A Survey of Stochastic Portfolio Theory

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Synopsis

The purpose of these lectures is to offer an overview of `Stochastic Portfolio Theory`, a rich and flexible framework for analyzing portfolio behavior and equity market structure. This theory was developed in the book by E.R. Fernholz (*Stochastic Portfolio Theory*, Springer 2002) and was studied further in the papers Fernholz (*Journal of Mathematical Economics*, 1999; *Finance & Stochastics*, 2001), Fernholz, Karatzas & Kardaras [FKK] (*Finance & Stochastics*, 2005), Fernholz & Karatzas [FK] (*Annals of Finance*, 2005), Banner, Fernholz & Karatzas [BFK] (*Annals of Applied Probability*, 2005), and Karatzas & Kardaras (preprint, 2006). It is descriptive as opposed to normative, is consistent with the observable characteristics of actual portfolios, and provides a theoretical tool which is useful for practical applications.

As a theoretical tool, this framework provides fresh insights into questions of market structure and arbitrage, and can be used to construct portfolios with controlled behavior. As a practical tool, Stochastic Portfolio Theory has been applied to the analysis and optimization of portfolio performance and has been the basis of successful investment strategies for close to 20 years.

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1 The basic Model for Portfolios

We shall place ourselves, throughout these lectures, within a model $\mathcal{M}$ for a financial market of the form

$$\mathrm{d}B(t) \;=\; B(t)\,r(t)\,\mathrm{d}t\,, \qquad\qquad B(0)=1\,, \tag{1.1}$$

$$\mathrm{d}S_i(t) \;=\; S_i(t)\left[b_i(t)\mathrm{d}t + \sum_{\nu=1}^{d} \sigma_{i\nu}(t)\mathrm{d}W_\nu(t)\right], \qquad S_i(0)=s_i>0\,,\, i=1,\ldots,m\,,$$

consisting of a money-market $B(\cdot)$ and of m stocks, whose prices $S_1(\cdot),\cdots,S_m(\cdot)$ are driven by the $d-$dimensional Brownian motion $W(\cdot)=\big(W_1(\cdot),\cdots,W_d(\cdot)\big)'$ with $d \geq m$. Contrary to a usual assumption imposed on such models, here it is `not crucial that the filtration` $\mathbb{F}=\{\mathcal{F}_t\}_{0\leq t\leq T}$ `be the one generated by the Brownian motion itself`.

Thus, and until further notice, we shall take $\mathbb{F}$ to contain (possibly strictly) the Brownian filtration $\mathbb{F}^W=\big\{\mathcal{F}^W_t\big\}_{0\leq t\leq T}$ with $\mathcal{F}^W_t := \sigma(W(s),\, 0\leq s\leq t)\,.$

We shall assume that the interest-rate process $r(\cdot)$, the vector-valued process $b(\cdot)=\big(b_1(\cdot),\ldots,b_m(\cdot)\big)'$ of *rates of return*, and the $(m\times d)-$matrix-valued process $\sigma(\cdot)=\big(\sigma_{i\nu}(\cdot)\big)_{1\leq i\leq m,\,1\leq\nu\leq d}$ of *volatilities*, are all $\mathbb{F}-$progressively measurable, and satisfy the integrability conditions $\int_0^T |\,r(t)|\,\mathrm{d}t<\infty$ as well as

$$\sum_{i=1}^{m}\int_0^T\left(\big|\,b_i(t)\big| + \sum_{\nu=1}^{d}\big(\sigma_{i\nu}(t)\big)^2\right)\mathrm{d}t \;<\; \infty \tag{1.2}$$

almost surely, for every $T\in(0,\infty)$. With the notation

$$a_{ij}(t) \;:=\; \sum_{\nu=1}^{d}\sigma_{i\nu}(t)\sigma_{j\nu}(t) \;=\; \Big(\sigma(t)\sigma'(t)\Big)_{ij} \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\,\langle\,\log S_i,\log S_j\,\rangle(t) \tag{1.3}$$

for the non-negative definite matrix-valued *variation-covariation rate* (or simply "variance-covariance") *process* $a(\cdot)=\big(a_{ij}(\cdot)\big)_{1\leq i,j\leq m}$, as well as

$$\underbrace{\gamma_i(t) \;:=\; b_i(t)-\frac{1}{2}a_{ii}(t)}\,, \qquad\qquad i=1,\ldots,m\,, \tag{1.4}$$

we may use Itô's rule to solve (1.1), in the form

$$\underbrace{\mathrm{d}\Big(\log S_i(t)\Big) \;=\; \gamma_i(t)\,\mathrm{d}t\,+\sum_{\nu=1}^{d}\sigma_{i\nu}(t)\,\mathrm{d}W_\nu(t)}\,, \quad i=1,\ldots,m\,, \tag{1.5}$$

or equivalently:

$$S_i(t) \;=\; s_i\cdot\exp\left\{\int_0^t\gamma_i(u)\,\mathrm{d}u\,+\sum_{\nu=1}^{d}\int_0^t\sigma_{i\nu}(u)\,\mathrm{d}W_\nu(u)\right\}, \quad 0\leq t<\infty\,.$$

We shall refer to the quantity of (1.4) as the *rate of growth* of the i^{th} stock, because of the a.s. relationship

$$\lim_{T\to\infty} \frac{1}{T}\left(\log S_i(T) - \int_0^T \gamma_i(t)\,\mathrm{d}t\right) \;=\; 0\,, \qquad i=1,\cdots,m\,, \tag{1.6}$$

valid when the individual stock variances $a_{ii}(\cdot)$ are bounded, uniformly in (t,ω); this follows from the strong law of large numbers and from the representation of (local) martingales as time-changed Brownian motions.

1.1 Definition: Portfolio Rules. A *portfolio* $\pi(\cdot) = \big(\pi_1(\cdot),\ldots,\,\pi_m(\cdot)\big)'$ is an $\mathbb{F}-$progressively measurable process with values in the set

$$\Delta_+^m \;:=\; \Big\{(\pi_1,\ldots,\pi_m)\in\mathbb{R}^m \;\Big|\; \pi_1\ge 0,\ldots,\pi_m\ge 0 \quad\text{and}\quad \sum_{i=1}^m \pi_i = 1\Big\}\,. \tag{1.7}$$

An *extended portfolio* $\pi(\cdot) = \big(\pi_1(\cdot),\ldots,\pi_m(\cdot)\big)'$ is an $\mathbb{F}-$progressively measurable process, bounded uniformly in (t,ω), that takes values in the set $\Delta^m := \big\{(\pi_1,\ldots,\pi_m)\in\mathbb{R}^m \;\big|\; \sum_{i=1}^m \pi_i = 1\,\big\}$. For future reference, let us introduce also the notation $\Delta_{++}^m := \big\{(\pi_1,\ldots,\pi_m)\in\Delta_+^m \mid \pi_1>0,\ldots,\pi_m>0\,\big\}$.

Thus, a portfolio corresponds to a trading strategy that is fully invested at all times in the equity (stock) market, never sells stock short, and never invests in or borrows from the money market. An extended portfolio can sell one or more stocks short (though certainly not all) but is never allowed to borrow from, or invest in, the money market. When the need arises to differentiate portfolios from their extended counterparts, we shall add the adjectives *strict* (or "long-only") for emphasis.

The interpretation is that $\pi_i(t)$ represents the proportion of wealth $V(t)\equiv V^{w,\pi}(t)$ invested at time t in the i^{th} stock, so the quantities

$$h_i(t) \;=\; \pi_i(t)\,V^{w,\pi}(t)\,, \qquad i=1,\cdots,m \tag{1.8}$$

are the dollar amounts invested at any given time t in the individual stocks.

The wealth process $V^{w,\pi}(\cdot)$ that corresponds to an extended portfolio $\pi(\cdot)$ and initial capital $w>0$, satisfies the stochastic equation

$$\begin{aligned}\frac{\mathrm{d}V^{w,\pi}(t)}{V^{w,\pi}(t)} \;&=\; \sum_{i=1}^m \pi_i(t)\cdot\frac{\mathrm{d}S_i(t)}{S_i(t)} \;=\; \pi'(t)\,\big[\,b(t)\,\mathrm{d}t \,+\, \sigma(t)\,\mathrm{d}W(t)\,\big]\\ &=\; b^\pi(t)\mathrm{d}t \,+\, \sum_{\nu=1}^d \sigma_\nu^\pi(t)\,\mathrm{d}W_\nu(t)\,, \quad V(0)=w\,,\end{aligned} \tag{1.9}$$

and

$$b^\pi(t) \;:=\; \sum_{i=1}^m \pi_i(t)b_i(t)\,, \qquad \sigma_\nu^\pi(t) := \sum_{i=1}^m \pi_i(t)\sigma_{i\nu}(t) \tag{1.10}$$

for $\nu = 1, \dots, d$. These quantities are, respectively, the rate-of-return and the volatility coëfficients, that correspond to the portfolio $\pi(\cdot)$.

By analogy with (1.5) we can write the solution of the equation (1.10) as

$$\underbrace{d\Big(\log V^{w,\pi}(t)\Big) \;=\; \gamma^{\pi}(t)\,\mathrm{d}t \,+ \sum_{\nu=1}^{d} \sigma^{\pi}_{\nu}(t)\,\mathrm{d}W_{\nu}(t)}\;, \qquad V^{w,\pi}(0) = w \tag{1.11}$$

or equivalently:

$$V^{w,\pi}(t) \;=\; w \cdot \exp\left\{ \int_0^t \gamma^{\pi}(u)\,\mathrm{d}u \,+ \sum_{\nu=1}^{d} \int_0^t \sigma^{\pi}_{\nu}(u)\,\mathrm{d}W_{\nu}(u) \right\}, \quad 0 \le t \le T\,.$$

Here

$$\underbrace{\gamma^{\pi}(t) \;:=\; \sum_{i=1}^{m} \pi_i(t)\gamma_i(t) + \gamma^{\pi}_{*}(t)} \tag{1.12}$$

is the *growth rate*, and

$$\underbrace{\gamma^{\pi}_{*}(t) \;:=\; \frac{1}{2}\left(\sum_{i=1}^{m} \pi_i(t) a_{ii}(t) - \sum_{i=1}^{m}\sum_{j=1}^{m} \pi_i(t) a_{ij}(t) \pi_j(t) \right)} \tag{1.13}$$

is the **excess growth rate**, of the portfolio $\pi(\cdot)$.

As we shall see in Lemma 3.3 below, for a (strict, "long-only") portfolio rule, this excess growth rate is always non-negative – and is strictly positive for portfolios that do not concentrate their holdings in just one stock.

Again, the terminology "growth rate" is justified by the a.s. property

$$\lim_{T\to\infty} \frac{1}{T}\left(\log V^{w,\pi}(T) - \int_0^T \gamma^{\pi}(t)\,\mathrm{d}t \right) \;=\; 0 \tag{1.14}$$

which is valid, for instance, when all eigenvalues of the variance/covariance matrix-valued process $a(\cdot)$ of (1.3) are bounded away from infinity: i.e., when

$$\xi' a(t)\xi \;=\; \xi'\sigma(t)\sigma'(t)\xi \;\le\; K||\xi||^2\,, \quad \forall \;\; t \in [0,\infty) \quad \text{and} \quad \xi \in \mathbb{R}^m \tag{1.15}$$

holds almost surely, for some $K \in (0,\infty)$. We shall refer to (1.15) as the *uniform boundedness* condition on the volatility structure of $\mathcal{M}$.

Without further comment we shall write $V^{\pi}(\cdot) \equiv V^{1,\pi}(\cdot)$ for initial wealth $w = \$1$. Let us also note the following analogue of (1.11), namely

$$\mathrm{d}\Big(\log V^{\pi}(t)\Big) \;=\; \gamma^{\pi}_{*}(t)\,\mathrm{d}t \,+ \sum_{i=1}^{m} \pi_i(t) \cdot \mathrm{d}\big(\log S_i(t)\big)\,. \tag{1.16}$$

1.2 Definition: We shall use the reverse-order-statistics notation for the weights of an extended portfolio $\pi(\cdot)$, ranked at time t from largest down to smallest:

$$\max_{1\le i\le n} \pi_i(t) \;=:\; \pi_{(1)}(t) \;\ge\; \pi_{(2)}(t) \;\ge\; \ldots \;\ge\; \pi_{(n-1)}(t) \;\ge\; \pi_{(n)}(t) \;:=\; \min_{1\le i\le n} \pi_i(t)\,. \tag{1.17}$$

1.1 General Trading Strategies

For completeness of exposition and for later usage, let us go briefly in this subsection beyond (extended) portfolios and recall the notion of *trading strategies:* these are allowed to invest in (or borrow from) the money market, and to sell stocks short. Formally, they are $\mathbb{F}-$progressively measurable, $\mathbb{R}^m-$valued processes $h(\cdot) = \big(h_1(\cdot), \cdots h_m(\cdot)\big)'$ that satisfy the integrability condition

$$\sum_{i=1}^m \int_0^T \Big(\big|h_i(t)\big| \, \big|b_i(t) - r(r)\big| + h_i^2(t)\, a_{ii}(t) \Big) \mathrm{d}t \;<\; \infty\,, \qquad \text{a.s.}$$

for every $T \in (0,\infty)$. The interpretation is that the real-valued random variable $h_i(t)$ stands for the dollar amount invested by $h(\cdot)$ at time t in the i^{th} stock. If we denote by $\mathcal{V}^{w,h}(t)$ the wealth at time t corresponding to this strategy $h(\cdot)$ and to an initial capital $w > 0$, then $\mathcal{V}^{w,h}(t) - \sum_{i=1}^m h_i(t)$ is the amount invested in the money market, and we have the dynamics

$$\mathrm{d}\mathcal{V}^{w,h}(t) = \left(\mathcal{V}^{w,h}(t) - \sum_{i=1}^m h_i(t) \right) r(t)\,\mathrm{d}t + \sum_{i=1}^m h_i(t) \left\{ b_i(t)\mathrm{d}t + \sum_{\nu=1}^d \sigma_{i\nu}(t)\,\mathrm{d}W_\nu(t) \right\}$$

or equivalently

$$\frac{\mathcal{V}^{w,h}(t)}{B(t)} \;=\; w \;+\; \int_0^t \frac{h'(s)}{B(s)} \;\Big[\big(b(s) - r(s)\mathbb{I}\big)\mathrm{d}s + \sigma(s)\,\mathrm{d}W(s) \Big]\,, \quad 0 \le t \le T\,. \tag{1.18}$$

Here $\mathbb{I} = (1, \cdots, 1)'$ is the $m-$dimensional column vector with 1 in all entries.

As mentioned already, all quantities $h_i(\cdot)\,,\ 1 \le i \le m$ and $\mathcal{V}^{w,h}(t) - h'(\cdot)\mathbb{I}$ are allowed to take negative values. This possibility opens the door to the notorious *doubling strategies* of martingale theory (e.g. Karatzas & Shreve (1998), Chapter 1). In order to rule these out, we shall confine ourselves here to trading strategies $h(\cdot)$ that satisfy

$$\mathbb{P}\left[\mathcal{V}^{w,h}(t) \ge 0\,,\ \ \forall\ 0 \le t \le T \right] \;=\; 1\,. \tag{1.19}$$

Such strategies will be called **admissible** for the initial capital $w > 0$ on the time-horizon $[0,T]$; their collection will be denoted $\mathcal{H}(w;T)$.

We shall also find useful to look at the collection $\mathcal{H}_+(w;T) \subset \mathcal{H}(w;T)$ of *strongly admissible* strategies, with $\mathbb{P}\left[\mathcal{V}^{w,h}(t) > 0\,,\ \ \forall\ 0 \le t \le T \right] \;=\; 1\,.$

Each extended portfolio $\pi(\cdot)$ generates, via (1.8), a trading strategy $h(\cdot) \in \mathcal{H}_+(w) := \bigcap_{T>0} \mathcal{H}_+(w;T)$; and we have $\mathcal{V}^{w,h}(\cdot) \equiv V^{w,\pi}(\cdot)$.

2 The Market Portfolio

Suppose we normalize so that each stock has always just one share outstanding; then the stock price $S_i(t)$ can be interpreted as the capitalization of the i^{th} company at time t, and the quantities

$$S(t) := S_1(t) + \ldots + S_m(t) \qquad \text{and} \qquad \mu_i(t) := \frac{S_i(t)}{S(t)}\,, \quad i = 1, \ldots, m \tag{2.1}$$

as the total capitalization of the market and the relative capitalizations of the individual companies, respectively. Clearly $0 < \mu_i(t) < 1\,, \ \forall\, i = 1, \ldots, m$ and $\sum_{i=1}^m \mu_i(t) = 1$, so we may think of the vector process $\mu(\cdot) = \big(\mu_1(\cdot), \ldots, \mu_m(\cdot)\big)'$ as a portfolio rule that invests a proportion $\mu_i(t)$ of current wealth in the i^{th} asset at all times. The resulting wealth-process $V^{w,\mu}(\cdot)$ satisfies

$$\frac{\mathrm{d}V^{w,\mu}(t)}{V^{w,\mu}(t)} = \sum_{i=1}^m \mu_i(t) \cdot \frac{\mathrm{d}S_i(t)}{S_i(t)} = \sum_{i=1}^m \frac{\mathrm{d}S_i(t)}{S(t)} = \frac{\mathrm{d}S(t)}{S(t)}\,,$$

in accordance with (2.1) and (1.9). In other words,

$$V^{w,\mu}(\cdot) \equiv \frac{w}{S(0)} \cdot S(t)\;; \tag{2.2}$$

investing according to the portfolio $\mu(\cdot)$ is tantamount to ownership of the entire market, in proportion of course to the initial investment. For this reason, we shall call $\mu(\cdot)$ the **market portfolio**. By analogy with (1.11) we have

$$\mathrm{d}\Big(\log V^{w,\mu}(t)\Big) = \gamma^\mu(t)\,\mathrm{d}t + \sum_{\nu=1}^d \sigma^\mu_\nu(t)\,\mathrm{d}W_\nu(t)\,, \qquad V^{w,\mu}(0) = w\,, \tag{2.3}$$

and comparison of this last equation (2.3) with (1.5) gives the dynamics of the market-weights

$$\mathrm{d}\Big(\log \mu_i(t)\Big) = \big(\gamma_i(t) - \gamma^\mu(t)\big)\mathrm{d}t + \sum_{\nu=1}^d \big(\sigma_{i\nu}(t) - \sigma^\mu_\nu(t)\big)\mathrm{d}W_\nu(t) \tag{2.4}$$

for all stocks $i = 1, \ldots, m$, in the notation of (1.10), (1.12); equivalently,

$$\frac{\mathrm{d}\mu_i(t)}{\mu_i(t)} = \left(\gamma_i(t) - \gamma^\mu(t) + \frac{1}{2}\,\tau^\mu_{ii}(t)\right)\mathrm{d}t + \sum_{\nu=1}^d \big(\sigma_{i\nu}(t) - \sigma^\mu_\nu(t)\big)\,\mathrm{d}W_\nu(t)\,. \tag{2.5}$$

Here we introduce, for an arbitrary extended portfolio $\pi(\cdot)$, the quantities

$$\tau^\pi_{ij}(t) := \sum_{\nu=1}^d \big(\sigma_{i\nu}(t) - \sigma^\pi_\nu(t)\big)\big(\sigma_{j\nu}(t) - \sigma^\pi_\nu(t)\big) \tag{2.6}$$

$$= \big(\pi(t) - e_i\big)' a(t)\big(\pi(t) - e_j\big) = a_{ij}(t) - a^\pi_i(t) - a^\pi_j(t) + a^{\pi\pi}(t)$$

for $1 \le i,j \le m$, and set

$$a_i^\pi(t) := \sum_{j=1}^m \pi_j(t) a_{ij}(t), \qquad a^{\pi\pi}(t) := \sum_{i=1}^m \sum_{j=1}^m \pi_i(t) a_{ij}(t) \pi_j(t). \tag{2.7}$$

We shall call the matrix-valued process $\tau^\pi(\cdot) = \big(\tau_{ij}^\pi(\cdot)\big)_{1\le i,j\le m}$ of (2.6) the *variance/covariance process relative to the extended portfolio rule* $\pi(\cdot)$. It satisfies

$$\sum_{j=1}^m \tau_{ij}^\pi(t)\pi_j(t) = 0, \qquad \forall \quad i = 1, \cdots, m. \tag{2.8}$$

- The corresponding quantities

$$\tau_{ij}^\mu(t) := \sum_{\nu=1}^d \big(\sigma_{i\nu}(t) - \sigma_\nu^\mu(t)\big)\big(\sigma_{j\nu}(t) - \sigma_\nu^\mu(t)\big) = \frac{\mathrm{d}\,\langle \mu_i, \mu_j\rangle(t)}{\mu_i(t)\mu_j(t)\mathrm{d}t}, \quad 1 \le i,j \le m \tag{2.9}$$

of (2.6) for the market portfolio $\pi(\cdot) \equiv \mu(\cdot)$, are the `variances/covariances of the individual stocks relative to the entire market`. (For the second equality in (2.9), we have used the semimartingale decomposition of (2.5).)

3 Some Useful Properties

In this section we collect some useful properties of the relative variance/covariance matrix valued process in (2.6), for ease of reference in future usage.

For any given stock i and extended portfolio $\pi(\cdot)$, the *relative return process of the* i^{th} *stock versus* $\pi(\cdot)$ is the process

$$R_i^\pi(t) := \log\left(\frac{S_i(t)}{V^{w,\pi}(t)}\right)\Bigg|_{w=S_i(0)}, \qquad 0 \le t < \infty. \tag{3.1}$$

3.1 Lemma: *For any extended portfolio* $\pi(\cdot)$*, and for all* $1 \le i,j \le m$ *and* $t \in [0,\infty)$*, we have, almost surely:*

$$\tau_{ij}^\pi(t) = \frac{\mathrm{d}}{\mathrm{d}t}\langle R_i^\pi, R_j^\pi\rangle(t), \qquad \text{in particular,} \qquad \tau_{ii}^\pi(t) = \frac{\mathrm{d}}{\mathrm{d}t}\langle R_i^\pi\rangle(t) \ge 0, \tag{3.2}$$

and $\tau^\pi(t) = \big(\tau_{ij}^\pi(t)\big)_{1\le i,j\le m}$ *is a.s. nonnegative definite. Furthermore, if the variance/covariance matrix* $a(t)$ *is* positive *definite, then the matrix* $\tau^\pi(t)$ *has rank* $m-1$*, and its null space is spanned by the vector* $\pi(t)$*, almost surely.*

Proof: Comparing (1.5) with (1.11) we get the analogue

$$\mathrm{d}R_i^\pi(t) = \big(\gamma_i(t) - \gamma^\pi(t)\big)\mathrm{d}t + \sum_{\nu=1}^d \big(\sigma_{i\nu}(t) - \sigma_\nu^\pi(t)\big)\mathrm{d}W_\nu(t)$$

of (2.4), from which the first two claims follow.

