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FUNCTIONAL CENTRAL LIMIT THEOREMS IN BANACH SPACES

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Suppose that (X_{nj}) is a triangular array of random variables taking values in a Banach space E and that (B_n) is the corresponding sequence of random paths in E. Conditions are considered under which the distributions of B_n converge to a Gaussian measure on $C([0, 1]; E)$. Under stronger conditions on the array it is shown that if E is of type 2 the paths enjoy certain regularity properties, which are reflected in the convergence. The technique here is to factorise the integration procedure by which one passes from the array to the sequence of paths, using fractional integrals.

1. Introduction. Kuelbs (1973) has shown that if (X_{nj}) is a triangular array of random variables taking values in a Banach space E, if each row of the array consists of independent identically distributed random variables, and if the central limit theorem holds for the row sums, then an invariance principle (functional central limit theorem) holds for the array.

In Section 5 we show that similar results hold for more general arrays, in which the entries in each row need not be identically distributed, and indeed that the cylindrical convergence of the distributions of the sums to a measure on E is sufficient for the existence of a suitable measure on $C([0, 1]; E)$.

It is well known that 1-dimensional Brownian motion enjoys certain regularity properties, and in Section 7 we show that similar properties are enjoyed by the limits of vector-valued arrays in the case where E is a Banach space of type 2 or of cotype 2.

Finally Section 8 contains some rather elementary results which ensure that the central limit theorem does indeed hold.

Sections 2 to 4 contain some preliminary ideas and results which we shall need later, and Section 6 contains a discussion of the properties of Banach spaces of type 2 and of cotype 2 that we use in Sections 7 and 8.

I wish to thank the referee of an earlier version of this paper for pointing out its shortcomings so clearly.

2. Measures and cylinder measures. We shall assume basic results about measures on topological spaces and cylinder measures on topological vector spaces (cf. Badrikian (1970), Parthasarathy (1967) and Schwartz (1973)).

We recall that a cylinder measure λ on a Banach space E is *Gaussian* if $e'(\lambda)$ is a Gaussian measure for each e' in E'. In this paper, we suppose that all Gaussian measures and cylinder measures have zero mean.

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PROPOSITION 1. *A Radon probability measure γ on a Banach space E is Gaussian if $a(\gamma)$ is Gaussian for each a in a linear subspace A of E' which separates the points of E.*

PROOF. The natural map $j: E' \to L^0(\gamma)$ is continuous when E' is given the topology of uniform convergence on the compact sets of E, and A is dense in E' in this topology. Thus $j(E') \subseteq \overline{j(A)}$; since $j(A)$ is a Gaussian subspace of $L^0(\gamma)$, so is $j(E')$.

Fernique (1970) has shown that if γ is a Gaussian Radon measure on a Banach space then $\int e^{\alpha\|x\|^2}\gamma(dx) < \infty$ for some $\alpha > 0$. If $\varepsilon > 0$ and μ is a measure on a Banach space E, let

$$J_\varepsilon(\mu) = \inf\{s > 0 : \mu(\|x\| > s) \leqq \varepsilon\} .$$

Then Fernique's arguments easily show the following.

PROPOSITION 2. *If $\varepsilon > 0$ and $1 \leqq p < \infty$, there exist positive constants $K_{p,\varepsilon}$ and $k_{p,\varepsilon}$ such that if γ is a Gaussian Radon measure on a Banach space E,*

$$k_{p,\varepsilon} J_\varepsilon(\gamma) \leqq \left(\int \|x\|^p \gamma(dx)\right)^{1/p} \leqq K_{p,\varepsilon} J_\varepsilon(\gamma) .$$

The interest is of course in the second inequality. It follows immediately too that similar estimates can be made relating the different L^p norms.

3. Spaces of continuous functions. We shall be concerned with vector-valued functions on $[0, 1]$. If E is a Banach space, we denote by $C([0, 1]; E)$ the Banach space of all continuous E-valued functions on $[0, 1]$, with the usual norm. If $f \in C([0, 1]; E)$ and $t \in [0, 1]$, let $\pi_t(f) = f(t)$. It is easy to verify that a collection M of Radon probability measures on $C([0, 1]; E)$ is uniformly tight if

(a) given $\varepsilon > 0$ there exists $\delta > 0$ such that $\mu(\{f : \|f(t) - f(t')\| \leqq \varepsilon$ whenever $|t - t'| \leqq \delta\}) \geqq 1 - \varepsilon$ for each μ in M, and

(b) $\{\pi_t(\mu) ; \mu \in M\}$ is uniformly tight on E, for each t in $[0, 1]$.

