldsymbol{CF}, i)}{PV(\boldsymbol{CF}, i)} \approx -\frac{1}{1+i} D(\boldsymbol{CF}, i)\Delta i + \frac{1}{2(1+i)^2} C(\boldsymbol{CF}, i)(\Delta i)^2 .$$

In literature we can find a slightly different definition of the convexity, as an analogue to the modified duration:

$$C_{\text{mod}}(\boldsymbol{CF}, i) = \frac{PV''(\boldsymbol{CF}, i)}{PV(\boldsymbol{CF}, i)} .$$

Then the equation for the relative change of the present value may be expressed in terms of the modified measures as

$$(40) \quad \frac{PV(\boldsymbol{CF}, i + \Delta i) - PV(\boldsymbol{CF}, i)}{PV(\boldsymbol{CF}, i)} \approx -D_{\text{mod}}(\boldsymbol{CF}, i)\Delta i + \tfrac{1}{2} C_{\text{mod}}(\boldsymbol{CF}, i)(\Delta i)^2 .$$

3.7 Comparison of Investment Projects

As usual, investment projects will be represented by the corresponding expected cash flows. Since the future cash flows are uncertain, the results of decision making process are also uncertain. The detailed qualified analysis may reduce uncertainty, however.

We will deal with a set of competing projects. The decision maker may accept one or more projects and may even decide not to accept any. The projects are said to be *mutually exclusive* if at most one of the involved projects can be accepted. And they are said to be *independent* if an arbitrary number of the competing projects (including none of them) can be accepted.

There are two broad classes of investment projects that often arise in practise. In the first case, the investors use their own capital for the initial investment and they obtain incomes generated by the initial investment in successive periods. Such projects are characterized by negative payments in the initial period(s) and positive ones afterwards. Call them *class I projects.* In the second case, the investors take a loan at the beginning, make an investment, and then they acquire the benefits and also should pay back the loan. Such projects are characterized by positive payments in the initial period(s) and negative ones afterwards. Call them *class II projects.*

There is a variety of methods for decision making and we will mention only some of the principles. All the methods start with a careful analysis of the expected stream of payments including dividends, interest obtained or paid, salvage value of the assets at the end of the project's life, etc. The cost of capital (the valuation interest rate) should take into account the riskness (uncertainty) of the project.

3.7.1 Profitability Index

A simple indicator for a class I project $\boldsymbol{CF} = (CF_0, CF_1, \ldots, CF_T)$ is the *profitability index* defined by

$$PI(\boldsymbol{CF}, i) = -\frac{1}{CF_0} PV((CF_1, \ldots, CF_T), i). \tag{41}$$

This measure seems to be trivial but in fact it is, in some sense, equivalent to the measures based on the present value profile as we will see later. Among competing projects we select those with highest profitability indexes greater than one; we select none of them if all *PI*'s are less than one.

3.7.2 Payback Method

Another simple and rough method is the *payback method* applied again to class I projects. It is based on the *payback period* that is the number of periods required to recover the initial outflows. Formally, let us keep assumptions of Theorem 3.4.1. For a class I project we have $A_0 < 0$. Let k be the first index such that $A_k > 0$. Then the *payback period* is defined by

$$PB(\boldsymbol{CF}) = k - 1 - \frac{A_{k-1}}{CF_k}. \tag{42}$$

Here $k-1$ is the period just preceeding the full recovery, $-A_{k-1}$ is the uncovered cost at the beginning of this period, and CF_k (obviously positive) is the payment in the recovering period. If such a k does not exist, we set formally the payback period to infinity. Based on the payback method, we select the project(s) with the shortest payback period, or none of them if their payback periods all equal infinity.

A little better method based on this idea is the so called *discounted payback method.* Let i be a properly chosen project's cost of capital and define $A_j^{(i)} = \sum_{t=0}^{j} CF_t/(1+i)^t$. Assume again $A_0^{(i)} < 0$ and k the first index such that $A_k^{(i)} > 0$. Then the *discounted payback period* is defined as

$$PB(\boldsymbol{CF}, i) = k - 1 - \frac{A_{k-1}^{(i)}}{CF_k/(1+i)^k}. \tag{43}$$

If such a k does not exist we set formally the discounted payback period to infinity. The decisions based on the discounted payback method are the same as in case of the usual payback method.

3.7.2.1 Exercise and Problem. Analyze and try to prove the following conjecture. For a class I project of length T ($A_T > 0$), the discounted payback period approaches T as the interest rate approaches the internal rate of the project, $i \to IRR$.

3.7.3 Methods Based on the Present Value Profile

Typically, for class I projects the present value is a decreasing (and often also convex) function of the valuation interest rate i and the opposite is true for class II projects; the present value is an increasing (and often concave) function of i. However, this is not the rule as shown in the following counterexample.

3.7.3.1 A Counterexample

Consider an artificial cash flow $\boldsymbol{CF_Z} = (-6, -10, -4, -8, -3, -5, 18.5, 18.5)$. The assumptions of Theorem 3.4.1 are fulfilled. The only IRR is 0.006372. But $PV(\boldsymbol{CF_Z}, i)$ is decreasing for $i < 0.39$ and increasing for $i > 0.39$.

Hence the investor should take care of the individual present value profile, i.e., the graph of the present value in dependence on the interest rate involved.

The leading rule is simple; for a given i accept the project if its present value at this interest rate i is positive:

$$\boxed{\text{Accept if} \quad PV(\boldsymbol{CF}, i) > 0}\,.$$

For class I projects, the criterion of positive present value is equivalent to $PI(\boldsymbol{CF}, i) > 1$. In case of independent projects we select all the projects with the positive present values at the given interest rate. If the projects are mutually exclusive we select that with the highest present value. If we investigate a set of projects which are mutually exclusive dependent on the valuation interest rate we should select the project that is determined by the upper envelope of the present value profiles.

For one project, the critical point is *IRR*. If *PV* is a decreasing function of i then we accept the project if the valuation interest rate is less than IRR and reject it otherwise. Analogously, if *PV* is an increasing function of i, we accept the project if the valuation interest rate is greater than *IRR*. For projects which do not possess a monotonous present value profile, we should perform a more careful analysis.

For two or more projects, the critical points are not only the *IRR*'s of the individual projects but also their crossover rates. A *crossover rate* of two projects is such an interest rate for which the present values of the two projects are equal. Formally, let us consider two projects $\boldsymbol{CF_A}$ and $\boldsymbol{CF_B}$. The crossover rate i_{AB} is defined as a solution to the equation

$$PV(\boldsymbol{CF_A}, i_{AB}) = PV(\boldsymbol{CF_B}, i_{AB}).$$

Obviously, there may be more than one solution so that we must select that one with a reasonable economic interpretation. Since the present value is a linear function on the space of cash flows, we see that the crossover rate i_{AB} is in fact the internal rate of return determined by the difference between the two projects, IRR_{A-B}:

$$PV(\boldsymbol{CF_A} - \boldsymbol{CF_B}, IRR_{A-B}) = 0.$$

In the neighborhood of the crossover rate the investor should take care and carefully study also the sensitivity of the present value profiles with respect to the interest rate. This is best done by looking on the duration and possibly on the convexity. Such an analysis will be better understood from the example.

3.7.3.2 Example. Let us consider five projects:

(1) A: $\boldsymbol{CF_A} = (-1000, 300, 500, 200, 100)$

(2) B: $CF_B = (-1000, 47, 47, 47, 1047)$
(3) C: $CF_C = (-851.18586, 281.0005, 170.39716, 300, 200)$
(4) D: $CF_D = (-600, -500, -300, 400, 500, 600)$
(5) E: $CF_E = (1200, -400, -300, -200, -400)$.

Projects A, B, C, D are class I projects while E is a class II project. CF_B represents the cash flow of a four years coupon bond purchased for the par value 1000 giving the holder yearly coupons of 47 with redemption value 1000. The present value profiles of these projects are shown in Figure 4. Visually the present value profiles of the projects A and C coincide. The payback periods for the first four projects are

$$PB(CF_A) = 3.00 \quad PB(CF_B) = 3.82 \quad PB(CF_C) = 3.50 \quad PB(CF_D) = 4.83$$

and the discounted payback periods for two selected interest rates ($i = 0.02$, $i = 0.04$) are: $PB(CF_A, 0.02) = 3.40$, $PB(CF_B, 0.02) = 3.89$, $PB(CF_C, 0.02) = 3.70$, $PB(CF_D, 0.02) = 4.99$ and $PB(CF_A, 0.04) = 3.84$, $PB(CF_B, 0.04) = 3.97$, $PB(CF_C, 0.04) = 3.92$, $PB(CF_D, 0.04) = +\infty$. In case of independent projects, based on the discounted payback method we accept projects A, B, C, D if $i = 0$ or $i = 0.02$. For $i = 0.04$ we accept A, B, C and reject D. If the projects are mutually exclusive, we accept only A for all three values of i.

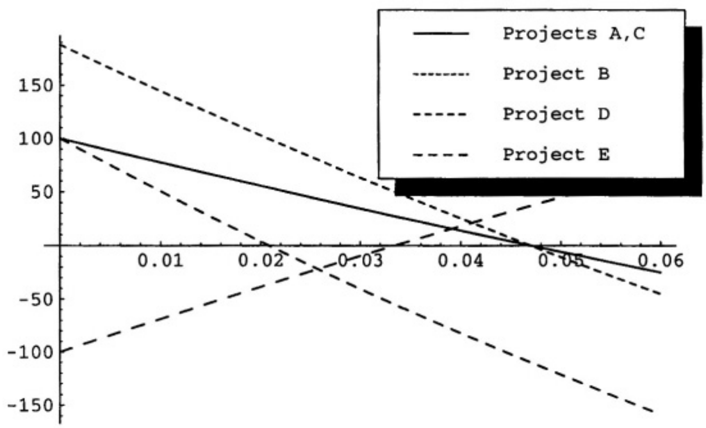

Figure 4: Present values of 5 projects

The present value is a decreasing function of i for projects A, B, C, D, and an increasing function for project E. Thus the acceptance region depends on the corresponding *IRR*'s:

$$IRR_A = 0.0471 \; IRR_B = 0.0470 \; IRR_C = 0.0472 \; IRR_D = 0.0208 \; IRR_E = 0.0333.$$

Consider first the case of independent projects. We accept A, B, C, D for $i \leq 0.021 (= IRR_D)$. For $0.021 < i \leq 0.033 (= IRR_E)$ we accept A, B, C. For $0.033 < i < 0.047$ (approximately) we accept A, B, C, E; and we accept only E for

$i > 0.047$.

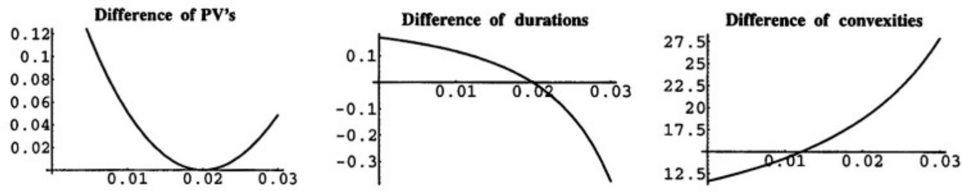

Figure 5: Characteristics of A and C

Second, consider mutually independent projects A, C, E only. Since the projects A and C have almost identical present value profiles, we must look first at the difference of their present values. In Figure 5 we have plots of $PV(\boldsymbol{CF_C}, i) - PV(\boldsymbol{CF_A}, i)$, $D(\boldsymbol{CF_C}, i) - D(\boldsymbol{CF_A}, i)$, and $C(\boldsymbol{CF_C}, i) - C(\boldsymbol{CF_A}, i)$. We see that $PV(\boldsymbol{CF_C}, i) \geq PV(\boldsymbol{CF_A}, i)$ and that the difference is negligible. We also have $PV(\boldsymbol{CF_A}, 0.02) = PV(\boldsymbol{CF_C}, 0.02)$ and $D(\boldsymbol{CF_A}, 0.02) = D(\boldsymbol{CF_C}, 0.02)$. Since the convexities fulfil the inequality $C(\boldsymbol{CF_C}, i) > C(\boldsymbol{CF_A}, i)$ we can decide in favor of project C against A. Further, the crossover rate for projects C and E is $IRR_{C-E} = 0.0391$. Thus to summarize, for $i \leq 0.0391$, we accept C and for $i > 0.0391$ we accept E, among the candidates A, C, E. If we consider all the five projects, then we obviously select B for $i < IRR_{B-E} = 0.041$ and E for greater values of i.

3.7.4 Internal Value

Suppose that the cash flow in question depends also on another variable or parameter y say, $\boldsymbol{CF} = \boldsymbol{CF}(y)$. For decision making, an important measure is the value of $y = y(i)$ such that the present value for a given interest rate i is zero. Call this value the *internal value* of the cash flow and denote it by HIV. (HIV has been introduced in [83] but we admit that such a simple indicator might have been known before.) Mathematically, HIV is defined implicitly by the relation

$$\boxed{\sum_{t=0}^{T} \frac{CF_t(\mathrm{HIV}(i))}{(1+i)^t} = 0\,.} \tag{44}$$

Often, the dependence of the present value on y is simple, for instance linear or quadratic. Hence, for a fixed interest rate, the analysis of the present value profile becomes more simple. Application of HIV is many-sided. Particularly, HIV is useful in valuation of all transactions where the foreign exchange rate appears, like currency swaps. In this case $y = FX$, the foreign exchange rate. The HIV can also be employed for the risk analysis of loans payable in foreign currencies or cash flows dependent on interest rates like LIBOR, etc. If more than one variable influence the cash flow involved, the above definition is still of use. The analysis is more complex in this case, however. Also a two-dimensional analysis if both the interest rate i and y vary is a rather complex task and needs a further research and analysis of particular situations.

3.8 Yield Curves

Generally, a *yield curve* plots interest rates paid on interest bearing securities against the time to maturity. Such a plot makes sense only for a class of comparable securities. Thus we may plot yield curves for government zero coupon bonds for maturities 1, 3, 6, 9, 12 months getting a completely different picture for AA rated firm's bonds for same maturities. Thus we should take into account the risk factors (cf. decomposition of interest rate) and also comparable taxation conditions.

Even for the same type of securities (like T-bills), the shape of the yield curve differs in time, i.e., the shape is different in years 2000 and 2001, say, *ceteris paribus*. This feature may be explained by many factors, like the change in spot riskless rate, inflation, and other exogenous factors. Another important feature is the internal need of the issuer for short, medium, or long financial funds.

Another problem arising with a yield curves' presentation is that the yields may be either *declared* or *actually observed* on the market. Here, by *declared yields* we mean the promised coupon rates for usual fixed coupon bonds while the *actually observed yields* are derived from the spot market price of the respective security, see I.4 for the calculation.

There is an obvious connection between the yield curve and the term structure of interest rates (cf. 2.5); for a given type of security (or a group of similar securities) with different maturities and for a given particular date t, the yield curve is the plot of the spot rates ${}_tR_n$, $n = 1, 2, \ldots$. The difference ${}_tR_n - {}_tR_1$ is called *yield spread*. Sometimes the *forward-rate curve* calculated from (2.5.13) is plotted.

A typical shape of the yield curve is *upward-sloping*, which simply means that the corresponding function is increasing and often concave. Such a yield curve is called *normal yield curve*. On the contrary, the yield curve which is *downward-sloping* (decreasing and often convex) is called *inverted yield curve*. Another shape often arising in practise is a *humped curve*; the yield curve increases at first and then decreases for longer maturities. Rarely we can meet a *flat*, i.e., *constant yield curve* or U-shaped curves. However, rather strange images, different from the above mentioned, can be met with in practise.

The shape and magnitude of the yield curve depend on many factors. Most important are the risk factors, the liquidity preference, and the expected inflation. Increasing risk factors (mainly default risk) cause approximately parallel upper shift of the yield curve. The higher the liquidity preference, the higher the liquidity premium for lending for longer time periods. With increasing expected inflation in future periods the longer-term rates become higher and vice versa. See 2.3 and 2.4 for explanation.

For financial decision making and also for analysis we often need yields for maturities which are not available on the market. Thus we must construct them from existing market data. To this purpose one may use purely numerical approaches like linear interpolation, e.g. Another recommended approach is based on regression models. Suppose that we have N comparable fixed or zero coupon bonds 1, … ,N with maturities $T_1, \ldots, T_N$ and observed yields $y_1, \ldots, y_N$, respectively. The postulated parametric regression model is (see [55], e.g.)

$$y_n = g(T_n; \theta) + \varepsilon_n, \qquad n = 1, \ldots, N, \tag{45}$$

where the hypothetical yield curve g of a known analytical form depends on an unknown vector parameter $\boldsymbol{\theta}$ which is to be estimated, and ε_n are disturbances with zero means. The estimate $\widehat{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$ is obtained as an argument of

$$\min_{\boldsymbol{\theta}} \sum_{n=1}^{N} |y_n - g(T_n; \boldsymbol{\theta})|^{\gamma} \tag{46}$$

for a properly chosen γ ($\gamma = 2$ for the least squares method and $\gamma = 1$ for the absolute deviation criterion, e.g.). There is also a variety of possible choices for the analytical form of g. Having the estimate $\widehat{\boldsymbol{\theta}}$, we may estimate the yield for a nonobserved maturity $T \neq T_n$, $n = 1, \ldots, N$ as

$$\widehat{y}_T = g(T; \widehat{\boldsymbol{\theta}}). \tag{47}$$

One of the simplest forms of g is a *polynomial function* of a small degree K

$$g(t; \boldsymbol{\theta}) = \sum_{k=0}^{K} \theta_k t^k \tag{48}$$

which leads to a polynomial regression. For $K = 3$ the corresponding function is a cubic function and 4 parameters are to be estimated. Due to bad experience with polynomial regression, other types of g are recommended.

One of the successful and recently frequently used models is the model of *cubic splines.* Assuming $T_1 < T_2 < \cdots < T_N$, we consider functions g such that (i) g is a piecewise cubic function, i.e., g equals

$$g_n(t) := \alpha_n + \beta_n t + \gamma_n t^2 + \delta_n t^3 \quad \text{for } t \in [T_{n-1}, T_n], \quad n = 2, \ldots, N, \tag{49}$$

(ii) g is twice continuously differentiable everywhere; this is (together with (i)) equivalent to

$$g_n(T_n) = g_{n-1}(T_n), \quad g'_n(T_n) = g'_{n-1}(T_n), \quad g''_n(T_n) = g''_{n-1}(T_n), \quad n = 2, \ldots, N.$$

We then choose the function $\widehat{g}$ from this class that minimizes a combination of the residual sum of squares and the integrated squared 2nd derivative of g:

$$\widehat{g} = \operatorname*{argmin}_{g} \left\{ \sum_{n=1}^{N} (y_n - g(T_n))^2 + \lambda \int_{T_1}^{T_N} (g''(t))^2 dt \right\}$$

with a smoothing constant $\lambda > 0$. The resulting $\widehat{g}$ represents a compromise between fit of data and smoothness of the fitting curve. Values of the smoothing constant λ cover ordinary least squares fitting by a straight line ($\lambda \to \infty$) as one extreme, and pure numerical interpolation by a piecewise cubic functions ($\lambda = 0$) as the other one. Details of the method together with an algorithm can be found in [150].

Another flexible model has been treated by Bradley and Crane in [24] (see also Example II.5.4.4):

$$g(t; \alpha, \beta, \gamma) = \alpha t^{\beta} e^{\gamma t}. \tag{50}$$

This model should be taken with care, however, because with wide range of observed maturities severe discrepancies may appear, see the Example below and Figure 8 in II.6. After the logarithmic transform and the reparametrization α^* the last equation becomes

$$\ln g(t; \alpha^*, \beta, \gamma) = \alpha^* + \beta \ln t + \gamma t \tag{51}$$

which is linear in parameters and these may be simply estimated by ordinary least squares method.

Two alternative techniques of modeling the term structure of a coupon bond will be discussed in 4.1.3.

3.8.1 Example. Consider declared interest rates for term deposits of the Czech saving company as in February 1999:

Maturity (in days)	7	14	30	60	90	120	150	180	210	240	270	290
Interest rate (p.a.)	5.4	5.4	6.2	6.1	6.1	6.00	6.00	5.9	5.9	5.9	5.9	5.8

The yield curve is humped. Let us make a comparison of three estimating procedures: (i) fitting by a cubic function, (ii) fitting by cubic splines, (iii) fitting by (50). For (i), (iii) there are no alternatives while in case (ii), we have experimentally chosen the smoothing constant as to get the best fit from the optical point of view. The estimated curves along with the original rates are plotted in Figure 6. We see that for such a pattern it is difficult to fit the data satisfactorily by simple analytic models. Particularly, fitting by the cubic function may lead to a dangerous conclusion, i.e., that for longer maturities the yield curve rises again. This is not the only exception. Another example (not presented here) shows that even the polynomial interpolation of a very nice smooth yield curve observed at discrete times (years) 1, … ,30, resembling a parabola, by a polynomial of the degree 29 reveals unrealistic values for some points within the intervals. We strongly recommend not to use the polynomial fitting procedure.

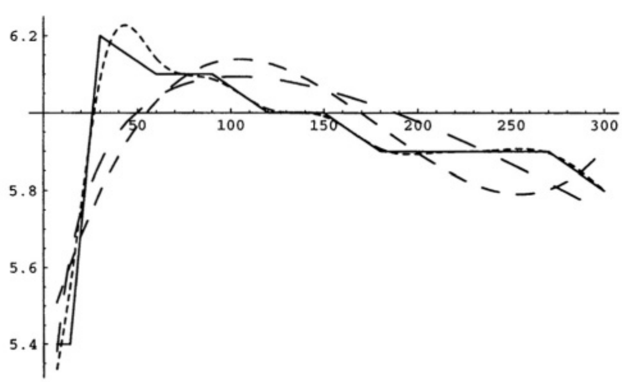

Figure 6: Fitting the yield curve
original rates (broken line) cubic splines (- - - -)
cubic function (– – –) Bradley-Crane (— — —)

I.4 RETURN, EXPECTED RETURN, AND RISK

return, rate of return, random walk hypothesis, Black-Scholes model, risk, parametric value at risk (VaR), nonparametric VaR

Warning

Some symbols used in the following text are very popular both in financial and financial mathematics literature, unfortunately with a different meaning. Particularly, symbols r or R may serve as typical representatives. Sometimes R or r means a return, sometimes the rate of return, sometimes the *expected rate of return,* sometimes the interest rate, etc. Also, there is an ambiguity in distinguishing between a random variable and the expected value of it. In financial literature, a random variable is often stressed by the wave, like $\widetilde{X}$, and the expected value is simply X, while in mathematics X is reserved for a random variable and EX stands for its expected value. The reader is politely asked to pay attention what the respective symbols mean.

4.1 Return

The concept of return should be considered in a dynamic setup; by *return* of a financial asset we mean the difference between the wealth (in monetary units) at the end and the beginning of the period under consideration. Consequently, this leads to the following definition of the *rate of return ROR,* say:

$$ROR = \frac{\text{wealth at the end of the period} - \text{wealth at the beginning of the period}}{\text{wealth at the beginning of the period}}.$$

Suppose that the (market) price of the underlying asset (security) at time t is P_t. Following the above idea, we may simply define the rate of return as

$$R_t = \frac{P_{t+1} - P_t}{P_t}. \tag{1}$$

As in the case of interest rate, we will alternatively use a percentage or a decimal form of the rate of return; $R_t = 0.1$ and $R_t = 10\%$ mean the same. Taking into account the accumulation (multiplicative) effect and an analogy with the force of interest, we can define another measure $r_t^\star$ as a rate of return by

$$1 + R_t = \frac{P_{t+1}}{P_t} =: \exp(r_t^\star), \tag{2}$$

that is,

$$r_t^\star = \ln(1 + R_t) = \ln\frac{P_{t+1}}{P_t} = \ln P_{t+1} - \ln P_t =: p_{t+1} - p_t, \tag{3}$$

by definition. Note that p_t's as defined above are often called *logarithmic prices.* For small values of the rate of return, $r_t^\star$ does not differ from R_t too much. By

Taylor expansion, $\ln(1+R_t) = R_t - R_t^2/2 + \ldots$, so that the difference is of order $O(R_t^2)$. Thus, for $R_t = 0.05$ we have $r_t^\star = 0.04879$, e.g. For higher values of R_t the difference increases. The rate of return R^T for the time horizon T is then defined by the relation

$$1 + R^T = \prod_{t=1}^{T}(1+R_t) = \exp\left(\sum_{t=1}^{T} r_t^\star\right) = \frac{P_T}{P_0}. \tag{4}$$

In case of securities, let us denote by R_t the rate of return for the period t, P_t the (market) price of the respective security at the end of period t, and D_t the dividend paid for the time interval $[t, t+1]$. Then

$$R_t = \frac{P_{t+1} + D_{t+1} - P_t}{P_t} = \frac{D_{t+1}}{P_t} + \frac{P_{t+1} - P_t}{P_t}\ . \tag{5}$$

The first part of the rate of return, D_{t+1}/P_t, represents the so called *dividend yield,* or in case of coupon bonds, *coupon yield,* while the second part, $(P_{t+1} - P_t)/P_t$, represents the *capital yield.* Note that the dividend is usually paid rarely in comparison with the time period considered, once, or twice a year, say. For a correct expression of the rate of return we should incorporate the corresponding part of the dividend into the formula (1). If we consider the time period of one week with the yearly paid dividend D, we substitute $D_t := D/52$, e.g. In the theory, we must also distinguish between expected returns (ex ante) based on subjective probabilities and returns coming from historical data (ex post).