Now suppose that $a(t)$ is positive definite. For any $x \in \mathbb{R}^m \setminus \{0\}$ and with $\eta := \sum_{i=1}^m x_i$, we compute from (2.4):

$$x'\tau^\pi(t)x \;=\; x'a(t)x \;-\; 2x'a(t)\pi(t)\cdot\eta \;+\; \pi'(t)a(t)\pi(t)\cdot\eta^2\,.$$

. If $\sum_{i=1}^m x_i = 0$, then $x'\tau^\pi(t)x \;=\; x'a(t)x > 0$.
. If on the other hand $\eta := \sum_{i=1}^m x_i \neq 0$, we consider the vector $y := x/\eta$ that satisfies $\sum_{i=1}^m y_i = 1$, and observe that $\eta^{-2}\cdot x'\tau^\pi(t)x$ is equal to

$$y'\tau^\pi(t)y \;=\; y'a(t)y \;-\; 2y'a(t)\pi(t) \;+\; \pi'(t)a(t)\pi(t) \;=\; \big(y-\pi(t)\big)'a(t)\big(y-\pi(t)\big)\,,$$

thus zero if and only if $y = \pi(t)$, or equivalently $x = \eta\cdot\pi(t)$. □

3.2 Lemma: *For any two extended portfolios* $\pi(\cdot)$, $\rho(\cdot)$, *we have*

$$\mathrm{d}\left(\log\frac{V^\pi(t)}{V^\rho(t)}\right) \;=\; \gamma_*^\pi(t)\,\mathrm{d}t \;+\; \sum_{i=1}^m \pi_i(t)\cdot\mathrm{d}\left(\log\frac{S_i(t)}{V^\rho(t)}\right)\,. \tag{3.3}$$

In particular,

$$\mathrm{d}\left(\log\frac{V^\pi(t)}{V^\mu(t)}\right) \;=\; \gamma_*^\pi(t)\,\mathrm{d}t \;+\; \sum_{i=1}^m \pi_i(t)\cdot\mathrm{d}\left(\log\mu_i(t)\right) \tag{3.4}$$

$$=\; \big(\gamma_*^\pi(t)-\gamma_*^\mu(t)\big)\,\mathrm{d}t \;+\; \sum_{i=1}^m \big(\pi_i(t)-\mu_i(t)\big)\cdot\mathrm{d}\left(\log\mu_i(t)\right)\,.$$

Proof: The equation (3.3) follows from (1.16), and the first equality in (3.4) is the special case $\rho(\cdot)\equiv\mu(\cdot)$. The second follows from the observation, borne out of (2.4), that $\sum_{i=1}^m \mu_i(t)\,\mathrm{d}(\log\mu_i(t)) = \sum_{i=1}^m \mu_i(t)(\gamma_i(t)-\gamma^\mu(t))\,\mathrm{d}t = -\gamma_*^\mu(t)\,\mathrm{d}t$.

3.3 Lemma: *For any two extended portfolios* $\pi(\cdot)$, $\rho(\cdot)$ *we have the* numéraire-invariance *property*

$$\gamma_*^\pi(t) \;=\; \frac{1}{2}\left(\sum_{i=1}^m \pi_i(t)\tau_{ii}^\rho(t) - \sum_{i=1}^m\sum_{j=1}^m \pi_i(t)\pi_j(t)\tau_{ij}^\rho(t)\right)\,. \tag{3.5}$$

In particular, recalling (2.8), we obtain

$$\underbrace{\gamma_*^\pi(t) \;=\; \frac{1}{2}\sum_{i=1}^m \pi_i(t)\tau_{ii}^\pi(t)}\;; \tag{3.6}$$

whereas from (3.6), (3.2) and Definition 1.1, we get

$$\gamma_*^\pi(t) \;\geq\; 0 \quad \text{for any (strict) portfolio } \pi(\cdot)\,. \tag{3.7}$$

Proof: From (2.6): $\sum_{i=1}^m \pi_i(t)\tau_{ii}^\rho(t) = \sum_{i=1}^m \pi_i(t)a_{ii}(t) - 2\sum_{i=1}^m \pi_i(t)a_i^\rho(t) + a^{\rho\rho}(t)$, as well as

$$\sum_{i=1}^m \sum_{j=1}^m \pi_i(t)\tau_{ij}^\rho(t)\pi_j(t) = \sum_{i=1}^m \sum_{j=1}^m \pi_i(t)a_{ij}(t) - 2\sum_{j=1}^m \pi_j(t)a_j^\rho(t) + a^{\rho\rho}(t),$$

and (3.5) follows from (1.13). □

For the market portfolio, the equation of (3.6) becomes

$$\underbrace{\gamma_*^\mu(t) = \frac{1}{2}\sum_{i=1}^m \mu_i(t)\,\tau_{ii}^\mu(t)}\,; \tag{3.8}$$

the summation on the right-hand-side is the average, according to the market weights of individual stocks, of these stocks' variances relative to the market. Thus, (3.8) gives an interpretation of the excess growth rate of the market portfolio, as a measure of `"intrinsic volatility available in the market"`.

3.4 Exercise: For any extended portfolio rule $\pi(\cdot)$, show

$$\mathrm{d}\left(\log\frac{V^\pi(t)}{V^\mu(t)}\right) = \sum_{i=1}^m \frac{\pi_i(t)}{\mu_i(t)}\,\mathrm{d}\mu_i(t) - \frac{1}{2}\sum_{i=1}^m\sum_{j=1}^m \pi_i(t)\pi_j(t)\tau_{ij}^\mu(t)\,. \tag{3.9}$$

(*Hint:* Recall (3.4) in conjunction with (2.4), (2.5) and the numéraire-invariance property (3.5)).

3.5 Lemma: *Assume that the variance/covariance process* $a(\cdot)$ *of (1.3) satisfies the following* strong non-degeneracy condition: *all its eigenvalues are bounded away from zero, i.e., there exists an* $\varepsilon \in (0,\infty)$ *such that*

$$\xi' a(t)\xi = \xi'\sigma(t)\sigma'(t)\xi \geq \varepsilon\|\xi\|^2, \quad \forall\ t\in[0,\infty) \quad \text{and} \quad \xi\in\mathbb{R}^m \tag{3.10}$$

holds almost surely. Then for every extended portfolio $\pi(\cdot)$ *and* $0 \le t < \infty$, *we have in the notation of (1.17) the a.s. inequalities*

$$\varepsilon\Big(1-\pi_i(t)\Big)^2 \leq \tau_{ii}^\pi(t)\,, \qquad i = 1,\cdots,m\,. \tag{3.11}$$

If the portfolio rule $\pi(\cdot)$ *is strict, we have also*

$$\frac{\varepsilon}{2}\Big(1-\pi_{(1)}(t)\Big) \leq \gamma_*^\pi(t)\,. \tag{3.12}$$

Proof: With $e_i = (0,\cdots,0,1,0,\cdots,0)'$ the i^{th} unit vector in $\mathbb{R}^n$, we have $\tau_{ii}^\pi(t) = \big(\pi(t)-e_i\big)' a(t)\,\big(\pi(t)-e_i\big) \geq \varepsilon\,\|\pi(t)-e_i\|^2 = \varepsilon\left[\big(1-\pi_i(t)\big)^2 + \sum_{j\neq i}\pi_j^2(t)\right]$

from (2.6) and (3.10), and (3.11) follows. Back into (3.6), and with $\pi_i(t) \geq 0$ $\forall \; i = 1, \cdots, m$, this lower estimate gives

$$\begin{aligned}
\gamma_*^\pi(t) &\geq \frac{\varepsilon}{2} \cdot \sum_{i=1}^m \pi_i(t) \left[\big(1 - \pi_i(t)\big)^2 + \sum_{j \neq i} \pi_j^2(t) \right] \\
&= \frac{\varepsilon}{2} \cdot \left[\sum_{i=1}^m \pi_i(t) \big(1 - \pi_i(t)\big)^2 + \sum_{j=1}^m \pi_j^2(t) \big(1 - \pi_j(t)\big) \right] \\
&= \frac{\varepsilon}{2} \cdot \sum_{i=1}^m \pi_i(t) \big(1 - \pi_i(t)\big) \geq \frac{\varepsilon}{2} \big(1 - \pi_{(1)}(t)\big). \qquad \Box
\end{aligned}$$

3.6 Lemma: *Assume that the* uniform boundedness *condition (1.15) holds; then for every (strict) portfolio rule* $\pi(\cdot)$*, and for* $0 \leq t < \infty$*, we have in the notation of (1.17) the a.s. inequalities*

$$\tau_{ii}^\pi(t) \leq K \big(1 - \pi_i(t)\big) \big(2 - \pi_i(t)\big), \qquad i = 1, \cdots, m \tag{3.13}$$

$$\gamma_*^\pi(t) \leq 2K \Big(1 - \pi_{(1)}(t)\Big). \tag{3.14}$$

Proof: As in the previous proof, we get $\tau_{ii}^\pi(t) \leq K \left[\big(1 - \pi_i(t)\big)^2 + \sum_{j \neq i} \pi_j^2(t) \right] \leq K \left[\big(1 - \pi_i(t)\big)^2 + \sum_{j \neq i} \pi_j(t) \right] = K\big(1 - \pi_i(t)\big)\big(2 - \pi_i(t)\big)$ as claimed in (3.13), and this leads to

$$\begin{aligned}
\gamma_*^\pi(t) &\leq K \cdot \sum_{i=1}^m \pi_i(t) \big(1 - \pi_i(t)\big) \\
&= K \cdot \left[\pi_{(1)}(t) \big(1 - \pi_{(1)}(t)\big) + \sum_{k=2}^m \pi_{(k)}(t) \big(1 - \pi_{(k)}(t)\big) \right] \\
&\leq K \cdot \left[\big(1 - \pi_{(1)}(t)\big) + \sum_{k=2}^m \pi_{(k)}(t) \right] = 2K \big(1 - \pi_{(1)}(t)\big). \qquad \Box
\end{aligned}$$

4 Portfolio Optimization

We can formulate already some interesting optimization problems.

Problem #1: Quadratic criterion, linear constraint (Markowitz, 1952). *Minimize the portfolio variance* $a^{\pi\pi}(t) = \sum_{i=1}^m \sum_{j=1}^m \pi_i(t) a_{ij}(t) \pi_j(t)$*, among all portfolios* $\pi(\cdot)$ *with rate-of-return* $b^\pi(t) = \sum_{i=1}^m \pi_i(t) b_i(t) \geq b_0$ *at least equal to a given constant.*

Problem #2: Quadratic criterion, quadratic constraint. *Minimize the portfolio variance*

$$a^{\pi\pi}(t) \;=\; \sum_{i=1}^{m}\sum_{j=1}^{m} \pi_i(t) a_{ij}(t) \pi_j(t)$$

among all portfolios $\pi(\cdot)$ *with growth-rate at least equal to a given constant* γ_0:

$$\sum_{i=1}^{m} \pi_i(t) \left(\gamma_i(t) + \frac{1}{2} a_{ii}(t) \right) \;\geq\; \gamma_0 + \frac{1}{2} \sum_{i=1}^{m}\sum_{j=1}^{m} \pi_i(t) a_{ij}(t) \pi_j(t) \,.$$

Problem #3: *Maximize the probability of reaching a given "ceiling"* $\mathfrak{c}$ *before reaching a given "floor"* $\mathfrak{f}$, *with* $0 < \mathfrak{f} < w < \mathfrak{c} < \infty$. More specifically, maximize $\mathbb{P}\left[\mathfrak{T}_{\mathfrak{c}} < \mathfrak{T}_{\mathfrak{f}} \right]$, with $\mathfrak{T}_{\xi} := \inf\{ t \geq 0 \,|\, V^{w,\pi}(t) = \xi \}$ for $\xi \in (0, \infty)$.

In the case of constant coëfficients γ_i and a_{ij}, the solution to this problem comes as follows: one looks at the mean-variance, or *signal-to-noise*, ratio

$$\frac{\gamma_\pi}{a^{\pi\pi}} \;=\; \frac{\sum_{i=1}^{m} \pi_i(\gamma_i + \frac{1}{2} a_{ii})}{\sum_{i=1}^{m}\sum_{j=1}^{m} \pi_i a_{ij} \pi_j} - \frac{1}{2}\,,$$

and finds a portfolio π that maximizes it (Pestien & Sudderth (1985)).

Problem #4: *Minimize the expected time* $\mathbb{E}\left[\mathfrak{T}_{\mathfrak{c}} \right]$ *until a given "ceiling"* $\mathfrak{c} \in (w, \infty)$ *is reached.*

Again with constant coëfficients, it turns out that it is enough to maximize the drift in the equation for $\log V^{w,\pi}(\cdot)$, namely

$$\gamma_\pi \;=\; \textstyle\sum_{i=1}^{m} \pi_i \left(\gamma_i + \frac{1}{2} a_{ii} \right) - \frac{1}{2} \sum_{i=1}^{m}\sum_{j=1}^{m} \pi_i a_{ij} \pi_j\,,$$

the portfolio growth-rate (Heath, Orey, Pestien & Sudderth (1987)).

Problem 5: *Maximize the probability* $\mathbb{P}\left[\mathfrak{T}_{\mathfrak{c}} < T \wedge \mathfrak{T}_{\mathfrak{f}} \right]$ *of reaching a given "ceiling"* $\mathfrak{c}$ *before reaching a given "floor"* $\mathfrak{f}$ *with* $0 < \mathfrak{f} < w < \mathfrak{c} < \infty$, *by a given "deadline"* $T \in (0, \infty)$.

Always with constant coëfficients, suppose there is a portfolio $\hat{\pi} = (\hat{\pi}_1, \ldots, \hat{\pi}_m)'$ that maximizes *both* the signal-to-noise ratio *and* the variance,

$$\frac{\gamma_\pi}{a^{\pi\pi}} \;=\; \frac{\sum_{i=1}^{m} \pi_i(\gamma_i + \frac{1}{2} a_{ii})}{\sum_{i=1}^{m}\sum_{j=1}^{m} \pi_i a_{ij} \pi_j} - \frac{1}{2} \qquad \text{and} \qquad a^{\pi\pi} \;=\; \sum_{i=1}^{m}\sum_{j=1}^{m} \pi_i a_{ij} \pi_j\,,$$

respectively, over all $\pi_1 \geq 0, \ldots, \pi_m \geq 0$ with $\sum_{i=1}^{m} \pi_i = 1$. Then this portfolio $\hat{\pi}$ is optimal for the above criterion (Sudderth & Weerasinghe (1989)).

This is a big assumption; it is satisfied, for instance, under the (very stringent, and unnatural...) condition that, for some $G > 0$, we have

$$b_i \;=\; \gamma_i + \frac{1}{2} a_{ii} \;=\; -G\,, \quad \text{for all } i = 1, \ldots, m\,.$$

`Open Question:` As far as I can tell, nobody seems to know the solution to this problem, if such "simultaneous maximization" is not possible.

5 Relative Arbitrage, and Its Consequences

The notion of arbitrage is of paramount importance in Mathematical Finance. We present in this section an allied notion, that of *relative arbitrage*, and explore some of its consequences. In later sections we shall encounter very specific, "descriptive" conditions on market structure, that lead to this form of arbitrage.

5.1 Definition: Relative Arbitrage. Given any two extended portfolios $\pi(\cdot)$, $\rho(\cdot)$ with the same initial capital $V^\pi(0) = V^\rho(0) = 1$, we shall say that $\pi(\cdot)$ *represents, relative to* $\rho(\cdot)$,

• *an arbitrage opportunity over the fixed, finite time-horizon* $[0,T]$, if there exists a constant $q = q_{\pi,\rho,T} > 0$ such that

$$\mathbb{P}\,[\,V^\pi(T) \geq q\,V^\rho(T)\,,\ \forall\ 0 \leq t \leq T\,] \,=\, 1 \tag{5.1}$$

and we have

$$\mathbb{P}\,[\,V^\pi(T) \geq V^\rho(T)\,] \,=\, 1 \qquad \text{and} \qquad \mathbb{P}\,[\,V^\pi(T) > V^\rho(T)\,] > 0\,; \tag{5.2}$$

• *a superior long-term growth opportunity*, if

$$\mathcal{L}^{\pi,\rho} \,:=\, \liminf_{T\to\infty}\ \frac{1}{T}\log\left(\frac{V^\pi(T)}{V^\rho(T)}\right) \,>\, 0 \quad \text{holds a.s.} \tag{5.3}$$

5.1 Strict Local Martingales

Let us place ourselves now, and for the remainder of this section, within the market model $\mathcal{M}$ of (1.1) and under the conditions (1.2), (1.15). We shall assume further, that there exists a *market price of risk* (or "relative risk") $\vartheta : [0,\infty)\times\Omega \to \mathbb{R}^d$; namely, an $\mathbb{F}-$progressively measurable process with

$$\sigma(t)\,\vartheta(t) \,=\, b(t) - r(t)\mathbb{I}\,,\ \forall\ 0 \leq t \leq T \qquad \text{and} \qquad \int_0^T ||\vartheta(t)||^2\,\mathrm{d}t \,<\, \infty \tag{5.4}$$

a.s., for each $T \in (0,\infty)$. Here $\mathbb{I} = (1,\cdots,1)'$ is the vector in $\mathbb{R}^n$ with all entried equal to 1. (If the volatility matrix $\sigma(\cdot)$ has full rank, namely m, we can take $\vartheta(t) = \sigma'(t)\big(\sigma(t)\sigma'(t)\big)^{-1}[\,b(t) - r(t)\mathbb{I}\,]$.) In terms of this process, we can then define the exponential local martingale and supermartingale

$$Z(t) \,:=\, \exp\left\{-\int_0^t \vartheta'(s)\,\mathrm{d}W(s) - \frac{1}{2}\int_0^t ||\vartheta(s)||^2\,\mathrm{d}s\right\}\,, \quad 0 \leq t \leq T \tag{5.5}$$

(a martingale, if and only if $\mathbb{E}(Z(T)) = 1$), and the "shifted Brownian Motion"

$$\widehat{W}(t) \,:=\, W(t) + \int_0^t \vartheta(s)\,\mathrm{d}s\,, \qquad 0 \leq t \leq T\,. \tag{5.6}$$

5.2 Proposition: A Strict Local Martingale. *Under the assumptions of this subsection, suppose there exists a time-horizon* $T \in (0, \infty)$ *such that relative arbitrage exists on* $[0, T]$ *. Then the process* $Z(\cdot)$ *of (5.5) is a* strict *local martingale: we have* $\mathbb{E}[Z(T)] < 1$ *.*

Proof: Assume, by way of contradiction, that $E[Z(T)] = 1$. Then from the Girsanov Theorem the recipe $\mathbb{Q}_T(A) := E[\, Z(T) \cdot 1_A \,]$ defines an equivalent probability measure on $\mathcal{F}(T)$, under which $\widehat{W}(\cdot)$ of (5.6) is Brownian motion and the discounted stock prices $S_i(\cdot)/B(\cdot)$ martingales, for $i = 1, \cdots, m$ (because (1.15) is assumed to hold). To wit, this $\mathbb{Q}_T$ is an *Equivalent Martingale Measure (EMM)* for the model. For any extended portfolio $\pi(\cdot)$, we get then from (5.6) and (1.9):

$$\mathrm{d}(V^{w,\pi}(t)/B(t)) \,=\, (V^{w,\pi}(t)/B(t)) \cdot \pi'(t)\, \sigma(t)\, \mathrm{d}\widehat{W}(t)\,, \qquad V^{w,\pi}(0) = w > 0\,,$$

and because of (1.15) the wealth process $V^{w,\pi}(\cdot)$ is a square-integrable martingale; thus so is the difference $D(\cdot) := V^{w,\pi}(\cdot) - V^{w,\rho}(\cdot)$ for any other extended portfolio $\rho(\cdot)$. But this gives $\mathbb{E}^{\mathbb{Q}_T}\big(D(T)\big) = D(0) = 0$, a conclusion inconsistent with the consequences $\mathbb{Q}_T\big(D(T) \geq 0\big) = 1$ and $\mathbb{Q}_T\big(D(T) > 0\big) > 0$ of (5.2). $\square$

- Now let us consider the "deflated" stock-price and wealth processes

$$\widehat{S}_i(t) \,:=\, \frac{Z(t)}{B(t)}\, S_i(t)\,, \quad i = 1, \cdots, m \qquad \text{and} \qquad \widehat{\mathcal{V}}^{w,h}(t) \,:=\, \frac{Z(t)}{B(t)}\, \mathcal{V}^{w,h}(t) \tag{5.7}$$

for $0 \leq t \leq T$, for arbitrary admissible trading strategy $h(\cdot)$ and initial capital $w > 0$. These processes satisfy, respectively, the dynamics

$$\mathrm{d}\widehat{S}_i(t) \,=\, \widehat{S}_i(t) \cdot \sum_{\nu=1}^{d} \big(\sigma_{i\nu}(t) - \vartheta_\nu(t)\big)\, \mathrm{d}W_\nu(t)\,, \quad \widehat{S}_i(0) \,=\, s_i\,, \tag{5.8}$$

$$\mathrm{d}\widehat{\mathcal{V}}^{w,h}(t) \,=\, \left(\frac{Z(t)\, h'(t)}{B(t)}\, \sigma(t) - \widehat{\mathcal{V}}^{w,h}(t)\, \vartheta'(t) \right) \mathrm{d}W(t)\,, \quad \widehat{\mathcal{V}}^{w,h}(0) \,=\, w\,. \tag{5.9}$$

(In other words, the ratio $Z(\cdot)/B(\cdot)$ continues to play its usual rôle as "deflator" of prices in such a market, even when $Z(\cdot)$ is just a local martingale.)