We shall also consider subspaces of $C([0, 1]; E)$ consisting of functions satisfying a regularity condition. If $0 < \alpha \leqq 1$, let $\phi_\alpha(t) = \Gamma(\alpha)^{-1}t^{\alpha-1}$ for $0 < t \leqq 1$. Then $\phi_\alpha \in L^1(0, 1)$ and if $\alpha > 0$, $\beta > 0$ and $\alpha + \beta \leqq 1$,

$$\int_0^t \phi_\alpha(s)\phi_\beta(t - s)\, ds = \phi_{\alpha+\beta}(t) .$$

If $f \in L^1(0, 1)$, we define the fractional integral

$$(T_\alpha f)(t) = \int_0^t f(s)\phi_\alpha(t - s)\, ds .$$

Note that $T_1 f$ is the indefinite integral of f, and that if $\alpha > 0$, $\beta > 0$ and $\alpha + \beta \leqq 1$, $T_{\alpha+\beta} = T_\alpha T_\beta$. In particular each T_α is a one-one compact linear mapping of $C[0, 1]$ into itself, and in the usual way we also obtain a one-one continuous linear mapping T_α of $C([0, 1]; E)$ into itself. We denote by $C^\alpha([0, 1]; E)$ the space $T_\alpha(C([0, 1]; E))$ with the norm defined by $\|T_\alpha f\|_\alpha = \|f\|$. The spaces C^α provide a measure of regularity of the functions involved; for example if

$0 < \beta < \alpha$ there is a constant $K_{\alpha\beta}$ such that if $f \in C^\alpha([0, 1]; E)$ then

$$\sup_{t \neq s} \frac{\|f(t) - f(s)\|}{|t - s|^\beta} \leqq K_{\alpha\beta}\|f\|_\alpha .$$

4. Triangular arrays. Let us describe the setting and terminology which we use. Let E be a Banach space, and for each n suppose that $X_{n1}, \cdots, X_{nj_n}$ is a sequence of independent E-valued integrable random variables on a probability space (Ω_n, P_n). Suppose further that for each n we are given an increasing sequence $(t_{nj})_{j=0}^{j_n}$ of numbers such that $t_{n0} = 0$, $t_{nj_n} = 1$. We set $v_{nj}^2 = t_{nj} - t_{n,j-1}$ and denote the interval $[t_{n,j-1}, t_{nj}]$ by I_{nj}. In the usual way we construct a sequence (B_n) of random variables by setting

$$B_n(t) = \sum_{k=1}^{j-1} X_{nk} + (t - t_{n,j-1})v_{nj}^{-2} X_{nj} ,$$

for t in I_{nj}. We consider B_n as taking values in $C^\gamma([0, 1]; E)$ for any $0 \leqq \gamma < 1$.

We shall suppose that, for all n and j,

(1) $$E(X_{nj}) = 0$$

and

(2) $$0 \leqq \sigma_{nj}^2 = E(\|X_{nj}\|^2) \leqq v_{nj}^2 .$$

If we are to prove a central limit theorem for dist (B_n), we must certainly have a classical central limit theorem for each of the sequences $(e'(B_n(t)))$. By the Lindeberg-Feller central limit theorem (see Chung (1974), Theorem 7.2.1), this is so if and only if the two following conditions hold. First, for each t in $[0, 1]$ (with t in I_{nj}) and e' in E' there exists $\lambda(t, e')$ such that

(3) $$\sum_n (t, e') = \sum_{k=1}^{j-1} \sigma_{nk}^2(e') + (t - t_{n,j-1})^2 v_{nj}^{-4} \sigma_{nj}^2(e') \to \lambda(t, e') ,$$

where $\sigma_{nk}(e') = E((e'(X_{nk}))^2)$. Secondly, we have a Lindeberg condition: for each $\eta > 0$

(4) $$\sum_{j=1}^{j_n} E((e'(X_{nj}))^2 I(|e'(X_{nj})| > \eta)) \to 0$$

as $n \to \infty$.

Thus if conditions (1)–(4) are satisfied there exists for each t in $[0, 1]$ a Gaussian cylinder measure $\gamma(t)$ on E, with $e'(\gamma(t))$ distributed $N(0, \lambda(t, e'))$, such that $B_n(t)$ converges cylindrically in distribution to $\gamma(t)$. More generally if $0 \leqq s < t \leqq 1$ there exists a Gaussian cylinder measure $\delta(s, t)$ on E such that $B_n(t) - B_n(s)$ converges cylindrically in distribution to $\delta(s, t)$. This leads to the following questions.

(i) When is $\gamma(t)$ a Radon measure?

(ii) If it is, does $B_n(t)$ converge in distribution to $\gamma(t)$?

(iii) When is there a Radon measure γ on $C([0, 1]; E)$ (or on $C^\gamma([0, 1]; E)$ such that $\pi_t(\gamma) = \gamma(t)$?

(iv) If there is, does B_n converge in distribution to γ?

5. Functional central limit theorems. Kuelbs (1973) makes the interesting remark that in the setting in which he works "the central limit theorem implies the invariance principle." We shall see that this is largely so in our more general setting. First we observe that if $\gamma(1)$ is a Radon measure then $\gamma(t)$ is a Radon measure for $0 < t \leqq 1$. Since $\gamma(1) = \gamma(t)*\delta(t, 1)$, this is an immediate consequence of the following proposition.