For an asset paying no dividends the rate of return becomes

$$R_t = \frac{P_{t+1} - P_t}{P_t} = \frac{P_{t+1}}{P_t} - 1. \tag{6}$$

4.1.1 Random Walk Hypothesis

Under the *random walk hypothesis* the logarithmic prices follow the model

$$p_{t+1} - p_t = \mu + \varepsilon_{t+1}, \quad t = 0, 1, \ldots \tag{7}$$

where ε_t's are either uncorrelated (weak form) or independent (strong form) identically distributed random variables (shortly iid for the latter case) with $E\,\varepsilon_t = 0$ and $\operatorname{var}\varepsilon_t = \sigma^2$, and μ represents a drift or trend. Next we will suppose that the ε_t's are iid. It follows that the $r_t^\star$'s are iid random variables under the random walk hypothesis. Since for $T \in \mathbb{N}$

$$p_T = p_0 + \mu T + \sum_{t=1}^{T} \varepsilon_t$$

we have $E\,p_T = p_0 + \mu T$ and $\operatorname{var} p_T = \sigma^2 T$. In the stationary case $\mu = 0$ there are only random fluctuations about the initial logarithmic price p_0. For the original prices P_t's we have

$$P_{t+1} = P_t e^{\mu + \varepsilon_{t+1}}$$

or

$$P_T = P_0 \exp\left(\mu T + \sum_{t=1}^{T} \varepsilon_t\right).$$

The ratios P_1/P_0, P_2/P_1, ... ,P_T/P_{T-1} are therefore iid random variables. Also the returns R_0, ... ,R_T are iid under the above assumptions. The case of normally distributed ε's will be treated in the next Section.

Sometimes it is supposed that the original price process is driven by

$$P_{t+1} - P_t = \mu + \varepsilon_{t+1}$$

with analogous assumptions on ε_t's.

Sometimes even an unrealistic assumption is made that the P_t's are independent identically distributed. However, the independence of P_t's does not generally guarantee the independence of the returns R_t's. Just look on the covariance between two successive rates of return:

$$\begin{aligned}(8)\quad \operatorname{cov}(R_t, R_{t-1}) &= \operatorname{cov}(P_{t+1}/P_t, P_t/P_{t-1}) = \\ & E(P_{t+1}/P_{t-1}) - E(P_{t+1}/P_t)E(P_t/P_{t-1}) = \\ \text{(independence)} &= EP_{t+1}E(1/P_{t-1}) - EP_{t+1}EP_tE(1/P_t)E(1/P_{t-1}) = \\ & \text{(identically distributed)} = EP_tE(1/P_t)(1 - EP_tE(1/P_t))\end{aligned}$$

which could hardly be zero.

4.1.2 A Simple Model for Price Development

The model presented in this Section serves as a background for more complicated models like Black-Scholes model for option valuation etc. We need only two assumptions concerning an efficient market: (i) all the past history of the price development is reflected in the present price; (ii) the response of the market on any new piece of information is immediate. Assumption (i) resembles a Markov property.

Let $\Delta t > 0$ and denote $\Delta P := P_{t+\Delta t} - P_t$, $P := P_t$ for a moment, P_0 being a starting price. In the model it is supposed that the return, $\Delta P/P$ in our case, can be decomposed into a deterministic and a stochastic part in the following way:

$$\boxed{\frac{\Delta P}{P} = \mu \Delta t + \sigma \Delta W.} \tag{9}$$

Here the first term $\mu\Delta t$ is the deterministic part, μ is called *drift* or a *trend coefficient* while the second part is a stochastic term with so called *volatility, standard error* or *diffusion* σ and $\Delta W := W(t + \Delta t) - W(t)$ standing for the increment of a standard Wiener process. In more general models, both μ and σ may be also functions of P and t. Recall that the *Wiener process* $\{W(t), t \geq 0\}$ is a stochastic process with continuous trajectories such that $W(0) = 0$ with probability 1, for s, t positive the distribution of $W(t) - W(s)$ is normal $N(0, |t - s|)$, and for any $0 < t_0 < t_1 < \cdots < t_n < \infty$ the random variables $W(t_0)$, $W(t_1) - W(t_0)$,

$\ldots, W(t_n) - W(t_{n-1})$ (the *increments*) are independent. See Part III for more details. Since the distribution of ΔW is $N(0, \Delta t)$, (9) may be written in the form

(10) $$\Delta P = \mu P \Delta t + \sigma P \varepsilon \sqrt{\Delta t}$$

where ε is an $N(0, 1)$ random variable so that the return $\Delta P/P$ possesses the normal distribution $N(\mu\Delta t, \sigma^2 \Delta t)$. This formula is useful for discrete modeling and simulation. Formally, for $\Delta t \to 0$, we obtain the *stochastic differential equation* (SDE, see Theorem 12.6, p. 223 in [93])

(11) $$\boxed{\frac{dP}{P} = \mu dt + \sigma dW.}$$

This equation describes the so called *geometrical Brownian motion,* see Part III 2.2.12. We will now make use of Itô formula to characterize the development of logarithmic prices. For $f = f(P, t)$ the Itô formula reads (see Part III, Corollary 2.2.9)

(12) $$df = \left(\frac{\partial f}{\partial P} \mu P + \frac{1}{2} \frac{\partial^2 f}{\partial P^2} \sigma^2 P^2 + \frac{\partial f}{\partial t} \right) dt + \frac{\partial f}{\partial P} \sigma P dW.$$

Put $f(P) := \ln P$. The first and second derivatives of f with respect to P are $1/P$ and $-1/P^2$, respectively. After some algebra we obtain the solution to (11) for the logarithmic prices:

(13) $$d \ln P = (\mu - \tfrac{1}{2}\sigma^2) dt + \sigma dW.$$

The discrete version of the last equation is (recall that $\ln P = p$)

(14) $$\ln P_{t+\Delta t} - \ln P_t = \ln(P_{t+\Delta t}/P_t) = p_{t+\Delta t} - p_t = (\mu - \tfrac{1}{2}\sigma^2)\Delta t + \sigma\varepsilon\sqrt{\Delta t}$$

with ε distributed as $N(0, 1)$ again.

The solution to the SDE for the price process with given initial value P_0 is

(15) $$P_t = P_0 \exp\{(\mu - \tfrac{1}{2}\sigma^2)t + \sigma W(t)\}$$

(see also 3.1.1 in Part III) so that

$$\mathcal{L}(p_t - p_0) = N((\mu - \tfrac{1}{2}\sigma^2)t, \sigma^2 t)$$

(see 3.1.2 in Part III) and therefore

(16) $$\mathcal{L}(P_t/P_0) = LN((\mu - \tfrac{1}{2}\sigma^2)t, \sigma^2 t),$$

where by the symbol $LN(m, s^2)$ we mean the distribution of the random variable $\exp\{N(m, s^2)\}$, the *log-normal distribution* with parameters m and s^2 which are not its mean and variance, respectively. The density of $LN(m, s^2)$ is

(17) $$g(x; m, s^2) = \begin{cases} \frac{1}{x\sqrt{2\pi s^2}} \exp\{-\frac{1}{2} \left(\frac{\ln x - m}{s}\right)^2\} & x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

The mean of $LN(m, s^2)$ is $E\,LN(m, s^2) = \exp(m + \frac{1}{2}s^2)$, and the variance is $\operatorname{var} LN(m, s^2) = \exp(m + \frac{1}{2}s^2)(\exp(s^2) - 1)$.

As a consequence of (16) we can deduce that the conditional distribution of P_t given P_0 is

(18) $$\mathcal{L}(P_t|P_0) = LN(\ln P_0 + (\mu - \tfrac{1}{2}\sigma^2)t, \sigma^2 t).$$

After some algebra we obtain the conditional expectation and variance:

$$E\,(P_t|P_0) = P_0 e^{\mu t}, \quad \operatorname{var}(P_t|P_0) = P_0^2 e^{2\mu t}(e^{\sigma^2 t} - 1).$$

4.1.3 Important Remark

In this Part, unless otherwise stated, by *returns* we will mean either returns or rates of returns without further specification. Either of the return defined above will be considered as a random variable denoted by ρ, ρ_t, or $\rho(t)$ for the respective time period.

4.1.4 Expected Return

Often, return is a nonnegative random variable but this is not the rule. Let us denote F the distribution function of ρ. The *expected return* of ρ is the expected value

$$r := E\rho = \int_{-\infty}^{\infty} x dF(x).$$

4.2 Risk Measurement

Here we will restrict our explanation only to cases of *quantitative measures* of risk. All of the measures discussed here are based on the variance of the random variable in question, the return in our case.

4.2.1 Standard Deviation – Volatility

Basically, the *risk* of the return is defined as the standard deviation of ρ:

$$\sigma = \sqrt{E(\rho - E\rho)^2} = \sqrt{E\rho^2 - (E\rho)^2}.$$

A *riskless asset* is an asset with $\sigma = 0$ so that the return is a constant with probability one.

In literature we can find an analogous measure based on the variance of the return, called *volatility*. This term is used either for the variance or for the standard deviation of the return or of another stochastic financial variable.

4.2.2 Example. Let us consider two assets, A and B. Suppose that the rates of return randomly depend on the state of the economy in the way showed in Figure 7. Obviously, both assets have the same expected rate of return, r_A and r_B, respectively:

$$r_A = 0.3 \cdot 100 + 0.4 \cdot 15 - 0.3 \cdot 70 = 15 \ [\%],$$

$$r_B = 0.3 \cdot 20 + 0.4 \cdot 15 + 0.3 \cdot 10 = 15 \ [\%].$$

		Return [%]	
State of Economy	**Probability**	**A**	**B**
Boom	0.3	100	20
Normal	0.4	15	15
Recession	0.3	-70	10

Figure 7: States of Economy

Their respective variances are

$$\sigma_A^2 = 0.3 \cdot 100^2 + 0.4 \cdot 15^2 + 0.3 \cdot (-70)^2 - 15^2 = 4335\ [\%^2], \quad \sigma_B^2 = 15\ [\%^2],$$

so that the risks are $\sigma_A = 65.84[\%]$ and $\sigma_B = 3.87[\%]$. We conclude that the investment into asset B is less risky than into A.

4.2.3 Value at Risk

Another useful and recommended measure of risk is called *Value at Risk,* shortly *VaR.* (Distinguish between symbols VaR – Value at Risk and var – the variance.) *Value at Risk at confidence level* $1-\alpha$, shortly VaR_α is defined by the relation

$$P(\rho < -\mathrm{VaR}_\alpha) = \alpha. \tag{19}$$

In words, $-\mathrm{VaR}_\alpha$ is the cut-off point under which the return will attain values only with some given (small) probability α. Thus $-\mathrm{VaR}_\alpha$ is the 100α per cent quantile of the distribution of ρ. Different financial institutions use different levels of confidence; the Bankers Trust 99 per cent, J P Morgan 95 per cent, Citibank 95.4 per cent, e.g. Otherwise the confidence level is stated in reverse form, 1 per cent, 5 per cent, etc., but the meaning is the same; the maximum possible loss will be more than VaR with probability α or it will be less than VaR with probability $1-\alpha$.

4.2.3.1 Parametric VaR

Let us start with the so called *parametric* VaR. Suppose that the random return possesses a distribution from a *location-scale* family of distributions. Let $G(x)$ be a distribution function free of any other parameters and suppose that the distribution function $F_{\mu,\sigma}(x)$ of the return ρ is of the form

$$F_{\mu,\sigma}(x) = G\left(\frac{x-\mu}{\sigma}\right) \tag{20}$$

where μ is a real number called *location parameter* and $\sigma > 0$ is called *scale parameter.* In what follows, we deal with distributions for which the location parameter μ is equal to the expected return and the scale parameter σ is equal to the standard deviation. If we denote the 100α per cent quantile of the distribution function $G(x)$ as u_α, then we get for VaR_α

$$P(\rho < -\mathrm{VaR}_\alpha) = P\left(\frac{\rho-\mu}{\sigma} < \frac{-\mathrm{VaR}_\alpha - \mu}{\sigma}\right) = G\left(\frac{-\mathrm{VaR}_\alpha - \mu}{\sigma}\right) = \alpha \tag{21}$$

and therefore

$$\boxed{\mathrm{VaR}_\alpha = -\mu - \sigma u_\alpha.} \tag{22}$$

Sometimes, the last quantity is called *absolute value at risk* while

$$\mathrm{VaR}_{\alpha\,\mathrm{rel}} = -\sigma u_\alpha \tag{23}$$

is called *relative value at risk.* Parameters μ and σ are usually unknown (even if the analytical form of the distribution G is supposed to be known) and typically they should be estimated by their sample counterparts.

Such an approach is good if we want to calculate VaR for the return based on data coming from the respective period. If we need VaR for a subsequent period or periods, we must take into account that both the mean μ and the volatility parameter σ can change in time. There are two simple models with a theoretical background that overcome this problem.

Firstly, we assume that the mean return does not change in time, but the variance of it is proportional to time. So if we consider the prospective return after T periods after the original parameters had been obtained, we suppose that the variance is

$$\sigma_T^2 = T\sigma^2. \tag{24}$$

It follows that the value at risk may now be computed from the formula

$$\text{VaR}_\alpha = -\mu - \sigma\sqrt{T}\,u_\alpha. \tag{25}$$

Secondly, if we suppose that the mean is proportional to time, i.e., $\mu_T = T\mu$, then the formula for VaR becomes

$$\text{VaR}_\alpha = -T\mu - \sigma\sqrt{T}\,u_\alpha. \tag{26}$$

Often it is supposed that G is the distribution function of the standard normal distribution Φ. Numerous examples show that this is not a frequent case in practise, however.

Formulas (20) to (26) relate to **return** ρ and to its characteristics μ, σ. If ρ is the **rate of return** instead, and μ, σ its characteristics, then the value at risk in (20) – (26) is expressed in terms of the initial investment P_0 as unit, in other words, the maximum possible loss (in dollars) is $-\mu P_0 - \sigma u_\alpha P_0$.

4.2.3.2 Nonparametric VaR

If only little is known about the analytical (parametric) form of the returns' distributions but a sufficient amount of (historical) data is available, then a proper method for the risk analysis may be based on a nonparametric approach. Suppose that the observed returns during a given period (one year, say) are $R_1, \ldots R_T$. For data based on daily closing prices from a stock exchange we have about $T = 250$ observations yearly, e.g. We rank the observed returns to get the ordered random sample

$$R_{(1)} \le \cdots \le R_{(T)}.$$

Instead of the theoretical quantile u_α in the above considerations we will use the *empirical* αth *quantile* $\widehat{u}_\alpha$ defined for $0 < \alpha < 1$ by

$$\widehat{u}_\alpha = \begin{cases} R_{(\lfloor T\alpha \rfloor + 1)} & \text{if } T\alpha \ne \text{ an integer} \\ \frac{1}{2}[R_{(T\alpha)} + R_{(T\alpha+1)}] & \text{if } T\alpha \text{ is an integer} \end{cases}. \tag{30}$$

For a chosen confidence level α we may state that the return will not fall under $\widehat{u}_\alpha$ with probability $1 - \alpha$. Similarly, the conclusion for VaR in case of a loss follows. This may be accepted as true for a one-period prospective, in the above case for one year ahead. The extension to more than one period needs some kind of speculation, however. Some regression techniques for a trend investigation may be helpful in this case.

4.2.4 Remark (The Distribution of R and Related Quantities)

The simplest assumption in accordance with the random walk hypothesis is that the R's are iid and moreover that they are normally distributed. The empirical studies reveal that this is often not the case. Usually we meet a violation of the zero *skewness* and zero *excess* property of the supposed normal distribution. Sometimes the problem of symmetry is not too severe for returns, but an important violation may be observed with other characteristics. Concerning excess, the difference between the theoretical value for a normal random variable (equal to 0) and the actually observed values sometimes appears to be significant. In [159], p. 45, the reader may find an analysis of the excess of stock returns which shows that the distribution of the respective returns is far from normal. See also [109].

4.2.5 Stress Testing

Often it is of interest for an investor to know what will happen if the market conditions attain their extremes, either in positive or negative direction from the investor's point of view. Of course, the more unfavorable, the more important they are for the investor's decision making, and they resemble VaR (in the sense of maximum possible loss) to some extent. A possible method to see what will happen is based on a *scenario analysis*. *Stress testing* starts with a construction of scenarios covering the extreme situations involved. The scenarios may be developed either from historical experience (*historical scenarios*) or from a theoretical model of the further development of the characteristic in question (*hypothetical scenarios*).

Stress testing catches the dynamics. It is therefore a task for the decision maker to state the limits or maximum likely changes for the periods of time under investigation. There are some recommendations. For example (see [89]), the Derivatives Policy Group suggests the following guidelines for the extreme movements of the variables involved in derivative's products (all given in basic points) for a one month's period; parallel yield curve shift ±100, yield curve twisting (change in shape) ±25, stock exchange index change ±10, foreign exchange rate change ±6, volatility change ±20.

A computational problem can arise with stress testing. If the time horizon covers T periods, say, and we consider four possible outstanding values of a variable in question (typically maximum, minimum, mean, and median), we have to generate 4^T scenarios and afterwards to evaluate the desired indicator or measure. For a typical ten years' currency swap described in Example 1.6.4.4 with the interests paid semiannually we have $T = 20$, so that the total number of scenarios is $4^{20} = 1,099,511,627,776$, a pretty large number of scenarios to be analysed. Note that actually two variables affect the resulting cash flow in Example 1.6.4.4; the exchange rate and LIBOR. Hence in fact there are even more than four possibilities at every period, at least at the initial and the final period.

To avoid this trouble, usually only a few (relative to the total number) of scenarios are selected and the desired measures evaluated. The typical trajectories then cover the most optimistic and most pessimistic (worst–case) scenarios consisting of all maximum and minimum values together with the average or median trajectory.

Another reduction of the size of the problem may be reached by a careful selection from the whole set of scenarios. A useful technique of such a selection is a

Monte Carlo simulation approach. We sample a number of scenarios at random and evaluate the desired characteristics of every sampled scenario. Such characteristics create a random sample and its useful descriptive statistics can be calculated. Since these statistics are obtained from a large number of characteristics, thousands say, we may employ the standard statistical inference based on a normal distribution's assumption, using the central limit theorem's argument. Note that the Monte Carlo simulation is generally a very useful device for the risk analysis.

I.5 VALUATION OF SECURITIES

valuation of different securities (bonds, options, forwards, and futures), arbitrage, hedging, put-call parity, Black-Scholes formula, binomial model

5.1 Coupon Bonds

Consider a simple coupon bond, coupons fixed, see 1.6.1.5. For the sake of simplicity **assume that the coupons are paid annually**. The cash flow to the holder of the bond is $-P, C, C, \ldots, C, C+F$, where P is the value invested into purchasing the bond. At the time of issuing, the issuer sells the bond for its face value F. If this is not the case, there is something wrong with the initial setup of the coupon rate. Usually bond valuation does not consider the initial cost of purchasing the bond, P, and rather takes into account only the future cash flow resulting from the coupon payments and the redemption of the face value at the maturity date, so that the corresponding cash flow becomes $C, C, \ldots, C, C+F$. Moreover, the history of the past payments is of no interest for the holder, and he or she values the security on the basis of the expected future cash flow only.

More formally, let us suppose that the time of valuation is t while the maturity time is $T, t < T$. The coupon payments take place in times $T - \lfloor T-t \rfloor, T - \lfloor T-t \rfloor + 1$, ... ,$T$. At time T there is the additional payment of the face value F. Altogether we have $\lfloor T-t \rfloor + 1$ payments. With the valuation interest rate i, and the corresponding discount factor $v = 1/(1+i)$, we can express the present value of the above cash flow sometimes called the *dirty, gross, fair,* or *full price* or *value* of the bond as

$$(1) \quad PV = Cv^{T-\lfloor T-t \rfloor - t} + Cv^{T-\lfloor T-t \rfloor - t+1} + \cdots + (C+F)v^{T-t} = C \sum_{j=0}^{\lfloor T-t \rfloor} v^{T-\lfloor T-t \rfloor - t+j} + Fv^{T-t}.$$

This formula provides a correct expression of the present value of the bond. There is one point to be discussed, however. If $T-t$ is an integer, the above formula assumes the immediate payment of the coupon at time t. In practise this is hardly the case because the issuer states the clause of so called *ex-coupon*. It means, that after some date, called *ex-coupon date,* the bond is traded without the first forthcoming coupon and the coupon payment belongs to the former holder of the bond. Thus it is more realistic to adapt (1) to

$$(2) \qquad PV = Cv + Cv^2 + \cdots + (C+F)v^{T-t} = C \sum_{j=1}^{T-t} v^j + Fv^{T-t}.$$

Sometimes it is useful to invert the time by setting $n = T - t$. In this case, n means the time to maturity (n need not be an integer). Then

$$(3) \qquad PV = Cv^{\{n\}} + Cv^{\{n\}+1} + \cdots + (C+F)v^n = Cv^{\{n\}} \sum_{j=0}^{\lfloor n \rfloor} v^j + Fv^n$$

with the first term missing if n is an integer. The value $Cv^{\{n\}}$ is called *accrued interest.* Using the simple interest method, the accrued interest can be expressed as $C/(1+\{n\}i)$. The two values slightly differ, of course. Accrued interest is a reward to the seller of the bond compensating the loss of the next forthcoming coupon. The difference between the dirty price and the accrued interest is called *pure price, pure value, net value* of the bond which therefore takes the form

$$PV_p = Cv^{\{n\}}\sum_{j=1}^{\lfloor n\rfloor} v^j + Fv^n, \tag{5}$$

which is also quoted in the financial press.

A very important measure of a bond is the so called yield to maturity. Let us suppose that the market price of the bond is *MP.* Consider the value of the bond expressed in terms of interest rate i, either from (1) or (3), $PV(i)$, *ceteris paribus.* Then the *yield to maturity, YTM,* is defined as a solution to the equation

$$MP = PV(YTM). \tag{6}$$

Since *YTM* is in fact the internal rate of return and the assumptions of Theorem 3.4.1 are fulfilled, there is just one *YTM.*

Another very simple but frequently used measure of a bond is its *current yield*:

$$\text{Current yield} = \frac{c}{MP}. \tag{7}$$

Note that so far we have supposed that the coupons are paid annually. We will discuss other than annual frequency of coupons later.

For further analysis it is convenient to suppose that n is an integer. Then the value of the bond (immediately after the coupon payment), now identical with the net value, becomes

$$PV(c,F,n,i) = C\sum_{j=1}^{n} v^j + Fv^n = C\frac{v(1-v^n)}{1-v} + Fv^n = \frac{1}{i}F(c+(1+i)^{-n}(i-c)). \tag{8}$$

Note, that in ancient literature this formula is used for calculation of the net value of the bond if n is not an integer. In this case, the net value is calculated as the linear interpolation between values $P_0 = PV(c,F,\lfloor n\rfloor+1,i)$ and $P_1 = PV(c,F,\lfloor n\rfloor,i)$. The interpolated value is $P_n = P_0+(1-\{n\})(P_1-P_0)$. The dirty value is calculated as $P_n+(1-\{n\})C$, the term $(1-\{n\})C$ standing for the accrued interest, without taking into account discounting.

From formula (8) we can immediately deduce that the net value of the bond at the maturity date equals its par value: $PV(c,F,0,i) = F$. Further, for the valuation interest rate i equal to the coupon rate c, $i=c$, the net value of the bond is equal to the par value independently of the time to maturity: $PV(c,F,n,c) = F$. For $i > c$, $PV(c,F,n,i)$ is a decreasing function of n and $PV(c,F,n,i) < F$ for

$n > 0$. The reverse is true for $i < c$, so that the net value is an increasing function of n and $PV(c, F, n, i) > F$. Hence, in case $i > c$, the bond is called a *discount bond* while in case $i < c$ the bond is called a *premium bond*. Thus, an increase of the interest rate will cause the value of the bond to fall, whereas a decrease of this rate will cause it to rise. As n approaches 0 (this means, to the maturity date), the net value of the bond approaches its par value F. An analysis of (8) also shows that, *ceteris paribus*, bonds with longer maturities are more sensitive to changes of i than those with shorter ones.