5.3 Exercise: Strict Local Martingales Galore. Verify the claims of (5.8), (5.9). Then show that, in the setting of Proposition 5.2, the deflated stock-prices $\widehat{S}_i(\cdot)$ of (5.7) are *strict* local martingales: $\mathbb{E}\,[\, \widehat{S}_i(T) \,] < s_i$ holds for every $i = 1, \cdots, m$.

5.4 Proposition: Non-Existence of EMM. *In the context of Proposition 5.2, no Equivalent Martingale Measure can exist for this model if the filtration is generated by the driving Brownian Motion* $W(\cdot)$: $\mathbb{F} = \mathbb{F}^W$.

Proof: If $\mathbb{F} = \mathbb{F}^W$ and $\mathbb{Q}$ is equivalent to $\mathbb{P}$ on $\mathcal{F}_T$, then the martingale representation property of the Brownian filtration gives $(\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P})|_{\mathcal{F}_t} = Z(t)$, $0 \leq t \leq$

T for some process $Z(\cdot)$ of the form (5.5) and some progressively measurable $\vartheta(\cdot)$ with $\int_0^T ||\vartheta(t)||^2 \, \mathrm{d}t < \infty$ a.s.; and Itô's rule leads to the extension

$$\frac{\mathrm{d}\widehat{S}_i(t)}{\widehat{S}_i(t)} = \left(b_i(t) - r(t) - \sum_{\nu=1}^d \sigma_{i\nu}(t)\vartheta_\nu(t) \right) \mathrm{d}t + \sum_{\nu=1}^d \big(\sigma_{i\nu}(t) - \vartheta_\nu(t) \big) \, \mathrm{d}W_\nu(t)$$

of (5.8) for the deflated stock-prices of (5.7). But if all the $S_i(\cdot)/B(\cdot)$'s are $\mathbb{Q}-$martingales, then the $\widehat{S}_i(\cdot)$'s are all $\mathbb{P}-$martingales, and this leads to the first property $\sigma(\cdot)\,\vartheta(\cdot) = b(\cdot) - r(\cdot)\mathbb{I}$ in (5.4). We repeat now the argument of Proposition 5.2 and arrive at a contradiction with (5.2), the existence of relative arbitrage on $[0, T]$. $\square$

Let us introduce now the decreasing function

$$f(t) := \frac{1}{S(0)} \cdot \mathbb{E}\left[\frac{Z(t)}{B(t)} \, S(t) \right], \qquad 0 < t \le T \tag{5.10}$$

which satisfies $f(0) := 1 > f(t) > 0$ from Exercise 5.3.

5.5 Exercise: With Brownian filtration $\mathbb{F} = \mathbb{F}^W$, $m = d$ and an invertible volatility matrix $\sigma(\cdot)$, consider the maximal attainable relative return

$$\mathfrak{R}(T) := \sup \big\{ \, r > 1 \,|\, \exists \, h(\cdot) \in \mathcal{H}(1;T) \ \text{ s.t. } \big(\mathcal{V}^{1,h}(T)/V^{1,\mu}(T) \big) \ge r \, , \text{ a.s.} \big\} \tag{5.11}$$

in excess of the market, over the interval $[0,T]$. Show that this quantity can be computed then in terms of the function of (5.10), as $\mathfrak{R}(T) = 1/f(T)$.

5.6 Exercise: The shortest time to beat the market by a given amount. Let us place ourselves under the assumptions of Exercise 5.5, but now assume that relative arbitrage exists on $[0,T]$ for *every* $T \in (0, \infty)$; cf. section 7. For a given "exceedance level" $r > 1$, consider the shortest length of time

$$\mathbf{T}(r) := \inf \big\{ \, T > 0 \,|\, \exists \, h(\cdot) \in \mathcal{H}(1;T) \ \text{ s.t. } \big(\mathcal{V}^{1,h}(T)/V^{1,\mu}(T) \big) \ge r \, , \text{ a.s.} \big\} \tag{5.12}$$

required to guarantee a return of at least r times the market. Show that this quantity is given by the number $\mathbf{T}(r) = \inf \big\{ \, T > 0 \,|\, f(T) \le 1/r \, \big\}$, the inverse of the decreasing function $f(\cdot)$ of (5.10) evaluated at $1/r$.

5.7 Exercise: *(An observation of C. Kardaras (2006).)* Show that in the Definition 5.1 of relative arbitrage, the requirement (5.1) is somewhat superfluous, in the following sense: If one can find a portfolio $\pi(\cdot)$ that satisfies the domination properties (5.2) relative to some "benchmark" portfolio $\rho(\cdot)$, then there exists another portfolio $\widehat{\pi}(\cdot)$ that satisfies both (5.1) and (5.2) relative to the same benchmark $\rho(\cdot)$.

(*Hint:* Consider the more conservative strategy of investing a portion $w \in (0,1)$ of the initial capital \$1 in $\pi(\cdot)$, and the remaining proportion $1 - w$ in $\rho(\cdot)$.)

5.8 Exercise: Can the counterparts of (5.11), (5.12) be computed when one is not allowed to use general strategies $h(\cdot) \in \mathcal{H}(1;T)$, but rather only strict-sense portfolios $\pi(\cdot)$?

6 Diversity

The notion of diversity for a financial market corresponds to the intuitive (and descriptive) idea, that no single company can ever be allowed to dominate the entire market in terms of relative capitalization. To make this notion precise, let us say that the model $\mathcal{M}$ of (1.1), (1.2) is `diverse` on the time-horizon $[0,T]$, if there exists a number $\delta \in (0,1)$ such that the quantities of (2.1) satisfy a.s.

$$\mu_{(1)}(t) < 1 - \delta\,, \qquad \forall\;\; 0 \le t \le T \tag{6.1}$$

in the order-statistics notation of (1.17). In a similar vein, we say that $\mathcal{M}$ is `weakly diverse` on the time-horizon $[0,T]$, if for some $\delta \in (0,1)$ we have

$$\frac{1}{T}\int_0^T \mu_{(1)}(t)\,dt \;<\; 1-\delta \tag{6.2}$$

almost surely. We say that $\mathcal{M}$ is `uniformly weakly diverse` on $[T_0,\infty)$, if there exists a $\delta \in (0,1)$ such that (6.2) holds a.s. for every $T \in [T_0,\infty)$.

• It follows directly from (3.14) of Lemma 3.6 that, under the condition (1.15), the model $\mathcal{M}$ of (1.1), (1.2) is diverse (respectively, weakly diverse) on the time-interval $[0,T]$, if there exists a number $\zeta > 0$ such that

$$\gamma_*^\mu(t) \;\ge\; \zeta\,, \quad \forall\;\; 0 \le t \le T \qquad \left(\text{respectively,}\quad \frac{1}{T}\int_0^T \gamma_*^\mu(t)\,\mathrm{d}t \;\ge\; \zeta\right) \tag{6.3}$$

holds almost surely. And (3.12) of Lemma 3.5 shows that, under the condition (3.10), the conditions of (6.3) are satisfied if diversity (respectively, weak diversity) holds on the time-interval $[0,T]$.

• As we shall see in section 9, diversity can ensured by a strongly negative rate of growth for the largest stock, resulting in a sufficiently strong repelling drift (e.g., a log-pole-type singularity) away from an appropriate boundary, as well as non-negative growth-rates for all the other stocks.

• **`If all the stocks in`** $\mathcal{M}$ **`have the same growth rate (`** $\gamma_i(\cdot) \equiv \gamma(\cdot)$ **`,`** $\forall\;\; 1 \le i \le m$ **`) and (1.15) holds, then we have almost surely:`**

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \gamma_*^\mu(t)\,\mathrm{d}t \;=\; 0\,. \tag{6.4}$$

In particular, *such an equal growth-rate market* $\mathcal{M}$ *cannot be diverse, even weakly, for long time-horizons,* provided that (3.10) is also satisfied.

Here is a quick argument: recall that for $S(\cdot) := S_1(\cdot) + \cdots + S_m(\cdot)$ we have

$$\lim_{T\to\infty} \frac{1}{T}\left(\log S(T) - \int_0^T \gamma^\mu(t)\mathrm{d}t\right) = 0\,, \quad \lim_{T\to\infty} \frac{1}{T}\left(\log S_i(T) - \int_0^T \gamma(t)\mathrm{d}t\right) = 0$$

a.s., from (1.14), (1.6) and $\gamma_i(\cdot) \equiv \gamma(\cdot)\,,\,\forall\, 1 \le i \le m$. But then we have also

$$\lim_{T\to\infty} \frac{1}{T}\left(\log S_{(1)}(T) - \int_0^T \gamma(t)\mathrm{d}t\right) = 0\,, \quad \text{a.s.}$$

for the biggest stock $S_{(1)}(\cdot) := \max_{1\le i\le m} S_i(\cdot)$, and note the inequalities $S_{(1)}(\cdot) \le S(\cdot) \le m\, S_{(1)}(\cdot)$. Therefore,

$$\lim_{T\to\infty} \frac{1}{T}\big(\log S_{(1)}(T) - \log S(T)\big) = 0\,, \quad \text{thus} \quad \lim_{T\to\infty} \frac{1}{T}\int_0^T \big(\gamma^\mu(t) - \gamma(t)\big)\mathrm{d}t = 0\,.$$

But $\gamma^\mu(t) = \sum_{i=1}^m \mu_i(t)\gamma(t) + \gamma_*^\mu(t) = \gamma(t) + \gamma_*^\mu(t)$, because of the assumption of equal growth rates, and (6.4) follows. If (3.10) also holds, then (3.12) and (6.4) imply

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \big(1 - \mu_{(1)}(t)\big)\,\mathrm{d}t \;=\; 0\,,$$

so weak diversity fails on long time-horizons: once in a while a single stock dominates the *entire market*, then recedes; sooner or later another stock takes its place as absolutely dominant leader; and so on.

6.1 Exercise: Coherence. We say that the market model $\mathcal{M}$ of (1.1), (1.2) is *coherent*, if the relative capitalizations of (2.1) satisfy

$$\lim_{T\to\infty} \frac{1}{T} \log \mu_i(T) \;=\; 0 \qquad \text{almost surely, for each } \; i = 1, \cdots, m \tag{6.5}$$

(i.e., if "none of the stocks declines too rapidly"). Under the condition (1.15), show that coherence is equivalent to each of the following two conditions:

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \big(\gamma_i(t) - \gamma^\mu(t)\big)\,\mathrm{d}t \;=\; 0 \quad \text{a.s., for each } \; i = 1, \cdots, m\,; \tag{6.6}$$

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \big(\gamma_i(t) - \gamma_j(t)\big)\,\mathrm{d}t \;=\; 0 \quad \text{a.s., for each pair } \; 1 \le i, j \le m\,. \tag{6.7}$$

6.2 Exercise: Argue that if all the stocks in the market $\mathcal{M}$ have constant growth rates, and if (1.15), (3.10) hold, then $\mathcal{M}$ cannot be diverse, even weakly, over long time-horizons.

7 Diversity leads to Arbitrage

We provide now examples which demonstrate the following principle: *if the model $\mathcal{M}$ of (1.1), (1.2) is weakly diverse over the interval $[0,T]$, and if (3.10) holds, then $\mathcal{M}$ contains arbitrage opportunities relative to the market portfolio, at least for sufficiently large time-horizons $T \in (0,\infty)$.*

The first such examples involve heavily the so-called **diversity-weighted portfolio** $\pi^{(p)}(\cdot) = \big(\pi_1^{(p)}(\cdot), \ldots, \pi_m^{(p)}(\cdot)\big)'$, defined for arbitrary but fixed $p \in (0,1)$ in terms of the market portfolio $\mu(\cdot)$ of (2.1) by

$$\pi_i^{(p)}(t) := \frac{\big(\mu_i(t)\big)^p}{\sum_{j=1}^m \big(\mu_j(t)\big)^p}\,, \quad \forall \;\; i = 1, \ldots, m\,. \tag{7.1}$$

Compared to $\mu(\cdot)$, the portfolio $\pi^{(p)}(\cdot)$ in (7.1) decreases the proportion(s) held in the largest stock(s) and increases those placed in the smallest stock(s), while preserving the relative rankings of all stocks. It is relatively easy to implement in practice, as it involves only observable quantities (the relative market weights) and needs no parameter estimation or optimization. The actual performance of this portfolio relative to the S&P 500 index over a 22-year period is discussed in detail by Fernholz (2002), Chapter 6.

We show below that if the model $\mathcal{M}$ is weakly diverse on a finite time-horizon $[0,T]$, then the value-process $V^{\pi^{(p)}}(\cdot)$ of the portfolio in (7.1) satisfies

$$V^{\pi^{(p)}}(T) > V^{\mu}(T) \cdot \Big(m^{-1/p}\; e^{\,\varepsilon\delta T/2} \Big)^{1-p} \tag{7.2}$$

almost surely. In particular,

$$\mathbb{P}\Big[V^{\pi^{(p)}}(T) > V^{\mu}(T) \Big] = 1\,, \quad \text{provided that} \quad T \geq \frac{2}{p\varepsilon\delta} \cdot \log m\,, \tag{7.3}$$

and $\pi^{(p)}(\cdot)$ is an arbitrage opportunity relative to the market $\mu(\cdot)$, in the sense of (5.1)-(5.2). The significance of such a result, for practical long-term portfolio management, cannot be overstated.

✠ *What conditions on the coëfficients $b(\cdot)$, $\sigma(\cdot)$ of $\mathcal{M}$ are sufficient for guaranteeing diversity, as in (6.1), over the time-horizon $[0,T]$?* For simplicity, assume that (3.10) and (1.15) both hold. Then certainly $\mathcal{M}$ cannot be diverse if $b_1(\cdot) - r(\cdot), \ldots, b_m(\cdot) - r(\cdot)$ are bounded uniformly in (t,ω), or even if they satisfy a condition of the Novikov type

$$\mathbb{E}\left[\exp\left\{ \frac{1}{2} \int_0^T \Big|\Big| b(t) - r(t)\mathbb{I} \Big|\Big|^2 dt \right\} \right] < \infty\,. \tag{7.4}$$

The reason is that, under all these conditions (3.10), (1.15) and (7.4), the process $\vartheta(\cdot) = \sigma'(\cdot)\big(\sigma(\cdot)\sigma(\cdot)\big)^{-1}[\,b(\cdot) - r(\cdot)\mathbb{I}\,]$ satisfies the requirements (5.4), and the resulting exponential local martingale $Z(\cdot)$ of (5.5) is a true martingale – contradicting Proposition 5.2, at least for sufficiently large $T > 0$.

Proof of (7.3): Let us start by introducing the function

$$\mathbf{G}_p(x) := \Big(\sum_{i=1}^m x_i^p \Big)^{1/p}, \qquad x \in \Delta_{++}^m\,, \tag{7.5}$$

which we shall interpret as a "measure of diversity"; see below. An application of Itô's rule to the process $\{\mathbf{G}_p(\mu(t)),\ 0 \le t < \infty\}$ leads after some computation, and in conjunction with (3.9) and the numéraire-invariance property (3.5), to the expression

$$\underbrace{\log\left(\frac{V^{\pi^{(p)}}(T)}{V^{\mu}(T)}\right) = \log\left(\frac{\mathbf{G}_p(\mu(T))}{\mathbf{G}_p(\mu(0))}\right) + \int_0^T \mathfrak{g}(t)\mathrm{d}t\,,} \qquad \mathfrak{g}(t) := (1-p)\,\gamma_*^{\pi^{(p)}}(t) \tag{7.6}$$

for the wealth $V^{\pi^{(p)}}(\cdot)$ of the diversity-weighted portfolio $\pi^{(p)}(\cdot)$ of (7.1). One big advantage of the expression (7.6) is that is free of stochastic integrals, and thus lends itself to pathwise (almost sure) comparisons.

7.1 Exercise: `Verify the computation (7.6).`

For the function of (7.5), we have the simple bounds

$$1 = \sum_{i=1}^m \mu_i(t) \le \sum_{i=1}^m \big(\mu_i(t)\big)^p = \Big(\mathbf{G}_p(\mu(t))\Big)^p \le m^{1-p}$$

(minimum diversity occurs when the entire market is concentrated in one stock, and maximum diversity when all stocks have the same capitalization), so that the function of (7.5) satisfies

$$\log\left(\frac{\mathbf{G}_p(\mu(T))}{\mathbf{G}_p(\mu(0))}\right) \ge -\frac{1-p}{p}\cdot\log m\,. \tag{7.7}$$

This shows that $V^{\pi^{(p)}}(\cdot)/V^{\mu}(\cdot)$ is bounded from below by the constant $m^{-(1-p)/p}$, so (5.1) is satisfied for $\rho(\cdot) \equiv \mu(\cdot)$ and $\pi(\cdot) \equiv \pi^{(p)}(\cdot)$.

♠ On the other hand, we have already remarked that the biggest weight of the portfolio $\pi^{(p)}(\cdot)$ in (7.1) does not exceed the largest market weight:

$$\pi_{(1)}^{(p)}(t) := \max_{1\le i\le m} \pi_i^{(p)}(t) = \frac{\big(\mu_{(1)}(t)\big)^p}{\sum_{k=1}^m \big(\mu_{(k)}(t)\big)^p} \le \mu_{(1)}(t) \tag{7.8}$$

(check this, and that the reverse inequality holds for the smallest weights: $\pi_{(m)}^{(p)}(t) := \min_{1\le i\le m} \pi_i^{(p)}(t) \ge \mu_{(m)}(t)$).

We have assumed that the market is weakly diverse over $[0,T]$, namely, that there is some $0<\delta<1$ for which $\int_0^T \big(1-\mu_{(1)}(t)\big)\,dt > \delta\,T$ holds almost surely. From (3.12) and (7.8), this implies

$$\int_0^T \gamma_*^{\pi^{(p)}}(t)\,\mathrm{d}t \ge \frac{\varepsilon}{2}\cdot\int_0^T \Big(1-\pi_{(1)}^{(p)}(t)\Big)\,\mathrm{d}t \ge \frac{\varepsilon}{2}\cdot\int_0^T \Big(1-\mu_{(1)}(t)\Big)\,\mathrm{d}t > \frac{\varepsilon}{2}\cdot\delta\,T$$

a.s. In conjunction with (7.7), this leads to to (7.2) and (7.3) via

$$\log\left(\frac{V^{\pi^{(p)}}(T)}{V^{\mu}(T)}\right) > (1-p)\left[\frac{\varepsilon T}{2}\cdot\delta - \frac{1}{p}\cdot\log m\right]. \qquad \Box \tag{7.9}$$

If $\mathcal{M}$ is uniformly weakly diverse and strongly non-degenerate over an interval $[T_*, \infty)$, then (7.9) implies that the market portfolio will lag significantly behind the diversity-weighted one, over long time-horizons; i.e., (5.3) holds:

$$\mathcal{L}^{\pi^{(p)},\mu} = \liminf_{T\to\infty} \frac{1}{T} \log\Big(V^{\pi^{(p)}}(T) \,/\, V^{\mu}(T)\Big) \geq (1-p)\varepsilon\delta/2 > 0\,, \qquad \text{a.s.}$$

7.2 Exercise (Fernholz 2002): Under the conditions of this section, consider the portfolio with weights

$$\pi_i(t) = \left(\frac{2-\mu_i(t)}{\mathbf{G}(\mu(t))} - 1\right)\mu_i(t)\,, \quad 1 \le i \le m\,, \quad \text{where} \quad \mathbf{G}(x) := 1 - \frac{1}{2}\sum_{i=1}^{m} x_i^2$$

for $x \in \Delta_{++}^m$. Show that this portfolio leads to arbitrage relative to the market over sufficiently long time-horizons $[0,T]$, namely with $T \geq (2m/\varepsilon\delta^2)\log 2$. (*Hint:* Establish an analogue of (7.6) for this new portfolio $\pi(\cdot)$ and function G, in which $\mathfrak{g}(t) = \left(\sum_{i=1}^m \mu_i^2(t)\tau_{ii}^{\mu}(t)\right)/(2\mathbf{G}(\mu(t)))$.

8 Mirror Portfolios, Short-Horizon Arbitrage

In the previous section we saw that in weakly diverse markets which satisfy the strict non-degeneracy condition (3.10), one can construct explicitly simple arbitrages relative to the market over sufficiently long time-horizons. The purpose of this section is to demonstrate that, under these same conditions, such arbitrages exist indeed over *arbitrary* time-horizons, no matter how small.

For any given extended portfolio $\pi(\cdot)$ and real number $q \neq 0$, define the *$q-$mirror image of $\pi(\cdot)$ with respect to the market portfolio*, as

$$\widetilde{\pi}^{[q]}(\cdot) := q\,\pi(\cdot) + (1-q)\,\mu(\cdot)\,.$$

This is clearly an extended portfolio; and it is strict, as long as $\pi(\cdot)$ itself is strict and $0 < q < 1$. If $q = -1$, we call $\widetilde{\pi}^{[-1]}(\cdot) = 2\mu(\cdot) - \pi(\cdot)$ the "mirror image" of $\pi(\cdot)$ with respect to the market.

By analogy with (2.6), let us define the *relative covariance of $\pi(\cdot)$ with respect to the market*, as

$$\tau_{\mu\mu}^{\pi}(t) := \big(\pi(t)-\mu(t)\big)' a(t)\big(\pi(t)-\mu(t)\big)\,, \qquad 0 \le t \le T\,.$$

8.1 Exercise: Recall from (2.8) the fact $\tau^{\mu}(t)\mu(t) \equiv 0$, and establish the elementary properties $\tau_{\mu\mu}^{\pi}(t) = \pi'(t)\tau^{\mu}(t)\pi(t) = \tau_{\pi\pi}^{\mu}(t)$, $\tau_{\widetilde{\pi}^{[q]}\widetilde{\pi}^{[q]}}^{\mu}(t) = q^2\,\tau_{\pi\pi}^{\mu}(t)$.

