

# The Pricing of Exotic Options by Monte-Carlo Simulations in a Lévy Market with Stochastic Volatility

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### **Abstract**

Recently, stock price models based on Lévy processes with stochastic volatility were introduced. The resulting vanilla option prices can be calibrated almost perfectly to empirical prices. Under this model, we will price exotic options, like the barrier, lookback and cliquet options, by Monte-Carlo simulation. The sampling of paths is based on a compound Poisson approximation of the Lévy process involved. The precise choice of the terms in the approximation is crucial and investigated in detail. In order to reduce the standard error of the Monte-Carlo simulation, we make use of the technique of control variates. It turns out that there are significant differences with the classical Black-Scholes prices.

# 1 Introduction

The most famous continuous-time model for stock prices or indices is the celebrated Black-Scholes model (BS-model) [11]. It uses the Normal distribution to fit the log-returns of the underlying: the price process of the underlying is given by the geometric Brownian Motion

$$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right),$$

where  $\{W_t, t \geq 0\}$  is standard Brownian motion, i.e.  $W_t$  follows a Normal distribution with mean 0 and variance  $t$ . Under this model pricing formulae for a variety of options are available. We are particularly interested in the pricing of so-called exotic options of European nature, i.e. the payoff function can be path-dependent, however there is a fix maturity date and no-early exercise is allowed.

Path-dependent options have become popular in the OTC market in the last decades. Examples of these exotic path-dependent options are lookback options and barrier options. The lookback call option has the particular feature of allowing its holder to buy the stock at the minimum it has achieved over the life of the option. The payoff of a barrier options depends on whether the price of the underlying asset crosses a given threshold (the barrier) before maturity. The simplest barrier options are "knock in" options which come into existence when the price of the underlying asset touches the barrier and "knock-out" options which come out of existence in that case. For example, an up-and-out call has the same payoff as a regular "plain vanilla" call if the price of the underlying assets remains below the barrier over the life of the option but becomes worthless as soon as the price of the underlying asset crosses the barrier. Under the Black-Scholes framework closed-form option pricing formulae for the above types of barrier and lookback options are available (see for example [24]). We also focus on cliquet options, these options depend on the relative returns of the asset over a series of predetermined periods. These options are popular in mutual funds with capital protection. However, even in the BS-model, no closed-form pricing formulae exists for the pricing of these types of derivatives.

It is well known however that the log-returns of most financial assets are asymmetrically distributed and have an actual kurtosis that is higher than that of the Normal distribution. The BS-model is thus a very poor model to describe stock price dynamics. In real markets traders are well aware that the future probability distribution of the underlying asset may not be lognormal and they use a volatility smile adjustment. The smile-effect is decreasing with time to maturity. Moreover, smiles are frequently asymmetric. To price a set of European vanilla options, one uses for every strike  $K$  and for every maturity  $T$  another volatility parameter  $\sigma$ . This is fundamentally wrong since this implies that only one underlying stock/index is modeled by a number of completely different stochastic processes. Moreover, one has no guarantee that the chosen volatility parameters can be used to price exotic options.

In order to deal with the non-Gaussian character of the log-returns, in the late 1980s and in the 1990s several other models, based on more sophisticated distributions, were proposed. In these models the stock price process is now the exponential of a so-called Lévy process. As the Brownian motion, a Lévy process has stationary and independent increments; the distribution of the increments now has to belong to the general class of infinitely divisible laws. The choice of this law is crucial in the modeling and it should reflect the stochastic behavior of the log-returns of the asset.

Madan and Seneta [28] have proposed a Lévy process with Variance Gamma (VG) distributed increments. We mention also the Hyperbolic model [20] proposed by Eberlein and Keller and their generalizations [30]. In the same year Barndorff-Nielsen proposed the Normal Inverse Gaussian (NIG) Lévy process [3]. The CGMY model was introduced in [16] as a generalization of the VG model. Finally, we mention the Meixner process which was introduced in [37] (see also [38], [39], [40] and [22]). All models give a much better fit to historical data. Also, one is able to calibrate model prices of vanilla options to market options much better than under the BS-model. Overall one thus observes a significant improvement over the BS-model.

Several attempts have been made to obtain closed-form expressions to price exotic options under these Lévy models. However finding explicit formulae for exotic options in the more sophisticated Lévy market is very hard. Barrier options under a Lévy market were considered by [14]. The results rely on the Wiener-Hopf decomposition and one uses analytic techniques. Similar and totally general results by probabilistic methods for barrier and lookback options are described by [41]. The numerical calculations needed are of high complexity: numerical integrals with dimension 3 or 4 are needed, together with numerical inversion methods; it is not clear at all whether these pricing techniques are more accurate than Monte-Carlo based pricing.

Moreover, although there is a significant improvement in accuracy with respect to the BS-model, there still is a discrepancy between model prices and market prices. The main feature which these Lévy models are missing, is the fact that the volatility or more general the environment is changing stochastically over time. Stochastic volatility is a stylized feature of financial time series of log-returns of asset prices.

In order to deal with this problem, one starts from the Black-Scholes setting and makes the volatility parameter itself stochastic. Different choices can be made to describe the stochastic behavior of the volatility. We mention the Cox-Ingersoll-Ross (CIR) process and the models of Barndorff-Nielsen and Shephard (see [6], [7], [8], and [9]) based on Ornstein-Uhlenbeck processes (OU-processes). We will not follow this approach, but focus on the introduction of the stochastic environment through the stochastic time change as proposed in [17]. This technique can not only be used starting from the BS-model, but also from the Lévy models. In these stochastic volatility models one makes the (business) time (of the Lévy process) stochastic. In times of high volatility time is running fast, and in periods with low volatility the time is going slow. For this rate of time process, one proposes in [17] the classical example of a mean-reverting positive

process: the CIR process. These models are called the Lévy Stochastic Volatility models (Lévy-SV models). In [17] and [40] it was shown that by following this procedure, one can almost perfectly calibrate Lévy-SV model option prices to market prices. Finding explicit formulae for exotic options is almost hopeless in these models. However, once you have calibrated the model to a basic set of options, it is possible to price other (exotic) options using Monte-Carlo simulations. Moreover, the complexity of the simulations does not increase drastically; besides the Lévy process, one only has to simulate the time-changing process, which is in our case the classical and easily simulated CIR process. The Lévy process can be simulated based on a compound-Poisson approximation. Special care has to be taken for the very small jumps; as proposed in [2] these small jumps can in some cases be approximated by a Brownian motion.

Throughout this paper we make use of a data set with the mid-prices of a set of European call options on the SP500-index at the close of the market on the 18h of April 2002. At this day the SP500 closed at 1124.47. The short rate was at that time equal to  $r = 0.019$  per year and we had a dividend yield of  $q = 0.012$  per year.

First, we look at the exotic options we want to price, together with general pricing techniques for vanilla options. In Section 3, we give an overview of some popular Lévy processes. We focus on the VG-process, the NIG-process and the Meixner process. In the next section, we will use these Lévy processes in the construction of Lévy-SV models. Basically, a Lévy-SV model exist of a combination of a Lévy processes with a stochastic time changing process. As in the paper [17], our rate of time change is the CIR process. We explain a procedure to simulate all these ingredients of the Lévy-SV models. Next, we calibrate the different Lévy-SV models to our data set of market prices. The calibration procedure gives us the risk-neutral parameters of our model which we will use to produce a significant number of stock price paths. Finally, we will perform a number of simulations and compute option prices for all the mentioned models. In order to reduce the standard error of the Monte-Carlo simulation, we make use of the technique of control variates. This technique is particularly useful in this setting, since we can make use of the vanilla call prices available in the market as control variates.

## 2 Pricing of Derivatives

Throughout the text we will denote by  $r$  the daily interest rate and  $q$  the dividend yield per year. We assume a fixed planning horizon  $T$ . Our market consist of one riskless asset (the bond) with price process given by  $B = \{B_t = e^{rt}, 0 \leq t \leq T\}$  and one risky asset (the stock) with price process  $S = \{S_t, 0 \leq t \leq T\}$ . We focus on European-type derivatives, i.e. no early exercise is possible. Given our market model, let  $G(\{S_u, 0 \leq u \leq T\})$  denote the payoff of the derivative at its time of expiry  $T$ .

