Dieter Sondermann

Introduction to Stochastic Calculus for Finance

A New Didactic Approach

With 6 Figures

Springer

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To Freddy, Hans and Marek, who patiently helped me to a deeper understanding of stochastic calculus.

Preface

There are by now numerous excellent books available on stochastic calculus with specific applications to finance, such as Duffie (2001), Elliott-Kopp (1999), Karatzas-Shreve (1998), Lamberton-Lapeyre (1995), and Shiryaev (1999) on different levels of mathematical sophistication. What justifies another contribution to this subject? The motivation is mainly pedagogical. These notes start with an elementary approach to continuous time methods of Itô's calculus due to Föllmer. In an fundamental, but not well-known paper published in French in the Seminaire de Probabilité in 1981 (see Foellmer (1981)), Föllmer showed that one can develop Itô's calculus without probabilities as an exercise in real analysis. [1]

The notes are based on courses offered regularly to graduate students in economics and mathematics at the University of Bonn choosing "financial economics" as special topic. To students interested in finance the course opens a quick (but by no means "dirty") road to the tools required for advanced finance. One can start the course with what they know about real analysis (e.g. Taylor's Theorem) and basic probability theory as usually taught in undergraduate courses in economic departments and business schools. What is needed beyond (collected in Chap. 1) can be explained, if necessary, in a few introductory hours.

The content of these notes was also presented, sometimes in condensed form, to MA students at the IMPA in Rio, ETH Zürich, to practi-

[1] An English translation of Föllmer's paper is added to these notes in the Appendix. In Chap. 2 we use Föllmer's approach only for the relative simple case of processes with continuous paths. Föllmer also treats the more difficult case of jump-diffusion processes, a topic deliberately left out in these notes.

tioners in the finance industry, and to PhD students and professors of mathematics at the Weizmann institute. There was always a positive feedback. In particular, the pathwise Föllmer approach to stochastic calculus was appreciated also by mathematicians not so much familiar with stochastics, but interested in mathematical finance. Thus the course proved suitable for a broad range of participants with quite different background.

I am greatly indebted to many people who have contributed to this course. In particular I am indebted to Hans Föllmer for generously allowing me to use his lecture notes in stochastics. Most of Chapter 2 and part of Chapter 3 follows closely his lecture. Without his contribution these notes would not exist. Special thanks are due to my assistants, in particular to Rüdiger Frey, Antje Mahayni, Philipp Schönbucher, and Frank Thierbach. They have accompanied my courses in Bonn with great enthusiasm, leading the students with engagement through the demanding course material in tutorials and contributing many useful exercises. I also profited from their critical remarks and from comments made by Freddy Delbaen, Klaus Schürger, Michael Suchanecki, and an unknown referee. Finally, I am grateful to all those students who have helped in typesetting, in particular to Florian Schröder.

Bonn, June 2006 *Dieter Sondermann*

Contents

Introduction

The lecture notes are organized as follows: Chapter 1 gives a concise overview of the theory of Lebesgue and Stieltjes integration and convergence theorems used repeatedly in this course. For mathematic students, familiar e.g. with the content of Bauer (1996) or Bauer (2001), this chapter can be skipped or used as additional reference .

Chapter 2 follows closely Föllmer's approach to Itô's calculus, and is to a large extent based on lectures given by him in Bonn (see Foellmer (1991)). A motivation for this approach is given in Sect. 2.1. This section provides a good introduction to the course, since it starts with familiar concepts from real analysis.

In Chap. 3 the Girsanov transformation is treated in more detail, as usually contained in mathematical finance textbooks. Sect. 3.2 is taken from Revuz-Yor (1991) and is basic for the following applications to finance.

The core of this lecture is Chapter 4, which presents the fundamentals of "financial economics" in continuous time, such as the *market price of risk*, the *no-arbitrage principle*, the *fundamental pricing rule* and its invariance under *numeraire changes*. Special emphasis is laid on the economic interpretation of the so-called *"risk-neutral" arbitrage measure* and its relation to the "real world" measure considered in general equilibrium theory, a topic sometimes leading to confusion between economists and financial engineers.

Using the general Girsanov transformation, as developed in Sect. 3.2, the rather intricate problem of the *change of numeraire* can be treated in a rigorous manner, and the so-called "two-country" or "Siegel" paradox serves as an illustration. The section on Feynman-Kac relates the martingal approach used explicitly in these notes to the more classical approach based on partial differential equations.

In Chap. 5 the preceding methods are applied to term structure models. By looking at a term structure model in continuous time in the general form of Heath-Jarrow-Morton (1992) as an infinite collection of assets (the zerobonds of different maturities), the methods developed in Chap. 4 can be applied without modification to this situation. Readers who have gone through the original articles of HJM may appreciate the simplicity of this approach, which leads to the basic results of HJM in a straightforward way. The same applies to the now quite popular *Libor Market Model* treated in Sect. 5.5 .

Chapter 6 presents some more advanced topics of stochastic calculus such as *local times* and the *generalized Itô formula*. The basic question here is: Does one really need the apparatus of Itô's calculus in finance? A question which is tantamount to : are charts of financial assets in reality of unbounded variation? The answer is YES, as any practitioner experienced in "delta-hedging" can confirm. Chapter 6 provides the theoretical background for this phenomenon.

1

Preliminaries

Recommended literature : (Bauer 1996), (Bauer 2001)

We assume that the reader is familiar with the following basic concepts:

$(\Omega, \mathcal{F}, P)$ is a probability space, i.e.

$\mathcal{F}$ is a σ-algebra of subsets of the nonempty set Ω

P is a σ-additive measure on $(\Omega, \mathcal{F})$ with $P[\Omega] = 1$

X is a random variable on $(\Omega, \mathcal{F}, P)$ with values in $\overline{\mathbb{R}} := [-\infty, \infty]$, i.e. X is a map $X : \Omega \longrightarrow \overline{\mathbb{R}}$ with $[X \leq a] \in \mathcal{F}$ for all $a \in \mathbb{R}$

1.1 Brief Sketch of Lebesgue's Integral

The Lebesgue integral of a random variable X can be defined in three steps.

(a) For a discrete random variable of the form $X = \sum_{i=1}^{n} \alpha_i \, \mathbf{1}_{A_i}$, $\alpha_i \in \mathbb{R}$, $A_i \in \mathcal{F}$ the integral (resp. the expectation) of X is defined as

$$E[X] := \int_\Omega X(\omega) \, dP(\omega) := \sum_i \alpha_i \; P[A_i].$$

Note: In the following we will drop the argument ω in the integral and write shortly $\int_\Omega X \, dP$.

Let $\mathcal{E}$ denote the set of all discrete random variables.

(b) Consider the set of all random variables which are monotone limits of discrete random variables, i.e. define

$$\mathcal{E}^* := \left\{ X \ : \ \exists \quad u_1 \leq \ldots \ , \ u_n \in \mathcal{E} \ , \ u_n \uparrow X \right\}$$

Remark: X random variable with $X \geq 0 \Longrightarrow X \in \mathcal{E}^*$.

For $X \in \mathcal{E}^*$ define

$$\int_\Omega X \, dP := \lim_{n \longrightarrow \infty} \int_\Omega u_n \, dP.$$

(c) For an arbitrary random variable X consider the decomposition $X = X^+ - X^-$ with

$$X^+ := \sup(X, 0) \quad , \quad X^- := \sup(-X, 0).$$

According to (b), $X^+ \ , \ X^- \in \mathcal{E}^*$.

If either $E[X^+] < \infty$ or $E[X^-] < \infty$, define

$$\int_\Omega X \, dP := \int_\Omega X^+ \, dP - \int_\Omega X^- \, dP.$$

Properties of the Lebesgue Integral:

- *Linearity* : $\int_\Omega (\alpha \ X + \beta \ Y) \, dP = \alpha \int_\Omega X \, dP + \beta \int_\Omega Y \, dP$
- *Positivity* : $X \geq 0$ implies $\int X \, dP \geq 0$ and

$$\int X \, dP > 0 \Longleftrightarrow P[X > 0] > 0.$$

- *Monotone Convergence* (*Beppo Levi*).
  Let (X_n) be a monotone sequence of random variables (i.e. $X_n \leq X_{n+1}$) with $X_1 \geq C$. Then
  $$X := \lim_n X_n \in \mathcal{E}^*$$
  and
  $$\lim_{n \longrightarrow \infty} \int_\Omega X_n \, dP = \int_\Omega \lim_{n \longrightarrow \infty} X_n \, dP = \int_\Omega X \, dP.$$
- *Fatou's Lemma*
  (i) For any sequence (X_n) of random variables which are bounded from below one has
  $$\int_\Omega \lim_{n \longrightarrow \infty} \inf X_n \, dP \leq \lim_{n \longrightarrow \infty} \inf \int_\Omega X_n \, dP.$$
  (ii) For any sequence (X_n) of random variables bounded from above one has
  $$\int_\Omega \lim_{n \longrightarrow \infty} \sup X_n \, dP \geq \lim_{n \longrightarrow \infty} \sup \int_\Omega X_n \, dP.$$
- *Jensen's Inequality*
  Let X be an integrable random variable with values in $\mathbb{R}$ and $u : \overline{\mathbb{R}} \longrightarrow \bar{\mathbb{R}}$ a convex function.
  Then one has
  $$u(\mathrm{E}[X]) \leq \mathrm{E}[u(X)].$$

  Jensen's inequality is frequently applied, e.g. to $u(X) = |X|$, $u(X) = e^X$ or $u(X) = [X - a]^+$.

L^p-Spaces $(1 \leq p < \infty)$

$\mathrm{L}^p(\Omega)$ denotes the set of all real-valued random variables X on $(\Omega, \mathcal{F}, P)$ with $\mathrm{E}[|X|^p] < \infty$ for some $1 \leq p < \infty$. For $X \in \mathrm{L}^p$, the L^p-norm is defined as
$$||X||_p := \Big(E[|X|^p]\Big)^{\frac{1}{p}}.$$

The L^p-norm has the following properties:

(a) *Hölder's Inequality*
Given $X \in \mathrm{L}^p(\Omega)$ and $Y \in \mathrm{L}^q(\Omega)$ with $\frac{1}{p} + \frac{1}{q} = 1$, one has

$$\int_\Omega |X| \cdot |Y| \, dP \leq \Big(\int_\Omega |X|^p \, dP \Big)^{\frac{1}{p}} \cdot \Big(\int_\Omega |Y|^q \, dP \Big)^{\frac{1}{q}} \, dP < \infty,$$

In particular, since $|X \cdot Y| \leq |X| \cdot |Y|$, implies $X \cdot Y \in \mathrm{L}^1(\Omega)$.

(b) $\mathrm{L}^p(\Omega)$ is a normed vector space. In particular, $X, Y \in \mathrm{L}^p$ implies $X + Y \in \mathrm{L}^p$ and one has

$$||X + Y||_p \leq ||X||_p + ||Y||_p. \quad \text{(triangle inequality)}$$

(c) $\mathrm{L}^q \subset \mathrm{L}^p$ for $p < q$.

Important special case: $p = 2$.

On L^2, the vector space of quadratically integrable random variables, there exists even a scalar product defined by

$$\langle X, Y \rangle := \int_\Omega X \cdot Y \, dP$$

Hence one has

$$||X||_2 = \sqrt{\langle X, X \rangle}$$

and Hölder's inequality takes the form

$$\langle X, Y \rangle = \int_\Omega X \cdot Y \, dP \leq ||X||_2 \cdot ||Y||_2.$$

1.2 Convergence Concepts for Random Variables

The strength of the Lebesgue integral, as compared with the Riemann integral, consists in limit theorems - notably 'Lebesgue's Theorem' - which allow to study the limit of random variables and their integrals. Without the limit theorems - provided by the Lebesgue integration theory - stochastic analysis would be impossible.

In this section we collect the basic convergence concepts for sequences of random variables and their relationships.

Definition 1.2.1. *Let $(X_n)_{n\in\mathbb{N}}$, X be random variables on $(\Omega, \mathcal{F}, P)$.*

(a) The sequence (X_n) converges to X P-almost surely if

$$P\Big[\{\omega \,:\, X_n(\omega) \longrightarrow X(\omega)\}\Big] = 1.$$

We will then write $X_n \longrightarrow X$ P-a.s.

(b) The sequence (X_n) converges in probability if, for every $\epsilon > 0$

$$\lim_{n\longrightarrow\infty} P\Big[|X_n - X| > \epsilon\Big] = 0.$$

We will then write $P - \lim X_n = X$.

(c) Let (X_n) be in $L^p(\Omega)$ for some $p \in [1, \infty)$.
The sequence (X_n) converges to X in L^p if

$$\lim_{n\longrightarrow\infty} ||X_n - X||_p = \lim_{n\longrightarrow\infty} \Big(E[|X_n - X|^p]\Big)^{\frac{1}{p}} = 0.$$

We will then write $X_n \longrightarrow X$ in L^p or $X_n \xrightarrow{L^p} X$. (X is then also in L^p).

Still another convergence concept for random variables is that of *weak convergence*, also called *convergence in distribution*. Since here only the distributions of a random variable matter, the random variables X_n may be defined on different probability spaces. Let

$$X_n \,:\, (\Omega_n, \mathcal{F}_n, P^n) \longrightarrow E \text{ and } X \,:\, (\Omega, \mathcal{F}, P) \longrightarrow E$$

be random variables with values in a metric space E (e.g. $E = \mathbb{R}$ or $E = C[0,T]$ the space of all continuous real-valued functions on $[0,T]$).

Definition 1.2.2. *The sequence (X_n) converges to X weakly (or in distribution) if, for every continuous bounded function $f : E \longrightarrow \mathbb{R}$,*

$$\lim_{n\longrightarrow\infty} \int_{\Omega_n} f(X_n)\, dP_n = \lim_{n\longrightarrow\infty} \int_{\Omega} f(X)\, dP.$$

We will then write $X_n \longrightarrow X$ weakly or $X_n \stackrel{\mathcal{D}}{\longrightarrow} X$.

Relations between the different notions of convergence

(a) a.s.-convergence and convergence in probability

(i) $X_n \longrightarrow X$ P-a.s. $\Longrightarrow P - \lim X_n = X$

(ii) $P - \lim X_n = X \Longrightarrow \exists$ subsequence $(X_{n'})$ of (X_n) with $X_{n'} \longrightarrow X$ $P-$a.s.

(b) Convergence in probability and L^1-convergence

Assume $X_n \longrightarrow X$ in L^1.

$$\left| \int_{\Omega} X_n\, dP - \int_{\Omega} X\, dP \right| \quad \leq \quad \int_{\Omega} |X_n - X|\, dP \longrightarrow 0$$

and hence

$$\lim_{n\longrightarrow\infty} \int_{\Omega} X_n\, dP \quad = \quad \int_{\Omega} \lim_{n\longrightarrow\infty} X_n\, dP$$

Thus L^1-convergence allows to exchange limit and integration, a most important property for stochastic calculus.

Clearly L^1-convergence implies convergence in probability. The following simple example shows that the converse does not hold.

Example:
Let $\Omega = [0,1]$, $\mathcal{F} =$ Borel-σ-Algebra and $P =$ Lebesgue measure. Consider the sequence $X_n(\omega) := n \cdot \mathbf{1}_{[0,1/n]}$. Then $X_n \longrightarrow 0$ P-a.s., hence also in probability. But $\int_{\Omega} X_n\, dP = 1$, for all n.

The above example shows that an additional condition is needed which prevents the X_n from growing too fast. A sufficient condition (which is also necessary) is the following

Definition 1.2.3. *The sequence* (X_n) *is called* *uniformly integrable* *if*

$$\lim_{C\longrightarrow\infty} \sup_n \int_{|X_n|>C} |X_n|\, dP = 0.$$

Sufficient conditions for uniform integrability are the following:

1. $\sup\limits_n \mathrm{E}[|X|^p] < \infty$ for some $p > 1$,
2. There exists a random variable $Y \in \mathrm{L}^1$ such that $|X_n| \leq Y$ P-a.s. for all n.

Condition 2. is Lebesgue's 'dominated convergence' condition.

The relation between L^1-convergence and convergence in probability is now given by

Proposition 1.2.4. (Lebesgue) *The following are equivalent:*

1. $P - \lim X_n = X$ *and* (X_n) *is uniformly integrable,*
2. $X_n \longrightarrow X$ *in* L^1.

Application: (Changing the order of differentiation and integration)

Let $X : \mathbb{R}\times\Omega \longrightarrow \mathbb{R}$ be a family of random variables $X(t,\cdot)$, which is, for P-a.e. $\omega \in \Omega$, differentiable in t. If there exists a random variable $Y \in \mathrm{L}^1(\Omega)$ such that

$$|\dot{X}(t,\omega)| \leq Y(\omega) \quad P\text{-a.s.}$$

then the function $t \longrightarrow \int_\Omega X(t,\omega)\, dP(\omega)$ is differentiable in t and its derivative is $\int_\Omega \dot{X}(t,\omega)\, dP(\omega)$.

(c) Convergence in distribution and convergence in probability

Convergence in probability always implies convergence in distribution, i.e.

$$P - \lim X_n = X \Longrightarrow X_n \xrightarrow{\mathcal{D}} X.$$

The converse only holds if the limit X is P-a.s. constant.

1.3 The Lebesgue-Stieltjes Integral

From an elementary statistics course the following concepts and notations should be well-known.

Consider a real-valued random variable X on $(\Omega, \mathcal{F}, P)$ and a Borel-measurable mapping $f : \mathbb{R} \longrightarrow \mathbb{R}$. I.e. we have

$(\Omega, \mathcal{F}, P) \xrightarrow{X} (\mathbb{R}, \mathcal{B}, P_X) \xrightarrow{f} \mathbb{R}$ with

$P_X[B] := P[X^{-1}(B)]$ *distribution of X*

$F_X(x) := P_X\Big[\,]-\infty, x]\,\Big] = P[X \le x]$ *distribution function of X.*

Then the (Lebesgues-Stieltjes) integral $\int\limits_{\mathbb{R}} f(x)\; dF_X(x)$ is well defined due to the following integral transformation formula:

Proposition 1.3.1.

$$\int\limits_{\Omega} f \circ X \; dP = \int\limits_{\mathbb{R}} f \; dP_X = \int\limits_{\mathbb{R}} f(x)\; dF_X(x).$$

Proof. Let $f = \mathbf{1}_B$ be the characteristic function of the Borel set $B \in \mathcal{B}$. Then by definition of P_X and F_X one has

$$\int\limits_{\Omega} f \circ X \; dP = \int \mathbf{1}_{X^{-1}(B)} dP = P[X^{-1}(B)] = P_X(B) = \int\limits_{\mathbb{R}} f \; dP_X.$$

By linearity of the integral operator the relation is then also true for all step functions $f \in \mathcal{E}$. By Beppo Levi's monotone convergence theorem it extends to all $f \in \mathcal{E}^*$ and hence to all integrable functions $f = f^+ - f^-$ with $f^+, f^- \in \mathcal{E}^*$. □

Corollary 1.3.2. $f \circ X \in L^1(\Omega, P) \Longrightarrow f \in L^1(\mathbb{R}, P_X)$.

Hence integration on Ω is reduced to integration on $\mathbb{R}$. In particular, the moments of a random variable X can be computed as Lebesgue-Stieltjes integral with respect to F_X via

$$f(x) = x^r \Longrightarrow E[X^r] = \int_{\mathbb{R}} x^r \, dF_X.$$

We recall two well-known facts from elementary statistics.

Properties of $F = F_X$:

(i) F is isotone, i.e. $x \leq y \Longrightarrow F(x) \leq F(y)$,
(ii) F is right continuous,
(iii) $\lim_{x \longrightarrow -\infty} F(x) = 0; \quad \lim_{x \longrightarrow \infty} F(x) = 1.$

Remark 1.1. *(i) implies that F has left limits. Together with (ii) this property is often called 'càdlàg' (from the French "continu à droite - limites à gauche").*

Proposition 1.3.3. *X is a real random variable on $(\Omega, \mathcal{F}, P) \Longleftrightarrow F_X$ satisfies* (i) - (iii)

Then, for any distribution function F and any Lebesgue-integrable real function f, the Lebesgue-Stieltjes Integral $\int_{\mathbb{R}} f \, dF$ is well-defined and known from elementary statistics courses.

Generalization to functions of finite variation

We now consider real-valued right-continuous functions A on the time interval $[0, \infty[$. The value of A at time t is denoted by $A(t)$ or A_t (Note that the integration variable x is now replaced by t).
Let Π be the set of all finite subdivisions π of the interval $[0, t]$ with $0 = t_0 < t_1 < \ldots < t_n = t$. Consider the sum

$$V_t^\pi := \sum_{i=0}^{n-1} |A_{t_{i+1}} - A_{t_i}|$$

Definition 1.3.4. *The function A is of* <u>*finite variation*</u> *if, for every t,*

$$V_t(A) = \sup_{\pi \in \Pi} V_t^\pi < +\infty.$$

The function $t \longrightarrow V_t$ is called the *total variation of* A. Let $\mathrm{FV}(\mathbb{R}_+)$ denote the set of all real-valued right-continuous functions on $\mathbb{R}_+ = [0, \infty[$ of finite variation.

Proposition 1.3.5. *Every* $A \in \mathrm{FV}(\mathbb{R}_+)$ *is the difference of two isotone càdlàg functions.*

Proof. Obviously

$$A_t = \frac{1}{2}(V_t + A_t) - \frac{1}{2}(V_t - A_t) = A_t^+ - A_t^-$$

Both terms are also right-continuous and clearly isotone, hence càdlàg. □

As a result the function A has left limits at every $t \in]0, \infty[$. We write $A_{t-} = \lim_{s \nearrow t} A_s$ and set $A_{0-} = 0$.
In exactly the same way as a distribution function F_X defines a measure P_X on $(\mathbb{R}, \mathcal{B})$ via $P_X\Big[]-\infty, x]\Big] = F_X(x)$, every $A \in \mathrm{FV}(\mathbb{R}_+)$ defines a measure μ_A on $(\mathbb{R}_+, \mathcal{B})$ given by

$$\mu([0, t]) = A_t.$$

Note: Of course μ_A is no longer a probability measure and may take negative values. Such a measure is called a signed measure.

Likewise as for distribution functions one has

$$\mu([0, t[) = A_{t-}$$

and

$$\mu(\{t\}) = \mu_{A^+} - \mu_{A^-} = \Delta A_t$$

is the mass of μ concentrated in point t. Proposition 1.3.5 leads to the decomposition

$$\mu_A = \mu_{A^+} - \mu_{A^-}$$

into two positive measures. Hence for any $\mathcal{B}$-measurable real-valued function f on $\mathbb{R}_+$, the Lebesgue-Stieltjes integral is well defined as

$$\int f\, d\mu = \int f\, d\mu^+ - \int f\, d\mu^- = \int f(s)\, \mu(ds) = \int f(s)\, dA(s).$$

Definition 1.3.6. $\int\limits_0^t f_s\, dA_s := \int \mathbf{1}_{]0,t]}(s)\, f_s\, dA_s$ *is called the integral of f with respect to A integrated over the interval* $]0,t]$.

In particular, it follows

$$\int\limits_0^t dA_s = \mu([0,t]) - \mu(\{0\}) = A_t - A_0.$$

1.4 Exercises

Sect. 1.1

1. Show that the Definition 1.1(a) is independent of the representation of $X \in \mathcal{E}$.
   (Hint: If $X = \sum\limits_{i=1}^{n} \alpha_i\, \mathbf{1}_{A_i} = \sum\limits_{J=1}^{m} \beta_j\, \mathbf{1}_{B_j}$ use a joint partition of Ω as new representation)

2. Show that the Definition 1.1(b) is independent of the approximating sequence of $X \in \mathcal{E}^*$.

Sect. 1.2

1. Let $\Omega = [0,1]$ be the unit interval with $\mathcal{F}$ the σ-algebra of Borel sets and P the Lebesgue measure. Consider the following sequence $X_n(\omega) : [0,1] \longrightarrow \mathbb{R}$:

$$\begin{aligned}
X_1(\omega) =& \mathbf{1}_{[0,1/2]}, \text{ i.e. } X_1(\omega) = \begin{cases} 1, & \omega \in [0,1/2] \\ 0, & \omega \in [1/2,1] \end{cases} \\
X_2(\omega) =& \mathbf{1}_{[1/2,1]} \\
X_3(\omega) =& \mathbf{1}_{[0,1/3]}, X_4(\omega) = \mathbf{1}_{[1/3,2/3]}, X_5(\omega) = \mathbf{1}_{[2/3,1]} \\
X_6(\omega) =& \mathbf{1}_{[0,1/4]}, \ldots, X_9(\omega) = \mathbf{1}_{[3/4,1]} \quad \text{etc.}
\end{aligned}$$

   Show that (X_n) converges in probability, but does not converge P-a.s.

2. Consider $(\Omega, \mathcal{F}, P)$ as in exercise 1. Define the sequence X_n by

$$X_n(\omega) = \begin{cases} 0, & \omega < \frac{1}{2} \\ 1, & \omega \geq \frac{1}{2} \end{cases} \text{ for } n \text{ even}$$

$$X_n(\omega) = \begin{cases} 1, & \omega < \frac{1}{2} \\ 0, & \omega \geq \frac{1}{2} \end{cases} \text{ for } n \text{ odd}$$

Show that X_n converges in distribution, but not in probability.

2

Introduction to Itô-Calculus

This chapter is based on Foellmer (1981) and follows closely Foellmer (1991). For the techniques used in this chapter we refer to Chap. 1, or to Bauer (1996) resp. Bauer (2001). Some results are quoted (without proof) from Protter (1990) and Revuz-Yor (1991).

The first elementary applications to option pricing in this chapter deal with the standard Black-Scholes model (Black-Scholes (1973)), first by means of the classical PDE approach (Sect. 2.5), then by using the martingale approach (Sect. 2.9).

2.1 Stochastic Calculus vs. Classical Calculus

Let $X : [0, \infty] \rightarrowtail \mathbb{R}$ be a real-valued function $X(t) = X_t$. For example the function X_t can describe the speed or the acceleration of a solid body in dependence of time t. But X_t can also represent the price of a security over time, called the *chart* of the security X. However, there is a fundamental difference between the two interpretations. In the first case X as a function of t is a "smooth" function, not only continuous (natura non facit saltus!), but also (sufficiently often) differentiable. For this class of functions the well-known tools of classical calculus apply. Using the notation $\dot{X}_t := \frac{dX_t}{dt}$ for the differentiation of X_t w.r.t. time t, as common in physics, the basic relation between differentiation and integration can be stated as

$$X_t = X_0 + \int_0^t \dot{X}_s ds$$

or

$$dX_t = \dot{X}_t dt.$$

Let $F \in C^2(\mathbb{R})$ be a twice continuously differentiable real-valued function on the real line $\mathbb{R}$. Then Taylor's theorem states

$$\triangle F(X_t) = F(X_{t+\triangle t}) - F(X_t) = F'(X_t)\triangle X_t + \frac{1}{2}F''(X_{\widetilde{t}})(\triangle X_t)^2$$

with $\triangle X_t = X_{t+\triangle t} - X_t$ and some $\widetilde{t} \in [t, t+\triangle t]$.

Taking the limit for $\triangle t \to 0$ gives

$$dF(X_t) = F'(X_t)dX_t$$

or, equivalently,

$$F(X_t) = F(X_0) + \int_0^t F'(X_s)dX_s$$

since, for a smooth function X_t, $\triangle X_t \xrightarrow[\triangle t \to 0]{} dX_t = \dot{X}_t dt$, and the terms of higher order, which are of order $(dt)^2$, disappear.

However, this classical relation is no longer applicable for real-valued functions occurring in mathematical finance. When in the 19th century the German mathematician Weierstraß constructed a real-valued function which is continuous, but nowhere differentiable, this was considered as nothing else but a mathematical curiosity. Unfortunately, this "curiosity" is at the core of mathematical finance. Charts of exchange rates, interest rates, and liquid assets are practically continuous, as the nowadays available high frequency data show. But they are of unbounded variation in every given time interval, as argued in Chap. 6 of these notes. In particular, they are nowhere differentiable, thus the Weierstraß function depicts a possible finance chart [1]. Therefore classical calculus requires an extension to functions of unbounded variation, a task for long time overlooked by mathematicians. This gap was filled by the development of stochastic calculus, which can be considered as the theory of differentiation and integration of stochastic processes.

[1] However, as pointed out to me by Hans Föllmer, the Weierstraß function shows deterministic cyclical behavior, hence as a finance chart it is only acceptable to strong believers in business cycles.

As already mentioned in the preface, there are now numerous books available developing stochastic calculus with emphasis on applications to financial markets on different levels of mathematical sophistication. But here we follow the fundamentally different approach due to Foellmer (1981), who showed that one can develop Itô's calculus without probabilities as an exercise in real analysis.

What extension of the classical calculus is needed for real-valued functions of unbounded variation? Simply, when forming the differential $dF(X_t)$ the second term of the Taylor formula can no longer be neglected, since the term $(\Delta X_t)^2$, the *quadratic variation* of X_t, does not disappear for $\Delta t \to 0$. Thus for functions of unbounded variation the differential is of the form

$$dF(X_t) = F'(X_t)\, dX_t + \frac{1}{2}\, F''(X_t)\, (dX_t)^2 \tag{1}$$

or, in explicit form,

$$F(X_t) = F(X_0) + \int_0^t F'(X_s)\, dX_s + \frac{1}{2} \int_0^t F''(X_s)\, (dX_s)^2 \tag{2}$$

where $(dX_t)^2$ is the infinitesimal quadratic variation of X.

Ironically, it was not the newly appearing second term which created the main difficulty in developing stochastic calculus. For functions of finite quadratic variation this F''-term is a well-defined classical Lebesgue-Stieltjes integral. The real challenge was to give a precise meaning to the first integral, where both the argument of the integrand and the integrator are of unbounded variation on any arbitrarily small time interval. This task was first [2] solved by Itô, hence the name *Itô formula* for the relation (1) and *Itô integral* for the first integral in (2). For a lucid overview over the historic development of the subject see e.g. Foellmer (1998).

[2] Only recently it was discovered that the "Itô" formula was already found in the year 1940 by the German-French mathematician Wolfgang Döblin. For the tragic fate and the mathematical legacy of W. Döblin see Bru and Yor (2002).

Using the path-wise approach of Föllmer one can derive both the Itô formula and the Itô integral without any recourse to probability theory. By looking at a stochastic process "path by path" one can give a precise meaning to the expressions (1) and (2) by using only elementary tools of classical real analysis.[3] Only later probability theory is needed, when we consider the interplay of all paths of stochastic processes like diffusions and semimartingales.

2.2 Quadratic Variation and 1-dimensional Itô-Formula

Following Foellmer (1981) we start with a fixed sequence $(\tau_n)_{n=1,2,\dots}$ of finite partitions

$$\tau_n = \{0 = t_0 < t_1 < \ldots < t_{i_n} < \infty\}$$

of $[0, \infty)$ with $t_{i_n} \underset{n}{\rightarrow} \infty$ and $|\tau_n| = \sup_{t_i \in \tau_n} |t_{i+1} - t_i| \underset{n}{\rightarrow} 0$.

Definition 2.2.1. *Let X be a real-valued continuous function on $[0, \infty)$. If, for all $t \geq 0$, the limit*

$$\langle X \rangle_t = \lim_n \sum_{\substack{t_i \in \tau_n \\ t_i \leq t}} (X_{t_{i+1}} - X_{t_i})^2 \tag{3}$$

exists, the function $t \longrightarrow \langle X \rangle_t$ is called the quadratic variation of X.