After some algebra we get a formula for the duration corresponding to the net value expressed in terms of the discount factor:

$$D_n = \frac{1}{1-v} + \frac{n-1-nv(1+c)}{1-v+cv(v^{-n}-1)} \tag{9}$$

or in terms of the valuation interest rate:

$$D_n = 1 + \frac{1}{i} - \frac{1+i+n(c-i)}{i+c((1+i)^n-1)}. \tag{10}$$

For $i = c$ the expression for the duration simplifies to

$$D_n = \frac{1+c-(1+c)^{1-n}}{c}. \tag{11}$$

5.1.1 Example. Suppose we have a bond with par value $F = 1$ and coupon $C = 0.1$ (all in thousands CZK) so that the coupon rate is $c = 0.1$, that is 10 per cent p.a.

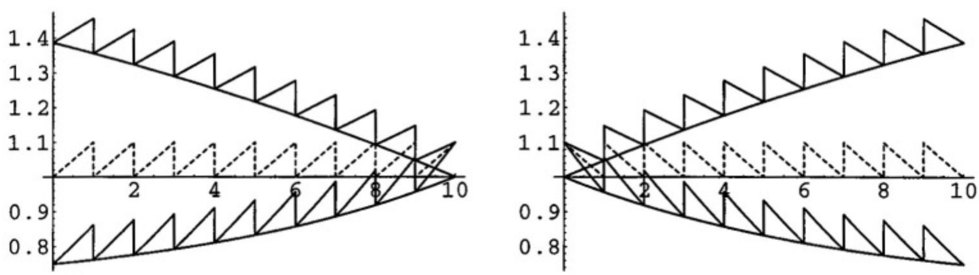

Figure 8: Value of the bond: left – normal time, right – inverted time
sawed – full price, smooth – pure price, dashed – full price for $i = c$

If the market interest rate is 0.05, the bond is a premium one, if it is 0.15, the bond is a discount one. The dependence of the price of the bond on time to maturity is graphically illustrated in Figure 8. The decreasing function on the left part of the figure corresponds to the price of the bond with the valuation interest rate $i = 0.05$, etc.

5.1.2 Exercise. Investigate modifications of the above formulas in case that the coupon payments appear semiannually, i.e., with frequency 2, which is perhaps the most frequent case.

5.1.3 Remark (Construction of the Yield Curve of Coupon Bonds)

The simplest way is to take a set of similar coupon bonds with different maturities and their calculated yields to maturity. Then some method of fitting discussed above may be applied. An alternative approach is known as *bootstrapping*. (Do not confuse with the same term used in statistics!) The idea consists in valuating every coupon payment (and also the principal) using the corresponding spot interest rate. Thus the present value formula (8) for the valuation of the bond with maturity T at time t is now

$$PV = \frac{C}{1+{}_tR_1} + \frac{C}{(1+{}_tR_2)^2} + \cdots + \frac{C+F}{(1+{}_tR_T)^T}. \tag{12}$$

Suppose for the simplicity that there are exactly T coupon bonds with maturities $1, \ldots, T$, fixed coupons $C_1, \ldots, C_T$, face values $F_1, \ldots, F_T$, and (observed) market prices $MP_1, \ldots, MP_T$. For the bond j, the present value is expressed as

$$PV_j = \frac{C_j}{1+{}_tR_1} + \cdots + \frac{C_j+F_j}{(1+{}_tR_j)^j}, \quad j = 1, \ldots, T. \tag{13}$$

For the first bond, the yield to maturity ${}_tR_1$ is, according to (6), calculated from the relation

$$MP_1 = \frac{C_1+F_1}{1+{}_tR_1}. \tag{14}$$

For the second bond we use or bootstrap the information from the first bond (${}_tR_1$ already ascertained) using the relation

$$MP_2 = \frac{C_2}{1+{}_tR_1} + \frac{C_2+F_2}{(1+{}_tR_2)^2},$$

and by recursion, having known ${}_tR_1, \ldots, {}_tR_{j-1}$, we calculate the yield ${}_tR_j$ from the relation

$$MP_j = \sum_{k=1}^{j-1} \frac{C_j}{(1+{}_tR_k)^k} + \frac{C_j+F_j}{(1+{}_tR_j)^j}. \tag{15}$$

Care should be taken if the maturities are not equally spaced. In any case, some fitting procedure is almost always necessary.

5.1.4 Callable Bonds

A callable bond means that the issuer has the right to call back the bond prior to the designed maturity. In fact, in our terminology, the issuer is the holder of a call option which has some value itself. Therefore, from the point of view of the holder of the bond, the price of the callable bond is

$$\boxed{\text{price of the bond without callable feature} - \text{price of the call option.}}$$

The value of this call option may be derived by standard methods given later in 5.2.5 and Part III. Similarly, *putable bonds* can be issued and valued, see II.6.4.

5.1.5 Remark (Amortized Bonds)

An *amortized bond* is characterized by constant installment payments of the principal and interest like a loan, see 3.3.8. Suppose that the cash flow of the usual coupon bond is $\boldsymbol{CF_c} = (C, \ldots, C, C + F)$ and that of the amortized bond $\boldsymbol{CF_a} = (C^*, \ldots, C^*)$, both of the same length. As an exercise, find C^* such that the two bonds are equivalent in the sense of the equality of their present values under the valuation interest rate equal to the coupon rate $c = C/F$. Remind that an amortized bond is less risky than a classical coupon bond (it immediately repays the principal) and hence, in practise, the risk premium offered by the issuer is not as high as in case of the usual coupon bond. The actual C^* is less than that of calculated on the above equivalence principle.

5.1.6 Remark (Simple Bonds under Uncertainty)

Suppose that a zero coupon bond pays F with probability p and pays ηF with probability $1 - p$ at maturity, where $\eta \in [0, 1)$ is called the *recovery rate* and $\lambda := 1 - \eta$ is called the *loss rate*. The case $\eta = 0$ is equivalent to the default. Suppose that the valuation interest rate is r. The fair value of the bond is the expected present value:

$$\overline{PV} = \frac{1}{1+r}[pF + (1-p)\eta F].$$

Further, let us consider a one-period coupon bond with coupon C and par value F which sells for par F now and pays $C + F$ with probability p and $\eta(C + F)$ with probability $1 - p$ at maturity, η having the same meaning as above. The question is what is the fair value of the coupon C under the valuation interest rate r. We equate the present value and the discounted expected future value

$$F = \frac{1}{1+r}[p(C+F) + (1-p)\eta(C+F)],$$

solve for C, and get the fair value of the coupon

$$C = \frac{F[1 + r - p - (1-p)\eta]}{p + (1-p)\eta}.$$

As an exercise, extend the above one-period case to a multiperiod one.

5.2 Options

We start with the valuation of options since the ideas of their pricing are general enough to be used for the valuation of other derivative securities. The key concepts, arbitrage and hedging play a crucial role in the mathematical modeling.

5.2.1 Arbitrage

All the models treated in this book assume *no-arbitrage principle*, in other words, the absence of arbitrage opportunities. By an *arbitrage opportunity* we mean any of the two situations:

(1) At the same time, the same asset is sold at different prices at different places. Nowadays, this can hardly happen in the financial world since the

information from one stock exchange is available on the stock exchange on the opposite side of the globe within a second.

(2) With zero investment at time 0 there is no probability of loss but there is a possibility of a riskless profit at time 1. More rigorously, an *arbitrage opportunity* in this case is a self-financing trading strategy with no initial investment, and a positive probability of positive profit and zero probability of negative profit later on (cf. III.3.3 and III.3.3.1).

Arbitrage opportunity is often characterized as a "money pump" and no-arbitrage principle by the slogan: "There is no such thing like a free lunch.".

5.2.2 Hedging

Hedging may be compared to insurance. It provides an insurance against unfavorable development of the market from the investor's point of view. Hedging may reduce the risk but, under no-arbitrage principle, risk cannot be fully eliminated. In principle, *hedging* consists of taking two opposite positions in the assets which are highly negatively correlated. The investor who hedges his/her position is called *hedger.* A *perfect hedge* means that the hedger combines an option and an underlying asset in such proportions that result in a riskless position and provide a riskless profit (equal to the riskless interest rate). See also III. 3.3.5. This is a rather idealized situation, however, since it does not take transaction costs into account.

5.2.3 Notation

We will assume the continuous-time world with constant riskless rate of interest (force of interest) r applied both to borrowing and lending. Symbols c and p will stand for the price of a European CALL and PUT, respectively. Analogously, symbols C and P will be used for prices of the respective American options. The price of the underlying asset (usually stock) will be denoted S and we will suppose that there are no liabilities like dividends connected with this asset during the period involved. We will also assume that S is a random variable or, more generally a stochastic process, an approach consistently adopted in Part III. Finally let K denote the strike price and T the expiry date. If necessary, we add subscript t to stress the dependence of the respective quantity on time, S_t, S_T, c_t, etc. If the option is exercised, denote the time of exercising by τ, $\tau \leq T$. For Europeans, $\tau = T$.

The payoff of an exercised call option is

$$(S_\tau - K)^+ \tag{16}$$

and that of a put option is

$$(K - S_\tau)^+. \tag{17}$$

(16) and (17) are called *terminal payoffs.* At any given $t < T$, the value $S_t - K$ for a CALL and $K - S_t$ for a PUT is called the *intrinsic value* of the respective option. This is the value which the option would have if it were exercised at time t. If the intrinsic value is positive, zero, or negative, we say that the option is *in the money*, *at the money,* or *out of money,* respectively.

5.2.4 PUT – CALL Parity

Let us consider the portfolio long one asset, long one PUT, and short one CALL. It means that we have bought one asset plus one PUT on that asset and sold one CALL on the same asset. Both the options on the asset in the portfolio are European with the same expiry date T and the same strike price K. The value of the portfolio at time $t \leq T$ is therefore

(18) $$\Pi_t = S_t + p_t - c_t.$$

Look what will happen at the expiry date. The value of the portfolio becomes

(19) $$\Pi_T = S_T + (K - S_T)^+ - (S_T - K)^+.$$

If $S_T \leq K$, then $\Pi_T = S_T + K - S_T - 0 = K$, and if $S_T > K$, then $\Pi_T = S_T + 0 - (S_T - K) = K$. We conclude that such a portfolio is riskless and leads to the certain gain K. What is the value of the portfolio at time $t < T$? Since the future value is K, the present value is $\Pi_t = Ke^{-r(T-t)}$ (for riskless investment we have used the riskless interest rate r). Thus we have obtained so called *put-call parity relation*:

(20) $$S_t + p_t = Ke^{-r(T-t)} + c_t, \quad t \leq T.$$

This is an example of risk elimination. Note that this formula cannot be applied to American options due to the early exercise feature.

5.2.5 Option Pricing

5.2.5.1 Natural Boundaries

The limited liability condition says that all option prices are non-negative. Since American options have all features like Europeans plus the right of an early exercise, they must be worth at least the Europeans:

$$C_t \geq c_t, \qquad P_t \geq p_t.$$

Further, from the put-call parity relation it follows that

$$c_t \geq (S_t - Ke^{-r(T-t)})^+.$$

For an American CALL we have

$$C_t \geq (S_t - K)^+.$$

The proof is by the contrary; suppose that $0 \leq C_t < S_t - K$. Then we can buy the CALL at C_t, immediately exercise it and thus get a riskless profit $S_t - K - C_t$ which is in a contradiction to the no-arbitrage principle.

5.2.5.2 Exercise. The quantities which are not explicitly mentioned remain constant. Prove the following propositions:

(1) If $t_1 \leq t_2$ then $C_{t_1} \geq C_{t_2}$ and $P_{t_1} \geq P_{t_2}$.
(2) If $K_1 \leq K_2$ then $c_t(K_1) \geq c_t(K_2)$ and $p_T(K_1) \geq p_t(K_2)$. The same holds for the Americans.
(3) If $S_1 \leq S_2$ then $c_t(S_1) \leq c_t(S_2)$ and $p_t(S_1) \geq p_t(S_2)$. The same holds for the Americans.

5.2.5.3 The Black–Scholes Formula

Let us consider a European call option on a stock, the current price of which is known and equal to S_t. Since the payoff at the expiry date T is $(S_T - K)^+$, the present value of this payoff is

$$e^{-r(T-t)}(S_T - K)^+.$$

Next we adopt the so called *risk-neutral* valuation. Under this approach we do not consider any risk preferences of the investors. Since the higher the level of risk aversion, the higher the expected return μ will be for a risky asset, by excluding the risk preferences we conclude that the only correct risk-neutral μ is $\mu = r$, the riskless rate. At this point it is important to emphasize that by the above choice we **do not assert** that the conditional distribution of S_T given S_t is that for which $\mu = r$! It seems to be reasonable to take the conditional expected value of the discounted payoff given the current value of the underlying asset S_t as the value of the option but with the expectation taken with respect to the riskless rate r:

$$c_t = e^{-r(T-t)}\, E^*\left((S_T - K)^+ | S_t\right). \tag{21}$$

where E^* stands for the expected value in a risk-neutral world. In Black-Scholes approach we suppose that the conditional distribution of S_T given S_t adjusted for risk-neutrality (see formula (18) in 4.1.2) is log-normal

$$\mathcal{L}^*(S_T|S_t) = LN(\ln S_t + (r - \frac{1}{2}\sigma^2)(T-t), \sigma^2(T-t)). \tag{22}$$

To evaluate (21) under assumption (22) we first calculate the expected value

$$E(m, s^2) = E\,(X - K)^+$$

where the random variable X possesses a log-normal distribution $LN(m, s^2)$ with the probability density function given by formula (17) in 4.1.2. After some algebra we get

$$E(m, s^2) = \int_K^\infty (x - K) g(x; m, s^2) dx = \tag{23}$$

$$e^{m+\frac{1}{2}s^2}\,\Phi\left(\frac{m + s^2 - \ln K}{s}\right) - K\Phi\left(\frac{m - \ln K}{s}\right),$$

where Φ stands for the distribution function of the standard normal distribution $N(0, 1)$. Substituting $m \to \ln S_t + (r - \frac{1}{2}\sigma^2)(T-t)$ and $s^2 \to \sigma^2(T-t)$ into the expression for $E(m, s^2)$ gives

$$E^*\left((S_T - K)^+ | S_t\right) = S_t e^{r(T-t)}\Phi(d_1) - K\Phi(d_2), \tag{24}$$

where

$$d_1 = \frac{\ln(S_t/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 = \frac{\ln(S_t/K) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}. \tag{25}$$

Altogether, going back to (21) we have derived the *Black-Scholes formula* for the value of a European call option:

$$c_t = S_t\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2). \tag{26}$$

An elementary application of the put-call parity relation provides the value of a European put option

$$p_t = Ke^{-r(T-t)}\Phi(-d_2) - S_t\Phi(-d_1), \tag{27}$$

where we used $\Phi(x) = 1 - \Phi(-x)$.

5.2.5.4 The Binomial Option Pricing Model

We will now assume that the stock price changes only at the equally spaced time instants t, $t+1$, ... The time unit may be arbitrary (month, day, hour, ...). Further let us suppose that if the stock price is S_t at time t, then at time $t+1$ it may take only one of two values, dS_t or uS_t with probability p or $1-p$, respectively. Thus

$$P(S_{t+1} = uS_t|S_t) = p, \quad P(S_{t+1} = dS_t|S_t) = 1 - p =: q. \tag{28}$$

Also suppose that the changes are mutually independent and the probabilities do not depend on time. By the no-arbitrage principle we may suppose that the riskless interest rate r fulfills $d < 1 + r < u$. (Suppose on the contrary that $1 + r < d < u$, e.g. Then the investor could borrow any amount of cash at the riskless rate r, buy the stocks and sell them for at least d after one period. Such a strategy would lead to a riskless profit $d - r - 1$.) Note that the usual assumption is $d < 1 < u$ so that the price can move up and down. Next we state a relationship among p, u, d, and r in a risk-neutral world. The expected return from holding the stock should be the same as the riskless return resulting from the investment S_t at the riskless rate r. Since

$$E(S_{t+1}|S_t) = puS_t + qdS_t,$$

by the above argument we conclude that

$$puS_t + qdS_t = (1 + r)S_t$$

and this is fulfilled for

$$p = \pi := \frac{1 + r - d}{u - d}. \tag{29}$$

Due to our assumptions, $\pi \in (0, 1)$ is a probability called the *risk-neutral probability*. We can also obtain the risk neutral probability by the following construction. Consider the so called *replicating portfolio* consisting of A stocks and B riskless bonds which gives the same payoff as one European call option on the stock with the strike price K and expiry date $t + 1$. The terminal value of the option is

$$c_{t+1}^u := (uS_t - K)^+ \quad \text{with probability } p$$

and

$$c_{t+1}^d := (dS_t - K)^+ \text{ with probability } q.$$

Thus A and B must satisfy the system of equations

$$AuS_t + B(1+r) = c_{t+1}^u, \quad AdS_t + B(1+r) = c_{t+1}^d.$$

The solution to this system is

$$A = \frac{c_{t+1}^u - c_{t+1}^d}{(u-d)S_t}, \quad B = \frac{uc_{t+1}^d - dc_{t+1}^u}{(1+r)(u-d)}. \tag{30}$$

From the obvious inequality $uc_{t+1}^d \leq dc_{t+1}^u$ we observe $B \leq 0$ so that the replicating portfolio always involves borrowing cash and buying the stock in the above proportions. The present value of the CALL is, after substitution from (30), given by

$$c_t = AS_t + B = \frac{(1+r-d)c_{t+1}^u + (u-1-r)c_{t+1}^d}{(1+r)(u-d)} = \frac{1}{1+r}(\pi c_{t+1}^u + (1-\pi)c_{t+1}^d), \tag{31}$$

where π is the risk-neutral probability defined in (29).

Up to now we have considered a one-period model. Let us look on a simple generalization for a multi-period model. After two periods we obviously have

$$P(S_{t+2} = u^2 S_t | S_t) = p^2, \quad P(S_{t+2} = udS_t | S_t) = 2pq, \quad P(S_{t+2} = d^2 S_t | S_t) = q^2.$$

Generally, after $T-t$ periods ($T > t$, T integer)

$$P(S_T = u^j d^{T-t-j} S_t | S_t) = \binom{T-t}{j} p^j q^{T-t-j}, \quad j = 0, \ldots, T-t. \tag{32}$$

This is the *binomial model* describing the probability distribution of the stock price after $T-t$ periods. By $\mathrm{Bi}(n,p)$ we will denote the *binomial distribution* with parameters n, p, i.e., the distribution of a random variable X such that $P(X = j) = \binom{n}{j} p^j q^{n-j}$, $j = 0, 1, \ldots, n$.

Consider now a European call option with strike price K and expiry date T. Using the same argument as in the derivation of (21), but with discrete discounting, the value of the option at time t is given by

$$c_t = (1+r)^{t-T} E((S_T - K)^+ | S_t). \tag{33}$$

In a risk-neutral world we should have used the risk-neutral probability π but the option price can be expressed for an arbitrary p. We have

$$c_t = (1+r)^{t-T} \sum_{j=0}^{T-t} \binom{T-t}{j} p^j q^{T-t-j} (u^j d^{T-t-j} S_t - K)^+. \tag{34}$$

Let J be the smallest non-negative integer such that $u^J d^{T-t-j} S_t \geq K$. Put

$$p^\star := \frac{up}{1+r} \qquad q^\star := \frac{dq}{1+r}.$$

Then

$$(35) \quad c_t = S_t \sum_{j=J}^{T-t} \binom{T-t}{j} (p^\star)^j (q^\star)^{T-t-j} - K(1+r)^{t-T} \sum_{j=J}^{T-t} \binom{T-t}{j} p^j q^{T-t-j}.$$

If $p = \pi$, the risk-neutral probability, then $p^\star + q^\star = 1$ so that in this case we can express (35) in the form

$$(36) \qquad c_t = S_t P(\mathrm{Bi}(T-t, p^\star) \geq J) - K(1+r)^{t-T} P(\mathrm{Bi}(T-t, p) \geq J).$$

With the binomial model, a number of questions arise. We have seen that even under the assumption of the risk-neutral probability there are some degrees of freedom in choice of u and d. We just mention how to handle the unknown parameters appearing in the model. Some ideas are based on comparing the parameters of the discrete model to those of the continuous one. Another popular relationship between u and d is $u = 1/d$. The choice $p = \frac{1}{2}$ is also popular. Such assumptions reduce the dimension of the respective parametric space and open space for a broad discussion. See [172] and [105] for more details.

Like in classical probability theory, also here there is a close connection between the binomial model and its limiting counterpart, the normal distribution model, as a consequence of the central limit theorem for iid random variables with finite positive variances. See [144] for more details.

Since the binomial model is discrete, it enables a straightforward modeling by Monte Carlo simulation. The simulation models take the advantage of the fact that on different stages of the dynamic simulation, numerous specific features and movements of the real life problems may be incorporated. Note that some of the mentioned movements, particularly shocks, may hardly be considered in a theoretical model.

5.2.5.5 Options on Assets Paying Dividends

So far we have considered the underlying stock that does not pay any dividend. We can modify the above results also for a dividend-paying stock. If a stock pays a dividend during the life time of the option, the payment of the dividend causes the stock price to fall by an amount equal to the dividend. The *dividend yield* y is expressed as a proportion of the stock price. For the purposes of this Part, we will suppose that the dividend yield is constant and understood as continuous like the force of interest. Hence during the time interval $(t, t + \Delta t)$ the stock pays $y S_t \Delta t$. For European options we may still use Black-Scholes' type formulas (25), (26) but now with

$$(37) \qquad d_{1,2} = \frac{\ln(S_t/K) + (r - y \pm \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}.$$

The formula for a PUT is given by (27) but with $d_{1,2}$ from (37).

5.2.5.6 Valuing American Options

The following widely used, but, in our opinion, questionable argument enables to value an American option on an asset which does not pay dividends. From the inequalities of 5.2.5.1 we have

$$C_t \geq c_t \geq S_t - Ke^{-r(T-t)} > S_t - K.$$

If the option is exercised at time $\tau < T$, then its value immediately becomes $S_\tau - K$ which is less than the lower bound if the option is still alive. It follows that an American call option will never be exercised prior to its expiry date and hence the value of an American CALL should be the same as that of the corresponding European CALL.

There is no such argument for American put options and/or American options on dividend paying assets. Further information on the topic can be found in [105], [116], [172], e.g.

5.2.5.7 Comparative Statics – The Greeks

In option pricing formulas there are actually five variables (also called parameters), S_t, K, r, $T-t$, and σ. The sensitivity to the option prices on these variables plays a crucial role in financial decision making and is measured by partial derivatives. Since traditionally these sensitivities are denoted by Greek letters, they are often called *Greeks.*

In what follows we will suppose that the options involved are European and that their prices are driven by the Black-Scholes formula (26) and (27). Note that the definitions in the form of derivatives given below can be used in a more general setup. Also note that the respective sensitivities for PUTs can be usually simply calculated using the put-call parity relation (20). Let V denote the value of either a CALL or a PUT.

Delta. The delta Δ is defined by

$$\Delta = \frac{\partial V}{\partial S_t}. \tag{38}$$

After some algebra we get the delta for a call and a put option:

$$\Delta_c = \Phi(d_1), \quad \Delta_p = \Delta_c - 1 = -\Phi(-d_1). \tag{39}$$

Obviously, since $\Delta_c > 0$, the value of a CALL is always an increasing function of S_t. The reverse is true for a PUT. The concept of delta is used for so called *delta hedging*. Suppose we are long one asset and short A call options on that asset. The value of such a portfolio is therefore $\Pi_t = S_t - Ac_t$. We wish to determine A so as to make the value of the portfolio invariant with respect to (small) changes in the asset price, i.e.,

$$\frac{\partial \Pi_t}{\partial S_t} = 0.$$

It follows that the desired $A = 1/\Delta_c$, the so called *hedge ratio*. Nevertheless, since delta is changing in time, the portfolio should be rebalanced frequently and the hedge should be a *dynamic hedge*. A dynamic hedge can be rather costly, particularly in case the transaction costs are not negligible. So dynamic hedging strategies (as well as other strategies dependent on a frequent trading in time) are good for market makers or brokers and others with low transaction costs. Observe that the lower the asset price in comparison with K, the higher the hedge ratio. A similar measure of the sensitivity is the *elasticity* of the call price with respect to the asset price defined by $E_c = \Delta_c S_t / c_t$. Note that always $E_c > 1$ (prove as an exercise). Hence the call option is more risky than the underlying asset. An analogue to delta is the duration.

Gamma. The gamma Γ is the second derivative of V with respect to the asset price:

$$\Gamma = \frac{\partial^2 V}{\partial S_t^2}. \tag{40}$$

Gamma for CALL and PUT is the same:

$$\Gamma = \Gamma_c = \Gamma_p = \frac{\varphi(d_1)}{S_t \sigma \sqrt{T-t}} \tag{41}$$

where φ is the probability density function of the standard normal distribution $N(0, 1)$. Since $\Gamma > 0$, the values of both types, CALL or PUT, are convex functions of S_t. Observe that gamma resembles the convexity introduced in I.3.6.

