(2.9) Proposition. $P\left[\lim_{\varepsilon\downarrow 0}\sqrt{\frac{\pi\varepsilon}{2}}\eta_t(\varepsilon) = L_t \text{ for every } t\right] = 1.$

Proof. Let $\varepsilon_k = 2/\pi k^2$; then $\bar{n}([\varepsilon_k, \infty[) = k$ and the sequence $\left\{N_t^{\varepsilon_{k+1}} - N_t^{\varepsilon_k}\right\}$ is a sequence of independent Poisson r.v.'s with parameter t. Thus, for fixed t, the law of large numbers implies that a.s.

$$\lim_n \frac{1}{n} N_t^{\varepsilon_n} = \lim_n \sqrt{\frac{\pi\varepsilon_n}{2}} N_t^{\varepsilon_n} = t.$$

As N_t^ε increases when ε decreases, for $\varepsilon_{n+1} \le \varepsilon < \varepsilon_n$,

$$\sqrt{\frac{\pi\varepsilon_{n+1}}{2}} N_t^{\varepsilon_n} \le \sqrt{\frac{\pi\varepsilon}{2}} N_t^{\varepsilon} \le \sqrt{\frac{\pi\varepsilon_n}{2}} N_t^{\varepsilon_{n+1}}$$

and plainly

$$P\left[\lim_{\varepsilon\downarrow 0}\sqrt{\frac{\pi\varepsilon}{2}} N_t^\varepsilon = t\right] = 1.$$

We may find a set Σ of probability 1 such that for $w \in \Sigma$,

$$\lim_{\varepsilon\downarrow 0}\sqrt{\frac{\pi\varepsilon}{2}} N_t^\varepsilon(w) = t$$

for every rational t. Since N_t^ε increases with t, the convergence actually holds for all t's. For each $w \in \Sigma$, we may replace t by $L_t(w)$ which ends the proof.

Remarks. 1°) A remarkable feature of the above result is that $\eta_t(\varepsilon)$ depends only on the set of zeros of B up to t. Thus we have an approximation procedure for L_t, depending only on Z. This generalizes to the local time of regenerative sets (see Notes and Comments).

2°) The same kind of proof gives the approximation by downcrossings seen in Chap. VI (see Exercise (2.10)).

(2.10) Exercise. 1°) Prove that

$$n\left(\sup_{t<R}|u(t)| \ge x\right) = 1/x.$$

[Hint: If $A_x = \left\{u : \sup_{t<R} u(t) \ge x\right\}$, observe that L_{T_x} is the first jump time of the Poisson process $N_t^{A_x}$ and use the law of L_{T_x} found in Sect. 4 Chap. VI.]

2°) Using 1°) and the method of Proposition (2.9), prove, in the case of BM, the a.s. convergence in the approximation result of Theorem (1.11) Chap. VI.

3°) Let $a > 0$ and set $M_a(u) = \sup_{t\le g_{T_a}} u(t)$ where, as usual, g_{T_a} is the last zero of the Brownian path before the time T_a when it first reaches a. Prove that M_a is uniformly distributed on $[0, a]$. This is part of Williams' decomposition theorem (see Sect. 4 Chap. VII).

[Hint: If J_a (resp. J_y) is the first jump of the Poisson process N^{A_a} (resp. N^{A_y}), then $P[M_a < y] = P\left[J_y = J_a\right]$; use Lemma (1.13).]

(2.11) Exercise. Prove that the process X defined, in the notation of Proposition (2.5), by $X_t(w) = |e_s(t - \tau_{s-}(w), w)| - s$, if $\tau_{s-}(w) \le t \le \tau_s(w)$, is the BM

$$\beta_t = \int_0^t \operatorname{sgn}(B_s)dB_s$$

and that $Y_t(w) = s + |e_s(t - \tau_{s-}(w), w)|$ if $\tau_{s-}(w) \le t \le \tau_s(w)$, is a $\mathrm{BES}^3(0)$.

(2.12) Exercise. 1°) Let $A \in \mathscr{U}_\delta$ be such that $n(A) < \infty$. Observe that the number $C^A_{d_t}$ of excursions belonging to A in the interval $[0, d_t]$ (i.e. whose two end points lie between 0 and d_t) is defined unambiguously and prove that $E\left[C^A_{d_t}\right] = n(A)E[L_t]$.

2°) Prove that on $\{d_s < t\}$,

$$C^A_{d_t} - C^A_{d_s} = C^A_{d_t - d_s} \circ \theta_{d_s}, \quad L_{d_t} - L_{d_s} = L_{d_t - d_s} \circ \theta_{d_s},$$

and consequently that $C^A_{d_t} - n(A)L_t$ is a $(\mathscr{F}_t)$-martingale.

[Hint: Use the strong Markov property of BM.]

\# **(2.13) Exercise (Scaling properties).** 1°) For any $c > 0$, define a map s_c on $\mathbf{W}$ or U_δ by

$$s_c(w)(t) = w(ct)/\sqrt{c}.$$

Prove that $e_t(s_c(w)) = s_c\left(e_{t\sqrt{c}}(w)\right)$ and that for $A \in \mathscr{U}_\delta$

$$n\left(s_c^{-1}(A)\right) = n(A)/\sqrt{c}.$$

[Hint: See Exercise (2.11) in Chap. VI.]

2°) **(Normalized excursions)** We say that $u \in U$ is normalized if $R(u) = 1$. Let U^1 be the subset of normalized excursions. We define a map ν from U to U^1 by

$$\nu(u) = s_{R(u)}(u).$$

Prove that for $\Gamma \subset U^1$, the quantity

$$\gamma(\Gamma) = n_+\left(\nu^{-1}(\Gamma) \cap (R \ge c)\right)/n_+(R \ge c)$$

is independent of $c > 0$. The probability measure γ may be called the *law of the normalized Brownian excursion*.

3°) Show that for any Borel subset S of $\mathbb{R}_+$,

$$n_+\left(\nu^{-1}(\Gamma) \cap (R \in S)\right) = \gamma(\Gamma)n_+(R \in S)$$

which may be seen as displaying the independence between the length of an excursion and its form.

4°) Let e^c be the first positive excursion e such that $R(e) \ge c$. Prove that

$$\gamma(\Gamma) = P\left[\nu(e^c) \in \Gamma\right].$$

[Hint: Use Lemma (1.13).]

\# **(2.14) Exercise.** Let $\Lambda_t(\varepsilon)$ be the total length of the excursions with length $< \varepsilon$, strictly contained in $[0, t[$. Prove that

$$P\left[\lim_{\varepsilon \downarrow 0} \sqrt{\frac{\pi}{2\varepsilon}} \Lambda_t(\varepsilon) = L_t \quad \text{for every } t\right] = 1.$$

[Hint: $\Lambda_t(\varepsilon) = -\int_0^\varepsilon x\eta_t(dx)$ where η_t is defined in Proposition (2.9).]

\# **(2.15) Exercise.** Let $S_t = \sup_{s \le t} B_s$ and $n_t(\varepsilon)$ the number of flat stretches of S of length $\ge \varepsilon$ contained in $[0, t]$. Prove that

$$P\left[\lim_{\varepsilon \downarrow 0} \sqrt{\frac{\pi\varepsilon}{2}} n_t(\varepsilon) = S_t \quad \text{for every } t\right] = 1.$$

(2.16) Exercise (Skew Brownian motion). Let (Y_n) be a sequence of independent r.v.'s taking the values 1 and -1 with probabilities α and $1-\alpha$ $(0 \le \alpha \le 1)$ and independent of B. For each w in the set on which B is defined, the set of excursions $e_t(w)$ is countable and may be given the ordering of $\mathbb{N}$. In a manner similar to Proposition (2.5), define a process X^α by putting

$$X_t^\alpha = Y_n \left| e_s(t - \tau_{s-}(w), w) \right|$$

if $\tau_{s-}(w) \le t < \tau_s(w)$ and e_s is the n-th excursion in the above ordering. Prove that the process thus obtained is a Markov process and that it is a skew BM by showing that its transition function is that of Exercise (1.16), Chap. I. Thus, we see that the skew BM may be obtained from the reflecting BM by changing the sign of each excursion with probability $1-\alpha$. As a result, a Markov process X is a skew BM if and only if $|X|$ is a reflecting BM.

* **(2.17) Exercise.** Let $A_t^+ = \int_0^t 1_{(B_s > 0)} ds$, $A_t^- = \int_0^t 1_{(B_s < 0)} ds$.

1°) Prove that the law of the pair $L_t^{-2}(A_t^+, A_t^-)$ is independent of t and that $A_{\tau_t}^+$ and $A_{\tau_t}^-$ are independent stable r.v.'s of index 1/2.

[Hint: $A_{\tau_t}^+ + A_{\tau_t}^- = \tau_t$ which is a stable r.v. of index 1/2.]

2°) Let a^+ and a^- be two positive real numbers, S an independent exponential r.v. with parameter 1. Prove that

$$E\left[\exp\left(-L_S^{-2}\left(a^+ A_S^+ + a^- A_S^-\right)\right)\right] = \int_0^\infty \exp(-\phi(s))\phi'(s)ds$$

where $\phi(s) = \frac{1}{\sqrt{2}}\left[\left(s^2 + a^+\right)^{1/2} + \left(s^2 + a^-\right)^{1/2}\right]$. Prove that, consequently, the pair $L_t^{-2}\left(A_t^+, A_t^-\right)$ has the same law as $\frac{1}{4}(T^+, T^-)$ where T^+ and T^- are two independent r.v.'s having the law of τ_1 and derive therefrom the arcsine law of Sect. 2 Chap. VI (see also Exercise (4.20) below). The reader may wish to compare this method with that of Exercise (2.33) Chap. VI.

(2.18) Exercise. Prove that the set of Brownian excursions can almost-surely be labeled by $\mathbb{Q}_+$ in such a way that $q < q'$ entails that e_q occurs before $e_{q'}$.

[Hint: Call e_1 the excursion straddling 1, then $e_{1/2}$ the excursion straddling $g_1/2, \dots$]

§3. Excursions Straddling a Given Time

From Sect. 2 in Chap. VI, we recall the notation

$$g_t = \sup\{s < t : B_s = 0\}, \qquad d_t = \inf\{s \geq t : B_s = 0\},$$

and set

$$A_t = t - g_t, \qquad \Lambda_t = d_t - g_t.$$

Plainly, $d_t > g_t$ if and only if there is an excursion which straddles t and in that case, A_t is the age of the excursion at time t and Λ_t is its length. We have $\Lambda_t = R(i_{g_t})$.

(3.1) Lemma. *The map $t \to g_t$ is right-continuous.*

Proof. Let $t_n \downarrow t$; if there exists an n such that $g_{t_n} < t$, then $g_{t_m} = g_t$ for $m \geq n$; if $g_{t_n} \geq t$ for every n, then $t \leq g_{t_n} \leq t_n$ for every n, hence t is itself a zero of B and $g_t = t = \lim_n g_{t_n}$.

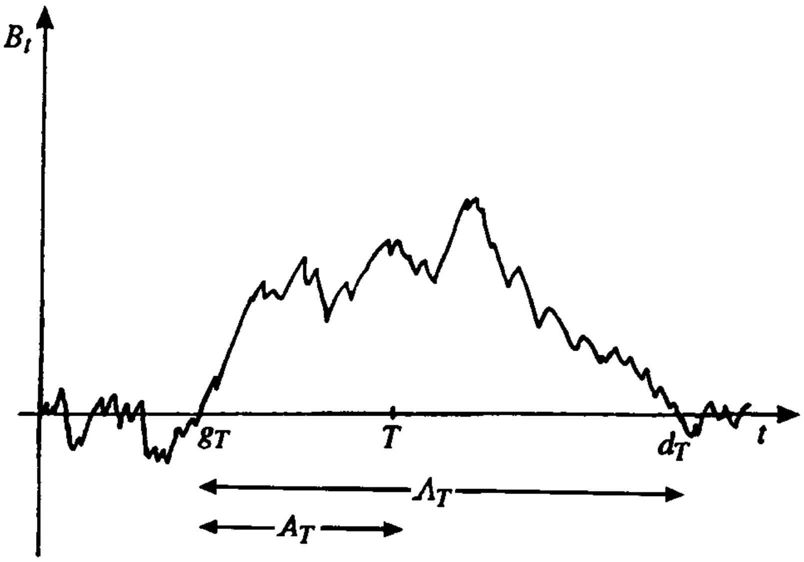

Fig. 8.

For a positive r.v. S we denote by $\mathscr{F}_S$ the σ-algebra generated by the variables H_S where H ranges through the optional processes. If S is a stopping time, this coincides with the usual σ-field $\mathscr{F}_S$. Let us observe that in general $S \leq S'$ does not entail $\mathscr{F}_S \subset \mathscr{F}_{S'}$, when S and S' are not stopping times; one can for instance find a r.v. $S \leq 1$ such that $\mathscr{F}_S = \mathscr{F}_\infty$.

Before we proceed, let us recall that by Corollary (3.3), Chap. V, since we are working in the Brownian filtration $(\mathscr{F}_t)$, there is no difference between optional and predictable processes.

(3.2) Lemma. *The family $(\check{\mathscr{F}}_t) = (\mathscr{F}_{g_t})$ is a subfiltration of $(\mathscr{F}_t)$ and if T is a $(\check{\mathscr{F}}_t)$-stopping time, then $\mathscr{F}_{g_T} \subset \check{\mathscr{F}}_T \subset \mathscr{F}_T$.*

Proof. As g_t is $\mathscr{F}_t$-measurable, the same reasoning as in Proposition (4.9) Chap. I shows that, for a predictable process Z, the r.v. Z_{g_t} is $\mathscr{F}_t$-measurable whence $\check{\mathscr{F}}_t \subset \mathscr{F}_t$ follows.

Now choose u in $\mathbb{R}_+$ and set $Z'_t = Z_{g_{t\wedge u}}$; thanks to Lemma (3.1) and to what we have just proved, Z' is $(\mathscr{F}_t)$-optional hence predictable. Pick $v > u$; if $g_v \le u$, then $g_u = g_v$ and since $g_{g_t} = g_t$ for every t,

$$Z'_{g_v} = Z_{g_v} = Z_{g_u},$$

and if $g_v > u$, then $Z'_{g_t} = Z_{g_u}$. As a result, each of the r.v.'s which generate $\check{\mathscr{F}}_u$ is among those which generate $\check{\mathscr{F}}_v$. It follows that $\check{\mathscr{F}}_u \subset \check{\mathscr{F}}_v$, that is, $(\check{\mathscr{F}}_t)$ is a filtration.

Let now T be a $(\check{\mathscr{F}}_t)$-stopping time. By definition, the σ-algebra $\mathscr{G}_{g_T}$ is generated by the variables Z_{g_T} with Z optional; but $Z_{g_T} = \left(Z_{g_\cdot}\right)_T$ and $Z_{g_\cdot}$ is $\check{\mathscr{F}}_{g_t}$-optional because of Lemma (3.1) which entails that Z_{g_T} is $\check{\mathscr{F}}_T$-measurable, hence that $\mathscr{F}_{g_T} \subseteq \check{\mathscr{F}}_T$. On the other hand, since $\check{\mathscr{F}}_t \subset \mathscr{F}_t$, the time T is also a $(\mathscr{F}_t)$-stopping time from which the inclusion $\check{\mathscr{F}}_T \subset \mathscr{F}_T$ is easily proved. □

We now come to one of the main results of this section which allows us to compute the laws of some particular excursions when n is known. If F is a positive $\mathscr{U}$-measurable function on U, for $s > 0$, we set

$$q(s, F) = n(R > s)^{-1} \int_{\{R>s\}} F\, dn \equiv n(F \mid R > s).$$

We recall that $0 < n(R > s) < \infty$ for every $s > 0$.

(3.3) Proposition. *For every fixed $t > 0$,*

$$E\left[F\left(i_{g_t}\right) \mid \check{\mathscr{F}}_t\right] = q(A_t, F) \qquad \text{a.s.},$$

and for a $(\check{\mathscr{F}}_t)$-stopping time T,

$$E\left[F\left(i_{g_T}\right) \mid \check{\mathscr{F}}_T\right] = q(A_T, F) \quad \text{a.s. on the set} \quad \{0 < g_T < T\}.$$

Proof. We know that, a.s., t is not a zero of B hence $0 < g_t < t$ and $q(A_t, F)$ is defined; also, if $s \in G_w$ and $s < t$, we have $s = g_t$ if and only if $s + R \circ \theta_s > t$. As a result, g_t is the only $s \in G_w$ such that $s < t$ and $s + R \circ \theta_s > t$. If Z is a positive $(\mathscr{F}_t)$-predictable process, we consequently have

$$E\left[Z_{g_t} F\left(i_{g_t}\right)\right] = E\Big[\sum_{s \in G_w} Z_s F(i_s) 1_{\{R\circ\theta_s > t-s>0\}}\Big].$$

We may replace $R\circ\theta_s$ by $R(i_s)$ and then apply the master formula to the right-hand side which yields

$$E\left[Z_{g_t} F\left(i_{g_t}\right)\right] = E\left[\int_0^\infty ds \int Z_{\tau_s}(w) F(u) 1_{\{R(u) > t - \tau_s(w) > 0\}} n(du)\right].$$

Since by Proposition (2.8), for every $x > 0$, we have $n(R > x) > 0$, the right-hand side of the last displayed equality may be written

$$E\left[\int_0^\infty ds\ Z_{\tau_s}(w)n(R > t - \tau_s(w))q(t - \tau_s(w), F)\right].$$

And, using the master formula in the reverse direction, this is equal to

$$E\Big[\sum_{s \in G_w} Z_s q(t-s, F) 1_{\{R\circ\theta_s > t-s>0\}}\Big] = E\left[Z_{g_t} q(t - g_t, F)\right]$$

which yields the first formula in the statement.

To get the second one, we consider a sequence of countably valued $(.\check{\mathscr{F}}_t)$-stopping times T_n decreasing to T. The formula is true for T_n since it is true for constant times. Moreover, on $\{0 < g_T < T\}$, one has $\{g_{T_n} = g_T\}$ from some n_0 onwards and $\lim_n 1_{\{g_{T_n} < T_n\}} = 1$; therefore, for bounded F,

$$\begin{aligned} E\left[F\left(i_{g_T}\right) \mid .\check{\mathscr{F}}_T\right] &= \lim_n E\left[F\left(i_{g_{T_n}}\right) 1_{\{g_{T_n} < T_n\}} \mid .\check{\mathscr{F}}_{T_n} \mid .\check{\mathscr{F}}_T\right] \\ &= \lim_n E\left[q\left(A_{T_n}, F\right) 1_{\{g_{T_n} < T_n\}} \mid .\check{\mathscr{F}}_T\right] = q(A_T, F) \end{aligned}$$

because $\lim_n A_{T_n} = A_T$, the function $q(\cdot, F)$ is continuous and A_T is $.\check{\mathscr{F}}_T$-measurable. The extension to an unbounded F is easy. □

The foregoing result gives the conditional expectation of a function of the excursion straddling t with respect to the past of BM at the time when the excursion begins. This may be made still more precise by conditioning with respect to the length of this excursion as well. In the sequel, we write $E[\cdot \mid .\check{\mathscr{F}}_t, A_t]$ for $E[\cdot \mid .\check{\mathscr{F}}_t \vee \sigma(A_t)]$. Furthermore, we denote by $\nu(\cdot; F)$ a function such that $\nu(R; F)$ is a version of the conditional expectation $n(F \mid R)$. This is well defined since n is σ-finite on the σ-field generated by R and by Proposition (2.8), the function $r \to \nu(r; F)$ is unique up to Lebesgue equivalence. We may now state

(3.4) Proposition. *With the same hypothesis and notation as in the last proposition,*

$$E\left[F\left(i_{g_t}\right) \mid .\check{\mathscr{F}}_t, A_t\right] = \nu(A_t; F),$$

and

$$E\left[F\left(i_{g_T}\right) \mid .\check{\mathscr{F}}_T, A_T\right] = \nu(A_T; F) \quad on\ \{0 < g_T < T\}.$$

Proof. Let ϕ be a positive Borel function on $\mathbb{R}_+$; making use of the preceding result, we may write

$$\begin{aligned} E\left[Z_{g_t}\phi(A_t)F\left(i_{g_t}\right)\right] &= E\left[Z_{g_t}\phi\left(R\left(i_{g_t}\right)\right)F\left(i_{g_t}\right)\right] \\ &= E\left[Z_{g_t}q\left(A_t, \phi(R)F\right)\right]. \end{aligned}$$

But, looking back at the definition of q, we have

$$q(\cdot, \phi(R)F) = q(\cdot, \phi(R)\nu(R; F)),$$

so that using again the last proposition, but in the reverse direction, we get

$$E\left[Z_{g_t}\phi(\Lambda_t)F\left(i_{g_t}\right)\right] = E\left[Z_{g_t}\phi(\Lambda_t)\nu(\Lambda_t; F)\right]$$

which is the desired result. The generalization to stopping times is performed as in the preceding proof. □

We now prove an independence property between some particular excursions and the past of the BM up to the times when these excursions begin. A $(\mathscr{F}_t^0)$-stopping time T is said to be *terminal* if $T = t + T \circ \theta_t$ a.s. on the set $\{T > t\}$; hitting times, for instance, are terminal times. For such a time, $T = g_T + T \circ \theta_{g_T}$ a.s. on $\{g_T < T\}$. A time T may be viewed as defined on U by setting for $u = i_0(w)$,

$$T(u) = T(w) \quad \text{if } R(w) \geq T(w), \qquad T(u) = +\infty \quad \text{otherwise.}$$

By Galmarino's test of Exercise (4.21) in Chap. I, this definition is unambiguous. If $T(u) < \infty$, the length Λ_T of the excursion straddling T may then also be viewed as defined on U. Thanks to these conventions, the expressions in the next proposition make sense.