Remark 2.1. *Following Foellmer (1981) we start with the study of the paths of a real-valued stochastic process, i.e. a real-valued function $t \longrightarrow X_t(\omega)$ for a fixed $\omega \in \Omega$. It will be shown later, that Definition 2.2.1 and the following results are, for almost all $\omega \in \Omega$, independent of the particular choice of the partition sequence (τ_n).*

[3] The Föllmer approach also applies to jump-diffusion processes. In these notes we restrict ourselves deliberately to continuous processes.

Remark 2.2. *Unlike the total variation V_t of X, for the quadratic variation one has*

$$\langle X \rangle_t \neq \sup_{\pi} \sum_{t_i \in \pi} (X_{t_{i+1}} - X_{t_i})^2$$

where the supremum is taken over all partitions
$\pi = (0 = t_0 < \ldots < t_n = t)$ *of* $[0, t]$. *As will be shown, for the Brownian motion one has* $\langle X \rangle_t = t$ *for almost all paths. However, the right-hand side equals* $+\infty$*, for almost all paths (see Lévy (1965)).*

Proposition 2.2.2. $X \in \mathrm{FV}(\mathbb{R}_+) \Longrightarrow \langle X_t \rangle \equiv 0 \quad$ *for all* $t \geq 0$

Proof. One has, for any $t \geq 0$,

$$\sum_{t \geq t_i \in \tau_n} (X_{t_{i+1}} - X_{t_i})^2 \leq \sup |X_{t_{i+1}} - X_{t_i}| \cdot \sum |X_{t_{i+1}} - X_{t_i}|.$$

$V_t(X) < \infty$ implies that the second term is bounded. By continuity of X the first term converges to zero with $|t_{i+1} - t_i| \underset{n}{\rightarrow} 0$. □

The above proposition implies that functions X_t with positive quadratic variation $\langle X \rangle_t$ are of unbounded total variation. Hence the integral $\int f(X_t)\, dX_t$ cannot be defined as a 'classical' Lebesgue-Stieltjes integral.

However $t \longrightarrow \langle X \rangle_t$ is a positive and isotone function and thus belongs to $\mathrm{FV}(\mathbb{R}_+)$. Hence, as shown in Sect. 1.3,

$$\mu([0, t]) := \langle X \rangle_t$$

defines a positive measure μ on $(\mathbb{R}_+, \mathcal{B})$ and the integral

$$\int f(s)\, d\mu(s) = \int f(s)\, d\langle X \rangle_s$$

with respect to the quadratic variation $\langle X \rangle$ is well-defined as Lebesgue-Stieltjes integral.

Remark 2.3. *The convergence (3) can be interpreted as weak convergence of measures* μ_n *defined by*

$$\mu_n = \sum_{t_i \in \tau_n} \Big(X_{t_{i+1}} - X_{t_i} \Big)^2 \delta_{t_i}$$

where δ_{t_i} *denotes the Dirac measure with total mass one in* $t = t_i$. *Then the sequence* (μ_n) *converges weakly to the measure* μ *with* $d\mu = d\langle X \rangle$.

Lemma 2.2.3. *Let f be a real-valued continuous function on $[0,t]$. Then one has*

$$\sum_{\substack{t_i \in \tau_n \\ t_i \le t}} f(t_i)\,(X_{t_{i+1}} - X_{t_i})^2 = \int f\, \mathbf{1}_{[0,t]}\, d\mu_n$$

$$\xrightarrow[n\uparrow\infty]{} \int f\, \mathbf{1}_{[0,t]}\, d\mu = \int_0^t f(s)\, d\langle X\rangle_s.$$

Quadratic Variation of the Brownian motion

Definition 2.2.4. *A real-valued stochastic process $(B_t)_{0\le t<\infty}$ on a probability space $(\Omega,\mathcal{F},P)$ is called* <u>*'standard Brownian motion'*</u> *, if*

(i) $B_0 = 0$
(ii) $t \longrightarrow B_t(\omega)$ *is a continuous function P-a.s.*
(iii) the increments $B_t - B_s$ are independent and have normal distribution $\mathrm{N}(0, t-s)$*, for any* $0 \le s < t$.

Theorem 2.2.5. (Lévy) *For P-almost all paths $t \longrightarrow B_t(\omega)$ one has*

$$\langle B\rangle_t(\omega) = t \quad \forall\, t \ge 0, \tag{4}$$

Proof. It suffices prove the claim for a fixed $t_0 \in \mathbb{Q}_+$, the set of non-negative rational numbers. Then, since $\mathbb{Q}_+$ is countable, $\langle B\rangle_t(\omega) = t$ P-a.s for all $t \in \mathbb{Q}_+$ which by P-a.s continuity of the paths implies P-a.s convergence for all $t \in \mathbb{R}_+$.

Consider the sequence

$$X_n := \sum_{\substack{t_i \in \tau_n \\ t_i \le t_0}} \underbrace{(B_{t_{i+1}} - B_{t_i})}_{=:Y_i}{}^2 = \sum_i Y_i^2.$$

By condition (iii) the Y_i are independent with normal distribution $\mathrm{N}(0, \Delta t_{i+1})$. One has

$$\begin{aligned} \mathrm{E}[Y_i^2] &= \sigma^2(Y_i) = \Delta t_{i+1} \\ \sigma^2(Y_i^2) &= \mathrm{E}[Y_i^4] - \mathrm{E}[Y_i^2]^2 = 3\sigma^4(Y_i) - (\Delta t_{i+1})^2 = 2(\Delta t_{i+1})^2. \end{aligned}$$

From the Central Limit Theorem it follows that, for sufficiently large n, the X_n are approximately distributed with normal distribution

$$\mathrm{N}\Big(\sum_i \Delta t_i \,,\; 2\sum_i (\Delta t_i)^2\Big)$$

Clearly, for $n \to \infty$, these distributions converge to $\mathrm{N}(t_0, 0)$. Hence X_n converges weakly to the constant t_0, which (see section 1.2.(c)) implies

$$P - \lim_{(\tau_n)_{n=1,2,\ldots}} X_n = t_0.$$

But this implies (see Sect. 1.2.(a)) that there exists a subsequence $(\tau_{n'})$ of (τ_n) such that

$$\langle B \rangle_t = \lim X_{n'} = t_0 \; P - a.s..$$

□

An immediate consequence of Lévy's theorem is the following

Corollary 2.2.6. *For any* $t > 0$ *, the paths of the Brownian motion are of unbounded variation on the interval* $[0, t]$*.*

Theorem 2.2.7. *(Itô's formula in* $\mathbb{R}^1$*) : Let* $X : [0, \infty) \longrightarrow \mathbb{R}^1$ *be a continuous function with continuous quadratic variation* $\langle X \rangle_t$*, and* $F \in C^2(\mathbb{R}^1)$ *a twice continuously differentiable real function. Then for any* $t \geq 0$

$$F(X_t) = F(X_0) + \int_0^t F'(X_s)\, dX_s + \frac{1}{2} \int_0^t F''(X_s)\, d\langle X \rangle_s \tag{5}$$

where

$$\int_0^t F'(X_s)\, dX_s = \lim_{n \uparrow \infty} \sum_{\substack{t_i \in \tau_n \\ t_i \leq t}} F'(X_{t_i})\, (X_{t_{i+1}} - X_{t_i}) \tag{6}$$

= *limit of non-anticipating Riemann sums*
(i.e F' *is evaluated at the left end of the interval)*
= *Itô integral of* $F'(X_t)$ *w.r.t.* X_t

Remark 2.4. *The Itô formula is often written in short notation as*

$$dF(X) = F'(X)\,dX + \frac{1}{2}\,F''(X)\,d\langle X\rangle \tag{7}$$

called the stochastic differential of $F(X)$. This is nothing else as an equivalent notation for the relation (5). To make it meaningful the two integrals in (5) must be well-defined. This is the case for the second integral which is well-defined as Lebesgue-Stieltjes integral, since the quadratic variation $\langle X\rangle_t$ is of finite variation (see Sect. 1.3). The important contribution of Itô consists in developing a well-defined concept for integrals of the first type, where the integrator is of unbounded variation. The existence of the limit (6) is shown in the following proof.

Remark 2.5. *For non-continuous functions X we refer to Foellmer (1981) or Protter (1990)*

Proof of the Theorem: Let $t > 0$, $t_i \in \tau_n$, $t_i \leq t$. By Taylor's theorem one has

$$\begin{aligned} F(X_{t_{i+1}}) - F(X_{t_i}) = F'(X_{t_i})\,&\underbrace{(X_{t_{i+1}} - X_{t_i})}_{\Delta X_{t_i}} \\ &+ \frac{1}{2}\,F''(X_{\tilde{t}_i})\,(\Delta X_{t_i})^2 \qquad \tilde{t}_i \in (t_i, t_{i+1}) \\ &= F'(X_{t_i})\,\Delta X_{t_i} + \frac{1}{2}\,F''(X_{t_i})\,(\Delta X_{t_i})^2 \\ &+ \underbrace{\frac{1}{2}\Big\{F''(X_{\tilde{t}_i}) - F''(X_{t_i})\Big\}\,(\Delta X_{t_i})^2}_{R_n(t_i)} \end{aligned}$$

Define $\delta_n = \max_{t_i \in \tau_n, t_i \leq t} |\Delta X_{t_i}|$. Since F'' is uniformly continuous on $X[0,t]$, it follows

$$\begin{aligned} |R_n(t_i)| &\leq \frac{1}{2} \max_{\substack{|x-y| \leq \delta_n \\ x,y \in X[0,t]}} |F''(x) - F''(y)|\,(\Delta X_{t_i})^2 \\ &\leq \epsilon_n (\Delta X_{t_i})^2 \end{aligned}$$

for some $\epsilon_n > 0$, which converges to zero as $\delta_n \to 0$.

For $n \longrightarrow \infty$ it follows

a) $$\Big| \sum_{t \geq t_i \in \tau_n} R_n(t_i) \Big| \leq \epsilon_n \cdot \underbrace{\sum_{t \geq t_i \in \tau_n} (\Delta X_{t_i})^2}_{\text{bounded}} \xrightarrow[n\uparrow\infty]{} 0 \ .$$

b) $$\sum_{t \geq t_i \in \tau_n} (F(X_{t_{i+1}} - F(X_{t_i})) \xrightarrow[n\uparrow\infty]{} F(X_t) - F(X_0)$$

c) $$\sum \frac{1}{2} F''(X_{t_i}) \, (\Delta X_{t_i})^2 \xrightarrow[n\uparrow\infty]{} \frac{1}{2} \int_0^t F''(X_s) \, d\langle X_s \rangle.$$

(Observe that the left-hand side of b) is an alternating sum, hence all intermediate members cancel, and the sums in c) converge to the Lebesgue-Stieltjes integral.)

Hence also $\sum F'(X_{t_i}) \, \Delta X_{t_i}$ must converge and there exists

$$\lim_n \sum_{t \geq t_i \in \tau_n} F'(X_{t_i}) \, \Delta X_{t_i} =: \int_0^t F'(X_s) \, dX_s.$$

□

Corollary 2.2.8. *In the classical case* ($\langle X \rangle \equiv 0$ *or* $X \in \mathrm{FV}$) *Itô's formula reduces to*

$$F(X_t) = F(X_0) + \int_0^t F'(X_s) \, dX_s$$

or in short notation, for $X \in C^1$,

$$dF(X) = F'(X) \, dX = F'(X) \, \dot{X} \, dt.$$

Examples:

1) $F(x) = x^n$ implies

$$X_t^n = X_0^n + n \int_0^t X_s^{n-1}\, dX_s + \frac{n(n-1)}{2} \int_0^t X_s^{n-2}\, d\langle X \rangle_s,$$

or in short notation

$$d(X^n) = n\, X^{n-1}\, dX + \frac{n(n-1)}{2}\, X^{n-2}\, d\langle X \rangle.$$

In particular, for $n = 2$ and $X_t = B_t$ standard Brownian motion, it follows

$$B_t^2 = 2 \int_0^t B_s.\, dB_s + \underbrace{\int_0^t d\langle B \rangle_s}_{=t}.$$

2) $F(x) = e^X$ implies

$$d(e^X) = e^X\, dX + \frac{1}{2}\, e^X\, d\langle X \rangle\ ,$$

(i.e. $dF = F\, dX$ no longer holds for $F(X_t) = e^{X_t}$ with $\langle X \rangle \not\equiv 0$) .

3) $F(x) = \log x$ implies

$$d(\log X) = \frac{dX}{X} - \frac{1}{2X^2}\, d\langle X \rangle.$$

Proposition 2.2.9. *Let $X_t = M_t + A_t$ with X and M continuous and $A \in \mathrm{FV}$. Then the QV $\langle X \rangle$ exists if and only if $\langle M \rangle$ exists, and $\langle X \rangle = \langle M \rangle$.*

Proof.

$$\sum (\Delta X)^2 = \sum (\Delta M)^2 + \underbrace{\sum (\Delta A)^2}_{\underset{n}{\longrightarrow} 0} + 2 \sum \Delta M \cdot \Delta A$$

$$\left| \sum \Delta M \cdot \Delta A \right| \le \underbrace{\sup_{t_i \in \tau_n} |M_{t_{i+1}} - M_{t_i}|}_{\underset{n}{\longrightarrow} 0} \cdot \underbrace{\sum |A_{t_{i+1}} - A_{t_i}|}_{\text{bounded}}.$$

□

Proposition 2.2.10. *For $F \in C^1$ the quadratic variation of $F(X_t)$ is given by*

$$\langle F(X) \rangle_t = \int_0^t F'(X_s)^2 \, d\langle X \rangle_s$$

Proof. Consider $t > 0$, $t_i \in \tau_n$, $t_i \leq t$. By Taylor's theorem one has, for some $\widetilde{t_i} \in (t_i, t_{i+1})$,

$$\begin{aligned} |F(X_{t_{i+1}}) - F(X_{t_i})|^2 &= F(X_{t_i})^2 (\Delta X_{t_i})^2 \\ &\quad + \frac{1}{2} \, (F'(X_{\widetilde{t_i}}) - F'(X_{t_i}))^2 (\Delta X_{t_i})^2. \end{aligned}$$

Since F' is uniformly continuous on $X[t, 0]$, it follows (see proof of Theorem 2.2.7)

$$\sum_{t \geq t_i \in \tau_n} (F(X_{t_{i+1}}) - F(X_{t_i}))^2 = \underbrace{\sum F'(X_{t_i})^2 \, (\Delta X_{t_i})^2}_{\xrightarrow[\text{Lemma 2.2.3}]{} \int_0^t F'(X_s)^2 d\langle X_s \rangle} + \epsilon_n \underbrace{\sum (\Delta X_{t_i})^2}_{\xrightarrow[n \uparrow \infty]{} 0}$$

□

Corollary 2.2.11. *For $f \in C^1$ the Itô integral*

$$M_t = \int_0^t f(X_s) \, dX_s$$

has the quadratic variation

$$\langle M \rangle_t = \int_0^t f^2(X_s) \, d\langle X \rangle_s.$$

Proof. (for the case that there exists a primitive function F with $F' = f$) In this case the Itô formula for $F(X_t)$ is

$$F(X_t) = F(X_0) + M_t + \frac{1}{2} \int_0^t f'(X_s) d\langle X \rangle_s.$$

Thus, from Propositions 2.2.9 and 2.2.10, it follows

$$\Longrightarrow \langle M\rangle_t = \langle F(X)\rangle_t = \int_0^t (F')^2\,(X_s)\; d\langle X\rangle_s .$$

□

Example: $M_t = \int_0^t B_s\; dB_s$ has, according to Corollary 2.2.11, the quadratic variation

$$\langle M\rangle_t = \int_0^t B_s^2\; d\langle B\rangle_s = \int_0^t B_s^2\; ds \Longrightarrow d\langle M\rangle_t = B_t^2\; dt.$$

2.3 Covariation and Multidimensional Itô-Formula

Consider two functions $X, Y \in C^0[0,\infty)$ with continuous quadratic variations $\langle X\rangle$ and $\langle Y\rangle$ w.r.t. to the (fixed) partition sequence (τ_n). We shall see later that all concepts developed w.r.t. to (τ_n) are independent of the choice of this particular sequence.

Definition 2.3.1. *If for any* $t \geq 0$

$$\langle X, Y\rangle_t = \lim_{n\uparrow\infty} \sum_{\substack{t_i \in \tau_n \\ t_i \leq t}} (X_{t_{i+1}} - X_{t_i})(Y_{t_{i+1}} - Y_{t_i})$$

exists, then the map $t \to \langle X, Y\rangle_t$ *is called the covariation of* X *and* Y.

Proposition 2.3.2. $\langle X, Y\rangle$ *exists if, and only if,* $\langle X + Y\rangle$ *exists, and one has*

$$\langle X, Y\rangle = \frac{1}{2}\Big(\langle X + Y\rangle - \langle X\rangle - \langle Y\rangle\Big) \qquad \text{(Polarization formula).}$$

Proof. Follows immediately from

$$\begin{aligned}\langle X+Y\rangle_t &= \lim \sum (\Delta X + \Delta Y)^2 \\ &= \lim\left(\sum \Delta X^2 + \sum \Delta Y^2 + 2\sum \Delta X \Delta Y\right)\end{aligned}$$

□

Remark 2.6. *The polarization formula is obviously equivalent to*

$$\langle X+Y\rangle = \langle X\rangle + \langle Y\rangle + 2\langle X,Y\rangle \tag{8}$$

(compare the variance of the sum of two random variables).

Example Let $X_t(w), Y_t(w)$ be two independent Brownian motions on $(\Omega, \mathcal{F}, P)$. Then one has

$$\langle X,Y\rangle_t(w) = 0 \quad P\text{-a.s.} \quad \forall\, t \geq 0.$$

This follows from the fact that $\dfrac{X_t+Y_t}{\sqrt{2}}$ is again the Brownian motion

Hence $\langle X+Y\rangle_t = 2t$ which by (8) implies $\langle X,Y\rangle \equiv 0$.

Remark 2.7. *The covariation $\langle X,Y\rangle$ is the distribution function of a signed measure $\mu = \mu^+ - \mu^-$ on $[0,\infty)$ (see Sect. 1.3).*

Proposition 2.3.3. *Consider $f, g \in C^1$ and their Itô integrals*

$$Y_t = \int_0^t f(X_s)\, dX_s$$

$$Z_t = \int_0^t g(X_s)\, dX_s$$

w.r.t. to X_t. Then their covariation is $\langle Y,Z\rangle_t = \displaystyle\int_0^t f(X_s)\, g(X_s)\, d\langle X\rangle_s$

Proof. From Corollary 2.2.11 it follows

$$\begin{aligned}\langle Y+Z\rangle_t &= \Big\langle \int_0^t (f+g)(X_s)\,dX_s \Big\rangle_t \\ &= \int_0^t (f+g)^2(X_s)\,d\langle X\rangle_s \\ &= \langle Y\rangle_t + \langle Z\rangle_t + 2\int_0^t (f\cdot g)\,(X_s)\,d\langle X\rangle_s.\end{aligned}$$

The proposition thus follows from the polarization formula resp. formula (8). □

Let now $X = (X^1,\dots,X^d) : [0,\infty) \longrightarrow \mathbb{R}^d$ be continuous (i.e $X \in C^0[0,\infty)^d$) with continuous covariation

$$\langle X^k, X^l\rangle_t = \begin{cases} \langle X^k\rangle_t & k=l \\ \frac{1}{2}\Big(\langle X^k+X^l\rangle_t - \langle X^k\rangle_t - \langle X^l\rangle_t\Big) & k\neq l \end{cases}$$

Example: Brownian motion on $\mathbb{R}^d$ realized on $\Omega = C[0,\infty)^d$, i.e

$$B_t = (B_t^1,\dots,B_t^d) \quad P = \prod_{i=1}^d P_i \qquad P_i = \text{Wiener measure}$$

$$\Longrightarrow \langle B^k, B^l\rangle_t = t\,\delta_{kl}\; P\text{-a.s.}, \qquad \text{where } \delta_{kl} = \begin{cases} 1 & k=l \\ 0 & k\neq l \end{cases}$$

(For existence of B_t and the construction of the Wiener measure on the path space $C^0[0,\infty)^d$ see e.g. Bauer (1996))

Given $F \in C^2(\mathbb{R}^d)$, we use the following notations:

$$\nabla F(x) = \Big(\frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_d}\Big)(x) = \Big(F_{x_1}(x), \dots, F_{x_d}(x)\Big) = \text{ gradient of } F$$

$$\Delta F(x) = \sum_{i=1}^{d} F_{x_i,x_i}(x) = \text{ Laplace-operator, i.e. } \Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$$

$$dF(x) = (\underbrace{\nabla F(x), dx}_{\text{scalar product}}) = \sum_i F_{x_i}(x) \underbrace{dx_i}_{\dot{x}_i\, dt} \quad \text{classical differential.}$$

Theorem 2.3.4. (d-dimensional Itô-formula)*: For* $F \in C^2(\mathbb{R}^d)$ *one has*

$$F(X_t) = F(X_0) + \underbrace{\int_0^t \nabla F(X_s)\, dX_s}_{\text{Itô integral}} + \frac{1}{2} \sum_{k,l=1}^{d} \int_0^t F_{x_k,x_l}(X_s)\, d\langle X^k, X^l \rangle_s,$$

and the limit $\lim_n \sum_{\substack{t_i \in \tau_n \\ t_i \le t}} \Big(\nabla F(X_{t_i}), (X_{t_{i+1}} - X_{t_i})\Big) =: \int_0^t \nabla F(X_s)\, dX_s$

exists.

Proof. The proof is analogous to that of Prop. 2.2.7 by applying the d-dimensional Taylor-formula to the discrete increments of F. □

In differential form the Itô-formula can be written as

$$dF(X_t) = \Big(\nabla F(X_t), dX_t\Big) + \frac{1}{2} \sum_{k,l} \frac{\partial^2 F}{\partial x_k\, \partial x_l}\, (X_t)\, d\langle X^k, X^l \rangle_t$$

which is the chain rule for stochastic differentials.

Example: For the d-dimensional Brownian motion $B_t = (B_t^1, \dots, B_t^d)$, $\langle B^k, B^l \rangle_t = t\, \delta_{kl}$ implies

$$dF(B_t) = \Big(\nabla F(B_t), dB_t\Big) + \frac{1}{2}\,\Delta F(B_t)\,dt.$$

Corollary 2.3.5. (Product rule for Itô calculus):
For X, Y with continuous $\langle X\rangle, \langle Y\rangle, \langle X,Y\rangle$ it follows

$$d(X\cdot Y) = X\,dY + Y\,dX + d\langle X,Y\rangle$$

i.e.

$$X_t\,Y_t = X_0\,Y_0 + \int_0^t X_s\,dY_s + \int_0^t Y_s\,dX_s + \langle X,Y\rangle_t. \tag{9}$$

Proof. Apply Itô's formula for $d = 2$ to the function $F(X,Y) = X\cdot Y \in C^2(\mathbb{R}^2)$ □

Remark 2.8. *For $Y \in$ FV it follows that $\langle X+Y\rangle = \langle X\rangle$, and hence $\langle Y\rangle = 0$. Thus (9) takes the form*

$$\int_0^t Y_s\,dX_s = X_t\,Y_t - X_0\,Y_0 - \int_0^t X_s\,dY_s \tag{10}$$

which is the classical partial integration formula. Observe that X_t may be of infinite variation, since the integral on the right-hand side is well-defined as Stieltjes integral, for $Y_t \in FV$. By this observation Wiener first obtained a "stochastic" integral for the integrator $X_t = B_t$ and $Y_t = h(t)$ a deterministic function, the so-called Wiener integral.

Corollary 2.3.6. (Itô's formula for a time-dependent function)
For $F \in C^{2,1}(\mathbb{R}^2_+)$ and $X : \mathbb{R}_+ \longrightarrow \mathbb{R}$ continuous with continuous $\langle X\rangle$ one has

1) $$F(X,t) = F(X_0,0) + \int_0^t F_x(X_s,s)\,dX_s + \int_0^t F_t(X_s,s)\,ds + \frac{1}{2}\int_0^t F_{xx}(X_s,s)\,d\langle X\rangle_s$$

2) $$F(X,\langle X\rangle_t) = F(X_0,0) + \int_0^t F_x(X_s,\langle X\rangle_s)\,dX_s + \int_0^t \Big(\frac{1}{2}\;F_{xx} + F_t\Big)\,(X_s,\langle X_s\rangle)\,d\langle X\rangle_s.$$

Proof. Apply Itô's formula for $d = 2$, to $F(X, Y)$ and choose $Y_t = t$ resp. $Y_t = \langle X \rangle_t$. □

Remark 2.9. *The transformation* $t \longrightarrow \langle X \rangle_t$ *is called time change according to the "interior clock" of the process* X_t*. In particular, if* $F(X, t)$ *satisfies the differential equation (dual heat equation)*

$$\frac{1}{2}\, F_{xx} + F_t = 0 \tag{11}$$

it follows

$$F(X, \langle X \rangle_t) = F(X_0, 0) + \int_0^t F_x(X_s, \langle X \rangle_s)\, dX_s. \tag{12}$$

2.4 Examples

1) $dG = \alpha\, G\, dX \quad (X_0 = 0)$

We show that the above stochastic differential equation (short: SDE) has the solution

$$G_t = G_0 + \int_0^t \alpha\, G_s\, dX_s = G_0\, \mathcal{E}(\alpha\, X_t)$$

where $\mathcal{E}(X_t) := \exp\{X_t - \frac{1}{2}\, \langle X \rangle_t\}$ is the so called *stochastic exponential* or the *Doléans-Dade exponential* of Y_t

Proof. $F(x, t) = G_0\, \exp\left\{\alpha x - \frac{1}{2}\, \alpha^2 t\right\}$ satisfies the dual heat equation (11). Hence by (12), it follows

$$G_t = F(X_t\, , \langle X \rangle_t) = G_0 \mathrm{E}(\alpha X_t)$$

$$\Longrightarrow G_t - G_0 = F(X_t\, , \langle X \rangle_t) - F(X_0, 0)$$

$$\underset{(12)}{=} \int_0^t F_x(X_s\, , \langle X \rangle_s)\, dX_s = \int_0^t \alpha\, G_s\, dX_s.$$

□

Remark 2.10. *Clearly, for* $\langle X\rangle_t \equiv 0$ *one obtains the classical solution* $G_t = G_0 \cdot e^{\alpha X_t}$.

2) $dG = \mu\, G\, dt + \sigma\, G\, dX$

Here a drift term μ is added. Using the previous result, it is easy to check that

$$G_t = G_0\, \mathcal{E}(\sigma\, X_t)\, e^{\mu\, t} = G_0\, \exp\left\{\mu\, t + \sigma\, X_t - \frac{1}{2}\, \sigma^2 \langle X\rangle_t\right\}$$

is a solution of the above SDE.

3) $\dfrac{dS_t}{S_t} = \mu(t)\, dt + \sigma(t)\, dB_t$

The above SDE defines a *diffusion* or *Itô process*. It is the standard model used in finance for the returns of a security price process S_t with infinitesimal drift $\mu(t)dt$ and stochastic noise $\sigma(t)dB_t$, where $\sigma(t)$ is called the *volatility* of S_t.

We show that the SDE has the solution

$$S_t = S_0\, \exp\left\{\int_0^t \left(\mu(s) - \frac{1}{2}\, \sigma^2(s)\right) ds + \int_0^t \sigma(s)\, dB_s\right\}. \qquad (13)$$

Proof. We give a proof using Itô's product formula. The process (13) can be written as

$$S_t = S_0\, \exp\left\{\int_0^t \mu(s)\, ds\right\} \cdot \mathcal{E}(M_t) = Y_t \cdot Z_t$$

with $M_t = \int_0^t \sigma(s)\, dB_s$ and $\langle M\rangle_t = \int_0^t \sigma^2(s)\, ds$.
Since Y_t is of finite variation, the product rule implies

$$\begin{aligned} dS_t &= Y_t\, dZ_t + Z_t\, dY_t + d\langle Y_t, Z_t\rangle \\ &= Y_t\, Z_t\, dM_t + Z_t\, Y_t\, \mu(t)\, dt + 0 \\ &= S_t\, \sigma(t)\, dB_t + S_t\, \mu(t)\, dt. \end{aligned}$$

□

2.5 First Application to Financial Markets

We consider a financial market with only one security without interest and divided payments. This market is modelled as follows:

$(\Omega, (\mathcal{F}_t)_{t\geq 0}, P)$ is a filtered probability space, i.e., $(\mathcal{F}_t)$ is a family of σ-algebras with $\mathcal{F}_s \subset \mathcal{F}_t \quad$ for $s < t$, representing the information available at time t.

$X_t = X_t(\omega)$ is the *price process of the security adapted to the filtration* $(\mathcal{F}_t)$, i.e. X_t is $\mathcal{F}_t$-measurable, for all $t \geq 0$.

$\phi_t = \phi_t(\omega)$ is another stochastic process adapted to $\mathcal{F}_t$, called a *portfolio strategy.*

$\phi_t = \phi_t(\omega)$ denotes the number of shares of the security held by an investor at time t in state ω. Adaptation to $(\mathcal{F}_t)$ means that the investment decision can only be based on the information available at time t.

Given the portfolio strategy ϕ_t, the value of the portfolio at time t is of the form

$$V_t = \phi_t X_t + \eta_t = V(X_t, t) \tag{14}$$

where η_t is a money account, yielding no interest.

A portfolio strategy (short p.s.) is called *self-financing* if, after an initial investment $V_0 = \eta_0$, all changes in the value of the portfolio V_t are only due to the accumulated gains (or losses) resulting from price changes of X_t. Formally this means

Definition 2.5.1. *The p.s.* ϕ_t *is self-financing* $\underset{def.}{\longleftrightarrow} V_t = V_0 + \int_0^t \phi_s \, dX_s$

$$\longleftrightarrow dV = \phi \, dX.$$

Recall that the Itô integral in (2.5.1) is defined as

$$\int_0^t \phi_s \, dX_s := \lim_n \sum_{t \geq t_i \in \tau_n} \phi(t_i) \, (X_{t_{i+1}} - X_{t_i}),$$

which means that $\phi(t_i)$ has to be fixed at the beginning of each investment interval. In other words: the Itô integral is non-anticipating, and hence the appropriate concept for finance (in contrast to other stochastic integrals like e.g. the Stratonovich integral).

Applying Itô's formula to the value process V yields

$$\begin{aligned} dV &= V_x\, dX + \dot{V}\, dt + \frac{1}{2}\, V_{xx}\, d\langle X\rangle \\ &= \phi\, dX + \dot{V}\, dt + \frac{1}{2}\, V_{xx}\, d\langle X\rangle \end{aligned}$$

Hence ϕ is self-financing if, and only if, V satisfies the differential equation

$$\dot{V}\, dt + \frac{1}{2}\, V_{xx}\, d\langle X\rangle = 0 \tag{15}$$

for all $t > 0$, where $\dot{V} = \frac{\partial}{\partial t} V(x,t)$.

Consequence: Let $H = F(X_T)$ be a contingent claim (e.g. a "call" option $H = (X_T - K)^+$). If there exists a self-financing p.s. ϕ_t with $V_T = H$ then the arbitrage price $V_t = V(X_t,t)$ of H at the time t satisfies the partial differential equation (PDE) (15) with boundary condition

$$V(X_T, T) = H$$

and for any $0 \le t \le T$

$$V(X_t,t) = V(X_0,0) + \int_0^t V_x(X_s,s)\, dX_s$$

Remark 2.11. *For $X_t = S_t$ as defined in example 2.4.3 (Black-Scholes model), one has $d\langle X\rangle_t = \sigma_t^2\, X_t^2\, dt$ and (15) is equivalent to the (PDE)*

$$\dot{V} + \frac{1}{2}\, \sigma^2\, X^2\, V_{xx} = 0$$

This is the classical approach to option pricing, as pioneered by Black-Scholes (1973) and Merton (1973), which leads to the solution of PDE's under boundary conditions.