Theta. The theta Θ is the time derivative of V:

$$\Theta = \frac{\partial V}{\partial t}. \tag{42}$$

The calculation is a bit cumbersome but useful exercise (a good idea is to use some CAS (Computer Algebra System) like Mathematics®):

$$\Theta_c = -Ke^{-r(T-t)} \left(\frac{\sigma}{2\sqrt{T-t}} \varphi(d_2) + \Phi(d_2) \right). \tag{43}$$

Alternatively, using the identity

$$K\varphi(d_2) = S_t e^{r(T-t)} \varphi(d_1) \tag{44}$$

we get it in the form:

$$\Theta_c = -\frac{\sigma S_t}{2\sqrt{T-t}} \varphi(d_1) - Kre^{-r(T-t)} \Phi(d_2). \tag{45}$$

We see that Θ_c is always negative. From the put-call parity relation we obtain

$$\Theta_p = \Theta_c + Kre^{-r(T-t)}. \tag{46}$$

Nothing can be said about the sign of the last expression.

Rho. The rho P expresses the dependence on the riskless rate:

$$\mathrm{P} = \frac{\partial V}{\partial r}; \tag{47}$$

for a CALL it takes the form

$$\mathrm{P}_c = K(T-t)e^{-r(T-t)}\Phi(d_2) \tag{48}$$

and for a PUT

$$\mathrm{P}_p = -K(T-t)e^{-r(T-t)}\Phi(-d_2). \tag{49}$$

Immediately we see that $\mathrm{P}_c > 0$ and $\mathrm{P}_p < 0$.

Vega. The vega $\mathcal{V}$ measures the sensitivity of the option price with respect to the volatility σ of the underlying asset:

$$\mathcal{V} = \frac{\partial V}{\partial \sigma}. \tag{50}$$

For both types of options $\mathcal{V}$ is the same:

$$\mathcal{V} = \mathcal{V}_c = \mathcal{V}_p = S_t\sqrt{T-t}\varphi(d_1). \tag{51}$$

Sometimes it is also of interest to investigate the sensitivity to the strike price but for unknown reasons the corresponding Greek is missing. Nevertheless we have

$$\frac{\partial c_t}{\partial K} = -e^{-r(T-t)}\Phi(d_2) \tag{52}$$

which is always negative and

$$\frac{\partial p_t}{\partial K} = e^{-r(T-t)}\Phi(-d_2) \tag{53}$$

which is always positive, both these conclusions in accordance with an intuitive insight.

5.2.5.8 Exercise. Derive formulas (41), (48), (51), (52), (53).

5.2.5.9 Volatility and Implied Volatility

The parameter σ, the volatility, is of vital importance in option pricing. Since it is difficult to speculate on its value, usually some estimates must be used.

One of the most frequently used estimates, called the *historical volatility,* is based on past data. In practise, this estimator is, in fact, the usual sample standard deviation, for instance, of quantities $\ln(S_{t+1}/S_t)$, $t = 1, \ldots, T-1$ in the Black-Scholes model. A care must be taken however: The time steps for such a calculation must be in accordance with time units in which the other quantities are measured.

More sophisticated estimation procedures are based on models of the stochastic behavior of volatility. See [26] for a review and [127] for a bootstrap estimation of volatility.

Since the other parameters in the formulas for option pricing are known at time t, and also the market value V_t^M of the option is known, then, after substituting these known values of the parameters into the Black-Scholes formula (26), we can determine the unknown volatility. The corresponding equation reads

$$V_t^M = f(S_t, K, \sigma, T, t, r), \tag{54}$$

where f is the function resulting from the Black-Scholes formula. A solution $\widehat{\sigma}$ to (54) is called *implied volatility.* Equation (54) is to be solved for an unknown σ given the values of all remaining quantities. We see that volatility cannot be explicitly expressed from (54) so that a solution must be found numerically. Moreover, it is not clear, how many solutions to the mentioned equation exist. If there are more than one, we should carefully analyze them with respect to a reasonable financial interpretation.

Modern computer algebra systems provide the users with a variety of routines and financial application libraries which can be used for the above analysis. See [147], [148], and http://www.wolfram.com for a possible approach. Some specific cases may be found in the series of papers of Benninga and Wiener: [9], [10], [11], [12], [13], [14].

5.2.5.10 Example. Let us consider 6 options, 3 CALL's and 3 PUT's, on a Volkswagen stock priced at EUR 70.72 April 23, 1999, expiring 3rd Friday, June 1999 with strike prices $K_1 = 67.5$, $K_2 = 70.0$, $K_3 = 72.5$. The actual prices for the respective CALLs were 6.31, 4.92, 3.77 and those for the PUTs 2.92, 4.08, 5.48. We have $T - t = 58/360$, $r = 0.05$. The implied volatilities computed using function `FindRoot` in Mathematica are 0.38, 0.38, 0.35 for CALLs and 0.41, 0.42, 0.43 for PUTs, respectively.

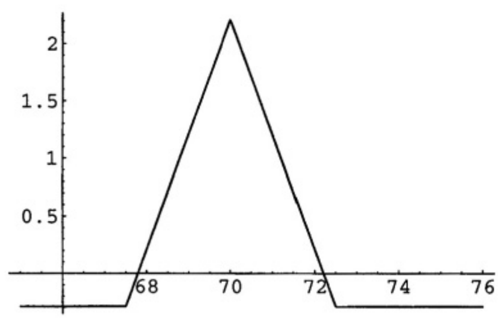

Figure 9: Payoff at expiry of a butterfly spread

Let us further consider the portfolio consisting of four the above options: long CALL with strike K_1, long PUT with strike K_3, short CALL with strike K_2, and short PUT with strike K_2. The value of that portfolio as a function of the stock price S at expiry with today's prices of the options is:

$$V(S) = -6.31 - 5.48 + 4.92 + 4.08 + (S - K_1)^+ + (K_3 - S)^+ - (S - K_2)^+ - (K_2 - S)^+.$$

This is an example of a combination of options, particularly the so called *butterfly spread.* See Figure 9 for the payoff of this portfolio.

5.2.5.11 Exercise. Examine and plot the payoffs of the following combinations of options (all on the same asset) at expiry:

(1) long CALL with strike K_1 and short CALL with strike $K_2 > K_1$ and prices $c_2 < c_1$ (*bullish spread*),
(2) long one CALL with strike K_1, long one CALL with strike $K_3 > K_1$, short two CALLs with strike $K_2 = (K_1 + K_3)/2$ and prices $c_1 > c_2 > c_3$. This is also a *butterfly spread.*
(3) long one CALL, long one PUT with the same strike K, called a bottom *straddle,*
(4) reverse of (3): short one CALL, short one PUT with the same strike K, called a *top straddle.*

Explain the motivation for the above strategies.

5.3 Forwards and Futures

Valuing both forwards and futures is practically the same from the mathematical point of view and since there is no option but obligation to deliver, it is simpler than that of valuing options. The seller must deliver the asset at time T. We can derive the forward price by the non-arbitrage principle. The seller borrows an amount S_t (=the price of the underlying asset at time t) at riskless rate and buys the asset. Hence the *forward price* must be

$$F_t = S_t e^{r(T-t)}$$

otherwise there will be a riskless profit or loss in contradiction to the nonexistence of the arbitrage. The asset may pay a dividend or need to be stored (like gold, grain, or oil) with some additional costs. If the corresponding rate is y, then the forward price becomes

$$F_t = S_t e^{(r-y)(T-t)}.$$

Note that $y > 0$ in case of dividends and $y < 0$ if there are some additional costs.

I.6 MATCHING OF ASSETS AND LIABILITIES

matching, immunization, dedicated bond portfolio, static model, dynamic model, stochastic model

6.1 Matching and Immunization

In what follows in this Chapter, by *assets* we mean the inflows and by *liabilities* the outflows of a company. The main purpose of *matching* is to balance assets and liabilities in such a way that the deficiency is either zero or as small as possible. Perhaps only of theoretical value is the case of *absolute matching*; let a_t and ℓ_t be the total assets and liabilities at time t, $t = 0, \dots, T$, respectively. If $a_t = \ell_t$ for all t we say that assets and liabilities are *absolutely matched.* This does not sound realistic, however, so that an alternative approach is needed. The most frequent method is to match the discounted cash flows and/or other characteristics of assets and liabilities.

Suppose that the liabilities (assets) are represented by a cash flow $\boldsymbol{l} \in \mathbb{L}^{T+1}$ ($\boldsymbol{a} \in \mathbb{L}^{T+1}$), see 3.1. The principle of matched present values of assets and liabilities at force of interest δ is then expressed as

$$PV(\boldsymbol{l}, \delta) = PV(\boldsymbol{a}, \delta). \tag{1}$$

This identity can only be satisfied for finite number of δ's with the exception of $\boldsymbol{l} = \boldsymbol{a}$, the absolute matching. In practise, one can choose the force of interest δ_0 which he or she believes will be most likely for the period of time under consideration. Then the matching condition for the present values is

$$PV(\boldsymbol{l}, \delta_0) = PV(\boldsymbol{a}, \delta_0). \tag{2}$$

Since δ_0 is only an estimate of δ, there is a danger that for some other forces of interest, even close to the estimated one, the present value of liabilities will exceed that of assets. So it is a good idea for an investor to *immunize* his or her position by imposing further conditions expressed in terms of derived characteristics of cash flows. The condition

$$D(\boldsymbol{l}, \delta_0) = D(\boldsymbol{a}, \delta_0) \tag{3}$$

requires the same duration of assets and liabilities and the condition

$$C(\boldsymbol{l}, \delta_0) < C(\boldsymbol{a}, \delta_0) \tag{4}$$

guarantees that at least in a small neighborhood of δ_0, $PV(\boldsymbol{a}, \delta_0) > PV(\boldsymbol{l}, \delta_0)$ will hold. If we change condition (4) and require instead

$$C(\boldsymbol{l}, \delta_0) = C(\boldsymbol{a}, \delta_0), \tag{5}$$

we can give an explicit solution to the problem.

Suppose $\boldsymbol{l} \in \mathbb{L}^{T+1}$ is a given vector of liabilities, fixed in the sequel. Let $\mathbb{A} \subseteq \mathbb{L}^{T+1}$ be a 3-dimensional linear subspace of available assets generated by the base assets $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ which form the basis of $\mathbb{A}$. Let us denote $\mathbf{E} := (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, a $(T + 1) \times 3$ matrix. Thus every $\boldsymbol{a} \in \mathbb{A}$ may be expressed as $\boldsymbol{a} = \mathbf{Ex}$, where the coefficients $\mathbf{x} = (x_1, x_2, x_3)^\top$ are uniquely determined. Let us further denote

$$\mathbf{d}_j := \frac{\partial^j}{\partial \delta^j}(1, e^{-\delta}, e^{-2\delta}, \ldots, e^{-T\delta})^\top \Big|_{\delta=\delta_0},$$

the derivatives of the discount factors taken at $\delta = \delta_0$, $j = 0, 1 \ldots$, and $\mathbf{D} = (\mathbf{d}_0, \mathbf{d}_1, \mathbf{d}_2)$, a $(T + 1) \times 3$ matrix. The three conditions (2), (3), and (5) may now be rewritten in the form

$$\mathbf{d}_j^\top \boldsymbol{a} = \mathbf{d}_j^\top \boldsymbol{l}, \qquad j = 0, 1, 2$$

or equivalently

$$\mathbf{D}^\top \boldsymbol{a} = \mathbf{D}^\top \boldsymbol{l}.$$

If we substitute $\boldsymbol{a} = \mathbf{Ex}$, we get the system of three linear equations for unknown $\mathbf{x}$:

$$\mathbf{D}^\top \mathbf{E}\mathbf{x} = \mathbf{D}^\top \boldsymbol{l}$$

which possesses the unique solution

$$\mathbf{x} = (\mathbf{D}^\top \mathbf{E})^{-1} \mathbf{D}^\top \boldsymbol{l} \tag{6}$$

provided the inverse exists. Exactly the same formula holds true if we, instead of three matching conditions, impose matching conditions employing higher order derivatives.

6.2 Dedicated Bond Portfolio

An important application of the idea of matching assets to liabilities is the investment strategy known as a *dedicated bond portfolio*, (see [60] e.g.) which deals with a proper selection of available bonds. In general, we may think about allocation of funds among arbitrary investment opportunities represented by their expected cash flows. A stochastic version of this problem is treated in Part II.1.2.

6.2.1 Static Model

Suppose that we have the time horizon $t = 1, \ldots, T$ with investment opportunities represented by cash flows $\boldsymbol{CF}_1, \ldots, \boldsymbol{CF}_N$, $\boldsymbol{CF}_n = (CF_{n1}, \ldots, CF_{nT})^\top$, $n = 1, \ldots N$ and initial acquisition costs (i.e., the cost for buying these cash flows) $\mathbf{c} = (c_1, \ldots, c_N)^\top$. It means that c_n is the cost of the investment at time $t = 0$ resulting in the expected future cash flow $\boldsymbol{CF}_n$, $n = 1, \ldots, N$. Further, let $\boldsymbol{l} = (l_1, \ldots, l_T)^\top$ be the expected liabilities over the considered time horizon. Let the initial wealth of the investor be $W = 1$. The objective is to create a portfolio $\mathbf{x} = (x_1, \ldots, x_N)^\top$ (to find the weights in other words, $\mathbf{1}^\top \mathbf{x} = 1$) consisting of the above cash flows so as to minimize the total acquisition costs $\mathbf{c}^\top\mathbf{x}$ subject to $J+1$ matching conditions

$$\sum_{n=1}^{N} x_n PV^{(j)}(\boldsymbol{CF}_n, \delta) = PV^{(j)}(\boldsymbol{l}, \delta), \qquad j = 0, \ldots, J \tag{7}$$

where $PV^{(j)}$ stands for the jth derivative of the present value with respect to the force of interest δ. For $j = 0$ it means the perfect match of present values both of assets and liabilities, for $j = 1$ and 2 the perfect match of durations and convexities, respectively, etc. Further imposed conditions on portfolio may be of the type

$$\mathbf{b}_L \leq \mathbf{x} \leq \mathbf{b}_U. \tag{8}$$

The lower limit $\mathbf{b}_L$ may represent the reasonable amounts of investment while the upper limit $\mathbf{b}_U$ may take into account some legal requirements. For example, in the Czech Republic, pension funds are not allowed to invest more than 10 per cent into one asset. In our terms, it means that the respective $x_n \leq 0.1$. We also add the natural condition $\mathbf{x} \geq \mathbf{0}$. Altogether, we have a problem of linear programming:

$$\boxed{\text{Find } \min \mathbf{c}^\top\mathbf{x} \quad \text{under restrictions (7), (8),} \quad \mathbf{x} \geq \mathbf{0}.} \tag{9}$$

If we abandon condition (8), the theory says that the optimal solution $\mathbf{x}^*$ will consist of at most $J+1$ (=number of conditions) positive weights. For $J > N-1$ there may be no solution to the problem. However, this case is of theoretical interest only, since in practise we usually ask just for matching up to convexity, $J = 2$, in this case.

Since there is an uncertainty about the valuation force of interest δ, we usually need to solve the above problem for a set (scenarios) of expected interest rates and to discuss the solutions from the fundamental point of view.

6.2.2 Dynamic Model

In the above model, the only dynamics involved has been included via present values. In practise, the liability schedule is often determined at any time instant, $t = 1, \ldots, T$. This may be the case of obligatory balances, reserves, or solvency margins. At time t, the inflow is

$$a_t = \sum_{n=1}^{N} x_n CF_{nt}, \tag{10}$$

so that the necessary conditions to meet the liabilities at any time now read

$$\sum_{n=1}^{N} x_n CF_{nt} \geq l_t, \qquad t = 1, \ldots, T. \tag{11}$$

It is a good idea for the investor, even under condition (11), to reinvest a possible surplus. Suppose that i_t is the short-term reinvestment interest rate for the period

$(t, t+1)$, and s_t^+ is the surplus at time t. Then the inequality condition (11) becomes the equality

$$\sum_{n=1}^{N} x_n CF_{nt} + (1 + i_{t-1}) s_{t-1}^+ - s_t^+ = l_t, \qquad t = 1, \dots, T, \tag{12}$$

with the initial surplus s_0^+, if any. Again, the optimal solution is given by solving the linear program

Find $\min(\mathbf{c}^\top \mathbf{x} + s_0^+)$ under restrictions (8), (12), $\mathbf{x} \geq \mathbf{0}$, $\mathbf{s}^+ \geq \mathbf{0}$,

where $\mathbf{s}^+ := (s_1^+, \dots, s_T^+)^\top$.

6.2.3 Discussion of the Restrictions

Note that if the short-terms interest rates are higher than the interest rates coming from the investment into *CF*'s, the solution will naturally result in $\mathbf{x} = \mathbf{0}$ and some positive s^+'s.

6.3 A Stochastic Model of Matching

Here we give a simple stochastic version of the model given in 6.1. Suppose that the force of interest δ is now a random variable. Denote $\mathbf{d} := (1, e^{-\delta}, \dots, e^{-T\delta})^\top$, the vector of discount factors. Note that if δ possesses a normal distribution then $e^{-j\delta}$'s posses log-normal distributions. Then the *surplus S* is also a random variable that may be expressed as

$$S = PV(\boldsymbol{a}, \delta) - PV(\boldsymbol{l}, \delta) = \boldsymbol{a}^\top \mathbf{d} - \boldsymbol{l}^\top \mathbf{d} = (\boldsymbol{a} - \boldsymbol{l})^\top \mathbf{d} \tag{13}$$

and the *expected surplus* is $ES = (\boldsymbol{a} - \boldsymbol{l})^\top E\mathbf{d}$. The elements of $E\mathbf{d}$ are the moments of the log-normal random variable $e^{-\delta}$. We will find the assets $\boldsymbol{a}$ which minimize the *mean squared error* ES^2. Put $\mathbf{V} := E\mathbf{d}\mathbf{d}^\top$. We have then

$$ES^2 = E(\boldsymbol{a}^\top \mathbf{d} - \boldsymbol{l}^\top \mathbf{d})^2 = E((\mathbf{E}\mathbf{x} - \boldsymbol{l})^\top \mathbf{d}\mathbf{d}^\top (\mathbf{E}\mathbf{x} - \boldsymbol{l})) = (\mathbf{E}\mathbf{x} - \boldsymbol{l})^\top \mathbf{V} (\mathbf{E}\mathbf{x} - \boldsymbol{l}). \tag{14}$$

This is obviously a convex function in $\mathbf{x}$ so that the minimum can be found by putting the gradient equal to zero:

$$\frac{\partial ES^2}{\partial \mathbf{x}} = 2\mathbf{E}^\top \mathbf{V} \mathbf{E} \mathbf{x} - 2\mathbf{E}^\top \mathbf{V} \boldsymbol{l} =: \mathbf{0}. \tag{15}$$

The solution is

$$\mathbf{x} = (\mathbf{E}^\top \mathbf{V} \mathbf{E})^{-1} \mathbf{E}^\top \mathbf{V} \boldsymbol{l} \tag{16}$$

provided the inverse exists. Thus the assets are in the form

$$\boldsymbol{a} = \mathbf{E} (\mathbf{E}^\top \mathbf{V} \mathbf{E})^{-1} \mathbf{E}^\top \mathbf{V} \boldsymbol{l}. \tag{17}$$

I.7 INDEX NUMBERS AND INFLATION

construction of index numbers, properties of index numbers, stock exchange indexes, inflation, retail price index

In this Chapter we will use the following notation. Let $\boldsymbol{a} = (a_1, \ldots, a_n)^\top$, $\boldsymbol{b} = (b_1, \ldots, b_n)^\top \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$, $\mathbf{1} := (1, 1, \ldots, 1)^\top$. The symbols "·" (more often omitted), "$*$", "/" will mean: $\boldsymbol{a}^\top \cdot \boldsymbol{b} := \boldsymbol{a}^\top \boldsymbol{b} := \sum_{i=1}^n a_i b_i$ (the scalar product), but $\boldsymbol{a}\boldsymbol{b}^\top$ is the $n \times n$ matrix with elements $a_i b_j$, $\boldsymbol{a} * \boldsymbol{b} := (a_1 b_1, \ldots, a_n b_n)^\top$, $\boldsymbol{a}/\boldsymbol{b} := (a_1/b_1, \ldots a_n/b_n)^\top$, $\alpha/\boldsymbol{a} := (\alpha/a_1, \ldots, \alpha/a_n)^\top$, $\boldsymbol{a}/\alpha = (a_1/\alpha, \ldots, a_n/\alpha)^\top$, $\boldsymbol{a} + \alpha = (a_1 + \alpha, \ldots, a_n + \alpha)^\top$, $\boldsymbol{a}^\alpha = (a_1^\alpha, \ldots, a_n^\alpha)^\top$.

7.1 Construction of Index Numbers

Index numbers (or simply indexes) serve as a means for the comparison of the same complex event either among territories at the same time (cross-section) or on the same territory in different times (time series), see [18]. Without loss of generality we will compare the events over time. The well-known indexes are *RPI* (*Retail Price Index,* used in UK; the USA equivalent of RPI is *CPI, Consumer Price Index*) and many of the stock exchange indexes like PX (Prague Stock Exchange), FTSE (Financial Times Stock Exchange), Dow Jones, Standard and Poor's (New York Stock Exchange). All of the mentioned stock exchange indexes appear in various modifications. Let us consider a complex event *A,* say the cost of living, which may only be observed via some particular events $A_1, \ldots, A_N$ like consumption of food, household expenditures, etc. Usually *N* is large so that only $n \leq N$ representatives out of $A_1, \ldots, A_N$ can be used for computation purposes. We can renumber the representatives to become $A_1, \ldots, A_n$. In period t, let the indicator of the particular event A_i be p_t^i with weight q_t^i, $t = 0, \ldots, T$, $i = 1, \ldots, n$. Denote $\boldsymbol{p}_t := (p_t^1, \ldots, p_t^n)^\top$, $\boldsymbol{q}_t := (q_t^1, \ldots, q_t^n)^\top$. Index t may be the index of a region or of a time period, e.g. As we note above, we will consider t as time. Similarly, the $\boldsymbol{p}$'s will usually stand for the prices while the $\boldsymbol{q}$'s for the corresponding quantities or weights. It is the goal of the theory of index numbers to find a scalar characteristics of changes of a global price level over time. The *price index* is a number which shows how the complex event *A* changes over time with changing prices $\boldsymbol{p}$'s, while *quantity index* measures the influence of changes in the quantities $\boldsymbol{q}$'s.

Let s be the initial (base) period and t be the current period, I_{st} the price index describing the change in price level from time s to t. From the historical point of view, the first attempt resulted in the following naive index

(1) $$I_{st} = \frac{\sum_{i=1}^n p_t^i}{\sum_{i=1}^n p_s^i} = \frac{\mathbf{1}^\top \boldsymbol{p}_t}{\mathbf{1}^\top \boldsymbol{p}_s},$$

which has the disadvantage that it depends on the quantity units in which the prices are given. Another index suggested by Edgeworth is simply the geometric mean of the corresponding ratios of prices:

(2) $$I_{st}^G = \sqrt[n]{\prod_{i=1}^n \left(\frac{\boldsymbol{p}_t}{\boldsymbol{p}_s}\right)_i}$$

which has the same unfavorable property as (1) but there is an idea behind, i.e., if the ratios p_t^i/p_s^i are random variables possessing a log-normal distribution then the geometric mean would be a good estimator. A better approach starts with the idea that the price index should be a weighted arithmetic mean of the ratios p_t^i/p_s^i with weights $\boldsymbol{w} = (w_1, \dots, w_n)^\top$:

$$(3) \qquad I_{st} = \sum_{i=1}^{n} w_i \frac{p_t^i}{p_s^i} = \boldsymbol{w}^\top \frac{\boldsymbol{p}_t}{\boldsymbol{p}_s}.$$

A suitable (and generally accepted) choice of weights takes the form

$$(4) \qquad \boldsymbol{w} = \frac{\boldsymbol{p}_\sigma * \boldsymbol{q}_\tau}{\boldsymbol{p}_\sigma^\top \boldsymbol{q}_\tau}.$$

For $\sigma = \tau = s$ we get the *Laspèyres price index*

$$(5) \qquad I_{st}^L = \frac{(\boldsymbol{p}_s * \boldsymbol{q}_s)^\top}{\boldsymbol{p}_s^\top \boldsymbol{q}_s} \frac{\boldsymbol{p}_t}{\boldsymbol{p}_s} = \frac{\boldsymbol{p}_t^\top \boldsymbol{q}_s}{\boldsymbol{p}_s^\top \boldsymbol{q}_s}$$

and putting $\sigma = s$, $\tau = t$ we get *Paasche price index*

$$(6) \qquad I_{st}^P = \frac{(\boldsymbol{p}_s * \boldsymbol{q}_t)^\top}{\boldsymbol{p}_s^\top \boldsymbol{q}_t} \frac{\boldsymbol{p}_t}{\boldsymbol{p}_s} = \frac{\boldsymbol{p}_t^\top \boldsymbol{q}_t}{\boldsymbol{p}_s^\top \boldsymbol{q}_t}.$$

The meaning of the so called *aggregate* $\boldsymbol{p}_\sigma^\top \boldsymbol{q}_\tau$ is clear; it is the price of a consumer's basket if he or she buys quantities $\boldsymbol{q}_\tau$ for prices $\boldsymbol{p}_\sigma$. The index numbers are defined as the ratios of these aggregates.