PROPOSITION 3. *Suppose that λ and μ are symmetric probability cylinder measures on a Banach space E and that $\nu = \lambda*\mu$ is a Radon measure on E. Then λ and μ are Radon measures.*

PROOF. Given $\varepsilon > 0$ there exists a convex compact set K such that $\nu(K) \geqq 1 - \varepsilon/2$. If π_N is a continuous projection onto a finite-dimensional quotient E/N of E then $\pi_N(\nu)(\pi_N(K)) \geqq 1 - \varepsilon/2$. Since $\pi_N(\nu) = \pi_N(\lambda)*\pi_N(\mu)$, $\pi_N(\lambda)(\pi_N(K)) \geqq 1 - \varepsilon$, by the lemma on page 12 of Kahane (1968). The result now follows from Prokhorov's theorem (Bourbaki (1969), Chapter IX. 4, Théorème 1, page 52).

Before establishing a functional central limit theorem, we first obtain a central limit theorem for dist $(B_n(t))$ from the convergence of dist $(B_n(1))$.

I wish to thank Professor V. Mandrekar for suggesting the use of Fourier transforms in the proof of this theorem; this suggestion led to a considerable improvement in the result obtained.

THEOREM 1. *Suppose that (X_{nj}) is a triangular array satisfying* (1)—(4), *and that the convergence in* (4) *is uniform on the unit ball of E'. Suppose also that*

$$M(t) = \sup \{\textstyle\sum_n (t, e') : n = 1, 2, 3, \cdots, \|e'\| \leqq 1\} < \infty \tag{5}$$

for each t in $[0, 1]$. *If $\gamma(1)$ is a Radon measure and if $B_n(1)$ converges in distribution to $\gamma(1)$, then $B_n(t)$ converges in distribution to $\gamma(t)$ for each* $0 < t < 1$.

PROOF. If $t \in I_{nj}$, let $B_n'(t) = \sum_{k=1}^{j-1} X_{nk}$ and let $B_n''(t) = \sum_{k=1}^{j} X_{nk}$. By Theorem 2.2 (page 59) of Parthasarathy (1967), the sequences $(B_n'(t))$ and $(B_n''(t))$ are shift compact. Since $B_n(t)$ is a convex combination of $(B_n'(t))$ and $(B_n''(t))$, the sequence $(B_n(t))$ is shift compact. Let $f_n(t, e') = E(\exp(ie'(B_n(t))))$ and let $\phi(t, e') = \frac{1}{2}\exp(-(\lambda(t, e'))^2)$. Then by Theorem 4.5 (page 171) of Parthasarathy (1967) it is sufficient to show that $f_n(t, e')$ converges to $\phi(t, e')$ uniformly on the unit ball of E'. To see this we follow through the proof of the Lindeberg–Feller central limit theorem (see Chung (1974), page 212, Exercise 4).

Given $\varepsilon > 0$, let $\eta = 3\varepsilon/M(t)$. Then $|f_n(t, e') - \phi(t, e')| \leqq R_1 + R_2$, where

$$R_1 = \frac{1}{6} \int_{|e'(B_n(t))|<\eta} |e'(B_n(t))|^3 \, dP_n < \varepsilon/2 ,$$

by the choice of η, and

$$R_2 = \int_{|e'(B_n(t))|<\eta} |e'(B_n(t))|^2 \, dP < \varepsilon/2 ,$$

for sufficiently large n, which does not depend on e' (for $\|e'\| \leqq 1$), since the convergence in (4) is supposed to be uniform in such e'. This establishes the theorem.

Note that $\lambda(t, e') \leqq \int ||x||^2\gamma_t(dx)$, which is finite, by the results of Fernique (1970). Thus condition (5) is satisfied if the convergence in (3) is also uniform on the unit ball of E'.

It is now a straightforward matter to adapt the arguments given by Parthasarathy (1967), pages 222–224, as was done by Kuelbs (1973), to establish

THEOREM 2. *Suppose that* (X_{nj}) *is a triangular array satisfying the conditions of Theorem* 1. *If* $\gamma(1)$ *is a Radon measure and the sequence* $(B_n(1))$ *converges in distribution to* $\gamma(1)$, *then there exists a Gaussian Radon measure* γ *on* $C([0, 1]; E)$ *such that* $\pi_t(\gamma) = \gamma(t)$ *for* $0 \leqq t \leqq 1$, *and such that* (B_n) *converges in distribution on* $C([0, 1]; E)$ *to* γ.

Let us remark that if each (X_{nj}) is symmetric then it is not necessary to assume the uniformity of convergence in (4) or condition (5) for Theorems 1 and 2 to hold. For the sequence $(B_n(t))$ is shift compact and symmetric, and so the shift may be taken to be 0; thus (dist $(B_n(t))$) is relatively compact.

We now use this theorem to show that even if $B_n(1)$ does not converge to $\gamma(1)$, the fact that $\gamma(1)$ is a Radon measure is sufficient to ensure the existence of a Radon measure γ on $C([0, 1]; E)$ with the property that $\pi_t(\gamma) = \gamma(t)$.