Assume now that there exists an additional security $Y_t = e^{rt}$, i.e a bond with fixed compounded interest rate $r > 0$. Consider the following portfolio strategy:

- buy one contingent claim H at price V,
- sell $\phi = V_x$ shares of security X.

The value of this portfolio is

$$\Pi = V - V_x\, X.$$

According to Itô's formula it follows

$$d\Pi = dV - V_x\, dX = \dot{V}\, dt + \frac{1}{2}\, V_{xx}\, \sigma^2 X^2\, dt. \tag{16}$$

But, whereas the return $dX_t(\omega)$ depends on ω, the right-hand side of (16) does not, since $X_t^2(\omega)$ is fixed at t. Hence the portfolio Π is riskless and by the no-arbitrage principle its return must equal the riskless interest rate, i.e.

$$d\,\Pi = r\,\Pi\, dt. \tag{17}$$

Combining (16) and (17) gives the PDE

$$\dot{V} + r\, X\, V_x + \frac{1}{2}\, \sigma^2\, X^2\, V_{xx} = r \cdot V \quad \text{(Black-Scholes PDE)}. \tag{18}$$

This PDE holds for any derivative of the form $F(X_T)$. A simple example is a forward contract on X_T fixed at time $t = 0$ at price K. It is easy to check that

$$F(t) = X_t - K\, e^{-r(T-t)} = V(X_t, t)$$

is the (unique) solution of (18) under the boundary constraint

$$F(T) = V(X_T, T) = X_T - K.$$

2.6 Stopping Times and Local Martingales

In the previous sections we have concentrated on real-valued functions and shown how Itô's calculus comes into play for functions of unbounded variations. The concepts of integration and differentiation of such functions are quite independent of any probability concept and should indeed be considered as part of the calculus of real-valued functions[4]. But such functions were considered as rather exotic and uninteresting for practical applications and thus neglected in the 'classical' calculus[5]. Only the study of stochastic processes brought up the need for such an extension of the classical calculus. As will be shown in the following sections, the paths of non-trivial stochastic processes are of unbounded variation.

We now consider stochastic processes X defined on a probability space $(\Omega, \mathcal{F}, P)$ with real-valued continuous paths $X_t(\omega)$. Of course all concepts developed for real-valued functions in the previous sections apply to the paths of such processes.

In the following $(\Omega, (\mathcal{F}_t)_{t\geq 0}, P)$ always denotes a probability space with right-continuous filtration satisfying the usual conditions:

(i) $\mathcal{F}_t$ is a σ-algebra for all $t \geq 0$
(ii) $\mathcal{F}_s \subset \mathcal{F}_t$ for $s < t$
(iii) $\mathcal{F}_s = \bigcap_{s<t} \mathcal{F}_t$ for all $s \geq 0$.

A stochastic process $X = (X_t)_{t\geq 0}$ on $(\Omega, (\mathcal{F}_t))$ is called *adapted* if, for any t, X_t is $\mathcal{F}_t$-measurable, i.e.

$$\{X_t \leq \alpha\} \in \mathcal{F}_t \quad \forall\, \alpha \in \mathbb{R}.$$

or, shortly, $X_t \in \mathcal{F}_t$.

[4] This was first observed by Foellmer (1981) in his fundamental article 'Calcul de Itô sans probabilité' (see the Appendix for an English translation of this article).

[5] The first example of a function of unbounded variation is due to the 19th-century mathematician Weierstraß, who constructed a continuous real-valued function which is nowhere differentiable.

Definition 2.6.1. *A stochastic process M on $(\Omega, (\mathcal{F}_t), P)$ is called a martingale, if*

(i) (M_t) *is adapted and integrable* $($*i.e.* $M_t \in L^1(\Omega, \mathcal{F}_t, P) \quad \forall\, t \geq 0,\ )$
(ii) $\mathrm{E}[M_t|\mathcal{F}_s] = M_s$ *P-a.s., for all* $0 \leq s \leq t$.

Remark 2.12. *For a stochastic process X on a probability space $(\Omega, \mathcal{F}, P)$ consider the following filtration:*

$$\begin{aligned} \mathcal{F}_t^0 &:= \sigma(X_s : s \leq t) && \sigma\text{-algebra generated by the sets} \\ & && \{X_s \leq \alpha \ : \ s \leq t \ , \ \alpha \in \mathbb{R}\} \\ \mathcal{F}_t &:= \bigcap_{t<u} \mathcal{F}_u^0 && \text{right-continuous modification.} \end{aligned}$$

$(\mathcal{F}_t)$ is called the natural filtration generated by (X_t). Clearly, if X has right-continuous paths, $\mathcal{F}_t = \mathcal{F}_t^0$.

Lemma 2.6.2. (M_t) *martingale* $\Longrightarrow \mathrm{E}[(M_t - M_0)^2] = \mathrm{E}[M_t^2] - \mathrm{E}[M_0^2]$.

Proof. $(M_t - M_0)^2 = M_t^2 - M_0^2 - 2\, M_0\ (M_t - M_0)$

$$\Longrightarrow \mathrm{E}[(M_t - M_0)^2] = \mathrm{E}[M_t^2 - M_0^2] - 2\mathrm{E}\Big[M_0\ \underbrace{\mathrm{E}[M_t - M_0|\mathcal{F}_0]}_{=0}\Big]$$

□

Examples of martingales:

Let $B = (B_t)_{t \geq 0}$ be a Brownian motion on $(\Omega, \mathcal{F}, P)$ with natural filtration $(\mathcal{F}_t)_{t \geq 0}$.

(e.g. $\Omega = C[0, \infty) =$ all continuous paths
$(\mathcal{F}_t)$ filtration generated by (B_t)
$\mathcal{F} = \mathcal{F}_\infty = \sigma(B_t : t \geq 0)$
$P =$ Wiener measure)

Proposition 2.6.3. *The following processes are martingales with respect to the filtration* $(\mathcal{F}_t)$*:*

(1) B_t
(2) $B_t^2 - t$
(3) $S_t = S_0 \ \exp\left\{\sigma \ B_t - \frac{1}{2} \ \sigma^2 \ t\right\} = S_0 \ \mathcal{E}(\sigma \ B_t)$.

Proof. (i) The processes are clearly adapted. Integrability follows from

(1) $\mathrm{E}[B_t^2] = t < \infty \Longrightarrow B_t \in L^2 \subset L^1$,

(2) $$\mathrm{E}[|S_t|] = \mathrm{E}[S_t] = S_0 \ \exp\left\{-\frac{1}{2} \ \sigma^2 \ t\right\} \cdot \underbrace{\mathrm{E}[e^{\sigma B_t}]}_{=\exp\{(1/2)\ \sigma^2 \ t\}} = S_0 \in L^1.$$

Remark 2.13. $X \sim \mathrm{N}(\mu, \sigma^2) \Longrightarrow \mathrm{E}[e^X] = \exp\{\mu + \frac{1}{2} \ \sigma^2\}$.

(ii) (1) $B_{t+h} - B_t$ independent of B_t

$$\Longrightarrow \mathrm{E}\Big[B_{\underbrace{t+h}_{s}} - B_t \Big| \mathcal{F}_t\Big] = \mathrm{E}[B_{t+h} - B_t] = 0$$

$$\Longrightarrow \mathrm{E}[B_s|\mathcal{F}_t] = B_t + \underbrace{\mathrm{E}[B_s - B_t|\mathcal{F}_t]}_{=0} \quad s \geq t.$$

(2) $$\begin{aligned}\mathrm{E}[B_{t+h}^2 - B_t^2|\mathcal{F}_t] &= \mathrm{E}\Big[(B_{t+h} - B_t)^2 + 2 \ (B_{t+h} - B_t) \ B_t \mid \mathcal{F}_t\Big] \\ &= h + 2 \ B_t \ \underbrace{\mathrm{E}[B_{t+h} - B_t|\mathcal{F}_t]}_{=0} = h.\end{aligned}$$

(3) $$\begin{aligned}\mathrm{E}[S_{t+h}|\mathcal{F}_t] &= \mathrm{E}\Big[\underbrace{S_0 \ \mathcal{E}(\sigma \ B_t)}_{S_t} \ \mathcal{E}\Big(\sigma(B_{t+h} - B_t)\Big)\Big|\mathcal{F}_t\Big] \\ &= S_t \ \underbrace{\mathrm{E}\Big[\exp\Big\{\sigma(B_{t+h} - B_t) - \frac{1}{2} \ \sigma^2 \ h\Big\}\Big]}_{=1}.\end{aligned}$$

□

Stopping Times

Definition 2.6.4. *The random variable $T : \Omega \longrightarrow [0,\infty]$ is called a stopping time if*

$$[T \leq t] \in \mathcal{F}_t \quad (t \geq 0). \tag{19}$$

Lemma 2.6.5. *For a right-continuous filtration the condition (19) is equivalent to*

$$[T < t] \in \mathcal{F}_t \quad (t \geq 0).$$

Proof. $(\Longrightarrow)\ [T < t] = \bigcup_n \underbrace{[T \leq t - \frac{1}{n}]}_{\in \mathcal{F}_{t-1/n}} \in \mathcal{F}_t$

$$(\Longleftarrow)\ [T \leq t] = \bigcap_{\epsilon > 0} \underbrace{[T < t + \epsilon]}_{\in \mathcal{F}_{t+\epsilon}} \overset{\text{right-continuous}}{\Longrightarrow} [T \leq t] \in \mathcal{F}_t$$

□

Lemma 2.6.6. *Every stopping time is a decreasing limit of discrete stopping times.*

Proof. Consider the sequence

$$D_n = \left\{ K\ 2^{-n} \middle| K = 0, 1, 2, \ldots \right\}_{n=1,2,\ldots}$$

of dyadic partitions of the interval $[0,\infty)$. Define, for any n,

$$T_n(\omega) = \begin{cases} K\ 2^{-n} \text{ if } T(\omega) \in [(K-1)\ 2^{-n}\ ,\ K\ 2^{-n}) \\ \ +\infty \ \ \text{if } T(\omega) = \infty \end{cases}$$

Clearly,

$$[T_n \leq d] = [T_n < d] \in \mathcal{F}_d \quad \text{for } d = K\ 2^{-n} \in D_n$$

and

$$[T_n \leq t] = \bigcup_{t \geq d \in D_n} [T_n = d] \in \mathcal{F}_t.$$

Hence (T_n) are stopping times and $T_n(\omega) \downarrow T(\omega) \quad \forall\, \omega \in \Omega$.

□

Definition 2.6.7. *Let T be a stopping time.*

$$\mathcal{F}_T := \Big\{ A \in \mathcal{F}_\infty : A \cap \{T \le t\} \in \mathcal{F}_t \quad \forall\, t \ge 0 \Big\}$$

is called the σ-algebra of T-observable events.

Clearly, for discrete T_n, $\mathcal{F}_{T_n}$ is a σ-algebra. Hence, by Lemma 2.6.6,

$$\mathcal{F}_T = \bigcap_n \mathcal{F}_{T_n} \qquad \text{is a } \sigma\text{-algebra.}$$

$\mathcal{F}_T$ contains all events which are observable up to the stopping time T.

This is illustrated in the following Fig. 2.1, where $\Omega = \{\omega_1, \dots, \omega_8\}$ is the set of all paths from $t = 0$ to $t = 3$ and $\mathcal{F}_T = \sigma\{A_1, A_2, A_3\}$.

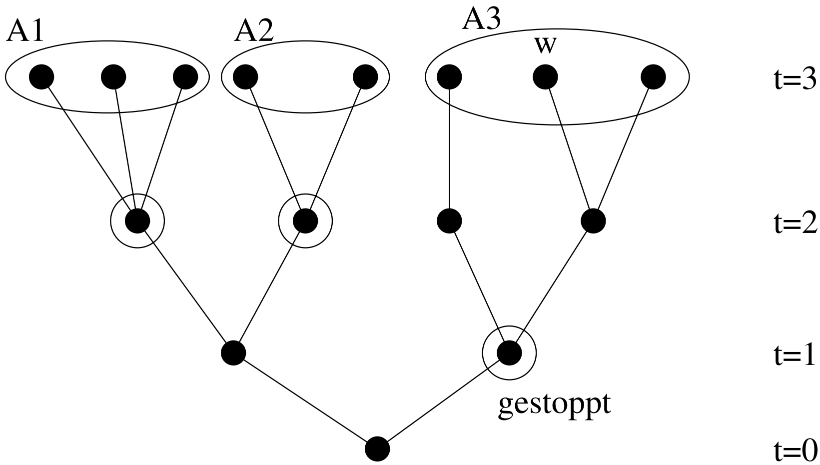


Fig. 2.1. Tree of $\mathcal{F}_T$-observable events

Remark 2.14. *One has*

$$\mathcal{F}_T = \sigma\{X_T : X \textit{ all adapted càdlàg processes}\}$$

(Protter (1990), p.6)

Lemma 2.6.8. *For an adapted, right-continuous process X the map*

$$\begin{aligned} X_T : \Omega &\longrightarrow (\mathbb{R}^d, \mathcal{B}^d) \\ \omega &\longmapsto X_T(\omega) := X_{T(\omega)}(\omega) \end{aligned}$$

is $\mathcal{F}_T$-measurable.

Proof. From Lemma 2.6.6 take $T_n \downarrow T$ and $d \in D_n$.

1) For $X_{T_n} \in \mathcal{F}_{T_n}$ and $B \in \mathcal{B}^d$, $t \geq 0$ it follows

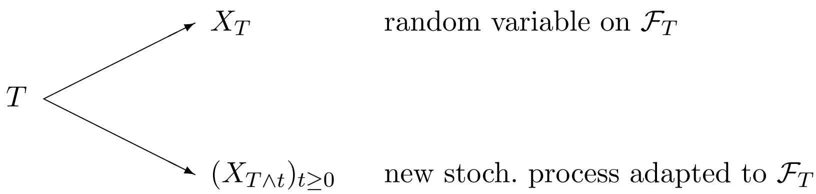
$$\{X_{T_n} \in B\} \cap \{T_n \leq t\} = \bigcup_{t \geq d \in D_n} \{X_{T_n} \in B\} \cap \{T_n = d\}$$
$$= \underbrace{\bigcup_{t \geq d \in D_n} \underbrace{\{X_d \in B\}}_{\in \mathcal{F}_d}}_{\in \mathcal{F}_t}.$$

2) (X_t) right-continuous $\Longrightarrow X_T = \lim_n X_{T_n} \quad \forall\, \omega \in \Omega$

$\Longrightarrow X_T$ $\mathcal{F}_T$-measurable with $\bigcap_n \mathcal{F}_{T_n} = \mathcal{F}_T$.

□

For any stopping time T, one has



One of the most useful theorems of probability theory is the following so-called '*Optional Stopping Theorem'*.

Theorem 2.6.9. *Let $(X_t)_{t\geq 0}$ be a real-valued process which is adapted, integrable and càdlàg.*

The following statements are equivalent:

(i) $(X_t)_{t\geq 0}$ *is a martingale,*

(ii) $\mathrm{E}[X_T] = \mathrm{E}[X_0]$ *for any bounded stopping time T*
(i.e. $T(\omega) \leq c \quad \forall\, \omega \in \Omega$) ,

(iii) $\mathrm{E}[X_T | \mathcal{F}_S] = X_S$ *for any bounded stopping times $S \leq T$.*

Proof. We use the following notation:

$$\int_A X_s \, dP =: \mathrm{E}[X_s \, \mathbf{1}_A] =: \mathrm{E}[X_s; A].$$

For $X_t \in L^1 \in (\Omega, \mathcal{F}_t, P)$ it then follows

$$(X_t) \text{ martingale } \Longleftrightarrow \mathrm{E}[X_s; A] = \mathrm{E}[X_t; A] \quad \forall\, s \le t \quad \forall\, A \in \mathcal{F}_s.$$

(i) $\Longrightarrow$ (ii):
Let $T \le c$, and let $T_n \downarrow T$ be a discretization with partition D_n (compare Lemma 2.6.6), and $A \in \mathcal{F}_{T_n}$. Then it follows

$$\begin{aligned}\mathrm{E}[X_{T_n}; A] &= \sum_{d \in D_n} \mathrm{E}[X_d \,;\ A \cap \underbrace{\{T_n = d\}}_{=\emptyset \text{ for } d>c}] \\ &= \sum_{d \in D_n} \mathrm{E}[X_c \,;\ A \cap \{T_n = d\}]\ , \text{ since } X \text{ is a martingale} \\ &= \mathrm{E}[X_c; A].\end{aligned}$$

In particular $\mathrm{E}[X_{T_n}] = \mathrm{E}[X_c; \Omega] = \mathrm{E}[X_0]$.
Since $X_T = \lim_n X_{T_n}$ P-a.s., Lebesgue's theorem implies $\mathrm{E}[X_T] = \mathrm{E}[X_0]$.

(ii) $\Longrightarrow$ (iii):
Let $A \in \mathcal{F}_s$, $S \le T \le c$. Define new stopping time

$$\hat{S}(\omega) := \begin{cases} S(\omega) & \omega \in A \\ T(\omega) & \omega \in A^c \end{cases}$$

Since $\hat{S}$ and T are bounded, (ii) implies

$$\mathrm{E}[X_{\hat{S}}] = \mathrm{E}[X_0] = \mathrm{E}[X_T] = \mathrm{E}[X_T; A] + \mathrm{E}[X_T; A^c].$$

On the other hand

$$\mathrm{E}[X_{\hat{S}}] = \mathrm{E}[X_S; A] + \mathrm{E}[X_T; A^c],$$

which together with the above equation implies $\mathrm{E}[X_S; A] = \mathrm{E}[X_T; A]$.

(iii) $\Longrightarrow$ (i):
Set $S \equiv s$ and $T \equiv t$

□

Remark 2.15. *By Lebesgue's theorem the optional stopping theorem also holds for finite stopping times $T < \infty$, P-a.s., for $(X_{T \wedge n})$ uniformly integrable.*

As an application of the optional stopping theorem we consider the hitting times of a Brownian motion for an interval $a \le 0 < b$ defined by

$$T_{a,b}(\omega) := \min\{t : B_t(\omega) \notin [a,b]\}.$$

Proposition 2.6.10. $P[B_{T_{a,b}} = b] = \dfrac{|a|}{b-a}$, $P[B_{T_{a,b}} = a] = \dfrac{b}{b-a}$,

$$\textit{and} \quad \mathrm{E}[T_{a,b}] = |a| \cdot b.$$

Proof.

1) $0 = \mathrm{E}[B_0] = \mathrm{E}[B_T] = b \cdot P[B_T = b] + a\,(1 - P[B_T = b])$

$$\Longrightarrow P[B_T = b] = \frac{-a}{b-a}.$$

2) $(B_t^2 - t)$ martingale

$$\Longrightarrow 0 = \mathrm{E}[B_0^2 - 0] = \mathrm{E}[B_{T\wedge n}^2 - T \wedge n]$$

$$\Longrightarrow \mathrm{E}[B_{T\wedge n}^2] = \mathrm{E}[T \wedge n] \uparrow \mathrm{E}[T] \quad \text{(Beppo-Levi)} .$$

On the other hand $\mathrm{E}[B_{T\wedge n}^2] \xrightarrow[n\uparrow\infty]{} \mathrm{E}[B_T^2]$, and it follows

$$\mathrm{E}[T] = \mathrm{E}[B_T^2] = b^2\, P[B_T = b] + a^2\, P[B_T = a]$$

$$\Longrightarrow \mathrm{E}[T] = |a| \cdot b.$$

□

Corollary 2.6.11. *For the hitting time of* $b > 0$, $T_b(\omega) = \inf\{t : B_t(\omega) > b\}$, *it follows*

$$T_b < \infty \ P\text{-}a.s.,\ \textit{but}\ \mathrm{E}[T_b] = +\infty.$$

Proof.

$$P[T_b < \infty] = \lim_{a\downarrow -\infty} P[T_{a,b}] = \lim \frac{|a|}{b+|a|} = 1$$

$$\mathrm{E}[T_b] \ge \mathrm{E}[T_{a,b}] = |a| \cdot b \xrightarrow[a\downarrow -\infty]{} \infty$$

□

Remark 2.16. *This result is rather discouraging for gamblers who follow the "realize-modest-gains" strategy, i.e. choose some $b > 0$, continue with fixed stake till the cumulated gains reach the bound b, then stop. If the game is "fair", then, for any $b > 0$, the gains will surpass b in finite time with probability one. But even for arbitrary small $b > 0$, the average waiting for this to happen is infinite. In the meantime the cumulated losses can become arbitrarily large.*

2.7 Local Martingales and Semimartingales

An important result of stochastic calculus is that Itô integrals with respect to a martingale are again martingales. This is, however, not quite correct: it is true for the class of *local martingales*, and more generally for *semimartingales.*

Definition 2.7.1. *An adapted càdlàg process $(M_t)_{t\geq 0}$ is called a local martingale, if there exist stopping times $T_1 \leq T_2 \leq \dots$ such that*

(i) $\sup_n T_n = \infty$ *a.s.*

(ii) $(M_{T_n \wedge t})$ *is a martingale for all n.*

Remark 2.17. *By defining new stopping times $T'_n = T_n \wedge n$ the localizing sequence in the above definition can always be assumed to be bounded (which implies that the martingales $(M_{T'_n \wedge t})$ are uniformly integrable, as required in some standard textbooks). Furthermore, if M is continuous, by setting $S_n = \inf\{t : |X_t| > n\}$ and $T'_n = T_n \wedge S_n$ one may assume the martingales to be bounded (see Revuz-Yor (1991),p.117).*

The following definition extends the Itô integral of stochastic integrands with respect to local martingales in a straightforward way.

Definition 2.7.2. *Let $(H_t)_{t\geq 0}$ be an adapted càdlàg process and $(X_t)_{t\geq 0}$ a continuous local martingale. If the following limit exists for all $t \geq 0$ P-a.s.*

$$M_t(\omega) = \lim_n \sum_{t \geq t_i \in \tau_n} H_{t_i}(\omega)\ (X_{t_{i+1}}(\omega) - X_{t_i}(\omega))$$

then $M_t = \int_0^t H_s\ dX_s$ *is called the stochastic integral of* (H_t) *with respect to* (X_t)*.*

Remark 2.18. *We will see soon that this definition also does not depend on the specific partition sequence* (τ_n)*.*

Theorem 2.7.3.

$$M_t = \int_0^t H_s\ dX_s \tag{20}$$

is a local martingale.

Proof. 1) First assume (X_t) and (H_t) are bounded, i.e.

$$|X_t(\omega)| \leq k \quad \text{and} \quad |H_t(\omega)| \leq l \quad \forall\, t, \omega.$$

We show $(M_t)_{t \geq 0}$ is a L^2-martingale. Define

$$M_t^n := \sum_{t \geq t_i \in \tau_n} H_{t_i}\ (X_{t_{i+1}} - X_{t_i}).$$

a) $$\begin{aligned} \mathrm{E}[(M_t^n)^2] &= \sum \mathrm{E}\Big[H_{t_i}^2\ (X_{t_{i+1}} - X_{t_i})^2\Big] \\ &\leq l^2 \sum_{t \geq t_i \in \tau_n} \mathrm{E}\Big[X_{t_{i+1}}^2 - X_{t_i}^2 - 2\ X_{t_i}\ (X_{t_{i+1}} - X_{t_i})\Big] \\ &= l^2\ \mathrm{E}\Big[X_{t_j}^2 - X_{t_0}^2\Big] \leq l^2\ c_0 < \infty \end{aligned}$$

where $t_j = \inf\{t_i \in \tau_n : t_i > t\}$, since $\mathrm{E}[X_{t_{i+1}} - X_{t_i}] = 0$ and the remaining term is an alternating sum.

$\Longrightarrow M_t^n \in L^2$ and bounded .

b) M_t^n is a martingale, for any $t = t_i \in \tau_n$

$$\mathrm{E}\Big[M_{t_{i+1}}^n - M_{t_i}^n | \mathcal{F}_{t_i}\Big] = H_{t_i}\ \mathrm{E}\Big[(X_{t_{i+1}} - X_{t_i}) | \mathcal{F}_{t_i}\Big] = 0$$

since X is a martingale.

c) Let $s < t$, $A_s \in \mathcal{F}_s$. To show:

$$\mathrm{E}[M_s; A_s] = \mathrm{E}[M_t; A_s]. \tag{21}$$

Choose $s_n, t_n \in \tau_n$ with $s < s_n < t < t_n$ and $s_n \downarrow s$, $t_n \downarrow t$.

$$s < s_n \longrightarrow A_s \in \mathcal{F}_{s_n} \underset{\text{b)}}{\longrightarrow} \mathrm{E}[M^n_{s_n}; A_s] = \mathrm{E}[M^n_{t_n}; A_s].$$

$$\left.\begin{array}{l} M^n_{s_n}(\omega) \longrightarrow M_s(\omega) \ P\text{-f.s.} \\ M^n_{t_n}(\omega) \longrightarrow M_t(\omega) \ P\text{-f.s.} \end{array}\right\} \underset{\text{Lebesgue}}{\Longrightarrow} (21)$$

2) The general case can be reduced to 1) by defining the following stopping times

$$S_n(\omega) = \inf\{t : |X_t(\omega)| > n\}$$
$$U_n(\omega) = \inf\{t : |H_t(\omega)| > n\}.$$

Consider the stopping time $V_n = S_n \wedge U_n \wedge T_n$ where T_n is a localizing sequence of bounded stopping times for (X_t). By assumption $(X_{T_n \wedge t})$ is a martingale. Since $V_n \leq T_n$, the stopping theorem implies that $(X_{V_n \wedge t})$ is a martingale, and hence by 1) also $(M_{V_n \wedge t})$. Clearly $V_n \uparrow \infty$. Hence (M_t) is a local martingale.

□

Corollary 2.7.4. *If* (X_t) *is a local martingale, then*

$X_t^2 - \langle X \rangle_t \quad (t \geq 0)$ *is a local martingale.*

Proof. $dX^2 \underset{\text{Itô}}{=} 2\ X\ dX + d\langle X \rangle$, i.e.

$$X_t^2 - \langle X \rangle_t = X_0^2 + \underbrace{2 \int_0^t X_s \, dX_s}_{\text{local martingale}} .$$

□

Corollary 2.7.5. *Let X be a continuous local martingale with $\langle X\rangle \equiv 0$ P-a.s. Then*

$$X_t(\omega) \equiv X_0(\omega) \quad P\text{-}a.s.,$$

i.e., continuous local martingales with paths of finite variation are trivial stochastic processes.

Proof. $M_t = X_t^2 - \langle X\rangle_t = X_t^2$ is a local martingale.

Let $T_n = T_n' \wedge T_n''$ be a joint localizing sequence for X_t and X_t^2.

$\Longrightarrow (X_{T_n\wedge t})\ ,\ (X^2_{T_n\wedge t})$ are martingales. Hence if follows:

$$\begin{aligned}
0 \le \mathrm{E}[(X_t - X_0)^2] &\underset{(X_t \text{ cont.})}{=} \mathrm{E}\Big[\lim_n (X_{T_n\wedge t} - X_0)^2\Big] \\
&\underset{(\text{Fatou})}{\le} \liminf_n \mathrm{E}\Big[(X_{T_n\wedge t} - X_0)^2\Big] \\
&\underset{(X_{T_n\wedge t} \text{ mart.})}{=} \liminf \mathrm{E}\Big[X^2_{T_n\wedge t} - X_0^2\Big] \\
&\underset{(X^2_{T_n\wedge t} \text{ mart.})}{=} 0
\end{aligned}$$

$$\Longrightarrow P[X_t = X_0 \quad \forall\, t \in \mathbb{Q}] = 1$$

$$\underset{(X_t \text{ cont.})}{\Longrightarrow} P[X_t = X_0 \quad \forall\, t] = 1$$

□

Now we are able to prove the

Independence of the calculus from (τ_n):

Let X be a continuous local martingale. Then

$$X_t^2 = X_0^2 + 2\int_0^t X_s\, dX_s + \langle X\rangle_t.$$

Assume there exist two partition sequences $(\tau_n^{(1)})$ and $(\tau_n^{(2)})$ with

$$X_t^2 - X_0^2 = \begin{cases} \text{Itô integral}^{(1)} + \langle X\rangle_t^{(1)} \\ \text{Itô integral}^{(2)} + \langle X\rangle_t^{(2)} \end{cases}$$

Then $M_t = \langle X\rangle_t^{(1)} - \langle X\rangle_t^{(2)}$, as the difference of two Itô integrals, is a local martingale with paths of finite variation. Hence by corollary 2.7.5

$$M_t(\omega) = M_0 = 0 \quad P - \text{a.s.}$$

But this implies

$$\langle X\rangle_t^{(1)} - \langle X\rangle_t^{(2)} \equiv 0 \quad P - \text{a.s.}$$

and

$$\text{Itô integral}^{(1)} \equiv \text{ Itô integral}^{(2)} \quad P - \text{a.s.}$$

The same argument applies to the stochastic integral $M_t = \displaystyle\int_0^t H_s \, dX_s$, since M_t is a continuous local martingale.

We end this section with several sufficient conditions for a local martingale to be actually a martingale.

Proposition 2.7.6. *Let M be a local martingale. The following conditions are sufficient for M to be a martingale (see Protter (1990) Theorem47 (p.35) and Theorem27, Corollary 3 (p.66)):*

1) $\mathrm{E}[\sup_{s\leq t} |M_s|] < \infty \quad \forall\, t \geq 0 \Longrightarrow$ *M is a martingale .*
2) $\mathrm{E}[\sup_t |M_t|] < \infty \Longrightarrow$ *M uniformly integrable martingale (in particular M bounded) .*
3) $\mathrm{E}[\langle M\rangle_t] < \infty \quad \forall\, t \geq 0 \Longrightarrow$ *M is a L^2-martingale and $M^2 - \langle M\rangle$ is a*

$$L^1\text{-}martingale\ .$$

Condition 3) is clearly satisfied by the Brownian motion $(B_t)_{t\geq 0}$, since $\mathrm{E}[\langle B\rangle_t] = t < \infty$. Hence B is a L^2-martingale and $B^2 - \langle B\rangle$ L^1-martingale.

Condition 3) has an interesting interpretation. Let M be a local martingale satisfying 3). Then it follows, for any $t > 0$,

$$M_t^2 - \langle M\rangle_t \quad \text{martingale} \Longrightarrow \mathrm{E}\Big[\underbrace{\mathrm{E}[M_t^2 - \langle M\rangle_t|\mathcal{F}_0]}_{M_0^2 - 0}\Big] = \mathrm{E}[M_0^2].$$

Thus $\quad \mathrm{E}[M_t^2] = \mathrm{E}[M_0^2] + \mathrm{E}[\langle M\rangle_t]$, which implies

$$\underbrace{\mathrm{Var}[M_t]}_{\text{local in } t} = \mathrm{E}[M_t^2] - (\underbrace{\mathrm{E}[M_t]}_{\mathrm{E}[M_0]})^2 = \underbrace{\mathrm{E}[\langle M\rangle_t]}_{\text{global on } [0,t]} \ .$$

I.e. the variance of M at time t equals the average quadratic variation of all paths of M on the interval $[0, t]$.

According to Theorem2.7.3 the class of local martingales is closed with respect to stochastic integration. This property extends to the larger class of semimartingales.

Definition 2.7.7. *A continuous stochastic process X is called a semi-martingale, if there exists a decomposition*

$$X_t = X_0 + M_t + A_t$$

with $M_0 = A_0 = 0$, M a local martingale and A a process of finite variation.