In practise, there are two ways of comparison: a) we compare the price level with the initial (base) period and afterwards we obtain $I_{01}^L, \dots, I_{0T}^L$ if we take the weights from the base period (*base-weighted index*) or $I_{01}^P, \dots, I_{0T}^P$ if we take the weights from the current period (*current-weighted index*), b) we create the chain of indexes $I_{01}, I_{12}, \dots, I_{T-1,T}$, with the same meaning as in a).

The index numbers should have some desirable and natural properties: (i) $I_{tt} = 1$, (ii) $I_{st}I_{ts} = 1$ (change of time), (iii) $\prod_{t=0}^{T-1} I_{t,t+1} = I_{0T}$ (chain rule). Neither Laspèyrese nor Paasche index generally fulfil (ii) or (iii). For example,

$$I_{01}^L I_{12}^L = \frac{\boldsymbol{p}_1^\top \boldsymbol{q}_0}{\boldsymbol{p}_0^\top \boldsymbol{q}_0} \frac{\boldsymbol{p}_2^\top \boldsymbol{q}_1}{\boldsymbol{p}_1^\top \boldsymbol{q}_1} \neq I_{02}^L = \frac{\boldsymbol{p}_2^\top \boldsymbol{q}_0}{\boldsymbol{p}_0^\top \boldsymbol{q}_0}.$$

The equality in the last expression is achieved if $\boldsymbol{q}_0 = \boldsymbol{q}_1$ which is not too realistic since the individuals adapt to price level. The ratio

$$(7) \qquad B_{02} = \frac{I_{01} I_{12}}{I_{02}}$$

is called *bias*. Let us examine the bias for the Laspèyres index number. Put $\mathbf{x} := \boldsymbol{p}_2/\boldsymbol{p}_1$, $\mathbf{y} := \boldsymbol{q}_1/\boldsymbol{q}_0$, and $\mathbf{f} = (\boldsymbol{p}_1 * \boldsymbol{q}_0)/\boldsymbol{p}_1^\top \boldsymbol{q}_0$. Obviously $\mathbf{1}^\top \mathbf{f} = 1$, so that f are weights. Then

$$(8) \qquad B_{02}^L = \frac{\boldsymbol{p}_1^\top \boldsymbol{q}_0}{\boldsymbol{p}_0^\top \boldsymbol{q}_0} \frac{\boldsymbol{p}_2^\top \boldsymbol{q}_1}{\boldsymbol{p}_1^\top \boldsymbol{q}_1} \frac{\boldsymbol{p}_0^\top \boldsymbol{q}_0}{\boldsymbol{p}_2^\top \boldsymbol{q}_0} = \frac{\boldsymbol{p}_2^\top \boldsymbol{q}_1}{\boldsymbol{p}_1^\top \boldsymbol{q}_0} / \left(\frac{\boldsymbol{p}_1^\top \boldsymbol{q}_1}{\boldsymbol{p}_1^\top \boldsymbol{q}_0} \frac{\boldsymbol{p}_2^\top \boldsymbol{q}_0}{\boldsymbol{p}_1^\top \boldsymbol{q}_0} \right) = \frac{(\mathbf{x} * \mathbf{y})^\top \mathbf{f}}{\mathbf{x}^\top \mathbf{f} \; \mathbf{y}^\top \mathbf{f}}.$$

Denote further $\bar{x} := \mathbf{x}^\top \mathbf{f}$, $\bar{y} := \mathbf{y}^\top \mathbf{f}$, the weighted means, $s_x := \sqrt{\mathbf{f}^\top (\mathbf{x} - \bar{x})^2}$, $s_y := \sqrt{\mathbf{f}^\top (\mathbf{y} - \bar{y})^2}$, the weighted standard errors,

$$r_{xy} := \frac{(\mathbf{x} * \mathbf{y})^\top \mathbf{f} - \bar{x}\bar{y}}{s_x s_y},$$

the correlation coefficient, $V_x := s_x/\bar{x}$, $V_y := s_y/\bar{y}$, the coefficients of variation. Then

$$B^L_{02} = \frac{s_x s_y r_{xy} + \bar{x}\bar{y}}{\bar{x}\bar{y}} = 1 + r_{xy} V_x V_y. \tag{9}$$

If $r_{xy} > 0$ which is the case if the demand increases, $\boldsymbol{q}_1/\boldsymbol{q}_0 \nearrow$, consequently the prices in the next period go up, $\mathbf{p}_2/\mathbf{p}_1 \nearrow$. We can conclude that the Laspèyres price index has positive bias. The reverse is true for the Paasche index. The basic ideas concerning this problem may be found in [18].

We give three examples of more sophisticated index numbers which avoid some lacks of the above indexes. The *Lowe price index* is defined by

$$I^{LW}_{st} = \frac{\boldsymbol{p}_t^\top \boldsymbol{q}}{\boldsymbol{p}_s^\top \boldsymbol{q}}, \tag{10}$$

where $\boldsymbol{q}$ are weights constant over time, possibly constructed artificially. The *Edgeworth–Marshall price index* takes the weights as the arithmetic mean of the weights of the compared periods $\frac{1}{2}(\boldsymbol{q}_s + \boldsymbol{q}_t)$:

$$I^{EM}_{st} = \frac{\boldsymbol{p}_t^\top (\boldsymbol{q}_s + \boldsymbol{q}_t)}{\boldsymbol{p}_s^\top (\boldsymbol{q}_s + \boldsymbol{q}_t)}. \tag{11}$$

The geometric mean of the Laspèyrese and Paasche index gives the *Fisher price index number*

$$I^F_{st} = \sqrt{\frac{\boldsymbol{p}_t^\top \boldsymbol{q}_s}{\boldsymbol{p}_s^\top \boldsymbol{q}_s} \frac{\boldsymbol{p}_t^\top \boldsymbol{q}_t}{\boldsymbol{p}_s^\top \boldsymbol{q}_t}}. \tag{12}$$

Lowe and Fisher indexes have already the desirable properties (i), (ii), and (iii).

7.2 Stock Exchange Indicators

Most of the stock exchange or market indicators are constructed in a similar way as the Laspèyres price index. There are some exceptions, however. We start with one of the oldest indicators, the *Dow Jones Industrial Average* (DJIA) which monitors 30 best stocks (called *blue chips*) traded on *the New York Stock Exchange* (NYSE). It is defined by:

$$\text{DJIA}_t = \frac{1}{D_t} \sum_{i=1}^{30} p_t^i = \frac{1}{D_t} \mathbf{1}^\top \boldsymbol{p}_t,$$

where D_t is called *divisor.* Originally the divisor (in 1928) was just the number of involved stocks, $D_{1928} = 30$. Later it served to ensure continuity of the corresponding time series due to mergers, splits, replacement of the companies in the index, etc. In 1991, $D_{1991} = 0.559$. This phenomenon may be recognized as the change of representatives and the problem of continuity can be generally settled down in the following way. Let t_0 be the time of change. Let $I_1, \ldots, I_{t_0}$ be the values of the indicators based on old representatives, and $I'_{t_0}, \ldots, I'_T$ the values based on new representatives. To ensure the continuity, the following relation must hold:

$$I_{t_0} = C_{t_0} I'_{t_0}.$$

The indicators based on new representatives are afterwards multiplied by C_{t_0}, the *continuity factor,* until a further change of representatives. Hence the series will look like

$$I_1, \ldots, I_{t_0} = C_{t_0} I'_{t_0}, C_{t_0} I'_{t_0+1}, \ldots,$$

till the next change of the representatives. Most of the indicators are also adjusted (multiplied by a factor) to commence with the initial value 100 or 1000, say.

Other market indicators use the weights; the market prices $\boldsymbol{p}_t$ are weighted by the numbers of shares outstanding $\boldsymbol{q}_t$. Therefore the value of the indicator is

$$C_{t_0} \boldsymbol{p}_t^\top \boldsymbol{q}_t,$$

where C_{t_0} is a proper continuity factor. A popular composite index of this type is *Standard & Poor's 500* (S & P 500) consisting of 400 industrial, 20 transportation, 40 utility, and 40 financial stocks. Another one is *NYSE Composite Index* which consists of about 1600 stocks. Finally, let us mention a sample of other frequently used indicators which are constructed similarly; *NASDAQ* (the National Association of Security Dealers Automatic Quotation), *AMEX* (American Stock Exchange) and non-American indexes *Nikkei* (Tokyo), *FTSI* (Financial Times Share Index, London), *DAX 30* (Germany), *PX 50* (Prague).

7.3 Inflation

In 2.4 we have seen that inflation has an important impact on the determination of the interest rate. *Inflation* means an increase of the general price level and, as a consequence, a decrease of the purchasing power of money. An opposite to inflation is the *deflation* which can occasionally also be observed as in the United Kingdom in the period 1920–1935. Inflation is measured by the *retail price index* (*RPI,* United Kingdom) or by the *consumer price index* (*CPI*, USA).

Usually, the retail price index is constructed as a slight modification to the Laspèyres price index by a government statistical office and its construction is a rather complex task. The weights are derived from the sample surveys of the composition of a consumer basket. They have to change from time to time. At the beginning of last century, the consumer basket consisted mainly of the essentials; in the United Kingdom the weight of food was 60 per cent in 1914; some sixty years later it was only 25 per cent and it decreased to 16 per cent in 1990.

7.3.1 Example (RPI in the Czech Republic). Thousands of goods are grouped in 10 main groups and the total of weights is 1000. The groups and their weights in 1993 were: food 327.1, housing 143.7, transport 104.8, leisure 97.5, clothing 90.9, household goods and services 77.2, other goods and services 50.5, public catering and accommodation 47.2, health care 44.2, education 16.9. Thus the importance of food in the index was 32.71 per cent. Denote these weights as $\boldsymbol{q}_{1993}$. Let us look on the situation in August 1997 (denoted as 8/1997). The current monthly inflation is calculated from the index

$$I_{7/1997,8/1997} = \frac{\boldsymbol{p}_{8/1997}^{\top}\boldsymbol{q}_{1993}}{\boldsymbol{p}_{7/1997}^{\top}\boldsymbol{q}_{1993}} = 1.007\,,$$

i.e., 100.7 per cent with 7/1997 set to 100 per cent. Thus the monthly inflation was 0.7 per cent. In comparison to August 1996

$$I_{8/1996,8/1997} = \frac{\boldsymbol{p}_{8/1997}^{\top}\boldsymbol{q}_{1993}}{\boldsymbol{p}_{8/1996}^{\top}\boldsymbol{q}_{1993}} = 1.099$$

or 109.9 per cent. Comparison with the yearly average of 1994 is calculated as

$$I_{1994,8/1997} = \frac{\boldsymbol{p}_{8/1997}^{\top}\boldsymbol{q}_{1993}}{\boldsymbol{p}_{1994}^{\top}\boldsymbol{q}_{1993}} = 1.319\,.$$

Finally, the yearly inflation for the period September 1996 to August 1997 (9/1996–8/1997) compared to the same period of the past year is calculated from

$$I_{9/1995-8/1996,9/1996-8/1997} = \frac{\boldsymbol{p}_{9/1996-8/1997}^{\top}\boldsymbol{q}_{1993}}{\boldsymbol{p}_{9/1995-8/1996}^{\top}\boldsymbol{q}_{1993}} = 1.079$$

so that the current yearly inflation was 7.9 per cent.

I.8 BASICS OF UTILITY THEORY

utility function, marginal utility, risk aversion, certainty equivalent

8.1 The Concept of Utility

Utility in economic theory means a degree of satisfaction or welfare coming from an economic activity, from possession or consumption of goods. In financial world, by utility we usually mean the welfare originated from investment. Suppose that we have an N-dimensional set of investment opportunities $\mathcal{X}$. For $\mathbf{x} = (x_1, \ldots, x_N)^\top \in \mathcal{X}$, x_n will be understood as the volume of the investment into the nth investment. In *utility theory* we suppose that there is an ordering relation on $\mathcal{X} \times \mathcal{X}$ denoted by $\succsim$. If $\mathbf{x}$, $\mathbf{y} \in \mathcal{X}$, then $\mathbf{x} \succsim \mathbf{y}$ means that $\mathbf{x}$ is *weakly preferred* to $\mathbf{y}$. If $\mathbf{x} \succsim \mathbf{y}$ but not $\mathbf{y} \succsim \mathbf{x}$ we say that $\mathbf{x}$ is *preferred* to $\mathbf{y}$, and write $\mathbf{x} \succ \mathbf{y}$. If $\mathbf{x} \succsim \mathbf{y}$ and $\mathbf{y} \succsim \mathbf{x}$ we say that $\mathbf{x}$ is *equivalent* to $\mathbf{y}$ and write $\mathbf{x} \sim \mathbf{y}$. It is reasonable to assume that $\forall\ \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{X}$ either $\mathbf{x} \succsim \mathbf{y}$ or $\mathbf{y} \succsim \mathbf{x}$ (*completeness*), $\mathbf{x} \succsim \mathbf{x}$ (*reflexivity*), and $(\mathbf{x} \succsim \mathbf{y} \wedge \mathbf{y} \succsim \mathbf{z}) \Rightarrow \mathbf{x} \succsim \mathbf{z}$ (*transitivity*).

8.2 Utility Function

If there exists a real valued function $U : \mathcal{X} \to \mathbb{R}$ such that

$$(U(\mathbf{x}) > U(\mathbf{y}) \Leftrightarrow \mathbf{x} \succ \mathbf{y}) \wedge (U(\mathbf{x}) = U(\mathbf{y}) \Leftrightarrow \mathbf{x} \sim \mathbf{y}),$$

it is called *ordinal utility function,* shortly *utility function,* and the underlying theory is known as *ordinal utility theory.* Obviously from $\mathbf{x} \succsim \mathbf{y}$ it follows that $U(\mathbf{x}) \geq U(\mathbf{y})$. For any given $c \in \mathbb{R}$, the set $\mathcal{I}_c = \{\mathbf{x} \in \mathcal{X} : U(\mathbf{x}) = c\}$ is called the *indifference set*. The corresponding plot is called the *indifference surface* or *indifference curve.* This means that from the point of view of an investor, all the investment opportunities from $\mathcal{I}_c$ provide the same degree of satisfaction and the investor cannot distinguish among them.

Finally note that for decision making the utility function contains only the information on ordering. In most cases there is no interpretation of a specific value of $U(\mathbf{x})$. If we consider any increasing function of $U(\mathbf{x})$, the conclusions remain the same. Such an *invariance property* may be an advantage in calculations.

8.2.1 Example. Let us consider the utility function ($N = 2$)

$$u(x_1, x_2) = \sqrt{x_1} + \sqrt{2x_2}.$$

The indifference curves for $c = 2, 2.5, 3$ are shown in Figure 10.

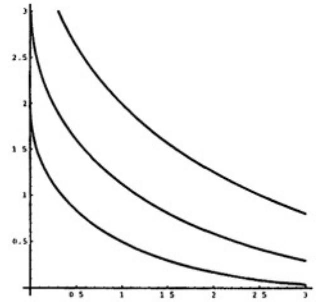

Figure 10: Indifference Curves

In the rightward direction the utility increases.

8.3 Characteristics of Utility Functions

Obviously, the utility function is increasing in the sense of preferences. The slope of the indifference curve can be expressed in terms of the respective derivative $\mathrm{d}x_j/\mathrm{d}x_i$. If $U(\mathbf{x}) = c$ then the total differential

$$\mathrm{d}U = \frac{\partial U}{\partial x_1}\mathrm{d}x_1 + \cdots + \frac{\partial U}{\partial x_N}\mathrm{d}x_N = 0$$

and if we let all $x_k's$ but x_i and x_j constant we get

(1) $$S_{ij} := \frac{\mathrm{d}x_j}{\mathrm{d}x_i} = -\frac{\partial U(\mathbf{x})/\partial x_i}{\partial U(\mathbf{x})/\partial x_j}.$$

S_{ij} gives the *marginal rate of substitution*; the increase by one unit of i must be compensated by the increase of S_{ij} units of j. Since usually $S_{ij} < 0$, the increase of i results in a decrease of j by $|S_{ij}|$ units. (Remark, that for unknown reasons in literature the marginal rate of substitution is defined with the opposite sign as $-\mathrm{d}x_j/\mathrm{d}x_i$.)

A simple observation of the behavior of a rational investor leads to the *law of diminishing marginal utility*:

With increasing amount of investment the additional satisfaction or utility will decrease, *ceteris paribus.*

The explanation is simple. Suppose that in situation A you invest USD 10,000 and get 10 per cent return, i.e. USD 1,000. In situation B you have already invested USD 1,000,000 and have got 10 per cent return again, i.e. USD 100,000. If you invest some additional USD 10,000 in situation B, your return will increase to USD 101,000. Surely these additional USD 1,000 in situation B will not be valued as the same amount in situation A.

Mathematically, the law of diminishing marginal utility says that the utility function describing reasonable principles of decision making is concave. We can summarize natural assumptions on a utility function:

The utility function is increasing, concave, and twice differentiable.

It should be emphasized that the invariance property (application of an increasing function to the utility function) does not preserve concavity in general.

In practise, utility functions usually depend on $\mathbf{x}$ through some aggregate functions which may be functions of another utility functions as is the following case. Let the costs of the respective investments be expressed by the *price vector* $\boldsymbol{p} = (p_1, \ldots, p_N)^\top$. Further let us suppose that the wealth of the investor is W so that he or she can choose an arbitrary $\mathbf{x}$ satisfying the condition $\boldsymbol{p}^\top \mathbf{x} \le W$. If we do not suppose an immediate consumption (which brings another problem), we may suppose $\boldsymbol{p}^\top \mathbf{x} = W$. Given a utility function $U = U(\mathbf{x})$ we can define another utility function

(2) $$U_1(W) = \max_{\mathbf{x}} \{U(\mathbf{x}) : \boldsymbol{p}^\top \mathbf{x} = W\}$$

which, given $\boldsymbol{p}$, depends only on W.

8.4 Some Particular Utility Functions

A broad class of utility functions are *separable* in the sense

$$U(\mathbf{x}) = \sum_{n=1}^{N} w_n U_n(x_n)$$

where U_n's are utility functions and w_n's are positive weights. Such a utility function is *additively separable* but since the equivalent utility function

$$\exp\{U(\mathbf{x})\} = \prod_{n=1}^{N} \exp\{w_n U_n(x_n)\}$$

is *multiplicatively separable* we do not need to stress the kind of separability.

An example of the above is

(3) $$U_\gamma(\mathbf{x}) = \frac{1}{\gamma} \sum_{n=1}^{N} a_n x_n^\gamma$$

for positive constants a_n and $0 < \gamma \le 1$. For $\gamma = 1$ this function is linear and for $\gamma < 1$ it is concave. For $\gamma \to 0_+$ it becomes

(4) $$U_0(\mathbf{x}) = \sum_{n=1}^{N} a_n \ln x_n$$

which is always concave. This utility function is evidently related to the *Cobb-Douglas production function*

(5) $$U_{CD}(\mathbf{x}) = \prod_{n-1}^{N} x_n^{a_n}$$

but this function is both increasing and concave only for $0 < a_n < 1$. Next we review some other frequently used types of utility functions. Some of them will be analyzed in the next Section. The utility function

$$U_H(W) = \frac{1-\gamma}{\gamma}\left(\frac{\beta W}{1-\gamma} + \eta\right)^\gamma \tag{6}$$

is defined for the values W satisfying $\beta W/(1-\gamma) + \eta > 0$, where $\beta > 0$, $\gamma \neq 1$, and η are parameters. The *exponential utility function*

$$U_E(W) = -\frac{1}{\eta}e^{-\eta W} \tag{7}$$

is defined for $W > 0$, where $\eta > 0$ is a parameter. The already mentioned *power utility function*

$$U_P(W) = \frac{1}{\gamma}W^\gamma \tag{8}$$

for $\gamma \to 0_+$ becomes the *logarithmic utility function*

$$U_L(W) = \ln W. \tag{9}$$

8.5 Risk Considerations

In mean-variance portfolio theory (see Chapter 9 for details) there are two decision variables involved: μ and σ, the expected return and risk, respectively. One possible choice among various utility functions is the *quadratic utility function*

$$U(\mu, \sigma; \kappa) = \mu - \kappa\sigma^2. \tag{10}$$

Since the objective is to maximize the utility with respect to some budget constraints, the parameter κ may be interpreted as a measure of the investor's risk aversion. The higher κ, the more adverse to risk the investor is. An analogy to (10) with parametrized expected return is

$$U(\mu, \sigma; \lambda) = \lambda\mu - \sigma^2. \tag{11}$$

Obviously, the lower λ, the higher the risk aversion, see II.3.2.1.

So far we have not dealt with random arguments of utility functions. The above mentioned *risk aversion* may be explained as follows. The investor may decide between two investment decisions; the first one results in a fixed certain amount W while the second one results in a random amount $W+\varepsilon$, where ε is a random variable with zero mean and a positive variance, $E\varepsilon = 0$, $\operatorname{var}\varepsilon > 0$. Since we suppose increasing utility functions it follows that with probability $P(\varepsilon > 0)$ the resulting utility will fulfill $U(W+\varepsilon) > U(W)$ but with $P(\varepsilon < 0)$ it will be $U(W+\varepsilon) < U(W)$. The expected value of the resulting amount is the same in both cases, equal to W, but the *risk averse investor* will prefer certain utility to the expected one:

$$EU(W+\varepsilon) < U(E(W+\varepsilon)) = U(W). \tag{12}$$

It is reasonable to suppose that the last inequality is true for all acceptable levels of W. By Jensen inequality, (12) is assured if U is strictly concave, i.e., $\forall\lambda \in (0,1)$ $\forall w^1, w^2$ $U(\lambda w^1 + (1-\lambda)w^2) > \lambda U(w^1) + (1-\lambda)U(w^2)$.

If the utility function of an investor is linear, then the investor is called *risk-neutral,* and if it is convex, the investor is called *risk loving* or *risk seeking.*

In what follows we will suppose that the investors are risk averse. The investor's aversion to risk can be measured in many ways but there are two measures of particular importance: absolute and relative risk aversion. Assume (see [143], p. 478) that the total investment of USD 10,000 is divided between (risky) stocks and the Treasury bills in equal proportions, USD 5,000 in stocks and USD 5,000 in T-bills. That is the decision of the investor with the initial wish to invest USD 10,000. Suppose now that there are USD 100,000 at the investor's disposal. If the investor increases the amount invested in stocks from USD 5,000 to USD 20,000, say, then he or she manifests the *decreasing absolute risk aversion.* This is the most common behavior of the investors. With increasing investor's wealth, the amount invested in risky assets also increases. Analogously, the *increasing absolute risk aversion* is characterized by the behavior of the investor who reduces the dollar investment into risky assets as his or her wealth increases. If the amount invested into stocks remains the same (not proportion but amount!), we speak of the *constant absolute risk aversion.*

A convenient measure of the absolute risk aversion based on an underlying utility function has been proposed by Arrow and Pratt and is known as the *Pratt-Arrow absolute risk aversion function*:

$$A(W) = -\frac{U''(W)}{U'(W)}. \tag{13}$$

The related *relative risk aversion function* is defined by

$$R(W) = WA(W). \tag{14}$$

8.5.1 Remark (HARA Utility Functions)

In (6) we have defined a class of utility functions $\{U_H\}$. The utility functions from this class are called *HARA utility functions* (*abbreviation for Hyperbolic Absolute Risk Aversion*). To find the reason just calculate the corresponding $A(W)$ from (13).

8.5.2 Exercise. Derive $A(W)$ and $R(W)$ for utility functions presented in 8.4 and comment the results.

8.6 Certainty Equivalent

We have seen that for risk averse investors the utility function is concave. A natural question arises, what certain amount is needed to achieve the same utility as the expected utility with a random wealth. In other words, let W be a random variable representing the wealth and W_c be an amount called *certainty equivalent.* By the principle, W_c is the amount that satisfies the equation

$$U(W_c) = EU(W). \tag{15}$$

Since for strictly concave utility functions $EU(W) < U(EW)$, surely the certainty equivalent satisfies $W_c < EW$.