First note that if K is any compact subset of E and $\mathscr{N}$ is the collection of closed finite-codimensional subspaces of E,

$$\begin{aligned}\int_K ||x||^2\gamma(1)(dx) &= \sup_{N\in\mathscr{N}} \int_{q_N(K)} ||x||^2 q_N(\gamma(1))(dx)\\ &\leqq \sup_{N\in\mathscr{N}} \lim_n \int_{q_N(K)} ||q_N(B_n(1))||^2\, dP_n\\ &\leqq 1\end{aligned}$$

(the equality being justified by Bourbaki (1952), Chapter IV. 1, Théorème 1, page 104) so that $\int_E ||x||^2\gamma(1)\, dx \leqq 1$. From this and from the arguments given earlier it follows that for each $0 \leqq t_1 < t_2 \leqq 1$ there is a Gaussian Radon measure $\delta(t_1, t_2)$ such that $\gamma(t_2) = \gamma(t_1)*\delta(t_1, t_2)$ and $\int_E ||x||^2\delta(t_1, t_2)(dx) \leqq t_2 - t_1$. Thus the assertion follows from the following theorem.

THEOREM 3. *Suppose that* $\{\delta(s, t) : 0 \leqq s < t \leqq 1\}$ *is a family of Gaussian Radon measures on a Banach space E such that* $\delta(s, t)*\delta(t, u) = \delta(s, u)$ *whenever* $0 \leqq s < t < u \leqq 1$ *and such that* $\int ||x||^2\delta(s, t)(dx) \leqq t - s$. *Then there exists a Gaussian Radon measure* γ *on* $C([0, 1]; E)$ *such that* $\pi(t) = \delta(0, t)$ *for each t in* $[0, 1]$.

PROOF. For each n, let $X_{n1}, \cdots, X_{nn}$ be independent E-valued random variables on some probability space with distributions $\delta(0, 1/n), \cdots, \delta((n-1)/n, 1)$. Then the triangular array (X_{nj}) satisfies all the conditions of Theorem 2.

6. Banach spaces of type 2 and cotype 2. While the results of the preceding section hold in a general Banach space, we shall find it necessary in the proofs of the next section to impose conditions on the Banach space in question.

Recall (cf. Hoffmann-Jørgensen (1972), Pisier (1973)) that a Banach space E is of *type* 2 if there exists a constant C such that whenever $x_1, \cdots, x_n$ are in E

and $\varepsilon_1, \cdots, \varepsilon_n$ are Bernoulli random variables

$$(E(||\sum_{i=1}^n \varepsilon_i x_i||^2))^{\frac{1}{2}} \leqq C(\sum_{i=1}^n ||x_i||^2)^{\frac{1}{2}} .$$

Subspaces and quotient spaces of Banach spaces of type 2 are of type 2 (and the same constant C will suffice). L^p spaces are of type 2 if $2 \leqq p < \infty$.

We shall need the following result.

PROPOSITION 4. *If E is a Banach space of type 2 and if $p \geqq 2$ there exists a constant K_p such that if $X_1, \cdots, X_n$ are independent E-valued random variables with $E(X_i) = 0$ and $E(||X_i||^p) \leqq 1$ for $1 \leqq i \leqq n$, and $a_1, \cdots, a_n$ are scalars, then*

$$(E(||\sum_{i=1}^n a_i X_i||^p))^{1/p} \leqq K_p(\sum_{i=1}^n a_i^2)^{\frac{1}{2}} .$$

PROOF. Let $X_1^*, \cdots, X_n^*$ be a symmetrization of $X_1, \cdots, X_n$. Then if $\varepsilon_1 \cdots, \varepsilon_n$ are Bernoulli random variables

$$\begin{aligned} E_\omega(||\sum_{i=1}^n a_i X_i(\omega)||^p) &\leqq E_\omega(||\sum_{i=1}^n a_i X_i^*(\omega)||^p) \\ &= E_t E_\omega(||\sum_{i=1}^n a_i \varepsilon_i(t) X_i^*(\omega)||^p) \\ &= E_\omega E_t(||\sum_{i=1}^n a_i \varepsilon_i(t) X_i^*(\omega)||^p) \\ &\leqq J_p E_\omega((E_t(||\sum_{i=1}^n a_i \varepsilon_i(t) X_i^*(\omega)||^2))^{p/2}) \\ &\leqq C^p J_p E_\omega(\sum_{i=1}^n a_i^2 ||X_i^*(\omega)||^2)^{p/2} \\ &\leqq 2^p C^p J_p(\sum_{i=1}^n a_i^2)^{p/2}, \end{aligned}$$

the first inequality coming from Hoffmann-Jørgensen (1972) Theorem 2.6, the second from the arguments of Kahane (1968) pages 15–17, the third from the fact that E is of type 2 and the fourth from Minkowski's inequality in $L^{p/2}$.

A Banach space is of *cotype* 2 if there is a constant C such that whenever $x_1, \cdots, x_n$ are in E and $f_1, \cdots, f_n$ are independent normalised Gaussian random variables,

$$(\sum_{i=1}^n ||x_i||^2)^{\frac{1}{2}} \leqq C(E(||\sum_{i=1}^n f_i x_i||^2))^{\frac{1}{2}} ,$$

or alternatively if whenever (f_i) is a sequence of independent normalised Gaussian random variables and (x_i) is a sequence in E such that $\sum_{i=1}^\infty f_i x_i$ converges almost surely, then $\sum_{i=1}^\infty ||x_i||^2 < \infty$. L^p spaces are of cotype 2 if $1 \leqq p \leqq 2$, and a Banach space is of type 2 and cotype 2 if and only if it is isomorphic to a Hilbert space.