If H is another semimartingale, for which $\int H\, dX$ exists, it follows

$$\int H\, dX = \underbrace{\int H\, dM}_{\text{local mart.}} + \underbrace{\int H\, dA}_{\in \mathrm{FV}} .$$

Hence $\int H\, dX$ is again a semimartingale.

2.8 Itô's Representation Theorem

Let B be a Brownian motion and H an adapted process. A sufficient condition for the stochastic integral $M_t = \int_0^t H_s\, dB_s$ to be a martingale (see Prop. 2.7.6 (3)) is

$$\mathrm{E}[\langle M\rangle_t] = \mathrm{E}\Big[\int_0^t H_s^2\, ds\Big] < \infty \quad \forall\, t$$

which implies that M is a L^2-martingale and $M^2 - \langle M \rangle$ a L^1-martingale.

The following theorem (see e.g. Revuz-Yor (1991) p.187) shows that the converse also holds.

Theorem 2.8.1. (*Itô's representation theorem*)

Let B a Brownian motion on $(\Omega, (\mathcal{F}_t), P)$ with "natural" filtration (generated by (B_t) and all P-null sets) and M a L^2-martingale on $(\mathcal{F}_t, P)$. Then there exists an adapted process H with

$$\mathrm{E}\Big[\int_0^t H_s^2 \, ds\Big] < \infty \quad t \geq 0$$

such that

$$M_t = M_0 + \int_0^t H_s \, dB_s \quad \forall \, t \geq 0.$$

2.9 Application to Option Pricing

Let B denote again a Brownian motion on $(\Omega, (\mathcal{F}_t), P)$ with natural filtration.
Let $(S_t)_{t \geq 0}$ be a price process adapted to $(\mathcal{F}_t)$ and C an option depending on the paths of $(S_t)_{0 \leq t \leq T}$, for some fixed T.

Examples: $C(\omega) = [S_T(\omega) - K]^+$ *call option*

$C(\omega) = \max\limits_{0 \leq t \leq T} S_t(\omega)$ *lookback option*

$C(\omega) = [S_T(\omega) - K]^+ \cdot \mathbf{1}_{\{S_t(\omega) > L \, \forall \, t \leq T\}}$ *knock-out call* $(L < \inf\{S_0, K\})$

Then C is a $\mathcal{F}_T$-measurable so-called contingent claim. Assume there exists a self-financing trading strategy $(\phi_t)_{0 \leq t \leq T}$ which generates C, i.e. there exists a value process

$$V_t = V_0 + \int_0^t \phi_s \, dS_s \quad 0 \le t \le T$$

with $\quad V_T = C$.

If S_t is a martingale, according to Theorem 2.7.3 V_t is a local martingale, and, under suitable conditions on C (see Prop. 2.7.6), a martingale. Hence

$$V_t = \mathrm{E}[V_T|\mathcal{F}_t] = \mathrm{E}[C|\mathcal{F}_t]$$

is the arbitrage price of C at time t.

In particular $\quad V_0 = \mathrm{E}[C]$.

Existence of (ϕ_t):

If C is square-integrable, i.e. $C \in \mathrm{L}^2(\mathcal{F}_T, P)$, by Itô's representation Theorem (see Theorem 2.8.1) there exists an adapted process $(H_t)_{0 \le t \le T}$ with

$$\mathrm{E}\Big[\int_0^T H_s^2 \, ds\Big] < \infty,$$

and

$$C = \mathrm{E}[C] + \int_0^T H_s \, dB_s \ .$$

Assume $S_t = S_0 \exp\left\{\sigma \, B_t - \frac{1}{2}\,\sigma^2 \, t\right\} = S_0 \cdot \mathcal{E}(\sigma \, B_t)$.

$\Longrightarrow S_t$ is $(\mathcal{F}_t)$-martingale with $dS_t = \sigma \, S_t \, dB_t$

$$\Longrightarrow C = \mathrm{E}[C] + \int_0^T H_s \, dB_s = \underbrace{\mathrm{E}[C]}_{\text{premium}} + \int_0^T \underbrace{\frac{H_s}{\sigma \, S_s}}_{\phi_s} \, dS_s \ .$$

For the call option $C = [S_T - K]^+$ it follows

$$\begin{aligned} \mathrm{E}[C] &= \int_{\Omega} [S_T(\omega) - K]^+ \, P(d\omega) = \int_{[S_T > K]} S_T \, dP - K \cdot P[S_T > K] \\ &= S_0 \, \Phi(g(K)) - K \cdot \Phi(h(K)) \end{aligned} \tag{22}$$

with

$$g(K) = \frac{\ln(S_0/K)}{\sigma\sqrt{T}} + \frac{1}{2}\,\sigma\sqrt{T}$$

$$h(K) = g(K) - \sigma\sqrt{T}$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}\,z^2} dz \quad \text{(standard normal distribution)}$$

which is the Black-Scholes formula for $r = 0$.

This is the martingale approach to option pricing.

Proof. (of equation (22))

$$\begin{aligned} F(x) &= P[S_T \le x] \qquad \text{distribution function of } S_T \\ &= P[\ln S_T \le \ln x] = P\Big[\ln S_0 + \sigma\, B_T - \frac{1}{2}\,\sigma^2\, T \le \ln x\Big] \\ &= P\Big[\sigma\, B_T \le \ln x - \ln S_0 + \frac{1}{2}\,\sigma^2\, T\Big] \\ &= P\Big[\frac{B_T}{\sqrt{T}} \le \underbrace{\frac{\ln(x/S_0)}{\sigma\sqrt{T}} + \frac{1}{2}\,\sigma\sqrt{T}}_{-h(x)}\Big] = \Phi(-h(x)). \end{aligned}$$

Hence $\quad K \cdot P[S_T > K] = K(1 - \Phi(-h(K)) = K \cdot \Phi(h(K))$

$$\begin{aligned} f(x) = F'(x) &= \Phi'(-h(x)) \frac{d}{dx}\,(-h(x)) \\ &= \varphi(h(x)) \frac{1}{x\,\sigma\sqrt{T}} = \text{ density of } F(x) \end{aligned}$$

with $\quad \varphi(h) = \dfrac{1}{\sqrt{2\pi}}\, e^{-\frac{h^2}{2}} \quad$ density of N(0,1).

Lemma 2.9.1. $g = h + \sigma\sqrt{T} \Longrightarrow \varphi(g(x)) = \varphi(h(x))\frac{x}{S_0}$

(The proof is left for exercise).

It follows

$$\int_{[S_T \geq K]} S_T \, dP = \int_K^\infty x \, f(x) \, dx = \int_K^\infty \frac{\varphi(h(x))}{\sigma\sqrt{T}} \, dx = S_0 \int_K^\infty \frac{\varphi(g(x))}{x \, \sigma\sqrt{T}} \, dx$$
$$= S_0 \Big[- \Phi(g(x)) \Big]_K^\infty = S_0 \, \Phi(g(K)).$$

□

Remark 2.19. *The above example illustrates the martingale approach to option pricing. It was pioneered by Harrison-Kreps (1979) and Harrison-Pliska (1981) and has proved as a powerful tool in finance. Whereas the original Black-Scholes approach leads to solving PDE's under boundary constraints, the martingale technique leads to option prices as expectations under the "martingale measure". This technique will be developed in detail in Chap. 4. The Feynman-Kac Theorem establishes a relation between these two apparently so different approaches (see Sect. 4.3).*

In the above example we have assumed that the security price process S_t is already a martingale. But in general S_t will only be a semimartingale under the given probability measure P. The technique of how to transform P into an "equivalent martingale measure" is developed in Chap. 3.

3

The Girsanov Transformation

The Girsanov transformation is an important tool in mathematical finance. We first give a heuristic introduction taken from Foellmer (1991) (see also Karatzas-Shreve (1988), Sect.3.5). Using only elementary facts of independent normally distributed random variables, it leads to the Doléans-Dade exponential as new density under a change of measure for the Brownian motion.

Sect. 3.2 deals with the Girsanov transformation in general form, as can be found in Revuz-Yor (1991). The proofs are straightforward applications of tools developed in Chap. 2. This section, which at first sight looks rather abstract, is basic for the applications to finance in Chapters 4 and 5, where the general Girsanov transformation is repeatedly used.

Sect. 3.3 treats the special case of the Brownian motion.

3.1 Heuristic Introduction

Following Foellmer (1991) we first give a heuristic derivation of the Girsanov transformation for the 1-dimensional Brownian motion based on elementary probability concepts. This heuristic will help to understand what happens under the Girsanov transformation.

Let X be a random variable (r.v.) on $(\Omega, \mathcal{F}, P)$ which is standard-normal, in short $X \sim \mathrm{N}(0,1)$.

$$\Longrightarrow P[X \leq a] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-\frac{1}{2}x^2} dx = \int_{-\infty}^{a} \underbrace{f(x)}_{n_{0,1}(x)} \, dx$$

Consider now another density function:

$\widehat{f}(x) = e^{(\mu x - \frac{1}{2}\mu^2)} f(x).$

$$\Longrightarrow \widehat{P}[X \leq a] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-\frac{1}{2}(x-\mu)^2} dx = \int_{-\infty}^{a} n_{\mu,1}(x) \, dx$$

$\Longrightarrow X \sim \mathrm{N}(\mu, 1)$ under $\widehat{P}$

Observe that the r.v. X has not been changed, but its distribution has changed under the new measure $\widehat{P}$.

Remark 3.1. *Clearly $\widehat{P}_X \sim P_X$ and the Radon-Nikodym derivative is*

$$\frac{d\widehat{P}_X}{dP_X}(x) = e^{(\mu x - \frac{1}{2}\mu^2)}.$$

Under $\widehat{P}$ the r.v. $\widehat{X} = X - \mu$ has now the distribution $\mathrm{N}(0,1)$, i.e.

P_X	$\equiv$	$\widehat{P}_{\widehat{X}}$
distribution of X under P	$\equiv$	distribution of $\widehat{X}$ under $\widehat{P}$

For $X \sim \mathrm{N}(0, \sigma^2)$ under P
$X \sim \mathrm{N}(\mu, \sigma^2)$ under $\widehat{P}$ ($\leftrightarrow \widehat{X} = X - \mu \sim \mathrm{N}(0, \sigma^2)$ under $\widehat{P}$)

it follows

$$\frac{d\widehat{P}_X}{dP_X}(x) = \frac{n_{\mu,\sigma^2}(x)}{n_{0,\sigma^2}(x)} = e^{\frac{1}{\sigma^2}(\mu x - \frac{1}{2}\mu^2)}.$$

Application to Brownian Motion

Let $(B_t)_{0\le t\le 1}$ be a BM on $(\Omega, (\mathcal{F})_t, P)$.

$\Longrightarrow B_t \sim \mathrm{N}(0,t)$ under P
and $\Delta B_t = B_{t+\Delta t} - B_t \sim \mathrm{N}(0,\Delta t)$, independent of B_t

Consider now a BM with drift

$$\widehat{B}_t = B_t - \int_0^t H_s \, ds$$

for some stochastic process $(H_s)_{0\le s\le 1}$.

Question: Under which measure $\widehat{P}$ is $(\widehat{B}_t)$ again a BM (without drift) ?

We discretize the unit interval $[0,1]$ by $t_i = \dfrac{i}{n} \quad (i = 0\ldots n)$.
It follows

$$B_{\frac{j}{n}} = \sum_{i=1}^{j} \underbrace{B_{\frac{i}{n}} - B_{\frac{i-1}{n}}}_{:=X_i\sim \mathrm{N}(0,\Delta t) \text{ under } P} \qquad j = 1\ldots n$$

$\prod_{i=1}^{n} \mathrm{N}(0,\Delta t)$ joint distribution of $X_1,\ldots,X_n$ (under P)

$$\widehat{X}_i = X_i - \underbrace{H_{\frac{i-1}{n}} \cdot \Delta t}_{\mu_i}$$

$\Longrightarrow \widehat{X}_i \sim \mathrm{N}(0,\Delta t)$ under $\widehat{P}$

with Radon-Nikodym-derivative $e^{\frac{1}{\Delta t}\,(\mu_i x_i - \frac{1}{2}\,\mu_i^2)}$ w.r.t. P.

Joint distribution of $\widehat{X}_i$ under $\widehat{P}$

$$
\begin{aligned}
d\widehat{P}^{(n)} &= \prod_{i=1}^{n} \exp\left\{ \frac{1}{\Delta t} \left(H_{\frac{i-1}{n}} \Delta t \cdot X_i - \frac{1}{2}\, H^2_{\frac{i-1}{n}} \Delta t^2 \right) \right\} dP^{(n)} \\
&= \exp\left\{ \sum_{i=1}^{n} H_{\frac{i-1}{n}} (B_{\frac{i}{n}} - B_{\frac{i-1}{n}}) - \frac{1}{2} \sum_{i=1}^{n} H^2_{\frac{i-1}{n}} \Delta t \right\} dP^{(n)} \\
\text{"} \xrightarrow[n\to\infty]{} \text{"} &\ \exp\left\{ \int_0^1 H_s \, dB_s - \frac{1}{2} \int_0^1 H_s^2 \, ds \right\} dP \qquad (1)
\end{aligned}
$$

$$
\Longrightarrow \quad \boxed{\begin{aligned} \widehat{P} &= \mathcal{E}(L_1)\ P \quad \text{with } L_1 = \int_0^1 H_s \, dB_s \\ &= \exp\{L_1 - \tfrac{1}{2} \langle L_1 \rangle\} \cdot P \end{aligned}}
$$

Remark 3.2. $\mathcal{E}(L_1) = \exp\{L_1 - \frac{1}{2} \langle L_1 \rangle\}$ *is called the "stochastic exponential" or the "Doléans-Dade" exponential.*

The limit process (1) is of course only a heuristic argument. A mathematically rigorous derivation follows in Sect. 3.2 and 3.3.

3.2 The General Girsanov Transformation

Let $(\Omega, (\mathcal{F}_t)_{0\le t}, P)$ satisfy the usual conditions, i.e. the filtration is right continuous, complete and $\mathcal{F}_\infty = \sigma(\bigcup_t \mathcal{F}_t)$.

$\mathcal{M}^{(c)}_{\text{loc}}(P) :=$ all (continuous) local martingales w.r.t. P

Definition 3.2.1. *X is called P-semimartingale* $\underset{\text{Def.}}{\Longleftrightarrow} X = M + A$

with $M \in \mathcal{M}_{\text{loc}}(P)$ and $A \in$ FV.

Let Q be another probability measure on $(\Omega, \mathcal{F}_\infty)$.

Definition 3.2.2.

(i) $Q << P \underset{\text{Def.}}{\Longleftrightarrow} P[A] = 0 \Longrightarrow Q[A] = 0 \quad \forall\, A \in \mathcal{F}_\infty$

(ii) $Q \sim P \underset{\text{Def.}}{\Longleftrightarrow} Q << P$ *and* $P << Q$

(i.e. Q *and* P *have identical null sets)*

Proposition 3.2.3. (Radon-Nikodym)
Let $Q << P$. *Then there exists* $Z \in L^1(\Omega, \mathcal{F}_\infty)$ *with* $Q = Z\ P$, *i.e.*

$$Q[A] = \int_A Z(\omega)\ P(d\omega) \quad \forall A \in \mathcal{F}_\infty$$

Notation: $\quad Z = \dfrac{dQ}{dP}$

$\Longrightarrow Z_t = \mathrm{E}_P[Z|\mathcal{F}_t] = \mathrm{E}_P\left[\dfrac{dQ}{dP}|\mathcal{F}_t\right]$

is a right continuous martingale, $\mathrm{E}_P[Z_t] \equiv 1$. Furthermore

(i) $Z_t(\omega) > 0 \quad Q$-a.s.
(ii) $Z_t = \dfrac{dQ_t}{dP_t} \quad$ on $\mathcal{F}_t$

Lemma 3.2.4. *Let* $Q \sim P$. *Then one has*

$$X \in \mathcal{M}^c_{\mathrm{loc}}(Q) \longleftrightarrow X\ Z \in \mathcal{M}^c_{\mathrm{loc}}(P).$$

Proof. Let $s < t$, $A \in \mathcal{F}_s$. Is X a martingale, then obviously

$$\mathrm{E}_Q[X_s; A] = \mathrm{E}_Q[X_t; A] \Longleftrightarrow \mathrm{E}_P[X_s\ Z_s; A] = \mathrm{E}_P[X_t\ Z_t; A]. \qquad (2)$$

Let $T_n \uparrow \infty$ be a localizing stopping time for X. It follows

$Q = Z_{T_n} P$ on $\mathcal{F}_{T_n} \Longrightarrow$ (2) for $X_{T_n}, X_{T_n}\ Z_{T_n}$

□

Notation: Following Protter (1990) we use the following short notation for stochastic integrals: $H \bullet X = \int H dX$. Evaluating these processes at t, we have

$$H \bullet X_t = \int_0^t H_s dX_s = H_0 X_0 + \int_{(0,t]} H_s dX_s$$

Theorem 3.2.5. (Girsanov)
Let $Q << P$ with $Z = \frac{dQ}{dP}$ continuous.
It follows

$$M \in \mathcal{M}^c_{\text{loc}}(P) \Longrightarrow \widetilde{M} = M - Z^{-1} \bullet \langle M, Z\rangle \in \mathcal{M}^c_{\text{loc}}(Q)$$

$$(\text{i.e.} \quad \widetilde{M}_t = M_t - \int_0^t \frac{1}{Z_s}\, d\langle M, Z\rangle_s).$$

Proof. According to Lemma 3.2.4 one has to show: $\widetilde{M}\ Z \in \mathcal{M}^c_{\text{loc}}(P)$

$$\begin{aligned}\widetilde{M}_t\ Z_t &= \widetilde{M}_0\ Z_0 + \int_0^t \widetilde{M}_s\ dZ_s + \int_0^t Z_s\ d\widetilde{M}_s + \langle \widetilde{M}, Z\rangle_t \\ &= \widetilde{M}_0\ Z_0 + \underbrace{\int_0^t \widetilde{M}_s\ dZ_s}_{\in \mathcal{M}_{\text{loc}}(P)} + \underbrace{\int_0^t Z_s\ dM_s}_{\in \mathcal{M}_{\text{loc}}(P)} \underbrace{- \langle M, Z\rangle_t + \langle \widetilde{M}, Z\rangle_t}_{=0 \text{ since } \widetilde{M} - M \in \text{FV}}\end{aligned}$$

□

Let now $Q \sim P$.

$$Q << P \longrightarrow Z = \frac{dQ}{dP} > 0 \quad Q\text{-a.s.}$$

$$P << Q \longrightarrow Z \quad \text{also } P\text{-a.s. strictly positive}$$

Lemma 3.2.6. *Any strictly positive process (Z_t) can be represented as*

$$Z_t = \exp\left\{L_t - \frac{1}{2}\ \langle L\rangle_t\right\} = \mathcal{E}(L)_t$$

with

$$L_t = \log Z_0 + \int_0^t Z_s^{-1}\ dZ_s.$$

Proof.

$$\text{Itô} \implies \log Z_t = \underbrace{\log Z_0 + \int_0^t Z_s^{-1}\, dZ_s}_{L_t} - \frac{1}{2} \underbrace{\int_0^t Z_s^{-2}\, d\langle Z\rangle_s}_{\langle L\rangle_t},$$

since by Corollary 2.2.11

$$M_t = \int_0^t f(X_s)\, dX_s \implies \langle M\rangle_t = \int_0^t f^2(X_s)\, d\langle X\rangle_s.$$

□

Proposition 3.2.7. (Girsanov for $Q \sim P$): *Let* $Q = \mathcal{E}(L) \cdot P$. *Then one has*

$$M \in \mathcal{M}^c_{\text{loc}}(P) \iff \widetilde{M} = M - \langle M, L\rangle \in \mathcal{M}^c_{\text{loc}}(Q)$$

and $\quad P = \mathcal{E}(-\widetilde{L}) \cdot Q.$

Proof. $\widetilde{M} = M - Z^{-1} \bullet \langle M, Z\rangle \in \mathcal{M}^c_{\text{loc}}(Q) \quad$ (Girsanov) , and

$L_t = \log Z_0 + \int_0^t \frac{1}{Z_s}\, dZ_s$, i.e. $L = Z^{-1} \bullet Z$. Thus

$$\langle M, L\rangle_t = \langle M, Z^{-1} \bullet Z\rangle_t = \int_0^t Z_s^{-1}\, d\langle M, Z\rangle_s = (Z^{-1} \bullet \langle M, Z\rangle)_t$$

↑

computational rule for $\langle X, Y\rangle$.

$$L \in \mathcal{M}^c_{\text{loc}}(P) \longrightarrow -\widetilde{L} = -L + \underbrace{\langle L, L\rangle}_{=\langle L\rangle} \in \mathcal{M}^c_{\text{loc}}(Q)$$

$P = \mathcal{E}(L)^{-1} \cdot Q$

$$\mathcal{E}(L)_t^{-1} = \exp\Big\{ - L_t + \frac{1}{2}\langle L\rangle_t\Big\} = \exp\Big\{ - \widetilde{L}_t - \frac{1}{2} \underbrace{\langle L\rangle_t}_{=\langle \widetilde{L}\rangle_t} \Big\} = \mathcal{E}(-\widetilde{L})_t.$$

Conversely one has

$$\widetilde{M} \in \mathcal{M}^c_{\text{loc}}(Q) \longrightarrow \widetilde{\widetilde{M}} = \widetilde{M} - \langle \widetilde{M}, -\widetilde{L} \rangle \in \mathcal{M}^c_{\text{loc}}(P)$$
$$= \widetilde{M} + \underbrace{\langle \widetilde{M}, \widetilde{L} \rangle}_{\langle M, L \rangle} = M.$$

□

Computational rule for covariation of stochastic integrals: (see Protter (1990) , Theorem29, p.68)

Proposition 3.2.8. *X, Y continuous semimartingales,*
H, K admissible integrands (w.r.t. X, Y)
Then one has

$$\langle H \bullet X, K \bullet Y \rangle_t = \int_0^t H_s \, K_s \, d\langle X, Y \rangle_s.$$

Remark 3.3. $X = M + A \quad Y = N + B \quad M, N \in \mathcal{M}^{(c)}_{\text{loc}}$

$\Longrightarrow \langle X, Y \rangle = \langle M, N \rangle$

In particular, it follows:

$$\langle H \bullet M, N \rangle_t = \int_0^t H_s \, d\langle M, N \rangle_s$$

and

$$\langle M \rangle = \left\langle \int_0^t f(X) \, dX_s \right\rangle = \langle f(X) \bullet X, f(X) \bullet X \rangle_t = \int_0^t f^2(X_s) \, d\langle X \rangle_s.$$

3.3 Application to Brownian Motion

Girsanov solves the following problem:

$$\text{Given}: \left.\begin{matrix} X \in \mathcal{M}_{\text{loc}}(P) \\ Q \sim P \end{matrix}\right\} \underset{\text{Girsanov}}{\Longrightarrow} \widetilde{X} \in \mathcal{M}_{\text{loc}}(Q)$$

We now consider the inverse problem:

Given a semimartigale $X = M + A$ w.r.t. P, i.e. $M \in \mathcal{M}_{\text{loc}}(P)$, $A \in \text{FV}$

$\exists\, Q \sim P: \quad X \in \mathcal{M}_{\text{loc}}(Q)$??

I.e., we are looking for a P-equivalent measure Q under which X is a martingale (so-called *equivalent martingale measure*).

Approach: $Q = \mathcal{E}(L) \cdot P$
Girsanov $\Longrightarrow \widetilde{M} = M - \langle M, L\rangle \in \mathcal{M}_{\text{loc}}(Q)$
$\widetilde{M} = X \Longleftrightarrow A = -\langle M, L\rangle$.

However, there are two problems:

1) Determination of L

2) Under what conditions is $Q = \mathcal{E}(L) \cdot P$ a probability measure ??

ad 2:

$$\begin{aligned} Q \text{ prob. measure } &\longleftrightarrow \mathrm{E}_P[\mathcal{E}(L)_t] \equiv 1 \\ &\longleftrightarrow \mathcal{E}(L)_t \text{ is a martingale (since } L_0 = 0). \end{aligned} \tag{3}$$

A sufficient condition for $\mathcal{E}(L)_t$ to be a martingale is the so-called Novikov condition (see e.g. Revuz-Yor (1991), p.308):

$$\mathrm{E}\Big[\exp\{\frac{1}{2}\, \langle L\rangle_t\}\Big] < \infty \qquad \forall\, t\,. \tag{4}$$

ad 1: Solution for semimartingales of Brownian motion

$$\text{Let } X_t = B_t + \underbrace{\int_0^t H_s\, ds}_{\text{Drift } = A}$$

with $(B_t)_{0\le t\le T}$ Brownian motion w.r.t. P and $H_t(\omega)$ adapted, càdlàg (e.g. $H_t = f(B_s; s \le t))$.

Claim: $L_t = -\int_0^t H_s \, dB_s$ solves the problem.

Proof.

$$-\langle B, L\rangle_t = \langle B, H \bullet B\rangle_t = \int_0^t H_s \, d\langle B\rangle_s = \int_0^t H_s \, ds = A$$

$\Longrightarrow X_t \in \mathcal{M}_{\text{loc}}(Q)$ for $Q = \mathcal{E}(L) \cdot P$.

□

Furthermore it follows

$$\langle X\rangle_t = \langle B\rangle_t = t < \infty \Longrightarrow (X_t) \quad L^2\text{-martingale.}$$

It even follows: (X_t) is a Brownian motion w.r.t. Q ! This follows from Lévy's theorem:

$$(X_t) \in \mathcal{M}^c_{\text{loc}}(P)\ ,\ \langle X\rangle_t = t \;\forall\, t \ge 0 \quad P\text{-a.s.} \Longrightarrow (X_t) \text{ is B.M. w.r.t. } P.$$

Example of a non-continuous local martingale with $\langle X\rangle_t = t$:

X_t Poisson-Process (with parameter λ); i.e.

$$P[X_t = K] = e^{-\lambda t}\frac{(\lambda\, t)^K}{K!} \quad , \quad E[X_t] = \lambda\, t = \text{Var}(X_t)$$

$\Longrightarrow M_t = X_t - \lambda\, t$ martingale and $M_t^2 - \lambda\, t$ martingale

$\Longrightarrow \langle M\rangle_t = \lambda\, t$

Thus for $\lambda = 1$, M_t is a martingale with $\langle M\rangle_t = t$, but not continuous.

Same procedure for d-dimensional Brownian motion:

Let $B_t = (B_t^1, \ldots, B_t^d)$ be d-dimensional Brownian motion, i.e. (B_t^i) $(i = 1, \ldots, d)$ are independent one-dimensional Brownian motions.

Proposition 3.3.1. (Lévy) *For an $(\mathcal{F}_t)$-adapted continuous d-dimensional process $X_t = (X_t^1, \ldots, X_t^d)$ with $X_0 = 0$ the following statements are equivalent:*

(i) X is a Brownian motion w.r.t. $(\mathcal{F}_t)$
(ii) X is a continuous local martingale mit $\langle X^i, X^j \rangle_t = \delta_{ij} \cdot t$ $\forall\, i, j = 1 \ldots d$.

Proof. see Revuz-Yor (1991), p.141 .

□

Proposition 3.3.2. (Girsanov for d-dimensional Brownian motion)

Let $X_t = B_t + \int_0^t H_s\, ds$ be a d-dimensional process with

$$\mathrm{E}\left[\exp\left\{\frac{1}{2}\int_0^T ||H_s||^2\, ds\right\}\right] < \infty \qquad \text{(Novikov condition)}. \tag{5}$$

Define Q as $\frac{dQ}{dP} = \mathcal{E}\Big(-\int_0^T H_s\, dB_s\Big)$.

Then under Q (X_t) is a d-dimensional Brownian motion for $0 \le t \le T$.

4

Application to Financial Economics

In this chapter the methods developed in Chapter 2 and 3 are applied to derive the fundamentals of "financial economics" in continuous time, such as the *market price of risk*, the *no-arbitrage principle*, the *fundamental pricing rule* and its invariance under *numeraire changes*, and the *forward measure*, a useful tool to deal with stochastic interest rates. Special emphasis is laid on the economic interpretation of the so-called "*risk-neutral" arbitrage measure* and its relation to the "real world" measure considered in general equilibrium theory, a topic sometimes leading to confusion between economists and financial engineers.

Using the general Girsanov transformation, as developed in Sect. 3.2, the rather intricate problem of the *change of numeraire* can be treated in a rigorous manner, and the so-called "two-country" or "Siegel" paradox serves as an illustration. The section on Feynman-Kac relates the martingale approach used explicitly in these notes to the more classical approach based on partial differential equations.

4.1 The Market Price of Risk and Risk-neutral Valuation

We consider an economy depending on d independent stochastic factors modelled by a d-dimensional Brownian motion $B_t = (B_t^1, \ldots, B_t^d)$ on the filtered probability space $(\Omega, (\mathcal{F}_t)_{0\le t\le T}, P)$.

Let X be a security with returns

$$\frac{dX_t}{X_t} = \mu_X(t)\,dt + \underbrace{\sigma_X(t) \circ dB_t}_{\text{scalar product in } \mathbb{R}^d} \qquad 0 \le t \le T \tag{1}$$

where $\mu_X(t)$, $\sigma_X(t) = (\sigma_X^1, ..., \sigma_X^d)$ are adapted processes on $(\Omega, (\mathcal{F}_t), P)$

We assume that there exists a Lebesgue-integrable real function $\phi_X(t)$ on $[0,T]$, such that $|\mu_X(t,\omega)| \le \phi_X(t)$ and $||\sigma_X(t,\omega)||^2 \le \phi_X(t)$ P-a.s.

Remark 4.1. *A process of the form (1) is a (positive) Itô-process. Every strictly positive diffusion process X_t can be written as an Itô-process by taking its logarithm (compare Lemma 3.2.6).*

Furthermore we consider the following processes:

$r(t) = r(t,\omega)$ an adapted and bounded process of the "spot" interest rate,

$$\beta(t) = \beta(t,\omega) = \exp \int_0^t r(s,\omega)\,ds \text{ the accumulation process .}$$

$\beta(t)$ is the "savings" account by rolling over one monetary unit at the prevailing spot rate. Clearly, $\beta(t)$ is an adapted continuous process of finite variation.

Definition 4.1.1. *Let $\lambda(t) = (\lambda^1, \ldots, \lambda^d)_t$ be an adapted process with*

$$\begin{aligned} \mu_X(t) &= r(t) + \lambda(t) \circ \sigma_X(t) \\ &= r(t) + \sum_{i=1}^{d} \lambda^i(t)\,\sigma_X^i(t). \end{aligned} \tag{2}$$

Then $\lambda(t)$ is called the process of the "market price of risk" and $\lambda^i(t)$ the price of the i-th risk-factor .

Remark 4.2. *For $d = 1$ the relation (2) can be written as*

$$\frac{\mathrm{E}\left[\frac{dX_t}{X_t}|\mathcal{F}_t\right] - r(t)\,dt}{\sqrt{\mathrm{Var}\left(\frac{dX_t}{X_t}|\mathcal{F}_t\right)}} = \lambda(t)\,dt$$

which can be interpreted as the (expected) excess return per risk unit.

Let now Y be another security with

$$\frac{dY_t}{Y_t} = \mu_Y(t)\,dt + \sigma_Y(t) \circ dB_t.$$

Let $\lambda_X(t)$ and $\lambda_Y(t)$ be the corresponding risk price processes of X and Y.