8.6.1 Example (Multiperiod Certainty Equivalent Model). Suppose a nonnegative random cash flow $\boldsymbol{CF} = (CF_1, \ldots, CF_T)$, the logarithmic utility function $U(W) = \ln(1 + W)$, and the valuation discount factor v. We are looking for a *certainty equivalent* cash flow $\mathbf{C} = (C_1, \ldots, C_T)$ which gives the holder the same utility in terms of the present value as the expected discounted random cash flow:

$$\sum_{t=1}^{T} U(C_t)v^t = \sum_{t=1}^{T} EU(CF_t)v^t \tag{16}$$

and which is "minimal" in the following sense:

$$\min_{\mathbf{C}} \sum_{t=1}^{T} C_t v^t, \tag{17}$$

see [100]. The solution is given by the method of Lagrange multipliers. The corresponding Lagrange function is

$$L(\mathbf{C}, \lambda) = \sum_{t=1}^{T} \left[C_t - \lambda(U(C_t) - EU(CF_t))\right] v^t.$$

The solution is found by setting the gradient of L equal to zero

$$\frac{\partial L(\mathbf{C})}{\partial \mathbf{C}} = \mathbf{0}$$

which is equivalent to the system of equations

$$1 - \frac{\lambda}{C_t + 1} = 0, \quad t = 1, \ldots, T.$$

We conclude that the certainty equivalent cash flow is constant, $C_t \equiv C^*$, and C^* can be found from (16):

$$C^* = \exp\left(\frac{\sum_{t=1}^{T} EU(CF_t)v^t}{\sum_{t=1}^{T} v^t}\right) - 1. \tag{18}$$

I.9 MARKOWITZ MEAN–VARIANCE PORTFOLIO

portfolio, efficient market, market portfolio, efficient portfolio, minimum-variance portfolio, Sharpe ratio, optimal portfolios of riskless and risky assets, separation theorem, tangency portfolio, geometry of minimum-variance portfolios

We have mentioned earlier that almost every investment is uncertain with respect to the gain obtained in the future. A natural question arises, is it possible to reduce the risk related to investment by some sophisticated procedure? The answer is yes, and the method is diversification. A very old rule says that you should divide your disposable funds (wealth) into three equal parts; one third put into deposits, one third invest into shares, and buy gold for the remaining third. This approach may seem to be naive but clearly it is a method for reducing risk. Here we deal with more exact, still elementary procedures, which give the investor hints how to diversify his or her funds. We deal with the classical topics concerning optimal portfolio selection, the rigorous treatment of which has been started by Markowitz [112]. In the explanation we will restrict ourselves to financial assets only (shares, bonds, derivatives) despite the fact that the ideas and results may be applied to real assets as well. More details and specific models are treated in Part II.

Although it is not necessary to assume too much for the purpose of the mathematical construction of an optimal portfolio, usually some reasonable and some artificial restrictive economical assumptions are made in this case, and follow Markowitz, Tobin, and Sharpe. A market is said to be the *efficient market* if it fulfills the assumptions below. (We add the comments to the assumptions in brackets: realistic – usually fulfilled, limited – may be fulfilled in most cases, restrictive – unlikely in most cases, unrealistic – hardly to be fulfilled.)

(1) The investors make decisions on their portfolios exclusively on the information based on the expected returns and covariance structure of returns, or in other words they have *homogeneous expectations* (realistic).
(2) The investors choose portfolios with the highest expected return among those with the same risk (rational behavior, realistic).
(3) The investors choose portfolios with the smallest risk among those with the same expected return (risk aversion, realistic).
(4) The assets are infinitely divisible (limited, because trading on a stock exchange is usually performed in lots – a *lot* means one hundred stocks, say – and there are extra costs for trading the fractions of lots).
(5) The investment horizon is one period of time (realistic).
(6) There are no transaction costs and taxes (limited, but the costs or taxes may partly be incorporated into the returns if they are linear functions of a traded volume).
(7) There exists just one riskless interest rate and all the investors can lend or borrow any amount of necessary funds at this riskless interest rate (unrealistic).
(8) All the assets in question are marketable (realistic).
(9) The investors can sell assets short (restrictive, mostly by legal regulations).

(10) No investor can affect the returns of the respective assets substantially (restrictive, since, in other words, it means that there is no investor with funds exceeding the other investors' funds too much).

(11) All necessary information (about means and covariances) are equally available to all the investors at the same time (restrictive).

Under these assumptions, the *market equilibrium* takes place since the investors have perfect knowledge of the market and behave in a rational way (they are risk averse).

9.1 Portfolio

By *portfolio* we mean a group of (financial) assets. A rational investor chooses his/her portfolio so as to maximize the expected return and to minimize the risk. More precisely, let us consider N assets, $1, \ldots, N$, say, and the wealth (disposable money) equal to 1. *Portfolio* is then the vector $\mathbf{x} = (x_1, \ldots, x_N)^\top$, where x_n represents the fraction of the unit wealth invested in the nth asset, $n = 1, \ldots, N$, so that $\mathbf{1}^\top \mathbf{x} = 1$. Generally, at the moment, we do not suppose $x_n \geq 0$ since the case $x_n < 0$ has an economic meaning. This is the case of *short sales*; the investor can sell a security that he or she does not own. It is equivalent to the borrowing of the respective asset, a kind of speculation. Further, let us suppose that the returns (alternatively the rates of return) of the N mentioned assets are random variables $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_N)^\top$ with the *expected returns* $\mathbf{r} = E\boldsymbol{\rho} = (r_1, \ldots, r_N)^\top$ and the covariance matrix $\mathbf{V} = (\sigma_{ij})$ where $\sigma_{ij} = \operatorname{cov}(\rho_i, \rho_j)$, $i, j = 1, \ldots, N$. Alternatively, we will denote the diagonal elements $\sigma_i^2 := \sigma_{ii}$, the σ_i's standing for standard deviations of the returns: $\sigma_i = \sqrt{\sigma_{ii}}$. For a given portfolio p, represented by weights $\mathbf{x}$, the *expected return on the portfolio* p is

$$r_p = \mathbf{r}^\top \mathbf{x} \tag{1}$$

and the *variance of the portfolio* (which is an abbreviation for the variance of the portfolio return) p is

$$\sigma_p^2 = \mathbf{x}^\top \mathbf{V} \mathbf{x}. \tag{2}$$

The *risk of the portfolio* p is simply the standard deviation

$$\sigma_p = \sqrt{\sigma_p^2}. \tag{3}$$

9.1.1 Example. Let us consider two assets A and B in the period of nine years with the corresponding returns (in per cent): 17, 13, 15, 20, 10, 16, 14, 12, 18 for the asset A and 13, 17, 15, 10, 20, 14, 16, 18, 12 for the asset B. The mean returns for both the assets based on these historical data are the same and equal to 15. The risks are also the same, 3.1225. If we invest all the unit wealth to any of the assets, we can expect a risky return 15 per cent. If we divide our unit wealth equally between the two assets, we have certain return 15 per cent over the given time interval with no risk. This is the case if the returns are perfectly negatively correlated, see Figure 11.

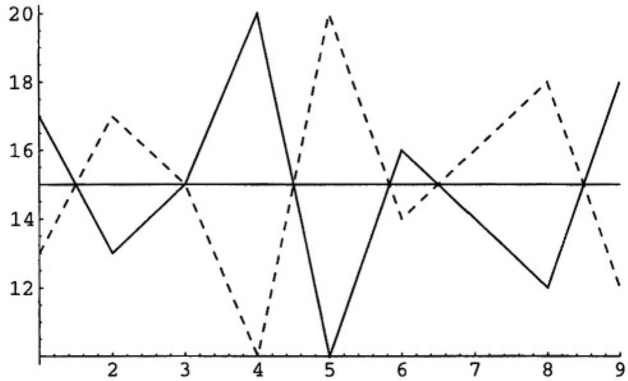

Figure 11: Returns of two assets

9.1.2 Market and Efficient Portfolio

A *market portfolio* is any portfolio in which all the assets come in the same fractions as they appear on the capital market, expressed by their capitalization (market value of the respective asset multiplied by the number of the assets). This is an abstract notion and in practise we usually substitute it by a properly chosen stock exchange index. A portfolio $\mathbf{x}^\star$ is called *efficient portfolio* if there is no other portfolio $\mathbf{x}$ such that

$$(\mathbf{r}^\top\mathbf{x}^\star < \mathbf{r}^\top\mathbf{x} \ \wedge \ \mathbf{x}^{\star\top}\mathbf{V}\mathbf{x}^\star \geq \mathbf{x}^\top\mathbf{V}\mathbf{x}) \vee (\mathbf{r}^\top\mathbf{x}^\star = \mathbf{r}^\top\mathbf{x} \ \wedge \ \mathbf{x}^{\star\top}\mathbf{V}\mathbf{x}^\star > \mathbf{x}^\top\mathbf{V}\mathbf{x}).$$

In other words, an efficient portfolio is a portfolio for which there is no other portfolio with the same or greater expected return and smaller risk.

9.2 Construction of Optimal Portfolios and Separation Theorems

There is a variety of problems concerning the choice of an "optimal portfolio". Our decisions here will be based just on the information about the expected returns and the covariance structure of the returns. This is known as *Markowitz approach,* see Part II for more details. Two basic problems appear in this context:

(i) to minimize $\frac{1}{2}\mathbf{x}^\top\mathbf{V}\mathbf{x}$ subject to $\mathbf{1}^\top\mathbf{x} = 1$, $\mathbf{r}^\top\mathbf{x} = \mu$, where μ is the prescribed expected return. In other words, the investor seeks the expected return μ with minimum risk. The corresponding portfolio is called *minimum-variance portfolio.* Note that minimizing the risk is equivalent to minimizing the variance.

(ii) to maximize the so called *Sharpe's ratio* or *Sharpe's measure of portfolio*

$$\text{(4)} \qquad \frac{\text{expected return on portfolio}}{\text{risk of portfolio}} = \frac{\mathbf{r}^\top\mathbf{x}}{\sqrt{\mathbf{x}^\top\mathbf{V}\mathbf{x}}}$$

subject to $\mathbf{1}^\top\mathbf{x} = 1$.

In any of the problems above the following cases may be considered: $\mathbf{x}$ arbitrary (short sales allowed), $\mathbf{x} \geq \mathbf{0}$ (short sales are not allowed), $\mathbf{V}$ positive definite (which implies that there is no riskless asset), or $\mathbf{V}$ just positive semidefinite. The latter case may occur if there exists a riskless asset or if the returns of two assets are perfectly correlated, e.g., in which case the matrix $\mathbf{V}$ is singular. Note that if $\mathbf{V}$ is singular then (ii) has no sense.

9.2.1 Example. Let us consider two assets with expected returns $\mathbf{r} = (8, 14)^{\mathsf{T}}$, $\sigma_{11} = 9$, $\sigma_{22} = 36$, $\sigma_{12} = \sigma_{21} = \varrho\sqrt{\sigma_{11}\sigma_{22}} = 18\varrho$ where ϱ is the correlation between returns ρ_1 and ρ_2 so that

$$\mathbf{V} = \begin{pmatrix} 9 & 18\varrho \\ 18\varrho & 36 \end{pmatrix}.$$

We will analyse the portfolio of the two assets with ϱ as a parameter. Obviously, the risks of the assets 1 and 2 are 3 and 6, respectively. Let $\mathbf{x} = (x_1, x_2)^{\mathsf{T}}$ denote a portfolio p. Since $x_1 + x_2 = 1$, we can express the expected return on the portfolio

$$r_p = 8x_1 + 14(1 - x_1) = 14 - 6x_1$$

and the variance of the portfolio

$$\sigma_p^2 = 9x_1^2 + 36\varrho x_1(1 - x_1) + 36(1 - x_1)^2.$$

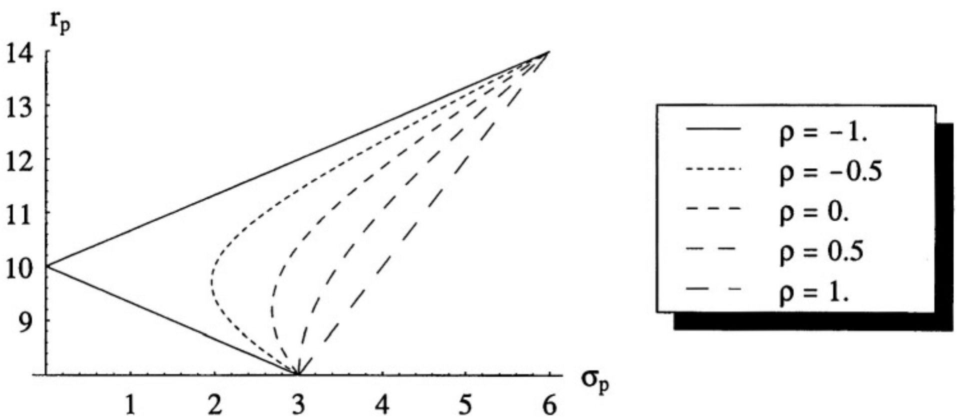

Figure 12: Efficient Frontiers

The dependence between the expected return and risk is usually plotted in the *risk – expected-return plane* or the *standard-deviation – expected-return plane* and the corresponding curves are called *efficient frontiers.* For selected values of ϱ, the dependence is illustrated in Figure 12 for $x_1 \in [0, 1]$. We see that zero risk can be attained only in the case of perfect negative correlation between the returns, $\varrho = -1$. Next we find the minimum-variance portfolio. Since for $|\varrho| < 1$ the portfolio variance σ_p^2 is a positive definite quadratic form, it suffices to find the root of the equation

$$\frac{d\sigma_p^2}{dx_1} = -18(-2\varrho + x_1(4\varrho - 5) + 4) := 0.$$

The solution to this equation is

$$x_1^\star = \frac{2(\varrho - 2)}{4\varrho - 5} \qquad x_2^\star = 1 - x_1^\star = \frac{2\varrho - 1}{4\varrho - 5}.$$

The corresponding expected return and variance of the portfolio are

$$r_p^\star = \frac{44\varrho - 46}{4\varrho - 5} \qquad \sigma_p^{2\star} = \frac{36(\varrho^2 - 1)}{4\varrho - 5}.$$

A simple analysis shows that for $\varrho \leq 0.5$ the corresponding $x_1^\star$ is in the range $[0,1]$, while for $\varrho > 0.5$ it exceeds 1, so that to reach the minimum risk it is necessary to sale short or to borrow the asset 2. In the extreme case $\varrho = 1$, $x_1^\star = 2$, so that the necessary additional fund is obtained by selling the asset 2 of value 1 short, $x_2^\star = -1$. With zero risk, the maximum expected return is only attainable for $\varrho = -1$ in which case the expected return is 10. This can rarely happen in the real world. We can also observe that for all risks but one there are two portfolios with the same risk but two different expected returns.

9.2.2 Remark

The reader may verify that in case of two assets with $\mathbf{V} = (\sigma_{ij})$, $i,j = 1,2$, the minimum-variance portfolio has

$$x_1^\star = \frac{\sigma_2(\sigma_2 - \varrho\sigma_1)}{\sigma_{11} - 2\varrho\sigma_1\sigma_2 + \sigma_{22}}.$$

Particularly, for $\varrho = -1$ we have $x_1^\star = \frac{\sigma_2}{\sigma_1+\sigma_2}$, for $\varrho = 0$ $x_1^\star = \frac{\sigma_{22}}{\sigma_{11}+\sigma_{22}}$, and for $\varrho = 1$ $x_1^\star = \frac{\sigma_2}{\sigma_2-\sigma_1}$ provided $\sigma_1 \neq \sigma_2$.

9.2.3 Example (Riskless Asset). Suppose that we have just two assets, one riskless with return r_0, and one risky (call it A), with expected return r_A and variance σ_A^2. Let us build a portfolio from these two assets with weights x_0 standing for the riskless asset and x_A for the risky asset, $x_0 + x_A = 1$. Then the expected return on the portfolio is $r_p = (1 - x_A)r_0 + x_A r_A$ with variance $\sigma_p^2 = x_A^2\sigma_A^2$. The dependence in the *risk – expected-return* plane is linear. For $0 \leq x_A \leq 1$ it means that the investor lends the portion $1 - x_A$ of his or her money at the interest rate r_0 while for $x_A > 1$, the investor borrows at the riskless interest rate. Borrowing money at the riskless interest rate seems not to be quite realistic, however. The Government are an exception.

9.2.4 General Solution (Risky Assets, Short Sales Allowed)

Here we will solve the problem
minimize $\frac{1}{2}\mathbf{x}^\top \mathbf{V}\mathbf{x}$ subject to $\mathbf{1}^\top \mathbf{x} = 1$, $\mathbf{r}^\top \mathbf{x} = \mu$, μ prescribed.

Suppose $\mathbf{V}$ positive definite. We exclude the case $\mathbf{r} = k\mathbf{1}$ for some constant k since in this case the solution is trivial; simply take just one asset n_0 for which $n_0 = \arg \min_{1\le n\le N} \sigma_n^2$. Now we can obtain the solution by the method of Lagrange multipliers. The Lagrange function for the problem is

$$L(\mathbf{x}, \lambda_1, \lambda_2) = \frac{1}{2}\mathbf{x}^\top \mathbf{V}\mathbf{x} + \lambda_1(1 - \mathbf{1}^\top \mathbf{x}) + \lambda_2(\mu - \mathbf{r}^\top \mathbf{x})$$

and the equation

$$\frac{\partial}{\partial \mathbf{x}} L = \mathbf{V}\mathbf{x} - \lambda_1 \mathbf{1} - \lambda_2 \mathbf{r} = \mathbf{0}$$

gives the optimal solution

$$\boxed{\mathbf{x}^* = \lambda_1 \mathbf{V}^{-1}\mathbf{1} + \lambda_2 \mathbf{V}^{-1}\mathbf{r}\,.} \tag{5}$$

Put $A := \mathbf{1}^\top \mathbf{V}^{-1}\mathbf{1}$, $B := \mathbf{1}^\top \mathbf{V}^{-1}\mathbf{r}$, $C := \mathbf{r}^\top \mathbf{V}^{-1}\mathbf{r}$, and $\Delta := AC - B^2$. Obviously, $A > 0$, $C > 0$, $\Delta > 0$. The last inequality is a simple consequence of the Schwarz inequality since we have supposed $\mathbf{1}$ and $\mathbf{r}$ linearly independent. The constants λ_1, λ_2 can be derived from the initial conditions:

$$1 = \mathbf{1}^\top \mathbf{x} = \lambda_1 A + \lambda_2 B\,,$$

$$\mu = \mathbf{r}^\top \mathbf{x} = \lambda_1 B + \lambda_2 C\,,$$

so that

$$\boxed{\lambda_1 = \frac{C - \mu B}{\Delta} \qquad \lambda_2 = \frac{\mu A - B}{\Delta}\,.} \tag{6}$$

Now we have to distinguish the two cases:

(a) $\underline{\mathbf{1}^\top \mathbf{V}^{-1}\mathbf{r} = 0}$

First, let us note that we can hardly meet this case in practise, but, from the theoretical point of view a for given $\mathbf{V}$ we can find a subspace of $\mathbf{r}$'s of dimension $N-1$ satisfying $\mathbf{1}^\top \mathbf{V}^{-1}\mathbf{r} = 0$. In this case

$$\lambda_1 = \frac{1}{\mathbf{1}^\top \mathbf{V}^{-1}\mathbf{1}}, \qquad \lambda_2 = \frac{\mu}{\mathbf{r}^\top \mathbf{V}^{-1}\mathbf{r}},$$

so that the minimum-variance portfolio is

$$\mathbf{x}^* = \frac{\mathbf{V}^{-1}\mathbf{1}}{\mathbf{1}^\top \mathbf{V}^{-1}\mathbf{1}} + \frac{\mu \mathbf{V}^{-1}\mathbf{r}}{\mathbf{r}^\top \mathbf{V}^{-1}\mathbf{r}}\,.$$

(b) $\underline{\mathbf{1}^\top \mathbf{V}^{-1}\mathbf{r} \neq 0}$

Put

$$\mathbf{x}^1 = \frac{\mathbf{V}^{-1}\mathbf{1}}{\mathbf{1}^\top\mathbf{V}^{-1}\mathbf{1}}, \qquad \mathbf{x}^2 = \frac{\mathbf{V}^{-1}\mathbf{r}}{\mathbf{1}^\top\mathbf{V}^{-1}\mathbf{r}}. \tag{7}$$

The minimum-variance portfolio may now be expressed in the form (with δ's dependent on μ)

$$\mathbf{x}^\star = \delta_1\mathbf{x}^1 + \delta_2\mathbf{x}^2$$

with $\delta_1 = A(C - \mu B)/\Delta =: \delta(\mu)$ and $\delta_2 = B(\mu A - B)/\Delta = 1 - \delta(\mu)$ where $\mathbf{x}^1$, $\mathbf{x}^2$ may be considered as the basis portfolios.

Note that

$$\mathbf{1}^\top\mathbf{x}^\star = 1 = \delta_1\mathbf{1}^\top\mathbf{x}^1 + \delta_2\mathbf{1}^\top\mathbf{x}^2 = \delta_1 + \delta_2 .$$

The optimal portfolio may be expressed in an alternative form:

$$\mathbf{x}^\star = \delta(\mu)\mathbf{x}^1 + (1 - \delta(\mu))\mathbf{x}^2 . \tag{8}$$

It is important to emphasize that the basis portfolios $\mathbf{x}^1$ and $\mathbf{x}^2$ are independent of the prescribed μ but the weight $\delta(\mu)$ does depend on μ.

9.2.5 Remark (Alternative Form of the Minimum-Variance Portfolio)

Put $\mathbf{z}_1 = \frac{1}{\Delta}(C\mathbf{V}^{-1}\mathbf{1} - B\mathbf{V}^{-1}\mathbf{r})$ and $\mathbf{z}_2 = \frac{1}{\Delta}(A\mathbf{V}^{-1}\mathbf{r} - B\mathbf{V}^{-1}\mathbf{1})$, where $\mathbf{z}_1$ is a portfolio. Then (8) may be expressed in the form

$$\mathbf{x}^\star = \mathbf{z}_1 + \mu\mathbf{z}_2. \tag{9}$$

9.2.6 Two Funds Separation Theorem. *Let $\mathbf{x}_a$, $\mathbf{x}_b$ be two minimum-variance portfolios with expected returns r_a, r_b, respectively, $r_a \neq r_b$. Then every minimum-variance portfolio $\mathbf{x}_c$ can be expressed in the form $\mathbf{x}_c = \alpha\mathbf{x}_a + (1-\alpha)\mathbf{x}_b$ for some α. Every portfolio of the form $\mathbf{x}_c = \alpha\mathbf{x}_a + (1-\alpha)\mathbf{x}_b$ is a minimum-variance portfolio.*

Proof. Let r_c denote the expected return on the minimum-variance portfolio $\mathbf{x}_c$. Choose α such that $r_c = \alpha r_a + (1 - \alpha)r_b$, that is, $\alpha = (r_c - r_b)/(r_a - r_b)$. The coefficients λ_1, λ_2 in (6) for portfolios a, b are

$$\lambda_{1i} = \frac{1}{\Delta}(C - r_iB), \qquad \lambda_{2i} = \frac{1}{\Delta}(r_iA - B), \quad i = a, b$$

and since $\mathbf{x}_c$ is also a minimum-variance portfolio, the above relations hold for $i = c$ as well. Now

$$\begin{aligned}
\mathbf{x}_c &= \lambda_{1c}\mathbf{V}^{-1}\mathbf{1} + \lambda_{2c}\mathbf{V}^{-1}\mathbf{r} = \frac{1}{\Delta}(C - r_cB)\mathbf{V}^{-1}\mathbf{1} + \frac{1}{\Delta}(r_cA - B)\mathbf{V}^{-1}\mathbf{r} = \\
&\frac{1}{\Delta}(C - \alpha r_aB - (1-\alpha)r_bB)\mathbf{V}^{-1}\mathbf{1} + \frac{1}{\Delta}(\alpha r_aA + (1-\alpha)r_bA - B)\mathbf{V}^{-1}\mathbf{r} = \\
&\frac{1}{\Delta}(\alpha(C - r_aB) - (1-\alpha)(C - r_bB)\mathbf{V}^{-1}\mathbf{1} + \frac{1}{\Delta}(\alpha(r_aA - B) + (1-\alpha)(r_bA - B))\mathbf{V}^{-1}\mathbf{r} = \\
&(\alpha\lambda_{1a} + (1-\alpha)\lambda_{1b})\mathbf{V}^{-1}\mathbf{1} + (\alpha\lambda_{2a} + (1-\alpha)\lambda_{2b})\mathbf{V}^{-1}\mathbf{r} = \\
&\alpha(\lambda_{1a}\mathbf{V}^{-1}\mathbf{1} + \lambda_{2a}\mathbf{V}^{-1}\mathbf{r}) + (1-\alpha)(\lambda_{1b}\mathbf{V}^{-1}\mathbf{1} + \lambda_{2b}\mathbf{V}^{-1}\mathbf{r}) = \\
&\alpha\mathbf{x}_a + (1-\alpha)\mathbf{x}_b.
\end{aligned}$$

The second assertion is obvious. □

9.2.7 Remark (Covariance Between the Returns of Two Minimum-Variance Portfolios)

Let $\mathbf{x}_a$, $\mathbf{x}_b$ be two minimum-variance portfolios with expected returns r_a, r_b, respectively. Then, after some algebra, we get the covariance

(10) $\operatorname{cov}(\mathbf{x}_a^\top \boldsymbol{\rho}, \mathbf{x}_b^\top \boldsymbol{\rho}) = \mathbf{x}_a^\top \mathbf{V} \mathbf{x}_b =$

$$(\lambda_{1a}\mathbf{V}^{-1}\mathbf{1} + \lambda_{2a}\mathbf{V}^{-1}\mathbf{r})^\top \mathbf{V}(\lambda_{1b}\mathbf{V}^{-1}\mathbf{1} + \lambda_{2b}\mathbf{V}^{-1}\mathbf{r}) = \frac{1}{\Delta}(Ar_a r_b - Br_a - Br_b + C).$$

As a consequence, the variance of a minimum-variance portfolio $\mathbf{x}$ with the expected return r is

$$\boxed{\sigma^2(r) := \operatorname{var} \mathbf{x}^\top \boldsymbol{\rho} = \frac{1}{\Delta}(Ar^2 - 2Br + C).} \tag{11}$$

The *global minimum-variance portfolio* $\mathbf{x}_G$ is defined as the portfolio for which the variance $\sigma^2(r)$ attains its minimum. We have

$$\frac{\partial \sigma^2(r)}{\partial r} = \frac{2Ar - 2B}{\Delta}.$$

Thus the expected return r_G corresponding to the global minimum-variance portfolio is

$$r_G = \frac{B}{A}$$

so that $\lambda_{1G} = 1/A$, $\lambda_{2G} = 0$ and

$$\mathbf{x}_G = \frac{\mathbf{V}^{-1}\mathbf{1}}{\mathbf{1}^\top \mathbf{V}^{-1}\mathbf{1}} \tag{12}$$

and the variance of the portfolio $\mathbf{x}_G$ is

$$\operatorname{var}(\mathbf{x}_G^\top \boldsymbol{\rho}) = \frac{1}{A}.$$

The usual graphical representation of the set of minimum-variance portfolios is in the so called *expected-return–variance plane* or in the *expected-return – standard-deviation plane*. The resulting plot is also known as *minimum-variance frontier*. From the expression for the variance of minimum-variance portfolios $\sigma^2(r)$ we immediately see that the dependence of the variance on any given expected return is expressed as a parabola while the dependence of the risk on any given expected return r is expressed as a hyperbola:

$$\sigma(r) = \sqrt{\frac{1}{\Delta}(Ar^2 - 2Br + C)}. \tag{13}$$

The focus of this hyperbola is at the point $x_G = B/A$ and $\sigma(r_G) = 1/\sqrt{A}$. The derivative of $\sigma(r)$ is

$$\sigma'(r) = \frac{Ar - B}{\sqrt{\Delta(Ar^2 - 2Br + C)}}.$$

Taking the limits of this expression as $r \to \infty$, $r \to -\infty$, we get the slopes of the asymptotes of the hyperbola, $\sqrt{A/\Delta}$, $-\sqrt{A/\Delta}$, respectively. For historical reasons, the plot is in the form where the standard deviation (risk) is on the horizontal axis while the expected return is on the vertical one. The asymptotes expressed as functions of σ are $r(\sigma) = B/A \pm \sigma\sqrt{\Delta/A}$ in this case.