(3.5) Proposition. *If T is a terminal $(\mathscr{F}_t^0)$-stopping time, then on $\{0 < g_T < T\}$,*

$$E\left[F\left(i_{g_T}\right) \mid \mathscr{F}_{g_T}\right] = n\left(F 1_{(R>T)}\right)/n(R > T) \equiv n(F \mid R > T)$$

and

$$E\left[F\left(i_{g_T}\right) \mid \mathscr{F}_{g_T}, \Lambda_T\right] = \nu\left(\Lambda_T; F 1_{(R>T)}\right)/\nu\left(\Lambda_T; 1_{(R>T)}\right).$$

Proof. For a positive predictable process Z, the same arguments as in Proposition (3.3) show that

$$\begin{aligned}
&E\left[Z_{g_T} F\left(i_{g_T}\right) 1_{(0<g_T<T)}\right] \\
&= E\left[\sum_{s \in G_w} 1_{(s<T)} Z_s F(i_s) 1_{(R(i_s)>T(i_s))}\right] \\
&= E\left[\int_0^\infty ds\ Z_{\tau_s} 1_{(\tau_s<T)} \int F(u) 1_{(R(u)>T(u))} n(du)\right] \\
&= E\left[\int_0^\infty ds\ Z_{\tau_s} 1_{(\tau_s<T)} n(R > T)\right] n\left(F 1_{(R>T)}\right)/n(R > T) \\
&= E\left[\ Z_{g_T} 1_{(0<g_T<T)}\right] n\left(F 1_{(R>T)}\right)/n(R > T)
\end{aligned}$$

which proves the first half of the statement. To prove the second half we use the first one and use the same pattern as in Proposition (3.4).

Remark. As the right-hand side of the first equality in the statement is a constant, it follows that any excursion which straddles a terminal time T is independent of the past of the BM up to time g_T. This is the independence property we had announced. We may observe that, by Proposition (3.3), this property does not hold with a fixed time t in lieu of the terminal time T (see however Exercise (3.11)).

We close this section with an interesting application of the above results which will be used in the following section. As usual, we set

$$T_\varepsilon(w) = \inf\{t > 0 : w(t) > \varepsilon\}.$$

On U, we have $\{T_\varepsilon < \infty\} = \{T_\varepsilon < R\}$, and moreover

(3.6) Proposition. $n\left(\sup_{s\le R(u)} u(s) > \varepsilon\right) = n(T_\varepsilon < \infty) = 1/2\varepsilon$.

Proof. Let $0 < x < y$. The time T_x is a terminal time to which we may apply the preceding proposition with $F = 1_{(T_y<R)}$; it follows that

$$P\left[T_y\left(\theta_{g_{T_x}}\right) < \infty\right] = n(T_y < \infty)/n(T_x < \infty).$$

The left-hand side of this equality is also equal to $P_x\left[T_y < T_0\right] = x/y$; as a result, $n(T_\varepsilon < \infty) = c/\varepsilon$ for a constant c which we now determine.

Proposition (2.6) applied to $H(s, \cdot; u) = 1_{(T_\varepsilon<\infty)}(u)1_{(s\le t)}$ yields

$$n(T_\varepsilon < \infty)E[L_t] = E\Big[\sum_{s\in G_w\cap[0,t]} 1_{(T_\varepsilon<\infty)}(i_s)\Big],$$

which, in the notation of Sect. 1 Chap. VI, implies that

$$cE[L_t] = E\left[\varepsilon\left(d_\varepsilon(t) \pm 1\right)\right];$$

letting ε tend to 0, by Theorem (1.10) of Chap. VI, we get $c = 1/2$.

Remark. This was also proved in Exercise (2.10).

(3.7) Exercise. Use the results in this and the preceding section to give the conditional law of d_1 with respect to g_1. Deduce therefrom another proof of 4°) and 5°) in Exercise (3.20) of Chap. III.

\# **(3.8) Exercise.** 1°) Prove that conditionally on $g_1 = u$, the process $(B_t, t \le u)$ is a Brownian Bridge over $[0, u]$. Derive therefrom that $B_{tg_1}/\sqrt{g_1}$, $0 \le t \le 1$ is a Brownian Bridge over $[0, 1]$ which is independent of g_1 and of $\left\{B_{g_1+u}, u \ge 0\right\}$ and that the law of g_1 is the arcsine law. See also Exercise (2.30) Chap. VI.

2°) The Brownian Bridge over $[0, 1]$ is a semimartingale. Let l^a be the family of its local times up to time 1. Prove that $L^a_{g_1}$ has the same law as $\sqrt{g_1}\, l^{a/\sqrt{g_1}}$ where g_1 is independent of the Bridge. In particular, $L^0_1 = \sqrt{g_1}\, l^0$; derive therefrom that l^0 has the same law as $\sqrt{2\mathbf{e}}$, where $\mathbf{e}$ is an exponential r.v. with parameter 1.

[Hint: See Sect. 6 Chap. 0.]

3°) Prove that the process $M_u = |B_{g_1+u(1-g_1)}| / \sqrt{1-g_1}$, $0 \le u \le 1$, is independent of the σ-algebra $\check{\mathscr{F}}_1$; M is called the *Brownian Meander* of length 1. Prove that M_1 has the law of $\sqrt{2\mathbf{e}}$ just as l^0 above.

[Hint: Use the scaling properties of n described in Exercise (2.13).]

4°) Prove that the joint law of (g_t, L_t, B_t) is

$$1_{(l\ge 0)} 1_{(s\le t)} \frac{l}{\sqrt{2\pi s^3}} \exp\left(-\frac{l^2}{2s}\right) \frac{|x|}{\sqrt{2\pi(t-s)^3}} \exp\left(-\frac{x^2}{2(t-s)}\right) ds\, dl\, dx.$$

[Hint: $(g_1, L_1, B_1) \stackrel{(d)}{=} \left(g_1, \sqrt{g_1}\, l^0, \sqrt{1-g_1}\, M_1\right)$ where g_1, l^0, M_1 are independent.]

* **(3.9) Exercise.** We retain the notation of the preceding exercise and put moreover $A_t = \int_0^t 1_{(B_s>0)} ds$ and $U = \int_0^1 1_{(\beta_s>0)} ds$ where β is a Brownian Bridge. We recall from Sect. 2 Chap. VI that the law of A_1 is the Arcsine law; we aim at proving that U is uniformly distributed on $[0, 1]$.

1°) Let T be an exponential r.v. with parameter 1 independent of B. Prove that A_{g_T} and $(A_T - A_{g_T})$ are independent. As a result,

$$A_T \stackrel{(d)}{=} T A_1 \stackrel{(d)}{=} T g_1 U + T\varepsilon(1 - g_1)$$

where ε is a Bernoulli r.v. and T, g_1, U, ε are independent.

2°) Using Laplace transform, deduce from the above result that

$$\frac{1}{2} N^2 U \stackrel{(d)}{=} \frac{1}{2} N^2 V$$

where N is a centered Gaussian r.v. with variance 1 which is assumed to be independent of U on one hand, and of V on the other hand, and where V is uniformly distributed on $[0, 1]$. Prove that this entails the desired result.

(3.10) Exercise. Prove that the natural filtration $(\mathscr{F}_t^g)$ of the process $g_{\cdot}$ is strictly coarser than the filtration $\left(\check{\mathscr{F}}_t\right)$ and is equal to $(\mathscr{F}_t^L)$.

(3.11) Exercise. For $a > 0$ let $T = \inf\{t : t - g_t = a\}$. Prove the independence between the excursion which straddles T and the past of the BM up to time g_T.

§4. Descriptions of Itô's Measure and Applications

In Sects. 2 and 3, we have defined the Itô measure n and shown how it can be used in the statements or proofs of many results. In this section, we shall give several precise descriptions of n which will lead to other applications.

Let us first observe that when a σ-finite measure is given on a function space, as is the case for n and U, a property of the measure is a property of the "law" of the coordinate process when governed by this measure. Moreover, the measure is the unique extension of its restriction to the semi-algebra of measurable rectangles; in other words, the measure is known as soon as are known the finite-dimensional distributions of the coordinate process.

Furthermore, the notion of homogeneous Markov process makes perfect sense if the time set is $]0,\infty[$ instead of $[0,\infty[$ as in Chap. III. The only difference is that we cannot speak of "initial measures" any longer and if, given the transition semi-group P_t, we want to write down finite-dimensional distributions $P\left[X_{t_1}\in A_1,\dots,X_{t_k}\in A_k\right]$ for k-uples $0<t_1<\dots<t_k$, we have to know the measures $\lambda_t = X_t(P)$. The above distribution is then equal to

$$\int_{A_1}\lambda_{t_1}(dx_1)\int_{A_2}P_{t_2-t_1}(x_1,dx_2)\dots\int_{A_k}P_{t_k-t_{k-1}}(x_{k-1},dx_k).$$

The family of measures λ_t is known as the *entrance law*. To be an entrance law, (λ_t) has to satisfy the equality $\lambda_t P_s = \lambda_{t+s}$ for every s and $t>0$. Conversely, given (λ_t) and a t.f. (P_t) satisfying this equation, one can construct a measure on the canonical space such that the coordinate process has the above marginals and therefore is a Markov process. Notice that the λ_t's may be σ-finite measures and that everything still makes sense; if μ is an invariant measure for P_t, the family $\lambda_t=\mu$ for every t is an entrance law. In the situation of Chap. III, if the process is governed by P_ν, the entrance law is (νP_t). Finally, we may observe that in this situation, the semi-group needs to be defined only for $t>0$.

We now recall some notation from Chap. III. We denote by Q_t the semi-group of the BM killed when it reaches 0 (see Exercises (1.15) and (3.29) in Chap. III). We recall that it is given by the density

$$q_t(x,y) = (2\pi t)^{-1/2}\left(\exp\left(-\frac{1}{2t}(y-x)^2\right)-\exp\left(-\frac{1}{2t}(y+x)^2\right)\right)1_{(xy>0)}.$$

We will denote by $\lambda_t(dy)$ the measure on $\mathbb{R}\backslash\{0\}$ which has the density

$$m_t(y) = \left(2\pi t^3\right)^{-1/2}|y|\exp(-y^2/2t)$$

with respect to the Lebesgue measure dy. For fixed y, this is the density in t of the hitting time T_y as was shown in Sect. 3 Chap. III.

Let us observe that on $]0,\infty[$

$$m_t = -\frac{\partial g_t}{\partial y} = \lim_{x\to 0}\frac{1}{2x}q_t(x,\cdot).$$

Our first result deals with the coordinate process w restricted to the interval $]0,R[$, or to use the devices of Chap. III we will consider – in this first result only – that $w(t)$ is equal to the fictitious point δ on $[R,\infty[$. With this convention we may now state

(4.1) Theorem. *Under n, the coordinate process $w(t)$, $t>0$, is a homogeneous strong Markov process with Q_t as transition semi-group and λ_t, $t>0$, as entrance law.*

Proof. Everything being symmetric with respect to 0, it is enough to prove the result for n_+ or n_-. We will prove it for n_+, but will keep n in the notation for the sake of simplicity.

The space $(U_\delta, \mathscr{U}_\delta)$ may serve as the canonical space for the homogeneous Markov process (in the sense of Chap. III) associated with Q_t, in other words Brownian motion killed when it first hits $\{0\}$; we call (Q_x) the corresponding probability measures on $(U_\delta, \mathscr{U}_\delta)$. As usual, we call θ_r the shift operators on U_δ.

Our first task will be to prove the equality

eq. (4.1) $$n\left((u(r) \in A) \cap \theta_r^{-1}(\Gamma)\right) = n\left(1_A(u(r))Q_{u(r)}(\Gamma)\right)$$

for $\Gamma \in \mathscr{U}$, $A \in \mathscr{B}(\mathbb{R}_+ - \{0\})$ and $r > 0$. Suppose that $n(u(r) \in A) > 0$ failing which the equality is plainly true. For $r > 0$, we have $\{u(r) \in A\} \subset \{r < R\}$ hence $n(u(r) \in A) < \infty$ and the expressions we are about to write will make sense. Using Lemma (1.13) for the process $e^{\{u(r)\in A\}}$, we get

$$n\left(1_A(u(r))\left(1_\Gamma \circ \theta_r\right)\right)/n(u(r) \in A) = P\left[e_S^{\{u(r)\in A\}}(w) \in \theta_r^{-1}(\Gamma)\right]$$

where P is the Wiener measure.

The time S which is the first jump time of the process $e^{\{u(r)\in A\}}$ is a $(\mathscr{F}_{\tau_t})$-stopping time; the times τ_{S-} and τ_S are therefore $(\mathscr{F}_t)$-stopping times. We set $T = \tau_{S-} + r$. The last displayed expression may be rewritten

$$P\left[\{B_T \in A\} \cap \{\widehat{B}_. \circ \theta_T \in \Gamma\}\right]$$

where $\widehat{B}_.$ stands for thc BM killed when it hits $\{0\}$. By the strong Markov property for the $(\mathscr{F}_t)$-stopping time T, this is equal to

$$E\left[1_{\{B_T\in A\}}Q_{B_T}[\Gamma]\right].$$

As a result

$$n\left((u(r) \in A) \cap \theta_r^{-1}(\Gamma)\right) = n(u(r) \in A)\int \gamma(dx)Q_x[\Gamma]$$

where γ is the law of B_T under the restriction of P to $\{B_T \in A\}$. For a Borel subset C of $\mathbb{R}$, make $\Gamma = \{u(0) \in C\}$ in the above formula, which, since then $Q_x[\Gamma] = 1_C(x)$, becomes

$$n(u(r) \in A \cap C) = n(u(r) \in A)\gamma(C);$$

it follows that $\gamma(\cdot) = n(u(r) \in A \cap \cdot)/n(u(r) \in A)$ which proves eq. (4.1).

Let now $0 < t_1 < t_2 < \ldots < t_k < t$ be real numbers and let $f_1, \ldots, f_k$, f be positive Borel functions on $\mathbb{R}$. Since $\widehat{B}$, the BM killed at 0, is a Markov process we have, for every x

eq. (4.2) $$Q_x\left[\prod_{i=1}^k f_i\left(w(t_i)\right) f(w(t))\right] = Q_x\left[\prod_{i=1}^k f_i\left(w(t_i)\right) Q_{t-t_k}f(w(t_k))\right].$$

Set $F = \prod_{i=2}^{k} f_i\left(w(t_i - t_1)\right)$. By repeated applications of equations (4.1) and (4.2), we may now write

$$\begin{aligned} n\left[\left(\prod_{i=1}^{k} f_i\left(w(t_i)\right)\right) f(w(t))\right] &= n\left[f_1\left(w(t_1)\right)\left(F \cdot f(w(t-t_1))\right) \circ \theta_{t_1}\right] \\ &= n\left[f_1\left(w(t_1)\right) Q_{w(t_1)}\left[F \cdot f\left(w(t-t_1)\right)\right]\right] \\ &= n\left[f_1\left(w(t_1)\right) Q_{w(t_1)}\left[F Q_{t-t_k} f\left((w(t_k - t_1)\right)\right]\right] \\ &= n\left[f_1\left(w(t_1)\right) F \circ \theta_{t_1} Q_{t-t_k} f(w(t_k))\right] \\ &= n\left[\prod_{i=1}^{k} f_i\left(w(t_i)\right) Q_{t-t_k} f(w(t_k))\right] \end{aligned}$$

which shows that the coordinate process w is, under n, a homogeneous Markov process with transition semi-group Q_t. By what we have seen in Chap. III, $\widehat{B}$ has the strong Markov property; using this in eq. (4.2) instead of the ordinary Markov property, we get the analogous property for n; we leave the details as an exercise for the reader.

The entrance law is given by $\lambda_t(A) = n(u(t) \in A)$ and it remains to prove that those measures have the density announced in the statement. It is enough to compute $\lambda_t([y, \infty[\,)$ for $y > 0$. For $0 < \varepsilon < y$,

$$\lambda_t([y, \infty[\,) = n(u(t) \geq y) = n(u(t) \geq y;\, T_\varepsilon < t)$$

where $T_\varepsilon = \inf\{t > 0 : u(t) > \varepsilon\}$. Using the strong Markov property for n just alluded to, we get

$$\begin{aligned} \lambda_t([y, \infty[\,) &= n\left(T_\varepsilon < t;\, Q_{u(T_\varepsilon)}(u(t - T_\varepsilon) \geq y)\right) \\ &= n\left(T_\varepsilon < t;\, Q_{t-T_\varepsilon}(\varepsilon, [y, \infty[\,)\right). \end{aligned}$$

Applying Proposition (3.5) with $F(u) = 1_{(T_\varepsilon(u) < t)} Q_{t - T_\varepsilon(u)}(\varepsilon, [y, \infty[\,)$ yields

$$\lambda_t([y, \infty[\,) = E\left[1_{(\widetilde{T}_\varepsilon < t)} Q_{t-\widetilde{T}_\varepsilon}(\varepsilon, [y, \infty[\,)\right] n(T_\varepsilon < R)$$

where $\widetilde{T}_\varepsilon = T_\varepsilon\left(i_{g_{T_\varepsilon}}\right)$. Using Proposition (3.6) and the known value of Q_t, this is further equal to

$$E\left[1_{(\widetilde{T}_\varepsilon < t)}\left(\Phi_{t-\widetilde{T}_\varepsilon}(y + \varepsilon) - \Phi_{t-\widetilde{T}_\varepsilon}(y - \varepsilon)\right)/2\varepsilon\right]$$

with $\Phi_t(y) = \int_{-\infty}^{y} g_t(z)dz$. If we let ε tend to zero, then $\widetilde{T}_\varepsilon$ converges to zero P-a.s. and we get $\lambda_t([y, \infty[\,) = g_t(y)$ which completes the proof.

Remarks. 1°) That (λ_t) is an entrance law for Q_t can be checked by elementary computations but is of course a consequence of the above proof.

2°) Another derivation of the value of (λ_t) is given in Exercise (4.9).

The above result permits to give *Itô's description of n* which was hinted at in the remark after Proposition (2.8). Let us recall that according to this proposition the density of R under n_+ is $(2\sqrt{2\pi r^3})^{-1}$. In the following result, we deal with the law of the Bessel Bridge of dimension 3 over $[0, r]$ namely $P^{3,r}_{0,0}$ which we will abbreviate to π_r. The following result shows in particular that the law of the normalized excursion (Exercise (2.13)) is the probability measure π_1.

(4.2) Theorem. *Under n_+, and conditionally on $R = r$, the coordinate process w has the law π_r. In other words, if $\Gamma \in \mathscr{U}_\delta^+$,*

$$n_+(\Gamma) = \int_0^\infty \pi_r(\Gamma \cap \{R = r\}) \frac{dr}{2\sqrt{2\pi r^3}} .$$

Proof. The result of Theorem (4.1) may be stated by saying that for $0 < t_1 < t_2 < \ldots < t_n$ and Borel sets $A_i \subset]0, \infty[$, if we set

$$\Gamma = \bigcap_{i=1}^n \{u(t_i) \in A_i\},$$

then

$$n_+(\Gamma) = \int_{A_1} m_{t_1}(x_1)dx_1 \int_{A_2} q_{t_2-t_1}(x_1, x_2)dx_2 \ldots \int_{A_n} q_{t_n-t_{n-1}}(x_{n-1}, x_n)dx_n.$$

On the other hand, using the explicit value for π_r given in Sect. 3 of Chap. XI, and taking into account the fact that $\Gamma \cap \{R < t_n\} = \emptyset$, the formula in the statement reads

$$\begin{aligned} n_+(\Gamma) &= \int_{t_n}^\infty \frac{dr}{2\sqrt{2\pi r^3}} \int_{A_1} 2\sqrt{2\pi r^3} m_{t_1}(x_1)dx_1 \int_{A_2} q_{t_2-t_1}(x_1, x_2)dx_2 \ldots \\ &\quad \ldots \int_{A_n} q_{t_n-t_{n-1}}(x_{n-1}, x_n) m_{r-t_n}(x_n)dx_n. \end{aligned}$$

But

$$\int_{t_n}^\infty \frac{dr}{2\sqrt{2\pi r^3}} 2\sqrt{2\pi r^3} m_{r-t_n}(x_n) = 1$$

as was seen already several times. Thus the two expressions for $n_+(\Gamma)$ are equal and the proof is complete.

This result has the following important

(4.3) Corollary. *The measure n is invariant under time-reversal; in other words, it is invariant under the map $u \to \hat{u}$ where*

$$\hat{u}(t) = u(R(u) - t)1_{(R(u) \geq t)}.$$

Proof. By Exercise (3.7) of Chap. XI, this follows at once from the previous result.