According to the fundamental theorem of asset pricing (see [18]) the arbi-

trage free price  $V_t$  of the derivative at time  $t \in [0, T]$  is given by

$$V_t = E_Q[e^{-r(T-t)}G(\{S_u, 0 \leq u \leq T\})|\mathcal{F}_t],$$

where the expectation is taken with respect to an equivalent martingale measure  $Q$  and  $\mathcal{F} = \{\mathcal{F}_t, 0 \leq t \leq T\}$  is the natural filtration of  $S = \{S_t, 0 \leq t \leq T\}$ . An equivalent martingale measure is a probability measure which is equivalent (it has the same null-sets) to the given (historical) probability measure and under which the discounted process  $\{e^{-(r-q)t}S_t\}$  is a martingale. Unfortunately for most models, in particular the more realistic ones, the class of equivalent measures is rather large and often covers the full no-arbitrage interval. In this perspective the BS-model, where there is an unique equivalent martingale measure, is very exceptional. Models with more than one equivalent measures are called incomplete.

## 2.1 Vanilla options

A pricing method which can be applied in general when the characteristic function of the risk-neutral stock price process is known, was developed by Carr and Madan [15] for the classical vanilla options. More precisely, let  $C(K, T)$  be the price of a European call option with strike  $K$  and maturity  $T$ . Let  $\alpha$  be a positive constant such that the  $\alpha$ th moment of the stock price exists. Carr and Madan then showed that

$$C(K, T) = \frac{\exp(-\alpha \log(K))}{\pi} \int_0^{+\infty} \exp(-iv \log(K)) \psi(v) dv \quad (1)$$

where

$$\psi(v) = \frac{e^{-rT} E[\exp(i(v - (\alpha + 1)i) \log(S_T))]}{\alpha^2 + \alpha - v^2 + i(2\alpha + 1)v}.$$

The Fast Fourier Transform can be used to invert the generalized Fourier transform of the call price. Put options can be priced using the put-call parity. This Fourier-method was generalized to other types of options, like power options and self-quanto options in [31].

## 2.2 Exotic options

### 2.2.1 Barrier and Lookback Options

Let us consider contracts of duration  $T$ , and denote the maximum and minimum process, resp., of a process  $Y = \{Y_t, 0 \leq t \leq T\}$  as

$$M_t^Y = \sup\{Y_u; 0 \leq u \leq t\} \text{ and } m_t^Y = \inf\{Y_u; 0 \leq u \leq t\}, \quad 0 \leq t \leq T.$$

Using risk-neutral valuation, we have that the time  $t = 0$  price of a lookback call option is given by

$$LC = e^{-rT} E_Q[S_T - m_T^S].$$

For single barrier options, we will focus on the following types:

- The down-and-out barrier is worthless unless its minimum remains above some "low barrier"  $H$ , in which case it retains the structure of a European call with strike  $K$ . Its initial price is given by:

$$DOB = e^{-rT} E_Q[(S_T - K)^+ 1(m_T^S > H)]$$

- The down-and-in barrier is a normal European call with strike  $K$ , if its minimum went below some "low barrier"  $H$ . If this barrier was never reached during the life-time of the option, the option is worthless. Its initial price is given by:

$$DIB = e^{-rT} E_Q[(S_T - K)^+ 1(m_T^S \leq H)]$$

- The up-and-in barrier is worthless unless its maximum crossed some "high barrier"  $H$ , in which case it retains the structure of a European call with strike  $K$ . Its price is given by:

$$UIB = e^{-rT} E_Q[(S_T - K)^+ 1(M_T^S \geq H)]$$

- The up-and-out barrier is worthless unless its maximum remains below some "high barrier"  $H$ , in which case it retains the structure of a European call with strike  $K$ . Its price is given by:

$$UOB = e^{-rT} E_Q[(S_T - K)^+ 1(M_T^S < H)]$$

We note that the value,  $DIB$ , of the down-and-in barrier call option with barrier  $H$  and strike  $K$  plus the value,  $DOB$ , of the down-and-out barrier option with same barrier  $H$  and same strike  $K$ , is equal to the value  $C$  of the vanilla call with strike  $K$ . The same is true for the up-and-out together with the up-and-in:

$$\begin{aligned} DIB + DOB &= \exp(-rT) E_Q[(S_T - K)^+ (1(m_T^S \geq H) + 1(m_T^S < H))] \\ &= \exp(-rT) E_Q[(S_T - K)^+] \\ &= C; \\ UIB + UOB &= \exp(-rT) E_Q[(S_T - K)^+ (1(M_T^S \geq H) + 1(M_T^S < H))] \\ &= \exp(-rT) E_Q[(S_T - K)^+] \\ &= C. \end{aligned} \tag{2}$$

We thus clearly see that the price of a vanilla call is correlated with the prices of the corresponding barrier options.

An important issue for barrier and lookback options is the frequency that the stock price is observed for purposes of determining whether the barrier has been reached. The above given formula assume a continuous observation. Often, the terms of the contract are modified and there are only a discrete number of observations, for example at the close of each trading day. [13] provide a way of adjusting the formulas under the Black-Scholes setting for the situation of discrete observations in case of lookback options. For barrier options the adjusting

is described in [12]: The barrier  $H$  is replaced by  $H \exp(0.582\sigma\sqrt{T/m})$  for an up-and-in or up-and-out option and by  $H \exp(-0.582\sigma\sqrt{T/m})$  for a down-and-in and down-and-out barrier, where  $m$  is the number of times the stock prices is observed;  $T/m$  is the time interval between observations. In the numerical calculations below, we have assumed a discrete number of observations, namely at the close of each trading day. Moreover, we have assumed a year consists of 250 trading days.

### 2.2.2 Cliquet Options

We also consider a more involved option, a cliquet option. These options are popular in mutual funds: investor's capital is protected and they benefit in a limited way of possible stock price rises. It has a payoff function which depends on the relative returns of the stock after a series of predetermined dates (in our case after 1, 2 and 3 years). These (yearly) returns are first floored with zero (capital is protected) and capped with a return  $cap$  (gains are limited). We will consider caps ranging in  $cap \in [0.05, 0.15]$ . The final payoff is the sum of the modified relative returns:

$$CLIQ = \exp(-rT)E_Q \left[ \sum_{i=1}^3 \min \left( \max \left( \frac{S_i - S_{i-1}}{S_{i-1}}, 0 \right), cap \right) \right]$$

## 3 Lévy Processes

Suppose  $\phi(z)$  is the characteristic function of a distribution. If for every positive integer  $n$ ,  $\phi(z)$  is also the  $n$ th power of a characteristic function, we say that the distribution is infinitely divisible. One can define for every such an infinitely divisible distribution a stochastic process,  $X = \{X_t, t \geq 0\}$ , called Lévy process, which starts at zero, has independent and stationary increments and such that the distribution of an increment over  $[s, s+t]$ ,  $s, t \geq 0$ , i.e.  $X_{t+s} - X_s$ , has  $(\phi(z))^t$  as characteristic function.

The function  $\psi(z) = \log \phi(z)$  is called the *characteristic exponent* and it satisfies the following *Lévy-Khintchine formula* [10]:

$$\psi(z) = i\gamma z - \frac{\sigma^2}{2} z^2 + \int_{-\infty}^{+\infty} (\exp(izx) - 1 - izx1_{\{|x|<1\}}) \nu(dx),$$

where  $\gamma \in \mathbb{R}$ ,  $\sigma^2 \geq 0$  and  $\nu$  is a measure on  $\mathbb{R} \setminus \{0\}$  with  $\int_{-\infty}^{+\infty} (1 \wedge x^2) \nu(dx) < \infty$ . We say that our infinitely divisible distribution has a triplet of Lévy characteristics  $[\gamma, \sigma^2, \nu(dx)]$ . The measure  $\nu(dx)$  is called the *Lévy measure* of  $X$ . From the Lévy-Khintchine formula, one sees that, in general, a Lévy process consists of three independent parts: a linear deterministic part, a Brownian part, and a pure jump part. The Lévy measure  $\nu(dx)$  dictates how the jumps occur. Jumps of sizes in the set  $A$  occur according to a Poisson process with parameter  $\int_A \nu(dx)$ . If  $\sigma^2 = 0$  and  $\int_{-1}^{+1} |x| \nu(dx) < \infty$  it follows from standard Lévy process theory [10] [36], that the process is of finite variation.