We assume that the underlying economy is in an equilibrium state, i.e. all price processes $X(t)$, $Y(t)$, $r(t)$ etc. are determined by supply and demand of the agents in this economy. In order to determine such an equilibrium one would have to know the consumption and investment behavior of each agent depending on his initial endowment, preferences over consumption streams, expectations and attitude towards risk, a formidable task. The power of financial economics consists in working with a much weaker condition, which certainly is necessary for an economy to be in equilibrium, namely the condition of "No arbitrage", i.e. the absence of arbitrage profits without taking risks. Such an arbitrage opportunity would exist if, at any time t and any state ω, one could form a riskless portfolio which has a higher return than the riskless interest rate $r(t, \omega)$. By "No arbitrage" we mean that such an arbitrage opportunity does not exist.

The fundamental consequence of the above assumption is the following theorem:

Theorem 4.1.2. *"No arbitrage" implies*

$$\lambda_X(t,\omega) = \lambda_Y(t,\omega) \quad \textit{for all } (t,\omega) \in [0,T) \times \Omega.$$

Proof. We give a proof for $d = 1$ (for $d > 1$ see Ingersoll (1987)).

Assume there exists (t,ω) with $\lambda_X(t,\omega) > \lambda_Y(t,\omega)$. We may assume that $0 < \sigma_X(t,\omega) < \sigma_Y(t,\omega)$ (otherwise replace ϕ by $-\phi$ in the following definition of ϕ).

Choose at time t the following portfolio of X and Y :

$$\phi_X = \frac{\sigma_Y}{(\sigma_Y - \sigma_X)\cdot X_t} \quad , \quad \phi_Y = -\frac{\sigma_X}{(\sigma_Y - \sigma_X)\cdot Y_t}.$$

The value of this portfolio is

$$V_t(\phi) = \phi_X\, X_t + \phi_Y\, Y_t = 1$$

and its return

$$\begin{aligned} dV_t &= \phi_X\, dX_t + \phi_Y\, dY_t \\ &= \underbrace{\frac{1}{\sigma_Y - \sigma_X}\Big[(\sigma_Y\, \mu_X - \sigma_X\, \mu_Y)}_{\mu_V} dt + \underbrace{(\sigma_Y\, \sigma_X - \sigma_X\, \sigma_Y)\, dB_t}_{\equiv 0}\Big] \end{aligned}$$

Hence V_t has a riskless return

$$\begin{aligned} \mu_V &= \frac{1}{\sigma_Y - \sigma_X}\Big[\sigma_Y(r_t + \lambda_X\, \sigma_X) - \sigma_X(r_t + \lambda_Y\, \sigma_Y)\Big] \\ &= r_t + \underbrace{\frac{\sigma_Y \cdot \sigma_X}{\sigma_Y - \sigma_X}}_{>0}\, (\underbrace{\lambda_X - \lambda_Y}_{>0}) > r_t \end{aligned}$$

which is in contradiction to the "No arbitrage" condition.

□

Question: When is the process $\lambda(t)$ P-a.s. uniquely determined ?

Assume there exist n securities $X^1, \ldots, X^n$ with

$$\sum(t) = \begin{pmatrix} \sigma_{X_1}(t) \\ \cdot \\ \cdot \\ \cdot \\ \sigma_{X_n}(t) \end{pmatrix} \quad (n \times d)\text{-matrix of volatilities}$$

and

$$\mu(t) \quad = \begin{pmatrix} \mu_{X_1}(t) \\ \cdot \\ \cdot \\ \cdot \\ \mu_{X_n}(t) \end{pmatrix}, e = \begin{pmatrix} 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{pmatrix}.$$

No arbitrage implies

$$\mu(t) = r(t) \cdot e + \sum(t) \circ \lambda(t).$$

Hence $\lambda(t, \omega)$ is uniquely determined if and only if the stochastic matrix $\sum(t, \omega)$ has full rank d, for all t, P-a.s.

Clearly a necessary condition is $n \geq d$, i.e. there must be at least as many securities as stochastic factors in the economy. A sufficient condition for $d = 1$ is $\sigma_X(t, \omega) > 0$ for all t, ω.

Elimination of $\lambda(t)$ by means of Girsanov:

Assume that the Novikov condition

$$\mathrm{E}\Big[\exp\{\frac{1}{2}\int_0^T ||\lambda(s)||^2 \, ds\}\Big] < \infty \tag{3}$$

is fulfilled. Define

$$L_t = -\int_0^t \lambda(s) \circ dB_s \quad 0 \leq t \leq T$$
$$P^* = \mathcal{E}(L) \cdot P \quad , \ P^* \sim P.$$

Then the Girsanov theorem implies for each security price process X_t:

$$\frac{dX_t}{X_t} = \mu_X(t)\,dt + \sigma_X(t)\,dB_t = r(t)\,dt + \sigma_X(t)\,(dB_t + \lambda(t)\,dt)$$
$$= r(t)\,dt + \sigma_X(t)\,dB_t^*$$

and B_t^* is a d-dimensional Brownian motion under P^*.

Remark 4.3. *Clearly P^* is unique if $\lambda(t)$ is unique.*

Interpretation of P and P^*:

We can interpret the original measure P as the (objective) "real-world" measure or, in the equilibrium interpretation, as the (subjective) expectation of the "representative investor". For the equilibrium price process X_t with respect to P one has

$$\mathrm{E}_P\Big[\frac{dX}{X}|\mathcal{F}_t\Big] = r(t)\,dt + \lambda(t)\,\sigma_X(t)\,dt \quad (\textbf{under } P) \qquad (4)$$

with $\lambda(t) :=$ market price of risk (varies with P !).

The measure P^* can be interpreted as the expectation of a risk-neutral investor, since

$$\mathrm{E}_{P^*}\Big[\frac{dX}{X}|\mathcal{F}_t\Big] = r(t)\,dt + \text{ zero-premium } (\textbf{under } P^*). \qquad (5)$$

The relation (5) has been called by Cox-Ingersoll-Ross (1981) the "Local Expectation Hypothesis", a terminology which has led to some confusion. Note that the equilibrium process has **not** been changed, it is the **same** under both measures P and P^*. Girsanov's Theorem allows us to replace the relation (4) through the equivalent simpler relation (5). In particular, no assumption has been made about the existence of risk-neutral investors. In a real economy neither a "representative" nor a "risk-neutral" investor will exist, since both assumptions would prevent the existence of a (stable) equilibrium. The great advantage of the representation (5) under the (martingale) measure P^* is that we do not have to know anything about the individual expectations P and the investors' attitude towards risk.

In summary: X_t has not been changed. It is the same equilibrium price process as under P, but in simpler representation under P^*. P^* is called the "equivalent risk-neutral measure" or the "P-equivalent martingale measure".

4.2 The Fundamental Pricing Rule

Let $Z_t > 0$ be the price process of an arbitrary security without dividend payments in $[0, T]$ with dynamics

$$\frac{dZ_t}{Z_t} = \mu_Z(t)\, dt + \sigma_Z(t) \circ dB_t \qquad \text{(Itô-process under } P\text{)}.$$

with $\mu_Z(t)$, $\sigma_Z(t)$ adapted processes as defined in Sect. 4.1 .

Remark 4.4. *Any strictly positive semimartingale of the Brownian motion can be written as an Itô-process (see Lemma 3.2.6).*

Theorem 4.2.1. *The discounted process*

$$\widehat{Z}_t = \exp\Big\{ -\int_0^t r(s)\, ds \Big\}\, Z_t = \frac{Z_t}{\beta_t}$$

is a martingale under the risk-neutral measure P^*.

Proof. Under P^* one has $\dfrac{dZ_t}{Z_t} = r(t)\, dt + \sigma_Z(t)\, dB_t^*$.
By Itô it follows $d\widehat{Z}_t = \widehat{Z}_t\, \sigma_Z(t)\, dB_t^*$, hence $\widehat{Z}_t$ is a local martingale under P^*, with solution $\widehat{Z}_t = Z_0 \cdot \mathcal{E}(N_t)$ where $N_t = \int_0^t ||\sigma_Z(s)||^2\, dB_s$.
But $\langle N \rangle_t = \int_0^t ||\sigma_Z(s)||^2\, ds \leq \int_0^t \phi_Z(s)\, ds < \infty$ P^*-a.s., which implies:

$$\mathrm{E}^*\Big[\exp\{\tfrac{1}{2}\langle N \rangle_t\} \Big] \leq \exp\{\frac{1}{2} \int_0^t \phi_Z(s)\, ds\} < \infty \quad \forall\, t$$

and the Novikov condition (see Sect. 3.3 (4)) is satisfied. Hence $\widehat{Z}_t$ is a martingale under P^*. □

Theorem 4.2.2. (Fundamental Pricing Rule)

$$\begin{aligned} Z_t &= \mathrm{E}^*\Big[\exp\Big\{ -\int_t^T r(s)\, ds \Big\} \cdot Z_T \,\Big|\, \mathcal{F}_t \Big] \quad 0 \leq t \leq T \\ &= \mathrm{E}^*\Big[\frac{\beta_t}{\beta_T}\, Z_T \,\Big|\, \mathcal{F}_t \Big]. \end{aligned}$$

Proof. Since $\widehat{Z}_T$ is a P^* - martingale, it follows:

$$\widehat{Z}_t = \mathrm{E}^*[\widehat{Z}_T \mid \mathcal{F}_t] = \mathrm{E}^*\Big[\exp\Big\{-\int_0^T r(s)\,ds\Big\}\,Z_T \,\Big|\, \mathcal{F}_t\Big]$$

$$Z_t = \mathrm{E}^*[\beta_t\,\widehat{Z}_T \mid \mathcal{F}_t] = \mathrm{E}^*\Big[\frac{\beta_t}{\beta_T}\,Z_T \,\Big|\, \mathcal{F}_t\Big].$$

□

Examples:

1. $Z_t = P_t(T)$ zero-bond with maturity T, i.e. $P_T(T) = 1$.

$$\Longrightarrow P_t(T) = \mathrm{E}^*\Big[\exp\Big\{-\int_t^T r(s)\,ds\Big\} \,\Big|\, \mathcal{F}_t\Big]$$

2. $Z_t = (X_T - K)^+$ call option on security X

$$\Longrightarrow Z_t = \mathrm{E}^*\Big[\exp\Big\{-\int_t^T r(s)\,ds\Big\}\,(X_T - K)^+ \,\Big|\, \mathcal{F}_t\Big]$$

special case: $d = 1$, $r(t,\omega) \equiv r$, $\sigma_Z(t,\omega) \equiv \sigma$

$$\Longrightarrow Z_t = X_t \cdot \Phi(d_1) - K \cdot \exp\{-(\underbrace{T-t}_{s}) \cdot r\}\,\Phi(d_2)$$

with $$d_{1,2} = \frac{\ln(X_t/K \cdot e^{-rs})}{\sigma\sqrt{s}} \pm \frac{1}{2}\,\sigma\sqrt{s}$$ (*Black-Scholes-formula*) .

The proof can be given as exercise (or see Sect. 2.9).

Hint: B^* is a Brownian motion under P^*, i.e.

$$P^*[B^*_T \le x] = \Phi\Big(\frac{x}{\sqrt{T}}\Big) \Longleftrightarrow B^*_T \sim \mathrm{N}(0,T) \quad \text{under } P^*$$

$$\Longleftrightarrow \frac{B^*_T}{\sqrt{T}} \sim \mathrm{N}(0,1).$$

Dividend paying securities

Let $Z(t)$ be the price process of a security with continuous dividend payments given by the dividend rate $d_Z(t)$ which is proportional to $Z(t)$.[1] Assume that the dynamics of the (ex dividend) process is given by

$$\frac{dZ_t}{Z_t} = \mu_Z(t)\;dt + \sigma_Z(t) \circ dB_t \quad \text{(ex-dividend process)} \tag{6}$$

Let $\widetilde{Z}(t)$ denote the process with accumulated dividends, i.e.

$$\frac{d\widetilde{Z}_t}{\widetilde{Z}_t} = \underbrace{(\mu_Z(t) + d_Z(t))}_{\mu_{\widetilde{Z}}(t)}\;dt + \sigma_Z(t) \circ dB_t \quad \text{(cum-dividend process)} \tag{7}$$

Denote by $\xi(t) = exp\{\int_0^t d_Z(s)\,ds\}$ the accumulated dividend process. One easily checks that $\widetilde{Z}_t = \xi(t) \cdot Z(t)$ is a solution of (7). Since $\widetilde{Z}_t$ does not pay dividends, the Fundamental Pricing Rule implies

$$\widetilde{Z}_t = \mathrm{E}^*\left[\exp\left\{-\int_t^T r(s)\;ds\right\} \cdot \widetilde{Z}_T \,\middle|\, \mathcal{F}_t\right] \quad 0 \le t \le T,$$

which implies

$$\begin{aligned} Z_t = \frac{\widetilde{Z}(t)}{\xi(t)} &= \mathrm{E}^*\left[\exp\left\{-\int_t^T r(s)\;ds\right\} \frac{\xi(T)}{\xi(t)} Z_T \,\middle|\, \mathcal{F}_t\right] \\ &= \mathrm{E}^*\left[\exp\left\{-\int_t^T (r(s) - d_Z(s))\;ds\right\} Z_T \,\middle|\, \mathcal{F}_t\right]. \end{aligned}$$

Remark 4.5. *Both $r(t)$ and $d_Z(t)$ may be stochastic.*

[1] Although this is an unrealistic assumption for a single stock, which rather pays lump sum dividends, it is a good approximation to reality for stock indices, and fits exactly the case where $Z(t)$ is a currency or an exchange rate (see the following sections).

4.3 Connection with the PDE-Approach (Feynman-Kac Formula)

In Sect. 4.2 general valuation formulas for securities and derivatives were derived by martingale methods, which lead to the computation of conditional expectations. The classical approach of Black-Scholes-Merton is based on the solution of partial differential equations under boundary conditions. A connection between these two at first sight fundamentally different approaches is provided by the so-called *Feynman-Kac formula.* If a number of technical conditions is satisfied, the Feynman-Kac formula enables one to switch back and forth between the martingale and the PDE-approach. In other words, the two seemingly so different approaches appear as two different sides of one coin.

An essential condition for Feynman-Kac is, however, that the underlying security processes have to be markovian diffusions, i.e. to be of the form

$$dX_t = \mu(X_t, t)\; dt + \sigma(X_t, t)\; dB_t \tag{8}$$

with (B_t) a Brownian motion under the measure P.

Feynman-Kac considers the following problem:

Find a solution $f(x, t) \in C^{2,1}(\mathbb{R} \times [0, T])$ for the PDE

$$A\, f(x, t) - r(x, t)\; f(x, t) = 0 \quad (x, t) \in \mathbb{R} \times [0, T) \tag{9}$$

under the boundary condition

$$f(x, T) = g(x) \quad x \in \mathbb{R} \tag{10}$$

where

$$A\, f(x, t) = f_t(x, t) + f_x(x, t)\; \mu(x, t) + \frac{1}{2}\, \sigma^2(x, t)\; f_{xx}(x, t). \tag{11}$$

Remark 4.6. *$A\,f$ is the so-called "infinitesimal generator" of $f(x, t)$, defined as the expected rate of change of $f(X_t, t)$, given $X_t = x$, i.e.*

$$A\, f(x, t) = \lim_{\Delta \to 0} \frac{1}{\Delta} \Big(\mathrm{E}[f(X_{t+\Delta}, t + \Delta) | X_t = x] - f(x, t) \Big).$$

From Itô's formula applied to $df(X_t, t)$ and the local martingale property of the Itô-integral one easily derives (11).

The Feynman-Kac solution of (9) till (11), if it exists, is given by the conditional expectation

$$f(x,t) = \mathrm{E}_P\Big[\exp\Big\{-\int_t^T r(X_s,s)\,ds\Big\}\,g(X_T)\Big|X_t = x\Big] \qquad (12)$$

where X is a solution of (8) with $X_t = x$.

For the necessary technical conditions (besides differentiability a number of Lipschitz- and polynomial boundary conditions are required) we refer to Duffie (2001).

Under additional assumptions the solution (12) is unique. In this case the Feynman-Kac formula also allows switching back: the conditional expectation (12) can be computed as the solution of the PDE (9) under the boundary constraint (10).

Application: Let X_t be a security process with return

$$\frac{dX_t}{X_t} = \mu_X(t)\,dt + \sigma_X(t)\,dB_t,$$

where μ_X and σ_X are deterministic integrable functions of t. Consider a contingent claim $Z_T = g(X_T)$ with $g : \mathbb{R} \longrightarrow \mathbb{R}$ continuous. By Girsanov there exists $P^* \sim P$ with

$$\frac{dX_t}{X_t} = r(t)\,dt + \sigma_X(t)\,dB_t^* \qquad (13)$$

Observe: In contrast to the martingale approach $r(t)$ and $\sigma_X(t)$ have to be non-stochastic.

According to the Fundamental Pricing Rule (see Sect. 4.2) one has

$$Z_t = \mathrm{E}_{P^*}\Big[\exp\Big\{-\int_t^T r(s)\,ds\Big\}\,g(X_T)\Big|X_t = x\Big]. \qquad (14)$$

Setting $Z_t = f(x,t) \in C^{2,1}$ and

$$\mu(X_t,t) = r(t)\cdot X_t \quad , \quad \sigma(X_t,t) = \sigma_X(t)\cdot X_t,$$

it follows from Feynman-Kac (when the technical FK-conditions are satisfied):

$Z_t = f(x,t)$ is solution of the PDE

$$-r(t)\ f(x,t) + f_t(x,t) + r(t)\ x\ f_x(x,t) + \frac{1}{2}\ \sigma_X^2(t)\ x^2\ f_{xx}(x,t) = 0 \quad (15)$$

under the boundary constraint

$$f(x,T) = g(x). \quad (16)$$

Thus the solution of (14) is reduced to the solution of the partial differential equation (15) under the boundary constraint (16).

(Check the equivalence of equation (15) with the Black-Scholes PDE in Sect. 2.5 (18)).

4.4 Currency Options and Siegel-Paradox

In nostalgic remembrance of our dear deceased Deutschmark we consider the exchange rate USD/DM. (It is a good exercise to transfer this and the following sections to the exchange rate EUR/USD).

We use the following notation:

$r^{\mathrm{DM}}(t)$ = DM spot rate at t (valid for the period $[t, t+dt]$)
$r^{\$}(t)$ = \$ spot-rate at t
X_t = exchange rate \$/DM (price of 1 \$ in DM at t)

Note that X_t is a security with continuous dividend payment $r^{\$}(t)$. According to Sect. 4.2 it follows

$$\frac{dX_t}{X_t} = (r^{\mathrm{DM}}(t) - r^{\$}(t))\ dt + \sigma_X(t)\ dB_t^* \quad \text{under } P^*$$

where P^* is the risk-neutral expectation of a German investor.

Consider now the process $Y_t = X_t^{-1} \cdot 100$ which is (or was!) the exchange as quoted in the US for 1 DM in cents.

Y_t is a security with continuous dividend $r^{\mathrm{DM}}(t)$. Note that $\sigma_Y(t) = -\sigma_X(t)$ since $\ln Y_t = -\ln X_t + c_0$.

Question: Does the following relation hold

$$\frac{dY_t}{Y_t} = (r^{\$}(t) - r^{\mathrm{DM}}(t))\, dt + \sigma_Y(t)\, dB_t^* \quad \text{under } P^*$$

for a risk-neutral American investor ?

Answer: From Itô it follows

$$\frac{dY_t}{Y_t} = \frac{dX_t^{-1}}{X_t^{-1}} = \left(d\frac{1}{X_t}\right) \cdot X_t$$

$$d\frac{1}{X_t} = -\frac{1}{X_t^2}\, dX_t + \frac{1}{2} \cdot 2 \cdot \frac{1}{X_t^3}\, d\langle X \rangle_t$$

with

$$d\langle X \rangle_t = \sigma_X^2(t)\, X_t^2\, dt.$$

From $\sigma_Y(t) = -\sigma_X(t)$ it then follows

$$\begin{aligned} \frac{dY_t}{Y_t} &= -\frac{dX_t}{X_t} + \sigma_X^2(t)\, dt \\ &= \left(r^{\$}(t) - r^{\mathrm{DM}}(t) + \sigma_Y^2(t)\right) dt + \sigma_Y(t)\, dB_t^* \end{aligned}$$

Thus a “risk-neutral” American investor requires a positive risk premium of $\sigma_Y^2(t)$.

This is the so-called ”Siegel”- or ”Two-Country-Paradox” (see Siegel (1972), McCulloch (1975)). The solution to this Paradox will be given in Sect. 4.6.

4.5 Change of Numeraire

A fundamental law in economics - the “Walras law” - implies that in equilibrium only relative prices can be determined. In order to obtain absolute prices one has to choose one commodity as “numeraire”. Which commodity is chosen as numeraire is a question of convenience. The situation is similar in financial economics. However, here a non-trivial technical complication arises: the risk-neutral measure P^* depends in a crucial way on the choice of the numeraire.

Let $X_t = (X_t^0, \dots, X_t^n)_{(0 \le t \le T)}$ be the price process of $n+1$ securities of the form

$$\frac{dX_t^i}{X_t^i} = \mu_i(t)\; dt + \sigma_i(t)\; dB_t \quad \text{under } P$$

with $\mu_i(t),\ \sigma_i(t)\ (i = 0, \dots, n)$ adapted processes on $(\Omega, (\mathcal{F}_t), P)$ satisfying the boundary condition as defined in Sect. 4.1 .

Usually (X_t^0) is chosen as numeraire, e.g. $dX_t^0 = r(t)\; X_t^0\; dt$ (the accumulation process). Then absolute security prices, measured in units of X^0, are given by the normed process

$$\widehat{X}_t = \Big(1, \frac{X^1}{X^0}, \dots, \frac{X^n}{X^0}\Big)_t .$$

Consider a portfolio strategy $(\phi)_t = (\phi^0, \dots, \phi^n)_t$. Then $V_t(\phi) = \phi_t \circ X_t$ is the value process with respect to X_t. Similarly $\widehat{V}_t(\phi) = \phi_t \circ \widehat{X}_t = V_t(\phi) \cdot (X_t^0)^{-1}$ is the value process with respect to $\widehat{X}_t$.

Recall:

$$(\phi_t) \text{ is self-financing w.r.t. } X_t \underset{\text{def}}{\iff} dV_t(\phi) = \phi_t \circ dX_t$$

$$\begin{aligned} \iff \quad V_t(\phi) &= \phi_0 \circ X_0 + \int_0^t \phi_s \circ dX_s \\ &= (\phi \bullet X)_t \quad 0 \le t \le T \end{aligned}$$

Recall: (meaning of the different 'dots')

$\cdot$ scalar multiplication
$\circ$ scalar product
$\bullet$ stochastic integral

Proposition 4.5.1. *ϕ_t is self-financing with respect to X_t if, and only if, ϕ_t is self-financing with respect to $\widehat{X}_t$.*

Proof.

$(\Longrightarrow)$ By assumption one has $dV_t = \phi_t \circ dX_t$.

Let $Y_t = (X_t^0)^{-1}$. According to the rules for the covariation one has

$$d\langle V, Y\rangle_t = d\langle \phi \bullet X, Y\rangle_t = \phi_t \circ d\langle X, Y\rangle_t \, .$$

Thus according to Itô's product rule

$$\begin{aligned} d\widehat{V}_t &= d(V_t \cdot Y_t) = V_t\, dY_t + Y_t\, dV_t + d\langle V, Y\rangle_t \\ &= (\phi_t \circ X_t)\, dY_t + Y_t(\phi_t \circ dX_t) + \phi_t \circ d\langle X, Y\rangle_t \\ &= \phi \circ \Big(X_t\, dY_t + Y_t\, dX_t + d\langle X, Y\rangle_t \Big) \\ &= \phi \circ d(X_t \cdot Y_t) = \phi \circ d\widehat{X}_t \end{aligned}$$

($\Longleftarrow$) The procedure is analogous.

□

Under the assumptions of Sect. 4.1, "no arbitrage" and the Girsanov theorem applied to the "market price of risk" process $\lambda(t)$ under the Novikov condition (3) together with Prop. 4.2.1 imply:

$\exists\, P^* \sim P : \widehat{X}_t$ martingale under P^*, and B^*_t B.M. w.r.t. P^* .

Let $C_T \in \mathcal{F}_T$ be a contingent claim with $C_T = V_T(\phi)$ for some self-financing portfolio strategy ϕ. Consider $\widehat{C}_T = C_T/X^0_T = \widehat{V}_T(\phi)$. Since $d\widehat{V} = \phi\, d\widehat{X}$, $\widehat{V}$ is a local P^*-martingale. If it is a martingale, then it follows:

$$\begin{aligned} \widehat{C}_t &= \mathrm{E}^*[\widehat{V}_T(\phi) \,|\, \mathcal{F}_t] = \mathrm{E}^*[\widehat{C}_T \,|\, \mathcal{F}_t] \\ C_t &= \widehat{C}_t \cdot X^0_t = \mathrm{E}^*\Big[\frac{X^0_t}{X^0_T} \cdot C_T \,\Big|\, \mathcal{F}_t\Big]. \end{aligned}$$

Question: Does the analysis depend on the choice of the numeraire ? Let now X^i be the numeraire. Consider

$$\widetilde{X}_t = \Big(\frac{X^0}{X^i}, \ldots, 1, \ldots, \frac{X^n}{X^i}\Big)_t \quad \text{security prices measured in } X^i \ ,$$

$$\widetilde{V} = \frac{V}{X^i} \quad , \quad \widetilde{C} = \frac{C}{X^i}.$$

According to Prop. 4.5.1 ϕ is also self-financing with respect to $\widetilde{X}$.

Proposition 4.5.2. *$\widetilde{X}_t$ is a martingale under $Q^*_i = \mathcal{E}(N_i) \cdot P^*$ with*

$$N_i(t) = \int_0^t \Big(\underbrace{\sigma_i(s) - \sigma_0(s)}_{\sigma_{\widehat{X}_i}} \Big)\, dB^*_s \quad (0 \le t \le T)$$

Remark 4.7. *For $dX^0 = X^0 r(t)dt$ one has $\sigma_0 = 0$.*

Proof. By assumption on σ_i and σ_0, there exist Lebesgue-integrable functions such that $||\sigma_i(s,\omega) - \sigma_0(s,\omega)||^2 \leq \phi_i(s) + \phi_0(s) = \psi(s)$ P-a.s., which implies

$$\langle N_i \rangle_t = \int_0^t ||\sigma_i - \sigma_0||^2 \, ds \leq \int_0^t \psi(s) ds < \infty \quad \forall \, t,$$

and the Novikov condition (see Sect. 3.3 (4)) is fulfilled. Thus $N_i(t)$ is a martingale $\Longrightarrow Z_t = \mathcal{E}(N_i(t))$ is a martingale. $\Rightarrow Q_i^*$ is a probability measure.

Hence (see Lemma 3.2.4) for any $j = 0, 1, \ldots, n$,

$$\widetilde{X}^j \in \mathcal{M}^c_{loc}(Q_i^*) \longleftrightarrow \widetilde{X}^j \cdot Z \in \mathcal{M}^c_{loc}(P^*).$$

We know $\dfrac{d\widehat{X}^i}{\widehat{X}^i} = \sigma_{\widehat{X}^i}(t) \, dB_t^*$ where B_t^* is a Brownian motion under P^*

$$\Longrightarrow \widehat{X}_t^i = \widehat{X}_0^i \cdot \mathcal{E}(N_i(t)). \tag{17}$$

Thus it follows

$$\begin{aligned} \widetilde{X}^j \cdot Z = \frac{X^j}{X^i} \cdot Z &= \widehat{X}^j \cdot (\widehat{X}^i)^{-1} \cdot Z \\ &= \widehat{X}^j \cdot (\widehat{X}_0^i)^{-1} \cdot \mathcal{E}(N_i)^{-1} \cdot \mathcal{E}(N_i) \in \mathcal{M}^c_{loc}(P^*). \end{aligned}$$

Therefore with (X^i) as numeraire, we have

$$\frac{d\widetilde{X}_t^j}{\widetilde{X}_t^j} = \underbrace{\sigma_{\widetilde{X}^j}(t)}_{=\sigma_j - \sigma_i} \, dB_t^i \qquad \text{where } B_t^i \text{ Brownian motion under } Q_i^*.$$

Again $\int_0^t ||\sigma_{\widetilde{X}^j}(s)||^2 \, ds \leq \int_0^t \widetilde{\psi}(s) ds < \infty$ implies that $\widetilde{X}^j$ is indeed a martingale.

□

Conclusion: P^* is a martingale measure with respect to (X^0) as numeraire if, and only if, Q_i^* is a martingale measure with respect to (X^i) as numeraire.

Connection $B_t^* \longleftrightarrow B_t^i$:

$$B_t^* \in \mathcal{M}^c(P^*)\ ,\ Q_i^* = \mathcal{E}(N_i)\cdot P^* \underset{\text{Girsanov}}{\Longrightarrow} B_t^i = B_t^* - \langle B^*, N_i\rangle_t \in \mathcal{M}^c(Q_i^*)$$

$$\Longrightarrow B_t^i = B_t^* - \int_0^t \underbrace{\sigma_{\widehat{X}_i}(s)}_{\sigma_i-\sigma_0}\, d\langle B\rangle_s = B_t^* + \int_0^t (\sigma_0 - \sigma_i)(s)\, ds.$$

Thus we have $\quad \widetilde{C}_t = \mathrm{E}_i^*\Big[\frac{C_T}{X_T^i}\,\Big|\,\mathcal{F}_t\Big] \quad$ with X_t^i as numeraire.

The decisive result is now given by the following proposition:

Proposition 4.5.3. $\qquad X_t^i \cdot \widetilde{C}_t = X_t^0 \cdot \widehat{C}_t.$

(i.e., the valuation of the contingent claim C_T does not depend on the numeraire.)

Lemma 4.5.4. (Bayes rule)*: Let $Q = Z_T \cdot P$ on $\mathcal{F}_T$ and $Y \in L^1(\Omega, \mathcal{F}_t, Q)$. Then, for any $s < t \le T$, it follows*

$$\mathrm{E}_Q[Y|\mathcal{F}_s] = \frac{1}{Z_s}\ \mathrm{E}_P[Y\cdot Z_t\,|\,\mathcal{F}_s].$$

Proof. Let $A \in \mathcal{F}_s$. Since $\dfrac{dQ_t}{dP_t} = Z_t = \mathrm{E}_P[Z_T\,|\,\mathcal{F}_t]$ it follows

$$\begin{aligned}\mathrm{E}_Q[\mathbf{1}_A\, Y] = \mathrm{E}_P[\mathbf{1}_A\, Y\cdot Z_t] &= \mathrm{E}_P\Big[\mathbf{1}_A\, \mathrm{E}_P[Y\cdot Z_t\,|\,\mathcal{F}_s]\Big] \\ &= \mathrm{E}_Q\Big[\mathbf{1}_A \cdot \frac{1}{Z_s}\ \mathrm{E}_P[Y\cdot Z_t\,|\,\mathcal{F}_s]\Big].\end{aligned}$$

□

Proof of Prop. 4.5.3:. According to (17) one has $Z_t = (\widehat{X}_0^i)^{-1}\cdot\widehat{X}_t^i$. Thus Lemma 4.5.4 implies

$$\begin{aligned}\widetilde{C}_t = \mathrm{E}_i^*[\widetilde{C}_T|\mathcal{F}_t] = \frac{1}{Z_t}\ \mathrm{E}^*[\widetilde{C}_T\ Z_T\,|\,\mathcal{F}_t] &= \frac{\widehat{X}_0^i}{\widehat{X}_t^i}\ \mathrm{E}^*\Big[\frac{C_T}{X_T^i}\cdot\frac{\widehat{X}_T^i}{\widehat{X}_0^i}\,\Big|\,\mathcal{F}_t\Big] \\ &= \frac{1}{\widehat{X}_t^i}\ \mathrm{E}^*\Big[\underbrace{\frac{C_T}{X_T^i}\cdot\frac{X_T^i}{X_t^0}}_{\widehat{C}_T}\,\Big|\,\mathcal{F}_t\Big] = \frac{X_t^0}{X_t^i}\ \widehat{C}_t.\end{aligned}$$

□

4.6 Solution of the Siegel-Paradox

The techniques developed in the previous section solve the Siegel-Paradox. The spot price process for a portfolio in Deutschmark and Dollars is given by

$$Z_t = (X_t, Y_t) = (\beta_t^{\mathrm{DM}}, e_t\, \beta_t^{\$}) \qquad \text{with}$$

$$\begin{aligned} e_t \quad &= \text{exchange rate \$/DM} \\ \beta_t^{\mathrm{DM}} &= \exp \int_0^t r^{\mathrm{DM}}(s)\, ds \\ \beta_t^{\$} \quad &= \exp \int_0^t r^{\$}(s)\, ds. \end{aligned}$$

From the view point of the German investor (X_t) is the numeraire. Thus

$$\widehat{Z}_t = (1, \widehat{Y}_t) \quad \text{with } \widehat{Y}_t = e_t \frac{\beta_t^{\$}}{\beta_t^{\mathrm{DM}}}$$

and

$$\frac{d\widehat{Y}_t}{\widehat{Y}_t} = \sigma_{\widehat{Y}}(t)\, dB_t \quad , \ \sigma_{\widehat{Y}} = \sigma_Y - \sigma_X = \sigma_e$$

under the martingale measure P^* which is the expectation of the risk-neutral German investor.