This can be used to give another proof of the time-reversal result of Corollary (4.6) in Chap. VII.

(4.4) Corollary. *If B is a* BM(0) *and for $a > 0$, $T_a = \inf\{t : B_t = a\}$, if Z is a* $\mathrm{BES}^3(0)$ *and $\sigma_a = \sup\{t : Z_t = a\}$, then the processes $Y_t = a - B_{T_a - t}$, $t < T_a$ and Z_t, $t < \sigma_a$ are equivalent.*

Proof. We retain the notation of Proposition (2.5) and set $\beta_t = L_t - |B_t| = s - |e_s(t - \tau_{s-})|$ if $\tau_{s-} \le t \le \tau_s$. We know from Sect. 2 Chap. VI that β is a standard BM.

If we set $Z_t = L_t + |B_t| = s + |e_s(t - \tau_{s-})|$ if $\tau_{s-} \le t \le \tau_s$, Pitman's theorem (see Corollary (3.8) of Chap. VI) asserts that Z is a $\mathrm{BES}^3(0)$.

For $a > 0$ it is easily seen that

$$\tau_a = \inf\{t : L_t = a\} = \inf\{t : \beta_t = a\};$$

moreover

$$\tau_a = \sup\{t : Z_t = a\}$$

since $Z_{\tau_a} = L_{\tau_a} + |B_{\tau_a}| = a$ and for $t > \tau_a$ one has $L_t > a$.

We now define another Poisson point process with values in $(U, \mathscr{U})$ by setting

$$\begin{aligned} \widetilde{e}_s &= e_s \quad \text{if } s > a, \\ \widetilde{e}_s(t) &= e_{a-s}\left(R(e_{a-s}) - t\right), \qquad 0 \le t \le R(e_{a-s}), \quad \text{if } s \le a. \end{aligned}$$

In other words, for $s \le a$, $\widetilde{e}_s = \hat{e}_{a-s}$ in the notation of Corollary (4.3). Thus, for a positive $\mathscr{B}(\mathbb{R}_+) \times \mathscr{U}_\delta$-measurable function f,

$$\sum_{0<s\le a} f(s, \widetilde{e}_s) = \sum_{0<s\le a} f(a - s, \hat{e}_s),$$

and the master formula yields

$$E\left[\sum_{0<s\le a} f(s, \widetilde{e}_s)\right] = E\left[\int_0^a ds \int f(a - s, \hat{u}) n(du)\right];$$

by Corollary (4.3), this is further equal to

$$E\left[\int_0^a ds \int f(s, u) n(du)\right].$$

This shows that the PPP $\widetilde{e}$ has the same characteristic measure, hence the same law as e. Consequently the process $\widetilde{Z}$ defined by

$$\widetilde{Z}_t = s + |\widetilde{e}_s(t - \widetilde{\tau}_{s-})| \qquad \text{if } \widetilde{\tau}_{s-} \le t \le \widetilde{\tau}_s,$$

has the same law as Z. Moreover, one moment's reflection shows that

$$\widetilde{Z}_t = a - \beta(\tau_a - t) \qquad \text{for } 0 \le t \le \tau_a$$

which ends the proof. □

Let us recall that in Sect. 4 of Chap. X we have derived Williams' path decomposition theorem from the above corollary (and the reversal result in Proposition (4.8) of Chap. VII). We will now use this decomposition theorem to give another description of n and several applications to BM. We denote by M the maximum of positive excursions, in other words M is a r.v. defined on U_δ^+ by

$$M(u) = \sup_{s \le R(u)} u(s).$$

The law of M under n_+ has been found in Exercise (2.10) and Proposition (3.6) and is given by $n_+(M \ge x) = 1/2x$.

We now give *Williams' description* of n. Pick two independent $\mathrm{BES}^3(0)$ processes ρ and $\widetilde{\rho}$ and call T_c and $\widetilde{T}_c$ the corresponding hitting times of $c > 0$. We define a process Z^c by setting

$$Z_t^c = \begin{cases} \rho_t, & 0 \le t \le T_c, \\ c - \widetilde{\rho}(t - T_c), & T_c \le t \le T_c + \widetilde{T}_c, \\ 0, & t \ge T_c + \widetilde{T}_c. \end{cases}$$

For $\Gamma \in \mathscr{U}_\delta^+$, we put $N(c, \Gamma) = P\left[Z^c_\cdot \in \Gamma\right]$. The map N is a kernel; indeed $\left(Z_t^c\right)_{t\ge 0} \stackrel{(d)}{=} \left(cZ^1_{t/c^2}\right)_{t \ge 0}$, thanks to the scaling properties of $\mathrm{BES}^3(0)$, so that N maps continuous functions into continuous functions on $\mathbb{R}$ and the result follows by a monotone class argument. By Proposition (4.8) in Chap. VII, the second part of Z^c might as well have been taken equal to $\widetilde{\rho}\left(T_c + \widetilde{T}_c - t\right)$, $T_c \le t \le T_c + \widetilde{T}_c$.

(4.5) Theorem. *For any $\Gamma \in \mathscr{U}_\delta^+$*

$$n_+(\Gamma) = \frac{1}{2}\int_0^\infty N(x, \Gamma)x^{-2}dx.$$

In other words, conditionally on its height being equal to c, the Brownian excursion has the law of Z^c.

Proof. Let $U_c = \{u : M(u) \ge c\}$; by Lemma (1.13), for $\Gamma \in \mathscr{U}_\delta^+$

$$n_+(\Gamma \cap U_c) = n_+(U_c)P[e^c \in \Gamma] = \frac{1}{2c}P[e^c \in \Gamma],$$

where e^c is the first excursion the height of which is $\ge c$. The law of this excursion is the law of the excursion which straddles T_c, i.e. the law of the process

$$Y_t = B_{g_{T_c}+t}, \qquad 0 \le t \le d_{T_c} - g_{T_c}.$$

By applying the strong Markov property to B at time T_c, we see that the process Y may be split up into two independent parts Y^1 and Y^2, with

$$Y_t^1 = B_{g_{T_c}+t}, \quad 0 \le t \le T_c - g_{T_c}; \qquad Y_t^2 = B_{T_c+t}, \quad 0 \le t \le d_{T_c} - T_c.$$

By the strong Markov property again, the part Y^2 has the law of $\widetilde{B}_t$, $0 \le t \le \widetilde{T}_0$, where $\widetilde{B}$ is a BM(c). Thus by Proposition (3.13) in Chap. VI, Y^2 may be described as follows: conditionally on the value M of the maximum, it may be further split up into two independent parts V^1 and V^2, with

$$V_t^1 = B_{T_c+t}, \quad 0 \le t \le T_M - T_c, \qquad V_t^2 = B_{T_M+t}, \quad 0 \le t \le d_{T_c} - T_M.$$

Moreover V^1 is a $\mathrm{BES}^3(c)$ run until it first hits M and V^2 has the law of $M - \rho_t$ where ρ is a $\mathrm{BES}^3(0)$ run until it hits M.

Furthermore, by Williams decomposition theorem (Theorem (4.9) Chap. VII), the process Y^1 is a $\mathrm{BES}^3(0)$ run until it hits c. By the strong Markov property for BES^3, if we piece together Y^1 and V^1, the process we obtain, namely

$$B_{g_{T_c}+t}, \qquad 0 \le t \le T_M - g_{T_c},$$

is a $\mathrm{BES}^3(0)$ run until it hits M.

As a result, we see that the law of e^c conditional on the value M of the maximum is that of Z^M. Since the law of this maximum has the density c/M^2 on $[c, \infty[$ as was seen in Proposition (3.13) of Chap. VI we get

$$P[e^c \in \Gamma] = c \int_c^\infty x^{-2} N(x, \Gamma) dx$$

which by the first sentence in the proof, is the desired result. □

To state and prove our next result, we will introduce some new notation. We will call $\mathscr{X}$ the space of real-valued continuous functions ω defined on an interval $[0, \zeta(\omega)] \subset [0, \infty[$. We endow it with the usual σ-fields $\mathscr{F}_t^0$ and $\mathscr{F}_\infty^0$ generated by the coordinates. The Itô measure n and the law of BM restricted to a compact interval may be seen as measures on $(\mathscr{X}, \mathscr{F}_\infty^0)$.

If μ and μ' are two such measures, we define $\mu \circ \mu'$ as the image of $\mu \otimes \mu'$ under the map $(\omega, \omega') \to \omega \circ \omega'$ where

$$\begin{aligned} \zeta(\omega \circ \omega') &= \zeta(\omega) + \zeta(\omega'), \\ \omega \circ \omega'(s) &= \omega(s), \quad \text{if } 0 \le s \le \zeta(\omega) \\ &= \omega(\zeta(\omega)) + \omega'(s - \zeta(\omega)) - \omega'(0) \quad \text{if } \zeta(\omega) \le s \le \zeta(\omega) + \zeta(\omega'). \end{aligned}$$

We denote by ${}^\vee\mu$ the image of μ under the time-reversal map $\omega \to \check{\omega}$ where

$$\zeta(\check{\omega}) = \zeta(\omega), \qquad \check{\omega}(s) = \omega(\zeta(\omega) - s), \qquad 0 \le s \le \zeta(\omega).$$

Finally, if T is a measurable map from $\mathscr{X}$ to $[0, \infty]$, we denote by μ^T the image of μ by the map $\omega \to k_T(\omega)$ where

$$\zeta(k_T(\omega)) = \zeta(\omega) \wedge T(\omega), \qquad k_T(\omega)(s) = \omega(s) \qquad \text{if } 0 \le s \le \zeta(\omega) \wedge T(\omega).$$

We also define, as usual:

$$T_a(\omega) = \inf\{t : \omega(t) = a\}, \qquad L_a(\omega) = \sup\{t : \omega(t) = a\}.$$

Although the law P_a of BM(a) cannot be considered as a measure on $\mathscr{X}$, we will use the notation $P_a^{T_0}$ for the law of BM(a) killed at 0 which we may consider as a measure on $\mathscr{X}$ carried by the set of paths ω such that $\zeta(\omega) = T_0(\omega)$. If S_3 is the law of BES$^3(0)$, we may likewise consider $S_3^{L_a}$ and the time-reversal result of Corollary (4.6) in Chap. VII then reads

$$\overset{\vee}{\left(P_a^{T_0}\right)} = S_3^{L_a}.$$

In the same way, the last result may be stated

$$n_+ = \frac{1}{2}\int_0^\infty a^{-2}\left(S_3^{T_a} \circ \overset{\vee}{\left(S_3^{T_a}\right)}\right) da.$$

The space U of excursions is contained in $\mathscr{X}$ and carries n; on this subspace, we will write R instead of ζ in keeping with the notation used so far and also use w or ω indifferently. We may now state

(4.6) Proposition.

$$\int_0^\infty n^u(\cdot \cap \{u < R\})du = \int_{-\infty}^{+\infty} \overset{\vee}{\left(P_a^{T_0}\right)} da.$$

Proof. Let (θ_t) be the usual translation operators and as above put

$$g_t(\omega) = \sup\{s < t : \omega(s) = 0\}, \qquad d_t(\omega) = \inf\{s \geq t : \omega(s) = 0\},$$

$$G_\omega = \{g_t(\omega), t \in \mathbb{R}_+\}.$$

We denote by E_0 the expectation with respect to the Wiener measure.

The equality in the statement will be obtained by comparing two expressions of

$$J = E_m\left[\int_0^\infty 1_{[T_0<t]} e^{-\lambda g_t}\, Y \circ k_{t-g_t} \circ \theta_{g_t} dt\right]$$

where $P_m = \int_{-\infty}^{+\infty} P_a da$, Y is a positive $\mathscr{F}_\infty^0$-measurable function and λ is > 0.

From Proposition (2.6) it follows that

$$\begin{aligned} E_0 &\left[\int_0^\infty e^{-\lambda g_t} Y \circ k_{t-g_t} \circ \theta_{g_t} dt\right] \\ &= E_0\left[\sum_{s\in G_\omega} e^{-\lambda s}\int_s^{d_s(\omega)} Y \circ k_{t-s} \circ \theta_s(\omega) dt\right] \\ &= E_0\left[\int_0^\infty e^{-\lambda s} dL_s\right] n\left(\int_0^R Y \circ k_u du\right). \end{aligned}$$

Using the strong Markov property for P_a we get

$$J = \int_{-\infty}^{+\infty} da E_a\left[e^{-\lambda T_0}\right] E_0\left[\int_0^\infty e^{-\lambda s} dL_s\right] n\left(\int_0^R Y \circ k_u du\right).$$

But $E_a\left[e^{-\lambda T_0}\right] = e^{-|a|\sqrt{2\lambda}}$ so that $\int_{-\infty}^{+\infty} da E_a\left[e^{-\lambda T_0}\right] = \sqrt{2/\lambda}$ and by the results in Sect. 2 of Chap. X, $E_0\left[\int_0^\infty e^{-\lambda s} dL_s\right] = 1/\sqrt{2\lambda}$. As a result

$$J = \lambda^{-1} \int_0^\infty n(Y \circ k_u 1_{(u<R)} du.$$

On the other hand, because $\zeta\left(k_{t-g_t} \circ \theta_{g_t}\right) = t$,

$$J = \int_0^\infty E_m^t\left[1_{[T_0<t]} e^{-\lambda g_t} Y \circ k_{t-g_t} \circ \theta_{g_t}\right] dt$$

and since obviously $\overset{\vee}{\left(P_m^t\right)} = P_m^t$,

$$J = \int_0^\infty E_m^t\left[1_{[T_0<t]} e^{-\lambda(t-T_0)} Z\right] dt$$

where $Z(\omega) = \left(Y \circ k_{T_0}\right)(\overset{\vee}{\omega})$; indeed $t - g_t$ is the hitting time of zero for the process reversed at t. Now the integrand under E_m^t depends only on what occurs before t and therefore we may replace E_m^t by E_m. Consequently

$$\begin{aligned} J &= E_m\left[Z \int_0^\infty 1_{[T_0<t]} e^{-\lambda(t-T_0)} dt\right] \\ &= \lambda^{-1} E_m[Z] = \lambda^{-1} \int_{-\infty}^{+\infty} E_a[Z] da. \end{aligned}$$

Comparing the two values found for J ends the proof.

Remark. Of course this result has a one-sided version, namely

$$\int_0^\infty n_+^u(\cdot \cap \{u < R\}) du = \int_0^\infty \overset{\vee}{\left(P_a^{T_0}\right)} da.$$

We may now turn to *Bismut's description* of n.

(4.7) Theorem. *Let $\bar{n}_+$ be the measure defined on $\mathbb{R}_+ \times U_\delta$ by*

$$\bar{n}_+(dt, du) = 1_{(0 \le t \le R(u))} dt\ n_+(du).$$

Then, under $\bar{n}_+$ the law of the r.v. $(t, u) \to u(t)$ is the Lebesgue measure da and conditionally on $u(t) = a$, the processes $\{u(s), 0 \le s \le t\}$ and $\{u(R(u) - s), t \le s \le R(u)\}$ are two independent BES^3 *processes run until they last hit a.*

The above result may be seen, by looking upon U_δ as a subset of $\mathscr{X}$, as an equality between two measures on $\mathbb{R}_+ \times \mathscr{X}$. By the monotone class theorem, it is enough to prove the equality for the functions of the form $f(t)H(\omega)$ where H belongs to a class stable under pointwise multiplication and generating the σ-field on $\mathscr{X}$. Such a class is provided by the r.v.'s which may be written

$$H = \prod_{i=1}^{n} \int_0^\infty e^{-\lambda_i s} f_i(\omega(s))ds$$

where λ_i is a positive real number and f_i a bounded Borel function on $\mathbb{R}$. It is clear that for each t, these r.v.'s may be written $Z_t \cdot Y_t \circ \theta_t$ where Z_t is $\mathscr{F}_t^0$-measurable. We will use this in the following

Proof. Using the Markov description of n proved in Theorem (4.1), we have

$$\begin{aligned}
&\int_{\mathbb{R}_+ \times U_\delta} f(t) Z_t(u) Y_t(\theta_t(u))\, \bar{n}_+(dt, du) \\
&= \int_0^\infty dt\ f(t) \int_{U_\delta} 1_{[t<R(u)]} Z_t(u) Y_t(\theta_t(u))\, n_+(du) \\
&= \int_0^\infty dt\ f(t) \int_{U_\delta} 1_{[1<R(u)]} Z_t(u) E^{T_0}_{u(t)}[Y_t] n_+(du) \\
&= \int_0^\infty dt\ f(t) \int_{U_\delta} Z_t(u) E^{T_0}_{u(t)}[Y_t] dn^t_+(\cdot \cap (t < R))
\end{aligned}$$

where $E_x^{T_0}$ is the expectation taken with respect to the law of BM(x) killed at 0. Using the one-sided version of Proposition (4.6) (see the remark after it), this is further equal to

$$\int_0^\infty da\ \check{E}_a^{T_0}\left[f(\zeta(\omega)) Z_\zeta(\omega) E^{T_0}_{\omega(\zeta)}[Y_\zeta]\right].$$

But for $\check{P}_a^{T_0}$, we have $\zeta = L_a$ a.s. hence in particular $\omega(\zeta) = a$ a.s. and for $P_a^{T_0}$, we have $\zeta = T_0$ a.s. so that we finally get

$$\int_0^\infty da\ \check{E}_a^{T_0}\left[f(L_a) Z_{L_a}\right] E_a^{T_0}\left[Y_{T_0}\right].$$

Using the time-reversal result of Corollary (4.4), this is precisely what was to be proved.

\# **(4.8) Exercise. (Another proof of the explicit value of λ_t in Theorem (4.1)).** Let f be a positive Borel function on $\mathbb{R}_+$ and U_p the resolvent of BM. Using the formula of Proposition (2.6), prove that

$$U_p f(0) = E\left[\int_0^\infty e^{-p\tau_s} ds\right] \int_0^\infty e^{-pu} \lambda_u(f) du.$$

Compute $\lambda_u(f)$ from the explicit form of U_p and the law of τ_s.

* **(4.9) Exercise.** Conditionally on $(R = r)$, prove that the Brownian excursion is a semimartingale over $[0, r]$ and, as such, has a family l^a, $a > 0$, of local times up to time r for which the occupation times formula obtains.

(4.10) Exercise. For $x \in \mathbb{R}_+$, let s_x be the time such that e_{s_x} is the first excursion for which $R(e_s) > x$. Let L be the length of the longest excursion e_u, $u < s_x$. Prove that

$$P[L < y] = (y/x)^{1/2} \qquad \text{for } y \leq x.$$

(4.11) Exercise. (Watanabe's process and Knight's identity). 1°) Retaining the usual notation, prove that the process $Y_t = S_{\tau_t}$ already studied in Exercise (1.9) of Chap. X, is a homogeneous Markov process on $[0, \infty[$ with semi-group T_t given by

$$T_0 = I, T_t f(x) = e^{-t/2x} f(x) + \int_x^\infty e^{-t/2y} \frac{t}{2y^2} f(y) dy.$$

[Hint: Use the description of BM by means of the excursion process given in Proposition (2.5)]. In particular,

$$P\left[S_{\tau_t} \leq a\right] = \exp(-t/2a).$$

Check the answers given in Exercise (1.27) Chap. VII.

* 2°) More generally, prove that

$$E\left[\exp\left(-\lambda^2 \tau_t^+/2\right) 1_{(S_{\tau_t} \leq a)}\right] = \exp(-\lambda t \coth(a\lambda)/2)$$

where $\tau_t^+ = \int_0^{\tau_t} 1_{(B_s > 0)} ds$.

3°) Deduce therefrom *Knight's identity*, i.e.

$$E\left[\exp\left(-\lambda^2 \tau_t^+/2S_{\tau_t}^2\right)\right] = 2\lambda/\sinh(2\lambda).$$

Prove that consequently,

$$\tau_t^+/S_{\tau_t}^2 \overset{(d)}{=} \inf\{s : U_s = 2\},$$

where U is a $\mathrm{BES}^3(0)$.

[Hint: Prove and use the formula

$$\int \left(1 - \exp(-R/2) 1_{(M \leq x)}\right) dn^+ = (\coth x)/2.$$

where $M = \sup_{t<R} w(t)$.]

4°) Give another proof using time reversal.

(4.12) Exercise. (Continuation of Exercise (2.13) on normalized excursions). Let p be the density of the law of M (see Exercise (4.13)) under γ. Prove that

$$\int_0^\infty x p(x) dx = \sqrt{\pi/2},$$

that is: the mean height of the normalized excursion is $\sqrt{\pi/2}$.

[Hint: Use 3°) in Exercise (2.13) to write down the joint law of R and M under n as a function of p, then compare the marginal distribution of R with the distribution given in Proposition (2.8).]

(4.13) Exercise. 1°) Set $M(w) = \sup_{t<R} w(t)$; using the description of n given in Theorem (4.1), prove that

$$n_+(M \geq x) = \lim_{s\to 0}\left(\int_0^x \lambda_s(dy) Q_y\left[T_x < T_0\right] + \int_x^\infty \lambda_s(dy)\right)$$

and derive anew the law of M under n_+, which was already found in Exercise (2.10) and Proposition (3.6).