### 3.1 Examples of Lévy Processes

### 3.2 The Variance Gamma Process

The characteristic function of the VG( $\sigma, \nu, \theta$ ) law is given by

$$\phi_{VG}(u; \sigma, \nu, \theta) = (1 - iu\theta\nu + \sigma^2\nu u^2/2)^{-1/\nu}.$$

This distribution is infinitely divisible and one can define the VG-process  $X^{(VG)} = \{X_t^{(VG)}, t \geq 0\}$  as the process which starts at zero, has independent and stationary increments and where the increment  $X_{s+t}^{(VG)} - X_s^{(VG)}$  over the time interval  $[s, t+s]$  follows a VG( $\sigma, \nu/t, t\theta$ ) law. Clearly (take  $s = 0$  and note that  $V_0 = 0$ ),

$$\begin{aligned} E[\exp(iuX_t^{(VG)})] &= \phi_{VG}(u; \sigma\sqrt{t}, \nu/t, t\theta) \\ &= (\phi_{VG}(u; \sigma, \nu, \theta))^t \\ &= (1 - iu\theta\nu + \sigma^2\nu u^2/2)^{-t/\nu}. \end{aligned}$$

In [25], it was shown that the VG-process may also be expressed as the difference of two independent Gamma processes. This characterization allows the Lévy measure to be determined:

$$\nu_{VG}(dx) = \begin{cases} C \exp(Gx)|x|^{-1}dx & x < 0 \\ C \exp(-Mx)x^{-1}dx & x > 0 \end{cases},$$

where

$$\begin{aligned} C &= 1/\nu > 0 \\ G &= \left( \sqrt{\frac{\theta^2\nu^2}{4} + \frac{\sigma^2\nu}{2}} - \frac{\theta\nu}{2} \right)^{-1} > 0 \\ M &= \left( \sqrt{\frac{\theta^2\nu^2}{4} + \frac{\sigma^2\nu}{2}} + \frac{\theta\nu}{2} \right)^{-1} > 0. \end{aligned}$$

The Lévy measure has infinite mass, and hence a VG-process has infinitely many jumps in any finite time interval. Since  $\int_{-\infty}^{+\infty} |x|\nu_{VG}(dx) < \infty$ , a VG-process has paths of finite variation. A VG-process has no Brownian component and its Lévy triplet is given by  $[\gamma, 0, \nu_{VG}(dx)]$ , where

$$\gamma = \frac{-C(G(\exp(-M) - 1) - M(\exp(-G) - 1))}{MG}$$

With the parameterization in terms of  $C, G$  and  $M$ , the characteristic function of  $X_1^{(VG)}$  reads as follows:

$$\phi_{VG}(u; C, G, M) = \left( \frac{GM}{GM + (M - G)iu + u^2} \right)^C.$$

In this notation we will refer to the distribution by the notation VG( $C, G, M$ ).

The class of Variance Gamma distributions was introduced by Madan and Seneta [27] in the late 1980s as a model for stock returns. There (and in [28] and [26]) the symmetric case ( $\theta = 0$ ) was considered. In [25], the general case with skewness is treated.

### 3.3 The Normal Inverse Gaussian Process

The Normal Inverse Gaussian (NIG) distribution with parameters  $\alpha > 0$ ,  $-\alpha < \beta < \alpha$  and  $\delta > 0$ ,  $\text{NIG}(\alpha, \beta, \delta)$ , has a characteristic function [3] given by:

$$\phi_{\text{NIG}}(u; \alpha, \beta, \delta) = \exp\left(-\delta\left(\sqrt{\alpha^2 - (\beta + iu)^2} - \sqrt{\alpha^2 - \beta^2}\right)\right).$$

Again, one can clearly see that this is an infinitely divisible characteristic function. Hence we can define the NIG-process  $X^{(\text{NIG})} = \{X_t^{(\text{NIG})}, t \geq 0\}$ , with  $X_0^{(\text{NIG})} = 0$ , stationary and independent NIG distributed increments: To be precise  $X_t^{(\text{NIG})}$  has a  $\text{NIG}(\alpha, \beta, t\delta)$  law.

The Lévy measure for the NIG process is given by

$$\nu_{\text{NIG}}(dx) = \frac{\delta\alpha}{\pi} \frac{\exp(\beta x) K_1(|x|)}{|x|} dx,$$

where  $K_\lambda(x)$  denotes the modified Bessel function of the third kind with index  $\lambda$  (see [1]).

A NIG-process has no Brownian component and its Lévy triplet is given by  $[\gamma, 0, \nu_{\text{NIG}}(dx)]$ , where

$$\gamma = (2\delta\alpha/\pi) \int_0^1 \sinh(\beta x) K_1(\alpha x) dx.$$

The density of the  $\text{NIG}(\alpha, \beta, \delta)$  distribution is given by

$$f_{\text{NIG}}(x; \alpha, \beta, \delta) = \frac{\alpha\delta}{\pi} \exp\left(\delta\sqrt{\alpha^2 - \beta^2} + \beta x\right) \frac{K_1(\alpha\sqrt{\delta^2 + x^2})}{\sqrt{\delta^2 + x^2}}.$$

The NIG distribution was introduced by Barndorff-Nielsen [3]. See also [4] [32] [33] and [34].

### 3.4 The Meixner Process

The density of the Meixner distribution ( $\text{Meixner}(\alpha, \beta, \delta)$ ) is given by

$$f_{\text{Meixner}}(x; \alpha, \beta, \delta) = \frac{(2 \cos(\beta/2))^{2\delta}}{2\alpha\pi\Gamma(2\delta)} \exp\left(\frac{bx}{a}\right) \left| \Gamma\left(\delta + \frac{ix}{a}\right) \right|^2,$$

where  $\alpha > 0$ ,  $-\pi < \beta < \pi$ ,  $\delta > 0$ .

The characteristic function of the  $\text{Meixner}(\alpha, \beta, \delta)$  distribution is given by

$$\phi_{\text{Meixner}}(u; \alpha, \beta, \delta) = \left( \frac{\cos(\beta/2)}{\cosh \frac{\alpha u - i\beta}{2}} \right)^{2\delta}$$

Clearly, the Meixner( $\alpha, \beta, \delta$ ) distribution is infinitely divisible and we can associate with it a Lévy process which we call the Meixner process. More precisely, a Meixner process  $X^{(Meixner)} = \{X_t^{(Meixner)}, t \geq 0\}$  is a stochastic process which starts at zero, i.e.  $X_0^{(Meixner)} = 0$ , has independent and stationary increments, and where the distribution of  $X_t^{(Meixner)}$  is given by the Meixner distribution Meixner( $\alpha, \beta, \delta t$ ).

It is easy to show that our Meixner process  $X^{(Meixner)} = \{X_t^{(Meixner)}, t \geq 0\}$  has no Brownian part and a pure jump part governed by the Lévy measure

$$\nu(dx) = \delta \frac{\exp(\beta x/\alpha)}{x \sinh(\pi x/\alpha)} dx;$$

The first parameter in the Lévy triplet equals

$$\gamma = \alpha \delta \tan(\beta/2) - 2\delta \int_1^\infty \sinh(\beta x/\alpha) / \sinh(\pi x/\alpha) dx$$

Because  $\int_{-\infty}^{+\infty} |x| \nu(dx) = \infty$  the process is of infinite variation.

The Meixner process was introduced in [37] (see also [38]) and originates from the theory of orthogonal polynomials. Later on it was suggested to serve for fitting stock returns in [22]. This application in finance was worked out in [39] and [40].

## 4 The Lévy-Stochastic Volatility Model

### 4.1 The Lévy-Stochastic Volatility Model: Theory

It has been observed that the volatilities estimated (or more general the parameters of uncertainty) change stochastically over time and are clustered as can be seen in Figure 1, where the absolute log-returns of the SP500-index over a period of 32 years is plotted. One clearly sees that there are periods with high absolute log-returns and periods with lower absolute log-returns.

In order to incorporate such an effect Carr, Madan, Geman and Yor [17] proposed the following: One increase or decrease the level of uncertainty by speeding up or slowing down the rate at which time passes. Moreover, in order to build clustering and to keep time going forward one employs a mean-reverting positive process as a measure of the local rate of time change. They use as the rate of time change the classical example of a mean-reverting positive stochastic process: the CIR process  $y_t$  that solves the SDE

$$dy_t = \kappa(\eta - y_t)dt + \lambda y_t^{1/2} dW_t,$$

where  $W = \{W_t, t \geq 0\}$  is standard Brownian motion.