8.2 Exercise: Compute the wealth of $\widetilde{\pi}^{[q]}(\cdot)$ relative to the market, as

$$\log\left(\frac{V^{\widetilde{\pi}^{[q]}}(T)}{V^{\mu}(T)}\right) = q\,\log\left(\frac{V^{\pi}(T)}{V^{\mu}(T)}\right) + \frac{q(1-q)}{2}\int_0^T \tau_{\pi\pi}^{\mu}(t)\,\mathrm{d}t\,.$$

8.3 Exercise: Suppose that the extended portfolio $\pi(\cdot)$ satisfies

$$\mathbb{P}\big(V^{\pi}(T)/V^{\mu}(T) \ge \beta \big) = 1 \qquad \text{or} \qquad \mathbb{P}\big(V^{\pi}(T)/V^{\mu}(T) \le 1/\beta \big) = 1$$

and

$$\mathbb{P}\left(\int_0^T \tau^{\mu}_{\pi\pi}(t)\, \mathrm{d}t \,\ge\, \eta \right) \,=\, 1$$

for some real numbers $T > 0$, $\eta > 0$ and $0 < \beta < 1$. Then there exists another extended portfolio $\widehat{\pi}(\cdot)$ with $\mathbb{P}\big(V^{\widehat{\pi}}(T) < V^{\mu}(T) \big) = 1$.

8.1 A "Seed" Portfolio

Now let us consider $\pi = e_1 = (1, 0, \cdots, 0)'$ and the market portfolio $\mu(\cdot)$; we shall fix a real number $q > 1$ in a moment, and define the extended portfolio

$$\widehat{\pi}(t) \,:=\, \widetilde{\pi}^{[q]}(t) \,=\, q\, e_1 \,+\, (1-q)\, \mu(t)\,, \qquad 0 \le t < \infty \tag{8.1}$$

which takes a long position in the first stock and a short position in the market. In particular, $\widehat{\pi}_1(t) \,=\, q \,+\, (1-q)\, \mu_1(t)$ and $\widehat{\pi}_i(t) \,=\, (1-q)\, \mu_i(t)$ for $i = 2, \cdots, m$. Then we have

$$\log\left(\frac{V^{\widehat{\pi}}(T)}{V^{\mu}(T)} \right) \,=\, q \cdot \left[\log\left(\frac{\mu_1(T)}{\mu_1(0)} \right) \,-\, \frac{q-1}{2} \int_0^T \tau^{\mu}_{11}(t)\, dt \right] \tag{8.2}$$

from Exercise 8.2. But taking $\beta := \mu_1(0)$ we have $(\mu_1(T)/\mu_1(0)) \le 1/\beta$; and if the market is weakly diverse on $[0,T]$ and satisfies the strict non-degeneracy condition (3.10), we obtain from (3.11) and the Cauchy-Schwarz inequality

$$\int_0^T \tau^{\mu}_{11}(t)\, dt \,\ge\, \varepsilon \int_0^T \big(1 - \mu_{(1)} \big)^2\, dt \,>\, \varepsilon \delta^2 T \,=:\, \eta\,. \tag{8.3}$$

From Exercise 8.3, *the market portfolio represents then an arbitrage opportunity with respect to the extended portfolio $\widehat{\pi}(\cdot)$ of (8.1)*, provided that for any given $T \in (0, \infty)$ we select

$$q \,>\, q(T) \,:=\, 1 \,+\, (2/\varepsilon\delta^2 T) \cdot \log\left(1/\mu_1(0) \right)\,. \tag{8.4}$$

♣ The extended portfolio $\widehat{\pi}(\cdot)$ of (8.1) can be used as a "seed", to create *all-long* portfolios that outperform the market portfolio $\mu(\cdot)$, *over any given time-horizon* $T \in (0, \infty)$. The idea is to embed $\widehat{\pi}(\cdot)$ in a sea of market portfolio, swamping the short positions while retaining the essential portfolio characteristics. Crucial in these constructions is the a.s. comparison, a consequence of (8.2):

$$V^{\widehat{\pi}}(t) \,\le\, \left(\frac{\mu_1(t)}{\mu_1(0)} \right)^q \cdot V^{\mu}(t)\,, \qquad 0 \le t < \infty\,. \tag{8.5}$$

8.2 Relative Arbitrage on Arbitrary Time-Horizons

To implement this idea, consider a strategy $h(\cdot)$ that invests $q/(\mu_1(0))^q$ dollars in the market portfolio, and -1 dollar in the extended portfolio $\widehat{\pi}(\cdot)$ of (8.1) at time $t=0$, and makes no change thereafter. The number $q>1$ is chosen again as in (8.4). The wealth generated by this strategy, with initial capital $z := q/(\mu_1(0))^q - 1 > 0$, is

$$\mathcal{V}^{z,h}(t) \;=\; \frac{q\,V^\mu(t)}{(\mu_1(0))^q} - V^{\widehat{\pi}}(t) \;\geq\; \frac{V^\mu(t)}{(\mu_1(0))^q}\left[\,q-(\mu_1(t))^q\,\right] \;>\; 0\,, \qquad 0\leq t<\infty\,, \tag{8.6}$$

thanks to (8.5) and $q>1>(\mu_1(t))^q$. This process $\mathcal{V}^{z,h}(\cdot)$ coincides with the wealth $V^{z,\eta}(\cdot)$ generated by an extended portfolio $\eta(\cdot)$ with weights

$$\eta_i(t) \;=\; \frac{1}{\mathcal{V}^{z,h}(t)}\left[\frac{q\,\mu_i(t)}{(\mu_1(0))^q}\cdot V^\mu(t)-\widehat{\pi}_i(t)\cdot V^{\widehat{\pi}}(t)\right], \qquad i=1,\cdots,m \tag{8.7}$$

that satisfy $\sum_{i=1}^m \eta_i(t) = 1$. Now we have $\widehat{\pi}_i(t) = -(q-1)\,\mu_i(t) < 0$ for $i=2,\cdots,m$, so the quantities $\eta_2(\cdot),\ldots,\eta_m(\cdot)$ are strictly positive. To check that $\eta(\cdot)$ is an all-long portfolio, we have to verify $\eta_1(t)\geq 0$; but the dollar amount invested by $\eta(\cdot)$ in the first stock at time t, namely

$$\frac{q\,\mu_1(t)}{(\mu_1(0))^q}\cdot V^\mu(t)-\left[\,q-(q-1)\mu_1(t)\,\right]\cdot V^{\widehat{\pi}}(t)$$

dominates $\frac{q\,\mu_1(t)}{(\mu_1(0))^q}\cdot V^\mu(t)-\left[q-(q-1)\mu_1(t)\right]\cdot\left(\frac{\mu_1(t)}{\mu_1(0)}\right)^q V^\mu(t)$, or equivalently

$$\frac{V^\mu(t)\mu_1(t)}{(\mu_1(0))^q}\cdot\left[\,(q-1)\;(\mu_1(t))^q+q\,\left\{1-(\mu_1(t))^{q-1}\right\}\,\right] \;>\; 0\,,$$

again thanks to (8.5) and $q>1>(\mu_1(t))^q$. Thus $\eta(\cdot)$ is indeed an all-long (strict) portfolio.

On the other hand, $\eta(\cdot)$ *outperforms at $t=T$ a market portfolio that starts with the same initial capital*; this is because $\eta(\cdot)$ is long in the market $\mu(\cdot)$ and short in the extended portfolio $\widehat{\pi}(\cdot)$, which underperforms the market at $t=T$. Indeed, from Exercise 8.3 we have

$$V^{z,\eta}(T) \;=\; \frac{q}{(\mu_1(0))^q}\,V^\mu(T)-V^{\widehat{\pi}}(T) \;>\; zV^\mu(T)=V^{z,\mu}(T) \quad \text{a.s.}$$

¶ Note, however, that as $T\downarrow 0$, the initial capital $z(T)=q(T)/(\mu_1(0))^{q(T)}-1$ required to do all of this, increases without bound: It may take a huge amount of initial investment to realize the "extra basis point's worth of relative arbitrage" over a very short time-horizon – confirming of course, if confirmation is needed, that `time is money`... .

9 A Diverse Market Model

The careful reader might have been wondering, whether the theory we have developed so far may turn out to be vacuous. *Do there exist market models of the form* (1.1), 1.2) *that are diverse, at least weakly?* This is of course a very legitimate question.

Let us mention then, rather briefly, an example of such a market model $\mathcal{M}$ which is diverse over any given time-horizon $[0,T]$ with $0 < T < \infty$, and indeed satisfies the conditions of subsection 4.1 as well. For the details of this construction we refer to [FKK] (2005).

With given $\delta \in (1/2, 1)$, equal numbers of stocks and driving Brownian motions (that is, $d = m$), constant volatility matrix σ that satisfies (3.10), and non-negative numbers $g_1, \dots, g_m$, we take a model

$$\mathrm{d}\big(\log S_i(t)\big) \;=\; \gamma_i(t)\,\mathrm{d}t \,+\, \sum_{\nu=1}^{m} \sigma_{i\nu}\mathrm{d}W_\nu(t)\,, \qquad 0 \le t \le T \tag{9.1}$$

in the form (1.5) for the vector $\mathfrak{S}(\cdot) = \big(S_1(\cdot), \cdots, S_m(\cdot)\big)'$ of stock prices. With the usual notation $S(t) = \sum_{j=1}^{m} S_j(t)$, its growth rates are specified as

$$\gamma_i(t) \;:=\; g_i\, 1_{\mathcal{Q}_i^c}(\mathfrak{S}(t)) - \frac{M}{\delta} \cdot \frac{1_{\mathcal{Q}_i}(\mathfrak{S}(t))}{\log\big((1-\delta)S(t)/S_i(t)\big)}\,. \tag{9.2}$$

In other words, $\gamma_i(t) = g_i \ge 0$ if $\mathfrak{S}(t) \notin \mathcal{Q}_i$ (the i^{th} stock does not have the largest capitalization); and

$$\gamma_i(t) = -\frac{M}{\delta} \cdot \frac{1}{\log\big((1-\delta)/\mu_i(t)\big)}\,, \qquad \text{if} \quad \mathfrak{S}(t) \in \mathcal{Q}_i \tag{9.3}$$

(the i^{th} stock *does* have the largest capitalization). We are setting here

$$\mathcal{Q}_1 := \left\{ x \in (0,\infty)^m \;\middle|\; x_1 \ge \max_{2 \le j \le m} x_j \right\}, \quad \mathcal{Q}_m := \left\{ x \in (0,\infty)^m \;\middle|\; x_m > \max_{1 \le j \le m-1} x_j \right\},$$

$$\mathcal{Q}_i := \left\{ x \in (0,\infty)^m \;\middle|\; x_i > \max_{1 \le j \le i-1} x_j\,,\; x_i \ge \max_{i+1 \le j \le m} x_j \right\} \quad \text{for } i = 2, \dots, m-1\,.$$

With this specification (9.2), (9.3), "all stocks but the largest behave like geometric Brownian motions" (with growth rates $g_i \ge 0$ and variances $a_{ii} = \sum_{\nu=1}^{m} \sigma_{i\nu}^2$), whereas the log-price of the largest stock is subjected to a log-pole-type singularity in its drift, away from an appropriate right boundary.

One can then show that the resulting system of stochastic differential equations has a unique, strong solution (so the filtration $\mathbb{F}$ is now the one generated by the driving $m-$dimensional Brownian motion), and that the diversity requirement (6.1) is satisfied on any given time-horizon. Such models can be modified appropriately, to create ones that are weakly diverse but not diverse.

♠ Slightly more generally, in order to guarantee diversity it is enough to require

$$\min_{2\le k\le m} \gamma_{(k)}(t) \;\ge\; 0 \;\ge\; \gamma_{(1)}(t)\,, \qquad \min_{2\le k\le m} \gamma_{(k)}(t) - \gamma_{(1)}(t) + \frac{\varepsilon}{2} \;\ge\; \frac{M}{\delta}\cdot F(Q(t))\,,$$

where $Q(t) := \log\big((1-\delta)/\mu_{(1)}(t)\big)$.

Here the function $F:(0,\infty)\to(0,\infty)$ is taken to be continuous, and such that the associated scale function

$$U(x) \;:=\; \int_1^x \exp\left[-\int_1^y F(z)dz\right] dy\,, \quad x\in(0,\infty) \qquad \text{satisfies} \quad U(0+) \;=\; -\infty\,;$$

for instance, we have $U(x)=\log x$ when $F(x)=1/x$ as above.

• Under these conditions, it can then be shown that the process $Q(\cdot)$ satisfies $\int_0^T (Q(t))^{-2}\,dt < \infty$ a.s. – and this leads to the a.s. square-integrability $\sum_{i=1}^m \int_0^T (b_i(t))^2\,dt < \infty$ of the induced rates of return for individual stocks

$$b_i(t) \;=\; \frac{1}{2}\,a_{ii} + g_i\,1_{\mathcal{Q}_i^c}(\mathfrak{S}(t)) - \frac{M}{\delta}\cdot\frac{1_{\mathcal{Q}_i}(\mathfrak{S}(t))}{\log\big((1-\delta)S(t)/S_i(t)\big)}\,, \qquad i=1,\cdots,m\,.$$

This property is, of course, very crucial: it guarantees that the market-price-of-risk process $\vartheta(\cdot) := \sigma^{-1}\,b(\cdot)$ is locally square-integrable a.s., so the exponential local martingale $Z(\cdot)$ of (5.5) is well defined. Thus the results of Proposition 5.2 and Exercise 5.3 are applicable to this model.

For additional examples and an interesting probabilistic construction, see Osterrieder & Rheinländer (2006).

10 Hedging and Optimization without EMM

Let us broach now the issue of hedging contingent claims in a market such as that of subsection 5.1, and over a time-horizon $[0,T]$ for which (5.2) is satisfied.

Consider first a *European contingent claim*, that is, an $\mathcal{F}(T)-$measurable random variable $Y:\Omega\to[0,\infty)$ with

$$0 \;<\; y \;:=\; E\,[\,Y Z(T)\,/\,B(T)\,] \;<\; \infty \tag{10.1}$$

in the notation of (5.5). From the point of view of the "seller" of the contingent claim (e.g., stock option), this random amount represents a liability that has to be covered with the right amount of initial funds at time $t=0$ and the right trading strategy during the interval $[0,T]$, so that at the end of the period (time $t=T$) the initial funds have grown enough, to cover the liability without risk. Thus, the seller is very interested in the *upper hedging price*

$$\mathcal{U}^Y(T) := \inf\big\{\, w>0 \mid \exists\; h(\cdot)\in\mathcal{H}(w;T) \;\text{ s.t. }\; \mathcal{V}^{w,h}(T)\ge Y\,,\; \text{a.s.}\,\big\}\,, \tag{10.2}$$

the smallest amount of initial capital that makes such riskless hedging possible.

The standard theory of Mathematical Finance assumes that $\mathfrak{M}$, the set of equivalent martingale measures (EMM) for the model $\mathcal{M}$, is non-empty; then computes $\mathcal{U}^Y(T)$ as

$$\mathcal{U}^Y(T) = \sup_{\mathbb{Q}\in\mathfrak{M}} \mathbb{E}^{\mathbb{Q}}\left[\, Y \,/\, B(T)\,\right],$$

the supremum of the claim's discounted expected value over this set of probability measures. In our context an EMM will typically not exist (that is, $\mathfrak{M} = \emptyset$ as in Proposition 5.4), so the approach breaks down and the problem seems hopeless.

Not quite, though: there is still a long way one can go, simply by utilizing the availability of the strict local martingale $Z(\cdot)$ (and of the associated "deflator" $Z(\cdot)\,/B(\cdot)$), as well as the properties (5.8), (5.9) of the processes in (5.7). For instance, if the set on the right-hand side of (10.2) is not empty, then for any $w > 0$ in this set and for any $h(\cdot) \in \mathcal{H}(w;T)$, the local martingale $\widehat{\mathcal{V}}^{w,h}(\cdot)$ of (5.7) is non-negative, thus a supermartingale. This gives

$$w \,\geq\, \mathbb{E}\left[\,\mathcal{V}^{\,w,h}(T)Z(T)\,/\,B(T)\,\right] \,\geq\, \mathbb{E}\left[\,YZ(T)\,/\,B(T)\,\right] \,=\, y\,,$$

and because $w > 0$ is arbitrary we deduce the inequality $\mathcal{U}^Y(T) \geq y$ (which holds trivially if the set of (10.2) is empty, since then $\mathcal{U}^Y(T) = \infty$).

10.1 Completeness without an EMM

To obtain the reverse inequality, we shall assume that $m = d$, i.e., that we have exactly as many sources of randomness as there are stocks in the market $\mathcal{M}$, and that the filtration $\mathbb{F}$ is generated by the driving Brownian Motion $W(\cdot)$ in (1.1). With these assumptions one can represent the non-negative martingale $M(t) := \mathbb{E}\left[\,YZ(T)\,/\,B(T)\,|\,\mathcal{F}(t)\,\right],\ 0 \leq t \leq T$ as a stochastic integral

$$M(t) \,=\, y \,+\, \int_0^t \psi'(s)\,dW(s) \,\geq\, 0\,, \qquad 0 \leq t \leq T \tag{10.3}$$

for some progressively measurable and a.s. square-integrable process $\psi : [0,T]\times\Omega \to \mathbb{R}^d$. Setting $V_*(\cdot) := M(\cdot)B(\cdot)/Z(\cdot)$ and $h_*(\cdot) := (B(\cdot)/Z(\cdot))\,a^{-1}(\cdot)\,\sigma(\cdot)\left[\,\psi(\cdot)+V_*(\cdot)\vartheta(\cdot)\,\right]$, then comparing (5.9) with (10.3), we observe $V_*(0) = y$, $V_*(T) = Y$ and $V_*(\cdot) \equiv \mathcal{V}^{\,y,h_*}(\cdot) \geq 0$, almost surely.

Therefore, the trading strategy $h_*(\cdot)$ is in $\mathcal{H}(y;T)$ and satisfies the `exact replication property` $\mathcal{V}^{\,y,h_*}(T) = Y$ a.s. This implies that y belongs to the set on the right-hand-side of (10.2), and so $y \geq \mathcal{U}^Y(T)$. But we have already established the reverse inequality, actually in much greater generality, so recalling (10.1) we get the *Black-Scholes-type formula*

$$\mathcal{U}^Y(T) \,=\, \mathbb{E}\left[\,YZ(T)/B(T)\,\right] \tag{10.4}$$

for the upper hedging price of (10.2), under the assumptions of the preceding paragraph. To wit, a market $\mathcal{M}$ that is weakly diverse – hence without an equivalent probability measure under which discounted stock-prices are (at least local) martingales – can nevertheless be *complete*. Similar observations have been made by Lowenstein & Willard (2000.a,b) and by Platen (2002, 2006).

10.2 Ramifications and Open Problems

10.1 Example: A European Call-Option. Consider the contingent claim $Y = \big(S_1(T) - q\big)^+$: this is a European call-option with strike $q > 0$ on the first stock. Let us assume also that the interest-rate process $r(\cdot)$ is bounded away from zero, namely that $\mathbb{P}[\, r(t) \geq r\,,\ \forall\, t \geq 0\,] = 1$ holds for some $r > 0$, and that the market $\mathcal{M}$ is weakly diverse on all sufficiently large time-horizons $T \in (0, \infty)$. Then for the hedging price of this contingent claim we have from Exercise 5.3, (10.4), Jensen's inequality, and $\mathbb{E}\,[\, Z(T)\,] < 1$:

$$
\begin{aligned}
S_1(0) \quad &> \quad \mathbb{E}\,[\, Z(T)S_1(T)/B(T)\,] \geq \mathbb{E}\,[\, Z(T)(S_1(T) - q)^+/B(T)\,] = \mathcal{U}^Y(T) \\
&\geq \quad \Big(\mathbb{E}\,[\, Z(T)S_1(T)/B(T)\,] - q \cdot E\,\Big(Z(T) \cdot e^{-\int_0^T r(t)\,dt}\Big)\Big)^+ \\
&\geq \quad \Big(\mathbb{E}\,[\, Z(T)S_1(T)/B(T)\,] - q\, e^{-rT}\, \mathbb{E}\,[Z(T)]\Big)^+ \\
&\geq \quad \Big(\mathbb{E}\,[\, Z(T)S_1(T)/B(T)\,] - q\, e^{-rT}\Big)^+ , \qquad \text{thus}
\end{aligned}
$$

$$
0 \leq \mathcal{U}^Y(\infty) := \lim_{T \to \infty} \mathcal{U}^Y(T) = \lim_{T \to \infty} \downarrow\, \mathbb{E}\big(Z(T)S_1(T)/B(T)\big) < S_1(0). \tag{10.5}
$$

The upper hedging price of the option is *strictly less* than the capitalization of the underlying stock at time $t = 0$, and tends to $\mathcal{U}^Y(\infty) \in [0, S_1(0))$ as the horizon increases without limit.

If $\mathcal{M}$ is weakly diverse uniformly over some $[T_0, \infty)$, then the limit in (10.5) is actually zero: *a European call-option that can never be exercised has zero hedging price*. Indeed, for every fixed $p \in (0,1)$ and $T \geq \frac{2 \log m}{p \varepsilon \delta} \vee T_0$, and with the normalization $S(0) = 1$, the quantity

$$
\mathbb{E}\left(\frac{Z(T)}{B(T)} S_1(T)\right) \leq \mathbb{E}\left(\frac{Z(T)}{B(T)} V^{\mu}(T)\right) \leq \mathbb{E}\left(\frac{Z(T)}{B(T)} V^{\pi^{(p)}}(T)\right) \cdot m^{\frac{1-p}{p}}\, e^{-\varepsilon\delta(1-p)T/2}
$$

is dominated by $m^{\frac{1-p}{p}} \cdot e^{-\varepsilon\delta(1-p)T/2}$, from (7.2), (2.2) and the supermartingale property of $Z(\cdot)V^{\pi^{(p)}}(\cdot)/B(\cdot)$. Letting $T \to \infty$ we obtain $\mathcal{U}^Y(\infty) = 0$.

10.2 Remark: Note the sharp difference between this case and the situation where an equivalent martingale measure exists on every finite time-horizon; namely, when both $Z(\cdot)$ and $Z(\cdot)S_1(\cdot)/B(\cdot)$ are martingales. Then we have $\mathbb{E}[\, Z(T)S_1(T)/B(T)\,] = S_1(0)$ for all $T \in (0, \infty)$, and $\mathcal{U}^Y(\infty) = S_1(0)$: as the time-horizon increases without limit, the hedging price of the call-option approaches the current stock value (Karatzas & Shreve (1998), p.62).

10.3 Remark: The above theory extends to the case $d > m$ of *incomplete markets*, and more generally to *closed, convex constraints* on portfolio choice as in Chapter 5 of Karatzas & Shreve (1998), under the conditions of (5.4). See the paper Karatzas & Kardaras (2006) for a treatment of these issues in a general semimartingale setting.