2°) Prove that $M_x = \sup\left\{B_t, t < g_{T_x}\right\}$ is uniformly distributed on $[0, x]$ (a part of Williams' decomposition theorem).

[Hint: If M_x is less than y, for $y \leq x$, the first excursion which goes over x is also the first to go over y.]

3°) By the same method as in 1°), give another proof of Proposition (2.8).

\# **(4.14) Exercise. (An excursion approach to Skorokhod problem).** We use the notation of Sect. 5 Chap. VI; we suppose that ψ_μ is continuous and strictly increasing and call ϕ its inverse.

1°) The stopping times

$$T = \inf\left\{t : S_t \geq \psi_\mu(B_t)\right\} \quad \text{and} \quad T' = \inf\{t : |B_t| \geq L_t - \phi(L_t)\}$$

have the same law.

2°) Prove that, in the notation of this chapter, the process $\{s, e_s\}$ is a PPP with values in $\mathbb{R}_+ \times U_\delta$ and characteristic measure $ds\, dn(u)$.

3°) Let $\Gamma_x = \{(s, u) \in \mathbb{R}_+ \times U_\delta : 0 \leq s \leq x \text{ and } M(u) \geq s - \phi(s)\}$ and $N_x = \sum_s 1_{\Gamma_x}(s, e_s)$. Prove that $P\left[L_{T'} \geq x\right] = P\left[N_x = 0\right]$ and derive therefrom that $\phi(S_T) = B_T$ has the law μ.

4°) Extend the method to the case where ψ_μ is merely right-continuous.

(4.15) Exercise. If $A = L^z$, $z > 0$, prove that the Lévy measure m_A of A_{τ_t} defined in Proposition (2.7) is given by

$$m_A(\,]x, \infty[\,) = (2z)^{-1}\exp(-x/2z), \quad x > 0.$$

* **(4.16) Exercise.** 1°) Using Proposition (3.3) prove, in the notation thereof, that if f is a function such that $f(|B_t|)$ is integrable,

$$E\left[f(|B_t|) \mid \check{\mathscr{F}}_t\right] = A_t^{-1}\int_0^\infty \exp\left(-y^2/2A_t\right) y f(y)dy.$$

[Hint: Write $B_t = B_{g_t+t-g_t} = B_{A_t}\left(i_{g_t}\right)$.]

2°) By applying 1°) to the functions $f(y) = y^2$ and $|y|$, prove that $t - 2g_t$ and $\sqrt{\frac{\pi}{2}(t-g_t)} - L_t$ are $(\check{\mathscr{F}}_t)$-martingales.

3°) If f is a bounded function with a bounded first derivative f' then

$$f(L_t) - \sqrt{\frac{\pi}{2}(t-g_t)}\, f'(L_t)$$

is a $(\check{\mathscr{F}}_t)$-martingale.

(4.17) Exercise. Let $\tau_a = \inf\{t : L_t = a\}$. Prove that the processes $\{B_t, t \leq \tau_a\}$ and $\left\{B_{\tau_a - t}, t \leq \tau_a\right\}$ are equivalent.

[Hint: Use Proposition (2.5) and Corollary (4.3).]

\# **(4.18) Exercise.** In the notation of Proposition (4.6) and Theorem (4.7), for P_0-almost every path, we may define the local time at 0 and its inverse τ_t; thus $P_0^{\tau_t}$ makes sense and is a probability measure on $\mathscr{X}$.

1°) Prove that

$$\int_0^\infty P_0^t dt = \int_0^\infty P_0^{\tau_s} ds \circ \int_0^\infty n^u(\cdot \cap (u < R)) du.$$

This formula in another guise may also be derived without using excursion theory as may be seen in Exercise (4.26). We recall that it was proved in Exercise (2.29) of Chap. VI that

$$\int_0^\infty P_0^{\tau_s} ds = \int_0^\infty Q_u \frac{du}{\sqrt{2\pi u}}$$

where Q_u is the law of the Brownian Bridge over the interval $[0, u]$.

2°) Call M^t the law of the Brownian meander of length t defined by scaling from the meander of length 1 (Exercise (3.8)) and prove that

$$n^t(\cdot \cap (t < R)) = M^t / \sqrt{2\pi t}.$$

As a result

$$(*) \qquad \int_0^\infty M^t \frac{dt}{\sqrt{2\pi t}} = \int_0^\infty S_3^{L_a} da.$$

3°) Derive from (*) Imhof's relation, i.e., for every t

$$(+) \qquad M^t = (\pi t/2)^{1/2} X_t^{-1} S_3^t$$

where $X_t(\omega) = \omega(t)$ is the coordinate process.

[Hint: In the left-hand side of $(*)$ use the conditioning given $L_a = t$, then use the result of Exercise (3.2) Chap. XI.]

By writing down the law of (ζ, X_ζ) under the two sides of $(*)$, one also finds the law of X_t under M^t which was already given in Exercise (3.8).

4°) Prove that (+) is equivalent to the following property: for any bounded continuous functional F on $C([0, 1], \mathbb{R})$,

$$M^1(F) = \lim_{r \downarrow 0} (\pi/2)^{1/2} E_r \left[F 1_{[T_0 > 1]}\right] / r$$

where P_r is the probability measure of BM(r) and T_0 is the first hitting time of 0. This question is not needed for the sequel.

[Hint: Use Exercise (1.22) Chap. XI.]

5°) On $C([0, 1], \mathbb{R})$, set $\mathscr{B}_t = \sigma(X_s, s \leq t)$. Prove that for $0 \leq t \leq 1$,

$$S_3^1\left[(\pi/2)^{1/2} X_1^{-1} \mid \mathscr{B}_t\right] = X_t^{-1} \phi\left((1-t)^{-1/2} X_t\right)$$

where $\phi(a) = \int_0^a \exp(-y^2/2)dy$. Observe that this shows, in the fundamental counterexample of Exercise (2.13) Chap. V, how much $1/X_t$ differs from a martingale.

[Hint: Use the Markov property of BES^3.]

6°) Prove that, under M^1, there is a Brownian motion β such that

$$X_t = \beta_t + \int_0^t \left(\frac{\phi'}{\phi}\right)\left(\frac{X_s}{\sqrt{1-s}}\right)\frac{ds}{\sqrt{1-s}}, \qquad 0 \le t \le 1,$$

which shows that the meander is a semimartingale and gives its decomposition in its natural filtration.

[Hint: Apply Girsanov's theorem with the martingale of 5°).]

** **(4.19) Exercise.** If $B_s = 0$, call $D(s)$ the length of the longest excursion which occured before time s. The aim of this exercise is to find the law of $D(g_t)$ for a fixed t. For $\beta > 0$, we set

$$c_\beta = \int_0^\infty (1 - e^{-\beta t})(2\pi t^3)^{-1/2}dt, \qquad d_\beta(x) = \int_x^\infty e^{-\beta t}(2\pi t^3)^{-1/2}dt$$

and

$$\phi_s(x, \beta) = E\left[1_{(D(\tau_s)>x)}\exp(-\beta\tau_s)\right].$$

1°) If $L_\beta(x) = E\left[\int_0^\infty \exp(-\beta t)1_{(D(g_t)>x)}dt\right]$, prove that

$$\beta L_\beta(x) = c_\beta \int_0^\infty \phi_s(x, \beta)ds.$$

2°) By writing

$$\phi_t(x, \beta) = E\left[\sum_{s\le t}\left\{1_{(D(\tau_s)>x)}\exp(-\beta\tau_s) - 1_{(D(\tau_{s-})>x)}\exp(-\beta\tau_{s-})\right\}\right],$$

prove that ϕ satisfies the equation

$$\phi_t(x, \beta) = -\left(c_\beta + d_\beta(x)\right)\int_0^t \phi_s(x, \beta)ds + d_\beta(x)\int_0^t e^{-c_\beta s}ds.$$

3°) Prove that

$$\beta L_\beta(x) = d_\beta(x)/\left(c_\beta + d_\beta(x)\right).$$

[Hint: $\{D(\tau_s) > x\} = \{D(\tau_{s-}) > x\} \cup \{\tau_s - \tau_{s-} > x\}$.]

4°) Solve the same problem with $D(d_t)$ in lieu of $D(g_t)$.

5°) Use the scaling property of BM to compute the Laplace transforms of $(D(g_1))^{-1}$ and $(D(d_1))^{-1}$.

** **(4.20) Exercise.** Let A be an additive functional of BM with associated measure μ and S_θ an independent exponential r.v. with parameter $\theta^2/2$.

1°) Use Exercise (4.18) 1°) to prove that for $\lambda > 0$,

$$E_0\left[\exp\left(-\lambda A_{S_\theta}\right)\right]$$
$$= \frac{\theta^2}{2}\int_0^\infty E_0\left[\exp\left(-\lambda A_{\tau_s} - \frac{\theta^2}{2}\tau_s\right)\right] ds \int_{-\infty}^{+\infty} E_a\left[\exp\left(-\lambda A_{T_0} - \frac{\theta^2}{2}T_0\right)\right] da.$$

2°) If ϕ and ψ are suitable solutions of the Sturm-Liouville equation $\phi'' = 2\left(\lambda\mu + \frac{\theta^2}{2}\right)\phi$, then

$$E\left[\exp(-\lambda A_{S_\theta})\right] = \left(\theta^2/2\phi'(0+)\right)\int_{-\infty}^{+\infty}\psi(a)da.$$

3°) With the notation of Theorem (2.7) Chap. VI find the explicit values of the expressions in 1°) for $A_t = A_t^+$ and derive therefrom another proof of the arcsine law. This question is independent of 2°).

[Hint: Use the independence of $A^+_{\tau_s}$ and $A^-_{\tau_s}$, the fact that $\tau_s = A^+_{\tau_s} + A^-_{\tau_s}$ and the results in Propositions (2.7) and (2.8).]

** **(4.21) Exercise (Lévy-Khintchine formula for BESQ^δ).** If l^a is a family of local times of the Brownian excursion (see Exercise (4.9)), call M the image of n^+ under the map $w \to l^a_R(w)$. The measure M is a measure on $\mathbf{W}_+ = C(\mathbb{R}_+, \mathbb{R}_+)$. If $f \in \mathbf{W}_+$ and X is a process we set

$$X_f = \int_0^\infty f(t)X_t dt.$$

1°) With the notation of Sect. 1 Chap. XI prove that for $x \geq 0$

$$Q^0_x\left[\exp(-X_f)\right] = \exp\left\{-x\int(1-\exp(-\langle f,\phi\rangle))M(d\phi)\right\}$$

where $\langle f,\phi\rangle = \int_0^\infty f(t)\phi(t)dt$.

[Hint: Use the second Ray-Knight theorem and Proposition (1.12).]

2°) For $\phi \in \mathbf{W}_+$ call ϕ^s the function defined by $\phi^s(t) = \phi\left((t-s)^+\right)$ and put $N = \int_0^\infty M_s ds$ where M_s is the image of M by the map $\phi \to \phi^s$. Prove that

$$Q^2_0\left[\exp(-X_f)\right] = \exp\left\{-2\int(1-\exp(-\langle f,\phi\rangle))N(d\phi)\right\}.$$

[Hint: Use 1°) in Exercise (2.7) of Chap. XI and the fact that for a BM the process $|B| + L$ is a BES$^3(0)$.]

The reader is warned that M_s has nothing to do with the law M^s of the meander in Exercise (4.18).

3°) Conclude that

$$Q^\delta_x\left[\exp(-X_f)\right] = \exp\left\{-\int(1-\exp(-\langle f,\phi\rangle))(xM+\delta N)(d\phi)\right\}.$$

4°) Likewise prove a similar Lévy-Khintchine representation for the laws $Q^{\delta}_{x\to 0}$ of the squares of Bessel bridges ending at 0; denote by M_0 and N_0 the corresponding measures, which are now defined on $C([0,1];\mathbb{R}_+)$.

5°) For a subinterval I of $\mathbb{R}_+$, and $x, y \in I$, with $x < y$, let $P_{x,y}$ be the probability distribution on $C^+(I)$ of a process $X_{x,y}$ which vanishes off the interval (x, y), and on (x, y), is a BES$Q^4_{y-x}(0,0)$ that is

$$X_{x,y}(v) = (y-x)Z\left(\frac{v-x}{y-x}\right)1_{(x\le v\le y)} \qquad (v \in I)$$

where Z has distribution $Q^{4,1}_{0,0}$.

Prove that the Lévy measures encountered above may be represented by the following integrals:

$$M = \frac{1}{2}\int_0^{\infty} y^{-2}P_{0,y}dy; \quad N = \frac{1}{2}\int_0^{\infty} dx \int_x^{\infty} (y-x)^{-2}P_{x,y}dy$$

$$M_0 = \frac{1}{2}\int_0^{1} y^{-2}P_{0,y}dy; \quad N_0 = \frac{1}{2}\int_0^{1} dx \int_x^{1} (y-x)^{-2}P_{x,y}dy.$$

\# **(4.22) Exercise.** Let ϕ and f be positive Borel functions on the appropriate spaces. Prove that

$$\int n_+(de)\int_0^{R(e)} \phi(s)f(e_s)ds = 2\int n_+(de)\phi(R(e))\int_0^{R(e)} f(2e_s)ds.$$

[Hint: Compute the left member with the help of Exercise (4.17) 2°) and the right one by using Theorem (4.1).]

* **(4.23) Exercise.** Prove that Theorem (4.7) is equivalent to the following result. Let ξ be the measure on $\mathbb{R}_+ \times \mathbf{W} \times \mathbf{W}$ given by

$$\xi(dt, dw, dw') = 1_{(t>0)}dt\ S_3(dw)S_3(dw')$$

and set $L(w) = \sup\{s : w(s) = t\}$. If we define an U-valued variable e by

$$e_s(w, w') = \begin{cases} w(s) & \text{if } 0 \le s \le L(w) \\ w'(L(w) + L(w') - s) & \text{if } L(w) \le s \le L(w) + L(w') \\ 0 & \text{if } s \ge L(w) + L(w'), \end{cases}$$

then the law of (L, e) under ξ is equal to $\bar{n}_+$.

** **(4.24) Exercise (Chung-Jacobi-Riemann identity).** Let B be the standard BM and T an exponential r.v. with parameter $1/2$, independent of B.

1° Prove that for every positive measurable functional F,

$$E\left[F(B_u; u \le g_T) \mid L_T = s\right] = e^s E\left[F(B_u; u \le \tau_s)\exp(-\tau_s/2)\right],$$

and consequently that

$$E\left[F\left(B_u; u \le g_T\right)\right] = \int_0^\infty E\left[F\left(B_u; u \le \tau_s\right)\exp\left(-\tau_s/2\right)\right]ds.$$

2°) Let S^0, I^0 and l^0 denote respectively the supremum, the opposite of the infimum and the local time at 0 of the standard Brownian bridge $(b(t); t \le 1)$. Given a $\mathcal{N}(0,1)$ Gaussian r.v. N independent of b, prove the three variate formula

$$P\left[|N|S^0 \le x; |N|I^0 \le y; |N|l^0 \in dl\right] = \exp(-l(\coth x + \coth y)/2)dl.$$

3°) Prove as a result that

$$P\left[|N|S^0 \le x; |N|I^0 \le y\right] = 2/(\coth x + \coth y)$$

and that, if $M^0 = \sup\{|b(s)|; s \le 1\}$,

$$P\left[|N|M^0 \le x\right] = \tanh x.$$

Prove Csáki's formula:

$$P\{S^0/S^0 + I^0 \le v\} = (1-v)(1 - \pi v\cot(\pi v)) \quad (0 < v < 1)$$

[Hint: Use the identity:

$$2v^2\int_0^\infty d\lambda\left(\frac{\sinh(v\lambda)}{v\sinh(\lambda)}\right)^2 = 1 - \pi v\cot(\pi v). \quad].$$

4°) Prove the Chung-Jacobi-Riemann identity:

$$\left(S^0 + I^0\right)^2 \stackrel{(d)}{=} \left(M^0\right)^2 + \left(\widetilde{M}^0\right)^2$$

where $\widetilde{M}^0$ is an independent copy of M^0.

5°) Characterize the pairs (S, I) of positive r.v.'s such that

i) $P[|N|S \le x; |N|I \le y] = 2/(h(x) + h(y))$ for a certain function h,

ii) $(S+I)^2 \stackrel{(d)}{=} M^2 + \widetilde{M}^2$,

where M and $\widetilde{M}$ are two independent copies of $S \vee I$.

(4.25) Exercise. (Brownian meander and Brownian bridges). Let $a \in \mathbb{R}$, and let Π_a be the law of the Brownian bridge $(B_t, t \le 1)$, with $B_0 = 0$ and $B_1 = a$. Prove that, under Π_a, both processes $(2S_t - B_t, t \le 1)$ and $(|B_t| + L_t, t \le 1)$ have the same distribution as the Brownian meander $(m_t, t \le 1)$ conditioned on $(m_1 \ge |a|)$.

[Hint: Use the relation (+) in Exercise (4.18) together with Exercise (3.20) in Chap. VI.]

In particular, the preceding description for $a = 0$ shows that, if $(b_t, t \le 1)$ is a standard Brownian bridge, with $\sigma_t = \sup_{s \le t} b_s$, and $(l_t, t \le 1)$ its local time at 0, then

$$(m_t, t \le 1) \stackrel{(d)}{=} (2\sigma_t - b_t, t \le 1) \stackrel{(d)}{=} (|b_t| + l_t, t \le 1).$$

Prove that under the probability measure $(\sigma_1/c) \cdot \Pi_0$ (resp. $(l_1/c) \cdot \Pi_0$) the process $(2\sigma_t - b_t, t \le 1)$ (resp. $(|b_t| + l_t, t \le 1)$) is a BES^3.

(4.26) Exercise. 1°) With the notation of this section set $J = \int_0^\infty P_0^t \, dt$ and prove that

$$J = \int_{-\infty}^{\infty} da \int_0^\infty P_0^{\tau_s^a} ds = \int_{-\infty}^{\infty} da \int_0^\infty \left(P_0^{T_a} \circ P_0^{\tau_s^0}\right) ds.$$

[Hint: Use the generalized occupation times formula of Exercise (1.13) Chapter VI.]

2°) Define a map $\omega \to \tilde{\omega}$ on $\mathscr{X}$ by

$$\zeta(\tilde{\omega}) = \zeta(\omega) \quad \text{and} \quad \tilde{\omega}(t) = \omega(0) + \omega(\zeta(\omega)) - \check{\omega}(t),$$

and call $\tilde{\mu}$ the image by this map of the measure μ. Prove that $\tilde{J} = J$ and that

$$(\mu \circ \mu')\tilde{} = \tilde{\mu}' \circ \tilde{\mu}$$

for any pair (μ, μ') of measures on $\mathscr{X}$.

3°) Prove that $P_0^{\tau_s^0} = \left(P_0^{\tau_s^0}\right)\tilde{}$ and conclude that

$$J = \left(\int_0^\infty P_0^{\tau_s^0} ds\right) \circ \left(\int_{-\infty}^{\infty} \check{\left(P_0^{T_a}\right)} da\right).$$

[Hint: See Exercise (2.29) Chapter VI.]

Notes and Comments

Sect. 1. This section is taken mainly from Itô [5] and Meyer [4].

Exercise (1.19) comes from Pitman-Yor [8].

Sect. 2. The first breakthrough in the description of Brownian motion in terms of excursions and Poisson point processes was the paper of Itô [5]. Although some ideas were already, at an intuitive level, in the work of Lévy, it was Itô who put the subject on a firm mathematical basis, thus supplying another cornerstone to Probability Theory. Admittedly, once the characteristic measure is known all sorts of computations can be carried through as, we hope, is clear from the exercises of the following sections. For the results of this section we also refer to Maisonneuve [6] and Pitman [4].

The approximation results such as Proposition (2.9), Exercise (2.14) and those already given in Chap. VI were proved or conjectured by Lévy. The proofs were given and gradually simplified in Itô-McKean [1], Williams [6], Chung-Durrett [1] and Maisonneuve [4].

Exercise (2.17) may be extended to the computation of the distribution of the multidimensional time spent in the different rays by a Walsh Brownian motion (see Barlow et al. [1] (1989)).

Sect. 3. In this section, it is shown how the *global excursion* theory, presented in Section 2, can be applied to describe the laws of *individual excursions*, i.e. excursions straddling a given random time T. We have presented the discussion only for stopping times T w.r.t. the filtration $(\check{\mathscr{F}}_t) = (\mathscr{F}_{g_t})$, and terminal $(\mathscr{F}_t)$ stopping times. See Maisonneuve [7] for a general discussion. The *canevas* for this section is Getoor-Sharpe [5] which is actually written in a much more general setting. We also refer to Chung [1]. The filtration $(\mathscr{F}_{g_t})$ was introduced and studied in Maisonneuve [6].

The Brownian Meander of Exercise (3.8) has recently been much studied (see Imhof ([1] and [2]), Durrett et al [1], Denisov [1] and Biane-Yor [3]). It has found many applications in the study of Azéma's martingale (see Exercise (4.16) taken from Azéma-Yor [3]).

Sect. 4. Theorems (4.1) and (4.2) are fundamental results of Itô [5]. The proof of Corollary (4.4) is taken from Ikeda-Watanabe [2].