The economic time elapsed in  $t$  units of calendar time is then given by  $Y_t$  where

$$Y_t = \int_0^t y_s ds$$

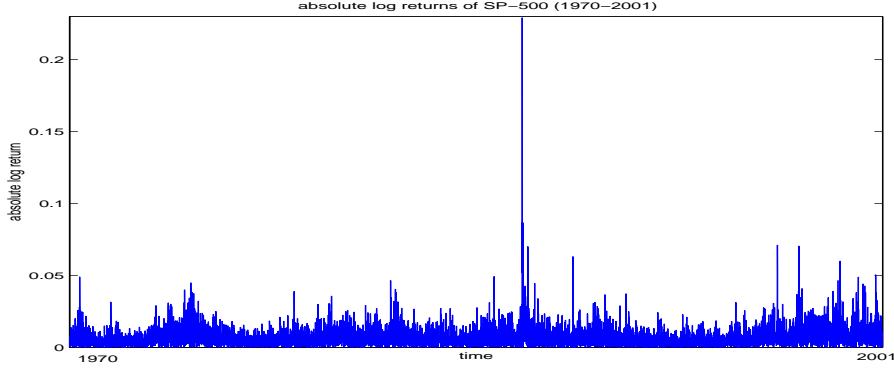


Figure 1: Volatility clusters: absolute log-returns SP500-index between 1970 and 2001

The characteristic function of  $Y_t$  is explicitly known:

$$\phi(u, t) = \frac{\exp(\kappa^2 \eta t / \lambda^2) \exp(2y(0)iu / (\kappa + \gamma \coth(\gamma t / 2)))}{(\cosh(\gamma t / 2) + \kappa \sinh(\gamma t / 2) / \gamma)^{2\kappa \eta / \lambda^2}},$$

where

$$\gamma = \sqrt{\kappa^2 - 2\lambda^2 i u}$$

Note, that for  $c > 0$ ,  $\tilde{y} = cy = \{cy_t, t \geq 0\}$ , satisfies the SDE

$$d\tilde{y}_t = \kappa(c\eta - \tilde{y}_t)dt + \sqrt{c}\lambda\tilde{y}_t^{1/2}dW_t, \quad (3)$$

and the initial condition is  $\tilde{y}_0 = cy_0$ .

The (risk-neutral) price process of the stock  $S = \{S_t, 0 \leq t \leq T\}$  is now modeled as follows:

$$S_t = S_0 \frac{\exp((r - q)t)}{E[\exp(X_{Y(t)})]} \exp(X_{Y(t)}), \quad (4)$$

where  $q$  is the dividend yield and  $X = \{X_t, 0 \leq t \leq T\}$  is a Lévy process with

$$E[\exp(iuX_t)] = \exp(t\psi_X(u)).$$

The characteristic function for the log of our stock price is given by:

$$\phi(u) = E[\exp(iu \log(S_t))] = \exp(iu((r - q)t + \log S_0)) \frac{\phi(-i\psi_X(u), t)}{\phi(-i\psi_X(-i), t) i^u}. \quad (5)$$

The characteristic function is important for the pricing of vanilla options (see formula (1)). Recall that in these methods we only needed the characteristic function of  $\log(S_t)$ . By the above formula, explicit formulae are at hand.

Note that if our Lévy process  $X = \{X_t, t \geq 0\}$  is a VG-process, we have for  $c > 0$ , that  $\tilde{X} = \{X_{ct}, t \geq 0\}$  is again a Lévy process of the same class, with the same parameters except the  $C$ -parameter, which is now multiplied with the constant  $c$ . The same can be said for the NIG and the Meixner process. The parameter which takes into account the same time-scaling property is now the  $\delta$ -parameter. In combination with (3) this means that in these cases there is one redundant parameter. We therefore, can set  $y_0 = 1$ , and scale the present rate of time change to one. More precisely, we have that that the characteristic function  $\phi(u)$  of (5) satisfies:

$$\begin{aligned}\phi_{VG-CIR}(u; C, G, M, \kappa, \eta, \lambda, y_0) &= \phi_{VG-CIR}(u; Cy_0, G, M, \kappa, \eta/y_0, \lambda/\sqrt{y_0}, 1) \\ \phi_{Meixner-CIR}(u; \alpha, \beta, \delta, \kappa, \eta, \lambda, y_0) &= \phi_{Meixner-CIR}(u; \alpha, \beta, \delta y_0, \kappa, \eta/y_0, \lambda/\sqrt{y_0}, 1) \\ \phi_{NIG-CIR}(u; \alpha, \beta, \delta, \kappa, \eta, \lambda, y_0) &= \phi_{NIG-CIR}(u; \alpha, \beta, \delta y_0, \kappa, \eta/y_0, \lambda/\sqrt{y_0}, 1)\end{aligned}$$

Also, instead of setting the  $y_0$  parameter equal to one, other involved parameters, e.g.  $\delta$  or  $C$ , can be scaled to 1.

Actually, this time-scaling effect lies at the heart of the idea of incorporating stochastic volatility through making time stochastic. Here, it comes down to the fact that instead of making the volatility parameter (of the BS-model) stochastic, we are making the parameter  $C$ , in the VG case, or the parameter  $\delta$ , in the NIG and the Meixner case, stochastic (via the time). Note that this effect, does not only influences the standard deviation (or volatility) of the processes; also the skewness and the kurtosis are now fluctuating stochastically.

## 4.2 Calibration to Market Data

Using formula (1) one can easily compute plain vanilla option prices under the above Lévy-SV models. By this one can calibrate model prices to markets prices for example in the least-squared sense. In Figures 2-4, one can see that the Lévy-SV models give a very good fit to the empirical option prices of our SP-500 data set. The o-signs are market prices the +-signs are model prices. In Table 1 an overview is given of the risk-neutral parameters coming out of the calibration procedure.

For comparative purposes, one computes the average absolute error as a percentage of the mean price. This statistic, which we will denote with  $ape$ , is an overall measure of the quality of fit:

$$ape = \frac{1}{\text{mean option price}} \sum_{\text{options}} \frac{|\text{Market price} - \text{Model price}|}{\text{number of options}}$$

Other measures which give also an estimate of the goodness of fit are the average absolute error ( $aae$ ), the root mean square error ( $rmse$ ) and the average relative percentage error ( $arpe$ ):

$$aae = \sum_{\text{options}} \frac{|\text{Market price} - \text{Model price}|}{\text{number of options}}$$

$$rmse = \sqrt{\sum_{options} \frac{(\text{Market price} - \text{Model price})^2}{\text{number of options}}}$$

$$arpe = \frac{1}{\text{number of options}} \sum_{options} \frac{|\text{Market price} - \text{Model price}|}{\text{Market price}}$$

In Table 2 an overview of these measures of fit are given.

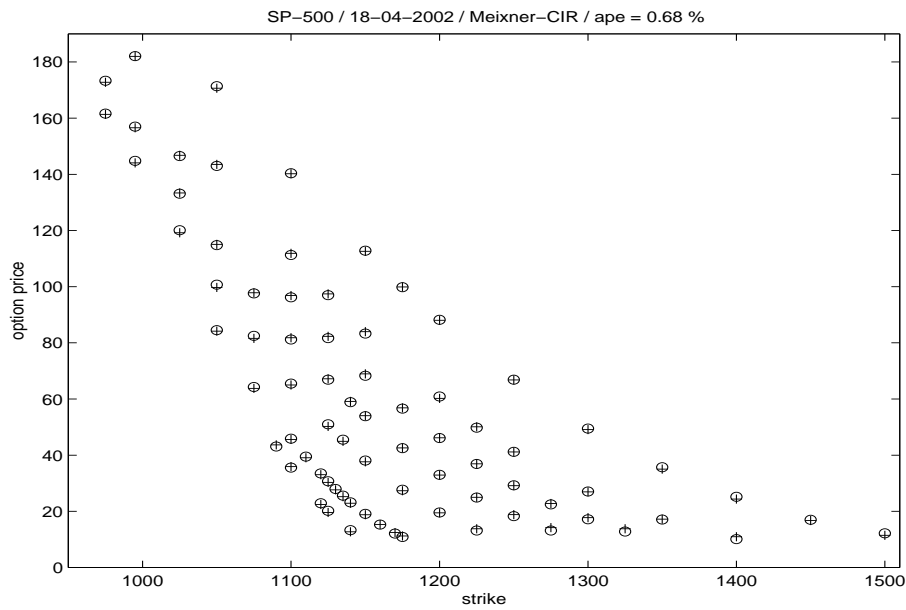


Figure 2: Meixner-CIR calibration on SP500 options (o's are market prices, +'s are model prices)