10.4 Exercise: Argue that the "put-call parity" property $\mathcal{U}^Y - \mathcal{U}^X = S_1(0) - q$, which is valid for the put and call options $X = (q - S_1(T))^+$ and $Y = (S_1(T) - q)^+$ when a unique EMM exists on $[0,T]$, fails when the deflated stock-prices are strict local martingales as in Exercise 5.3.

`10.5 Open Question:` Develop a theory for pricing *American contingent claims* under the assumptions of the present section. As Constantinos Kardaras (2006) observes, in the absence of an EMM (in fact, even when there exists an equivalent probability measure under which discounted prices are *local* martingales), it may not be optimal to exercise only at maturity $t = T$ an American call option written on a non-dividend-paying stock: early exercise may be advantageous, and this is closely related to the property $\mathcal{U}^Y(\infty) = 0$ above.

Can one then characterize, or compute, the optimal exercise time?

10.3 Utility Maximization in the Absence of EMM

Suppose we are given initial capital $w > 0$, a finite time-horizon $T > 0$, and a utility function $u : (0,\infty) \to \mathbb{R}$ (strictly increasing, strictly concave, of class $\mathcal{C}^1$, with $u'(0) := \lim_{x \downarrow 0} u'(x) = \infty$, $u'(\infty) := \lim_{x \to \infty} u'(x) = 0$ and $u(0) := \lim_{x \downarrow 0} u(x)$). The problem is to compute the maximal expected utility

$$\mathfrak{u}(w) := \sup_{h(\cdot) \in \mathcal{H}(w;T)} \mathbb{E}\left[u\big(\mathcal{V}^{w,h}(T)\big) \right]$$

from terminal wealth; to decide whether the supremum is attained; and if so, to identify a strategy $\widehat{h}(\cdot) \in \mathcal{H}(w;T)$ that attains it. We place ourselves under the assumptions of the present section, including those of subsection 10.1.

10.6 Exercise: Show that the answer to this question is given by the replicating strategy $\widehat{h}(\cdot) \in \mathcal{H}_+(w;T)$ for the contingent claim

$$\Upsilon = I\big(\Xi(w)\, D(T)\big), \qquad D(\cdot) := Z(\cdot)/B(\cdot),$$

in the sense $\mathcal{V}^{w,\widehat{h}}(T) = \Upsilon$ a.s. Here $I : (0,\infty) \to (0,\infty)$ is the inverse of the strictly decreasing "marginal utility" function $u' : (0,\infty) \to (0,\infty)$, and $\Xi : (0,\infty) \to (0,\infty)$ the inverse of the strictly decreasing function $\mathcal{W}(\cdot)$ given by

$$\mathcal{W}(\xi) := \mathbb{E}\left[D(T)\, I\left(\xi\, D(T)\right) \right], \quad 0 < \xi < \infty,$$

which we are assuming to be $(0,\infty)-$valued.

- In the case of the logarithmic utility function $u(x) = \log x$, $x \in (0,\infty)$, show that the "log-optimal" trading strategy $h_*(\cdot) \in \mathcal{H}_+(w;T)$ and its associated wealth process $V_*(\cdot) \equiv \mathcal{V}^{w,h_*}(\cdot)$ are given, respectively, by

$$h_*(t) = V_*(t) \cdot a^{-1}(t)[\, b(t) - r(t)\mathbb{I}\,], \quad V_*(t) = w/D(t) \tag{10.6}$$

for $0 \le t \le T$. Note also that the discounted log-optimal wealth process satisfies

$$\mathrm{d}\big(V_*(t)/B(t)\big) = \big(V_*(t)/B(t)\big)\, \vartheta'(t) \left[\vartheta(t)\,\mathrm{d}t + \mathrm{d}W(t) \right]. \tag{10.7}$$

Remark: Note that no assumption is been made regarding the existence of an EMM (i.e., that $Z(\cdot)$ should be a martingale). See Karatzas, Lehoczky, Shreve & Xu (1991) for more information on this problem, and on its much more interesting *incomplete market* version $d > m$, under the assumption that the volatility matrix $\sigma(\cdot)$ is of full (row) rank and without assuming the existence of EMM.

The log-optimal trading strategy of (10.6) has some obviously desirable features, discussed in the next exercise. But unlike the functionally-generated portfolios of the next section, it needs for its implementation knowledge of the variance/covariance structure and of the mean rates of return; these are very hard to estimate in practice.

10.7 Exercise: The "Numéraire" Property. Assume that the log-optimal strategy of (10.6) is defined for all $0 \le t < \infty$. Show that it has then the following *"numéraire property"*

$$\mathcal{V}^{w,h}(\cdot)\,/\,\mathcal{V}^{w,h_*}(\cdot) \qquad \text{is a supermartingale,} \;\; \forall \;\; h(\cdot) \in \mathcal{H}_+(w)\,, \tag{10.8}$$

and deduce the *asymptotic growth optimality* property

$$\limsup_{t\to\infty} \frac{1}{t}\left(\log\left(\frac{\mathcal{V}^{w,h}(t)}{\mathcal{V}^{w,h_*}(t)}\right)\right) \le 0 \quad \text{a.s.,} \quad \forall \;\; h(\cdot) \in \mathcal{H}_+(w)\,.$$

10.8 Exercise (Platen (2006)): Show that the equation for $\Psi(\cdot) := V_*(\cdot)/B(\cdot) = w/Z(\cdot)$ in (10.7) can be written as

$$\mathrm{d}\Psi(t) \;=\; \alpha(t)\,\mathrm{d}t \;+\; \sqrt{\Psi(t)\,\alpha(t)}\,\cdot \mathrm{d}\mathfrak{B}(t)\,, \qquad \Psi(0) = w$$

where $\mathfrak{B}(\cdot)$ is one-dimensional Brownian motion, and $\alpha(t) := \Psi(\cdot)\,||\vartheta(\cdot)||^2$.

Then observe that $\Psi(\cdot)$ is a time-changed and scaled *squared Bessel process in dimension 4* (sum of squares of four independent Brownian motions); that is, $\Psi(\cdot) \;=\; \mathfrak{X}\big(A(\cdot)\big)/4$, where

$$A(\cdot) \;:=\; \int_0^{\cdot} \alpha(s)\,\mathrm{d}s \qquad \text{and} \qquad \mathfrak{X}(u) = 4(w+u) + 2\int_0^u \sqrt{\mathfrak{X}(v)}\;\mathrm{d}\mathfrak{b}(v)\,, \quad u \ge 0$$

in terms of yet another standard, one-dimensional Brownian motion $\mathfrak{b}(\cdot)$.

11 Functionally-Generated Portfolios

We shall introduce now a class of new portfolios, called *functionally-generated*, that generalize broadly the diversity-weighted ones of section 7. For these new portfolios one can derive a decomposition of their relative return analogous to that of (7.6), and this proves useful in the construction and study of arbitrages relative to the market. Just like (7.6), this new decomposition (11.2) does not involve stochastic integrals, and opens the possibility for making probability-one comparisons on given time horizons.

Suppose that $\mathbf{G} : \mathbb{R}^m \to (0,\infty)$ is a $\mathcal{C}^2-$function, such that the mapping $x \mapsto x_i \, D_i \log \mathbf{G}(x)$ is bounded on some open neighborhood U of Δ^m, for all $i = 1, \cdots, m$. Consider also the extended portfolio rule $\pi(\cdot)$ with weights

$$\underbrace{\pi_i(t) \, = \, \left[D_i \log \mathbf{G}(\mu(t)) + 1 - \sum_{j=1}^m \mu_j(t) \, D_j \log \mathbf{G}(\mu(t)) \right] \cdot \mu_i(t)} \, , \qquad 1 \le i \le m \, . \tag{11.1}$$

We call this the `extended portfolio generated by` $\mathbf{G}(\cdot)$. It can be shown that the wealth process of this extended portfolio, relative to the market, is given by the *master formula*

$$\underbrace{\log\left(\frac{V^\pi(T)}{V^\mu(T)}\right) \, = \, \log\left(\frac{\mathbf{G}(\mu(T))}{\mathbf{G}(\mu(0))}\right) \, + \, \int_0^T \mathfrak{g}(t) \, \mathrm{d}t} \, , \qquad 0 \le T < \infty \tag{11.2}$$

with *drift process* $\mathfrak{g}(\cdot)$ given by

$$\underbrace{\mathfrak{g}(t) \, := \, \frac{-1}{2\mathbf{G}(\mu(t))} \sum_{i=1}^m \sum_{j=1}^m D^2_{ij} \mathbf{G}(\mu(t)) \cdot \mu_i(t) \mu_j(t) \, \tau^\mu_{ij}(t) \, .} \tag{11.3}$$

The quantities of (11.1) depend only on the market weights $\mu_1(t), \cdots, \mu_m(t)$, not on the covariance structure of the market. Therefore (11.1) can be implemented, and its associated wealth-process $V^\pi(\cdot)$ observed through time, only in terms of the evolution of these market weights over $[0,T]$.

The covariance structure enters only in the computation of the drift term of (11.3). But it should be stressed that in order to compute the "cumulative effect" $\int_0^T \mathfrak{g}(t) \, \mathrm{d}t$ of this drift over a period of time $[0,T]$ using past data, there is no need to know or estimate this covariance structure at all: the equation (11.2) does this for us in the form $\int_0^T \mathfrak{g}(t) \, \mathrm{d}t \, = \log \big(V^\pi(T) \mathbf{G}(\mu(0)) / V^\mu(T) \mathbf{G}(\mu(T)) \big)$, which contains only observable quantities.

11.1 Exercise: `Establish the "master formula" (11.2).`

11.2 Exercise: Suppose the function $\mathbf{G}(\cdot)$ is `concave` – or, more precisely, its Hessian $D^2\mathbf{G}(x) = \big\{ D^2_{ij}\mathbf{G}(x) \big\}_{1 \le i,j \le m}$ has at most one positive eigenvalue for each $x \in \Delta^n$ and, if a positive eigenvalue exists, the corresponding eigenvector is orthogonal to Δ^n. Then the portfolio rule $\pi(\cdot)$ generated by $\mathbf{G}(\cdot)$ as in (11.1) is strict (i.e., each weight $\pi_i(\cdot)$ is non-negative), and the drift term $\mathfrak{g}$ is non-negative; if $\mathrm{rank}\big(D^2\mathbf{G}(x)\big) > 1$ holds for each $x \in \Delta^n$, then $\mathfrak{g}(\cdot)$ is positive.

For instance, the choice

- $\mathbf{G}(\cdot) \equiv 1$ generates the market-portfolio;
- $\mathbf{G}(x) = \varphi_1 x_1 + \cdots + \varphi_m x_m$ generates the portfolio that buys at time $t = 0$, and holds until time $t = T$, a fixed number of shares φ_i in each stock;

• $\mathbf{G}(x) = \big(x_1 \cdots x_m\big)^{1/m}$ generates the *equally-weighted* (or "value-line") portfolio $\eta_i(\cdot) \equiv 1/m\,,\ i = 1, \cdots, m$ with $\mathfrak{g}(\cdot) \equiv \gamma_*^{\eta}(\cdot)\,;$

• $\mathbf{G}(x) = \big(x_1^p + \cdots + x_m^p\big)^{1/p}$ for some $0 < p < 1$ generates the *diversity-weighted* portfolio $\pi^{(p)}(\cdot)$, of (7.1) with $\mathfrak{g}(\cdot) \equiv (1-p)\,\gamma_*^{\pi^{(p)}}(\cdot)\,;$

• Consider the *entropy function* $\mathbf{H}(x) := 1 - \sum_{i=1}^m x_i \log x_i\,,\ x \in \Delta_{++}^m$ and, for any given $c \in (0,\infty)$, its modification

$$\mathbf{G}_c(x) := c + \mathbf{H}(x)\,, \quad \text{which satisfies:} \qquad c < \mathbf{G}_c(x) \le c + \log m\,, \quad x \in \Delta_{++}^m\,. \tag{11.4}$$

This new, modified entropy function, generates an *entropy-weighted* portfolio $\varpi^c(\cdot)$ with weights and drift given, respectively, as

$$\varpi_i^c(t) \;=\; \frac{\mu_i(t)}{\mathbf{G}_c(\mu(t))}\Big(c - \log \mu_i(t)\Big)\,,\ \ 1 \le i \le m \quad \text{and} \quad \mathfrak{g}^c(t) = \frac{\gamma_*^{\mu}(t)}{\mathbf{G}_c(\mu(t))}\,. \tag{11.5}$$

11.1 Sufficient Intrinsic Volatility leads to Arbitrage

Broadly accepted practitioner wisdom upholds that "sufficient volatility creates opportunities in a market". We shall try to put this intuition on a precise quantitative basis in Example 11.3 below, by identifying the excess growth rate of the market portfolio – which also measures the market's "available *intrinsic* volatility", according to (3.8) and the discussion following it – as a quantity whose availability can lead to arbitrage opportunities relative to the market.

11.3 Example: Suppose now that in the market $\mathcal{M}$ of (1.1), (1.2) there exists a constant $\zeta > 0$ such that

$$\underbrace{\frac{1}{T}\int_0^T \gamma_*^{\mu}(t)\,\mathrm{d}t \;\ge\; \zeta} \tag{11.6}$$

holds almost surely. For instance, this is the case when the excess growth rate of the market portfolio is bounded away from zero: that is, when

$$\underbrace{\gamma_*^{\mu}(t) \;\ge\; \zeta\,, \quad \forall\ \ 0 \le t \le T} \tag{11.7}$$

holds almost surely, for some constant $\zeta > 0$.

Consider again the entropy-weighted portfolio of (11.5), namely

$$\varpi_i^c(t) \;=\; \frac{\mu_i(t)\big(c - \log \mu_i(t)\big)}{\sum_{j=1}^m \mu_j(t)\big(c - \log \mu_j(t)\big)}\,, \qquad i = 1, \cdots, m\,, \tag{11.8}$$

now written in a form that makes plain its over-weighting of the small capitalization stocks. From (11.2) and the inequalities of (11.4), one sees that the relative performance of this portfolio with respect to the market, is given by

$$\log\left(\frac{V^{\varpi^c}(T)}{V^\mu(T)}\right) = \log\left(\frac{\mathbf{G}_c(\mu(T))}{\mathbf{G}_c(\mu(0))}\right) + \int_0^T \frac{\gamma_*^\mu(t)}{\mathbf{G}_c(\mu(t))}\,\mathrm{d}t \tag{11.9}$$

$$> -\log\left(\frac{c+\mathbf{H}\big(\mu(0)\big)}{c}\right) + \frac{\zeta T}{c+\log m}$$

almost surely. Thus, for every time horizon

$$T > \mathcal{T}_*(c) := \frac{1}{\zeta}\,\big(c+\log m\big)\cdot\log\left(1+\frac{\mathbf{H}\big(\mu(0)\big)}{c}\right),$$

or for that matter any

$$T > \mathcal{T}_* = \frac{1}{\zeta}\,\mathbf{H}\big(\mu(0)\big) \tag{11.10}$$

(since $\lim_{c\to\infty}\mathcal{T}_*(c)=\mathcal{T}_*$) and with $c>0$ sufficiently large, the portfolio $\varpi^c(\cdot)$ of (11.8) satisfies (5.2) relative to the market, on the given time-horizon $[0,T]$. It is straightforward that (5.1) is also satisfied, with $q=c/(c+\mathbf{H}(\mu(0))$. ◇

In particular, we have $\mathcal{L}^{\varpi^c,\mu}\geq\zeta/(c+\log m)>0$ a.s., under the condition (11.7) and with the notation of (5.3). Note also that we have not assumed in this discussion any condition (such as (1.15) or (3.10)) on the volatility structure of the market, beyond the minimal assumption of (1.2).

Remark: Let us recall here our discussion of the conditions in (6.3): if the variance/covariance matrix $a(\cdot)$ has all its eigenvalues bounded away from both zero and infinity, then the condition (11.7) (respectively, (11.6)) is equivalent to diversity (respectively, weak diversity) on [0,T]. The point of these latter conditions is that they guarantee the existence of relative arbitrage even when volatilities are unbounded and diversity fails. In the next section we shall study a concrete example of such a situation.

✠ Figure 1 plots the cumulative market excess growth for the U.S. equities over most of the twentieth century. *This computation does not need any estimation of covariance structure:* from (11.9) we can express this cumulative excess growth

$$\int_0^\cdot \gamma_*^\mu(t)\,\mathrm{d}t = \int_0^\cdot \mathbf{G}_c\big(\mu(t)\big)\cdot\mathrm{d}\left(\log\left(\frac{V^{\varpi^c}(t)}{V^\mu(t)}\,\frac{\mathbf{G}_c(\mu(0))}{\mathbf{G}_c(\mu(t))}\right)\right),$$

just in terms of quantities that are observable in the market. The plot suggests that the U.S. market has exhibited a strictly increasing cumulative excess growth over this period.

`Open Question:` Is the condition (11.7) sufficient for the existence of relative arbitrage over *arbitrary* (as opposed to sufficiently long) time-horizons?

Remark: For $0<p\leq 1$, introduce the quantity

$$\gamma_*^{\pi,p}(t) := \frac{1}{2}\sum_{i=1}^m \big(\pi_i(t)\big)^p\,\tau_{ii}^\pi(t) \tag{11.11}$$

which generalizes the excess growth rate of a portfolio $\pi(\cdot)$, in the sense $\gamma_*^{\pi,1}(\cdot) \equiv \gamma_*^{\pi}(\cdot)$. It is shown in Proposition 3.8 of [FK] (2005) that, with $0 < p < 1$, the a.s. requirement

$$\Gamma(T) \,\leq\, \int_0^T \gamma_*^{p,\mu}(t)\,dt \,<\, \infty\,, \qquad \forall\;\, 0 \leq T < \infty\,, \tag{11.12}$$

where $\Gamma : [0,\infty) \to [0,\infty)$ is a continuous, strictly increasing function with $\Gamma(0) = 0$, $\Gamma(\infty) = \infty$, guarantees that the portfolio

$$\pi_i(t) \,:=\, p\,\frac{\big(\mu_i(t)\big)^p}{\sum_{j=1}^m \big(\mu_j(t)\big)^p} \,+\, (1-p)\,\mu_i(t)\,, \qquad i = 1, \cdots, m \tag{11.13}$$

is an arbitrage opportunity relative to the market, namely $\mathbb{P}\,[\,V^\pi(T) > V^\pi(T)\,] = 1$, over sufficiently long time-horizons: $T > \mathrm{T}_* := \Gamma^{-1}(m^{1-p}\log m/p)$.

`Open Question:` Does (11.12) guarantee the existence of relative arbitrage opportunities over arbitrary time-horizons?

`Open Question:` Is there a result on the existence of relative arbitrage, that generalizes both Example 11.3 and the result outlined in (11.12), (11.13)? What quantity(ies) might be involved, in place of the market excess growth or its generalization (11.12)? is there a "best" result of this type?

`Open Question:` We have presented a few portfolios that lead to arbitrage relative to the market; they are all functionally generated (F-G). Is there a "best" such example within that class? Are there similar examples of portfolios that are *not* functionally generated, nor trivial modifications thereof? How representative (or "dense") in this context is the class of F-G portfolios?

`Open Question:` Generalize the theory of F-G portfolios to the case of a market with a countable infinity ($m = \infty$) of assets, or to some other model with a variable, unbounded number of assets.

`Open Question:` What, if any, is the connection of F-G portfolios with the "universal portfolios" of Cover (1991) and Jamshidian (1992) ?

11.2 Selection by Rank, Leakage, and the Size Effect

An important generalization of the ideas and methods in this section concerns generating functions that record market weights not according to their name (or index) i , but according to their rank. In order to present this generalization, let us recall the order statistics notation of (1.17) and considerfor each $0 \leq t < \infty$ the random permutation $\big(p_t(1), \cdots, p_t(m)\big)$ of $(1, \cdots, m)$ with

$$\mu_{p_t(k)}(t) = \mu_{(k)}(t)\,, \qquad \text{and} \quad p_t(k) < p_t(k+1) \quad \text{if} \quad \mu_{(k)}(t) = \mu_{(k+1)}(t) \tag{11.14}$$

for $k = 1, \ldots, m$. In words: $p_t(k)$ is the name (index) of the stock with the k^{th} largest relative capitalization at time t, and ties are resolved by resorting to the lowest index. Using Itô's rule for convex functions of semimartingales (e.g. Karatzas & Shreve [KS] (1991), section 3.7), one can obtain the following analogue of (2.5) for the *ranked market-weights*

$$\frac{\mathrm{d}\mu_{(k)}(t)}{\mu_{(k)}(t)} = \left(\gamma_{p_t(k)}(t) - \gamma^{\mu}(t) + \frac{1}{2}\,\tau^{\mu}_{(kk)}(t)\right)\mathrm{d}t \,+\, \frac{1}{2}\left[\mathrm{d}\mathfrak{L}^{k,k+1}(t) - \mathrm{d}\mathfrak{L}^{k-1,k}(t)\right] + \sum_{\nu=1}^{d}\big(\sigma_{p_t(k)\nu}(t) - \sigma^{\mu}_{\nu}(t)\big)\, dW_\nu(t) \qquad (11.15)$$

for each $k = 1, \ldots, m-1$. Here the quantity $\mathfrak{L}^{k,k+1}(t) \equiv \Lambda_{\Xi_k}(t)$ is the semimartingale local time at the origin, accumulated by the non-negative process

$$\Xi_k(t) \,:=\, \log\big(\mu_{(k)}/\mu_{(k+1)}\big)\,(t)\,, \qquad 0 \le t < \infty \qquad (11.16)$$

by the calendar time t; it measures the cumulative effect of the changes that have occurred during $[0,t]$ between ranks k and $k+1$. We are also setting $\mathfrak{L}^{0,1}(\cdot) \equiv 0$, $\mathfrak{L}^{m,m+1}(\cdot) \equiv 0$ and $\tau^{\mu}_{(k\ell)}(\cdot) := \tau^{\mu}_{p_t(k)p_t(\ell)}(\cdot)$.

Speaking intuitively, and somewhat loosely, the quantity $\mathfrak{L}^{k,k+1}(t)$ (respectively, $\mathfrak{L}^{k-1,k}(t)$) accounts in (11.16) for the cumulative upward (resp., downward) "pressure" exerted on the k^{th}-ranked stock by its immediate follower (resp., leader) in the relative capitalization rank.