In the next theorem we extend a characterisation of Banach spaces of cotype 2 given by Maurey (1973). If H is a Hilbert space and γ is the canonical Gaussian cylinder measure on H, a linear mapping u from H into a Banach space F is said to be *γ-Radonifying* (*weakly γ-Radonifying*) if $u(\gamma)$ is a Radon measure on F (on F'', $\sigma(F'', F')$). Recall that a linear mapping u from a Banach space E into a Banach space F is *2-summing* if there exists a positive constant C such that

$$\sum_{i=1}^n ||u(x_i)||^2 \leqq C^2 \sup_{||e'|| \leqq 1} \sum_{i=1}^n |e'(x_i)|^2 ,$$

for any finite sequence $x_1, \cdots, x_n$ in E. Alternatively, if (x_i) is a sequence in

E such that $\sum_{i=1}^{\infty} |e'(x_i)|^2 < \infty$ for each e' in E', then $\sum_{i=1}^{\infty} ||u(x_i)||^2 < \infty$. A 2-summing mapping from a Hilbert space is always γ-Radonifying.

THEOREM 4. *If E is a Banach space, the following are equivalent*:

(i) *E is of cotype 2.*

(ii) *Every weakly γ-Radonifying map from a Hilbert space H into E is 2-summing.*

(iii) *Every γ-Radonifying map from a Hilbert space H into E is 2-summing.*

(iv) *If δ is a Gaussian Radon measure on E, there is a Hilbert space H, a continuous linear map u from H into E and a Gaussian Radon measure δ_1 on H such that $u(\delta_1) = \delta$.*

Maurey (1973) established the equivalence of (i) and (ii). Clearly (ii) implies (iii). We shall show that (iii) implies (i), that (iii) implies (iv) and that (iv) implies (iii).

Suppose that (iii) is satisfied. Let $H = l^2$ and let γ be the canonical Gaussian cylinder measure on l^2. Let $\lambda : l^2 \to L^2(\Omega, \mu)$ be an associated mapping, so that $(\lambda(e_1), \lambda(e_2), \cdots)$ is a sequence of independent normalised Gaussian random variables. Suppose that (x_i) is a sequence in E such that $\sum_{i=1}^{\infty} \lambda(e_i)x_i$ converges almost everywhere. Then $\sum_{i=1}^{\infty} \lambda(e_i)x_i \in L^2(\Omega, \mu; E)$, and if $e' \in E'$

$$||e'(\textstyle\sum_{i=1}^{\infty} \lambda(e_i)x_i)||_{L^2} = ||\textstyle\sum_{i=1}^{\infty} e'(x_i)\lambda(e_i)||_{L^2}$$
$$= \textstyle\sum_{i=1}^{\infty} |e'(x_i)|^2 \leqq ||\textstyle\sum_{i=1}^{\infty} \lambda(e_i)x_i||_{L^2(E)} .$$

Thus if for any finite sequence $\alpha = (\alpha_1, \cdots, \alpha_n, 0, 0, \cdots)$ we set $T(\alpha) = \sum \alpha_i x_i$, T extends to a continuous linear map from l^2 into E. Clearly $T(\gamma) = \text{dist}\,(\sum \lambda(e_i)x_i)$, so that T is γ-Radonifying. By hypothesis, T is 2-summing, so that since $\sum_{i=1}^{\infty} |\langle h, e_i\rangle|^2 < \infty$ for each h in H, $\sum_{i=1}^{\infty} ||Te_i||^2 = \sum_{i=1}^{\infty} ||x_i||^2 < \infty$. Thus E is of cotype 2.

Suppose again that (iii) is satisfied and that δ is a Gaussian Radon measure on E. Let $\lambda : E' \to L^2(E, \delta)$ be the natural mapping. We can write $\lambda = j\lambda_1$ where λ_1 maps E' into a Gaussian linear subspace G of $L^2(E, \delta)$ and j is the inclusion mapping. Then if γ is the standard Gaussian cylinder meaure on G', $\delta = \lambda_1'(\gamma)$. It is easy to verify that λ_1' maps G' into E, and, by hypothesis, λ_1' is 2-summing. Thus there exists a Hilbert space H, a Hilbert–Schmidt mapping $R : G' \to H$ and a continuous linear mapping u from H into E such that $\lambda_1' = uR$. $\delta = u(R(\gamma))$ and $R(\gamma)$ is a Gaussian Radon measure on H. Note that we may assume that u is compact.