9.2.8 Remark

With the exception $\mu = B/A$, there are two minimum-variance portfolios with the same risk but two different expected returns. If we have the prescribed expected return $\mu < B/A$ then a simple calculation shows that the minimum-variance portfolio with the prescribed expected return $\tilde{\mu} = 2B/A - \mu$ has the same variance, $\sigma^2(\mu) = \sigma^2(\tilde{\mu})$. Thus, in accordance with the definition of an efficient portfolio, the *set of efficient portfolios* consists of all minimum-variance portfolios with expected returns $\mu \geq B/A$. In literature, the minimum-variance portfolios with expected returns less than *B/A* are often called *inefficient portfolios.*

9.2.9 Remark (Orthogonal Minimum-Variance Portfolios)

Let us seek the condition for expected returns of two minimum-variance portfolios a, b with uncorrelated returns. (Verify that this problem has no solution if either of these portfolios is the global minimum-variance portfolio.) From the formula in Remark 2.8.4 it follows that $Ar_a r_b - Br_a - Br_b + C = 0$ so that $r_a = (Br_b - C)/(Ar_b - B)$, $r_b \neq B/A$, or equivalently, $r_b = (Br_a - C)/(Ar_a - B)$, $r_a \neq B/A$. Note that portfolio a is efficient if and only if b is inefficient.

9.2.10 Remark

The global minimum variance portfolio has a peculiar covariance property. We have $\operatorname{cov}(\boldsymbol{\rho}^\top \mathbf{x}_G, \boldsymbol{\rho}^\top \mathbf{x}) = 1/A$ for every portfolio $\mathbf{x}$. This is of course also valid for any single asset: $\operatorname{cov}(\boldsymbol{\rho}^\top \mathbf{x}_G, \rho_n) = 1/A$, $n = 1, \ldots, N$.

9.2.11 Maximum of Sharpe's Ratio (Risky Assets, Short Sales Allowed)

There is no straightforward approach to the problem of direct maximizing the Sharpe's ratio defined in (4). Instead, we will solve the problem

maximize the square of the Sharpe's ratio

$$\frac{(\mathbf{r}^\top \mathbf{x})^2}{\mathbf{x}^\top \mathbf{V} \mathbf{x}} \tag{14}$$

subject to $\mathbf{1}^\top \mathbf{x} = 1$, $\mathbf{V}$ positive definite.

It is important to emphasize that the two problems are not equivalent. The maximum of (14) may be reached for a portfolio giving negative expected return. Such a result is useless, of course. Despite the fact that such a case can be rarely met with on efficient markets, it is necessary to be careful when handling the emerging markets or in the cases where the investors are not risk averse or simply do not pay attention to the return to risk ratio.

To attack the problem first note that from the assumption of positive definiteness of the matrix $\mathbf{V}$ it follows that there exists a symmetric square root matrix $\mathbf{V}^{1/2}$. From Schwarz inequality it follows that

$$(\mathbf{r}^\top \mathbf{x})^2 = (\mathbf{r}^\top \mathbf{V}^{-1/2} \mathbf{V}^{1/2} \mathbf{x})^2 \leq (\mathbf{r}^\top \mathbf{V}^{-1} \mathbf{r})(\mathbf{x}^\top \mathbf{V} \mathbf{x})$$

so that $\mathbf{r}^\top \mathbf{V}^{-1} \mathbf{r}$ is the upper bound for the squared Sharpe's ratio, and the equality holds if and only if $\mathbf{x} = \lambda \mathbf{V}^{-1} \mathbf{r}$ for some λ. Since $\mathbf{x}$ should be a portfolio, it follows that $\lambda = 1/\mathbf{1}^\top\mathbf{V}^{-1}\mathbf{r}$ provided the denominator is nonzero. If the denominator equals zero then there is no solution to the problem. With this exception, the optimal portfolio is

$$\mathbf{x}^* = \frac{\mathbf{V}^{-1}\mathbf{r}}{\mathbf{1}^\top\mathbf{V}^{-1}\mathbf{r}}. \tag{15}$$

9.2.12 General Solution (Riskless and Risky Assets, Short Sales Allowed)

In the presence of a *riskless asset* (also called *riskfree asset*) we can not fully adopt the above theory since the covariance matrix between returns becomes singular. The modification of the previous results is possible, however. The portfolio selection problem may now be formulated in the following way. Suppose we have N risky assets 1,..., N with expected returns $\mathbf{r}$ as above, $\mathbf{r} \neq k\mathbf{1}$, and one riskless asset 0 with return r_0. Let $\tilde{\boldsymbol{\rho}} = (r_0, \boldsymbol{\rho}^\top)^\top$ denote the $(N + 1) \times 1$ vector of the returns. It is economically plausible to suppose that on efficient markets the riskless return r_0 is less than the expected return on any risky efficient portfolio. Since the global minimum-variance portfolio possesses the expected return $r_G = B/A$, we will therefore assume

$$r_0 < \frac{B}{A}.$$

The covariance matrix of returns of the risky assets $\mathbf{V}$ is again assumed to be positive definite. The unit wealth is allocated among $N + 1$ assets 0,1,..., N with weights $x_0, x_1, \ldots, x_N$, and we are seeking a portfolio p represented as $\tilde{\mathbf{x}} = (x_0, \mathbf{x}^\top)^\top$ which minimizes the squared risk (independent of the portion of the riskless asset)

$$\frac{1}{2}\mathbf{x}^\top\mathbf{V}\mathbf{x} \tag{16}$$

under the conditions

$$\mathbf{1}^\top\tilde{\mathbf{x}} = 1 \qquad x_0 r_0 + \mathbf{r}^\top\mathbf{x} = \mu \tag{17}$$

where μ is the prescribed expected return on the portfolio and symbol $\mathbf{1}$ now means the $(N + 1) \times 1$ vector of 1's. Taking into account that $x_0 = 1 - \mathbf{1}^\top\mathbf{x}$, the second condition may be rewritten as

$$(\mathbf{r} - r_0\mathbf{1})^\top\mathbf{x} = \mu - r_0 =: \mu_e \tag{18}$$

where μ_e is called *expected excess return,* that means the return over the return of the riskless asset. So we are forced to solve the problem of finding

$$\min \frac{1}{2}\mathbf{x}^\top\mathbf{V}\mathbf{x} \quad \text{under condition} \quad (\mathbf{r} - r_0\mathbf{1})^\top\mathbf{x} = \mu_e. \tag{19}$$

Weight x_0 is not involved since afterwards it will be calculated using $x_0 = 1 - \mathbf{1}^\top\mathbf{x}$. The Lagrange function for this problem is

$$L(\mathbf{x}, \gamma) = \frac{1}{2}\mathbf{x}^\top\mathbf{V}\mathbf{x} + \gamma(\mu_e - (\mathbf{r} - r_0\mathbf{1})^\top\mathbf{x})$$

and from the equation

$$\frac{\partial L}{\partial \mathbf{x}} = \mathbf{V}\mathbf{x} + \gamma(r_0\mathbf{1} - \mathbf{r}) = \mathbf{0}$$

we obtain the optimal solution

$$\boxed{\mathbf{x}^\star = \gamma \mathbf{V}^{-1}(\mathbf{r} - r_0\mathbf{1}) \qquad x_0^\star = 1 - \mathbf{1}^\top \mathbf{x}^\star} \tag{20}$$

with γ satisfying the condition

$$(\mathbf{r} - r_0\mathbf{1})^\top \gamma \mathbf{V}^{-1}(\mathbf{r} - r_0\mathbf{1}) = \mu_e$$

or

$$\gamma = \frac{\mu_e}{Ar_0^2 - 2Br_0 + C},$$

where A, B, C are defined in 2.8. Such a portfolio is the portfolio with minimum risk with prescribed expected excess return μ_e and will be called *minimum-variance portfolio.*

9.2.13 Two Funds Separation Theorem with Riskless and Risky Assets

Define $\widetilde{\mathbf{x}}^1 := (1, 0, \ldots, 0)^\top$, the portfolio consisting of riskless asset only, and by

$$\mathbf{x}^t = \frac{\mathbf{V}^{-1}(\mathbf{r} - r_0\mathbf{1})}{B - Ar_0}, \tag{21}$$

the so called *tangency portfolio,* and $\widetilde{\mathbf{x}}^2 := (0, \mathbf{x}^{t\top})^\top$.

9.2.14 Two Funds Separation Theorem. *Every minimum-variance portfolio can be expressed in the form*

$$\widetilde{\mathbf{x}}^\star = \delta \widetilde{\mathbf{x}}^1 + (1 - \delta)\widetilde{\mathbf{x}}^2$$

where

$$\delta = \delta(\mu_e) = 1 - \frac{\mu_e(B - Ar_0)}{Ar_0^2 - 2Br_0 + C}$$

Proof. The proof is obvious. □

9.2.15 Corollary

Every portfolio consisting of minimum-variance portfolios is a minimum-variance portfolio.

9.2.16 Remark (Covariance Between the Returns of Two Minimum-Variance Portfolios)

Let $\tilde{\mathbf{x}}_a$, $\tilde{\mathbf{x}}_b$ be two minimum-variance portfolios with expected excess returns μ_{ea}, μ_{eb}, respectively. With weights δ's given by Theorem 2.10.1 we get

$$\text{(22)}\quad \operatorname{cov}(\tilde{\mathbf{x}}_a^\top \tilde{\rho}, \tilde{\mathbf{x}}_b^\top \tilde{\rho}) = \operatorname{cov}((\delta_a \tilde{\mathbf{x}}^1 + (1-\delta_a)\tilde{\mathbf{x}}^2)^\top \tilde{\rho}, (\delta_b \tilde{\mathbf{x}}^1 + (1-\delta_b)\tilde{\mathbf{x}}^2)^\top \tilde{\rho}) = (1-\delta_a)(1-\delta_b)\operatorname{cov}(\rho^\top \mathbf{x}^t, \rho^\top \mathbf{x}^t) = \frac{\mu_{ea}\mu_{eb}}{Ar_0^2 - 2Br_0 + C}.$$

Thus the variance of a minimum-variance portfolio in the presence of a riskless asset with expected return μ becomes

$$\text{(23)}\quad \boxed{\sigma_0^2(\mu) = \frac{(\mu - r_0)^2}{Ar_0^2 - 2Br_0 + C}.}$$

9.2.17 Remark (Properties of Tangency Portfolio)

A simple calculation shows that the tangency portfolio has the expected return $r^t = (C - Br_0)/(B - Ar_0)$ so that the expected excess return is

$$\text{(24)}\quad \mu_{et} = \frac{Ar_0^2 - 2Br_0 + C}{B - Ar_0}.$$

As a consequence, the variance of the tangency portfolio is

$$\text{(25)}\quad \sigma_t^2 := \operatorname{var}(\rho^\top \mathbf{x}^t) = \frac{Ar_0^2 - 2Br_0 + C}{(B - Ar_0)^2}.$$

Note that, since both the numerator and denominator in (24) are positive, also the expected excess return is positive.

9.2.18 Assertion. *The tangency portfolio belongs to the set of efficient portfolios of risky assets.*

Proof. Since the expected return on the tangency portfolio is $r^t = (C - Br_0)/(B - Ar_0)$, we just calculate the Lagrange multipliers:

$$\lambda_{1t} = \frac{r_0}{B - Ar_0}, \qquad \lambda_{2t} = \frac{1}{B - Ar_0}.$$

□

9.2.19 Remark

For expected excess return $\mu_e \in [0, \mu_{et}]$ we get $\delta \in [0, 1]$. This is the case of no short sales either of the riskless asset or of the tangency portfolio. A very unrealistic case is the case of borrowing the riskless asset, i.e., $\delta < 0$ which leads to higher expected returns than the tangency portfolio provides.

9.2.20 Remark (Geometry of the Minimum-Variance Portfolios with a Riskless Asset)

The dependence of the variance on the expected return in the expected-return – variance plane is again a parabola but in the expected-return – standard-deviation plane it becomes straight line

$$\sigma_0(r) = \frac{r - r_0}{\sqrt{Ar_0^2 - 2Br_0 + C}}, \quad r > r_0. \tag{26}$$

We have already mentioned that the tangency portfolio is a member of the set of efficient portfolios of risky assets. The point corresponding to this portfolio in the expected-return – standard-deviation plane is

$$\mathbf{P}_t = \{\frac{C - Br_0}{B - Ar_0}, \frac{\sqrt{Ar_0^2 - 2Br_0 + C}}{B - Ar_0}\}.$$

The line connecting the points $\mathbf{P}_0 = \{r_0, 0\}$ and $\mathbf{P}_t$ may be expressed as

$$y(r) = \frac{1}{\sqrt{Ar_0^2 - 2Br_0 + C}}(r - r_0).$$

For the derivative of the standard deviation of the return of the minimum-variance portfolio consisting of risky assets only we have

$$\frac{\partial\sigma(r)}{\partial r} = \frac{Ar - B}{\sqrt{\Delta(Ar^2 - 2Br + C)}}$$

and if we substitute the expected return on the tangency portfolio,

$$r \to (C - Br_0)/(B - Ar_0)$$

into the last expression, we get the tangency

$$\frac{1}{\sqrt{Ar_0^2 - 2Br_0 + C}}.$$

So $\mathbf{P}_t$ is the tangency point of the hyperbola and therefore line $y(r)$ is the tangency line to the hyperbola.

9.2.21 Remark (Short Sales not Allowed)

If short sales are not allowed, we are not able to give an explicit solution to the problem. The solution may be found by solving the *quadratic optimization problem*:

$$\min \tfrac{1}{2}\mathbf{x}^\mathsf{T}\mathbf{V}\mathbf{x} \quad \text{under the conditions} \quad x_0 = 1 - \mathbf{1}^\mathsf{T}\mathbf{x} \geq 0 \quad (\mathbf{r} - r_0\mathbf{1})^\mathsf{T}\mathbf{x} \geq \mu_e. \tag{27}$$

I.10 CAPITAL ASSET PRICING MODEL

market portfolio, Sharpe-Lintner model, security market line, capital market line

10.1 Sharpe-Lintner Model

In this Chapter we keep the notation of 9.2.12 and the assumptions of an efficient market. *Capital Asset Pricing Model,* shortly *CAPM,* expresses the expected excess returns of the individual assets in terms of the market expected excess return.

10.1.1 Alternative Form of the Expected Excess Return

Denote $\mu_t := \mathbf{r}^\top \mathbf{x}^t$ the expected return on the tangency portfolio and $\boldsymbol{\sigma}_t$ the vector of covariances between excess returns of the risky assets and the excess return on the tangency portfolio. We have

(1) $$\boldsymbol{\sigma}_t = \operatorname{cov}(\boldsymbol{\rho} - r_0 \mathbf{1}, \boldsymbol{\rho}^\top \mathbf{x}^t) = \mathbf{V}\mathbf{x}^t = \frac{\mathbf{r} - r_0 \mathbf{1}}{B - A r_0}.$$

Hence the variance of the tangency portfolio is

(2) $$\sigma_t^2 = \mathbf{x}^{t\top} \mathbf{V} \mathbf{x}^t = \frac{\mathbf{r}^\top \mathbf{x}^t - r_0 \mathbf{1}^\top \mathbf{x}^t}{B - A r_0} = \frac{\mu_t - r_0}{B - A r_0}$$

so that $B - A r_0 = (\mu_t - r_0)/\sigma_t^2$. On the other hand, $\mathbf{r} - r_0 \mathbf{1} = (B - A r_0)\boldsymbol{\sigma}_t$ so that

(3) $$\mathbf{r} - r_0 \mathbf{1} = \frac{\boldsymbol{\sigma}_t}{\sigma_t^2}(\mu_t - r_0).$$

10.1.2 Market Portfolio

Under the assumptions of an efficient market all investors on the market select their portfolios from the set of efficient portfolios. The investors differ only in their risk aversion. Mathematically it is expressed by weight δ in Theorem 9.2.14. Higher values of δ reflect higher risk aversion. Thus the weighted portfolio (according to the individual investors' wealth) consisting of the individual investors' portfolios also belongs to the set of efficient portfolios. The aggregate demand for risky assets is in the proportions of the tangency portfolio. In equilibrium demand and supply are equal and the proportions (both for riskless and risky assets) create the so called *market portfolio.* In other words, the market portfolio is a wealth-weighted average of the individual investors' optimal portfolios. If there is no supply of the riskless asset, the market portfolio coincides with the tangency portfolio. In practise, the market portfolio is often approximated by a composition of a stock exchange index. Let us denote such a *market portfolio* $\widetilde{\mathbf{x}}^M$. It may be expressed in the form

(4) $$\widetilde{\mathbf{x}}^M = \delta_M \widetilde{\mathbf{x}}^1 + (1 - \delta_M)\widetilde{\mathbf{x}}^2$$

for some $0 < \delta_M < 1$. Put $\mathbf{x}^M = (1 - \delta_M)\mathbf{x}^t$, the part of the market portfolio corresponding to risky assets only. The return on the market portfolio M is therefore

$\rho^M = \delta_M r_0 + (1-\delta_M)\boldsymbol{\rho}^\top \mathbf{x}^t$, the expected return on M is $\mu_M = \delta_M r_0 + (1-\delta_M)\mu_t$, the variance of the return on M is $\sigma_M^2 = (1-\delta_M)^2\sigma_t^2$, the expected excess return on M is $\mu_{eM} = \mu_M - r_0 = (1-\delta_M)(\mu_t - r_0)$, and the vector of covariances between excess returns of the risky assets and the excess return on the market portfolio reads

$$(5)\quad \boldsymbol{\sigma}_M = \operatorname{cov}(\boldsymbol{\rho} - r_0\mathbf{1}, \boldsymbol{\rho}^\top \mathbf{x}^M) = (1-\delta_M)\mathbf{V}\mathbf{x}^t = (1-\delta_M)\frac{\mathbf{r} - r_0\mathbf{1}}{B - Ar_0} = (1-\delta_M)\boldsymbol{\sigma}_t.$$

Now we substitute into formula (3) and after cancelling the factor $(1-\delta_M)$ we get

$$(6)\qquad \boxed{\mathbf{r} - r_0\mathbf{1} = \frac{\boldsymbol{\sigma}_M}{\sigma_M^2}(\mu_M - r_0) = \boldsymbol{\beta}(\mu_M - r_0)}$$

where $\boldsymbol{\beta} = \boldsymbol{\sigma}_M/\sigma_M^2$. The last formula is known as *CAPM* also *Sharpe-Lintner model CAPM.* For individual assets the CAPM becomes

$$(7)\qquad \boxed{r_n - r_0 = \beta_n(\mu_M - r_0), \quad n = 1, \ldots, N}$$

with $\beta_n = \sigma_{nM}/\sigma_M^2$ or

$$(8)\qquad \boxed{r_n - r_0 = \frac{\sigma_{nM}}{\sigma_M^2}(\mu_M - r_0)\,,}$$

if we denote $\sigma_{nM} := \operatorname{cov}(\rho_n, \rho^M)$.

The concept of the market portfolio is a bit abstract. By definition, it is the wealth-weighted sum of the portfolio holdings of all investors. The weights can hardly be observed in practise however, so for calculation an observable indicator of the market performance is needed. Usually the market portfolio is approximated by some stock exchange index like DJIA, S&P 500, FTSI, etc. The stock exchange indexes serve as proxies for the market portfolio and the US Treasury bill rates proxy the riskless rate.

10.2 Security Market Line

The graphical representation of (7) and (8) is known as the *security market line, SML.* We see that (7) expresses the expected return on the nth asset as a function of β_n while (8) expresses the same quantity in dependence on the covariance σ_{nM}. We refer to (7) as to the β-version and to (8) as to the *covariance version* of the security market line. The quantity β for an asset or a portfolio is called *factor beta* and it plays an important role in equity (stock) valuation. For the market portfolio, the corresponding $\beta^M = 1$ and for the riskless asset $\beta_0 = 0$. Obviously, $\beta = 1$ also for any efficient portfolio. The factor beta may be considered as a risk factor. Assets with $\beta > 1$ are riskier than the market average and those with $\beta < 1$ are less risky than the market average.

For an arbitrary portfolio $\mathbf{x}$, the factor β_p of that portfolio is $\beta_p = \mathbf{x}^\top\boldsymbol{\beta}$, the weighted average of the respective β_n's, $\boldsymbol{\beta}$ defined in (6). While the variance is a risk measure of an efficient portfolio, beta may be considered as an indicator of the market risk of an individual security. For the asset n, we can express (7) in an alternative way

$$\rho_n = r_0 + \beta_n(\rho^M - r_0) + \varepsilon_n, \quad n = 1, \ldots, N \tag{9}$$

where the ε_n's are disturbances, $E\,\varepsilon_n = 0$, $\operatorname{var}\varepsilon_n = \sigma^2_{n\varepsilon}$. From this equation we get the expression for the variance of ρ_n:

$$\sigma^2_n = \beta^2_n\sigma^2_M + \sigma^2_{n\varepsilon} + 2\operatorname{cov}(\rho^M, \varepsilon_n). \tag{10}$$

Often it is supposed that ρ^M and ε_n are not correlated (questionable) so that (10) simplifies to

$$\sigma^2_n = \beta^2_n\sigma^2_M + \sigma^2_{n\varepsilon} \tag{11}$$

with the interpretation that the *total risk* σ^2_n is decomposed into the *market risk* $\beta^2_n\sigma^2_M$ and the *unique* or *specific risk* $\sigma^2_{n\varepsilon}$. Only the specific risk is diversifiable in the sense that by holding the asset n in a sufficiently large portfolio, the prevailing part of the risk of the whole portfolio is that of market risk. In practise, however, it is not necessary to hold a portfolio mirroring the whole market portfolio. A comparatively small portfolio of some tens of assets would eliminate most of the specific risk.