Applying Itô's product rule it follows:

$$\Longrightarrow \frac{de_t}{e_t} = (r_{\mathrm{DM}}(t) - r_{\$}(t))\, dt + \sigma_e(t)\, dB_t \qquad \textbf{under } P^*.$$

However, from the viewpoint of the American investor the numeraire is (Y_t).

$$\Longrightarrow \widetilde{Z}_t = (\widetilde{X}_t, 1) \quad \text{with } \widetilde{X}_t = f_t \cdot \frac{\beta_t^{\text{DM}}}{\beta_t^{\$}}\ ,\ f_t = \frac{1}{e_t} \quad \text{(exch. rate DM/\$)}$$

New martingale measure : $\quad Q^* = \mathcal{E}(N) \cdot P^* \quad N_t = \int_0^t \underbrace{\sigma_{\widehat{Y}}(s)}_{=\sigma_e}\, dB_s.$

$$\widetilde{X} \text{ martingale under } Q^* \Longrightarrow \frac{d\widetilde{X}_t}{\widetilde{X}_t} = \sigma_{\widetilde{X}}(t)\, d\widetilde{B}_t$$

with $\sigma_{\widetilde{X}} = \sigma_X - \sigma_Y = -\sigma_e = \sigma_f$ and $d\widetilde{B}_t = dB_t + \sigma_f\, dt$ is Brownian motion under Q^*.

$$\Longrightarrow \quad \frac{df_t}{f_t} = (r^{\$}(t) - r^{\text{DM}}(t))\, dt + \sigma_f(t)\, d\widetilde{B}_t \qquad \textbf{under } Q^*$$
$$= \left(r^{\$}(t) - r^{\text{DM}}(t) + \sigma_f^2(t)\right) dt + \sigma_f(t)\, dB_t \qquad \textbf{under } P^*$$

Thus also from the view point of the American investor there is no risk premium under the correct martingale measure Q^*.

4.7 Admissible Strategies and Arbitrage-free Pricing

The “No arbitrage” condition given in section 4.1 rules out arbitrage opportunities in a short (strictly speaking infinitesimal) time interval. As we have shown this condition implies that the “market price of risk” process $\lambda(t)$ must be identic for all security price processes in the considered market, which under the “Novikov” condition implies the existence of an equivalent martingale measure P^*. Conversely, if an equivalent martingale measure P^* exists, then according to the relation (5) infinitesimal arbitrage opportunities cannot exist.

However, the “no arbitrage” condition used so far does not rule out, that there may exist dynamic trading strategies ϕ which over a finite time horizon allow arbitrage profits.

Let $X_t = (X^0, \ldots, X^n)_{0 \le t \le T}$ be the price process of $n+1$ securities satisfying the conditions given in section 4.5 . Let $(\phi)_t = (\phi^0, \ldots, \phi^n)_t$ be a self-financing portfolio strategy, i.e. an adapted $(n+1)$-dimensional stochastic process with $V_t(\phi) = \phi_t \circ X_t$ and $dV_t = \phi_t \circ dX_t$. As shown in Prop. 4.5.1, the property of ϕ of being self-financing does not depend on the choice of the numeraire.

Definition 4.7.1. *An arbitrage opportunity is a self-financing portfolio strategy ϕ such that the wealth process $V_t(\phi)$ satisfies the following conditions:*

$$V_0(\phi) = 0, \quad P[V_T(\phi) \ge 0] = 1, \quad \textit{and} \quad P[V_T(\phi) > 0] > 0.$$

Such a portfolio or trading strategy would allow one to start with an initial investment of zero and without adding money in the time interval $[0, T]$ to receive a positive amount at time T with positive probability. Since even in the simplest case of the standard Black-Scholes model ($d = 1,\ \mu$ and $\sigma = const.,\ r = 0$) there exist self-financing strategies which allow arbitrage (see e.g. Harrison-Kreps (1979) or Duffie (2001)) we need some subclass of “admissible” strategies which rule out such arbitrage opportunities.

Consider the case where $X_t^0 = \beta(t) = \exp \int_0^t r(s)\ ds$ is chosen as numeraire, and the discounted process is $\widehat{X} = X/X^0$.

Let Φ^0 be the class of self-financing strategies which are integrable w.r.t. $\widehat{X}$ and for which there exists some constant k with

$$\widehat{V}_t(\phi) = V_t(\phi)/X_t^0 = \phi_t \circ \widehat{X}_t \geq k \qquad \forall\, t \in [0,T] \tag{18}$$

The condition (18) can be interpreted as a credit constraint, which means that short sales are allowed, but the wealth process must stay above a lower bound k, which may be negative. The strategies in Φ^0 are also called "tame" (w.r.t. X^0).

Remark 4.8. *If the short rate process $r(t)$ is bounded, then obviously the condition (18) is equivalent to: $V_t(\phi)$ is bounded below.*

Let $\mathcal{M}(\widehat{X})$ denote the set of equivalent martingale measures under which $\widehat{X}$ is a martingale.

Theorem 4.7.2. *If $\mathcal{M}(\widehat{X}) \neq \emptyset$, then there is no arbitrage in Φ^0.*

Proof. Consider $\phi \in \Phi^0$ with $V_0(\phi) = 0$ and $P[V_T(\phi) \geq 0] = 1$. Then $\widehat{V}_t(\phi) = \phi_t \circ \widehat{X}_t = V_t(\phi) \cdot (X_t^0)^{-1}$ is the value process with respect to $\widehat{X}_t$. Since X^0 is strictly positive, V_t is positive iff $\widehat{V}_t$ is positive. Since V_t is self-financing w.r.t. X, it is also self-financing w.r.t. $\widehat{X}$. Hence $\widehat{V}_t(\phi) = \phi \bullet \widehat{X}_t$ is a local martingale for any $P^* \in \mathcal{M}(\widehat{X})$. Since $\widehat{V}_t$ is by assumption bounded below, Fatou's lemma (see Sect. 1.1) applied to a sequence of localizing stopping times of $\widehat{V}_t$ implies $E^*[\widehat{V}_T] \leq E^*[\widehat{V}_0] = 0$, which together with $\widehat{V}_T \geq 0$ P^*-a.s. implies $P^*[\widehat{V}_T > 0] = 0$. Since $P^* \sim P$ this is equivalent to $P[V_T > 0] = 0$.

□

If $C_T \in L^1(\Omega, \mathcal{F}_T, P^*)$ is a (European) contingent claim, which settles at time T, then the fundamental pricing rule implies that its price Π_t under $P^* \in \mathcal{M}(\widehat{X})$ at any $0 \leq t \leq T$ is given by the process

$$\Pi_t = X_t^0 E_{P^*}[C_T/X_T^0 \,|\, \mathcal{F}_t] \tag{19}$$

If there is a unique $P^* \in \mathcal{M}(\widehat{X})$ this process is also uniquely determined. However, there may be many different equivalent martingale measures in $\mathcal{M}(\widehat{X})$, as is the case when the market is "incomplete". In this case the price process (19) is no longer uniquely determined by the fundamental pricing rule, but depends on the choice of an equivalent martingale measure. In particular this is the case if $n < d$, i.e. there are less securities in the market than sources of uncertainty. Thus in incomplete markets the price of an arbitrary contingent claim cannot be determined by "no arbitrage" arguments.

Definition 4.7.3. *A contingent claim $C_T \in L^1(\Omega, \mathcal{F}_T, P^*)$ is* <u>*attainable*</u> *if there exists an admissible strategy $\phi \in \Phi^0$ with $C_T = V_T(\phi)$ and $\widehat{V}_t(\phi) = \phi_t \circ X_t / X_t^0$ is a P^*-martingale for some $P^* \in \mathcal{M}(\widehat{X})$.*

If any integrable contingent claim is attainable, the market is called <u>*complete*</u>*.*

Proposition 4.7.4. *The price Π_t of any attainable contingent claim C_T is uniquely determined by no-arbitrage and is given by the relation (19), where the expectation is taken for arbitrary $P^* \in \mathcal{M}(\widehat{X})$.*

Proof. Let ϕ and ψ be two admissible strategies with $V_T(\phi) = V_T(\psi) = C_T$ and $P_1^*, P_2^* \in \mathcal{M}(\widehat{X})$. Then also $\widehat{V}_T(\phi) = \widehat{V}_T(\psi) = \widehat{C}_T$. Consider the two value processes

$$V_t(\phi) = X_t^0 E_{P_1^*}[\widehat{V}_T(\phi) \,|\, \mathcal{F}_t)] \tag{20}$$

$$V_t(\psi) = X_t^0 E_{P_2^*}[\widehat{V}_T(\psi) \,|\, \mathcal{F}_t)]\,. \tag{21}$$

Then $\widehat{V}_t(\phi) = \phi_t \circ \widehat{X}_t$ is a local P_2^* - martingale and, since it is bounded from below, by Fatou's lemma, a P_2^* - sub-martingale, which implies

$$\widehat{V}_t(\psi) = E_{P_2^*}[\widehat{V}_T(\psi) \,|\, \mathcal{F}_t)] = E_{P_2^*}[\widehat{V}_T(\phi) \,|\, \mathcal{F}_t)] \le \widehat{V}_t(\phi)\,. \tag{22}$$

By the same argument it also follows $\widehat{V}_t(\phi) \le \widehat{V}_t(\psi)$. Hence $\Pi_t = V_t(\phi) = V_t(\psi)$ is the unique arbitrage price of C_T. □

Unfortunately the condition of "tameness" depends on the choice of the numeraire. Thus an admissible strategy in Φ^0 may not be admissible for the numeraire X^i, although the property of self-financing is independent of the numeraire, as shown in Proposition 4.5.1. This is unsatisfactory from an economic point of view. For the property of a market to be "arbitrage free" or not should not depend on the choice of the numeraire.

A satisfactory solution of this problem, which requires very advanced technical methods, is beyond the scope of these notes. These problems have been dealt with in a number of papers by Delbaen and Schachermayer, see in particular Delbaen-Schachermayer (1995) and Delbaen-Schachermayer (1997).

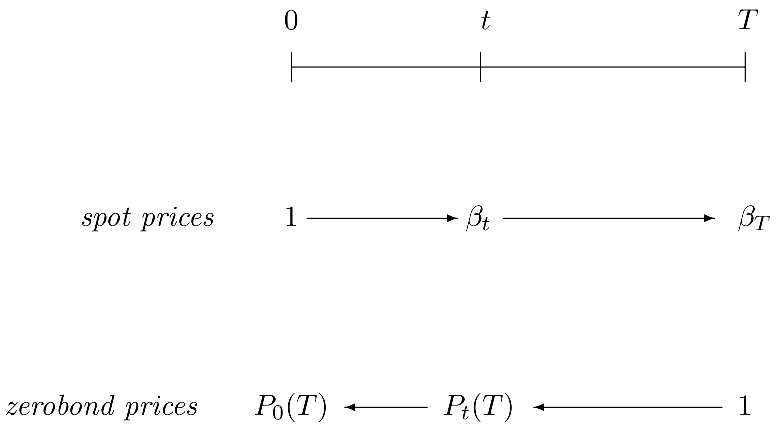
4.8 The "Forward Measure"

The "forward measure" is a useful tool when interest rates are stochastic. Then the accumulation factor

$$\beta_t(\omega) = \exp \int_0^t r(s,\omega)\, ds$$

is a stochastic process, making the computation of the conditional expectation in the Fundamental Pricing Rule (Theorem 4.2.2) cumbersome.

Consider the following connection between $\$_t$ (t = "today") and $\$_T$ (T = "tomorrow"):



Let $X_t = (X^0, X^1, \ldots)_t$ be price processes of securities. Choosing $X_t^0 = \beta_t$ as numeraire determines the martingale measure P^*, under which

$$X_t^i = \mathrm{E}^*\Big[\frac{\beta_t}{\beta_T}\, X_T^i \,|\, \mathcal{F}_t\Big] \quad i = 0, 1, 2 \ldots\ ,\ 0 \le t \le T.$$

Remark 4.9. *P^* is also called the "spot martingale measure", since the spot price process is chosen as numeraire.*

In particular one has

$$P_t(T) = \mathrm{E}^*\left[\frac{\beta_t}{\beta_T} \,|\, \mathcal{F}_t\right] \quad \text{since } P_T(T) \equiv 1.$$

Thus

$$\widehat{P}_t(T) = \frac{P_t(T)}{\beta_t} \quad \text{is a} \quad P^* - \textit{martingale}$$

and

$$\frac{dP_t(T)}{P_t(T)} = r(t)\, dt + \sigma(t,T) \circ dB_t^* \quad \text{under } P^*.$$

Remark 4.10. *Observe that $\sigma(t,t) = 0$ since $P_t(t) \equiv 1$.*

Change of numeraire from $\$_t$ to $\$_T$

Instead of β_t choose now $P_t(T)$ as numeraire. According to Sect. 4.5 the corresponding martingale measure P_T is given by

$$P_T = \mathcal{E}(L(T)) \cdot P^* \quad \text{with}$$

$$L_t(T) = \int_0^t \sigma_{\widehat{P}_t}(s) \circ dB_s^* = \int_0^t \sigma(s,T) \circ dB_s^*$$

and one has

$$B_t^T = B_t^* - \langle B_t^*, L_t \rangle = B_t^* - \int_0^t \sigma(s,T)\, ds.$$

Let $H_T \in \mathcal{F}_T$ be a contingent claim. Then under P^*:

$$H_t = \mathrm{E}^*\left[\frac{\beta_t}{\beta_T}\, H_T \,|\, \mathcal{F}_t\right] \text{ (Price in } \$_t.)$$

Under P_T it follows :

$$\widetilde{H}_T = \tfrac{H_T}{P_T(T)} = H_T$$

$$\widetilde{H}_t = E_T[H_T|\mathcal{F}_t] \qquad \text{(Price in } \$_T)$$

$$H_t = \widetilde{H}_t \cdot P_t(T) \qquad \text{(Price in } \$_t) \ .$$

Thus we have proved

$$H_t = \mathrm{E}^*\left[\frac{\beta_t}{\beta_T}\, H_T \,|\, \mathcal{F}_t\right] = P_t(T) \cdot \mathrm{E}_T[H_T \,|\, \mathcal{F}_t]$$

P_T = “forward measure” (see e.g. Geman-ElKaroui-Rochet (1995))

The connection with the definition of P_T sometimes given in the literature is established by the following proposition :

Proposition 4.8.1. $P_T[A] = \mathrm{E}^*\left[(\beta_T{\cdot}P_0(T))^{-1} \,|\, \mathcal{F}_t\right]{\cdot}P^*[A]$ *for any* $A \in \mathcal{F}_t$.

Proof. $\widehat{X}_t = \widehat{P}_t(T) = \dfrac{P_t(T)}{\beta_t}$ is P^*-martingale with

$$\frac{d\widehat{X}_t}{\widehat{X}_t} = \sigma(t,T) \circ dB_t = dL_t(T)$$

$\Longrightarrow \widehat{X}_t = \underbrace{\widehat{X}_0}_{=P_0(T)} \cdot\, \mathcal{E}(L_t(T)).$

For $A \in \mathcal{F}_t$ one has

$$\begin{aligned} P_T[A] &= \mathrm{E}^*[\mathcal{E}(L_T(T)) \,|\, \mathcal{F}_t] \cdot P^*[A] \\ &= \mathcal{E}(L_t(T)) \cdot P^*[A]. \end{aligned} \tag{23}$$

Since $\widehat{X}_t$ P^*-martingale, it follows

$$\mathrm{E}^*\left[\frac{1}{\beta_T} \,|\, \mathcal{F}_t\right] = \mathrm{E}^*[\widehat{X}_T|\mathcal{F}_t] = \widehat{X}_t = P_0(T) \cdot \mathcal{E}(L_t(T))$$

which together with (23) completes the proof. □

4.9 Option Pricing Under Stochastic Interest Rates

As an application of the forward measure we now study currency options when interest rates are stochastic.

Let $\$_t$ $(= e_t)$ be the process of the exchange rate \$/DM. This gives rise to the following price processes for a portfolio in DM/\$:

$(\beta_t^{\mathrm{DM}}, \$_t\, \beta_t^{\$})$	"spot" price processes (DM,\$ "today")	$0 \leq t \leq T$
$(P_t^{\mathrm{DM}}(T), \$_t \cdot P_t^{\$}(T))$	"forward" price processes (DM,\$ "tomorrow")	$0 \leq t \leq T$.

Now choose $P_t^{\mathrm{DM}}(T)$ as numeraire, i.e. instead of DM_t ("today") calculate in DM_T ("tomorrow"). This leads to the following discounted processes:

$$\widehat{Z}_t = (1, \widehat{Y}_t) \quad \text{with } \widehat{Y}_t = \$_t \cdot \frac{P_t^{\$}(T)}{P_t^{\mathrm{DM}}(T)} \qquad (T\text{-forward \$-price at } t)$$

By the Interest-rate-parity theorem, which is illustrated in the following diagram, $\widehat{Y}_t$ is, at time t, the forward price $F_t(T)$ of one dollar delivered at time T.

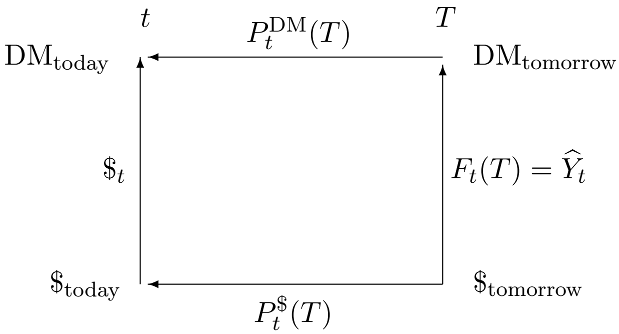


Diagram: **Interest-Rate-Parity Theorem**

Under the forward measure P_T, $\widehat{Y}_t$ is a martingale.

Clearly $\$_T = \widehat{Y}_T$. Thus it follows, that the price H_t of the call option $H_T = (\$_T - K)^+$ at time t is given by :

$$H_t = P_t^{\mathrm{DM}}(T)\, \mathrm{E}_T[(\widehat{Y}_T - K)^+ \,|\, \mathcal{F}_t] = P_t^{\mathrm{DM}}(T) \Big[\widehat{Y}_t\, \Phi(d_t^1) - K \cdot \Phi(d_t^2)\Big]$$
$$= \$_t \cdot P_t^{\$}(T) \cdot \Phi(d_t^1) - K \cdot P_t^{\mathrm{DM}}(T) \cdot \Phi(d_t^2)$$

where $d_t^{1,2} = \dfrac{\ln(\widehat{Y}_t/K)}{\eta_t} \pm \dfrac{1}{2}\, \eta_t$ with

$$\eta_t^2 = \int_t^T ||\sigma_{\widehat{Y}}(s)||^2 \, ds = \int_t^T ||\sigma_{\$}(s) + \sigma^{\$}(s,T) - \sigma^{\mathrm{DM}}(s,T)||^2 \, ds$$

($\sigma^{\$}$ resp. σ^{DM} are the volatilities of $P^{\$}(T)$ resp. $P^{\mathrm{DM}}(T)$)

Consider the following hedge strategy in T-forward contracts:

- Buy at $t : \Phi(d_t^1)$ Dollar at T
- Sell at $t :\ -K \cdot \Phi(d_t^2)$ DM at T

$\Longrightarrow V_t(\phi) = \Phi(d_t^1) \cdot \widehat{Y}_t - K \cdot \Phi(d_t^2)$ DM at T

$$\Longrightarrow V_T(\phi) = \left\{ \begin{array}{cc} \$_T - K & \text{for } \$_T = \widehat{Y} > K \\ 0 & \text{for } \$_T \leq K \end{array} \right\} \quad = H_T$$

Hence the strategy ϕ duplicates the contingent claim H_T.

Exercise: Show that the strategy ϕ is self-financing.

Hint: Consider the function $F(x,t) = x \cdot \Phi(d^1(x,t)) - K \cdot \Phi(d^2(x,t))$ where $d^{1,2}(x,t) = \dfrac{\ln(x/K)}{\eta_t} \pm \dfrac{1}{2}\, \eta_t$ with $\eta_t^2 = \displaystyle\int_t^T ||\sigma_X(s)||^2 \, ds.$

1) Show $x \cdot \varphi(d^1) = K \cdot \varphi(d^2)$ where $\varphi(z) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}\, z^2}$ (density of Φ)

2) Using the relation 1) show $F_x(x,t) = \Phi(d^1(x,t))$

3) Show $F_t(x,t) = x \cdot \varphi(d^1) \cdot \frac{d}{dt}\,\eta(t)$

4) Show $\frac{d}{dt}\,\eta(t) = -\frac{1}{2\eta} \cdot ||\sigma_X(t)||^2$

5) Show $F_t + \frac{1}{2}\,F_{xx}\,x^2\,||\sigma_X(t)||^2 = 0$

$\Longrightarrow V_t(\phi) = F(\widehat{Y}_t, t)$ is self-financing (see Sect. 2.5 (15) and Rem. 2.11)

$\Longrightarrow dV_t(\phi) = \Phi(d_t^1)\,d\widehat{Y}_t$

5

Term Structure Models

Term structure models are considered as one of the most complex and mathematically demanding subjects in finance. Contrary to the valuation of options, or more generally of contingent claims, the valuation of interest rate dependent instruments requires the study of many interacting markets, the so-called "term structure" of interest rates.

One of the first term structure models was the short-rate model of Vasicek (1977). By modelling the "short rate" over time, the dynamics of the "forward rate" is also implicitly determined. A drawback of short-rate models is that they cannot be fitted to the total term structure (they only catch the first point of the "yield curve"). This shortcoming was first overcome by the Ho-Lee model (see Ho-Lee (1986)), which starts with the present term structure as given by the zerobond prices. Both models are sketched in Sect. 5.4 as special HJM models.

A fundamental contribution was made by Heath-Jarrow-Morton (1992), who start by modelling the complete term structure as given by the (continuously compounded) forward rates. Most implementations of the HJM model are "Gaussian", meaning that the forward rates are assumed to be normally distributed. But this implies that interest rates may become negative, a rather undesirable property. This problem was overcome by the "log-normal" models. However, assuming that continuously compounded interest rates are log-normally distributed led to new difficulties: the rates explode over time and thus these models (see Black-Derman-Toy (1990) or Black-Karasinski (1991)) are unstable. This problem was solved by the (discrete) log-normal Sandmann-Sondermann model (see Sandmann-Sondermann (1993)). By switching from continuously compounded to "effective", i.e., annually compounded rates, they showed that such models have stable dynamics, and hence have a stable continuous limit (see Sandmann-Sondermann (1994)). This approach finally led to the so-called "LIBOR" or "Market" models due to Sandmann-Sondermann-Miltersen (1995) and Miltersen-Sandmann-Sondermann (1997), and further developed by Brace-Gatarek-Musiela (1997). These models have become quite popular in the finance industry, since they are stable and arbitrage-free, produce non-negative interest rates and, most importantly, reflect the market practice of Black's caplet formula (see Sect. 5.5 and 5.6).

For a comprehensive treatment of Fixed Income Markets and term structure models we refer to Musiela-Rutkowski (1997), who also give a detailed overview of the historical development of this complex subject.

By looking at a term structure model in continuous time in the general form of Heath-Jarrow-Morton (1992) as an infinite collection of assets (the zerobonds of different maturities), the methods developed in Chap. 4 can be applied without modification to this situation. Readers who have gone through the original articles of HJM may appreciate the simplicity of this approach, which leads to the basic results of HJM in a straightforward way. The same applies to the *Libor Market Model* treated in Sect. 5.5 .

5.1 Different Descriptions of the Term Structure of Interest Rates

There are three equivalent descriptions of the term structure, namely by means of

1) *Zerobonds*

$P(t,T)$ = price of one Euro, deliverable at T, at time t

2) *yields* (continuously compounded)

$$y(t,T) := -\frac{1}{T-t}\ \log P(t,T)$$

3) *forward rates* (continuously compounded)

$$f(t,T) := -\frac{\partial}{\partial T} \log P(t,T).$$

Relations between 1), 2) and 3) :

1) $\Longrightarrow$ 2) + 3) by definition

2) $\Longrightarrow P(t,T) = \exp\{-y(t,T)\,(T-t)\}$

$$f(t,T) = y(t,T) + (T-t)\,\frac{\partial}{\partial T}\,y(t,T) \tag{1}$$

3) $\Longrightarrow \; P(t,T) = \exp\Big\{-\int_t^T f(t,u)\,du\Big\}$

$$y(t,T) = \frac{1}{T-t}\int_t^T f(t,u)\,du.$$

Hence the term structure up to time T^* is completely described by any of the following families:

(i) $\Big\{P(t,T) \mid 0 \le T \le T^*\,,\ 0 \le t \le T\Big\}$

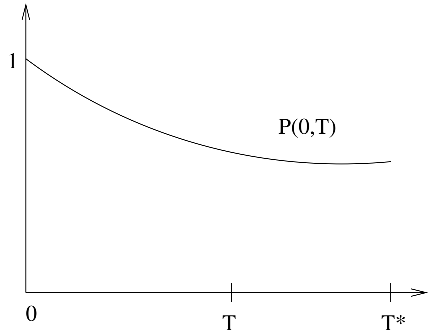
(ii) $\Big\{y(t,T) \mid 0 \le T \le T^*\,,\ 0 \le t \le T\Big\}$

(iii) $\Big\{f(t,T) \mid 0 \le T \le T^*\,,\ 0 \le t \le T\Big\}$.

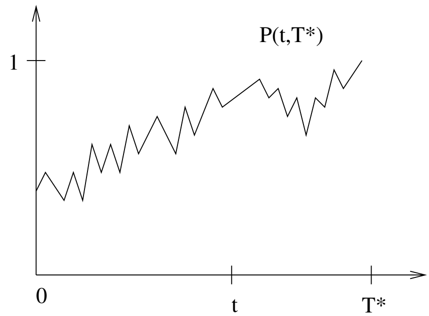
For fixed t, $T \longrightarrow P(t,T)\ ,\ t \le T \le T^*$ is a (smooth) curve of bond prices at t with different maturities.

For fixed T, $t \longrightarrow P(t,T)$ is the (stochastic) price process of a bond maturing at T.

This is illustrated in the diagrams of Fig. 5.1 on page 98.

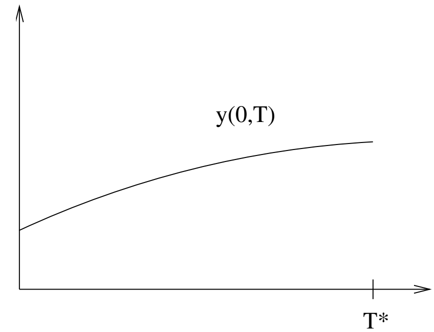


Bond prices now $(t = 0)$

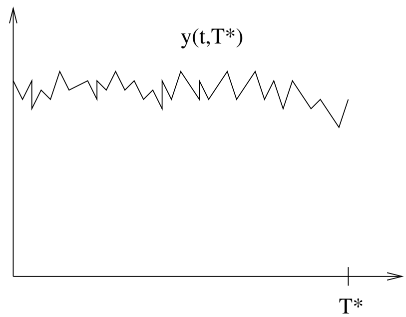


Price of a bond maturing at T^*

Similar for $y(t, T)$

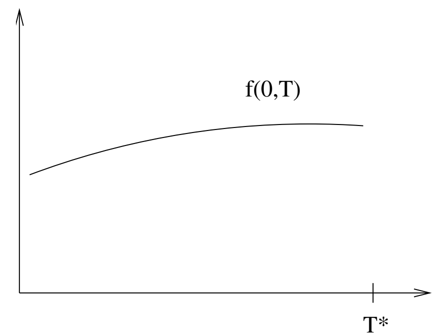


yield rates now $(t = 0)$

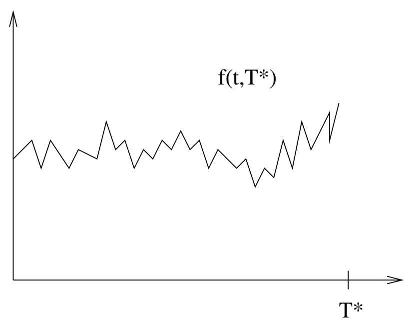


yields rates maturing at T^*

and for $f(t, T)$



forward rates now $(t = 0)$



forward rates maturing at T^*

Fig. 5.1. Different descriptions of the Term Structure

From (1) it follows

$$f(t,T) \begin{pmatrix} > \\ = \\ < \end{pmatrix} y(t,T) \Longleftrightarrow \text{ yield curve is } \begin{pmatrix} \text{rising} \\ \text{flat} \\ \text{falling} \end{pmatrix}$$

In particular, the relation (1) implies: the longer the maturity T, the more sensitive $f(t,T)$ reacts to twists in the yield curve.

Definition 5.1.1. *The instantaneous spot rate is defined by*

$$r(t) = f(t,t) = -\frac{\partial}{\partial T} (\log \ P(t,T))_{T=t}.$$

5.2 Stochastics of the Term Structure

Again we assume that there are d independent stochastic factors modelled by a d-dimensional Brownian motion.

$$B_t = (B_t^1, \ldots, B_t^d) \quad \text{on } (\Omega, (\mathcal{F}_t)_{0 \le t \le T^*}, P)$$

satisfying the usual conditions.

For each $T \le T^*$, consider the "asset" $X_t^T = P(t,T)$. As before we assume that, for any $0 \le T \le T^*$,

$$\frac{dP(t,T)}{P(t,T)} = \mu(t,T)\ dt + \underbrace{\sigma(t,T) \circ dB_t}_{\text{scalar product in } \mathbb{R}^d} \qquad 0 \le t \le T$$

with $\mu(t,T),\ \sigma(t,T)$ adapted processes, satisfying boundary conditions as defined in Sect. 4.1 .

Thus we have a continuum of "assets", whose dynamics are given by Borel-measurable mappings

$$\mu : C \times \Omega \longrightarrow \mathbb{R} \quad , \sigma : C \times \Omega \longrightarrow \mathbb{R}^d$$

with

$$C = \Big\{ (t,T) :\ 0 \le t \le T \le T^* \Big\}.$$

Assumption 5.2.1. (No-arbitrage)
There exists a d-dimensional adapted process $\lambda(t)_{0\leq t\leq T^*}$ *with*

$$E\Big[\exp\Big\{\frac{1}{2}\int_0^{T^*} ||\lambda(s)||^2\,ds\Big\}\Big] < \infty$$

such that, for all $0 \leq t \leq T \leq T^*$

$$\begin{aligned}\mu(t,T) &= r(t) + \lambda(t)\circ\sigma(t,T)\\ &= r(t) + \sum_{i=1}^{d} \lambda_i(t)\,\sigma_i(t,T).\end{aligned}$$

Remark 5.1.