Williams' description of the Itô measure is found in Williams [7] and Rogers-Williams [1] (see also Rogers [1]) and Bismut's description appeared in Bismut [3]. The formalism used in the proof of the latter as well as in Exercise (4.18) was first used in Biane-Yor [1]. The paper of Bismut contains further information which was used by Biane [1] to investigate the relationship between the Brownian Bridge and the Brownian excursion and complement the result of Vervaat [1].

Exercise (4.8) is due to Rogers [3]. Knight's identity (Knight [8]) derived in Exercise (4.11) has been explained in Biane [2] and Vallois [3] using a pathwise decomposition of the pseudo-Brownian bridge (cf. Exercise (2.29) Chap. VI); generalizations to Bessel processes (resp. perturbed Brownian motions) have been given by Pitman-Yor [9] (resp. [23]). The Watanabe process appears in Watanabe [2]. Exercise (4.14) is from Rogers [1]. Exercise (4.16) originates with Azéma [2] and Exercise (4.17) with Biane et al. [1]. Exercise (4.18) is taken partly from Azéma-Yor [3] and partly from Biane-Yor ([1] and [3]) and Exercise (4.19) from Knight [6]. Exercise (4.20) is in Biane-Yor [4] and Exercise (4.21) in Pitman-Yor [2]; further results connecting the Brownian bridge, excursion and meander are presented in Bertoin-Pitman [1].

With the help of the explicit Lévy-Khintchine representation of Q_x^δ obtained in Exercise (4.21), Le Gall-Yor [5] extend the Ray-Knight theorems on Brownian local times by showing that, for any $\delta > 0$, Q_0^δ is the law of certain local times processes in the space variable. In the same Exercise (4.21), the integral representations of M, N, M_0 and N_0 in terms of squares of BES^4 bridges are taken from Pitman [5]. Exercise (4.22) is in Azéma-Yor [3], and Exercise (4.23) originates from Bismut [3].

The joint law of the supremum, infimum and local time of the Brownian bridge is characterized in Exercise (4.24), taken from work in progress by Pitman and Yor. The presentation which involves an independent Gaussian random variable, differs from classical formulae, in terms of theta functions, found in the literature (see e.g. Borodin and Salminen [1]). Csáki's formula in question 3°) comes from Csáki [1] and is further discussed in Pitman-Yor [13]. Chung's identity of question 4°) remains rather mysterious, although Biane-Yor [1] and Williams [9] explain partly its relation to the functional equation of the Riemann zeta function. See also

Smith and Diaconis [1] for a random walk approach to the functional equation, and Biane-Pitman-Yor [1] for further developments.

Exercise (4.25) is a development and an improvement of the corresponding result found in Biane-Yor [3] for $a = 0$, and of the remark following Theorem 4.3 in Bertoin-Pitman [1]. The simple proof of Exercise (4.26) is taken from Leuridan [1].

Chapter XIII. Limit Theorems in Distribution

§1. Convergence in Distribution

In this section, we will specialize the notions of Sect. 5 Chap. 0 to the Wiener space $\mathbf{W}^d$. This space is a Polish space when endowed with the topology of uniform convergence on compact subsets of $\mathbb{R}_+$. This topology is associated with the metric

$$d(\omega, \omega') = \sum_1^{\infty} 2^{-n} \frac{\sup_{t \le n} |\omega(t) - \omega'(t)|}{1 + \sup_{t \le n} |\omega(t) - \omega'(t)|} .$$

The relatively compact subsets in this topology are given by Ascoli's theorem. Let

$$V^N(\omega, \delta) = \sup \left\{ |\omega(t) - \omega(t')|; \ |t - t'| \le \delta \text{ and } t, t' \le N \right\}.$$

With this notation, we have

(1.1) Proposition. *A subset Γ of $\mathbf{W}^d$ is relatively compact if and only if*

(i) the set $\{\omega(0), \omega \in \Gamma\}$ is bounded in $\mathbb{R}^d$;
(ii) for every N,

$$\lim_{\delta \downarrow 0} \sup_{\omega \in \Gamma} V^N(\omega, \delta) = 0.$$

In Sect. 5 Chap. 0, we have defined a notion of weak convergence for probability measures on the Borel σ-algebra of $\mathbf{W}^d$; the latter is described in the following

(1.2) Proposition. *The Borel σ-algebra on $\mathbf{W}^d$ is equal to the σ-algebra $\mathscr{F}$ generated by the coordinate mappings.*

Proof. The coordinate mappings are clearly continuous, hence $\mathscr{F}$ is contained in the Borel σ-algebra. To prove the reverse inclusion, we observe that by the definition of d, the map $\omega \to d(\omega, \omega')$ where ω' is fixed, is $\mathscr{F}$-measurable. As a result, every ball, hence every Borel set, is in $\mathscr{F}$.

Before we proceed, let us observe that the same notions take on a simpler form when the time range is reduced to a compact interval, but we will generally work with the whole half-line.

(1.3) Definition. *A sequence* (X^n) *of* $\mathbb{R}^d$*-valued continuous processes defined on probability spaces* $(\Omega^n, \mathscr{F}^n, P^n)$ *is said to* converge in distribution to a process X *if the sequence* $(X^n(P^n))$ *of their laws converges weakly on* $\mathbf{W}^d$ *to the law of* X. *We will write* $X^n \xrightarrow{(d)} X$.

In this definition, we have considered processes globally as $\mathbf{W}^d$-valued random variables. If we consider processes taken at some fixed times, we get a weaker notion of convergence.

(1.4) Definition. *A sequence* (X^n) *of (not necessarily continuous)* $\mathbb{R}^d$*-valued processes is said to converge to the process* X *in the* sense of finite distributions *if for any finite collection* $(t_1, \ldots, t_k)$ *of times, the* $\mathbb{R}^{dk}$*-valued r.v.'s* $\left(X^n_{t_1}, \ldots, X^n_{t_k}\right)$ *converge in law to* $\left(X_{t_1}, \ldots, X_{t_k}\right)$. *We will write* $X^n \xrightarrow{\text{f.d.}} X$.

Since the map $\omega \to (\omega(t_1), \ldots, \omega(t_k))$ is continuous on $\mathbf{W}^d$, it is easy to see that, if $X_n \xrightarrow{(d)} X$, then $X_n \xrightarrow{\text{f.d.}} X$. The converse is not true, and in fact continuous processes may converge in the sense of finite distributions to discontinuous processes as was seen in Sect. 4 of Chap. X and will be seen again in Sect. 3.

The above notions make sense for multi-indexed processes or in other words for $C\left((\mathbb{R}_+)^l, \mathbb{R}^d\right)$ in lieu of the Wiener space. We leave to the reader the task of writing down the extensions to this case (see Exercise (1.12)).

Convergence in distribution of a sequence of probability measures on $\mathbf{W}^d$ is fairly often obtained in two steps:

i) the sequence is proved to be weakly relatively compact;
ii) all the limit points are shown to have the same set of finite-dimensional distributions.

In many cases, one gets ii) by showing directly that the finite dimensional distributions converge, or in other words that there is convergence in the sense of finite distributions. To prove the first step above, it is usually necessary to use Prokhorov's criterion which we will now translate in the present context. Let us first observe that the function $V^N(\cdot, \delta)$ is a random variable on $\mathbf{W}^d$.

(1.5) Proposition. *A sequence* (P_n) *of probability measures on* $\mathbf{W}^d$ *is weakly relatively compact if and only if the following two conditions hold:*

i) for every $\varepsilon > 0$, *there exist a number* A *and an integer* n_0 *such that*

$$P_n[\,|\omega(0)| > A] \le \varepsilon, \qquad \text{for every } n \ge n_0;$$

ii) for every η, $\varepsilon > 0$ *and* $N \in \mathbb{N}$, *there exist a number* δ *and an integer* n_0 *such that*

$$P_n\left[V^N(\cdot, \delta) > \eta\right] \le \varepsilon \qquad \text{for every } n \ge n_0.$$

Remark. We will see in the course of the proof that we can actually take $n_0 = 0$.

Proof. The necessity with $n_0 = 0$ follows readily from Proposition (1.1) and Prokhorov's criterion of Sect. 5 Chap. 0.

Let us turn to the sufficiency. We assume that conditions i) and ii) hold. For every n_0, the finite family $(P_n)_{n\le n_0}$ is tight, hence satisfies i) and ii) for numbers A' and δ'. Therefore, by replacing A by $A \vee A'$ and δ by $\delta \wedge \delta'$, we may as well assume that conditions i) and ii) hold with $n_0 = 0$. This being so, for $\varepsilon > 0$ and $N \in \mathbb{N}$, let us pick $A_{N,\varepsilon}$ and $\delta_{N,k,\varepsilon}$ such that

$$\sup_n P_n\left[\,|\,\omega(0)| > A_{N,\varepsilon}\right] \le 2^{-N-1}\varepsilon,$$
$$\sup_n P_n\left[V^N(\cdot,\delta_{N,k,\varepsilon}) > 1/k\right] \le 2^{-N-k-1}\varepsilon,$$

and set $K_{N,\varepsilon} = \{\omega : |\,\omega(0)| \le A_{N,\varepsilon},\quad V^N(\omega,\delta_{N,k,\varepsilon}) \le 1/k \text{ for every } k \ge 1\}$. By Proposition (1.1), the set $K_\varepsilon = \bigcap_N K_{N,\varepsilon}$ is relatively compact in $\mathbf{W}^d$ and we have $P_n\left(K_\varepsilon^c\right) \le \sum_N P_N\left(K_{N,\varepsilon}^c\right) \le \varepsilon$, which completes the proof. □

We will use the following

(1.6) Corollary. *If $X^n = \left(X_1^n,\dots,X_d^n\right)$ is a sequence of d-dimensional continuous processes, the set $(X^n(P^n))$ of their laws is weakly relatively compact if and only if, for each j, the set of laws $X_j^n(P^n)$ is weakly relatively compact.*

Hereafter, we will need a condition which is slightly stronger than condition ii) in Proposition (1.5).

(1.7) Lemma. *Condition ii) in Proposition (1.5) is implied by the following condition: for any N and $\varepsilon, \eta > 0$, there exist a number δ, $0 < \delta < 1$, and an integer n_0, such that*

$$\delta^{-1}P_n\left[\left\{\omega : \sup_{t\le s\le t+\delta} |\,\omega(s)-\omega(t)| \ge \eta\right\}\right] \le \varepsilon \text{ for } n \ge n_0 \text{ and for all } t \le N.$$

Proof. Let N be fixed, pick $\varepsilon, \eta > 0$ and let n_0 and δ be such that the condition in the statement holds. For every integer i such that $0 \le i < N\delta^{-1}$, define

$$A_i = \left\{\sup_{i\delta\le s\le (i+1)\delta\wedge N} |\,\omega(i\delta)-\omega(s)| \ge \eta\right\}.$$

As is easily seen $\left\{V^N(\cdot,\delta) < 3\eta\right\} \supset \bigcap_i A_i^c$, and consequently for every $n \ge n_0$, we get

$$P_n\left[V^N.(\cdot,\delta) \ge 3\eta\right] \le P_n\left[\bigcup_i A_i\right] \le (1+[N\delta^{-1}])\delta\varepsilon < (N+1)\varepsilon$$

which proves our claim. □

The following result is very useful.

(1.8) Theorem (Kolmogorov's criterion for weak compactness). *Let (X^n) be a sequence of $\mathbb{R}^d$-valued continuous processes such that*

i) the family $\{X_0^n(P^n)\}$ of initial laws is tight in $\mathbb{R}^d$,
ii) there exist three strictly positive constants α, β, γ such that for every $s, t \in \mathbb{R}_+$ and every n,

$$E_n\left[|X_s^n - X_t^n|^\alpha\right] \leq \beta|s-t|^{\gamma+1};$$

then, the set $\{X^n(P^n)\}$ of the laws of the X_n's is weakly relatively compact.

Proof. Condition i) implies condition i) of Proposition (1.5), while condition ii) of Proposition (1.5) follows at once from Markov inequality and the result of Theorem (2.1) (or its extension in Exercise (2.10)) of Chap. I. □

We now turn to a first application to Brownian motion. We will see that the Wiener measure is the weak limit of the laws of suitably interpolated random walks. Let us mention that the existence of Wiener measure itself can be proved by a simple application of the above ideas.

In what follows, we consider a sequence of independent and identically distributed, centered random variables ξ_k such that $E\left[\xi_k^2\right] = \sigma^2 < \infty$. We set $S_0 = 0$, $S_n = \sum_{k=1}^n \xi_k$. If $[x]$ denotes the integer part of the real number x, we define the continuous process X^n by

$$X_t^n = \left(\sigma\sqrt{n}\right)^{-1}\left(S_{[nt]} + (nt - [nt])\xi_{[nt]+1}\right).$$

(1.9) Theorem (Donsker). *The processes X^n converge in distribution to the standard linear Brownian motion.*

Proof. We first prove the convergence of finite-dimensional distributions. Let $t_1 < t_2 < \ldots < t_k$; by the classical central limit theorem and the fact that $[nt]/n$ converges to t as n goes to $+\infty$, it is easily seen that $\left(X_{t_1}^n, X_{t_2}^n - X_{t_1}^n, \ldots, X_{t_k}^n - X_{t_{k-1}}^n\right)$ converges in law to $\left(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_k} - B_{t_{k-1}}\right)$ where B is a standard linear BM. The convergence of finite-dimensional distributions follows readily.

Therefore, it is sufficient to prove that the set of the laws of the X_n's is weakly relatively compact. Condition i) of Proposition (1.5) being obviously in force, it is enough to show that the condition of Lemma (1.7) is satisfied.

Assume first that the ξ_k's are bounded. The sequence $|S_k|^4$ is a submartingale and therefore for fixed n

$$P\left[\max_{i\leq n}|S_i| > \lambda\sigma\sqrt{n}\right] \leq E\left[|S_n|^4\right]\left(\lambda\sigma\sqrt{n}\right)^{-4}.$$

One computes easily that $E\left[S_n^4\right] = nE\left[\xi_1^4\right] + 3n(n-1)\sigma^4$. As a result, there is a constant K independent of the law of ξ_k such that

$$\overline{\lim_{n\to\infty}}\, P\left[\max_{i\leq n}|S_i| > \lambda\sigma\sqrt{n}\right] \leq K\lambda^{-4}.$$

By truncating and passing to the limit, it may be proved that this is still true if we remove the assumption that ξ_k is bounded. For every $k \geq 1$, the sequence

$\{S_{n+k} - S_k\}$ has the same law as the sequence $\{S_n\}$ so that finally, there exists an integer n_1 such that

$$P\left[\max_{i\leq n} |S_{i+k} - S_k| > \lambda\sigma\sqrt{n}\right] \leq K\lambda^{-4}$$

for every $k \geq 1$ and $n \geq n_1$. Pick ε and η such that $0 < \varepsilon$, $\eta < 1$ and then choose λ such that $K\lambda^{-2} < \eta\varepsilon^2$; set further $\delta = \varepsilon^2\lambda^{-2}$ and choose $n_0 > n_1\delta^{-1}$. If $n \geq n_0$, then $[n\delta] \geq n_1$, and the last displayed inequality may be rewritten as

$$P\left[\max_{i\leq[n\delta]} |S_{i+k} - S_k| \geq \lambda\sigma\sqrt{[n\delta]}\right] \leq \eta\varepsilon^2\lambda^{-2}.$$

Since $\lambda\sqrt{[n\delta]} \leq \varepsilon\sqrt{n}$, we get

$$\delta^{-1}P\left[\max_{i\leq[n\delta]} |S_{i+k} - S_k| \geq \varepsilon\sigma\sqrt{n}\right] \leq \eta$$

for every $k \geq 1$ and $n \geq n_0$. Because the X_n's are linear interpolations of the random walk (S_n), it is now easy to see that the condition in Lemma (1.7) is satisfied for every N and we are done. □

To illustrate the use of weak convergence as a tool to prove existence results, we will close this section with a result on solutions to martingale problems. At no extra cost, we will do it in the setting of Itô processes (Definition (2.5), Chap. VII).

We consider functions a and b defined on $\mathbb{R}_+ \times \mathbf{W}^d$ with values respectively in the sets of symmetric non-negative $d \times d$-matrices and $\mathbb{R}^d$-vectors. We assume these functions to be progressively measurable with respect to the filtration $(\mathscr{F}_t^0)$ generated by the coordinate mappings $\omega(t)$. The reader is referred to the beginning of Sect. 1 Chap. IX. With the notation of Sect. 2 Chap. VII, we may state

(1.10) Theorem. *If a and b are continuous on $\mathbb{R}_+ \times \mathbf{W}^d$, then for any probability measure μ on $\mathbb{R}^d$, there exists a probability measure P on $\mathbf{W}^d$ such that*

i) $P[\omega(0) \in A] = \mu(A)$;
ii) for any $f \in C_K^2$, *the process* $f(\omega(t)) - f(\omega(0)) - \int_0^t L_s f(\omega(s))ds$ *is a* $(\mathscr{F}_t^0, P)$*-martingale, where*

$$L_s f(\omega(s)) = \frac{1}{2}\sum a_{ij}(s,\omega)\frac{\partial^2 f}{\partial x_i \partial x_k}(\omega(s)) + \sum_i b_i(s,\omega)\frac{\partial f}{\partial x_i}(\omega(s)).$$

Proof. For each integer n, we define functions a_n and b_n by

$$a_n(t,\omega) = a([nt]/n, \omega), \qquad b_n(t,\omega) = b([nt]/n, \omega).$$

These functions are obviously progressively measurable and we call L_s^n the corresponding differential operators.

Pick a probability space $(\Omega, \mathscr{F}, P)$ on which a r.v. X_0 of law μ and a $\mathrm{BM}^d(0)$ independent of X_0, say B, are defined. Let σ_n be a square root of a_n. We define inductively a process X^n in the following way; we set $X_0^n = X_0$ and if X^n is defined up to time k/n, we set for $k/n < t \le (k+1)/n$,

$$X_t^n = X_{k/n}^n + \sigma_n(k/n, X^n_\cdot)(B_t - B_{k/n}) + b_n(k/n, X^n_\cdot)(t - k/n).$$

Plainly, X^n satisfies the SDE

$$X_t^n = \int_0^t \sigma_n(s, X^n_\cdot)dB_s + \int_0^t b_n(s, X^n_\cdot)ds$$

and if we call P^n the law of X^n on $\mathbf{W}^d$, then $P^n[\omega(0) \in A] = \mu(A)$ and $f(\omega(t)) - f(\omega(0)) - \int_0^t L_s^n f(\omega(s))ds$ is a P^n-martingale for every $f \in C_K^2$.

The set (P^n) is weakly relatively compact because condition i) in Theorem (1.8) is obviously satisfied and condition ii) follows from the boundedness of a and b and the Burkholder-Davis-Gundy inequalities applied on the space Ω.

Let P be a limit point of (P^n) and $(P^{n'})$ be a subsequence converging to P. We leave as an exercise to the reader the task of showing that, since for fixed t the functions $\int_0^t L_s^n f(\omega(s))ds$ are equi-continuous on $\mathbf{W}^d$ and converge to $\int_0^t L_s f(\omega(s))ds$, then

$$E_P\left[\left(f(\omega(t)) - \int_0^t L_s f(\omega(s))ds\right)\phi\right] = \lim_{n'\to\infty} E_{P^{n'}}\left[\left(f(\omega(t)) - \int_0^t L_s^{n'} f(\omega(s))ds\right)\phi\right]$$

for every continuous bounded function ϕ. If $t_1 < t_2$ and ϕ is $\mathscr{F}_{t_1}^0$-measurable it follows that

$$E_P\left[\left(f(\omega(t_2)) - f(\omega(t_1)) - \int_{t_1}^{t_2} L_s f(\omega(s))ds\right)\phi\right] = 0$$

since the corresponding equality holds for $P^{n'}$ and $L_s^{n'}$. By the monotone class theorem, this equality still holds if ϕ is merely bounded and $\mathscr{F}_{t_1}^0$-measurable; as a result, $f(\omega(t)) - f(\omega(0)) - \int_0^t L_s f(\omega(s))ds$ is a P-martingale and the proof is complete. □

Remarks. With respect to the results in Sect. 2 Chap. IX, we see that we have dropped the Lipschitz conditions. In fact, the hypothesis may be further weakened by assuming only the continuity in ω of a and b for each fixed t. On the other hand, the existence result we just proved is not of much use without a uniqueness result which is a much deeper theorem.

\# **(1.11) Exercise.** 1°) If (X^n) converges in distribution to X, prove that $(X^n)^*$ converges in distribution to X^* where, as usual, $X_t^* = \sup_{s\le t}|X_s|$.

2°) Prove the reflection principle for BM (Sect. 3 Chap. III) by means of the analogous reflection principle for random walks. The latter is easily proved in the case of the simple random walk, namely with the notation of Theorem (1.9), $P[\xi_k = 1] = P[\xi_k = -1] = 1/2$.

* **(1.12) Exercise.** Prove that a family (P_λ) of probability measures on $C\left((\mathbb{R}_+)^k, \mathbb{R}\right)$ is weakly relatively compact if there exist constants α, β, γ, $p > 0$ such that $\sup_\lambda E_\lambda[\,|X_0|^p] < \infty$, and for every pair (s, t) of points in $(\mathbb{R}_+)^k$

$$\sup_\lambda E_\lambda\left[\,|X_s - X_t|^a\right] \le \beta|s - t|^{k+\gamma}$$

where X is the coordinate process.