Betas have to be estimated. The most common approach is based on linear regression from historical data. If we have T observations of returns ρ_{nt} on the asset n and on the market return ρ^M_t (represented mainly by the actual value of a stock exchange index), $t = 1, \ldots, T$, we can rewrite (9) in the form of regression equations

$$\rho_{nt} = r_0 + \beta_n(\rho^M_t - r_0) + \varepsilon_{nt}, \quad t = 1, \ldots T \tag{12}$$

for unknown parameter β_n if the riskless rate r_0 is supposed to be known or as

$$\rho_{nt} = \alpha_n + \beta_n\rho^M_t + \varepsilon_{nt}, \quad t = 1, \ldots T \tag{13}$$

for unknown parameters α_n, β_n, if the excess return is not directly observable. The estimate of beta obtained by the least squares principle is

$$\widehat{\beta}_n = \frac{\sum_{t=1}^T(\rho_{nt} - \overline{\rho}_n)(\rho^M_t - \overline{\rho^M})}{\sum_{t=1}^T(\rho^M_t - \overline{\rho^M})^2} \tag{14}$$

for model (12) where $\overline{\rho}_n$ and $\overline{\rho^M}$ denote the respective averages. Under (13), the estimate of β_n is (14) again, and for α_n the estimate is

$$\widehat{\alpha}_n = \overline{\rho}_n - \widehat{\beta}_n\overline{\rho^M}. \tag{15}$$

In equilibrium the returns of all securities would lie along the security market line. If this is not the case, there is something wrong either with their risk parameter beta or with their pricing. If the beta on an asset is correct and the return is below SML, the asset is overpriced. If the return is above SML, the asset is underpriced. (Explain this phenomenon as an exercise. Note that with increasing price of a security the return decreases and vice versa.) The difference between the actual and expected (given by SML) return is called *Jensen measure*.

Betas are published in financial press and in the so called *Beta books* both for individual companies and for industries. For industry like essentials usually $\beta < 1$. This is typical for goods and services the demand for which is irrespective of the economic cycle. Thus food manufacturing may have $\beta = 0.9$ while car industry $\beta = 1.27$, and tourism $\beta = 1.66$.

10.3 Capital Market Line

Let us have a portfolio p with expected return μ_p and standard deviation σ_p. In the presence of a riskless asset we can modify the *Sharpe's measure of portfolio*:

$$\frac{\mu_p - r_0}{\sigma_p} \tag{16}$$

and we will call it *modified Sharpe's measure of portfolio.*

10.3.1 Assertion. *All efficient portfolios have the same modified Sharpe's measure of portfolio.*

Proof. For an efficient portfolio p the expected excess return may be expressed as $\mu_p - r_0 = \delta_p r_0 + (1 - \delta_p)\mu_t - r_0 = (1 - \delta_p)(\mu_t - r_0)$ for some δ_p. Similarly, for the standard deviation we get $\sigma_p = (1 - \delta_p)\sigma_t$. Thus

$$\frac{\mu_p - r_0}{\sigma_p} = \frac{\mu_t - r_0}{\sigma_t}.$$

□

Since we can take any of the efficient portfolios as a numeraire, we choose the market portfolio *M*. From the above assertion it follows that mean the μ_p and the standard deviation σ_p of any efficient portfolio fulfill the relationship

$$\boxed{\mu_p = r_0 + \frac{\mu_M - r_0}{\sigma_M}\sigma_p.} \tag{17}$$

The dependence of the expected return on an efficient portfolio on its standard deviation is linear and its graphical representation is called *Capital Market Line, CML.*

The substantial difference between CML and SML is that CML expresses the excess return on the efficient portfolios while SML is valid for any security or portfolio.

I.11 ARBITRAGE PRICING THEORY

regression model, multifactor model, factor analysis, modified method of principal components

Arbitrage Pricing Theory, (*APT*), also known as *Arbitrage Pricing Model, APM*, serves as a generalization of the single factor CAPM to a multifactor model. The idea behind the APT is that the returns vary from their expected values due to unanticipated changes in production, inflation, term structure, and other economic factors. In the multifactor model it is supposed that the return on an asset is explained in terms of a linear combination of more factors or indexes. Note that in CAPM, the expected return on an asset is a linear function of the expected market return only. The development of APT is based on the assumptions of an efficient market, see I.9. A technical realization of APT uses two popular statistical methods; regression analysis and factor analysis.

11.1 Regression Model

We suppose that the return ρ_n on an asset n (n fixed in this Section) fulfills the usual model of linear regression

(1) $$\rho_n = \beta_{n1}F_1 + \cdots + \beta_{nm}F_m + \varepsilon_n$$

where $F_1, \ldots, F_m$ are explanatory variables independent of the asset return in question, ε_n is a zero mean random disturbance, and $\beta_{n1}, \ldots, \beta_{nm}$ are unknown parameters which are specific for the given asset. Usually an absolute term must be considered which can be simply done by setting $F_1 \equiv 1$. In the regression model we suppose that the values of F_i's are observable while the random deviate ε_n is not. If we have T observations of the vector $(\rho_n, F_1, \ldots, F_m)$ then (1) becomes

(2) $$\rho_{nt} = \beta_{n1}F_{1t} + \cdots + \beta_{nm}F_{mt} + \varepsilon_{nt}, \quad t = 1, \ldots, T.$$

Such observations are usually gathered historical data. It should be emphasized that $\beta_{n1}, \ldots, \beta_{nm}$ are characteristics of the underlying asset and $F_{1t}, \ldots, F_{mt}$ are independent of the asset but they take different values for different t's. In a simple regression model it is supposed that $E\varepsilon_{nt} = 0$ and $\operatorname{cov}(\varepsilon_{ns}, \varepsilon_{nt}) = \sigma_n^2\delta_{st}$, $s,t = 1, \ldots, T$, where $\delta_{st} = 1$ for $s = t$ and $\delta_{st} = 0$ for $s \neq t$, σ_n^2 being also an unknown parameter. Put $\boldsymbol{\rho}_n := (\rho_{n1}, \ldots, \rho_{nT})^\top$, $\boldsymbol{\beta}_n := (\beta_{n1}, \ldots, \beta_{nm})^\top$, $\mathbf{F} := (F_{it})$, $i = 1, \ldots, m$, $t = 1, \ldots, T$ an $m \times T$ matrix, and $\boldsymbol{\epsilon}_n := (\varepsilon_{n1}, \ldots, \varepsilon_{nT})^\top$. Then we can express (2) in the matrix form

(3) $$\boxed{\boldsymbol{\rho}_n = \mathbf{F}^\top\boldsymbol{\beta}_n + \boldsymbol{\epsilon}_n,}$$

$E\boldsymbol{\epsilon}_n = \mathbf{0}$, $\operatorname{cov}\boldsymbol{\epsilon}_n = \sigma_n^2\mathbf{I}_T$. Further let us suppose that $T > m$ and that $\mathbf{F}$ has the full rank, $r(\mathbf{F}) = m$ so that the inverse$(\mathbf{F}\mathbf{F}^\top)^{-1}$ exists. The ordinary least squares estimator of $\boldsymbol{\beta}_n$ is

(4) $$\widehat{\boldsymbol{\beta}}_n = (\mathbf{F}\mathbf{F}^\top)^{-1}\mathbf{F}^\top\boldsymbol{\rho}_n$$

with the covariance matrix $\operatorname{cov}\widehat{\boldsymbol{\beta}}_n = \sigma_n^2(\mathbf{F}\mathbf{F}^\top)^{-1}$. An unbiased estimator of σ_n^2 is

$$(5) \qquad \widehat{\sigma}_n^2 = \frac{1}{T-m-1}(\boldsymbol{\rho}_n - \mathbf{F}^\top\widehat{\boldsymbol{\beta}}_n)^\top(\boldsymbol{\rho}_n - \mathbf{F}^\top\widehat{\boldsymbol{\beta}}_n).$$

The last statistic is used for the construction of the confidence intervals for $\boldsymbol{\beta}_n$.

11.1.1 Remark

An empirical study of this type with $m = 7$ may be found in [143] together with further references. The variables, based on monthly observations are: $F_1 \equiv 1$, F_2 = monthly growth in industrial production, F_3 = change in expected inflation, F_4 = unexpected inflation, F_5 = risk premium as the difference in yields of corporate bonds and long-term Treasury bonds, F_6 = change in the term structure as the difference in yields of long-term Treasury bonds and (short-term) Treasury bills, F_7 = return on the market portfolio measured by the NYSE index.

11.1.2 Remark

In the preceding remark we have seen that one of the explanatory variables was the market return. Since we may always include this variable in APT consideration together with additional explanatory variables, we can not obtain worse fitting than that with CAPM. This is a well-known fact, the more parameters you have, the better fit you get. The number of explanatory variables has to be chosen with care, however, since including highly correlated variables brings the problems with multicollinearity etc. Refer to standard textbooks on regression analysis, like [180].

11.2 Factor Model

Instead of returns ρ_n we will now consider *standard scores* or *standardized returns*

$$(6) \qquad \rho_n := \frac{\rho_n - E\,\rho_n}{\sqrt{\operatorname{var}\rho_n}}, \quad n = 1, \ldots, N$$

at a given time instant. In the factor model we suppose that

$$(7) \qquad \rho_n = b_{n1}f_1 + \cdots + b_{nm}f_m + e_n, \quad n = 1, \ldots, N$$

where $f_1, \ldots, f_m, e_n$ are random variables with zero means. $f_1, \ldots, f_m$ are called *common factors* or *sensitivities*, e_n is called *unique* or *specific factor,* and $b_{n1}, \ldots, b_{nm}$ are called *factor loadings* on the asset n. Note that in this context, e_n is also known as the *idiosyncratic risk,* the *asset-specific* or *firm-specific* component. The crucial assumption of the factor model is that neither the common nor the specific factors can be directly observed, i.e., they are *unobservable.* It is also supposed that all the factors are mutually uncorrelated. Denote, as usually, $\boldsymbol{\rho} := (\rho_1, \ldots, \rho_N)^\top$, $\mathbf{B} := (b_{nj})$, $n = 1, \ldots, N$, $j = 1, \ldots, m$ the $N \times m$ matrix of factor loadings, $\mathbf{f} := (f_1, \ldots, f_m)^\top$ the vector of common factors, and $\mathbf{e} := (e_1, \ldots, e_N)^\top$ the vector of specific factors. The matrix form of (7) becomes

$$(8) \qquad \boxed{\boldsymbol{\rho} = \mathbf{B}\mathbf{f} + \mathbf{e}.}$$

This is the *factor model* of returns. We summarize the above assumptions and make some additional ones:

$$E\mathbf{f} = \mathbf{0}, \quad E\mathbf{e} = \mathbf{0}, \quad \operatorname{cov}(\mathbf{f}, \mathbf{e}) = E\mathbf{f}\mathbf{e}^{\top} = \mathbf{0},$$

(9) $$\operatorname{cov}\mathbf{f} = \mathbf{I}_m, \quad \operatorname{cov}\mathbf{e} = \operatorname{diag}(\psi_1^2, \ldots, \psi_N^2) =: \boldsymbol{\Psi}.$$

The last assumption means that for different assets, the specific factors are uncorrelated and may have different variances. Under these assumptions, the covariance matrix of $\boldsymbol{\rho}$ is

(10) $$\boxed{\mathbf{R} = \operatorname{cov}\boldsymbol{\rho} = \mathbf{B}\mathbf{B}^{\top} + \boldsymbol{\Psi}.}$$

Since we have supposed that $\boldsymbol{\rho}$ is a standardized random vector, **R** coincides with the *correlation matrix of standardized returns.* Hence the nth element of the diagonal of **R** can be expressed in the form

(11) $$1 = 1 - \psi_n^2 + \psi_n^2 =: h_n^2 + \psi_n^2.$$

The quantity h_n^2 is called *communality* and ψ_n^2 is called *uniqueness, specificity,* or *specific variance* of the respective asset.

Note that the decomposition (10) is far from being unique. For example, if **U** is any $m \times m$ orthogonal matrix then

(12) $$\mathbf{R} = \mathbf{B}^*\mathbf{B}^{*\top} + \boldsymbol{\Psi}$$

where $\mathbf{B}^* = \mathbf{BU}$ is called an *orthogonal rotation.*

The main *objective of the factor analysis* may be formulated in the following way: Given the correlation matrix **R**, find the number m of common factors, a matrix of factor loadings **B** and a diagonal matrix $\boldsymbol{\Psi}$ with nonnegative elements such that (10) holds. The number of common factors should be small. This is a natural requirement since with a high number of common factors we loose the possibility of their proper interpretation.

There is a plenty of statistical methods aimed for solving the above problem. We just briefly mention one of the simplest but frequently used method with a clear motivation. The method is known as the *modified method of principal components.* The theoretical background of this method is based on the Lemma below. First recall that every symmetric $N \times N$ matrix allows a *spectral decomposition*

(13) $$\mathbf{A} = \lambda_1 \mathbf{x}_1 \mathbf{x}_1^{\top} + \cdots + \lambda_N \mathbf{x}_N \mathbf{x}_N^{\top}$$

where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$ are the eigenvalues and $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N$ the orthonormal eigenvectors corresponding to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_N$. By the Euclidean norm of a matrix **A** we mean $\|\mathbf{A}\| = \sqrt{\sum\sum a_{ij}^2}$.

11.2.1 Lemma. *Let* $\mathbf{A}$ *be an* $N \times N$ *symmetric positive definite matrix,* $r(\mathbf{A}) = r$*, and let* $m \leq r$*. Then the solution to the problem*

$$\text{minimize } \{ \ \|\mathbf{A} - \mathbf{B}\mathbf{B}^{\top}\|, \quad \mathbf{B} \text{ an } N \times m \text{ matrix} \},$$

i.e., the best approximation of $\mathbf{A}$ *by* $\mathbf{B}\mathbf{B}^{\top}$ *in the sense of Euclidean norm, is given by* $\widehat{\mathbf{B}} = (\sqrt{\lambda_1}\mathbf{x}_1, \ldots, \sqrt{\lambda_m}\mathbf{x}_m)$.

Proof. The proof can be found in textbooks on matrix calculus. □

The estimation procedure starts with a guess of the number of common factors. A heuristic rule says that we take m equal to the number of the eigenvalues of $\mathbf{R}$ greater than or equal to one. Then we estimate the communalities. A good initial approximation is given by

$$1 - \widehat{\psi_n^2} = \max_{i \neq n} |r_{in}| \tag{14}$$

or by the square of the multiple correlation coefficient $r^2_{n \cdot 1,\ldots,n-1,n+1,\ldots,N}$ in the regression of the nth variable on the remaining $N-1$ variables. From (14) we form the estimate $\widehat{\mathbf{\Psi}}_0 = \text{diag}(\widehat{\psi_1^2}, \ldots, \widehat{\psi_N^2})$. Now we define the *reduced correlation matrix* by

$$\mathbf{R}_1 = \mathbf{R} - \widehat{\mathbf{\Psi}}_0. \tag{15}$$

Note that in the theoretical model (10) it is assumed that this matrix is positive semidefinite. It may not be the case for (15) since instead of $\mathbf{\Psi}$ we have used an estimate of it. Nevertheless, since we suppose $m \ll N$ we can expect that at least m eigenvalues of $\mathbf{R}_1$ are positive and we can therefore construct the best approximation of it based on Lemma 11.2.1:

$$\widehat{\mathbf{R}}_1 = \widehat{\mathbf{B}}_1 \widehat{\mathbf{B}}_1^{\top} \tag{16}$$

where $\widehat{\mathbf{B}}_1 = (\sqrt{\lambda_1}\mathbf{x}_1, \ldots, \sqrt{\lambda_m}\mathbf{x}_m)$ from the spectral decomposition of the (surely symmetric) matrix $\mathbf{R}_1 = \sum_{n=1}^{N} \lambda_n \mathbf{x}_n \mathbf{x}_n^{\top}$. We then obtain a new estimate of specificities

$$\widehat{\mathbf{\Psi}}_1 = \text{diag}(\mathbf{R} - \widehat{\mathbf{R}}_1). \tag{17}$$

We must take the diagonal only since $\mathbf{R} - \widehat{\mathbf{R}}_1$ may not be a diagonal matrix. We go back to (15), form the new reduced correlation matrix $\mathbf{R}_2 = \mathbf{R} - \widehat{\mathbf{\Psi}}_1$ and iteratively improve the estimates of $\mathbf{\Psi}$ and $\mathbf{B}$ until the differences in successive iterations are sufficiently small. Eventually we get the decomposition

$$\mathbf{R} = \widehat{\mathbf{B}}\widehat{\mathbf{B}}^{\top} + \widehat{\mathbf{\Psi}} \tag{18}$$

or an analogy to the original model (8)

$$\rho = \widehat{\mathbf{B}}\mathbf{f} + \mathbf{e} \tag{19}$$

with $\mathbf{f}$ still remaining an unknown vector of common factors. But with known matrix $\widehat{\mathbf{B}}$, we may look on (19) as on a linear regression model with unknown parameters $\mathbf{f}$. By ordinary least squares method we get an estimate of $\mathbf{f}$:

$$\widehat{\mathbf{f}} = (\widehat{\mathbf{B}}^{\top}\widehat{\mathbf{B}})^{-1}\widehat{\mathbf{B}}^{\top}\rho. \tag{20}$$

11.2.2 Remark (Principal Components)

Factor analysis is a generalization of *principal components.* In the method of principal components we directly use the spectral decomposition (13) of the covariance matrix of returns $\mathbf{\Sigma}$:

$$\mathbf{\Sigma} = \sum_{n=1}^{N} \lambda_n \mathbf{x}_n \mathbf{x}_n^\top .$$

The random variable $Y_n := \mathbf{x}^\top \boldsymbol{\rho}$ is called nth principal component, $n = 1, \ldots, N$. The principal components have some plausible properties: (i) they are uncorrelated, (ii) $\operatorname{var} Y_n = \lambda_n$, (iii) the total dispersion of returns measured by $\sigma^2 = \operatorname{tr} \mathbf{\Sigma}$ is explained by all the principal components since $\operatorname{tr} \mathbf{\Sigma} = \sum_{n=1}^{N} \sigma_{nn}$. The eigenvalues are supposed to be ordered, hence the first principal component explains the greatest part of the total dispersion etc. In practise, often only a few components (3, say) explain most of the total dispersion (95 per cent, say). We see that the model of principal components coincides with that of the factor model if we put $m := N$ and $\mathbf{\Psi} := \mathbf{0}$, i.e., if no specific factors are considered.

11.2.3 Remark

The interpretation of the common factors represented by their factor loadings is a difficult and fairly controversial procedure. Roughly speaking, only the first two common factors may be usually identified with a more or less clear interpretation. The first factor represents an overall performance of economy giving higher loadings to the assets with greater importance. The second factor, often interpreted as a *bipolar factor,* usually divides the assets into industries which may act in opposite directions: oil – gas, nuclear power plants – heat power plants, etc.

The interpretation of the factor loadings is easier if each asset is highly loaded on at most one factor, and if all factor loadings are either large or close to zero. The assets are then grouped into disjoint sets, each of which is associated with one factor. The factor i has an influence on those assets for which b_{ni} is large. Since $|b_{ni}| \leq 1$, by term "b_{ni} large" we mean b_{ni} close to 1 or -1. We have seen that the decomposition (10) is also valid for any orthogonal rotation of **B**. There is a lot of methods of rotation which, up to some extent, improve the interpretation of factors in the above sense. Generally, their principle is to find the orthogonal matrix $\mathbf{U}$ such that the rows of the transformed matrix $\widehat{\mathbf{B}}\mathbf{U}$ contain a few large loadings while the others are close to zero. The most popular orthogonal rotation method is *varimax.* There are also non-orthogonal methods (oblique rotations) like *quartimin.* Note that under oblique rotations the factors are no longer uncorrelated. All these methods are iterative and difficult from the computational point of view. See Rao's contribution in [109], pp. 489–505, for further discussion.

I.12 BIBLIOGRAPHICAL NOTES

Many of the topics treated in this Part are classical pieces of finance, financial mathematics, financial management, and partly of economics. Hence it is quite natural that there are hundreds of books on similar subjects but they differ in their viewpoint on the subjects. Also the material involved is treated on very different levels. Hence the following notes may cover only a small part of the vast existing literature on the related topics.

Money, capital, and securities. A thorough text on basic financial concepts and financial institutions is [138]. In [143] the reader will find both theory and many examples of investment management. [25] and [141] may serve as readable books on financial management together with accounting considerations which are not mentioned in this book, however. Only the most important securities (this applies particularly to derivatives) are mentioned. For more see [60], [88].

Interest rate. A simple arithmetics of interest rate is contained in [114], a deeper insight in [161]. The section on decomposition is based on various sources like [143], [25]. Inflation is also treated in 1.7.3. Term structure is important in fixed-income securities' analysis. In continuous case, various models of the term structure are known as *Vasicek mean reversion, Cox-Ingersoll-Ross, Merton* (*Ho-Lee*), *Hull-White, Heath-Jarrow-Morton,* and other models, see [43], [82], [105]. For modeling term structure in *Mathematica*® see [11] and [13].

Measures of cash flows. An elementary approach may be found in any book on financial arithmetics like [114] or on financial and investment management like [25], [141], [143]. A thorough discussion on the benefit to leasing is in [76]. The concept of duration has been ascribed to Frederick Macaulay. Yield curves are often treated in the context of term structure models.

Return, expected return, and risk. A comprehensive but still elementary treatise on return and random walk's hypothesis may be found in [26]. [159] is devoted to modeling returns as time series. Further recommended reading consists of [43], [85], [109]. The historical development of the log-normal model for a price development can be roughly traced as: Bachelier [4], Einstein [56], Merton [116, reprinted paper of 1973], Black-Scholes [23]. Concerning volatility, here we confine ourselves only to the case of a constant volatility. Stochastic models of volatility including popular GARCH are treated in [26], [105], [107], [109], e.g. VaR is ascribed to [118], despite in Statistics this statistic is known as the quantile for almost one hundred years. Some recent books on VaR and related topics are [42] and [89].

Valuation of securities. Valuation of coupon bonds is a simple application of the cash flows' measures but some literature do not handle the related cash flow properly. There is a vast literature on the derivatives' pricing, usually starting with the *Black-Scholes* model and covering a lot of generalizations. The pioneering works are [23], [81], [116]. Recommended reading with further references: [41], [82], [105]. An original approach based on so called *fundamental transform* taking into account stochastic volatility together with a *Mathematica*® code is presented by Lewis [107]. The valuation of stocks (value of the firm) is not explicitly covered here since it often depends on accounting principles which are beyond the scope of this book. We refer to [25], [36], and [143]. We should emphasize that the practical derivatives' valuation needs an extensive computing and some symbolic calculation is often necessary. See books [147], [162], [163], and papers on special related topics [9], [10], [12], all making use of *Mathematica*®.

Matching of assets and liabilities. Problem of matching of assets and liabilities likely originated in life insurance industry, see [114] for reference and description of *Redington's theory of immunization* of a life office. Further reading [143], pp. 638–658, [60], [178]. For a related actuarial model see [173].

Index numbers and inflation. Perhaps the first comprehensive study and theory of index numbers is the 1922 book *The Making of Index Numbers* by I. Fisher. Here we closely followed Bílý [18] who was one of the promoters of actuarial sciences and econometrics in former Czechoslovakia and before 1948 the chief official at the Ministry of Finance. Our notation has been adapted for the computational purposes.
There are hundreds of stock exchange indexes. If from related markets, they are usually highly correlated. See [61] and [143] for more information. An example of a relationship between stock prices and inflation is given in [3].

Basics of utility theory. The use of utility theory in modern financial decision making has origins in the von Neumann–Morgenstern theory. For a detailed analysis see [85]. Some particular observations are in [88], [116], and [178].

Markowitz mean-variance portfolio. The pioneering contribution to the modern portfolio theory is paper [112] of Markowitz. Many other authors elaborated his fundamental idea of portfolio diversification, let us mention [57], [85]. Basically, the Markowitz model is a one-period model. For multiperiod-selection models as well as for continuous-time models we refer to [116], [43], and [85]. Generalizations of portfolio separation theorems to more than two funds may be found in [85], [116], e.g. Useful nonlinear programming techniques suitable for portfolio selection algorithms via *Mathematica*® are in [17].

Capital asset pricing model. CAPM presented here is based on the mean-variance portfolio theory. For generalizations of the CAPM (consumption-based, continuous-time, intertemporal, and others) refer to [116].

Arbitrage pricing theory. Originally, the model has been developed as a multi-factor model (as a model of factor analysis) by S. A. Ross (Arbitrage theory of capital asset pricing, J. of Economic Theory 13 (1976), pp. 341–60). Due to its rather difficult tractability caused by a necessity of the interpretation of common factors, the regression form of APT with specified independent variables seems to be more convenient in practise, see [143], [109].

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