(1) As shown in Sect. 4.1, no arbitrage implies that the risk price process is the same for all "assets", i.e. does not depend on the maturity T*.*

(2) Since there is now a continuum of assets, but only d factors, NA imposes restrictions on the drift terms $\mu(t,T)$ *(term structure models with finitely many factors are "over"-complete).*

(3) $\lambda(t)$ *can be interpreted as a risk premium for long term investments, since*

$$E\Big[\frac{dP(t,T)}{P(t,T)}\,|\,\mathcal{F}_t\Big] = r(t)\,dt + \underbrace{\lambda(t)\circ\sigma(t,T)\,dt}_{\textit{excess return}}.$$

As before Girsanov's theorem allows us to eliminate $\lambda(t)$ as follows:

$$L_t := -\int_0^t \lambda(s)\circ dB_s$$

$$P^* = \mathcal{E}(L_{T^*})\,P \quad P^* \sim P \text{ on } \mathcal{F}_{T^*}$$

$$B_t^* = B_t + \int_0^t \lambda(s)\,ds \quad \text{BM under } P^*$$

$$\Longrightarrow \frac{dP(t,T)}{P(t,T)} = r(t)\,dt + \sigma(t,T) \circ dB_t^* \quad 0 \le t \le T \le T^*$$

$$\Longrightarrow E_{P^*}\Big[\frac{dP(t,T)}{P(t,T)}\;|\,\mathcal{F}_t\Big] = r(t)\,dt \qquad \text{(Local Expectation Hypothesis).}$$

Again let

$$\beta(t) = \exp\Big\{\int_0^t r(s)\,ds\Big\}$$

be the accumulation factor (rolling over at the spot rate) = *spot numeraire.*

As in Chap. 4 it follows, that the discounted price processes are martingales under P^*, i.e.

$$\hat{P}(t,T) = \frac{P(t,T)}{\beta(t)}$$

has dynamics

$$\frac{d\hat{P}(t,T)}{\hat{P}(t,T)} = \sigma(t,T) \circ dB_t^* \qquad \forall\, 0 \le t \le T \le T^*.$$

More generally, for any Itô-Process $(Z_t)_{0\le t\le T\le T^*}$ on $(\Omega, (\mathcal{F}_t), P)$, which is of the form

$$\frac{dZ_t}{Z_t} = \mu_Z(t)\,dt + \sigma_Z(t) \circ dB_t$$

we have for $\hat{Z}_t = \dfrac{Z_t}{\beta_t}$

$$\frac{d\hat{Z}_t}{\hat{Z}_t} = \sigma_Z(t) \circ dB_t^* \quad \text{under } P^*$$

which again implies the Fundamental Pricing Rule:

$$Z_t = E^*\Big[\exp\Big\{-\int_t^T r(s)\,ds\Big\} Z_T \Big| \mathcal{F}_t\Big] = \beta_t\, E^*[\hat{Z}_T \,|\, \mathcal{F}_t]\ \forall\, 0 \le t \le T \le T^*$$

In particular, for any contingent claim H_T which is $\mathcal{F}_T$-measurable (and thus may depend on the whole term structure up to time T), we have :

$$H_t = \beta_t \; E^* \left[\frac{H_T}{\beta_T} \middle| \mathcal{F}_t \right] = P(t,T) \; E_T[H_T | \mathcal{F}_t],$$

where E_T is the expectation w.r.t. the forward measure

$$P_T = E^*[(\beta_T \; P(0,T))^{-1} \,|\, \mathcal{F}_t] \; P^* \quad \text{on } \mathcal{F}_t$$

i.e. P_T is the martingale measure with $P(t,T)$ as numeraire (see Sect. 4.8).

5.3 The HJM-Model

We have completely described the term structure by taking the zero-bonds as building blocks. As we have seen all results, as derived in Chap. 4 for asset prices and their derivatives, carry over to describe the stochastic nature of the term structure. However, the term structure is usually described by interest rates instead of bond prices, e.g. by the movements of the yield curve or the forward rate curve.

Therefore HJM start with modelling the dynamics of the forward rates $f(t,T)$ and assume that

$$df(t,T) = \alpha(t,T) \; dt + \underbrace{\tau(t,T) \circ dB_t}_{\text{scalar product in } \mathbb{R}^d} \qquad 0 \le t \le T \le T^*$$

under the same conditions as in Sect. 5.2. Since the dynamics of the term structure is already completely described by the dynamics of $P(t,T)$, all what remains to do is to reveal the implied dynamics for the forward rates $f(t,T)$.

For the technical conditions on the processes α and τ needed for a mathematical rigorous derivation of the HJM-model (in particular conditions allowing the change of stochastic differentiation and integration) we refer to Heath-Jarrow-Morton (1992).

Again, since there is a continuum of rates but only finitely many factors, the drift terms $\alpha(t,T)$ must satisfy certain restrictions for the model to be consistent.

Theorem 5.3.1. *No arbitrage implies the existence of an adapted process* $\lambda(t)$ *such that, for any* $0 \le t \le T \le T^*$,

$$\alpha(t,T) = \tau(t,T) \circ (\lambda(t) - \sigma(t,T)) \tag{2}$$

where

$$\sigma(t,T) = -\int_t^T \tau(t,u)\, du.$$

Proof. We have the following relations between $f(t,T)$ and $P(t,T)$:

$$\ln P(t,T) = -\int_t^T f(t,u)\, du$$

$$f(t,T) = -\frac{\partial}{\partial T} \ln P(t,T).$$

No arbitrage (see Sect. 4.1 and 5.2) implies the existence of a risk price process $\lambda(t)$, which is independent of T, with

$$E\Big[\frac{dP(t,T)}{P(t,T)}|\mathcal{F}_t\Big] = r(t)\, dt + \lambda(t) \circ \sigma(t,T)\, dt$$

where $\sigma(t,T)$ is the volatility of $P(t,T)$. By Itô

$$\begin{aligned} d\ln P(t,T) &= \frac{dP(t,T)}{P(t,T)} - \frac{1}{2}\frac{1}{P(t,T)^2}\, d\langle P(t,T)\rangle_t \\ &= \Big(r(t) + \lambda(t) \circ \sigma(t,T) - \frac{1}{2}||\sigma(t,T)||^2\Big)\, dt + \sigma(t,T) \circ dB_t. \end{aligned}$$

The **crucial step** now is that the HJM conditions allow changing differentials. In doing so it follows:

$$\begin{aligned} df(t,T) &= -\frac{\partial}{\partial T}\, d\ln P(t,T) \\ &= -\Big(\lambda(t)\frac{\partial}{\partial T}\,\sigma(t,T) - \sigma(t,T)\frac{\partial}{\partial T}\,\sigma(t,T)\Big)\, dt - \underbrace{\frac{\partial}{\partial T}\,\sigma(t,T)}_{:=\tau(t,T)}\, dB_t. \end{aligned}$$

Hence

$$\alpha(t,T) = \lambda(t) \circ \tau(t,T) - \sigma(t,T) \circ \tau(t,T)$$

and

$$\sigma(t,T) = -\int_t^T \tau(t,u)\, du.$$

□

The relation (2) allows again the elimination of the risk process $\lambda(t)$ (which is the same for $f(t,T)$ and $P(t,T)$) by Girsanov.

Since we have already done this change of measure from P to P^* for the bond prices, we can directly derive the dynamics of $f(t,T)$ under P^* by the same technique as above.

Theorem 5.3.2. (HJM)
Under P^* the dynamics of $f(t,T)$ is

$$df(t,T) = \left(\tau(t,T) \circ \int_t^T \tau(t,u)\, du\right) dt + \tau(t,T) \circ dB_t^*.$$

Proof. Follows from

$$d \ln P(t,T) = \left(r(t) - \frac{1}{2}\, ||\sigma(t,T)||^2\right) dt + \sigma(t,T) \circ dB_t^*$$

and

$$\sigma(t,T) = -\int_t^T \tau(t,u)\, du.$$

□

Hence forward rates and spot rates under the (spot) martingale measure P^* are given by

$$f(t,T) = f(0,T) - \int_0^t \tau(s,T) \circ \sigma(s,T)\, ds + \int_0^t \tau(s,T) \circ dB_s^*$$

$$r(t) = f(0,t) - \int_0^t \tau(s,t) \circ \sigma(s,t)\, ds + \int_0^t \tau(s,t) \circ dB_s^*, \quad \text{where}$$

$$\sigma(s,T) = -\int_s^T \tau(s,u)\, du.$$

Attention: Observe that the volatility τ of the forward rates and the volatility σ of the zerobonds have opposite signs, since both depend on the same Brownian motion. Hence, if forward rates move up, zerobond prices move down, and vice versa.

5.4 Examples

Literature: Ho-Lee (1986), Vasicek (1977), Musiela-Rutkowski (1997).

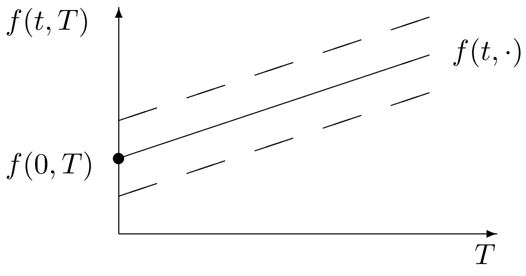
1. **Ho-Lee Model**

   $d = 1 \quad \tau(t,T) = \tau > 0 \quad \forall\, t, T$

   $\Longrightarrow df(t,T) = \alpha(t,T) + \tau\, dB_t$

   No arbitrage $\underset{\text{Theorem 5.3.1}}{\Longleftrightarrow}$ $\alpha(t,T) = \tau\, \lambda(t) + \tau^2 (T-t)$.

$$
\begin{aligned}
\text{Theorem 5.3.2} \Longrightarrow df(t,T) &= \tau^2 (T-t)\, dt + \tau\, dB_t^* \\
f(t,T) &= f(0,T) + \tau^2 \int_0^t (T-s)\, ds + \tau\, B_t^* \\
&= f(0,T) + \tau^2\, t(T - \frac{t}{2}) + \tau\, B_t^* \\
r(t) = f(t,t) &= f(0,t) + \frac{1}{2}\, \tau^2\, t^2 + \tau\, B_t^*.
\end{aligned}
$$



Ho-Lee turns any (flat) initial forward curve into an upward sloping curve at t .

Dynamics of $P(t,T)$ of the Ho-Lee model:

$$\frac{dP(t,T)}{P(t,T)} = r(t)\ dt + \sigma(t,T)\ dB_t^*$$
$$= \left(f(0,t) + \frac{1}{2}\ \tau^2\ t^2 + \tau\ B_t^*\right) dt - \tau(T-t)\ dB_t^*. \qquad (3)$$

Proposition 5.4.1.

$$P(t,T) = \frac{P(0,T)}{P(0,t)}\ \exp\left\{ -\frac{1}{2}\ \tau^2(T-t)\ T\ t - \tau(T-t)\ B_t^* \right\}.$$

Proof. To show

$$P(t,T) = P(0,T)\ e^{X_t}$$

with

$$X_t = \int_0^t f(0,s)\ ds - \frac{1}{2}\ \tau^2(T-t)\ T\ t - \tau(T-t)\ B_t^*$$

is the solution of (3).

Itô's formula implies:

$$dP(t,T) = P(0,T)\ \left(e^{X_t} dX_t + \frac{1}{2}\ e^{X_t} d\langle X\rangle_t\right)$$
$$= P(t,T)\ \left(dX_t + \frac{1}{2}\ d\langle X\rangle_t\right)$$

$$dX_t = f(0,t)\ dt - \frac{1}{2}\ \tau^2 (\underbrace{-T\ t + (T-t)\ T}_{(T-t)^2 - t^2})\ dt + \tau\ B_t^*\ dt - \tau(T-t)\ dB_t^*$$
$$= \left(f(0,t) + \frac{1}{2}\ \tau^2\ t^2 + \tau\ B_t^*\right) dt - \frac{1}{2}\ \underbrace{\tau^2(T-t)^2\ dt}_{d\langle X\rangle_t} - \tau(T-t)\ dB_t^*$$

$$\Longrightarrow \frac{dP(t,T)}{P(t,T)} = r(t)\ dt - \tau(T-t)\ dB_t^*.$$

□

2. **Vasicek-Model**

$$d = 1 \quad \tau(t,T) = \tau\, e^{-\alpha(T-t)} \quad \alpha, \tau > 0 \text{ fixed}$$

$$\Longrightarrow \sigma(t,T) = -\int_t^T \tau(t,u)\, du = \frac{\tau}{\alpha}\left(e^{-\alpha(T-t)} - 1\right).$$

$$\begin{aligned} \text{HJM} \longrightarrow df(t,T) &= -\tau(t,T)\,\sigma(t,T)\,dt + \tau(t,T)\,dB_t^* \\ &= \frac{\tau^2}{\alpha}\, e^{-\alpha(T-t)}\left(1 - e^{-\alpha(T-t)}\right) dt + \tau\, e^{-\alpha(T-t)}\, dB_t^* \\ &= \tau\, e^{-\alpha(T-t)}\left(-\frac{\tau}{\alpha}\,(e^{-\alpha(T-t)} - 1)\,dt + dB_t^*\right). \end{aligned}$$

$$\frac{dP(t,T)}{P(t,T)} = r(t)\,dt - \frac{\tau}{\alpha}\left(1 - e^{-\alpha(T-t)}\right) dB_t^*$$

It can be shown (see e.g. Musiela-Rutkowski (1997) Chapt.13) that the short rate is of the form

$$dr(t) = \left(a(t) - \alpha\, r(t)\right) dt + \tau\, dB_t^* \quad \text{(Ornstein-Uhlenbeck process)}$$

5.5 The “LIBOR Market” Model

<u>Literature</u>: see introduction to Chapter 5 .

Starting point are market rates (e.g. LIBOR = ”London Inter-Bank Offer Rate”).

$f(t,T,\delta)$ = nominal rate at t for period $[T, T+\delta]$, $\delta > 0$

$\delta = 0.25 \longrightarrow f(t,T,\delta)$ = 3-month forward LIBOR rate

Assumption 5.5.1. (MSS)

$$\frac{df(t,T,\delta)}{f(t,T,\delta)} = \mu(t,T)\,dt + \gamma(t,T) \circ dB_t.$$

Corresponding HJM-Model

$$dP(t,T) = P(t,T)\ (r(t) + \sigma(t,T) \circ dB^*(t))$$

$$df(t,T,dt) = -\tau(t,T) \circ \sigma(t,T)\ dt + \tau(t,T) \circ dB^*(t)$$

$$\sigma(t,T) = -\int_t^T \tau(t,u)\ du.$$

Switching to the forward measure by choosing $P(t,T)$ as numeraire (see Sect. 4.8 and 5.2), we get that

$$B_T^*(t) = B^*(t) + \int_0^t \sigma(s,T)\ ds$$

is Brownian motion under the forward measure

$$P_T = \mathcal{E}(L_T)\ P^* \quad \text{with} \quad L_T(t) = -\int_0^t \sigma(s,T) \circ dB^*(s).$$

The connection between the rates $f(t,T,\delta)$ and $P(t,T)$ is given by the forward contracts

$$F(t,T,\delta) := \frac{P(t,T+\delta)}{P(t,T)} = \Big[1 + \delta\ f(t,T,\delta)\Big]^{-1}.$$

Proposition 5.5.2.

$$\begin{aligned}\frac{dF(t,T,\delta)}{F(t,T,\delta)} &= (\sigma(t,T) - \sigma(t,T+\delta)) \circ dB_T^*(t) \\ &= -\Big(\int_T^{T+\delta} \tau(t,u)\ du\Big) \circ dB_T^*(t).\end{aligned}$$

Proof. Set $X = P(t,T+\delta)$, and $Y = P(t,T)$. Then Itô implies

$$\begin{aligned}\frac{d(XY^{-1})}{XY^{-1}} &= (\mu_X - \mu_Y + \sigma_Y \circ (\sigma_Y - \sigma_X))\ dt + (\sigma_X - \sigma_Y) \circ dB^*(t) \\ &= \Big(r(t) - r(t) + \sigma(t,T) \circ (\sigma(t,T) - \sigma(t,T+\delta))\Big)\ dt \\ &\quad + \Big(\sigma(t,T) - \sigma(t,T+\delta)\Big) \circ dB^*(t) \\ &= \Big(\sigma(t,T) - \sigma(t,T+\delta)\Big) \circ (\underbrace{dB^*(t) + \sigma(t,T)\ dt}_{dB_T^*(t)}).\end{aligned}$$

□

Theorem 5.5.3. *The relation between the volatilities* $\gamma(t,T)$ *and* $\sigma(t,T)$ *is given by*

$$\sigma(t,T+\delta) = \sigma(t,T) + \frac{\delta\, f(t,T,\delta)}{1+\delta\, f(t,T,\delta)} \cdot \gamma(t,T).$$

Proof. Since Girsanov does not change volatilities we have from the MSS-assumption

$$df(t,T,\delta) = \ldots dt + \ f(t,T,\delta) \cdot \gamma(t,T) \circ dB^*(t).$$

By Itô this implies

$$dF(t,T,\delta) = d(1+\delta\ f)^{-1} = -F^2 \delta\ df - \frac{1}{2}\ F^3\ d\langle f \rangle.$$

Hence

$$\begin{aligned} \frac{dF(t,T\delta)}{F(t,T,\delta)} &= -F\ \delta\ f\ \gamma(t,T)\ \circ dB^*(t) + \ldots dt \\ &= -\frac{\delta\ f}{1+\delta\ f}\ \gamma(t,T) \circ dB^*(t) + \ldots dt. \end{aligned}$$

By Prop. 5.5.2 we have

$$\frac{dF(t,T,\delta)}{F(t,T,\delta)} = (\sigma(t,T) - \sigma(t,T+\delta)) \circ dB^*_T(t).$$

Since Girsanov does not change the diffusion coefficients, we get the result by comparison of volatilities.

□

Remark 5.2. *The HJM-volatilities are no longer deterministic (even if the* $\gamma(t,T)$ *are assumed as deterministic), since they depend on* $f(t,T,\delta)$*, i.e.*

$$\sigma(t,T+\delta,\omega) = \sigma(t,T,f(t,T,\delta)(\omega)).$$

Starting with $\sigma(0,T) = 0$ and $f(0,T,\delta)$, $T = 0, \delta, 2\delta, \ldots$, they can be computed pathwise by a binomial lattice.

Theorem 5.5.4.

$$\frac{df(t,T,\delta)}{f(t,T,\delta)} = \gamma(t,T) \circ dB^*_{T+\delta}(t).$$

Proof.

$$(1 + \delta\, f(t,T,\delta))^{-1} = F(t,T,T+\delta) = \frac{P(t,T+\delta)}{P(t,T)}$$

$$\Longrightarrow f = \frac{1}{\delta}\Big(\frac{P(t,T)}{P(t,T+\delta)} - 1\Big).$$

Take $P(t,T+\delta)$ as numeraire

$\Longrightarrow$ $\dfrac{P(t,T)}{P(t,T+\delta)}$ is a martingale under $B^*_{T+\delta}$

$\Longrightarrow$ f and df/f are martingales under $B^*_{T+\delta}$

$\Longrightarrow$ $df = f \cdot \gamma(t,T) \circ dB^*_{T+\delta}$. □

The Market Caplet Formula

A "*caplet*" is defined as the payoff

$$C = \delta\, (f(T,T,\delta) - K)^+$$

payable at $T + \delta$.

By the Fundamental Pricing Rule we have

$$\begin{aligned} C_t &= \beta_t\, E^*\Big[\frac{C}{\beta_{T+\delta}}\Big|\mathcal{F}_t\Big] \\ &= \delta\, P(t,T+\delta)\, E_{T+\delta}[(f-K)^+|\mathcal{F}_t]. \end{aligned}$$

Since $\dfrac{df}{f} = \gamma(t,T) \circ dB^*_{T+\delta}$ is a lognormal martingale under $P_{T+\delta}$, $E_{T+\delta}[(f-K)^+]$ is the Black-Scholes formula for the call $(f-K)^+$.

Hence

$$C_t = \delta\, P(t,T+\delta)\, \Big[f(t,T,\delta)\, \Phi(d_1) - K\, \Phi(d_2)\Big]$$

$$d_{1,2}(t) = \frac{1}{\eta(t,T)}\Big(\ln\frac{f(t,T,\delta)}{K} \pm \frac{1}{2}\eta^2(t,T)\Big)$$

$$\eta^2(t,T) = \int_t^T \gamma^2(s,T)\, ds\,.$$

For a "*floorlet*" $F = \delta\,[K - f(t,T,\delta)]^+$ one obtains by the same method:

$$F_t = \delta\; P(t, T+\delta)\,\Big[K\;\Phi(-d_2) - f(t, T\delta)\;\Phi(-d_1)\Big].$$

5.6 Caps, Floors and Swaps

Consider a sequence $\mathcal{T} = \{T_0 < T_1 < \ldots < T_n\}$ of payment dates.

Let $L(t, T_i)$ denote the forward LIBOR rate for $[T_i, T_{i+1}]$, valid at $t\,,\ t \le T_0\,,\ (i = 0, \ldots, n-1)$.

Set $\delta_i = T_{i+1} - T_i$.

E.g.: 3-month LIBOR rates $\Longrightarrow \delta_i \approx 0.25$ (varies with calendar)

Definition 5.6.1. *A cap at cap-rate K on $L(t, T_i)$ is a collection of caplets of the form*

$$C(K, \mathcal{T}) = \Big\{\delta_i\;[L_i - K]^+, \quad i = 0 \ldots n-1\Big\}$$

which pays the amount $\delta_i\;[L(T_i, T_i) - K]^+$ at each T_{i+1} (payment in arrear) resp. $P(T_i, T_{i+1})\;\delta_i\;[L_i - K]^+$ at T_i.

Similarly a floor on $L(t, T_i)$ is given by

$$F(K, \mathcal{T}) = \Big\{\delta_i\;[K - L_i]^+, \quad i = 0 \ldots n-1\Big\}.$$

A swap at rate K is a sequence of payments

$$\mathit{Swap}\;(K, \mathcal{T}) = \Big\{\delta_i\;(L_i - K), \quad i = 0 \ldots n-1\Big\}$$

at each T_{i+1} (arrear swap).

Let $C_t(K),\ F_t(K),\ S_t(K)$ denote the price of these instruments at time $t \le T_0$.

The relation $a - b = [a-b]^+ - [b-a]^+$ immediately gives the Cap-Floor relation

$$C_t(K) = F_t(K) + S_t(K).$$

With the preceding result on caplets we obtain

$$C_t(K) = \sum_{i=0}^{n-1} \delta_i \; P(t, T_{i+1}) \; \Big(L(t, T_i) \; \Phi(d_1(t, T_i)) - K \; \Phi(d_2(t, T_i)) \Big)$$

$$d_{1,2}(t, T_i) = \frac{1}{\eta(t, T)} \; \Big(\ln \frac{L(t, T)}{K} \pm \frac{1}{2} \; \eta^2(t, T) \Big)$$

$$\eta^2(t, T) = \int_t^T \gamma^2(s, T) \; ds.$$

The price of a (forward) swap at t is given by

$$\begin{aligned} S_t(K) &= \sum_{i=0}^{n-1} \delta_i \; P(t, T_{i+1}) \; L(t, T_i) - \sum_{i=0}^{n-1} \delta_i \; K \; P(t, T_{i+1}) \\ &= P(t, T_0) - \Big(\underbrace{P(t, T_n) + \sum_{i=1}^{n} P(t, T_i) \; K \; (T_i - T_{i-1})}_{\text{Coupon bond with coupon}=K} \Big). \end{aligned}$$

The "forward swap rate" $K = K(t, \mathcal{T})$ is defined by the equation $S_t(K) = 0$. This gives

$$K(t, \mathcal{T}) = (P(t, T_0) - P(t, T_n)) \; \Big(\sum_{i=1}^{n} \delta_{i-1} \; P(t, T_i) \Big)^{-1}.$$

The "*swap rate*" $K(T_0, \mathcal{T})$ is the rate which assigns a zero price to a swap starting at T_0. Hence

$$\begin{aligned} K(T_0, \mathcal{T}) = 0 &\Longleftrightarrow \sum_{i=1}^{n} P(T_0, T_i) \; K \; (T_i - T_{i-1}) + P(T_0, T_n) = 1 \\ &\Longleftrightarrow \mathrm{CB}(K, \mathcal{T}) = 1. \end{aligned}$$

I.e. the swap rate is equal to the coupon rate of a coupon bond (with payment dates $\mathcal{T}$) quoted at "pari".

6

Why Do We Need Itô-Calculus in Finance?

As pointed out in Sect. 2.1, Itô's calculus is a necessary extension of real analysis to cope with functions of unbounded variation. Thus the question, whether we need stochastic calculus in finance, is tantamount to the question: are charts of stock prices, exchange or interest rates, in reality of unbounded variation? Such functions, like the Weierstraß function or a path of the Brownian motion, are pure mathematical constructs. Nobody can draw the graph of such a function, and even a computer can only give an approximate picture [1]. Why should such constructs represent what happens on the exchange markets?

The answer is given by contradiction. Let us **assume** that stock price movements **are in reality** of **finite** variation. Then clever people could make huge arbitrage profits by generating options, which are traded in the market at high premiums, at almost zero costs. Clearly a contradiction to reality!

This chapter requires some deeper results of stochastic analysis, like local times and generalized Itô formulas, taken from Carr-Jarrow (1990) and Revuz-Yor (1991). Section 6.4 should be of special interest to economists, since it presents another view on option pricing by means of Arrow-Debreu prices for contingent claims.

[1] A colleague once made the self-ironic remark: the older you get, the better you become in drawing paths of the Brownian motion.

6.1 The Buy-Sell-Paradox

Literature: Carr-Jarrow (1990)

Let (X_t) be a security price process in an economy with a non-stochastic interest rate $r(t)$ and bond price

$$P_t(T) = \exp\Big\{ - \int_t^T r(s)\, ds \Big\} \quad \text{as numeraire.}$$

$$\Longrightarrow F_t = \frac{X_t}{P_t(T)} \qquad T\text{-forward price.}$$

For a call $C_T = (X_T - K)^+$ consider the following hedging strategy

$$\phi_t(\omega) = \mathbf{1}_{\{F_t(\omega) > K\}} = \mathbf{1}_{\{X_t(\omega) > P_t(T)\cdot K\}}$$

This strategy is realized as follows: buy one share of the stock when up-crossing the strike price K, sell it when down-crossing the strike (see Fig. 6.1).

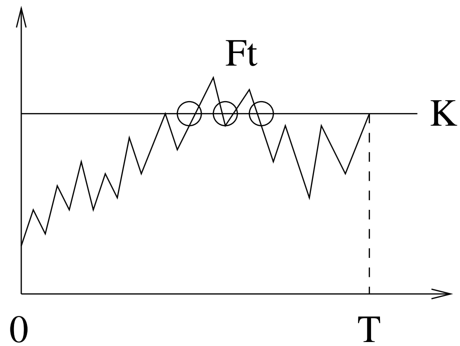


Fig. 6.1. Buy-Sell strategy

$$\begin{aligned} \Longrightarrow V_t(\phi) &= \mathbf{1}_{\{F_t > K\}} \cdot F_t - \mathbf{1}_{\{F_t > K\}} \cdot K \quad \$ \text{ at } T \\ &= \max\{F_t - K, 0\} = [F_t - K]^+ \end{aligned}$$

$$\Longrightarrow V_T(\phi) = [F_T - K]^+ = [X_T - K]^+ = C_T$$

The hedging strategy ϕ is "apparently" self-financing and generates C_T. Thus it follows:

$C_0^T = V_0(\phi) = [F_0 - K]^+$ initial investment in \$ deliverable at T

$\Longrightarrow C_0 = P_0(T) \cdot \underbrace{V_0(\phi)}_{C_0^T} = [X_0 - K \cdot P_0(T)]^+$ price in \$-today .

Thus for $X_0 \leq K \cdot P_0(T) \Longrightarrow$ **the option at time** $t = 0$ **has price zero.**

This is paradoxical, since also "out-of-the-money" options have positive prices. To solve this paradox we need the concept of *local times*, which is studied in the next two sections.

6.2 Local Times and Generalized Itô Formula

Literature: Revuz-Yor (1991), Chap. 6

Let $B_t(\omega)$ be a the path of the (1-dimensional) Brownian motion.

$I = [a, b] \subset \mathbb{R}$; λ Lebesgue measure on $\mathbb{R}$.

Definition 6.2.1.

$$\Gamma_t(I, \omega) := \int_0^t \mathbf{1}_I(B_s(\omega))\, ds = \lambda\{s \leq t : B_s(\omega) \in I\}$$

is called the <u>*"occupation time in I till time t"*</u> *(see Fig. 6.2 on p.116).*

For all (t, ω), $\Gamma_t(\cdot, \omega)$ defines a measure on the Borel sets $\mathcal{B}$ in $\mathbb{R}$.

$A \in \mathcal{B}$ $\lambda(A) = 0 \Longrightarrow \Gamma_t(A) = 0$ P-a.s.

$\Longrightarrow \Gamma_t(\cdot, \omega)$ has density $L_t(x, \omega)$ w.r.t. the Lebesgue measure λ, given by

$$
\begin{aligned}
L_t(x,\omega) &= \lim_{\epsilon\downarrow 0} \frac{1}{2\epsilon}\ \Gamma_t(I_x^\epsilon,\omega) = \lim_{\epsilon\downarrow 0} \frac{1}{2\epsilon}\ \lambda\{s \le t : |B_s(\omega) - x| \le \epsilon\} \\
&= \underline{\text{“local time in } x \text{ till } t\text{”}}
\end{aligned}
$$

where $I_x^\epsilon = [x - \epsilon, x + \epsilon]$.

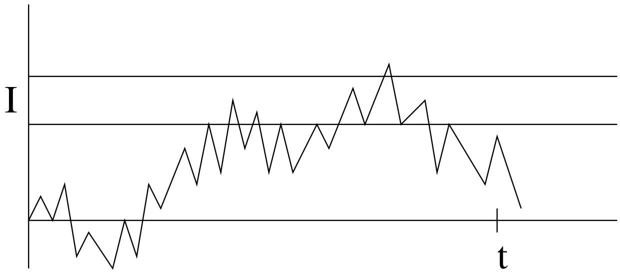


Fig. 6.2. Occupation time of the path in interval I

Consequence: $g : \mathbb{R} \longrightarrow \mathbb{R}$ measurable

$$
\Longrightarrow \int_0^t g(B_s(\omega))\ ds = \int_{-\infty}^{\infty} g(x)\ L_t(x,\omega)\ dx. \qquad (1)
$$

Proof. As in Prop. 1.3.1 it suffices to consider, for any $A \in \mathcal{B}$, the characteristic function $g = \mathbf{1}_A$

$$
\int_0^t \mathbf{1}_A(B_s)\ ds = \Gamma_t(A,\omega) = \int_{\mathbb{R}} \mathbf{1}_A(x) \cdot L_t(x)\ dx.
$$

□

Generalized Itô formula for convex functions

Let $F : \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function, i.e.,

$F(tx + (1-t)y) \le t\ F(x) + (1-t)\ F(y)$ for $0 \le t \le 1$

(e.g. $F(x) = [x - a]^+$).