* **(1.13) Exercise.** Let β_s^n, $s \in [0, 1]$ and γ_t^n, $t \in [0, 1]$ be two independent sequences of independent standard BM's. Prove that the sequence of doubly indexed processes

$$X_{(s,t)}^n = n^{-1/2} \sum_1^n \beta_s^i \gamma_t^i$$

converges in distribution to the Brownian sheet. This is obviously an infinite-dimensional central-limit theorem.

(1.14) Exercise. In the setting of Donsker's theorem, prove that the processes

$$\left(\sigma\sqrt{n}\right)^{-1}\left(S_{[nt]} + (nt - [nt])\xi_{[nt]+1} - tS_n\right), \qquad 0 \le t \le 1,$$

converge in distribution to the Brownian Bridge.

(1.15) Exercise. Let (M^n) be a sequence of (super) martingales defined on the same filtered space and such that

i) the sequence (M^n) converges in distribution to a process M;
ii) for each t, the sequence $\left(M_t^n\right)$ is uniformly integrable.

Prove that M is a (super) martingale for its natural filtration.

* **(1.16) Exercise.** Let (M^n) be a sequence of continuous local martingales vanishing at 0 and such that $(\langle M^n, M^n\rangle)$ converges in distribution to a deterministic function a. Let P_n be the law of M^n.

1°) Prove that the set (P_n) is weakly relatively compact.

[Hint: One can use Lemma (4.6) Chap. IV.]

2°) If, in addition, the M^n's are defined on the same filtered space and if, for each t, there is a constant $\alpha(t)$ such that $\langle M^n, M^n\rangle_t \le \alpha(t)$ for each n, show that (P_n) converges weakly to the law W_a of the gaussian martingale with increasing process $a(t)$ (see Exercise (1.14) Chap. V).

[Hint: Use the preceding exercise and the ideas of Proposition (1.23) Chap. IV.]

3°) Let $(M^n) = \left(M_i^n, i = 1, \dots, k\right)$ be a sequence of multidimensional local martingales such that (M_i^n) satisfies for each i all the above hypotheses and, in addition, for $i \ne j$, the processes $\langle M_i^n, M_j^n\rangle$ converge to zero in distribution. Prove that the laws of M^n converge weakly to $W_{a_1} \otimes \dots \otimes W_{a_k}$.

[Hint: One may consider the linear combinations $\sum u_i M_i^n$.]

The two following exercises may be solved by using only elementary properties of BM.

* **(1.17) Exercise (Scaling and asymptotic independence).** 1°) Using the notation of the following section, prove that if β is a BM, the processes β and $\beta^{(c)}$ are asymptotically independent as c goes to 0.

[Hint: For every $A > 0$, $(\beta_{c^2 t}, t \leq A)$ and $(\beta_{c^2 A+u} - \beta_{c^2 A}, u \geq 0)$ are independent.]

2°) Deduce from 1°) that the same property holds as c goes to infinity. (See also Exercise (2.9).)

Prove that for $c \neq 1$, the transformation $x \to X^{(c)}$ which preserves the Wiener measure, is ergodic. This ergodic property is the key point in the proof of Exercise (3.20), 1°), Chap. X.

3°) Prove that if $(\gamma_t, t \leq 1)$ is a process whose law P^γ on $C([0, 1], \mathbb{R})$ satisfies

$$P^\gamma_{|\mathscr{F}_t} \ll W_{|\mathscr{F}_t} \quad \text{for every } t < 1,$$

then the two-dimensional process $V_t^{(c)} = ((\gamma_t^{(c)}, \gamma_t), t \leq 1)$ converges in law as c goes to 0 towards $((\beta_t, \gamma_t), t \leq 1)$, where β is a BM which is independent of γ.

[Hint: Use Lemma (5.7) Chap. 0.]

4°) Prove that the law of $\gamma^{(c)}$ converges in total variation to the law of β i.e. the Wiener measure. Can the convergence in 3°) be strengthened into a convergence in total variation?

5°) Prove that $V^{(c)}$ converges in law as c goes to 0 whenever γ is a BB, a Bessel bridge or the Brownian meander and identify the limit in each case.

* **(1.18) Exercise. (A Bessel process looks eventually like a BM).** Let R be a $\text{BES}^\delta(r)$ with $\delta > 1$ and $r \geq 0$. Prove that as t goes to infinity, the process $(R_{t+s} - R_t, s \geq 0)$ converges in law to a BM^1.

[Hint: Use the canonical decomposition of R as a semimartingale. It may be necessary to write separate proofs for different dimensions.]

§2. Asymptotic Behavior of Additive Functionals of Brownian Motion

This section is devoted to the proof of a limit theorem for stochastic integrals with respect to BM. As a corollary, we will get (roughly speaking) the growth rate of occupation times of BM.

In what follows, B is a standard linear BM and L^a the family of its local times. As usual, we write L for L^0. The Lebesgue measure is denoted by m.

(2.1) Proposition. *If f is integrable,*

$$\lim_{n\to\infty} n \int_0^\cdot f(nB_s)ds = m(f)L \qquad a.s.,$$

and, for each t, the convergence of $n\int_0^t f(nB_s)ds$ to $m(f)L_t$ holds in L^p for every $p \geq 1$. Both convergences are uniform in t on compact intervals.

Proof. By the occupation times formula

$$n\int_0^t f(nB_s)ds = \int_{-\infty}^{+\infty} f(a)L_t^{a/n}da.$$

For fixed t, the map $a \to L_t^a$ is a.s. continuous and has compact support; thus, the r.v. $\sup_a L_t^a$ is a.s. finite and by the continuity of $L_t^{\cdot}$ and the dominated convergence theorem,

$$\lim_n n\int_0^t f(nB_s)ds = m(f)L_t \qquad \text{a.s.}$$

Hence, this is true simultaneously for every rational t; moreover, it is enough to prove the result for $f \geq 0$ in which case all the processes involved are increasing and the proof of the first assertion is easily completed.

For the second assertion, we observe that

$$\left|\int_{-\infty}^{+\infty} f(a)L_t^{a/n}da\right| \leq \|f\|_1\left(\sup_a L_t^a\right)$$

and, since by Theorem (2.4) in Chap. XI, $\sup_a L_t^a$ is in L^p for every p, the result follows from the dominated convergence theorem.

The uniformity follows easily from the continuity of L_t^a in both variables. □

The following is a statement about the asymptotic behavior of additive functionals, in particular occupation times. The convergence in distribution involved is that of processes (see Sect. 1), not merely of individual r.v.'s.

(2.2) Proposition. *If A is an integrable* CAF,

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}}A_{n\cdot} = \nu_A(1)L_{\cdot} \qquad \textit{in distribution.}$$

Proof. Since (see Exercise (2.11) Chap. VI) $L_{n\cdot}^a \overset{(d)}{=} \sqrt{n}L_{\cdot}^{a/\sqrt{n}}$, it follows that

$$\frac{1}{\sqrt{n}}A_{n\cdot} = \frac{1}{\sqrt{n}}\int L_{n\cdot}^a\nu_A(da) \overset{(d)}{=} \int L_{\cdot}^{a/\sqrt{n}}\nu_A(da)$$

and the latter expression converges a.s. to $\nu_A(1)L_{\cdot}$ by the same reasoning as in the previous proposition. □

The above result is satisfactory for $\nu_A(1) \neq 0$; it says that a positive integrable additive functional increases roughly like $\nu_A(1)\sqrt{t}$. On the contrary, the case $\nu_A(1) = 0$ must be further investigated and will lead to a central-limit type theorem with interesting consequences. Moreover, measures with zero integral are important when one wants to associate a potential theory with linear BM.

If we refer to Corollary (2.12) in Chap. X, we see that we might as well work with stochastic integrals and that is what we are going to do. To this end, we need a result which will be equally very useful in the following section. It is an *asymptotic version of Knight's theorem* (see Sect. 1 Chap. V).

In what follows, $(M_j^n, 1 \leq j \leq k)$ will be a sequence of k-tuples of continuous local martingales vanishing at 0 and such that $\langle M_j^n, M_j^n\rangle_\infty = \infty$ for every n and j. We call $\tau_j^n(t)$ the time-change associated with $\langle M_j^n, M_j^n\rangle$ and β_j^n the DDS Brownian motion of M_j^n.

(2.3) Theorem. *If, for every t, and every pair (i, j) with $i \neq j$*

$$\lim_{n\to\infty} \langle M_i^n, M_j^n\rangle_{\tau_i^n(t)} = \lim_{n\to\infty} \langle M_i^n, M_j^n\rangle_{\tau_j^n(t)} = 0$$

in probability, then the k-dimensional process $\beta^n = \left(\beta_j^n, 1 \leq j \leq k\right)$ converges in distribution to a BM^k.

Proof. The laws of the processes β_j^n are all equal to the one-dimensional Wiener measure. Therefore, the sequence $\{\beta^n\}$ is weakly relatively compact and we must prove that, for any limit process, the components, which are obviously linear BM's, are independent.

It is no more difficult to prove the results in the general case than in the case $k = 2$ for which we introduce the following handier notation. We consider two sequences of continuous local martingales (M^n) and (N^n). We call $\mu^n(t)$ and $\eta^n(t)$ the time-changes associated with $\langle M^n, M^n\rangle$ and $\langle N^n, N^n\rangle$ respectively and β^n and γ^n the corresponding DDS Brownian motions.

If $0 = t_0 < t_1 < \ldots < t_p = t$ and if we are given scalars $f_0, \ldots, f_{p-1}$ and $g_0, \ldots, g_{p-1}$, we set

$$f = \sum_j f_j 1_{]t_j, t_{j+1}]}, \qquad \beta^n(f) = \sum_j f_j\left(\beta^n_{t_{j+1}} - \beta^n_{t_j}\right)$$

$$g = \sum_j g_j 1_{]t_j, t_{j+1}]}, \qquad \gamma^n(g) = \sum_j g_j\left(\gamma^n_{t_{j+1}} - \gamma^n_{t_j}\right).$$

Let us first observe that if we set

$$U_s^n = \int_0^s f\left(\langle M^n, M^n\rangle_u\right) dM_u^n, \qquad V_s^n = \int_0^s g\left(\langle N^n, N^n\rangle_u\right) dN_u^n,$$

then $\beta(f) = U_\infty^n$ and $\gamma^n(g) = V_\infty^n$. Therefore writing $E\left[\mathscr{E}\left(i\left(U^n + V^n\right)\right)_\infty\right] = 1$ yields

$$E[\exp(i(\beta^n(f) + \gamma^n(g))) \cdot H^n] = \exp\left(-\frac{1}{2}\int\left(f^2 + g^2\right)(t)dt\right)$$

where

$$\begin{aligned} H^n &= \exp\left(\int_0^\infty f\left(\langle M^n, M^n\rangle_s\right) g\left(\langle N^n, N^n\rangle_s\right) d\langle M^n, N^n\rangle_s\right) \\ &= \exp\left(\sum_{i,j} f_i g_j\left(\langle M^n, N^n\rangle_{\mu^n(t_{i+1})\wedge\eta^n(t_{j+1})} - \langle M^n, N^n\rangle_{\mu^n(t_i)\vee\eta^n(t_j)}\right)\right). \end{aligned}$$

The hypothesis entails plainly that H^n converges to 1 in probability; thus the proof will be finished if we can apply the dominated convergence theorem. But Kunita-Watanabe's inequality (Proposition (1.15) Chap. IV) and the time-change formulas yield

$$H^n \leq \exp(\|f\|_2 \|g\|_2)$$

and we are done. □

We will make a great use of a corollary to the foregoing result which we now describe. For any process X and for a fixed real number $h > 0$, we define the *scaled* process $X^{(h)}$ by

$$X_t^{(h)} = h^{-1} X(h^2 t).$$

The importance of the scaling operation has already been seen in the case of BM. If M is a continuous local martingale and β its DDS Brownian motion, then $\beta^{(h)}$ is the DDS Brownian motion of $h^{-1}M$ as is stated in Exercise (1.17) of Chap. V.

We now consider a family M_i, $i = 1, 2, \ldots, k$ of continuous local martingales such that $\langle M_i, M_i\rangle_\infty = \infty$ for every i and call β_i their DDS Brownian motions. We set $M_i^n = M_i/\sqrt{n}$ and call β_i^n the DDS Brownian motion of M_i^n. As observed above, $\beta_i^n(t) = \beta_i(nt)/\sqrt{n}$.

(2.4) Corollary. *The k-dimensional process $\beta^n = \left(\beta_i^n, i = 1, \ldots, k\right)$ converges in distribution to a* BM^k *as soon as*

$$\lim_{t\to\infty} \langle M_i, M_j\rangle_t / \langle M_i, M_i\rangle_t = 0$$

almost surely for every $i, j \leq k$ with $i \neq j$.

Proof. If $\tau_i(t)$ (resp. $\tau_i^n(t)$) is the time-change associated with $\langle M_i, M_i\rangle$ (resp. $\langle M_i^n, M_i^n\rangle$), then $\tau_i^n(t) = \tau_i(nt)$ and consequently $\langle M_i^n, M_j^n\rangle_{\tau_i^n(t)} = n^{-1}\langle M_i, M_j\rangle_{\tau_i(nt)}$. The hypothesis entails that $t^{-1}\langle M_i, M_j\rangle_{\tau_i(t)}$ converges a.s. to 0 as t goes to $+\infty$, so that the result follows from Theorem (2.3). □

The foregoing corollary has a variant which often comes in handy.

(2.5) Corollary. *If there is a positive continuous strictly increasing function ϕ on $\mathbb{R}_+$ such that*

i) $\phi(t)^{-1}\langle M_i, M_i\rangle_t \xrightarrow[t\to\infty]{(d)} U_i$, $i = 1, 2, \ldots, k$ *where U_i is a strictly positive r.v.,*

ii) $\phi(t)^{-1}\sup_{s\leq t}\left|\langle M_i, M_j\rangle_s\right| \xrightarrow[t\to\infty]{} 0$ *in probability for every $i, j \leq k$ with $i \neq j$,*

then the conclusion of Corollary (2.4) holds.

Proof. Again it is enough to prove that, for $i \neq j$, $t^{-1}\langle M_i, M_j\rangle_{\tau_i(t)}$ converges to 0 in probability and in fact we shall show that $t^{-1}X_t$ converges to 0 in probability where $X_t = \sup\left\{\left|\langle M_i, M_j\rangle_s\right|; s \leq \tau_i(t)\right\}$.

Hypothesis i) implies that $t^{-1}\phi(\tau_i(t))$ converges to U_i^{-1} in distribution. For $\lambda > 0$ and $x > 0$, we have, using the fact that $X_.$ is increasing,

$$\begin{aligned} P\left[X_{\tau_i(t)} > \lambda t\right] &\leq P\left[\phi(\tau_i(t)) > tx\right] + P\left[X_{\tau_i(t)} > \lambda t; \tau_i(t) < \phi^{-1}(tx)\right] \\ &\leq P\left[\phi(\tau_i(t)) > tx\right] + P\left[X_{\phi^{-1}(tx)} > \lambda t\right]. \end{aligned}$$

Pick $\varepsilon > 0$; since U_i is strictly positive we may choose x sufficiently large and $T > 0$, such that, for every $t \geq T$,

$$P\left[\phi(\tau_i(t)) > tx\right] \leq \varepsilon.$$

Hypothesis ii) implies that there exists $T' \geq T$ such that for every $t \geq T'$,

$$P\left[X_{\phi^{-1}(tx)} > \lambda t\right] \leq \varepsilon.$$

It follows that for $t \geq T'$,

$$P\left[X_{\tau_i(t)} > \lambda t\right] \leq 2\varepsilon,$$

which is the desired result. □

We now return to the problem raised after Proposition (2.2). We consider Borel functions f_i, $i = 1, 2, \ldots, k$ in $L^1(m) \cap L^2(m)$ which we assume to be pairwise orthogonal in L^2, i.e. $\int f_i f_j dm = 0$ for $i \neq j$. We set

$$M_i^n = \sqrt{n} \int_0^{\cdot} f_i(nB_s) dB_s.$$

(2.6) Theorem (Papanicolaou-Stroock-Varadhan). *The $(k+1)$-dimensional process $\left(B, M_1^n, \ldots, M_k^n\right)$ converges in distribution to $(\beta, \|f_i\|_2 \gamma_l^i, i = 1, 2, \ldots, k)$ where $\left(\beta, \gamma^1, \ldots, \gamma^k\right)$ is a BM^{k+1} and l is the local time of β at zero.*

Proof. We have

$$\langle M_i^n, M_j^n \rangle_t = n \int_0^t (f_i f_j)(nB_s) ds,$$

so that by Proposition (2.1), we have a.s.

$$\lim_{n\to\infty} \langle M_i^n, M_i^n \rangle_t = \|f_i\|_2^2 L_t; \text{ for } i \neq j, \lim_{n\to\infty} \langle M_i^n, M_j^n \rangle_t = \lim_{n\to\infty} \langle M_i^n, B \rangle_t = 0$$

uniformly in t on compact intervals. Thus it is not difficult to see that the hypotheses of Theorem (2.3) obtain; as a result, if we call B_i^n the DDS Brownian motion of M_i^n, the process $\left(B, B_1^n, \ldots, B_k^n\right)$ converges in distribution to $\left(\beta, \gamma^1, \ldots, \gamma^k\right)$. Now $\left(B, M_1^n, \ldots, M_k^n\right)$ is equal to $\left(B, B_i^n\left(\langle M_i^n, M_i^n \rangle\right), i = 1, \ldots, k\right)$ and it is plain that $\left(B, \langle M_i^n, M_i^n \rangle, i = 1, \ldots, k, B_i^n, i = 1, \ldots, k\right)$ converges in distribution to $\left(\beta, \|f_i\|_2^2 l, i = 1, \ldots, k, \gamma^i, i = 1, \ldots, k\right)$. The result follows. □

Remark. Instead of n, we could use any sequence (a_n) converging to $+\infty$ or, for that matter, consider the real-indexed family

$$M_i^\lambda = \sqrt{\lambda} \int_0^{\cdot} f_i(\lambda B_s) dB_s$$

and let λ tend to $+\infty$. The proof would go through just the same.

We will now draw the consequences we have announced for an additive functional A which is the difference of two integrable positive continuous additive functionals (see Exercise (2.22) Chap. X) and such that $\nu_A(1) = 0$. In order to be able to apply the representation result given in Theorem (2.9) of Chap. X, we will have to make the additional assumption that

$$\int |x||\nu_A|(dx) < \infty.$$

As in Sect. 3 of the Appendix, we set

$$F(x) = \int |x - y|\nu_A(dy).$$

The function F is the difference of two convex functions and its second derivative in the sense of distributions is $2\nu_A$. Let F'_- be its left derivative which is equal to $\nu_A(]-\infty, \cdot[)$. We have the

(2.7) Lemma. *The function F is bounded and F'_- is in $\mathscr{L}^1(m) \cap \mathscr{L}^2(m)$. Moreover*

$$\|F'_-\|_2^2 = I(\nu_A)$$

where $I(\nu_A) = -(1/2)\int\int |x - y|\nu_A(dx)\nu_A(dy)$ is called the energy of ν_A.

Proof. Since

$$\int |x - y||\nu_A|(dy) \leq |x|\|\nu_A\| + \int |y||\nu_A|(dy),$$

the integral $I(\nu_A)$ is finite and we may apply Fubini's theorem to the effect that

$$\begin{aligned} I(\nu_A) &= -\iint_{x>y} (x - y)\nu_A(dx)\nu_A(dy) \\ &= -\iiint_{x>z>y} \nu_A(dx)\nu_A(dy)dz \\ &= -\int_{-\infty}^{+\infty} dz \left(\int_z^{\infty} \nu_A(dx)\right)\left(\int_{-\infty}^{z} \nu_A(dy)\right). \end{aligned}$$

The set of z's such that $\nu_A(\{z\}) \neq 0$ is countable and for the other z's, it follows from the hypothesis $\nu_A(1) = 0$ that

$$\nu_A(]-\infty, z[) = -\nu_A(]z, \infty[).$$

Thus the proof of the equality in the statement is complete.

By the same token

$$\begin{aligned} \int_0^{\infty} |F'_-(x)|dx &= \int_0^{\infty} dx|\nu_A(\,]x, \infty[\,)| \leq \int_0^{\infty} |\nu_A|(\,]x, \infty[\,)dx \\ &= \int_0^{\infty} x|\nu_A|(dx) < \infty, \end{aligned}$$

and likewise
$$\int_{-\infty}^{0} |F'_{-}(x)|dx \le \int_{-\infty}^{0} |x||\nu_A|(dx) < \infty.$$
Consequently, F'_{-} is in $\mathscr{L}^1$ and it follows that F is bounded. □

We may now prove that additive functionals satisfying the above set of hypotheses are, roughly speaking, of the order of $t^{1/4}$ as t goes to infinity.