Finally, suppose that (iv) is satisfied. Let v be a continuous linear mapping from a Hilbert space H_1 into E such that $v(\gamma)$ is a Radon measure on E, where γ is the standard Gaussian cylinder measure on H_1. We must show that v is 2-summing. By hypothesis, there exists a Hilbert space H, a Gaussian Radon measure δ_1 on H and a continuous linear mapping $u : H \to E$ such that $u(\delta_1) = v(\gamma)$. We may clearly suppose that u is $1 - 1$. Now let $j : H' \to L^2(H, \delta_1)$ be the natural map, and let $k : H_1' \to G$ be an isometry of H_1' with a Gaussian subspace G of a space $L^2(\Omega, P)$, associated with the cylinder measure γ. Thus we have the diagram

$$\begin{array}{ccl} & \overset{u'}{\nearrow} & H' \xrightarrow{\ j\ } L^2(H, \delta_1) \\ E' & & \\ & \underset{v'}{\searrow} & H_1' \xrightarrow{\ k\ } G \subseteqq L^2(\Omega, P)\ . \end{array}$$

Now if $x_1, \cdots, x_n$ are in E', dist $(ju'(x_1), \cdots, ju'(x_n))$ is the same as dist $(kv'(x_1), \cdots, kv'(x_n))$, and so there is an isometry i from $ju'(E')$ into G (which may be extended by continuity to an isometry from $j(H')$ into G) such that $iju' = kv'$. Then if $w = (k^{-1}ij)'$, $uw = v$ and $w(\gamma) = \delta_1$. Thus w is Hilbert–Schmidt, so that $v = uw$ is 2-summing.

7. Functional central limit theorems in $C^\delta([0, 1]; E)$. We shall now show that if we impose stronger conditions on the moments, if the Banach space is of type 2 and if the random variables are symmetric then we can deduce a functional central limit theorem in a suitable $C^\delta([0, 1]; E)$ from the central limit theorem.

Symmetry enters in an essential way; we shall need the following result, which is an immediate corollary of Ryll-Nardzewski and Woyczyński (1974) and which extends the lemma on page 12 of Kahane (1968).

THEOREM 5. *If $X_1, \cdots, X_n$ are independent symmetric random variables taking values in a normed space E, if K is an absolutely convex Borel subset of E and if $|\lambda_i| \leqq 1$ for $1 \leqq i \leqq n$ then*

$$P(\textstyle\sum_{i=1}^n \lambda_i X_i \notin 8K) \leqq 8P(\sum_{i=1}^n X_i \notin K)\ .$$

We now come to the functional central limit theorem.

THEOREM 6. *Suppose that (X_{nj}) is an array of symmetric random variables satisfying conditions* (1)—(6), *and taking values in a Banach space E of type* 2. *Suppose that there exist $p > 2$ and a constant M such that*

$$(6) \qquad (E(\|X_{nj}\|^p))^{1/p} \leqq M(E(\|X_{nj}\|^2))^{\frac{1}{2}} \leqq Mv_{nj}$$

for all n and j. Suppose that $0 < \delta < \frac{1}{2} - 1/p$. If $B_n(1)$ converges in distribution to a Radon measure on E, then there exists a measure γ on $C^\delta([0, 1]; E)$ such that the sequence (B_n) converges in distribution on $C^\delta([0, 1]; E)$ to γ.

PROOF. The proof is related to one given by Garling (1973), Theorem 8. (Note though that the symmetry condition was inadvertently omitted there, and that the condition on δ should be as here.)

We define new sequences of random variables by setting

$$A_n(t) = v_{nj}^{-\frac{1}{2}} X_{nj} \qquad \text{for} \quad t \quad \text{in} \quad I_{nj}$$

and

$$C_n^{\alpha}(t) = (T_\alpha A_n)(t) \qquad \text{for} \quad 0 < \alpha \leqq 1\ .$$

Thus $B_n = T_\delta C_n^{1-\delta}$. In order to prove the theorem, it is sufficient to show that (dist (B_n)) is uniformly tight in $C^\delta([0, 1]; E)$; that is, we must show that (dist $(C_n^{1-\delta})$) is uniformly tight in $C([0, 1]; E)$.

Let us first choose α and β such that $\alpha > \frac{1}{2}$, $\beta > 1/p$ and $\alpha + \beta = 1 - \delta$. Suppose that $t \in I_{nj}$. Set

$$\alpha_{nk} = (\Gamma(\alpha)v_{nk})^{-1} \int_{I_{nk}} (t-s)^{\alpha-1}\, ds \qquad \text{for} \quad 1 \leqq k < j$$

and

$$\alpha_{nj} = (\Gamma(\alpha)v_{nj})^{-1} \int_{t_{n,j-1}} (t-s)^{\alpha-1}\, ds\, .$$

Then $C_n^\alpha(t) = \sum_{k=1}^{j} \alpha_{nk} v_{nk}^{-1} X_{nk}$, so that $(E(||C_n^\alpha(t)||^p))^{1/p} \leqq K_p(\sum_{k=1}^{j} \alpha_{nk}^2)^{\frac{1}{2}}$, by Proposition 4 and condition 7. But it follows easily from the Cauchy–Schwarz inequality that

$$\textstyle\sum_{k=1}^{j} \alpha_{nk}^2 \leqq (\Gamma(\alpha))^{-1}(2\alpha - 1)^{-1} t^{2\alpha-1}$$

so that there is a constant N independent of n such that

$$E((\textstyle\int_0^1 ||C_n^\alpha(t)||^p\, dt)) \leqq N\, ,$$

and so

$$P_n(\textstyle\int_0^1 ||C_n^\alpha(t)||^p\, dt \geqq N/\varepsilon) \leqq \varepsilon\, .$$

But $t^{\beta-1} \in L^{p'}$, and so a simple argument using Hölder's inequality shows that there exists $\eta > 0$ such that if $\int_0^1 ||C_n^\alpha(t)||^p\, dt \leqq N/\varepsilon$ and if $|t - t'| < \eta$ then $||C_n^{\alpha+\beta}(t) - C_n^{\alpha+\beta}(t')|| < \varepsilon$. Thus condition (a) of Section 3 is satisfied.