## 5 Monte Carlo simulation of SV-Lévy processes: Theory

### 5.1 Introduction

Basically, the method goes as follows: we simulate, say  $m$ , paths of our stock prices process and calculate for each path the value of the payoff function  $V_i$ ,  $i = 1, \dots, m$ . Then the Monte-Carlo estimate of the expected value of the payoff is

$$\hat{V} = \frac{1}{m} \sum_{i=1}^m V_i. \quad (6)$$

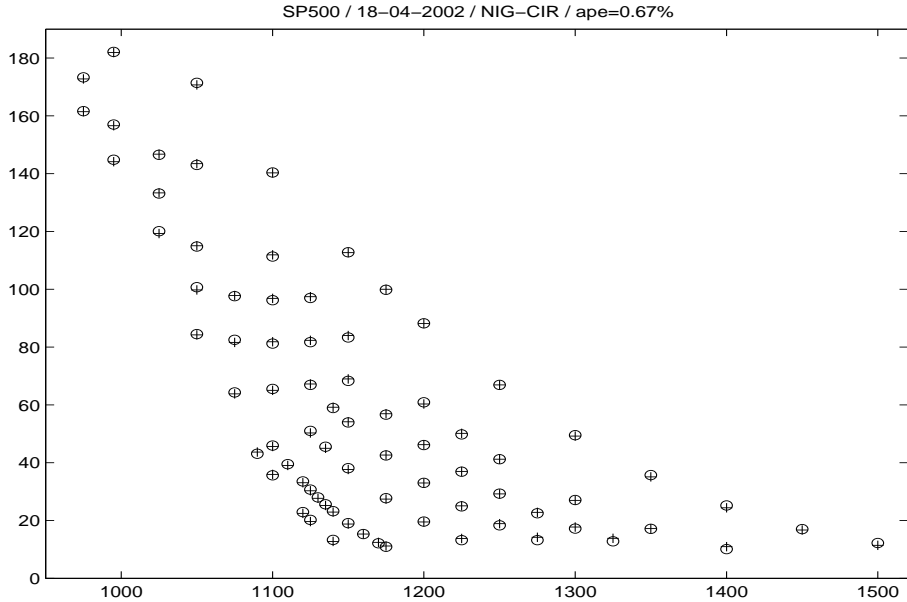


Figure 3: NIG-CIR calibration on SP500 options (o's are market prices, +'s are model prices)

The final option price is then obtained by discounting this estimate:  $\exp(-rT)\hat{V}$ . The standard error (SE) of the estimate is given by:

$$\sqrt{\frac{1}{(m-1)^2} \sum_{i=1}^m (\hat{V} - V_i)^2}.$$

The standard error decreases with the square root of the number of sample paths: to reduce the standard error by half, it is necessary to generate four times as many sample paths.

To simulate a stock price path, we first simulated our time change. For the CIR process, this is quite easy and classical: we follow the "Euler Scheme". Basically, we discretize the SDE as:

$$\Delta y_t = \kappa(\eta - y_t)\Delta t + \lambda y_t^{1/2} d\Delta W_t.$$

Next, we simulate our Lévy process upto time  $Y_T = \int_0^T y_s ds$ . To simulate a Lévy process, we exploit the well-known (compound-Poisson) approximation of the process, which we describe below in detail. It reduces the simulation of a Lévy process to the simulation of a number of independent Poisson process. Simulating independent Poisson distributed random numbers, and as such Poisson processes, is easy. We refer to [19].

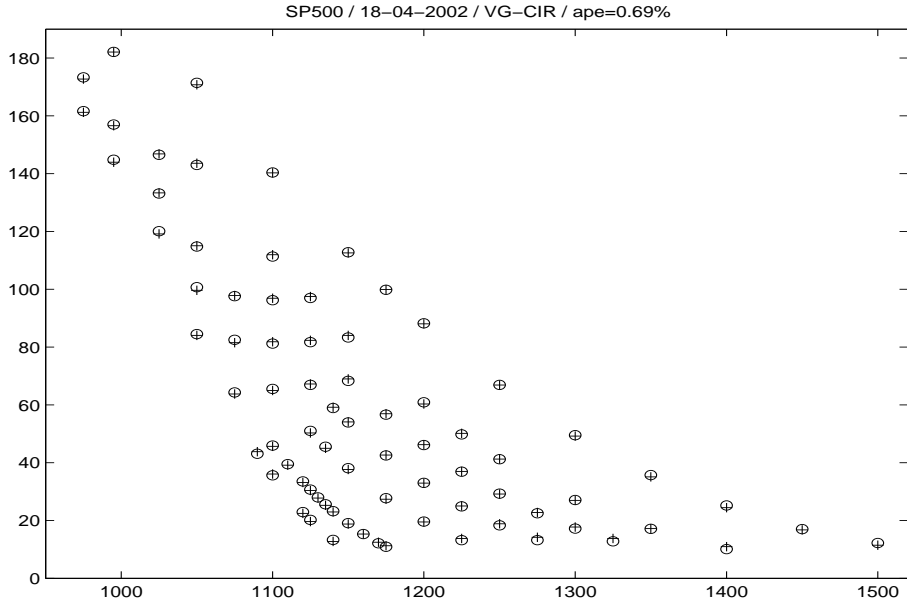


Figure 4: VG-CIR calibration on SP500 options (o's are market prices, +'s are model prices)

Finally, we rescale our Lévy path according to the path of our stochastic business time and plug this into our formula (4) for the stock price behavior.

## 5.2 The Compound-Poisson Approximation of a Lévy Process

The compound-Poisson approximation procedure is explained in detail for example in [35]. Further support can be given by [2] and [29]. The procedure goes as follows: Let  $X$  be a Lévy process with characteristic triplet  $[\gamma, \sigma^2, \nu(dx)]$ .

First, we will discretize the Lévy measure  $\nu(dx)$ . We choose some small  $0 < \epsilon < 1$ . Then we make a partition of  $\mathbb{R} \setminus [-\epsilon, \epsilon]$  of the following form. We choose real numbers  $a_0 < a_1 < \dots < a_k = -\epsilon$ ,  $\epsilon = a_{k+1} < a_{k+2} < \dots < a_{d+1}$ .

The jumps larger than  $\epsilon$  are approximated by a sum of independent Poisson processes in the following way: We take an independent Poisson process  $N_t^{(i)}$  for each interval,  $[a_{i-1}, a_i)$ ,  $1 \leq i \leq k$  and  $[a_i, a_{i+1})$ ,  $k+1 \leq i \leq d$ , with intensity  $\lambda_i$  given by the Lévy measure of the interval. Furthermore we choose a point  $c_i$  in each interval such that the variance of the Poisson process matches the part of the variance of the Lévy process corresponding to this interval.