A derivation of this result, under *appropriate conditions* that we choose not to broach here, can be found on pp.76-79 of Fernholz (2002); see also Banner & Ghomrasni (2006) for generalizations.

With this setup, we have then the following generalization of the master formula (11.2): consider a function $G : \mathbb{R}^m \to (0,\infty)$ exactly as assumed there, but now written in the form

$$G(x_1, \cdots, x_m) \,=\, \mathcal{G}\big(x_{(1)}, \cdots, x_{(m)}\big)\,, \qquad \forall \quad x \in \Delta^m$$

for some $\mathcal{G} \in \mathcal{C}^2(\Delta^m)$. Then with the shorthand $\mu_{(\cdot)}(t) \,:=\, \big(\mu_{p_t(1)}(t), \cdots, \mu_{p_t(m)}(t)\big)'$ and the notation

$$\Gamma(T) \,:=\, -\int_0^T \frac{1}{2\mathcal{G}\big(\mu_{(\cdot)}(t)\big)} \sum_{k=1}^{m}\sum_{\ell=1}^{m} D^2_{k\ell}\,\mathcal{G}\big(\mu_{(\cdot)}(t)\big)\cdot \mu_{(k)}(t)\mu_{(\ell)}(t)\tau^{\mu}_{(k\ell)}(t)\,\mathrm{d}t + \frac{1}{2}\sum_{k=1}^{m-1}\Big[\underline{\pi}_{\,p_t(k+1)}(t) - \underline{\pi}_{\,p_t(k)}(t)\Big]\;\mathrm{d}\mathfrak{L}^{k,k+1}(t)\,, \qquad (11.17)$$

one can show that the performance of the portfolio $\underline{\pi}(\cdot)$ in

$$\underline{\pi}_{\,p_t(k)}(t) \,=\, \left[D_k \log \mathcal{G}\big(\mu_{(\cdot)}(t)\big) + 1 - \sum_{j=1}^{m}\mu_{(\ell)}(t)\, D_\ell \log \mathcal{G}\big(\mu_{(\cdot)}(t)\big)\right]\cdot \mu_{(k)}(t)\,, \qquad (11.18)$$

$1 \leq k \leq m$, relative to the market, is given as

$$\log\left(\frac{V^{\underline{\pi}}(T)}{V^{\mu}(T)}\right) \;=\; \log\left(\frac{\mathcal{G}\big(\mu_{(\cdot)}(T)\big)}{\mathcal{G}\big(\mu_{(\cdot)}(0)\big)}\right) \;+\; \Gamma(T)\,, \qquad 0 \leq T < \infty\,. \tag{11.19}$$

We say that $\underline{\pi}(\cdot)$ is then the portfolio generated by the function $\mathcal{G}(\cdot)$. The details of the proof can be found in Fernholz (2002), pp.79-83.

¶ For instance, $G(x) = x_{(1)}$ gives the portfolio $\underline{\pi}_{\,p_t(k)}(\cdot) = \delta_{1k}$, $k = 1, \cdots, m$ that invests in the largest stock only. Its relative performance

$$\log\left(\frac{V^{\underline{\pi}}(T)}{V^{\mu}(T)}\right) \;=\; \log\left(\frac{\mu_{(1)}(T)}{\mu_{(1)}(0)}\right) \;-\; \frac{1}{2}\,\mathfrak{L}^{1,2}(T)\,, \qquad 0 \leq T < \infty$$

will suffer in the long run, if there are many changes in leadership: in order for the biggest stock to do well relative to the market, it must crush all competition!

11.4 Example: The Size Effect is the tendency of small stocks to have higher long-term returns relative to their larger brethren. The formula of (11.19) affords a simple, structural explanation of this observed phenomenon, as follows.

Fix an integer $n \in \{2, \cdots, m-1\}$ and consider $G_L(x) = x_{(1)} + \cdots + x_{(n)}$, $G_S(x) = x_{(n+1)} + \cdots + x_{(m)}$. These functions generate, respectively, the large-cap-weighted portfolio

$$\zeta_{p_t(k)}(t) \;=\; \frac{\mu_{(k)}(t)}{G_L(\mu(t))}\,, \quad k = 1, \cdots, n \quad \text{and} \quad \zeta_{p_t(k)}(t) \;=\; 0\,, \quad k = n+1, \cdots, m \tag{11.20}$$

and the small-cap-weighted portfolio

$$\eta_{p_t(k)}(t) \;=\; \frac{\mu_{(k)}(t)}{G_S(\mu(t))}\,, \quad k = n+1, \cdots, m \quad \text{and} \quad \eta_{p_t(k)}(t) \;=\; 0\,, \quad k = 1, \cdots, n\,, \tag{11.21}$$

respectively. According to (11.19), the performances of these portfolios relative to the market are given by

$$\log\left(\frac{V^{\zeta}(T)}{V^{\mu}(T)}\right) \;=\; \log\left(\frac{G_L(\mu(T))}{G_L(\mu(0)}\right) \;-\; \frac{1}{2}\int_0^T \zeta_{(n)}(t)\,\mathrm{d}\mathfrak{L}^{n,n+1}(t)\,, \tag{11.22}$$

$$\log\left(\frac{V^{\eta}(T)}{V^{\mu}(T)}\right) \;=\; \log\left(\frac{G_S(\mu(T))}{G_S(\mu(0)}\right) + \frac{1}{2}\int_0^T \eta_{(n)}(t)\,\mathrm{d}\mathfrak{L}^{n,n+1}(t)\,, \tag{11.23}$$

respectively. Therefore,

$$\log\left(\frac{V^{\eta}(T)}{V^{\zeta}(T)}\right) = \log\left(\frac{G_S(\mu(T))G_L(\mu(0))}{G_L(\mu(T)G_S(\mu(0)}\right) + \int_0^T \frac{\zeta_{(n)}(t) + \eta_{(n)}(t)}{2}\,\mathrm{d}\mathfrak{L}^{n,n+1}(t)\,. \tag{11.24}$$

If there is "stability" in the market, in the sense that the ratio of the relative capitalization of small to large stocks remains stable over time, then the first term on the right-hand side of (11.24) does not change much – whereas the second term keeps increasing and accounts for the better relative performance of the small stocks. Note that this argument does not invoke at all any putative assumption about "greater riskiness" of the smaller stocks.

The paper Fernholz & Karatzas (2006) studies conditions under which such stability in relative capitalizations prevails, and contains further discussion.

11.5 Remark: Estimation of Local Times. Hard as this might be to have guessed from the outset, the local times $\mathfrak{L}^{n,n+1}(\cdot) \equiv \Lambda_{\Xi_n}(\cdot)$ appearing in (11.15), (11.17) can be estimated in practice pretty accurately; indeed, (11.22) gives

$$\mathfrak{L}^{n,n+1}(\cdot) = \int_0^{\cdot} \frac{2}{\zeta_{(n)}(t)} \, \mathrm{d}\left(\log\left(\frac{G_L(\mu(t))}{G_L(\mu(0))}\,\frac{V^{\mu}(t)}{V^{\zeta}(t)}\right)\right), \quad n = 1, \cdots, m-1, \tag{11.25}$$

and the quantity on the right-hand side is completely observable.

11.6 Exercise: Leakage in a Diversity-Weighted Index of Large Stocks. With n and $\zeta(\cdot)$ as in Example 11.4 and fixed $p \in (0,1)$, consider the *diversity-weighted, large-capitalization index*

$$\pi^{\sharp}_{p_t(k)}(t) = \frac{\big(\mu_{(k)}(t)\big)^p}{\sum_{\ell=1}^{n}\big(\mu_{(\ell)}(t)\big)^p}\,, \quad 1 \le k \le n \quad \text{and} \quad \pi^{\sharp}_{p_t(k)}(t) = 0\,, \quad n+1 \le k \le m \tag{11.26}$$

generated by $G^{\sharp}_p(x) := \big(\sum_{\ell=1}^{n}\big(x_{(\ell)}\big)^p\big)^{1/p}$, by analogy with (7.5), (7.1).

Express the performance of (11.26) relative to the entire market as

$$\log\left(\frac{V^{\pi^{\sharp}}(T)}{V^{\mu}(T)}\right) = \log\left(\frac{G^{\sharp}_p(\mu(T))}{G^{\sharp}_p(\mu(0))}\right) + \int_0^T (1-p)\gamma^{\pi^{\sharp}}_*(t)\mathrm{d}t - \int_0^T \frac{\pi^{\sharp}_{(n)}(t)}{2}\,\mathrm{d}\mathfrak{L}^{n,n+1}(t)\,,$$

and relative to the large-cap-weighted portfolio $\zeta(\cdot)$ of (11.20) as

$$\mathrm{d}\log\left(\frac{V^{\pi^{\sharp}}(t)}{V^{\zeta}(t)}\right) = \mathrm{d}\log \mathbf{G}_p\big(\zeta_{(1)}(t), \cdots, \zeta_{(n)}(t)\big) + (1-p)\gamma^{\pi^{\sharp}}_*(t)\mathrm{d}t + \frac{1}{2}\big(\zeta_{(n)}(t) - \pi^{\sharp}_{(n)}(t)\big)\,\mathrm{d}\mathfrak{L}^{n,n+1}(t) \tag{11.27}$$

in the notation of (7.5). Because $\pi^{\sharp}_{(n)}(\cdot) \ge \zeta_{(n)}(\cdot)$ from (7.8) and the remark following it, the integral of the last term in (11.27) is monotonically decreasing. It represents the "leakage" that occurs when a capitalization-weighted portfolio is contained inside a larger market, and stocks cross-over (leak) from the cap-weighted to the market portfolio.

(*Hint:* Use the property $\mathbf{G}_p(s_1, \cdots, s_m) = s\,\mathbf{G}_p(\frac{s_1}{s}, \cdots, \frac{s_m}{s})$, $s := s_1 + \cdots + s_m$ of the function in (7.5), to get $G^{\sharp}_p(\mu(t))/G_L(\mu(t)) = \mathbf{G}_p\big(\zeta_{(1)}(t), \cdots, \zeta_{(n)}(t)\big)$.)

12 Stabilization by Volatility

We shall see in this section that the condition (11.7) is satisfied on every time-horizon $[0,T]$, $T \in (0,\infty)$ in an abstract market $\mathcal{M}$ with

$$\underbrace{\mathrm{d}\Big(\log S_i(t)\Big) \;=\; \frac{\alpha}{2\mu_i(t)}\,\mathrm{d}t \;+\; \frac{1}{\sqrt{\mu_i(t)}}\cdot \mathrm{d}W_i(t)}\,, \qquad i=1,\cdots,m\,. \tag{12.1}$$

Here $\alpha \geq 0$ is a given real constant, and $m \geq 2$ an integer. The theory developed by Bass & Perkins (2002) shows that the resulting system of stochastic differential equations

$$\mathrm{d}S_i(t) \;=\; \frac{1+\alpha}{2}\Big(S_1(t)+\ldots+S_m(t)\Big)\,\mathrm{d}t + \sqrt{S_i(t)\,\Big(S_1(t)+\ldots+S_m(t)\Big)}\,\mathrm{d}W_i(t)\,, \tag{12.2}$$

$i=1,\ldots,m$, for the Δ^m_{++}−valued diffusion process $\mathfrak{S}(\cdot) = \big(S_1(\cdot),\cdots,S_m(\cdot)\big)'$, determines uniquely its distribution; and that the conditions (1.2), (5.4) are satisfied by the processes $b_i(\cdot) = (1+\alpha)/2\mu_i(\cdot)$, $\sigma_{i\nu}(t) = (\mu_i(t))^{-1/2}\,\delta_{i\nu}$, $r(\cdot) \equiv 0$ and $\vartheta_\nu(\cdot) = 1/2\sqrt{\mu_\nu(\cdot)}$, $1 \leq i,\nu \leq m$. The reader might wish to observe that condition (3.10) is satisfied in this case, in fact with $\varepsilon = 1$; but (1.15) fails.

The model of (12.1) assigns to all stocks the same log-drift $\gamma_i(\cdot) \equiv 0$, and volatilities $\sigma_{i\nu}(t) = (\mu_i(t))^{-1/2}\,\delta_{i\nu}$ that are largest for the smallest stocks and smallest for the largest stocks. Not surprisingly then, individual stocks fluctuate rather widely in a market of this type; in particular, *diversity fails* on $[0,T]$; see Exercises 12.2 and 12.3.

12.1 Stability and Arbitrage Properties

Yet despite these fluctuations, the overall market has a *very stable behavior*. We call this phenomenon *stabilization by volatility* in the case $\alpha = 0$; and *stabilization by both volatility and drift* in the case $\alpha > 0$.

Indeed, the quantities $a^{\mu\mu}(\cdot)$, $\gamma^\mu_*(\cdot)$, $\gamma^\mu(\cdot)$ are computed from (2.7), (1.13), (1.12), respectively, as

$$a^{\mu\mu}(\cdot) \equiv 1\,, \quad \gamma^\mu_*(\cdot) \equiv \gamma_* := \frac{m-1}{2} > 0\,, \quad \gamma^\mu(\cdot) \;\equiv\; \gamma \;:=\; \frac{\big[\,(1+\alpha)m - 1\,\big]}{2} > 0\,. \tag{12.3}$$

This, in conjunction with (2.2), computes the total market capitalization

$$S(t) = S_1(t)+\ldots+S_m(t) \;=\; S(0)\cdot e^{\,\gamma t + \mathfrak{B}(t)}\,, \qquad 0 \leq t < \infty \tag{12.4}$$

as the exponential of the standard, one-dimensional Brownian motion $\mathfrak{B}(\cdot) := \sum_{\nu=1}^m \int_0^\cdot \sqrt{\mu_\nu(s)}\,\mathrm{d}W_\nu(s)$, plus drift $\gamma t > 0$. In particular, the overall market and the largest stock $S_{(1)}(\cdot) = \max_{1\leq i\leq m} S_i(\cdot)$ grow at the same, constant rate:

$$\lim_{T\to\infty}\left(\frac{1}{T}\log S(T)\right) \;=\; \lim_{T\to\infty}\left(\frac{1}{T}\log S_{(1)}(T)\right) \;=\; \gamma\,, \quad \text{a.s.} \tag{12.5}$$

On the other hand, according to Example 11.3 there exist in this model portfolios that lead to arbitrage opportunities relative to the market, at least on time-horizons $[0,T]$ with $T \in (\mathcal{T}_*, \infty)$, where

$$\mathcal{T}_* := \frac{2\mathbf{H}(\mu(0))}{m-1} \leq \frac{2\log m}{m-1}.$$

To wit: *we can have relative arbitrages in non-diverse markets with unbounded volatilities.* The last upper bound in the above expression becomes very small as the "size" m of the market increases, leading to the plausible conjecture that it should be possible to construct such relative arbitrages over *any* given time horizon with $T \in (0, \infty)$. The validity of this conjecture has been established recently by A. Banner & D. Fernholz (2006).

12.2 Bessel Processes

The crucial observation now, is that the solution of the system (12.1) can be expressed in terms of the squares of independent Bessel processes $\mathfrak{R}_1(\cdot), \ldots, \mathfrak{R}_m(\cdot)$ in dimension $\kappa := 2(1+\alpha) \geq 2$, and of an appropriate time-change:

$$S_i(t) = \mathfrak{R}_i^2\big(\Lambda(t)\big), \qquad 0 \leq t < \infty, \quad i = 1, \ldots, m, \tag{12.6}$$

where

$$\Lambda(t) := \frac{1}{4}\int_0^t S(u)\,\mathrm{d}u = \frac{S(0)}{4}\int_0^t e^{\gamma s + \mathcal{W}(s)}\,\mathrm{d}s, \qquad 0 \leq t < \infty \tag{12.7}$$

and

$$\mathfrak{R}_i(u) = \sqrt{S_i(0)} + \frac{\kappa - 1}{2}\int_0^u \frac{\mathrm{d}\xi}{\mathfrak{R}_i(\xi)} + \mathfrak{W}_i(u), \qquad 0 \leq u < \infty. \tag{12.8}$$

Here, the driving processes $\mathfrak{W}_i(\cdot) := \int_0^{\Lambda^{-1}(\cdot)} \sqrt{\Lambda'(t)}\,\mathrm{d}W_i(t)$ are independent, standard one-dimensional Brownian motions (e.g. [KS] (1991), pp.157-162). In a similar vein, we have the representation $S(t) = \mathfrak{R}^2\big(\Lambda(t)\big)$, $0 \leq t < \infty$ of the total market capitalization, in terms of the Bessel process

$$\mathfrak{R}(u) = \sqrt{S(0)} + \frac{m\kappa - 1}{2}\int_0^u \frac{\mathrm{d}\xi}{\mathfrak{R}(\xi)} + \mathfrak{W}(u), \qquad 0 \leq u < \infty \tag{12.9}$$

in dimension $m\kappa$, and of yet another one-dimensional Brownian motion $\mathfrak{W}(\cdot)$.

This observation provides a wealth of structure, which can be used then to study the asymptotic properties of the model (12.1).

12.1 Exercise: Justify the representations of (12.6)-(12.9).

12.2 Exercise: For the case $\alpha > 0$ ($\kappa > 2$), obtain the ergodic property

$$\lim_{u\to\infty}\left(\frac{1}{\log u}\int_0^u \frac{\mathrm{d}\xi}{\mathfrak{R}_i^2(\xi)}\right) = \frac{1}{\kappa - 2} = \frac{1}{2\alpha}, \qquad \text{a.s.}$$

(a consequence of the Birkhoff ergodic theorem and of the strong Markov property of the Bessel process), as well as the *Lamperti representation*

$$\mathfrak{R}_i(u) \;=\; \sqrt{s_i} \cdot e^{\,\alpha\vartheta + \mathfrak{B}_i(\vartheta)} \Big|_{\vartheta = \int_0^u \mathfrak{R}_i^{-2}(\xi)\,\mathrm{d}\xi}\,, \qquad 0 \le u < \infty$$

for the Bessel process $\mathfrak{R}_i(\cdot)$ in terms of the exponential of a standard Brownian motion $\mathfrak{B}_i(\cdot)$ with positive drift $\alpha > 0$. Deduce the a.s. properties

$$\lim_{u\to\infty} \left(\frac{\log \mathfrak{R}_i(u)}{\log u} \right) \;=\; \frac{1}{2}\,, \qquad \lim_{t\to\infty} \left(\frac{1}{t}\, \log S_i(t) \right) \;=\; \gamma\,, \tag{12.10}$$

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T a_{ii}(t)\,\mathrm{d}t \;=\; \lim_{T\to\infty} \frac{1}{T} \int_0^T \frac{\mathrm{d}t}{\mu_i(t)} \;=\; \frac{2\,\gamma}{\alpha} \;=\; m + \frac{m-1}{\alpha}\,, \tag{12.11}$$

for each $i = 1, \ldots, m$. In particular, all stocks grow at the same asymptotic rate $\gamma > 0$ as does the entire market, and the model of (12.1) is coherent in the sense of Exercise 6.1.

12.3 Exercise: In the case $\alpha = 0$ ($\kappa = 2$), show that

$$\lim_{u\to\infty} \left(\frac{\log \mathfrak{R}_i(u)}{\log u} \right) \;=\; \frac{1}{2} \qquad \text{holds in probability}\,,$$

but that we have almost surely:

$$\limsup_{u\to\infty} \left(\frac{\log \mathfrak{R}_i(u)}{\log u} \right) \;=\; \frac{1}{2}\,, \qquad \liminf_{u\to\infty} \left(\frac{\log \mathfrak{R}_i(u)}{\log u} \right) \;=\; -\infty\,.$$

Deduce, from this and (12.5), that

$$\lim_{t\to\infty} \left(\frac{1}{t}\, \log S_i(t) \right) \;=\; \gamma \qquad \text{holds in probability,} \tag{12.12}$$

but also that

$$\limsup_{t\to\infty} \left(\frac{1}{t}\, \log S_i(t) \right) \;=\; \gamma\,, \qquad \liminf_{t\to\infty} \left(\frac{1}{t}\, \log S_i(t) \right) \;=\; -\infty \tag{12.13}$$

hold almost surely. To wit, individual stocks can "crash" in this case, despite the overall stability of the market; and coherence now fails.

(*Hint:* Use the following zero-one law of Spitzer (*Transactions of the American Mathematical Society*, 1958): For a decreasing function $h(\cdot)$ we have

$$\mathbb{P}\Big(\mathfrak{R}_i(u) \;\ge\; u^{1/2}\, h(u) \quad \text{for all } \; u > 0 \; \text{ sufficiently large} \Big) \;=\; 1 \;\text{ or }\; 0\,,$$

depending on whether the series $\sum_{k=1}^\infty \big(k\,|\log h(k)|\,\big)^{-1}$ converges or diverges.)

12.4 Exercise: In the case $\alpha = 0$ ($\kappa = 2$), show that

$$\lim_{u \to \infty} \mathbb{P}\big[\, \mu_i\big(\Lambda^{-1}(u)\big) > 1 - \delta \,\big] \;=\; \delta^{\,m-1}$$

holds for every $i = 1, \ldots, m$ and $\delta \in (0,1)$; here $\Lambda^{-1}(\cdot) = 4 \int_0^{\cdot} \mathfrak{R}^{-2}(\xi)\, \mathrm{d}\xi$ is the inverse of the time-change $\Lambda(\cdot)$ in (12.7), and $\mathfrak{R}(\cdot)$ is the Bessel process in (12.9). Deduce that this model is not diverse on $[0, \infty)$.

12.5 Exercise: In the case $\alpha = 0$ ($\kappa = 2$), compute the exponential (strict) local martingale of (5.5) as

$$Z(t) \;=\; \left(\frac{\sqrt{s_1 \ldots s_m}}{\mathfrak{R}_1(u) \ldots \mathfrak{R}_m(u)} \; \exp\left[\frac{1}{2} \int_0^u \Big(\sum_{i=1}^m \mathfrak{R}_i^{-2}(\xi) \Big)\, \mathrm{d}\xi \right] \right) \Bigg|_{u = \Lambda(t)} .$$

12.6 Exercise: In the context of the volatility-stabilized model of this section, compute the variance of the diversity-weighted portfolio of (7.1) with $p = 1/2$:

$$\pi_i^{(p)}(t) \;=\; \frac{\sqrt{\mu_i(t)}}{\sum_{j=1}^m \sqrt{\mu_j(t)}}\,, \qquad i = 1, \cdots, m$$

Show that we have the relative arbitrage $\mathbb{P}\,[\, V^{\pi^{(p)}}(T) > V^{\mu}(T) \,] = 1$, at least on time-horizons $[0, T]$ with $T > (8 \log m)/(m-1)$.