We now turn to condition (b). Suppose that $0 < t \leqq 1$. Let us now set $\alpha = 1 - \delta$. Given $0 < \varepsilon < \frac{1}{2}$, chose $0 < s < t$ so that

$$K_p((\Gamma(\alpha))^{-2}(2\alpha - 1)^{-1}(t^{2\alpha-1} - s^{2\alpha-1}))^{\frac{1}{2}} \leqq \varepsilon^{p+2}\, .$$

Now suppose that $s \in I_{nl}$ and $t \in I_{nj}$. Then

$$\begin{aligned} C_n^\alpha(t) &= \textstyle\sum_{i=1}^{l} \alpha_{ni} v_{ni}^{-1} X_{ni} + \sum_{i=l+1}^{j} \alpha_{ni} v_{ni}^{-1} X_{ni} \\ &= E_n + F_n\, , \quad \text{say.} \end{aligned}$$

Arguing as before,

$$(E(||F_n||^p)^{1/p} \leqq K_p(\textstyle\sum_{i=l+1}^{j} \alpha_{ni}^2)^{\frac{1}{2}} \leqq \varepsilon^{p+2}$$

so that $P(||F_n|| > \varepsilon) \leqq \varepsilon^2 \leqq \varepsilon/2$.

On the other hand, if n is large enough, $t - t_{nl} > \frac{1}{2}(t - s)$, so that for $1 \leqq i \leqq l$, $\alpha_{nk} v_{nk}^{-1} \leqq (\Gamma(\alpha))^{-1}(\frac{1}{2}(t-s))^{-\delta}$. It follows from Theorem 5 and the uniform tightness of dist $(B_n(1))$ that there exists a compact set K such that $P_n(E_n \notin 8K) < \varepsilon/2$ for all n. Thus there is a finite set S such that $P_n(d(C_n^\alpha(t), S) > \varepsilon) < \varepsilon$ for each n, from which it follows that condition (b) is satisfied (cf. Parthasarathy (1967) page 49).

We can now improve on Theorem 3, in the case where E is of type 2.

THEOREM 7. *Suppose that* $\{\delta(s, t) : 0 \leqq s < t \leqq 1\}$ *is a family of Gaussian Radon measures on a Banach space E of type* 2, *satisfying the conditions of Theorem* 3. *Then if* $0 < \delta < \frac{1}{2}$ *there is a Gaussian Radon measure* γ *on* $C^\delta([0, 1]; E)$ *such that* $\pi_t(\gamma) = \delta(0, t)$ *for each t in* $[0, 1]$.

PROOF. If we construct a triangular array as in Theorem 3, we have only to verify that (7) holds for any $p > 2$. This follows from the remark following Proposition 2.

THEOREM 8. *Theorem 7 also holds for Banach spaces of cotype* 2.

PROOF. As $\delta(0, 1)$ is a Gaussian Radon measure there exists a Hilbert space H, a Gaussian Radon measure $d(0, 1)$ on H and a $1 - 1$ compact linear mapping u from H into E such that $u(d(0, 1)) = \delta(0, 1)$. Let B denote the unit ball of H. Then for each $\varepsilon > 0$ there exists $K_\varepsilon > 0$ such that $d(0, 1)(||x|| > K_\varepsilon) < \varepsilon/2$. Thus $\delta(0, 1)(K_\varepsilon u(B)) > 1 - \varepsilon/2$, and so $\delta(s, t)(K_\varepsilon u(B)) > 1 - \varepsilon$ for each $0 \leqq s < t \leqq 1$. In particular $\delta(s, t)(u(H)) = 1$ for each $0 \leqq s < t \leqq 1$; since we may clearly assume that H is separable, from which it follows that a subset A of H is Borel in H if and only if $u(A)$ is Borel in E, there exist for $0 \leqq s < t \leqq 1$ Gaussian Radon measures $d(s, t)$ on H such that $u(d(s, t)) = \delta(s, t)$. Applying Proposition 2,

$$\begin{aligned}(\textstyle\int_H ||x||^2 \, d(s, t) \, dx)^{\frac{1}{2}} &\leqq K_{2,\frac{1}{2}} J_{\frac{1}{2}}(d(s, t)) \\ &\leqq K_{2,\frac{1}{2}} ||u|| J_{\frac{1}{2}}(\delta(s, t)) \\ &\leqq K_{2,\frac{1}{2}} k_{2,\frac{1}{2}} ||u|| (\textstyle\int_E ||x||^2 \delta(s, t) \, dx)^{\frac{1}{2}} .\end{aligned}$$

Thus, as H is of type 2 we can apply Theorem 7 to the family of measures $\{d(s, t) : 0 \leqq s < t \leqq 1\}$ to obtain a measure g on $C^\delta([0, 1]; H)$; composition with u then produces the required measure on E.