VG-CIR						
$C$	$G$	$M$	$\kappa$	$\eta$	$\lambda$	$y_0$
11.9896	25.8523	35.5344	0.6020	1.5560	1.9992	1
NIG-CIR						
$\alpha$	$\beta$	$\delta$	$\kappa$	$\eta$	$\lambda$	$y_0$
18.4815	-4.8412	0.4685	0.5391	1.5746	1.8772	1
Meixner-CIR						
$\alpha$	$\beta$	$\delta$	$\kappa$	$\eta$	$\lambda$	$y_0$
0.1231	-0.5875	3.3588	0.5705	1.5863	1.9592	1

Table 1: Parameter estimation for Lévy SV models

Model	ape	aae	rmse	arpe
VG-CIR	0.69 %	0.4269	0.5003	1.33 %
NIG-CIR	0.67 %	0.4123	0.4814	1.32 %
Meixner-CIR	0.68 %	0.4204	0.4896	1.34 %

Table 2: *ape*, *aae*, *rmse* and *arpe* for Lévy SV models

### 5.2.1 Approximation of the Small Jumps by their Expected Value

Next, we look at the very small jumps. The first method is by just replacing them by their expected value. Putting all things together, we approximate  $X$  by a process  $X^{(d)}$ , consisting of a Brownian motion  $W = \{W_t, t \geq 0\}$  and  $d$  independent Poisson processes  $N^{(i)} = \{N_t^{(i)}, t \geq 0\}$ ,  $i = 1, \dots, d$  with parameter  $\lambda_i$ :

$$\begin{aligned}
X_t^{(d)} &= \gamma t + \sigma W_t + \sum_{i=1}^d c_i (N_t^{(i)} - \lambda_i t 1_{|c_i| < 1}), & (7) \\
\lambda_i &= \nu([a_{i-1}, a_i)) \text{ for } i \leq k, \\
&= \nu([a_i, a_{i+1})) \text{ for } k+1 \leq i \leq d, \\
c_i^2 \lambda_i &= \int_{a_{i-1}}^{a_i^-} x^2 \nu(dx) \text{ for } i \leq k, \\
&= \int_{a_i}^{a_{i+1}^-} x^2 \nu(dx) \text{ for } k+1 \leq i \leq d.
\end{aligned}$$

When the original process does not have a Brownian component ( $\sigma = 0$ ), then also the approximating process has not one.

### 5.2.2 Approximation of the Small Jumps by a Brownian Motion

A further improvement is to incorporate also the contribution from the variation of small jumps. Denote by

$$\sigma^2(\epsilon) = \int_{|x| < \epsilon} x^2 \nu(dx).$$

We let all (compensated) jumps smaller than  $\epsilon$  contribute to Brownian part of  $X$ . To be precise, we again approximate  $X$  by a process  $X^{(d)}$ , consisting of a Brownian motion  $W = \{W_t, t \geq 0\}$  and  $d$  independent Poisson processes  $N^{(i)} = \{N_t^{(i)}, t \geq 0\}$ ,  $i = 1, \dots, d$  with parameter  $\lambda_i$ . Only the Brownian part is different from above. We have now:

$$X_t^{(d)} = \gamma t + \tilde{\sigma} W_t + \sum_{i=1}^d c_i (N_t^{(i)} - \lambda_i 1_{|c_i| < 1} t), \quad (8)$$

where

$$\tilde{\sigma}^2 = \sigma^2 + \sigma^2(\epsilon),$$

and the  $\lambda_i$  and  $c_i$ ,  $i = 1, \dots, d$  as above.

Note that a Brownian term appears even when the original process does not have one ( $\sigma = 0$ ). In [2] a rigorous discussion is presented of when the latter approximation is valid. It turns out that this is the case if and only if for each  $\kappa > 0$

$$\lim_{\epsilon \rightarrow 0} \frac{\sigma(\kappa \sigma(\epsilon) \wedge \epsilon)}{\sigma(\epsilon)} = 1. \quad (9)$$

This condition is implied by

$$\lim_{\epsilon \rightarrow 0} \frac{\sigma(\epsilon)}{\epsilon} = \infty. \quad (10)$$

Moreover, if the Lévy measure of the original Lévy process does not have atoms in some neighborhood of the origin the condition (10) and condition (9) are equivalent. Results on the speed of convergence of the above approximation can be found in [2].

We conclude by noting that the Meixner and the NIG process satisfy the condition (9), but the VG does not. In the simulations below we thus use a Brownian motion in the approximation for the Meixner and the NIG, but not for the VG process.

### 5.3 On the Choice of the Approximating Poisson Processes

The choice of the intervals  $[a_{i-1}, a_i)$ ,  $1 \leq i \leq k$  and  $[a_i, a_{i+1})$ ,  $k+1 \leq i \leq d$  is crucial. We typically set  $k = 100$  and  $d = 2k$ , so we have the same number of Poisson processes reflecting a positive as a negative jump. Next, we look at three different ways to choose the intervals. First we look at equally spaced, then at equally weighted and finally at intervals with inverse linear boundaries. We illustrate this for the VG, NIG and Meixner processes, with parameters taken from Table 1.

#### 5.3.1 Equally Spaced Intervals

One can choose the intervals equally spaced, i.e.  $|a_{i-1} - a_i|$  is kept fix for all  $1 \leq i \leq d+1$ ,  $i \neq k+1$ . This choice is illustrated in Figure 5, where we plot for all Lévy processes  $\lambda_i$  versus  $c_i$ . A width equal to 0.001 was chosen and we zoomed in on the range  $[-0.05, 0.05]$ ;  $k = 100$ . Note the explosion near zero.

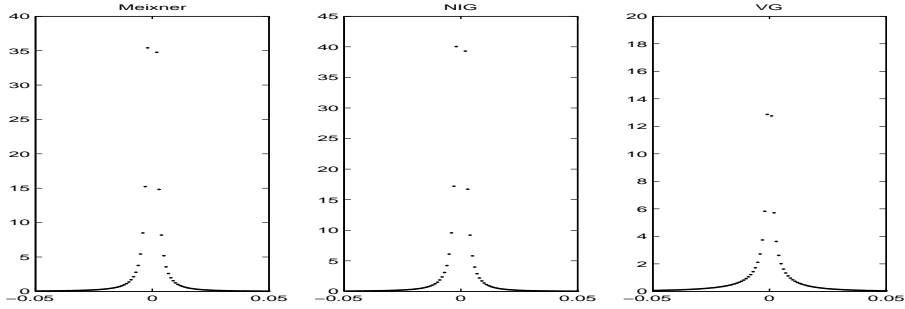


Figure 5: Equally spaced intervals

### 5.3.2 Equally Weighted Intervals

Here we opt to keep the intensities for the up-jumps and down-jumps corresponding to an interval constant. Thus, for equally weighted intervals, the Lévy measures of intervals on the negative part of the real line  $\nu([a_{i-1} - a_i])$  are kept fix for all  $1 \leq i \leq k$ . Similarly also the measure of intervals corresponding to up-jumps  $\nu([a_i - a_{i+1}])$  is kept fix for all  $k+1 \leq i \leq d$ . Note, that for this choice the outer intervals can become quite large.

### 5.3.3 Interval with Inverse Linear Boundaries

Finally, we propose the case where the boundaries are given by  $a_{i-1} = -\alpha i^{-1}$  and  $a_{2k+2-i} = \alpha i^{-1}$ ,  $1 \leq i \leq k+1$  and  $\alpha > 0$ . This leads to much gradually decreasing intensity parameters  $\lambda_i$  as can be seen from Figure 6, where  $\alpha = 0.2$  and  $k = 100$ . Moreover, there is no explosion to infinity near zero; the intensities come even down again. Note, that in Figure 6, we now show the whole range with  $c_i \in [-0.2, 0.2]$  for the same examples as above. Note also that in all cases the intensities of down jumps are slightly higher than those of the corresponding up jumps; this reflects the fact that log stock returns are negatively skewed.

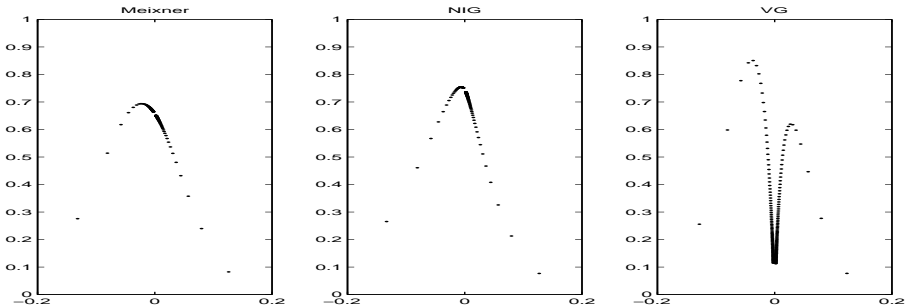


Figure 6:  $a_{i-1} = -0.2/i$  and  $a_{2k+2-i} = 0.2/i$ ,  $1 \leq i \leq k+1$

## 5.4 Variance Reduction by Control Variates

If we want to price exotic barrier and lookback options or other exotics (of European type), we often have information on vanilla options available. Note that we have obtained our parameters from calibration on market vanilla prices. In this case, where we thus have exact pricing information on related objects, we can use the variance reduction technique of control variates. The method is a highly speed-up method, but the implementation depends on the characteristics of the instruments being valued.