(2.8) Proposition. *If* $\nu_A(1) = 0$ *and* $\int |x||\nu_A|(dx) < \infty$, *the* 2*-dimensional process* $\left(n^{-1/2}B_{n\cdot}, n^{-1/4}A_{n\cdot}\right)$ *converges in distribution to* $\left(\beta, I(\nu_A)^{1/2}\gamma_l\right)$, *where* (β, γ) *is a* BM^2 *and* l *the local time of* β *at* 0.

Proof. By the representation result in Theorem (2.9) of Chap. X and Tanaka's formula,
$$n^{-1/4}A_{n\cdot} = n^{-1/4}[F(B_{n\cdot}) - F(0)] - n^{-1/4}\int_0^{n\cdot} F'_{-}(B_s)dB_s.$$
Since F is bounded, the first term on the right goes to zero as n goes to infinity and, therefore, it is enough to study the stochastic integral part.

Setting $s = nu$, we see that we might as well study the limit of
$$\left(n^{-1/2}B_{n\cdot}, n^{-1/4}\int_0^{\cdot} F'_{-}(B_{nu})dB_{nu}\right),$$
and since $B_{n\cdot} \overset{(d)}{=} \sqrt{n}B$, this process has the same law as
$$\left(B_{\cdot}, n^{1/4}\int_0^{\cdot} F'_{-}\left(\sqrt{n}B_u\right)dB_u\right).$$
Because F'_{-} is in $\mathscr{L}^1(m) \cap \mathscr{L}^2(m)$, it remains to apply the remark following Theorem (2.6). □

Remark. Propositions (2.2) and (2.8) are statements about the speed at which additive functionals of linear BM tend to infinity. In dimension $d > 2$, there is no such question as integrable additive functionals are finite at infinity but, for the planar BM, the same question arises and it was shown in Sect. 4 Chap. X that integrable additive functionals are of the order of $\log t$. However, as the limiting process is not continuous, one has to use other notions of convergence.

* **(2.9) Exercise.** 1°) In the situation of Theorem (2.3), if there is a sequence of positive random variables L_n such that

i) $\lim_n \langle M_i^n, M_i^n\rangle_{L_n} = +\infty$ in probability for each i;

ii) $\lim_n \sup_{s\le L_n} \left|\langle M_i^n, M_j^n\rangle_s\right| = 0$ in probability for each pair i, j with $i \neq j$,

prove that the conclusion of the Theorem holds.

2°) Assume now that there are only two indexes and write M^n for M_1^n and N^n for M_2^n. Prove that if there is a sequence (L_n) of positive random variables such that

i') $\lim_n \langle M^n, M^n \rangle_{L_n} = \infty$ in probability,
ii) $\lim_n \sup_{s \le L_n} |\langle M^n, N^n \rangle_s| = 0$,

then the conclusion of the Theorem holds.

3°) Deduce from the previous question that if $\beta_.$ is a BM, and if c converges to $+\infty$, then β and $\beta^{(c)}$ are asymptotically independent.

Remark however that the criterion given in Corollary (2.4) does not apply in the particular case of a pair $\left(M, \frac{1}{c}M\right)$ as $c \to \infty$. Give a more direct proof of the asymptotic independence of β and $\beta^{(c)}$.

* **(2.10) Exercise.** For f in $L^2 \cap L^1$, prove that for fixed t, the random variables $\sqrt{n}\int_0^t f(nB_s)dB_s$ converge weakly to zero in L^2 as n goes to infinity. As a result, the convergence in Theorem (2.6) cannot be improved to convergence in probability.

* **(2.11) Exercise.** Let $0 < a_1 < \ldots < a_k < \infty$ be a finite sequence of real numbers. Prove that the $(k+1)$-dimensional process

$$\left(B_t, \frac{\sqrt{n}}{2}\left(L_t^{a_i/n} - L_t^{a_{i-1}/n}\right), i = 1, 2, \ldots, k\right)$$

converges in distribution to

$$\left(\beta, \sqrt{a_i - a_{i-1}}\,\gamma_l^i, i = 1, 2, \ldots, k\right)$$

where $(\gamma^i, i = 1, 2, \ldots, k)$ is a k-dimensional BM independent of β and l is the local time of β at 0.

* **(2.12) Exercise.** 1°) Let

$$X(t, a) = \int_0^t 1_{[0,a]}(B_s)dB_s.$$

Prove that for $p \ge 2$, there exists a constant C_p such that for $0 \le s \le t \le 1$ and $0 \le a \le b \le 1$,

$$E\left[\,|X(t,b) - X(s,a)|^p\right] \le C_p\left((t-s)^{p/2} + (b-a)^{p/2}\right).$$

2°) Prove that the family of the laws P_λ of the doubly indexed processes

$$\left(B_t, \lambda^{1/2}\int_0^t 1_{[0,a]}(\lambda B_s)dB_s\right)$$

is weakly relatively compact.

[Hint. Use Exercise (1.12).]

3°) Prove that, as λ goes to infinity, the doubly-indexed processes

$$\left(B_t, \lambda^{1/2}\left(L_t^{a/\lambda} - L_t^0\right)/2\right)$$

converge in distribution to $\left(B_t, \mathbb{B}\left(L_t^0, a\right)\right)$, where $\mathbb{B}$ is a Brownian sheet independent of B.

[Hint: Use the preceding Exercise (2.11).]

4°) For $\nu > 0$, prove that

$$\varepsilon^{\nu} \int_{\varepsilon}^{\infty} a^{-(3/2+\nu)} \left(L_t^a - L_t^0\right) da \xrightarrow[\varepsilon \to 0]{(d)} 2 \int_0^{\infty} e^{-(\nu+1/2)u} \mathbb{B}\left(L_t^0, e^u\right) du.$$

5°) Let $\tau_x = \inf\{u : L_u^0 > x\}$; the processes $\lambda^{1/2}\left(L_{\tau_x}^{a/\lambda} - x\right)/2$ converge in distribution, as λ tends to $+\infty$, to the process $\sqrt{x}\gamma_a$ where γ_a is a standard BM. This may be derived from 3°) but may also be proved as a consequence of the second Ray-Knight theorem (Sect. 2 Chap. XI).

* **(2.13) Exercise.** With the notation of Theorem (1.10) in Chap. VI, prove that

$$\lim_{\varepsilon \to 0} \frac{1}{\sqrt{\varepsilon}} \left(\varepsilon d_\varepsilon(\cdot) - \frac{1}{2} L\right) = \gamma_l$$

in the sense of finite distributions, where as usual, l is the local time at 0 of a BM independent of γ.

[Hint: If $M_t^\varepsilon = \frac{1}{\sqrt{\varepsilon}} \int_0^t \theta_s^\varepsilon dB_s$ and P^ε is the law of $(B_t, L_t, M_t^\varepsilon)$, prove that the set $(P^\varepsilon, \varepsilon > 0)$ is relatively compact.]

* **(2.14) Exercise.** In the notation of this section, if (x_i), $i = 1, \dots, k$ is a sequence of real numbers, prove that $\left(B, \varepsilon^{-1/2}\left(L^{x_i+\varepsilon} - L^{x_i}\right), i = 1, \dots, k\right)$ converges in distribution as $\varepsilon \to 0$, to $\left(B, 2\beta^i_{L^{x_i}}, i = 1, \dots, k\right)$, where $\left(B, \beta^1, \dots, \beta^k\right)$ is a BM^{k+1}.

** **(2.15) Exercise.** Prove, in the notation of this section, that for any $x \in \mathbb{R}$, $\varepsilon^{-1/2}\left[\varepsilon^{-1} \int_0^{\cdot} 1_{[x,x+\varepsilon]}(B_s) ds - L^x\right]$ converges in distribution to $\left(2/\sqrt{3}\right) \beta_{L^x}$, as ε tends to 0. The reader will notice that this is the "central-limit" theorem associated with the a.s. result of Corollary (1.9) in Chap. VI.

[Hint: Extend the result of the preceeding exercise to $\left(L^{x_i+\varepsilon z} - L^{x_i}\right)$ and get a doubly indexed limiting process.]

* **(2.16) Exercise (A limit theorem for the Brownian motion on the unit sphere).** Let Z be a $BM^d(a)$ with $a \neq 0$ and $d \geq 2$; set $\rho = |Z|$. Let V be the process with values in the unit sphere of $\mathbb{R}^d$ defined by

$$Z_t = \rho_t V_{C_t}$$

where $C_t = \int_0^t \rho_s^{-2} ds$. This is the *skew-product* decomposition of BM^d.

1°) Prove that there is a BM^d, say B, independent of ρ and such that

$$V_t = V_0 + \int_0^t \sigma(V_s) dB_s - \frac{d-1}{2} \int_0^t V_s ds$$

where σ is the field of matrices given by

$$\sigma_{ij}(x) = \delta_{ij} - x_i x_j.$$

2°) If $X_t = \int_0^t \sigma(V_s)dB_s$, prove that $\langle X^i, X^j\rangle_t = \langle X^i, B^j\rangle_t$.
[Hint: Observe that $\sigma(x)x = 0$, $\sigma(x)y = y$ if $\langle x, y\rangle = 0$, hence $\sigma^2(x) = \sigma(x)$.]
3°) Show that

$$\lim_{t\to\infty} t^{-1}\langle X^i, B^j\rangle_t = \delta_{ij}(1 - d^{-1}) \qquad \text{a.s.}$$

4°) Prove that the $2d$-dimensional process $\left(c^{-1}B_{c^2t}, (2c)^{-1}\int_0^{c^2t} V_s ds\right)$ converges in distribution, as c tends to ∞, to the process

$$\left(B_t, d^{-1}\left(B_t + (d-1)^{-1/2}B'_t\right)\right)$$

where (B, B') is a BM^{2d}.

§3. Asymptotic Properties of Planar Brownian Motion

In this section, we take up the study of some asymptotic properties of complex BM which was initiated in Sect. 4 of Chap. X. We will use the asymptotic version of Knight's theorem (see the preceding section) which gives a sufficient condition for the DDS Brownian motions of two sequences of local martingales to be asymptotically independent. We will also have to envisage below the opposite situation in which these BM's are asymptotically equal. Thus, we start this section with a sufficient condition to this effect.

(3.1) Theorem. *Let (M_i^n), $i = 1, 2$, be two sequences of continuous local martingales and β_i^n their associated DDS Brownian motions. If $R_n(t)$ is a sequence of processes of time-changes such that the following limits exist in probability*

i) $\lim_n \langle M_1^n, M_1^n\rangle_{R_n(t)} = \lim_n \langle M_2^n, M_2^n\rangle_{R_n(t)} = t,$
ii) $\lim_n \langle M_1^n - M_2^n, M_1^n - M_2^n\rangle_{R_n(t)} = 0,$

then, $\lim_n \sup_{s\le t} \left|\beta_1^n(s) - \beta_2^n(s)\right| = 0$ *in probability.*

Proof. If T_i^n is the time-change associated with $\langle M_i^n, M_i^n\rangle$,

$$\begin{aligned}\left|\beta_1^n(t) - \beta_2^n(t)\right| \le & \left|M_1^n\left(T_1^n(t)\right) - M_1^n\left(R_n(t)\right)\right| + \left|M_1^n\left(R_n(t)\right) - M_2^n\left(R_n(t)\right)\right| \\ & + \left|M_2^n\left(R_n(t)\right) - M_2^n\left(T_2^n(t)\right)\right|\end{aligned}$$

By Exercise (4.14) Chap. IV, for fixed t, the left-hand side converges in probability to zero if each of the terms

$$\langle M_1^n, M_1^n\rangle_{R_n(t)}^{T_1^n(t)}, \quad \langle M_1^n - M_2^n, M_1^n - M_2^n\rangle_{R_n(t)}, \quad \langle M_2^n, M_2^n\rangle_{R_n(t)}^{T_2^n(t)}$$

converges in probability to zero. Since $\langle M_1^n, M_1^n\rangle_{T_1^n(t)} = t$, this follows readily from the hypothesis.

As a result, $\left|\beta_1^n - \beta_2^n\right| \xrightarrow{\text{f.d.}} 0$. On the other hand, Kolmogorov's criterion (1.8) entails that the set of laws of the processes $\beta_1^n - \beta_2^n$ is weakly relatively compact; thus, $\beta_1^n - \beta_2^n \xrightarrow{(d)} 0$. This implies (Exercise (1.11)) that $\sup_{s\le t}\left|\beta_1^n(s) - \beta_2^n(s)\right|$ converges in distribution, hence in probability, to zero. □

The following results are to be compared with Corollary (2.4). We now look for conditions under which the DDS Brownian motions are asymptotically equal.

(3.2) Corollary. *If M_i, $i = 1, 2$, are continuous local martingales and R is a process of time-changes such that the following limits exist in probability*

i) $$\lim_{u\to\infty} \frac{1}{u}\langle M_i, M_i\rangle_{R(u)} = 1 \text{ for } i = 1, 2,$$

ii) $$\lim_{u\to\infty} \frac{1}{u}\langle M_1 - M_2, M_1 - M_2\rangle_{R(u)} = 0,$$

then $\frac{1}{\sqrt{u}}(\beta_1(u\cdot) - \beta_2(u\cdot))$ converges in distribution to the zero process as u tends to infinity.

Proof. By the remarks in Sect. 5 Chap. 0, it is equivalent to show that the convergence holds in probability uniformly on every bounded interval. Moreover, by Exercise (1.17) Chap. V (see the remarks before Corollary (2.4)), $\frac{1}{\sqrt{u}}\beta_i(u\cdot)$ is the DDS Brownian motion of $\frac{1}{\sqrt{u}}M_i$. Thus, we need only apply Theorem (3.1) to $\left(\frac{1}{\sqrt{u}}M_1, \frac{1}{\sqrt{u}}M_2\right)$. □

The above corollary will be useful later on. The most likely candidates for R are mixtures of the time-changes μ_t^i associated with $\langle M^i, M^i\rangle$ and actually the following result shows that $\mu_t^1 \vee \mu_t^2$ will do.

(3.3) Proposition. *The following two assertions are equivalent:*

(i) $$\lim_{t\to\infty} \frac{1}{t}\langle M_1 - M_2, M_1 - M_2\rangle_{\mu_t^1\vee\mu_t^2} = 0 \textit{ in probability};$$

(ii) $$\lim_{t\to\infty} \frac{1}{t}\langle M_1, M_1\rangle_{\mu_t^2} = \lim_{t\to\infty} \frac{1}{t}\langle M_2, M_2\rangle_{\mu_t^1} = 1 \textit{ in probability},$$

and $$\lim_{t\to\infty} \frac{1}{t}\langle M_1 - M_2, M_1 - M_2\rangle_{\mu_t^1\wedge\mu_t^2} = 0 \textit{ in probability}.$$

Under these conditions, the convergence stated in Corollary (3.2) holds.

Proof. From the "Minkowski" inequality of Exercise (1.47) in Chap. IV, we conclude that

$$\left|\left(t^{-1}\langle M_1, M_1\rangle_{\mu_t^2}\right)^{1/2} - 1\right| \le \left(t^{-1}\langle M_1 - M_2, M_1 - M_2\rangle_{\mu_t^2}\right)^{1/2}.$$

By means of this inequality, the proof that i) implies ii) is easily completed.

To prove the converse, notice that

$$\begin{aligned}\langle M_1 - M_2, M_1 - M_2\rangle_{\mu_t^1\wedge\mu_t^2}^{\mu_t^1\vee\mu_t^2} &= \left|\langle M_1 - M_2, M_1 - M_2\rangle_{\mu_t^1} - \langle M_1 - M_2, M_1 - M_2\rangle_{\mu_t^2}\right| \\ &= \left|\langle M_1, M_1\rangle_{\mu_t^2} - \langle M_2, M_2\rangle_{\mu_t^2} + 2\left(\langle M_1, M_2\rangle_{\mu_t^1} - \langle M_1, M_2\rangle_{\mu_t^2}\right)\right|.\end{aligned}$$

Since by Kunita-Watanabe inequality

$$\left|\langle M_1, M_2\rangle_{\mu_t^1} - \langle M_1, M_2\rangle_{\mu_t^2}\right| \leq \left|t - \langle M_1, M_1\rangle_{\mu_t^2}\right|^{1/2} \left|t - \langle M_2, M_2\rangle_{\mu_t^1}\right|^{1/2},$$

the equivalence of ii) and i) follows easily. □

The foregoing proposition will be used under the following guise.

(3.4) Corollary. *If* $\langle M_1, M_1\rangle_\infty = \langle M_2, M_2\rangle_\infty = \infty$ *and*

$$\lim_{t\to\infty} \langle M_1 - M_2, M_1 - M_2\rangle_t / \langle M_i, M_i\rangle_t = 0 \textit{ almost-surely},$$

for $i = 1, 2$, *then the conclusion of Proposition (3.3) holds.*

Proof. The hypothesis implies that μ_t^i is finite and increases to $+\infty$ as t goes to infinity. Moreover

$$\langle M_1 - M_2, M_1 - M_2\rangle_{\mu_t^i} / \langle M_i, M_i\rangle_{\mu_t^i} = \frac{1}{t}\langle M_1 - M_2, M_1 - M_2\rangle_{\mu_t^i},$$

so that condition i) in the Proposition is easily seen to be satisfied. □

From now on, we consider a complex BM Z such that $Z_0 = z_0$ a.s. and pick points $z_1, \ldots, z_p$ in $\mathbb{C}$ which differ from z_0. For each j, we set

$$\chi_t^j = \int_0^t \frac{dZ_s}{Z_s - z_j} = \log\left|\frac{Z_t - z_j}{z_0 - z_j}\right| + i\theta_t^j,$$

where θ_t^j is the continuous determination of $\arg\left(\frac{Z_t - z_j}{z_0 - z_j}\right)$ which vanishes for $t = 0$. The process $(2\pi)^{-1}\theta_t^j$ is the "winding number" of Z around z_j up to time t; we want to further the results of Sect. 4 in Chap. X by studying the simultaneous asymptotic properties of the θ_t^j, $j = 1, \ldots, p$.

Let us set

$$C_t^j = \int_0^t \left|Z_s - z_j\right|^{-2} ds$$

and denote by T_t^j the time-change process which is the inverse of C^j. As was shown in Sect. 2 Chap. V, for each j, there is a complex BM $\zeta^j = \beta^j + i\gamma^j$ such that

$$\chi_t^j = \zeta_{C_t^j}^j, \qquad \beta^j + i\gamma^j = \chi_{T^j}^j.$$

We observe that up to a time-change, β^j is ≤ 0 when Z is inside the disk $D_j = D\left(z_j, |z_0 - z_j|\right)$ and ≥ 0 when Z is outside D_j.

We now recall the notation introduced in Chap. VI before Theorem (2.7). Let β be a standard linear BM and l its local time at 0. We put

$$M_t^+ = \int_0^t 1_{(\beta_s > 0)} d\beta_s, \qquad M_t^- = \int_0^t 1_{(\beta_s < 0)} d\beta_s,$$

and call $\alpha^{\pm}$ the time-changes associated with $\langle M^{\pm}, M^{\pm}\rangle$. Let β^+ and β^- be the positive and negative parts of β and put $\rho_t^{\pm} = \beta^{\pm}_{\alpha_t^{\pm}}$. By the results in Chapter VI, $(\delta^+, \delta^-) = \left(M^+_{\alpha^+}, M^-_{\alpha^-}\right)$ is a planar BM such that

$$\rho_t^{\pm} = \pm\delta_t^{\pm} + \frac{1}{2} l_{\alpha_t^{\pm}}.$$

The process $\delta^{\pm}$ is the DDS Brownian motion of $M^{\pm}$. Moreover, $\rho^{\pm}$ are reflecting BM's, $\left(\rho^{\pm}, \frac{1}{2} l_{\alpha_t^{\pm}}\right)$ have the same law as $(|\beta|, l)$ and

$$\frac{1}{2} l_{\alpha_t^+} = \sup_{s \le t}\left(-\delta_s^+\right), \qquad \frac{1}{2} l_{\alpha_t^-} = \sup_{s \le t}\left(\delta_s^-\right).$$

The processes $l_{\alpha_t^{\pm}}$ are the local times at 0 of $\rho^{\pm}$ (Exercise (2.14) Chap. VI).

(3.5) Proposition. *The processes ρ^+ and ρ^- are independent. Moreover, there are measurable functions f and g from* $\mathbf{W} \times \mathbf{W}$ *to* $\mathbf{W}$ *such that*

$$\beta = f(\rho^+, \rho^-) = g(\delta^+, \delta^-).$$

Proof. The first part follows from the independence of δ^+ and δ^-. To prove the second part, we observe that $\beta_t = \rho^+\left(\langle M^+, M^+\rangle_t\right) + \rho^-\left(\langle M^-, M^-\rangle_t\right)$; thus, it is enough to prove that $\langle M^{\pm}, M^{\pm}\rangle$ are measurable functions of $\rho^{\pm}$. Calling $L^{\pm}$ the local time of $\rho^{\pm}$ at zero, we have

$$l = L^+\left(\langle M^+, M^+\rangle\right) = L^-\left(\langle M^-, M^-\rangle\right).$$

Moreover, as $\langle M^+, M^+\rangle_t + \langle M^-, M^-\rangle_t = t$, one can guess that

$$\langle M^+, M^+\rangle_t = \inf\left\{s : L_s^+ > L_{t-s}^-\right\}$$

which is readily checked. Since ρ^+ and ρ^- are functions of δ^+ and δ^-, the proof is complete.