Furthermore, show that this diversity-weighted portfolio outperforms very significantly the market over long time-horizons:

$$\mathcal{L}^{\,\pi^{(p)},\mu} \;:=\; \liminf_{T \to \infty} \; \frac{1}{T} \log \left(\frac{V^{\pi^{(p)}}(T)}{V^{\mu}(T)} \right) \;=\; \liminf_{T \to \infty} \; \frac{1}{2T} \int_0^T \gamma_*^{\pi^{(p)}}(t)\, \mathrm{d}t \;\geq\; \frac{m-1}{8}\,,$$

almost surely. `Open Question:` Do the indicated limits exist? Can they be computed in closed form?

12.7 Exercise: For the equally-weighted portfolio $\eta_i(\cdot) \equiv 1/m$, $i = 1, \cdots, m$ in the volatility-stabilized model with $\alpha > 0$ verify, using (12.5), (12.10) and (12.11), that the limit

$$\mathcal{L}^{\,\eta,\mu} \;:=\; \lim_{T \to \infty} \; \frac{1}{T} \log \left(\frac{V^{\eta}(T)}{V^{\mu}(T)} \right) \;=\; \lim_{T \to \infty} \; \frac{1}{T} \int_0^T \gamma_*^{\eta}(t)\, \mathrm{d}t$$

of (5.3) exists a.s. and equals

$$\mathcal{L}^{\,\eta,\mu} \;=\; \frac{m-1}{2m} \left(1 + \frac{1 - (1/m)}{\alpha} \right) .$$

In other words: equal-weighting, with its bias towards the small-capitalization stocks, outperforms significantly the market over long time-horizons.

12.8 Exercise: Show that in the context of this section, the extended portfolio

$$\widehat{\pi}_i(t) := \frac{1+\alpha}{2} - \Big(\frac{m}{2}\,(1+\alpha) - 1 \Big)\, \mu_i(t) \;=\; \lambda \eta_i(t) + (1-\lambda)\mu_i(t)\,, \qquad i = 1, \cdots, m$$

with $\lambda = m(1+\alpha)/2 \geq 1$ (that is, long in the equally-weighted $\eta(\cdot)$ of Exercise 12.7, and short in the market), has the *numéraire property*

$$V^{\pi}(\cdot)/V^{\widehat{\pi}}(\cdot) \quad \text{is a supermartingale, for every extended portfolio } \pi(\cdot)\,.$$

`Open Question:` Does the a.s. limit $\mathcal{L}^{\,\widehat{\pi},\mu} := \lim_{T\to\infty} \frac{1}{T} \log \big(V^{\widehat{\pi}}(T)/V^{\mu}(T) \big)$ exist? If so, can its value be computed in closed form?

`Open Question:` For the diversity-weighted portfolio $\varpi_i^c(\cdot)$ of (11.8), compute in the context of the volatility-stabilized model the expression

$$\mathcal{L}^{\,\varpi^c,\mu} := \liminf_{T\to\infty} \frac{1}{T} \log \left(\frac{V^{\varpi^c}(T)}{V^{\mu}(T)} \right) \;=\; \gamma_* \cdot \liminf_{T\to\infty} \frac{1}{T} \int_0^T \frac{\mathrm{d}t}{c + \mathbf{H}(\mu(t))}$$

of (5.3), using (11.8) and (12.3). But note already from these expressions that

$$\mathcal{L}^{\,\varpi^c,\mu} \geq \frac{m-1}{2(c+\log m)} > 0 \quad \text{a.s.}\,,$$

which shows again a considerable outperforming of the market over long time-horizons. Do the indicated limits exist, as one would expect?

`Open Questions:` For fixed $t \in (0,\infty)$, determine the distributions of $\mu_i(t)$, $i = 1, \cdots, m$ and of the largest $\mu_{(1)}(t) := \max_{1\leq i\leq m} \mu_i(t)$ and smallest $\mu_{(m)}(t) := \min_{1\leq i\leq m} \mu_i(t)$ market weights.

What can be said about the behavior of the averages $\frac{1}{T}\int_0^T \mu_{(k)}(t)\, dt$, particularly for the largest ($k = 1$) and the smallest ($k = m$) stocks?

13 Ranked-Based Models

Size is one of the most important descriptive characteristics of financial assets. One can understand a lot about equity markets by observing and making sense of the continual ebb and flow of small-, medium- and large-capitalization stocks in their midst. A particularly convenient way to study this feature is by looking at the evolution of the *capital distribution curve* $\log k \mapsto \log \mu_{(k)}(t)$; that is, the logarithms of the market weights arranged in descending order, versus the logarithms of their respective ranks (see also (13.13) below for the steady-state counterpart). As shown on p. 95 of Fernholz (2002), this log-log plot has exhibited remarkable stability over the decades of the last century.

It is of considerable importance, then, to have available models which describe this flow of capital and exhibit stability properties for capital distribution that are in at least broad agreement with these observations.

The simplest model of this type assigns growth rates and volatilities to the various stocks, not according to their names (the indices i) but according to their *ranks* within the market's capitalization. More precisely, let us pick real numbers γ , $g_1, \ldots, g_m$, $\sigma_1 > 0, \ldots, \sigma_m > 0$ satisfying conditions that will be specified in a moment, and prescribe growth rates $\gamma_i(\cdot)$ and volatilities $\sigma_{i\nu}(\cdot)$

$$\gamma_i(t) \;=\; \gamma + \sum_{k=1}^{m} g_k \, 1_{\mathcal{Q}_i^{(k)}}(\mathfrak{S}(t))\,, \qquad \sigma_{i\nu}(t) \;=\; \gamma + \sum_{k=1}^{m} \sigma_k \, 1_{\mathcal{Q}_i^{(k)}}(\mathfrak{S}(t)) \qquad (13.1)$$

for $1 \le i, \nu \le m$ with $d = m$ and $\mathfrak{S}(\cdot) = \big(S_1(\cdot), \cdots, S_m(\cdot)\big)'$. Here $\big\{\mathcal{Q}_i^{(k)}\big\}_{1 \le i,k \le m}$ is a collection of polyhedral domains in $\mathbb{R}^m$, with the properties

$$\big\{\mathcal{Q}_i^{(k)}\big\}_{1 \le i \le m} \quad \text{is a partition of } \mathbb{R}^m\,, \quad \text{for each fixed } k\,,$$

$\big\{\mathcal{Q}_i^{(k)}\big\}_{1 \le k \le m}$ is a partition of $\mathbb{R}^m$, for each fixed i , and the interpretation:

$$\mathrm{y} = (y_1, \ldots, y_m) \in \mathcal{Q}_k^{(i)} \ \text{ means that } y_i \text{ is ranked } k^{\text{th}} \text{ among } y_1, \ldots, y_m\,.$$

(Ties are resolved by resorting to the lowest index i ; for instance, $\mathcal{Q}_i^{(1)} \equiv \mathcal{Q}_i$, $1 \le i \le m$ is the partition of $\mathbb{R}^m$ of section 9, right below (9.3); and so on.)

It is clear intuitively that, if such a model is to have some stability properties, it has to assign considerably higher growth rates to the smallest stocks than to the biggest ones. It turns out that the right conditions for stability are

$$g_1 < 0\,, \quad g_1 + g_2 < 0\,, \quad \ldots\,, \quad g_1 + \cdots + g_{m-1} < 0\,, \quad g_1 + \cdots + g_m = 0\,. \qquad (13.2)$$

These conditions are satisfied in the simplest model of this type, the *Atlas Model* that assigns

$$\gamma = g > 0\,, \quad g_k = -g \ \text{ for } \ k = 1, \ldots, m-1 \quad \text{and} \quad g_m = (m-1)g\,, \qquad (13.3)$$

thus $\gamma_i(t) = mg \cdot 1_{\mathcal{Q}_i^{(m)}}(\mathfrak{S}(t))$ in (13.1): zero growth rate goes to all the stocks but the smallest, which then becomes responsible for supporting the entire growth mg of the market.

These specifications amount to postulating that the log-capitalizations $Y_i(\cdot) := \log S_i(\cdot)\,,\ i = 1, \cdots, m$ satisfy the stochastic differential equations

$$dY_i(t) \,=\, \left(\gamma + \sum_{k=1}^{m} g_k \, 1_{\mathcal{Q}_k^{(i)}}(\mathfrak{Y}(t)) \right) dt \,+\, \sum_{k=1}^{m} \sigma_k \, 1_{\mathcal{Q}_k^{(i)}}(\mathfrak{Y}(t)) \, dW_i(t)\,, \qquad (13.4)$$

with $Y_i(0) = y_i = \log s_i$. As long as the vector $\mathfrak{Y}(\cdot) = \big(Y_1(\cdot), \cdots, Y_m(\cdot)\big)'$ is in the polyhedron $\mathcal{Q}_k^{(i)}$, the equation (13.3) posits that the coördinate process $Y_i(\cdot)$ evolves like a Brownian motion with drift $\gamma + g_k$ and variance σ_k^2.

The theory of Bass & Pardoux (*Probability Theory & Related Fields*, 1987) guarantees that this system has a weak solution, which is unique in distribution; once this solution has been constructed, we obtain stock capitalizations as $S_i(\cdot) = e^{Y_i(\cdot)}$ that satisfy (1.1), (13.1).

♠ An immediate observation from (13.3) is that the sum $Y(\cdot) := \sum_{i=1}^m Y_i(\cdot)$ of log-capitalizations satisfies

$$Y(t) \,=\, y \,+\, m\gamma\, t \,+\, \sum_{k=1}^{m} \sigma_k \, B_k(t)\,, \qquad 0 \le t < \infty$$

with $y := \sum_{i=1}^m y_i$, and $B_k(\cdot) := \sum_{i=1}^m \int_0^{\cdot} 1_{\mathcal{Q}_k^{(i)}}(\mathfrak{Y}(s))\, \mathrm{d}W_i(s)\,,\ k = 1, \ldots m$ independent scalar Brownian motions. The strong law of large numbers implies directly

$$\lim_{T \to \infty} \frac{1}{T} \sum_{i=1}^{m} Y_i(T) \,=\, m\gamma\,, \qquad \text{a.s.}$$

Then it takes a considerable amount of work (see Appendix in [BFK] 2005), in order to strengthen this result to

$$\lim_{T \to \infty} \frac{1}{T} \log X_i(T) \,=\, \lim_{T \to \infty} \frac{Y_i(T)}{T} \,=\, \gamma \quad \text{a.s., for every } \ i = 1, \ldots, m\,. \qquad (13.5)$$

13.1 Exercise: Use (13.5) to show that the model specified by (1.5), (13.1) is coherent in the sense of Exercise 6.1.

Remark: Taking Turns in the Various Ranks. From (13.4), (13.5) and the strong law of large numbers for Brownian motion, we deduce that the quantity $\sum_{k=1}^m g_k \left(\frac{1}{T} \int_0^T 1_{\mathcal{Q}_i^{(k)}}(\mathfrak{Y}(t))\, \mathrm{d}t \right)$ converges a.s. to zero, as $T \to \infty$. For the Atlas model in (13.3), this expression becomes $g\left(\frac{m}{T} \int_0^T 1_{\mathcal{Q}_i^{(m)}}(\mathfrak{Y}(t))\, \mathrm{d}t - 1 \right)$, and we obtain

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T 1_{\mathcal{Q}_i^{(m)}}(\mathfrak{Y}(t))\, \mathrm{d}t \,=\, \frac{1}{m} \quad \text{a.s., for every } \ i = 1, \ldots, m\,.$$

Namely, "each stock spends roughly $(1/m)^{\text{th}}$ of the time, acting as Atlas". Again, with considerable work, this is strengthened in [BFK] to

$$\underbrace{\lim_{T \to \infty} \frac{1}{T} \int_0^T 1_{\mathcal{Q}_i^{(k)}}\big(\mathfrak{Y}(t)\big)\, \mathrm{d}t \,=\, \frac{1}{m}\,, \quad \text{a.s., for every } \ 1 \le i, k \le m\,,} \qquad (13.6)$$

valid not just for the Atlas model, but under the more general conditions of (13.2). Thanks to the symmetry inherent in this model, "each stock spends roughly $(1/m)^{\text{th}}$ of the time in any given rank"; see Proposition 2.3 in [BFK].

13.1 Ranked Price Processes

For many purposes in the study of these models, it makes sense to look at the ranked log-capitalization processes

$$Z_k(t) := \sum_{i=1}^{m} Y_i(t) \cdot 1_{\mathcal{Q}_i^{(k)}}(\mathfrak{Y}(t)) , \quad 0 \le t < \infty \tag{13.7}$$

for $1 \le k \le m$. From these, we get the ranked capitalizations via $S_{(k)}(t) = e^{Z_k(t)}$, with notation similar to (1.17). Using an extended Tanaka-type formula, as we did in (11.15), it can be seen that the processes of (13.7) satisfy

$$Z_k(t) = Z_k(0) + (g_k + \gamma)\, t + \sigma_k\, B_k(t) + \frac{1}{2} \left[\mathfrak{L}^{k,k+1}(t) - \mathfrak{L}^{k-1,k}(t) \right] \tag{13.8}$$

for $0 \le t < \infty$ in that notation. Here, as in subsection 11.2, the continuous increasing process $\mathfrak{L}^{k,k+1}(\cdot) := \Lambda_{\Xi_k}(\cdot)$ is the semimartingale local time at the origin of the continuous, non-negative process $\Xi_k(\cdot) = Z_k(\cdot) - Z_{k+1}(\cdot) = \log\big(\mu_{(k)}(\cdot)/\mu_{(k+1)}(\cdot)\big)$ of (11.16) for $k = 1, \cdots, m-1$; and we make again the convention $\mathfrak{L}^{0,1}(\cdot) \equiv \mathfrak{L}^{m,m+1}(\cdot) \equiv 0$.

These local times play a big rôle in the analysis of this model. The quantity $\mathfrak{L}^{k,k+1}(T)$ represents again the cumulative amount of change between ranks k and $k+1$ that occurs over the time-interval $[0, T]$. Of course, in a model such as the one studied here, the intensity of changes in the lower ranks should be higher than in the top ranks.

This is borne out by experiment: as we saw in Remark 11.5 it turns out, somewhat surprisingly, that these local times can be estimated based only on observations of relative market weights and of the performance of simple portfolio rules over $[0, T]$; and that they exhibit a remarkably "linear" increase, with positive slopes that increase with k (see Fernholz (2002), Figure 5.2).

The analysis of the present model agrees with these observations: it follows from (13.5) and the dynamics of (13.8) that, for $k = 1, \ldots, m-1$, we have

$$\lim_{T\to\infty} \frac{1}{T}\, \mathfrak{L}^{k,k+1}(T) = \lambda_{k,k+1} := -2\,\big(g_1 + \ldots + g_k\big) > 0 , \quad \text{a.s.} \tag{13.9}$$

Our stability condition guarantees that these limits are positive, as they should be; and in typical examples, such as the Atlas model where $\lambda_{k,k+1} = kg$, they increase with k.

13.2 Some Asymptotics

A slightly more careful analysis of these local times reveals that the non-negative semimartingale $\Xi_k(\cdot)$ of (11.16) can be cast in the form of a *Skorohod problem*

$$\Xi_k(t) = \Xi_k(0) + \Theta_k(t) + \Lambda_{\Xi_k}(t) , \qquad 0 \le t < \infty ,$$

as the reflection, at the origin, of the semimartingale

$$\Theta_k(t) \,=\, (g_k - g_{k+1})\, t \,-\, \frac{1}{2}\,\big[\, \mathfrak{L}^{k-1,k}(t) + \mathfrak{L}^{k+1,k+2}(t) \,\big] \,+\, \mathbf{s}_k \cdot \widetilde{W}^{(k)}(t)\,,$$

where $\mathbf{s}_k := \big(\sigma_k^2 + \sigma_{k+1}^2\big)^{1/2}$ and $\widetilde{W}^{(k)}(\cdot) := \big(\sigma_k\, B_k(\cdot) - \sigma_{k+1}\, B_{k+1}(\cdot)\big)/s_k$ is standard Brownian Motion.

As a result of these observations and of (13.9), we conclude that the process $\Xi_k(\cdot)$ behaves asymptotically like Brownian motion with drift $-\lambda_{k,k+1} < 0$, variance σ_k^2, and reflection at the origin. Consequently,

$$\lim_{t\to\infty} \log\left(\frac{\mu_{(k)}(t)}{\mu_{(k+1)}(t)}\right) \,=\, \lim_{t\to\infty} \Xi_k(t) \,=\, \xi_k\,, \quad \text{in distribution} \tag{13.10}$$

where, for each $k = 1, \ldots, m-1$ the random variable ξ_k has an exponential distribution $\mathbb{P}(\xi_k > x) = e^{-r_k x}$, $x \ge 0$ with parameter

$$r_k := \frac{2\,\lambda_{k,k+1}}{\mathbf{s}_k^2} = -\frac{4(g_1 + \cdots + g_k)}{\sigma_k^2 + \sigma_{k+1}^2} \,>0\,. \tag{13.11}$$

13.3 The Steady-State Capital Distribution Curve

We also have from (13.10) the strong law of large numbers

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T g\big(\Xi_k(t)\big)\,\mathrm{d}t \,=\, \mathbb{E}\big[\, g(\xi_k)\,\big]\,, \quad \text{a.s.}$$

for every rank k, and every measurable function $g : [0,\infty) \to \mathbb{R}$ which satisfies $\int_0^\infty |g(x)|\, e^{-r_k x}\, dx < \infty$; see Khas'minskii (*Theory of Probability & Its Applications*, 1960). In particular,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \left(\log \frac{\mu_{(k)}(t)}{\mu_{(k+1)}(t)}\right) \mathrm{d}t \,=\, \mathbb{E}\big[\,\xi_k\,\big] \,=\, \frac{1}{r_k} \,=\, \frac{\mathbf{s}_k^2}{2\lambda_{k,k+1}}\,, \quad \text{a.s.} \tag{13.12}$$

This observation provides a tool for studying the *steady-state capital distribution curve*

$$\log k \;\longmapsto\; \lim_{T\to\infty} \frac{1}{T}\int_0^T \log\big(\mu_{(k)}(t)\big)\,\mathrm{d}t \,=:\, \mathfrak{n}(k)\,, \quad k = 1, \cdots, m-1 \tag{13.13}$$

alluded to at the beginning of this section (more on the existence of this limit in the next subsection). To estimate the **slope** $\mathfrak{q}(k)$ **of this curve** at the point $\log k$, we use (13.12) and the estimate $\log(k+1) - \log k \approx 1/k$, and obtain in the notation of (13.14):

$$\underbrace{\mathfrak{q}(k) \,\approx\, \frac{\mathfrak{n}(k) - \mathfrak{n}(k+1)}{\log k - \log(k+1)} \,=\, -\frac{k}{r_k} \,=\, \frac{k\big(\sigma_k^2 + \sigma_{k+1}^2\big)}{4(g_1 + \cdots + g_k)} \,<\, 0}\;. \tag{13.14}$$

♠ Consider, for instance, an Atlas model as in (13.3). With equal variances $\sigma_k^2 = \sigma^2 > 0$, this slope is the constant $-\sigma^2/2g$; the steady-state capital distribution curve can be approximated by a straight line. On the other hand, with variances $\sigma_k^2 = \sigma^2 + ks^2$ growing linearly with rank (as indeed suggested by Figure 5.5 in Fernholz (2002)), we get for large k the approximate slope

$$\mathfrak{q}(k) \approx -\frac{1}{2g}\left(\sigma^2 + ks^2\right), \qquad k = 1, \cdots, m-1$$

This suggest a decreasing, concave steady-state capital distribution curve, whose (negative) slope becomes more and more pronounced in magnitude with increasing rank, very much in accord with the features of Figure 5.1 in Fernholz (2002).

Remark: Estimation of Parameters in this Model. Let us remark that (13.9) provides a method for obtaining estimates $\widehat{\lambda}_{k,k+1}$ of the parameters $\lambda_{k,k+1}$, from the observable random variables $\mathfrak{L}^{k,k+1}(T)$ that measure cumulative change between ranks k and $k+1$; recall Remark 11.5 once again. Then estimates of the parameters g_k follow, as $\widehat{g}_k = \big(\widehat{\lambda}_{k-1,k} - \widehat{\lambda}_{k,k+1}\big)/2$; and the parameters $\mathbf{s}_k^2$ can be estimated from (13.12) and from the increments of the observable capital distribution curve of (13.13), namely $\widehat{\mathbf{s}}_k^2 = 2\,\widehat{\lambda}_{k,k+1}[\,\mathfrak{n}(k) - \mathfrak{n}(k+1)\,]$. Finally, we make the following selections for estimating the variances:

$$\widehat{\sigma}_k^2 = \frac{1}{4}\left(\widehat{\mathbf{s}}_{k-1}^2 + \widehat{\mathbf{s}}_k^2\right), \quad k = 2, \cdots, m-1, \qquad \widehat{\sigma}_1^2 = \frac{1}{2}\,\widehat{\mathbf{s}}_1^2, \quad \widehat{\sigma}_m^2 = \frac{1}{2}\,\widehat{\mathbf{s}}_{m-1}^2.$$

13.4 Stability of the Distribution of Capital

Let us now go back to (13.10); it can be seen that this leads to the convergence of the ranked market weights

$$\lim_{t\to\infty}\Big(\mu_{(1)}(t), \ldots, \mu_{(m)}(t)\Big) = (N_1, \ldots, N_m), \quad \text{in distribution} \tag{13.15}$$

to the random variables

$$N_m := \left(1 + e^{\xi_{m-1}} + \cdots + e^{\xi_{m-1}+\cdots+\xi_1}\right)^{-1} \quad \text{and} \quad N_k := N_m \cdot e^{\xi_{m-1}+\cdots+\xi_k} \tag{13.16}$$

for $k = 1, \ldots, m-1$. These are the long-term (steady-state) relative weights of the various stocks in the market, ranked from largest (N_1) to smallest (N_m). Again, we have from (13.15) the strong law of large numbers

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f\big(\mu_{(1)}(t), \ldots, \mu_{(m)}(t)\big)\,dt = \mathbb{E}\left[\,f(N_1, \ldots, N_m)\,\right], \quad \text{a.s.} \tag{13.17}$$

for every bounded and measurable $f : \Delta_{++}^m \to \mathbb{R}$. Note that (13.12) is a special case of this result, and that the function $\mathfrak{n}(\cdot)$ of (13.13) takes the form

$$\underbrace{\mathfrak{n}(k) = \mathbb{E}\left[\,\log(N_k)\,\right] = \sum_{\ell=k}^{m-1}\frac{1}{r_\ell} - \mathbb{E}\left[\,\log\left(1 + e^{\xi_{m-1}} + \cdots + e^{\xi_{m-1}+\cdots+\xi_1}\right)\right]}\,. \tag{13.18}$$

This is the good news; the bad news is that we do not know the joint distribution of the exponential random variables $\xi_1, \cdots, \xi_{m-1}$ in (13.10), so we cannot find that of $N_1, \cdots, N_m$ either. In particular, we cannot "pin down" the steady-state capital distribution function of (13.18), though we know precisely its increments $\mathfrak{n}(k+1) - \mathfrak{n}(k)$ and thus are able to estimate the slope of the steady-state capital distribution curve, as indeed we did in (13.14). In [BFK] a simple, certainty-equivalent approximation of the steady-state ranked market weights of (13.16) is carried out, and is used to study in detail the behavior of simple portfolio rules in such a model. We refer to this paper the reader who is interested in the details.