The idea is as follows. Let us assume that we wish to calculate some expected value,  $E[G] = E[G(\{S_t, 0 \leq t \leq T\})]$  of a (payoff) function  $G$  and that there is a related function  $H$  whose expectation  $E[H] = E[H(\{S_t, 0 \leq t \leq T\})]$  we know exactly. One has to think of  $G$  as the payoff function of the exotic option we want to price via Monte-Carlo and of  $H$  as the payoff function of the vanilla option whose price (and thus the expectation  $E[H]$ ) we observe in the market.

Suppose that for a sample path the value of the function  $G$  and  $H$  are positively correlated, e.g. the value of a up-and-in call is positively correlated with the value of a vanilla call with same strike price and time to expiry. This can be seen for example from Equation (2).

Define for some number  $b \in \mathbb{R}$  a new payoff function

$$\hat{G}(\{S_t, 0 \leq t \leq T\}) = G(\{S_t, 0 \leq t \leq T\}) + b(H(\{S_t, 0 \leq t \leq T\}) - E[H]).$$

Note that the expected value of the new function  $\hat{G}$  is the same as the expectation of the original function  $G$ . However there can be a significant difference in the variance: We have

$$\text{Var}[\hat{G}] = \text{Var}[G] - 2bcov[G, H] + b^2\text{Var}[H].$$

This variance is minimized if  $b = cov[G, H]/\text{Var}[H]$ . For this minimizing value of  $b$  we find

$$\begin{aligned} \text{Var}[\hat{G}] &= \text{Var}[G] \left( 1 - \frac{cov^2[G, H]}{\text{Var}[G]\text{Var}[H]} \right) \\ &= \text{Var}[G](1 - \text{corr}^2(G, H)) \\ &\leq \text{Var}[G]. \end{aligned}$$

So if the absolute value of the correlation between  $G$  and  $H$  is close to 1, the variance of  $\hat{G}$  will be very small. Clearly, if we find such a highly correlated function  $H$ , very large computational savings may be made.  $H$  is called the control variate. Note that the method is flexible enough to include several control variates.

The precise optimal value for  $b$  is not known but can be estimated from the same simulation. Special care has to be taken however since estimating parameters determining the result from the same simulation can introduce a bias. In the limit of very large numbers of iterations, this bias vanishes. A remedy for the problem of bias due to the estimation of  $b$  is to use an initial

simulation, possibly with fewer iterates than the main run, to estimate  $b$  in isolation. The control variate technique usually provides such a substantial speed-up in convergence that this initial parameter estimation is affordable.

To summarize, we give an overview of the procedure (with an initial estimation of  $b$ ). Recall we want to price an European exotic option expiring at time  $T$  with payoff function  $G(\{S_t, 0 \leq t \leq T\})$  and that we have a correlated option expiring also at time  $T$  with payoff  $H(\{S_t, 0 \leq t \leq T\})$  whose option price is observable in the market and given by

$$\exp(-rT)E[H(\{S_t, 0 \leq t \leq T\})] = \exp(-rT)E[H].$$

The expectation is under the markets risk-neutral pricing measure.

We proceed as follows:

1. Estimate the optimal  $b$ :

- a) Sample a significant number  $n$  of paths for the stock price  $S = \{S_t, 0 \leq t \leq T\}$  (see procedure below) and calculate for each path  $i$  :  $g_i = G(\{S_t, 0 \leq t \leq T\})$  and  $h_i = H(\{S_t, 0 \leq t \leq T\})$ .
- b) An estimate for  $b$  is

$$\hat{b} = \frac{1}{n} \left( \sum_{i=1}^n g_i h_i - E[H] \sum_{i=1}^n g_i \right).$$

2. Simulate a significant number  $m$  of paths for the stock price  $S = \{S_t, 0 \leq t \leq T\}$  (see procedure below) and calculate for each path  $i$ :  $g_i = G(\{S_t, 0 \leq t \leq T\})$  and  $h_i = H(\{S_t, 0 \leq t \leq T\})$ .
3. Calculate an estimation of the expected payoff by:

$$\hat{g} = \frac{1}{n} \left( \sum_{i=1}^m g_i - \hat{b}(h_i - E[H]) \right).$$

4. Discount the estimated payoff  $\hat{g}$  at the risk-free rate  $r$  to get an estimate of the value of the derivative: The option price is given by  $\exp(-rT)\hat{g}$ .

The simulation of the stock price process is summarized as follows:

- i) Simulate the rate of time-change process  $y = \{y_t, 0 \leq t \leq T\}$ .
- ii) Calculate from i), the time change  $Y = \{Y_t, 0 \leq t \leq T\}$ .
- iii) Simulate the Lévy process  $X = \{X_t, 0 \leq t \leq Y_T\}$ . Note that we sample over the period  $[0, Y_T]$ .
- iv) Calculate the time-changed Lévy process  $X_{Y_t}$ , for  $t \in [0, T]$ .
- v) Calculate the stock price process  $S = \{S_t, 0 \leq t \leq T\}$ .

In the Figure 7, one sees in the case of the Meixner-CIR combination a sample of all ingredients: the rate of time change  $y_t$ , the stochastic business time  $Y_t$ , the Lévy process  $X_t$ , the time changes Lévy process  $X_{Y_t}$ , and finally the stock price  $S_t$ .

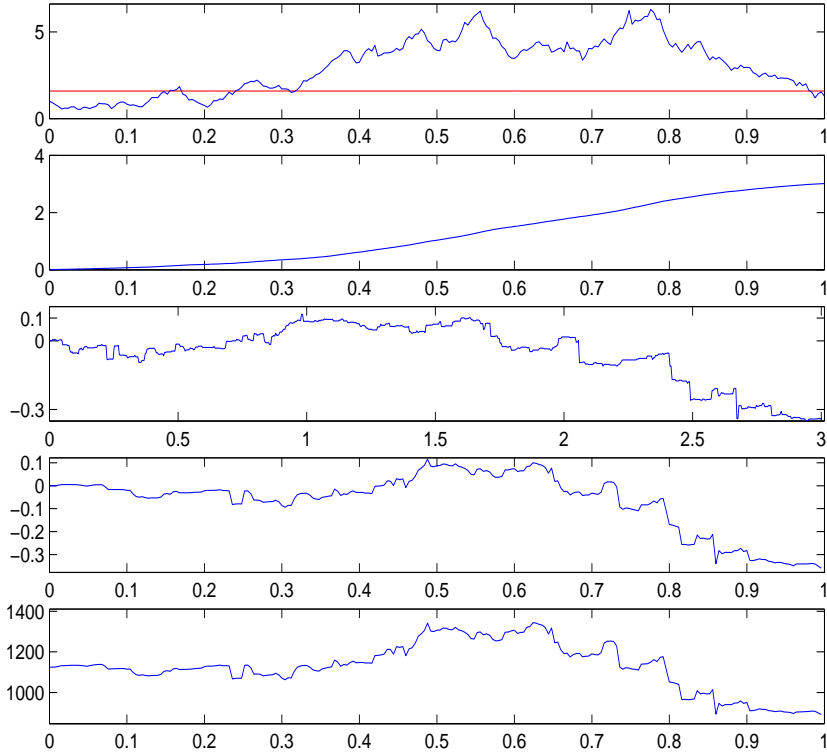


Figure 7: Simulation of  $y_t$ ,  $Y_t$ ,  $X_t$ ,  $X_{Y_t}$ , and  $S_t$

## 6 Monte Carlo Pricing of Exotics under a SV-Lévy Model

### 6.1 Barrier and Lookback Prices

We take for all barrier options the time to maturity  $T = 1$ , the strike  $K = S_0$  and the barrier  $H$  equal to

$$H_{UIB} = 1.1 \times S_0, \quad H_{UOB} = 1.3 \times S_0, \quad H_{DIB} = 0.95 \times S_0, \quad H_{DOB} = 0.8 \times S_0.$$

For all models, we make  $n = 10000$  simulations of paths covering a one year period. The time is discretised in 250 equally small time steps. We run 100 simulations to find an estimate for the optimal  $b$  of the control variate. We consider both Equally Weighted Intervals (EWI) and Interval with Inverse Linear Boundaries (IILB).