It would be of interest to know if Theorem 7 holds for any Banach space E.

8. The central limit theorem. Let us finally investigate the circumstances under which the central limit theorem holds for the triangular arrays which we consider. Hoffmann-Jørgensen (1974) has shown that a central limit theorem holds for every independent identically distributed sequence (X_i) of random variables with $E(||X_i||^2) < \infty$ if and only if E is a Banach space of type 2. Necessary and sufficient conditions are given by Parthasarathy (1967) page 200 for a central limit theorem to hold for triangular arrays (rather more general than those which we consider) taking values in a Hilbert space. Note that condition (3) of the theorem there is essentially a compactness one; it is necessary to impose some such condition, as the following example (described to me by A. Beck) shows. Let (e_n) be an orthonormal basis in a Hilbert space H, and for each n let $(X_{nj} : 1 \leqq j \leqq n)$ be a sequence of independent variables, with X_{nj} taking values $\pm n^{-\frac{1}{2}} e_j$ with probability $\frac{1}{2}$. Then clearly the sequence $(\text{dist}\,(\sum_{j=1}^n X_{nj}))$ is not uniformly tight in H and cannot therefore converge, while the array (X_{nj}) and the space H satisfy all the sorts of conditions which one usually imposes.

PROPOSITION 5. *Suppose that (Y_{nj}) is an array of random variables satisfying conditions* (1), (2) *and* (4), *and taking values in a Banach space E of type* 2. *Suppose that T is a compact linear operator from E into a Banach space F, and let $X_{nj} = TY_{nj}$. Then $S_n = \sum_{j=1}^{j_n} X_{nj}$ converges in distribution to a Gaussian Radon measure γ on F,* *provided that*

(3′) $E((f'(S_n))^2)$ *is convergent for all* f' *in* F' .

PROOF. As usual it is sufficient to show that the sequence (dist (S_n)) is uniformly tight. By Proposition 4,

$$E(\|\sum_{j=1}^{j_n} Y_{nj}\|^2) \leqq K_2^2 \sum_{j=1}^{j_n} E(\|Y_{nj}\|^2) \leqq K_2^2 ,$$

for all n, so that

$$P_n(\|\sum_{j=1}^{j_n} Y_{nj}\|^2 > K_2^2/\varepsilon) \leqq \varepsilon .$$

Thus if B is the unit ball of E and $K = \overline{T(B)}$,

$$P_n(S_n \notin (K_2/\varepsilon^{\frac{1}{2}})K) \leqq \varepsilon , \qquad \text{for all} \quad n.$$

This result has the possible disadvantage that it involves two Banach spaces and a compact operator. The final result is closer in spirit to the result in Parthasarathy (1967) mentioned above.

THEOREM 9. *Suppose that* (X_{nj}) *is a triangular array of random variables satisfying conditions* (1), (2) *and* (4), *and taking values in a Banach space* E *of type* 2. *Suppose that*

(3″) $E((e'(S_n))^2)$ *is convergent, for all* e' *in* E'

where $S_n = \sum_{j=1}^{j_n} X_{nj}$. *If for each* $\varepsilon > 0$ *there exists a finite dimensional subspace* F *of* E *such that*

(7) $\sum_{j=1}^{j_n} E((d(X_{nj}, F))^2) < \varepsilon$ *for* $n = 1, 2, \cdots$

then S_n *converges in distribution to a Gaussian Radon measure on* E.

PROOF. Recall that if E of type 2, so is E/F, with the same constant. Thus in Proposition 4 we can take the same constant K_2 for E and all its quotients. Given $\varepsilon > 0$, let $\eta = \varepsilon^3/2K_2^2$, and let F be the finite dimensional space corresponding to η.
Then

$$E((d(S_n, F))^2) \leqq K_2^2\eta \qquad \text{for} \quad n = 1, 2, \cdots$$

so that

$$P_n(d(S_n, F) > \varepsilon) \leqq K_2^2\eta/\varepsilon^2 = \varepsilon/2 .$$

Also

$$P_n(\|S_n\| > 2K_2/\varepsilon^{\frac{1}{2}}) \leqq \varepsilon/4 ,$$

so that if

$$K = \{x \in F : \|x\| \leqq 2K_2/\varepsilon^{\frac{1}{2}}) ,$$

$P_n(d(S_n, K) > \varepsilon$ for $n = 1, 2, \cdots$. By the remark on page 49 of Parthasarathy (1967) this ensures that (dist (S_n)) is uniformly tight.

Note also that the conditions of the theorem are satisfied if the array is generated in the usual way by an independent identically distributed sequence of random variables; this provides an alternative proof of Hoffmann-Jørgensen's central limit theorem.

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