`Major Open Question:` What can be said about the joint distribution of the long-term (steady-state) relative market weights of (13.16)? Can it be characterized, computed, or approximated in a good way? What can be said about the fluctuations of $\log(N_k)$ with respect to their means $\mathfrak{n}(k)$ in (13.18)?

`Research Question and Conjecture:` Study the steady-state capital distribution curve of the volatility-stabilized model in (12.1). With $\alpha > 0$, check the validity of the following conjecture: the slope $\mathfrak{q}(k) \approx (\mathfrak{n}(k) - \mathfrak{n}(k+1))/(\log k - \log(k+1))$ of the capital distribution $\mathfrak{n}(\cdot)$ at $\log k$, should be given as

$$\mathfrak{q}(k) \approx -4\gamma\, k\mathfrak{h}_k , \qquad \mathfrak{h}_k := \mathbb{E}\left(\frac{\log Q_{(k)} - \log Q_{(k+1)}}{Q_{(1)} + \cdots + Q_{(m)}} \right),$$

where $Q_{(1)} \geq \cdots \geq Q_{(m)}$ are the descending order statistics of a random sample from the chi-square distribution with $\kappa = 2(1+\alpha)$ degrees of freedom.

If this conjecture is correct, does $k\,\mathfrak{h}_k$ increase with k ?

14 Proofs of Selected Results

Exercise 5.3: Suppose that the processes $\widehat{S}_i(\cdot)$ are all martingales; then so is their sum, the process $\widehat{S}(\cdot) := Z(\cdot)S(\cdot)/B(\cdot)$. We normalize as $w = S(0) = 1$, so that $V^\mu(\cdot) \equiv S(\cdot)$. With $h(\cdot) \equiv V^\mu(\cdot)\mu(\cdot)$ and $\vartheta^\mu(t) := \sigma'(t)\mu(t) - \vartheta(t)$, the equation (5.9) takes then the form $\mathrm{d}\widehat{V}^\mu(t) = \widehat{V}^\mu(t)(\vartheta^\mu(t))'\,\mathrm{d}W(t)$, or equivalently

$$\widehat{V}^\mu(t) \;=\; \exp\left(\int_0^t (\vartheta^\mu(s))'\,\mathrm{d}W(s) - \frac{1}{2}\int_0^t ||\vartheta^\mu(s)||^2\,\mathrm{d}s\right), \tag{14.1}$$

and with $\widetilde{W}(t) := W(t) - \int_0^t \vartheta^\mu(s)\,ds$, $0 \le t \le T$ we get

$$\big(\widehat{V}^\mu(t)\big)^{-1} \;=\; \exp\left(-\int_0^t (\vartheta^\mu(s))'\,d\widetilde{W}(s) - \frac{1}{2}\int_0^t ||\vartheta^\mu(s)||^2\,ds\right).$$

The process $\widetilde{W}(\cdot)$ is Brownian motion under the equivalent probability measure $\widetilde{\mathbb{P}}_T(A) := \mathbb{E}\,[\,\widehat{V}^\mu(T)\cdot 1_A\,]$ on $\mathcal{F}(T)$, and Itô's rule gives

$$\mathrm{d}\left(\frac{V^\pi(t)}{V^\mu(t)}\right) \;=\; \left(\frac{V^\pi(t)}{V^\mu(t)}\right)\cdot\sum_{i=1}^m\sum_{\nu=1}^d \big(\pi_i(t) - \mu_i(t)\big)\,\sigma_{i\nu}(t)\,\mathrm{d}\widetilde{W}_\nu(t) \tag{14.2}$$

for an arbitrary extended portfolio $\pi(\cdot)$, in conjunction with $\mathrm{d}(V^\pi(t)/B(t)) = (V^\pi(t)/B(t))\pi'(t)\,\sigma(t)\,\mathrm{d}\widehat{W}(t)$ of (1.9) and (5.6). Then the boundedness condition (1.15) implies that the ratio $V^\pi(\cdot)/V^\mu(\cdot)$ is a martingale under $\widetilde{\mathbb{P}}_T$; in particular, $\mathbb{E}^{\widetilde{\mathbb{P}}_T}[\,V^\pi(T)/V^\mu(T)\,] = 1$. But if $\pi(\cdot)$ satisfies $\mathbb{P}[\,V^\pi(T) \ge V^\mu(T)\,] = 1$, we get $\widetilde{\mathbb{P}}_T[\,V^\pi(T) \ge V^\mu(T)\,] = 1$; in conjunction with $\mathbb{E}^{\widetilde{\mathbb{P}}_T}[\,V^\pi(T)/V^\mu(T)\,] = 1$, this leads to $\widetilde{\mathbb{P}}_T[\,V^\pi(T) = V^\mu(T)\,] = 1$, or equivalently $V^\pi(T) = V^\mu(T)$ a.s. $\mathbb{P}$, contradicting (5.2). Thus

$$\widehat{S}_j(t) \;=\; s_j\cdot\exp\left(\int_0^t (\vartheta^{(j)}(s))'\,dW(s) - \frac{1}{2}\int_0^t ||\vartheta^{(j)}(s)||^2\,ds\right) \tag{14.3}$$

of (5.8) is a strict local martingale, for some (at least one) $j \in \{1,\cdots,m\}$; we have set $\vartheta_\nu^{(k)}(t) := \sigma_{k\nu}(t) - \vartheta_\nu(t)$ for $\nu = 1,\cdots,d$, $k = 1,\cdots,m$.

Suppose now that the claim of the Exercise 5.3 fails, i.e., that $\widehat{S}_i(\cdot)$ is a martingale for some $i \ne j$. Then the measure $\mathbb{P}_T^{(i)}(A) := \mathbb{E}\,[\,(\widehat{S}_i(T)/s_i)\cdot 1_A\,]$ on $\mathcal{F}(T)$ is a probability, under which $\widetilde{W}^{(i)}(t) := W(t) - \int_0^t \vartheta^{(i)}(s)\,ds$, $0 \le t \le T$ is $\mathbb{R}^d$-valued Brownian motion. By analogy with (14.1)-(14.3) we have now

$$\big(\widehat{S}_i(t)\big)^{-1} \;=\; \frac{1}{s_i}\cdot\exp\left(-\int_0^t (\vartheta^{(i)}(s))'\,\mathrm{d}\widetilde{W}^{(i)}(s) - \frac{1}{2}\int_0^t ||\vartheta^{(i)}(s)||^2\,\mathrm{d}s\right),$$

and $\mathrm{d}\,(S_j(t)/S_i(t)) = (S_j(t)/S_i(t))\cdot\sum_{\nu=1}^d(\sigma_{j\nu}(t) - \sigma_{i\nu}(t))\,\mathrm{d}\widetilde{W}_\nu^{(i)}(t)$. Thus, thanks to condition (1.15), the process $S_j(\cdot)/S_i(\cdot)$ is a $\mathbb{P}_T^{(i)}$-martingale on

$[0,T]$, with moments of all orders. In particular,

$$\frac{S_j(0)}{S_i(0)} = \mathbb{E}^{\mathbb{P}_T^{(i)}}\left[\frac{S_j(T)}{S_i(T)}\right] = \mathbb{E}\left[\frac{Z(T)S_i(T)}{B(T)S_i(0)}\cdot\frac{S_j(T)}{S_i(T)}\right],$$

which contradicts $\mathbb{E}[Z(T)S_j(T)/B(T)] < S_j(0)$ and thus the strict local martingale property of $Z(\cdot)S_j(\cdot)/B(\cdot)$ under $\mathbb{P}$.

Exercise 8.2: Write the second equality in (3.4) with $\pi(\cdot)$ replaced by $\widetilde{\pi}^{[q]}(\cdot)$, and recall $\widetilde{\pi}^{[q]} - \mu = q\,(\pi-\mu)$. From the resulting expression, subtract the second equality in (3.4), now multiplied by q; the result is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\log\frac{V^{\widetilde{\pi}^{[q]}}(t)}{V^{\mu}(t)} - q\,\log\frac{V^{\pi}(t)}{V^{\mu}(t)}\right) = (q-1)\,\gamma_*^{\mu}(t) + \left(\gamma_*^{\widetilde{\pi}^{[q]}}(t) - q\,\gamma_*^{\pi}(t)\right).$$

But from the equalities of Exercise 8.1 and Lemma 3.3, we obtain

$$2\left(\gamma_*^{\widetilde{\pi}^{[q]}}(t) - q\,\gamma_*^{\pi}(t)\right) = \sum_{i=1}^{m}\left(\widetilde{\pi}^{[q]}(t) - q\,\pi_i(t)\right)\tau_{ii}^{\mu}(t) - \tau^{\mu}_{\widetilde{\pi}^{[q]}\widetilde{\pi}^{[q]}}(t) + q\,\tau^{\mu}_{\pi\pi}(t)$$

$$= (1-q)\sum_{i=1}^{m}\mu_i(t)\tau_{ii}^{\mu}(t) + q\,\tau^{\mu}_{\pi\pi}(t) - q^2\,\tau^{\mu}_{\pi\pi}(t) = (1-q)\,\left[\,2\,\gamma_*^{\mu}(t) + q\,\tau^{\mu}_{\pi\pi}(t)\,\right].$$

The desired equality now follows.

Exercise 8.3: If we have $\mathbb{P}\big(V^{\pi}(T)/V^{\mu}(T)\le 1/\beta\big)=1$, then we can just take $\widehat{\pi}(\cdot)\equiv\widetilde{\pi}^{[q]}(\cdot)$ with $q > 1+(2/\eta)\cdot\log(1/\beta)$, for then Exercise 8.2 gives

$$\log\left(\frac{V^{\widetilde{\pi}^{[q]}}(T)}{V^{\mu}(T)}\right) \le q\,\left[\log\left(\frac{1}{\beta}\right) + \frac{1-q}{2}\,\eta\right] < 0, \qquad \text{a.s.}$$

If, on the other hand, $\mathbb{P}\big(V^{\pi}(T)/V^{\mu}(T)\ge\beta\big)=1$ holds, then similar reasoning shows that it suffices to take $\widehat{\pi}(\cdot)\equiv\widetilde{\pi}^{[q]}(\cdot)$ with $q<\min\big(0,1-(2/\eta)\cdot\log(1/\beta)\big)$.

Exercise 11.1: To ease notation somewhat, let us set $g_i(t) := D_i\log\mathbf{G}(\mu(t))$ and $N(t) := 1-\sum_{j=1}^{m}\mu_j(t)g_j(t)$, so (11.1) reads: $\pi_i(t) = \big(g_i(t)+N(t)\big)\,\mu_i(t)$, for $i=1,\cdots m$. This way, the terms on the right-hand side of (3.9) become

$$\sum_{i=1}^{m}\frac{\pi_i(t)}{\mu_i(t)}\,\mathrm{d}\mu_i(t) = \sum_{i=1}^{m}g_i(t)\,\mathrm{d}\mu_i(t) + N(t)\cdot\mathrm{d}\left(\sum_{i=1}^{m}\mu_i(t)\right) = \sum_{i=1}^{m}g_i(t)\,\mathrm{d}\mu_i(t)$$

and

$$\sum_{i=1}^m \sum_{j=1}^m \pi_i(t)\pi_j(t)\tau_{ij}^\mu(t) \;=\; \sum_{i=1}^m \sum_{j=1}^m \big(g_i(t)+N(t)\big)\big(g_j(t)+N(t)\big)\mu_i(t)\mu_j(t)\tau_{ij}^\mu(t)$$

$$= \sum_{i=1}^m \sum_{j=1}^m g_i(t)\, g_j(t)\cdot \mu_i(t)\mu_j(t)\tau_{ij}^\mu(t)\,,$$

the latter thanks to (2.8) and Lemma 3.1. Thus, (3.9) gives

$$\mathrm{d}\left(\log \frac{V^\pi(t)}{V^\mu(t)}\right) \;=\; \sum_{i=1}^m g_i(t)\,\mathrm{d}\mu_i(t) - \frac{1}{2}\sum_{i=1}^m \sum_{j=1}^m g_i(t)\, g_j(t)\cdot \mu_i(t)\mu_j(t)\tau_{ij}^\mu(t)\,\mathrm{d}t\,. \tag{14.4}$$

On the other hand, $D^2_{ij}\log \mathbf{G}(x) = \big(D^2_{ij}\mathbf{G}(x)\,/\mathbf{G}(x)\big) - D_i \log \mathbf{G}(x)\, D_j \log \mathbf{G}(x)\,,$ so we get

$$\mathrm{d}\big(\log \mathbf{G}(\mu(t))\big) \;=\; \sum_{i=1}^m g_i(t)\,\mathrm{d}\mu_i(t) + \frac{1}{2}\sum_{i=1}^m \sum_{j=1}^m D^2_{ij}\log \mathbf{G}(\mu(t))\cdot \mathrm{d}\langle \mu_i,\mu_j\rangle(t)$$

$$= \sum_{i=1}^m g_i(t)\mathrm{d}\mu_i(t) + \frac{1}{2}\sum_{i=1}^m \sum_{j=1}^m \left\{ \frac{D^2_{ij}\mathbf{G}}{\mathbf{G}}\big(\mu(t)\big) - g_i(t)g_j(t)\right\} \mu_i(t)\mu_j(t)\tau_{ij}^\mu(t)\,\mathrm{d}t$$

by Itô's rule in conjunction with (2.9). Comparing this last expression with (14.4) and recalling (11.3), we deduce

$$\mathrm{d}\big(\log \mathbf{G}(\mu(t))\big) \;=\; \mathrm{d}\left(\log \frac{V^\pi(t)}{V^\mu(t)}\right) - \mathfrak{g}(t)\,\mathrm{d}t\,, \qquad \text{that is, (11.2).}$$

Exercise 7.1: For the function $\mathbf{G}_p(\cdot)$ of (7.5), we have $D_i\mathbf{G}_p(x) = \big(x_i\,/\mathbf{G}_p(x)\big)^{p-1}$ and

$$D_iD_j\mathbf{G}_p(x) \;=\; (1-p)\,\big(\mathbf{G}_p(x)\big)^{1-p}\left[\,(x_ix_j)^{p-1}\;\big(\mathbf{G}_p(x)\big)^{-p} - x_i^{p-2}\delta_{ij}\,\right].$$

This leads from (11.1) to the diversity-weighted portfolio of (7.1), and from (11.3) to

$$\mathfrak{g}(t) \;=\; \frac{1-p}{2}\left(\sum_{i=1}^m \pi_i^{(p)}(t)\tau_{ii}^\mu(t) - \sum_{i=1}^m \sum_{j=1}^m \pi_i^{(p)}(t)\pi_j^{(p)}(t)\tau_{ij}^\mu(t)\right) \;=\; (1-p)\,\gamma_*^{\pi^{(p)}}(t)$$

via the numéraire-invariance property (3.5).

Exercise 12.6: From (7.6) with $p=1/2$ we get

$$\log\left(\frac{V^{\pi^{(p)}}(T)}{V^\mu(T)}\right) \;=\; \log\left(\frac{\sum_{i=1}^m \sqrt{\mu_i(T)}}{\sum_{i=1}^m \sqrt{\mu_i(0)}}\right)^2 \;+\; \frac{1}{2}\int_0^T \gamma_*^{\pi^{(p)}}(t)\,\mathrm{d}t\,.$$

The first term on the right-hand side is bounded from below by $-\log m$. Concerning the second term, we have the excess growth rate computation

$$\sum_{i=1}^m \frac{\pi_i^{(p)}(t)}{\mu_i(t)} - \sum_{i=1}^m \frac{(\pi_i^{(p)}(t))^2}{\mu_i(t)} \;=\; \frac{\left(\sum_{k=1}^m \big(\sqrt{\mu_{(k)}(t)}\,\big)^{-1}\right)\left(\sum_{k=1}^m \sqrt{\mu_{(k)}(t)}\,\right) - m}{\left(\sum_{k=1}^m \sqrt{\mu_{(k)}(t)}\,\right)^2}$$

for the quantity $2\gamma_*^{\pi^{(p)}}(t)$. Clearly $\left(\sum_{k=1}^m \sqrt{\mu_{(k)}(t)}\,\right)^2 \le m$, so we obtain

$$2m\,\gamma_*^{\pi^{(p)}}(t) \;\ge\; \sum_{k=2}^m \frac{1}{\sqrt{\mu_{(k)}(t)}} \left(\sum_{\ell=1}^{k-1} \sqrt{\mu_{(\ell)}(t)} + \sum_{\ell=k+1}^m \sqrt{\mu_{(\ell)}(t)} \right)$$

$$\ge \sum_{k=2}^m \frac{1}{\sqrt{\mu_{(k)}(t)}} \cdot (k-1)\sqrt{\mu_{(k)}(t)} \;=\; \frac{m(m-1)}{2}\,.$$

The conclusions follow now easily.

Exercise 12.7: From (12.5) and (12.10) we get $\lim_{T\to\infty}(1/T)\log\mu_i(T) = 0$ a.s. for every $i = 1, \cdots, m$ (coherence). Here $\gamma_*^{\eta}(t) = ((m-1)/(2m^2))\sum_{i=1}^m(1/\mu_i(t))$ is the excess growth rate of the equally-weighted portfolio, and the master formula (11.2) with $\mathbf{G}(x) = \sqrt[m]{x_1\cdots x_m}$ gives

$$\log\left(\frac{V^\eta(T)}{V^\mu(T)}\right) \;=\; \frac{1}{m}\sum_{i=1}^m \log\left(\frac{\mu_i(T)}{\mu_i(0)}\right) \;+\; \frac{m-1}{2m^2}\sum_{i=1}^m \int_0^T \frac{\mathrm{d}t}{\mu_i(t)}\,.$$

The claim follows now from (12.11).

Exercise 12.8: An easy computation gives $2\gamma^\pi(t) = \sum_{i=1}^m \pi_i(t)[\,1+\alpha-\pi_i(t)\,]/\mu_i(t)$ for the growth rate of any extended portfolio $\pi(\cdot)$ in our context. Now, given any $(\mu_1,\cdots,\mu_m) \in \Delta_{++}^m$, it is checked that the expression $\sum_{i=1}^m \pi_i[\,1+\alpha-\pi_i\,]/\mu_i$ is maximized at $\widehat{\pi}_i = (\lambda/m) + (1-\lambda)\mu_i$ over $(\pi_1,\cdots,\pi_m) \in \Delta^m$. Therefore, $\gamma^{\widehat{\pi}}(\cdot) \ge \gamma^\pi(\cdot)$ pointwise, for every extended portfolio $\pi(\cdot)$.

As discussed in [KK] (2006), this leads to the numéraire property.

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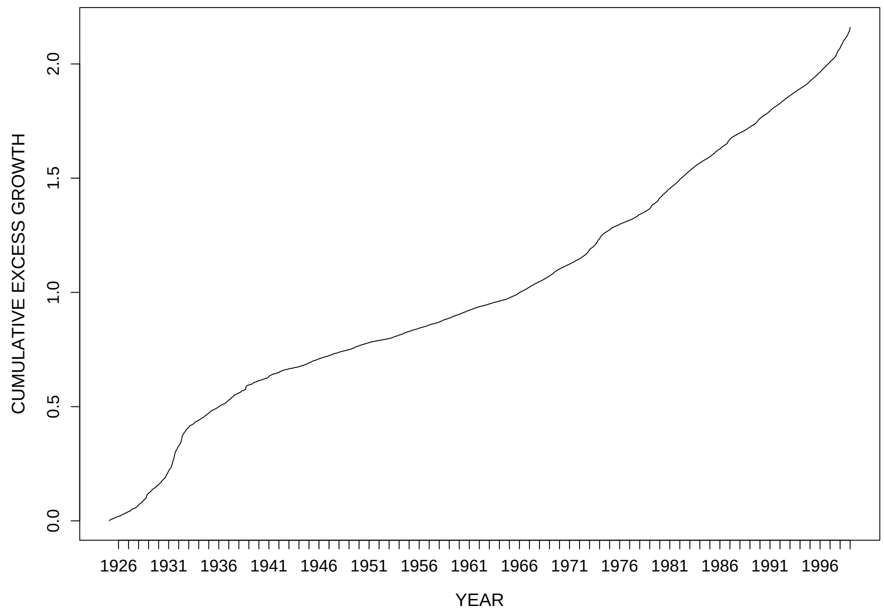

Figure 1: Cumulative excess growth $\int_0^{\cdot} \gamma_*^{\mu}(t)\, dt$ for the U.S. market during the period 1926 – 1999; reproduced here from [FK] (2005). The data used for this chart come from the monthly stock database of the Center for Research in Securities Prices (CRSP) at the University of Chicago. The market we construct consists of the stocks traded on the New York Stock Exchange (NYSE), the American Stock Exchange (AMEX) and the NASDAQ Stock Market, after the removal of all REITs, all closed-end funds, and those ADRs not included in the S&P 500 Index. Until 1962, the CRSP data included only NYSE stocks. The AMEX stocks were included after July 1962, and the NASDAQ stocks were included at the beginning of 1973. The number of stocks in this market varies from a few hundred in 1926 to about 7500 in 1999.