In Tables 3, we compare the price along all model considered together with Black-Scholes prices. The standard error of the simulation is given below the option prices in brackets. The volatility parameter in the BS-model is taken

equal to  $\sigma_{lse} = 0.1812$ ,  $\sigma_{min} = 0.1479$  and  $\sigma_{max} = 0.2259$ . These  $\sigma$ 's, which can be read off from Figure 8, correspond to the volatility giving rise to the least square-error of the Black-Scholes model prices compared with the empirical SP500 vanilla options, the minimal, and maximal implied volatility parameter over all strikes and maturities of our data set, respectively. The Black-Scholes

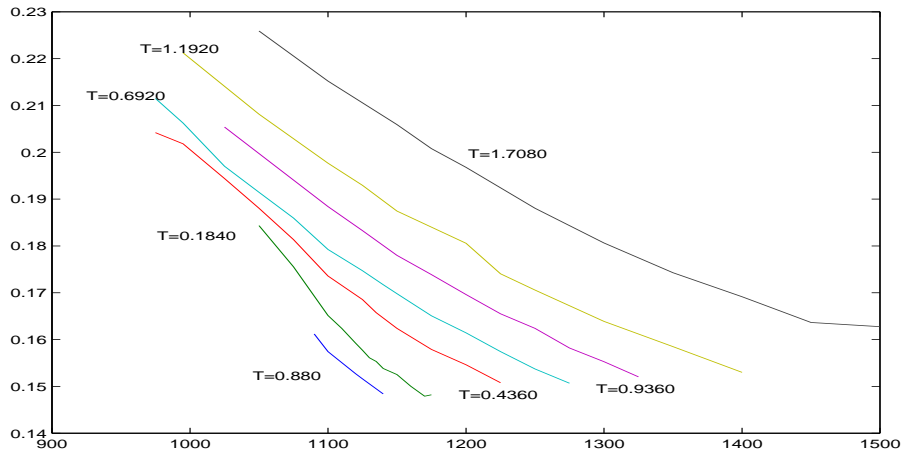


Figure 8: Implied Volatilities

barrier prices are for adjusted for the discrete observation of the stock prices as described above.

In Figure 9, one sees the effect of using control variates for the Monte-Carlo pricing of the UIB and the lookback option in the Meixner-CIR case. Similar figures can be obtained for the other options and cases; all show that the standard error is declining much faster in case of control variates then in the case without. In Figure 10 one sees how the Monte-Carlo prices converge over the number of iterations in the Meixner-CIR case. Note that in both figures we have logarithmic scales for the number of iterations.

## 6.2 Cliquet Option Prices

To price the cliquet option we have to rely even in the Black-Scholes world on Monte-Carlo simulations, since as far as we know there are no closed-formulae available. For the cliquet option, we calculated (using 50000 simulations) prices for cap's ranging from 0.05 to 0.15. We do this under the Black-Scholes model with volatility parameters  $\sigma_{lse} = 0.1812$ ,  $\sigma_{min} = 0.1479$  and  $\sigma_{max} = 0.2259$  and compare these with the Meixner-CIR-IILB prices in Figure 11. We clearly see that Black-Scholes model is significantly underpricing the option. We finally remark that the option prices under the other Lévy-SV models are almost identical with the Meixner-CIR-IILB case.

Model	LC	UIB	UOB	DIB	DOB
VG-CIR-EWI	135.27 (0.4942)	78.50 (0.2254)	63.18 (0.6833)	17.71 (0.5306)	86.07 (0.0811)
NIG-CIR-EWI	135.24 (0.4764)	79.08 (0.2161)	63.54 (0.6665)	16.47 (0.4924)	86.12 (0.0819)
Meixner-CIR-EWI	135.72 (0.4853)	78.57 (0.2239)	64.34 (0.7091)	17.28 (0.5168)	86.06 (0.0836)
VG-CIR-IILB	134.77 (0.4894)	78.66 (0.2224)	62.89 (0.7250)	17.42 (0.5259)	86.16 (0.0958)
NIG-CIR-IILB	135.48 (0.4817)	78.66 (0.2203)	63.27 (0.6841)	16.76 (0.5409)	86.18 (0.0609)
Meixner-CIR-IILB	134.83 (0.4712)	78.66 (0.2193)	63.87 (0.7087)	17.24 (0.5560)	86.08 (0.0794)
BS $\sigma_{min}$	128.64	65.66	46.26	21.78	69.17
BS $\sigma_{lse}$	155.12	81.66	39.98	32.88	83.50
BS $\sigma_{max}$	189.76	102.20	30.32	54.73	101.60

Table 3: Exotic Option prices

### 6.3 Conclusion

If we look at the prices of the exotic options in the Black-Scholes world, we observe that the BS-prices depend heavily on the choice of the volatility parameter and that it is not clear which value to take. For the Lévy-SV models the prices are very close to each other. We conclude that the BS-model is not at all appropriate to price exotics. Moreover there is evidence that the Lévy-SV models are much more reliable; they give a much better indication than the BS-model.

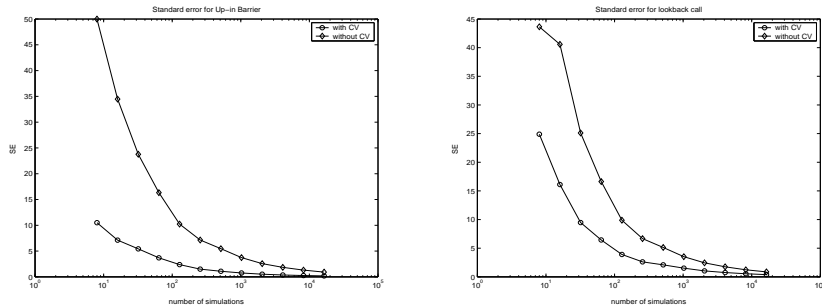


Figure 9: Standard error with and without control variates



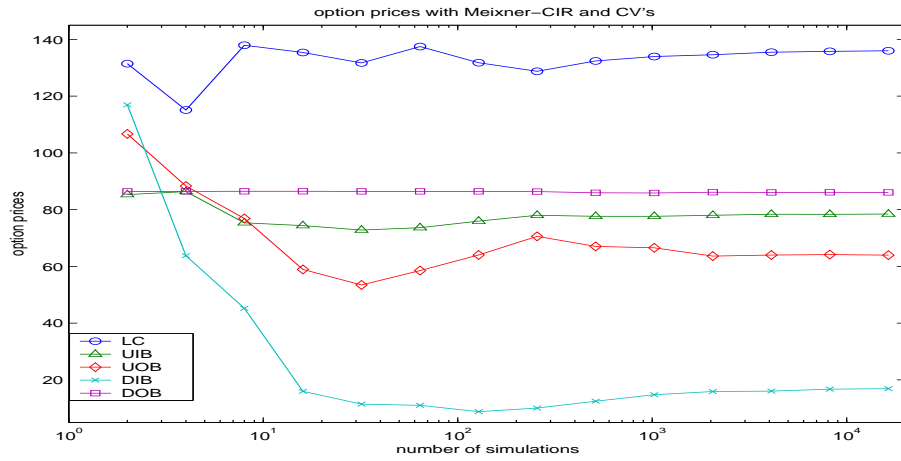


Figure 10: Convergence of options prices

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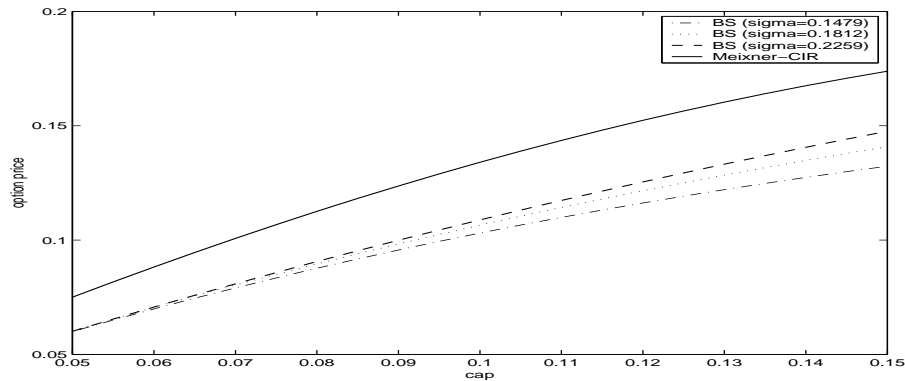


Figure 11: Cliquet prices for different cap's

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