Convexity $\Longrightarrow F$ is continuous, a.s. differentiable, and one has

$$F'_-(x) = \lim_{h\downarrow 0} \frac{F(x) - F(x-h)}{h} \quad , \quad \text{the left derivative of } F\text{, exists } \forall\, x$$

$F'_- \in \mathrm{FV} \Longrightarrow F'_-$ defines a measure μ on $(\mathbb{R}, \mathcal{B})$ by

$$\mu[a,b) = \int\limits_{[a,b)} dF'_- = F'_-(b) - F'_-(a).$$

Notation: $\mu = F''_- \cdot \lambda$ (2nd derivative measure of F)

Remark 6.1. $F \in C^2 \Longrightarrow \mu = F'' \cdot \lambda$. *For one has*

$$\int\limits_a^b F''(x)\, dx = F'_-(b) - F'_-(a) =: \int\limits_{[a,b)} \mu(dx).$$

For $F(x) = [x-a]^+$ one has $F'_-(x) = \begin{cases} 0 & x \le a \\ 1 & x > a \end{cases}$

Thus for $g : R \longrightarrow R$ it follows

$$\int\limits_{\mathbb{R}} g(x)\, \mu(dx) = \int\limits_{\mathbb{R}} g(x)\, dF'_-(x) = g(a) \text{ ,i.e. } dF'_- = \delta_a \text{ Dirac-measure in } a.$$

Theorem 6.2.2. (Generalized Itô Formula for convex functions)
Let $F : \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function. Then one has

$$F(B_t) = F(B_0) + \int\limits_0^t F'_-(B_s)\, dB_s + \frac{1}{2} \int\limits_{-\infty}^{\infty} L_t(x)\, \mu(dx)$$

where $\mu = F''_- \cdot \lambda$ *(2nd derivative measure of F) .*

Remark 6.2. $F \in C^2 \longrightarrow F'_- = F'$, $\mu(dx) = F''(x)\, dx$

$$\int\limits_{-\infty}^{\infty} L_t(x)\, F''(x)\, dx \underset{(1)}{=} \int\limits_0^t F''(B_s)\, ds$$

$\Longrightarrow$ *"classical" Itô formula.*

Generalization to Semi-Martingales

Let X be a continuous semimartingale, $A \subset \mathbb{R}$ a Borel set.

Definition 6.2.3. $\Gamma_t(A) = \int_0^t \mathbf{1}_A(X_s) \underbrace{d\langle X\rangle_s}_{\text{interior clock}}$
is called the *"occupation time of X in A till t"* .

Clearly $\Gamma_t(A)$ is a continuous process with monotonically increasing paths.

Proposition 6.2.4. *There exits a family of continuous, adapted, monotonically increasing processes* $(L_t^a)_{t\geq 0, a\in\mathbb{R}}$ *with*

$$L_t^a = \lim_{\epsilon\downarrow 0} \frac{1}{2\epsilon} \int_0^t \mathbf{1}_{\{|X_s - a|\leq\epsilon\}} \, d\langle X\rangle_s$$

and

$$\Gamma_t(A) = \int_A L_t^a \, da. \tag{2}$$

$L_t^a = L_t^a(X)(\omega)$ *is called the* *"local time of X in a till time t"* .

The above proposition implies:

(2) $\Longrightarrow \int_0^t g(X_s) \; d\langle X\rangle_s \; = \; \int_{-\infty}^{\infty} g(a) \; L_t^a \; da$ for any real measurable function g .

Corollary 6.2.5. $g \equiv 1 \underset{(2)}{\Longrightarrow} \langle X\rangle_t = \int_0^t d\langle X\rangle_s = \int_{-\infty}^{\infty} L_t^a(X) \, da.$

$\Longrightarrow X \in \mathrm{FV} \Longrightarrow L_t(X) \equiv 0.$

Let $U_t^\epsilon(a,\omega)$ denote the number of up-crossings by path $X_t(\omega)$ of the interval $[a, a+\epsilon]$ (see Fig. 6.3 on page 119).

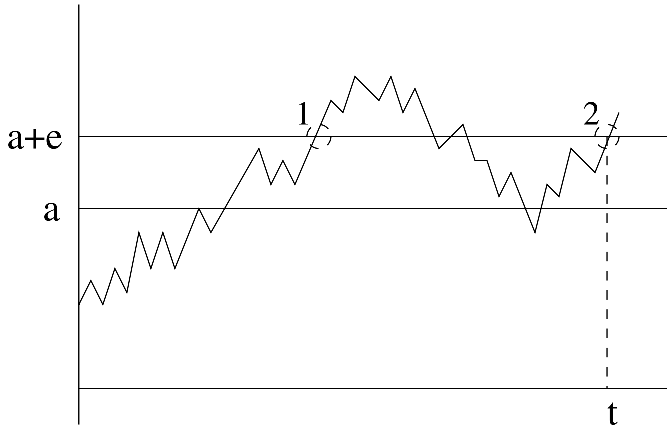


Fig. 6.3. $U_t^\epsilon(a,\omega) = \#$ up-crossings from $X_t(\omega)$ of the interval $[a, a+\epsilon]$

Proposition 6.2.6. (El-Karoui)

$$\lim_{\epsilon \downarrow 0} 2\epsilon\, U_t^\epsilon(a,\omega) = L_t^a(\omega) \quad P\text{-a.s.}$$

Consequence: $L_t^a(\omega) > 0 \Longrightarrow$ the path $X(\omega)$ crosses the a-line till time t infinitely often.

Theorem 6.2.7. (Itô-Tanaka formula)
Let X be a continuous semimartingale, F a real convex function. Then one has

$$F(X_t) = F(X_0) + \int_0^t F'_-(X_s)\, dX_s + \frac{1}{2} \int_{-\infty}^{\infty} L_t^a(X)\, F''_-(da)$$

with $F''_-(da)$ 2nd derivative measure of F (Lebesgue-Stieltjes integral).

Remark 6.3. *For $F \in C^2$ it follows*

$$F(X_t) = F(X_0) + \int_0^t F'(X_s)\, dX_s + \frac{1}{2} \underbrace{\int_{\mathbb{R}} L_t^a(X)\, F''(a)\, da}_{\underset{(Rem. 6.2)}{=} \int_0^t F''(X_s)\, d\langle X\rangle_s}$$

6.3 Solution of the Buy-Sell-Paradox

$g(x) = (x-K)^+$ is a convex function. Thus according to the Itô-Tanaka formula :

$$g(F_T) = (F_T-K)^+ = (F_0-K)^+ + \underbrace{\int_0^T g'_-(F_s)\, dF_s}_{\substack{= \int_0^T dV(\phi_s) \\ P_T\text{-Mart.} \\ \text{(self-financing)}}} + \underbrace{\frac{1}{2}\int_{-\infty}^{\infty} L_t^a(F)\, \underbrace{g''_-(da)}_{\delta_K(da)}}_{\substack{= \frac{1}{2} L_T^K(F) \\ = \lim_{\epsilon\downarrow 0} \epsilon \cdot U_T^\epsilon(K) \\ \text{(transaction costs)}}}$$

But for a process (F_t) of infinite variation, its local time L_T^K becomes positive, once it crosses the barrier K. Hence the "buy-sell" strategy is <u>not self-financing</u> !

The transaction costs $\frac{1}{2}\ L_T^K(F)$, which by Prop. 6.2.6 are equal to $\lim_{\epsilon\downarrow 0} \epsilon \cdot U_T^\epsilon(K)$, have a nice interpretation:

Assume that one tries to apply the Buy-Sell strategy in order to hedge the payoff $g(F_T) = (F_T - K)^+$, i.e., buy the stock at price K when up-crossing the barrier K, sell it again when down-crossing the barrier. But you cannot sell it at the same price. You need a so-called "cutout", you can place only limit orders of the form: buy at K, sell at $K - \varepsilon$ for some $\varepsilon > 0$. The smaller you choose ε, the more cutouts you will face, and in the limit the sum of these cutouts is just equal to the transaction costs (see Fig. 6.4 on p. 121) .

Taking the expectation under the forward measure P_T, it follows:

$$F_T \in \mathcal{M}(P_T) \Longrightarrow C_0^T = \mathrm{E}_T[(F_T - K)^+] = \underbrace{(F_0 - K)^+}_{\text{IV}} + \underbrace{\mathrm{E}_T\left[\frac{1}{2}\ L_T^K(F)\right]}_{\text{TV in } \$_{\text{tomorrow}}}$$

$$\Longrightarrow C_0 = P_0(T) \cdot C_0^T = \underbrace{(X_0 - K \cdot P_0(T))^+}_{\text{PIV}} + \underbrace{P_0(T)\ \mathrm{E}_T\left[\frac{1}{2}\ L_T^K\right] \quad \text{in } \$_{\text{today}}}_{\text{PV of expected local time}}$$

Here 'IV' stands for "interior value" and 'TV' for "time value". The prefix 'P' means "present".

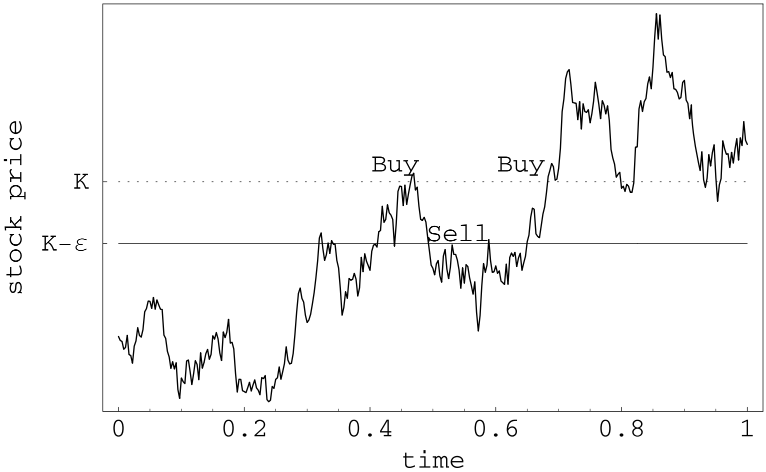


Fig. 6.4. Transaction costs caused by 'cutouts' ϵ

Remark 6.4. $(X_t) \in \mathrm{FV} \Longrightarrow L_T^K \equiv 0 \Longrightarrow \ TV = 0$
$\Longrightarrow$ *out-of-the money options are worthless*

However, this contradicts what we observe on the option markets which attach a positive time value also to out-of-the-money options. Hence it follows:

Consequence: $\Longrightarrow (X_t)$ is of infinite variation !

$\Longrightarrow$ Itô is a '**must**' for Option Pricing

6.4 Arrow-Debreu Prices in Finance

Let $Z_t = (X_t, Y_t)_{0 \leq t \leq T}$ be two security price processes with Y_t as numeraire.

1) $Y_t = \beta_t = e^{rt}$ numeraire: \$-today

2) $Y_t = P_t(T) = e^{-r(T-t)}$ numeraire: \$-tomorrow

3) $Y_t \equiv 1 (\Longleftrightarrow r = 0) \quad \Longrightarrow$ \$-today $\equiv$ \$-tomorrow

W.l.o.g. we can always assume that 3) holds by using the following transformations:

$$\widehat{Z}_t = \left(\widehat{X}_t = \frac{X_t}{Y_t}, 1\right) \quad ; \quad \widehat{H} = \frac{H}{Y_T} \in \mathcal{F}_T \quad \text{contingent claim}$$

$$\widehat{V}_t(\phi) = \frac{V_t(\phi)}{Y_t} \quad , \quad \widehat{H}_t = \mathrm{E}^*[\widehat{H}|\mathcal{F}_t]$$

$$H_t = \widehat{H}_t \cdot Y_t \quad \text{back-transformation.}$$

Consider the following special contingent claim $H_t(x) \in \mathcal{F}_t$

$$H_t(x) = \mathbf{1}_{\{X_t \le x\}} = \begin{cases} 1 & X_t \le x \\ 0 & X_t > x \end{cases}$$

(= **Arrow-Debreu security** contingent on value of X at time t).

AD-price at $t = 0$:

$$\mathrm{AD}(t,x) = \mathrm{E}^*[H_t(x)] = P^*[X_t \le x] = \text{ distribution of } X_t \text{ under } P^*.$$

$\mathrm{AD}(t,x)$ has density $f(t,x)$ defined by $f(t,x) = \lim\limits_{\epsilon \downarrow 0} \frac{1}{2\epsilon}\, \mathrm{E}^*[\mathbf{1}_{\{|X_t - x| \le \epsilon\}}]$, or

$$\begin{aligned} f(t,x)\, dx &= \mathrm{E}^*[\mathbf{1}_{I_x}(X_t)] \quad \text{with } I_x = (x - \frac{1}{2}\, dx\ ,\ x + \frac{1}{2}\, dx) \\ &= \text{density of AD-prices, given } X_t = x. \end{aligned}$$

Let $C(X_t)$ be a contingent claim on X_t. According to Prop. 1.3.1

$$\Longrightarrow \pi_0(C_t) = E^*[C(X_t)] = \int_{-\infty}^{\infty} C(x)\, dP_X^*(x) = \int_{-\infty}^{\infty} C(x)\, f(t,x)\, dx.$$

Example: $C(t,K) = (X_t - K)^+$ Call
$P(t,K) = (K - X_t)^+$ Put

By partial integration one obtains:

$$\pi_0(C) = \int_K^{\infty} (x-K)\, f(t,x)\, dx = \int_K^{\infty} (1 - \mathrm{AD}(t,x))\, dx$$

$$\pi_0(P) = \int_0^K (K-x)\, f(t,x)\, dx = \int_0^K \mathrm{AD}(t,x)\, dx. \qquad (3)$$

Consequence: $\mathrm{AD}(t,K) = \dfrac{d}{dK}\,\pi_0(P(t,K))$

$$f(t,K) = \frac{\partial^2}{\partial K^2}\,\pi_0(P(t,K)) = \frac{\partial^2}{\partial K^2}\,\pi_0(C(t,K))\ .$$

Interpretation of (3): Payoff at t

$$\int_0^K H_t(x)\,dx = \int_0^K \mathbf{1}_{\{X_t \le x\}} dx = \lambda\{0 \le x \le K : X_t \le x\}$$

$$= \begin{cases} K - X_t & X_t \le K \\ 0 & X_t > K. \end{cases}$$

Exercise: For the standard Black-Scholes model (μ, σ constant, $r = 0$) one has (compare Sect. 2.9):

$\mathrm{AD}(t,x) = 1 - \Phi(h(t,x)) = \Phi(-h(t,x))$ with density $f(t,x) = \dfrac{\varphi(h)}{x\,\sigma\sqrt{t}}$,

where $\Phi(\cdot)$ is the distribution function of $\mathrm{N}(0,1)$, $\varphi = \Phi'$, *and*

$h(t,x) = \frac{\ln(X_0/x)}{\sigma\sqrt{t}} - \frac{1}{2}\,\sigma\sqrt{t}.$

6.5 The Time Value of an Option as Expected Local Time

As shown in Sect. 6.2, the local time of the process X_t in K till maturity T is given by

$$L_T^K(\omega) = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_0^T \mathbf{1}_{\{|X_t(\omega) - K| \le \epsilon\}}\, d\langle X(\omega)\rangle_t.$$

Under suitable boundary conditions on $\mu_X(t), \sigma_X(t)$ one may commute limit and integration to obtain

$$\Longrightarrow \frac{1}{2}\,\mathrm{E}^*[L_T^K] = \frac{1}{2}\,\mathrm{E}^*\Big[\int_0^T \lim_{\epsilon} \frac{1}{2\epsilon}\,\mathbf{1}_{\{|X_t-K|\le\epsilon\}}\,d\langle X\rangle_t\Big]$$

$$= \frac{1}{2}\int_0^T f(t,K)\,d\langle X_t\rangle_{X_t=K}$$

$$= \frac{1}{2}\cdot\{\text{ AD-price of the quadr. variation of } X \text{ in } K \text{ till } T\},$$

which gives a new interpretation of the time value of an option.

For the standard Black-Scholes model one has:

$$d\langle X_t\rangle_{X_t=K} = \sigma^2\,K^2\,dt \qquad \text{and} \qquad f(t,K) = \frac{\varphi(h(t,K))}{K\,\sigma\sqrt{t}}$$

(see Exercise in Sect. 6.4). Hence it follows:

$$\Longrightarrow \frac{1}{2}\,\mathrm{E}^*[L_K^T] = \frac{1}{2}\,K^2\,\sigma^2\int_0^T f(t,K)\,dt = \frac{1}{2}\,K^2\sigma^2\int_0^T \frac{\varphi(h(t,K))}{K\cdot\sigma\sqrt{t}}\,dt$$

$$= \frac{1}{2}\,K\,\sigma\int_0^t \varphi(h(t,K))\cdot\frac{1}{\sqrt{t}}\,dt$$

Hence by Itô-Tanaka for $C_T = (X_T - K)^+$

$$C_0 = \mathrm{E}^*[(X_T-K)^+] = (X_0-K)^+ + \frac{1}{2}\,K\,\sigma\int_0^T \varphi(h(t,K))\cdot\frac{1}{\sqrt{t}}\,dt$$

(alternative BS-formula for forward price $X_t = F_t$ $(r = 0)$)

7

Appendix: Itô Calculus Without Probabilities

Séminaire de Probabilités XV 1979/80

ITO CALCULUS WITHOUT PROBABILITIES

by H. Föllmer

The aim of this note is to show that the Itô calculus can be developed "path by path" in the strict meaning of this term. We will derive Itô's formula as an exercise in analysis for a class of real functions of quadratic variation, including the construction of the stochastic integral $\int F'(X_{s-})dX_s$, by means of Riemann sums. Only afterwards we shall speak of probabilities in order to verify that for certain stochastic processes (semimartingales, processes of finite energy,...) almost all paths belong to this class.

Let x be a real function on $[0,\infty[$ which is right continuous and has left limits (also called *càdlàg*). We will use the following notation: $x_t = x(t)$, $\triangle x_t = x_t - x_{t-}$, $\triangle x_t^2 = (\triangle x_t)^2$.

We define a *subdivision* to be any finite sequence $\tau = (t_o, \cdots, t_k)$ such that $0 \leq t_o < \cdots < t_k < \infty$, and we put $t_{k+1} = \infty$ and $x_\infty = 0$. Let $(\tau_n)_{n=1,2,\cdots}$ be a sequence of subdivisions whose meshes converge to 0 on each compact interval. We say that x is of *quadratic variation along* (τ_n) if the discrete measures

$$\xi_n = \sum_{t_i \in \tau_n} (x_{t_{i+1}} - x_{t_i})^2 \varepsilon_{t_i} \tag{1}$$

converge weakly to a Radon measure ξ on $[0, \infty[$ whose atomic part is given by the quadratic jumps of x:

$$[x, x]_t = [x, x]_t^c + \sum_{s \le t} \triangle x_s^2, \tag{2}$$

where $[x, x]$ denotes the distribution function of ξ and $[x, x]^c$ its continuous part.

Theorem. *Let x be of quadratic variation along (τ_n) and F a function of class C^2 on $\mathbb{R}$. Then the Itô formula*

$$F(x_t) = F(x_o) + \int_0^t F(x_{s-}) dx_s + \frac{1}{2} \int_{]0,t]} F''(x_{s-}) d[x, x]_s \tag{3}$$

$$+ \sum_{s \le t} [F(x_s) - F(x_{s-}) - F'(x_{s-}) \triangle x_s - \frac{1}{2} F''(x_{s-}) \triangle x_s^2],$$

holds with

$$\int_0^t F'(x_{s-}) dx_s = \lim_n \sum_{\tau_n \ni t_i \le t} F'(x_{t_i})(x_{t_{i+1}} - x_{t_i}), \tag{4}$$

and the series in (4) is absolutely convergent.

Remark. Due to (2) the last two terms of (3) can be written as

$$\frac{1}{2} \int_0^t F''(x_{s-}) d[x, x]_s^c + \sum_{s \le t} [F(x_s) - F(x_{s-}) - F'(x_{s-}) \triangle x_s], \tag{5}$$

and we have

$$\int_0^t F''(x_{s-}) d[x, x]_s^c = \int_0^t F''(x_s) d[x, x]_s^c, \tag{6}$$

since x is a *càdlàg* function.

Proof. Let $t > 0$. Since x is right continuous we have

$$F(x_t) - F(x_o) = \lim_n \sum_{\tau_n \ni t_i \le t} [F(x_{t_{i+1}}) - F(x_{t_i})].$$

1) For the sake of clarity we first treat the particularly simple case where x is a continuous function. Taylor's formula can be written as

$$\sum_{\tau_n \ni t_i \le t} [F(x_{t_{i+1}}) - F(x_{t_i})] = \sum F'(x_{t_i})(x_{t_{i+1}} - x_{t_i}) + \frac{1}{2} \sum F''(x_{t_i})(x_{t_{i+1}} - x_{t_i})^2 + \sum r(x_{t_i}, x_{t_{i+1}}),$$

where

$$r(a,b) \le \varphi(|b-a|)(b-a)^2, \tag{7}$$

and where $\varphi(\cdot)$ is an increasing function on $[0, \infty[$ such that $\varphi(c) \to 0$ for $c \to 0\cdot$. For $n \uparrow \infty$ the second sum of the right hand side converges to

$$\frac{1}{2} \int_{[0,t]} F''(x_s) d[x,x]_s = \frac{1}{2} \int_{]0,t]} F''(x_{s-}) d[x,x]_s$$

due to the weak convergence of the discrete measures (ξ_n); note that by (2) the continuity of x implies the continuity of $[x,x]$. The third sum, which is dominated by

$$\varphi(\max_{\tau_n \ni t_i \le t} |x_{t_{i+1}} - x_{t_i}|) \sum_{\tau_n \ni t_i \le t} (x_{t_{i+1}} - x_{t_i})^2,$$

converges to 0 since x is continuous. Thus one obtains the existence of the limit (4) and Itô's formula (3).

2) Consider now the general case. Let $\varepsilon > 0$. We divide the jumps of x on $[0,t]$ into two classes: a finite class $C_1 = C_1(\varepsilon, t)$, and a class $C_2 = C_2(\varepsilon, t)$ such that $\sum_{s \in C_2} \triangle x_s^2 \leq \varepsilon^2$. Let us write

$$\sum_{\tau_n \ni t_i \leq t} [F(x_{t_{i+1}}) - F(x_{t_i})] = \sum{}_1 [F(x_{t_{i+1}} - F(x_{t_i})] + \sum{}_2 [F(x_{t_{i+1}} - F(x_{t_i})]$$

where $\sum_1$ indicates the summation over those $t_i \in \tau_n$ with $t_i \leq t$ for which the interval $]t_i, t_{i+1}]$ contains a jump of class C_1. We have

$$\lim_n \sum{}_1 [F(x_{t_{i+1}}) - F(x_{t_i})] = \sum_{s \in C_1} [F(x_s) - F(x_{s-})].$$

On the other hand, Taylor's formula allows us to write

$$\sum{}_2 [F(x_{t_{i+1}}) - F(x_{t_i})] =$$

$$\sum_{\tau_n \ni t_i \leq t} F'(x_{t_i})(x_{t_{i+1}} - x_{t_i}) + \frac{1}{2} \sum_{\tau_n \ni t_i \leq t} F''(x_{t_i})(x_{t_{i+1}} - x_{t_i})^2$$

$$- \sum{}_1 [F'(x_{t_i})(x_{t_{i+1}} - x_{t_i}) + \frac{1}{2} F''(x_{t_i})(x_{t_{i+1}} - x_{t_i})^2] + \sum{}_2 r(x_{t_i}, x_{t_{i+1}})$$

We will show below that the second sum on the right hand side converges to

$$\frac{1}{2} \int_{]0,t]} F''(x_{s-})[x,x]_s,$$

as $n \uparrow \infty$; see (9). The third sum converges to

$$\sum_{s \in C_1} [F'(x_{s-}) \triangle x_s + \frac{1}{2} F''(x_{s-}) \triangle x_s^2].$$

Due to the uniform continuity of F'' on the bounded set of values x_s $(0 \leq s \leq t)$ we can assume (7), and this implies

$$\limsup_n \sum{}_2 r(x_{t_i}, x_{t_{i+1}}) \leq \varphi(\varepsilon+)[x,x]_{t+}. \tag{8}$$

Let ε converge to 0. Then (8) converges to 0, and

$$\sum_{s\in C_1(\varepsilon,t)} [F(x_s) - F(x_{s-}) - F'(x_{s-})\triangle x_s] - \frac{1}{2}\sum_{s\in C_1(\varepsilon,t)} F''(x_{s-})\triangle x_s^2$$

converges to the series in (3). Furthermore the series converges absolutely since

$$\sum_{s\leq t} |F(x_s) - F(x_{s-}) - F'(x_{s-})\triangle x_s| \leq \text{ const } \sum_{s\leq t} \triangle x_s^2$$

by Taylor's formula. Thus we obtain the existence of the limit in (4) and Itô's formula (3).

3) Let us show that

$$\lim_n \sum_{\tau_n \ni t_i \leq t} f(x_{t_i})(x_{t_{i+1}} - x_{t_i})^2 = \int_{]0,t]} f(x_{s-})d[x,x]_s \tag{9}$$

for any continuous function f on $\mathbb{R}$. Let $\varepsilon > 0$, and denote by z the distribution function of the jumps in class $C_1 = C_1(\varepsilon, t)$, i. e.,

$$z_u = \sum_{C_1 \ni s \leq u} \triangle x_s \quad (u \geq 0).$$

We have

$$\lim_n \sum_{\tau_n \ni t_i \leq u} f(x_{t_i})(z_{t_{i+1}} - z_{t_i})^2 = \sum_{C_1 \ni s \leq u} f(x_{s-})\triangle x_s^2 \tag{10}$$

for each $u \geq 0$. Denote by ζ_n and η_n the discrete measures associated with z and $y = x - z$ in the sense of (1). By (10) the measures ζ_n converge weakly to the discrete measure

$$\zeta = \sum_{s\in C_1} \triangle x_s^2 \varepsilon_s.$$

Since the last sum of

$$\sum_{\tau_n \ni t_i \leq u} (x_{t_{i+1}} - x_{t_i})^2 = \sum (y_{t_{i+1}} - y_{t_i})^2 + \sum (z_{t_{i+1}} - z_{t_i})^2$$

$$+2\sum (y_{t_{i+1}} - y_{t_i})(z_{t_{i+1}} - z_{t_i})$$

converges to 0, the measures η_n converge weakly to the measure $\eta = \xi - \zeta$ whose atomic part has total mass $\leq \varepsilon^2$. Hence the function $f \circ x$ is almost surely continuous with respect to the continuous part of η, and this implies

$$\limsup_n \Big| \sum_{\tau_n \ni t_i \leq t} f(x_{t_i})(y_{t_{i+1}} - y_{t_i})^2 - \int_{]0,t]} f(x_{s-}) d\eta \Big| \leq 2 \parallel f \parallel_t \varepsilon^2 \quad (11)$$

where $\parallel f \parallel_t = \sup\{f(x_s); 0 \leq s \leq t\}$. Combining (10) and (11) we obtain (9), and this completes the proof. Let us emphasize that we have followed closely the "classical" argument; see Meyer [4]. The only new contribution is the use of *weak convergence*, which allows us to give a completely analytic version.

Remarks.

1) Let $x = (x^1, \cdots, x^n)$ be a *càdlàg* function on $[0, \infty[$ with values in $\mathbb{R}^n$. We say that x a *is of quadratic variation along* (τ_n) if this holds for all real functions $x^i, x^i + x^j$ $(1 \leq i, j \leq n)$. In this case we put

$$[x^i, x^j]_t = \frac{1}{2}([x^i + x^j, x^i + x^j]_t - [x^i, x^i]_t - [x^j, x^j]_t)$$
$$= [x^i, x^j]^c_t + \sum_{s \leq t} \triangle x^i_s \triangle x^j_s.$$

Then we have the Itô formula

$$F(x_t) = F(x_o) + \int_0^t DF(x_{s-}) dx_s + \frac{1}{2} \sum_{i,j} \int_0^t D_i D_j F(x_{s-}) d[x^i, x^j]^c_s$$
$$+ \sum_{s \leq t} [F(x_s) - F(x_{s-}) - \sum_i D_i F(x_{s-}) \triangle x^i_s] \quad (12)$$

for any function F of class C^2 on $\mathbb{R}^n$, where

$$\int_0^t DF(x_{s-})dx_s = \lim_n \sum_{\tau_n \ni t_i \leq t} < DF(x_{t_i}), x_{t_{i+1}} - x_{t_i} > \qquad (13)$$

($< \cdot, \cdot >=$ scalar product on $\mathbb{R}^n$). The proof is the same as above, but with more cumbersome notation.

2) The class of functions of quadratic variation is *stable with respect to* C^1 *- operations*. More precisely, if $x = (x^1, \cdots, x^n)$ is of quadratic variation along (τ_n) and F a continuously differentiable function on $\mathbb{R}^n$ then $y = F \circ x$ is of quadratic variation along (τ_n), with

$$[y, y]_t = \sum_{i,j} \int_0^t D_i F(x_s) D_j F(x_s) d[x^i, x^j]_s^c + \sum_{s \leq t} \triangle y_s^2. \qquad (14)$$

This is the analytic version of a result of Meyer for semimartingales, see [4] p. 359. The proof is analogous to the previous one.
Let us now turn to stochastic processes. Let $(X_t)_{t \geq 0}$ be a semimartingale. Then, for any $t \geq 0$, the sums

$$S_{\tau,t} = \sum_{\tau \ni t_i \leq t} (X_{t_{i+1}} - X_{t_i})^2 \qquad (15)$$

converge in probability to

$$[X, X]_t =< X^c, X^c >_t + \sum_{s \leq t} \triangle X_s^2$$

when the mesh of the subdivision τ converges to 0 on $[0, t]$; see Meyer [4] p. 358. For each sequence there exists thus a subsequence (τ_n) such that, almost surely,

$$\lim_n S_{\tau_n,t} = [X, X]_t \qquad (16)$$

for each rational t . This implies that *almost all paths are of quadratic variation along* (τ_n). Furthermore the relation (16) is valid for all $t \geq 0$ due to (9). The Itô formula (3), applied strictly pathwise, does not depend on the sequence (τ_n). In particular, we obtain the convergence in probability of the Riemann sums in (4) to the stochastic integral

$$\int_0^t F'(X_{s-})dX_s,$$

when the mesh of τ goes to 0 on $[0,t]$.

Remarks.

1) For *Brownian motion* and an arbitrary sequence of subdivisions (τ_n) with mesh tending to 0 on each compact interval, almost all paths are of quadratic variation along (τ_n). Indeed, by Lévy's theorem we have (16) without passing to subsequences.

2) For the above argument it suffices to know that the sums (15) converge in probability to an increasing process $[X,X]$ which has paths of the form (2). The class of *processes of quadratic variation* is clearly larger than the class of semimartingales: Just consider a deterministic process of quadratic variation which is of unbounded variation. Let us mention also the *processes of finite energy* $X = M + A$ where M is a local martingale and A is a process with paths of quadratic variation 0 along the dyadic subdivisions. These processes occur in the probabilistic study of Dirichlet spaces: see Fukushima [3].

3) For a semimartingale it is known how to construct the stochastic integral $\int H_{s-}dX_s$ (H càdlàg and adapted) pathwise as a limit of Riemann sums, in the sense that the sums converge almost surely outside an exceptional set which depends on H; see Bichteler [1]. We have just shown that for the particular needs of Itô calculus, where $H = f \circ X$ (f of class C^1), the exceptional set can be chosen in advance, independently of H. It is possible to go beyond the class C^1 by treating local times "path by path". But not too far beyond: Stricker [5] has just shown that an extension to continuous functions is only possible for processes with paths of finite variation.

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