Remark. To some extent, this is another proof of the fact that Brownian motion may be recovered from its excursion process (Proposition (2.5) Chap. XII), as ρ^+ and ρ^- may be seen as accounting respectively for the positive and negative excursions.

In the sequel, we are going to use simultaneously the above $\pm$ notational pattern for several BM's which will be distinguished by superscripts; the superscripts will be added to the $\pm$. For instance, if β^j is the real part of the process ζ^j defined above

$$\beta^j = g\left(\delta^{j+}, \delta^{j-}\right).$$

The following remark will be important.

(3.6) Lemma. *The process δ^{j+} is the* DDS *Brownian motion of the local martingale*

$$N_t^{j+} = \mathrm{Re} \int_0^t 1_{D_j^c}(Z_s) \frac{dZ_s}{Z_s - z_j}.$$

The same result holds for δ^{j-} with D_j instead of D_j^c and, naturally, the corresponding local martingale will be called N^{j-}.

Proof. It is easily seen that $N_t^{j+} = M_{T_t^j}^{j+}$ and since δ^{j+} is the DDS Brownian motion of M^{j+}, by Exercise (1.17) in Chap. V, it is also the DDS Brownian motion of N^{j+}. □

We now introduce some more notation pertaining to the imaginary part γ^j of ζ^j. We call γ_t^{j+} and γ_t^{j-} the DDS Brownian motions of the local martingales

$$\theta_t^{j+} = \int_0^t 1_{D_j^c}(Z_s) d\theta_s^j, \qquad \theta_t^{j-} = \int_0^t 1_{D_j}(Z_s) d\theta_s^j.$$

As in the previous proof, it is seen that

$$\langle \theta^{j+}, \theta^{j+} \rangle_t = \int_0^{C_t^j} 1_{(\beta_s^j \geq 0)} ds,$$

and, by the same reasoning, γ_t^{j+} is also the DDS Brownian motion of $\int_0^t 1_{(\beta_s^j \geq 0)} d\gamma_s^j$, namely

$$\gamma_t^{j+} = \int_0^{\alpha_t^{j+}} 1_{(\beta_s^j \geq 0)} d\gamma_s^j.$$

The same result holds for γ^{j-} with the obvious changes.

Moreover, it is plain that $\gamma^j = \gamma^{j+}(\langle M^{j+}, M^{j+} \rangle) + \gamma^{j-}(\langle M^{j-}, M^{j-} \rangle)$ so that, by Proposition (3.5), the knowledge of the four processes $(\rho^{j+}, \rho^{j-}, \gamma^{j+}, \gamma^{j-})$ is equivalent to the knowledge of (ρ^j, γ^j).

Our next result will make essential use of the scaling operation. Let us insist that for $h > 0$,

$$X^{(h)}(t) = h^{-1} X(h^2 t).$$

In particular, we denote by $\zeta^{j(h)}$ the Brownian motion

$$\left(\zeta^j\right)^{(h)}(t) = h^{-1} \zeta^j(h^2 t).$$

We must observe that the family $\left(\beta^{j\pm}, M^{j\pm}, \delta^{j\pm}, \rho^{j\pm}\right)$ of processes associated with the planar BM $\zeta^{j(h)}$ by the above scheme is actually equal to

$$\left(\beta^{j\pm(h)}, M^{j\pm(h)}, \delta^{j\pm(h)}, \rho^{j\pm(h)}\right).$$

Indeed, it is obvious for $\beta^{\pm}$ and we have

$$\begin{aligned} \int_0^t 1_{(\beta_s^{j(h)} > 0)} d\beta_s^{j(h)} &= \frac{1}{h} \int_0^t 1_{(\beta_{h^2 s}^j > 0)} d\beta_{h^2 s}^j \\ &= \frac{1}{h} \int_0^{h^2 t} 1_{(\beta_s^j > 0)} d\beta_s^j = M_t^{j+(h)}. \end{aligned}$$

As δ^{j+} is the DDS Brownian motion of M^{j+}, Exercise (1.17) in Chap. V tells us that $\delta^{j+(h)}$ is the DDS Brownian motion of $M_t^{j+(h)}$ which entails our claim in the case of $\delta^{\pm}$. Finally, the claim is also true for $\rho^{\pm}$ since it is a function of $\delta^{\pm}$.

We may now state

(3.7) Theorem. *The $2p$-dimensional process $\left(\zeta^{1(h)}, \dots, \zeta^{p(h)}\right)$ converges in distribution as h tends to infinity to a process $\left(\zeta^{1\infty}, \dots, \zeta^{p\infty}\right)$, the law of which is characterized by the following three properties:*

i) each $\zeta^{j\infty}$ is a complex BM;

ii) if we keep the same notational device as above with the obvious changes, then the processes $\rho^{j+\infty} + i\gamma^{j+\infty}$ are all identical;

iii) if we call $\rho^{+\infty} + i\gamma^{+\infty}$ the common value of the processes $\rho^{j+\infty} + i\gamma^{j+\infty}$, then the processes $\rho^{+\infty} + i\gamma^{+\infty}$, $\rho^{1-\infty} + i\gamma^{1-\infty}, \dots, \rho^{p-\infty} + i\gamma^{p-\infty}$, are independent.

Proof. By Corollary (1.6), the set of laws under consideration is weakly relatively compact. Therefore, all we have to prove is that every limit law satisfies i) through iii) of the statement.

We first observe that property i) is obvious. Next, to prove that the $\rho^{j+\infty}$ are identical, we may as well prove that the $\delta^{j+\infty}$ are identical. Now, by Corollary (5.8) Chap. 0, the processes $\delta^{j+\infty}$ are the limits in distribution of the processes $\delta^{j+(h)}$. Furthermore, by Lemma (3.6), the processes $\delta^{j+(h)}$ are the scaled DDS Brownian motions of the local martingales N^{j+}. Thus, it remains to prove that we can apply Corollary (3.4) to the local martingales N^{j+}. But by Sect. 2 Chap. V

$$\langle N^{j+}, N^{j+}\rangle_t = \int_0^t f^j(Z_s)ds$$

with $f^j(z) = \left|z - z_j\right|^{-2} 1_{D_j^c}(z)$ and likewise

$$\langle N^{j+} - N^{k+}, N^{j+} - N^{k+}\rangle_t = \int_0^t f^{jk}(Z_s)ds$$

with $f^{jk}(z) = \left|\frac{1}{z-z_j} 1_{D_j^c}(z) - \frac{1}{z-z_k} 1_{D_k^c}(z)\right|^2$. As the functions f^{jk} are integrable with respect to the Lebesgue measure in the plane, whereas the functions f^j are not, the ergodic theorem of Sect. 3 Chap. X (see Exercise (3.15) Chap. X) shows that the hypotheses of Corollary (3.4) are satisfied. As a result, the processes $\rho^{j+\infty}$ are identical. The same pattern of proof applies to the processes $\gamma^{j+\infty}$ without any changes. This proves ii).

We now turn to the proof of iii). By the same reasoning as in the proof of ii), it is enough to prove that

$$\left(\delta^{1+}, \gamma^{1+}, \delta^{1-}, \gamma^{1-}, \dots, \delta^{p-}, \gamma^{p-}\right)^{(h)}$$

converges in distribution to a BM^{2p+2}. By Lemma (3.6) and Exercise (1.17) in Chap. V, $\delta^{1+(h)}$ (resp. $\delta^{j-(h)}$) is the DDS Brownian motion of $\frac{1}{h}N^{1+}$ (resp. $\frac{1}{h}N^{j-}$) and likewise $\gamma^{1+(h)}$ (resp. $\gamma^{j-(h)}$) is the DDS Brownian motion of $\frac{1}{h}\theta^{1+}$ (resp. $\frac{1}{h}\theta^{j-}$). Thus, we need only apply Corollary (2.4) to the local martingales

$$\left(N^{1+}, \theta^{1+}, N^{1-}, \theta^{1-}, \ldots, N^{p-}, \theta^{p-}\right).$$

Let M be any of these martingales; then, as in the first part of the proof

$$\langle M, M\rangle_t = \int_0^t f(Z_s)ds$$

for a function f which is not integrable with respect to the Lebesgue measure. On the other hand, if M, N are two local martingales of the above list

$$\langle M, N\rangle_t = \int_0^t f(Z_s)ds$$

where, this time, f is integrable. For instance, for N^{j-} and N^{k-}, we get

$$f(z) = \left(z - z_j, z - z_k\right) \left|z - z_j\right|^{-2} |z - z_k|^{-2} 1_{D_j \cap D_k}(z)$$

which is integrable since

$$|f(z)| \leq \left|z - z_j\right|^{-1} |z - z_k|^{-1} 1_{D_j \cap D_k}(z);$$

the other cases are either trivial or similar. In any case, it is easily deduced from the ergodic theorem (see Exercise (3.15) Chap. X) that the hypotheses of Corollary (2.4) are satisfied. This completes the proof. □

The foregoing theorem allows to generalize Theorem (4.2) of Chap. X to several points. As in there, A_t will be an additive functional and we will assume that $\|A\| = 2\pi$.

(3.8) Theorem. *As t goes to infinity,*

$$\frac{2}{\log t}\left(\left(\theta_t^{j+}, \theta_t^{j-}\right), j = 1, \ldots, p, A_t\right)$$

converges in distribution to $\left((W^+, W^{j-}), j = 1, \ldots, p, \Lambda\right)$ where, for each j, the triple $\left(W^+, W^{j-}, \Lambda\right)$ has the law described in Theorem (4.2) of Chap. X, and, conditionally on Λ, the $p+1$ variables $\left(W^+, W^{j-}, j = 1, \ldots, p\right)$ are independent.

Proof. From Theorem (4.2) in Chap. X, we know that for each j, $\frac{2}{\log t}(\theta_t^{j+}, \theta_t^{j-}, A_t)$ converges to $(W^{j+}, W^{j-}, \Lambda)$; thus what we have to prove is the relationship between these triples when j varies, that is between W^{j+} and W^{j-}, $j = 1, \ldots, p$, given Λ. In the remark after Theorem (4.2) Chap. X, we pointed out that

$$\frac{2}{\log t}\left(\theta_t^{j+}, \theta_t^{j-}\right) - \left(\int_0^{T^j_{(\log t/2)}} 1_{(\beta_s^j \geq 0)} d\gamma_s^j, \int_0^{T^j_{(\log t/2)}} 1_{(\beta_s^j \leq 0)} d\gamma_s^j\right)$$

where $T_a^j = \inf\{t : \beta_t^j = a\}$, converges in probability to zero. With each planar BM $Z = X + iY$ we associate a bidimensional r.v. $W(Z)$ by setting

$$W(Z) = \left(\int_0^{T_1} 1_{(X_s \geq 0)} dY_s, \int_0^{T_1} 1_{(X_s \leq 0)} dY_s\right)$$

where $T_1 = \inf\{t : X_t = 1\}$. Thanks to the scaling properties of the family T_a^j, it is not hard to see that

$$\frac{2}{\log t}\left(\int_0^{T^j_{(\log t/2)}} 1_{(\beta_s^j \geq 0)} d\gamma_s^j, \int_0^{T^j_{(\log t/2)}} 1_{(\beta_s^j \leq 0)} d\gamma_s^j\right) = W\left(\zeta^{j(h)}\right)$$

if $h = \frac{1}{2}\log t$. By another application of Corollary (5.8) Chap. 0, it follows that $\frac{2}{\log t}((\theta_t^{j+}, \theta_t^{j-}), j = 1, \ldots, p)$ converges in distribution to $(W(\zeta^{j\infty}), j = 1, \ldots, p)$.

As a result, the r.v.'s W^{j+} which depend on $\beta^{j+\infty}$ alone are all equal to the same variable W^+. For the same reason, $T_1^{j\infty}$ does not depend on j. Furthermore, conditionally on A, each W^{j-} is independent of W^+, hence of $T_1^{j\infty}$, and becomes a function of $\rho^{j-\infty} + i\gamma^{j-\infty}$ alone. The independence follows from Theorem (3.7). □

We now record the asymptotic distribution for the windings θ^j themselves.

(3.9) Corollary. *The limiting distribution of* $\left(\frac{2}{\log t}\theta_t^j, j = 1, 2, \ldots, p\right)$ *is the law of* $\left(W_j = W^+ + W^{j-}, j = 1, \ldots, p\right)$ *which may be described as follows:*

i) $W^{j-} = HY_j$, *where*
ii) the r.v.'s Y_j *are independent Cauchy variables with parameter* 1 *which are also independent of the pair* (W^+, H);
iii) the Laplace-Fourier transform of the pair (W^+, H) *is given by*

$$E\left[\exp(-aH + ivW_+)\right] = [\cosh v + (a/v)\sinh v]^{-1}.$$

Proof. This is a reformulation of Corollary (4.4) in Chap. X with $H = A/2$. □

In Theorem (4.2) of Chap. X, we saw that the result is independent of the radius of the disk used to distinguish between "small" and "big" windings. In this section, we have used, for convenience sake, the disks D_j of radius $|z_0 - z_j|$, but it is, likewise, inessential. This is implied by the next result which will also be used in the proof of the last theorem of this section.

(3.10) Proposition. *If f is locally bounded and square-integrable with respect to the 2-dimensional Lebesgue measure, then*

$$\frac{1}{\log t} M_t = \frac{1}{\log t}\int_0^t f(Z_s) dZ_s$$

converges in probability to zero as t goes to infinity.

Proof. Since M is conformal,

$$\langle \mathrm{Re}M, \mathrm{Re}M\rangle_t = \langle \mathrm{Im}M, \mathrm{Im}M\rangle_t = \int_0^t |f|^2(Z_s)ds$$

and we know that $\frac{1}{\log t}\int_0^t |f|^2(Z_s)ds$ converges in distribution to a finite r.v. It follows that $\langle \mathrm{Re}M, \mathrm{Re}M\rangle_t/(\log t)^2$ converges in probability to zero and by Exercise (4.14) in Chap. IV, $\mathrm{Re}M_t/(\log t)$ and $\mathrm{Im}M_t/(\log t)$ converge in probability to zero. □

Remark. The assumption that f is locally bounded is only made to ensure that $\int_0^t |f|^2(Z_s)ds$ is P_0-a.s. finite for every $t > 0$.

The foregoing discussion entails further asymptotic results. We keep the same setting and notation and we write $\mathrm{Res}(f, a)$ for the residue of f at a.

(3.11) Theorem. *Let f be holomorphic in $\mathbb{C}\setminus\{z_1, \dots, z_p\}$ and Γ an open, relatively compact set such that $\{z_1, \dots, z_p\} \subset \Gamma$; then*

$$\frac{2}{\log t}\int_0^t f(Z_s)1_\Gamma(Z_s)dZ_s \xrightarrow[t\to\infty]{(d)} \sum_{j=1}^p \mathrm{Res}(f, z_j)\left\{\frac{\Lambda}{2} + iW^{j-}\right\}.$$

If f is moreover holomorphic at infinity with $\lim_{z\to\infty} f(z) = 0$, then

$$\frac{2}{\log t}\int_0^t f(Z_s)dZ_s$$

converges in distribution as $t \to \infty$, to

$$\sum_{j=1}^p \mathrm{Res}(f, z_j)\left\{\frac{\Lambda}{2} + iW^{j-}\right\} + \mathrm{Res}(f, \infty)\left\{\frac{\Lambda}{2} - 1 + iW^+\right\}.$$

Proof. By the preceding Proposition, we may as well suppose that Γ is the union of disjoint disks $\Gamma_j = D(z_j, \varepsilon_j)$ with $\varepsilon_j > 0$ and sufficiently small, so that we look for the limit of $\sum_{j=1}^p F_t^j$ with $F_t^j = \frac{2}{\log t}\int_0^t f(Z_s)1_{\Gamma_j}(Z_s)dZ_s$. Within Γ_j we may write $f(z) = h_j(z) + g_j\left(\frac{1}{z-z_j}\right)$ with h_j holomorphic in a neighborhood of Γ_j and

$$g_j(z) = \mathrm{Res}(f, z_j)z + \tilde{g}_j(z)z^2$$

for an entire function $\tilde{g}_j$. We set

$$H_t^j = \frac{2}{\log t}\int_0^t 1_{\Gamma_j}(Z_s)\tilde{g}_j\left(\frac{1}{Z_s - z_j}\right)\frac{dZ_s}{(Z_s - z_j)^2}.$$

By Proposition (3.10),

$$F_t^j - \mathrm{Res}(f, z_j)\left\{\frac{2}{\log t}\int_0^t 1_{\Gamma_j}(Z_s)\frac{dZ_s}{Z_s - z_j}\right\} - H_t^j$$

converges to zero in probability. We moreover claim that H_t^j converges to zero in probability.

Let $\widetilde{G}_j$ be the antiderivative of $\tilde{g}_j$ vanishing at 0. By Itô's formula for conformal martingales,

$$\widetilde{G}_j\left(\frac{1}{Z_t - z_j}\right) = \widetilde{G}_j\left(\frac{1}{z_0 - z_j}\right) - \int_0^t \tilde{g}_j\left(\frac{1}{Z_s - z_j}\right)\frac{dZ_s}{(Z_s - z_j)^2}.$$

Since Z_t converges to infinity in probability, the left hand side converges to $\widetilde{G}_j(0) = 0$ in probability. As a result

$$\frac{2}{\log t}\int_0^t \tilde{g}_j\left(\frac{1}{Z_s - z_j}\right)\frac{dZ_s}{(Z_s - z_j)^2}$$

converges to 0 in probability. But the real part of the conformal martingale

$$\int_0^t \tilde{g}_j\left(\frac{1}{Z_s - z_j}\right) 1_{\Gamma_j^c}(Z_s)\frac{dZ_s}{(Z_s - z_j)^2}$$

has a bracket equal to $\int_0^t \phi(Z_s)ds$ where

$$\phi(z) = \left|\tilde{g}_j\left((z - z_j)^{-1}\right)\right|^2 \left|z - z_j\right|^{-4} 1_{\Gamma_j^c}(z)$$

is integrable. By the same reasoning as in Proposition (3.10), our claim is proved.

The first statement is then an easy consequence of Theorem (3.8) and of Proposition (4.6) of Chap. X.

For the second statement, we write $f(z) = -\mathrm{Res}(f, \infty)\frac{1}{z} + \frac{1}{z^2}g(1/z)$ for $|z| \geq \eta$ and g holomorphic in a neighborhood of $\{|z| \leq 1/\eta\}$ where η has been chosen sufficiently large. We have to add to the previous limit that of

$$\frac{2}{\log t}\left\{-\mathrm{Res}(f, \infty)\int_0^t 1_{(Z_s \geq \eta)}\frac{dZ_s}{Z_s} + \int_0^t 1_{(Z_s \geq \eta)} g\left(\frac{1}{Z_s}\right)\frac{dZ_s}{Z_s^2}\right\}.$$

This first part converges in distribution to $-\mathrm{Res}(f, \infty)\left(\frac{A}{2} - 1 + iW^+\right)$ thanks to Theorem (3.8) and to Proposition (4.6) of Chap. X and the second part converges in probability to zero by the same reasoning as for $\tilde{g}_j$ above. □

* **(3.12) Exercise.** Let n be an integer and let τ_t be the time-change inverse of

$$\int_0^t |nZ_s|^{2((1/n)-1)} ds.$$

Prove that, with the notation of this section

$$\frac{2}{\log t}\left(\theta_{\tau_t}^1, \dots, \theta_{\tau_t}^p\right)$$

converges in law to $n\left(W^{1-} + W^+, \dots, W^{p-} + W^+\right)$.

[Hint: Use Theorem (3.7).]

** **(3.13) Exercise (Mutual windings).** Let $B^1, \dots, B^p$ be p complex BM's on a filtered probability space $(\Omega, \mathscr{F}, \mathscr{F}_t, P)$ which are correlated as follows: for every k and l, $k \neq l$, there exists a matrix $A_{k,l}$ such that for every u and v in $\mathbb{R}^2 (\simeq \mathbb{C})$,

$$\left(u, B_t^k\right)\left(v, B_t^l\right) - \left(u, A_{k,l} v\right) t, \quad t \geq 0,$$

is a martingale.

1°) Show that, if for every $k \neq l$, the matrix $A_{k,l}$ is not an orthogonal matrix and if $B_0^i \neq 0$ a.s. for every i, then

$$\frac{2}{\log t}\left(\theta_t^i, i \leq p\right) \xrightarrow[t \to \infty]{(d)} \left(C_i, i \leq p\right)$$

where the C_i's are independent Cauchy r.v.'s with parameter 1.

[Hint: Show that for $i \neq j$, $\int_0^t \left|d\langle \theta^i, \theta^j \rangle_s\right| / \log t$, $\int_0^t \left|d\langle \log|B^i|, \theta^j \rangle_s\right| / \log t$ and $\int_0^t \left|d\langle \log|B^i|, \log|B^j| \rangle_s\right| / \log t$ are bounded in probability as $t \to \infty$.]

2°) Let B be a BM^3 and $D^1, \dots, D^p$, p different straight lines which intersect at zero. Assume that B_0 is a.s. not in D^i for every i. Define the winding numbers θ_t^i, $i \leq p$, of B around D^i, $i \leq p$. Show that as a consequence of the previous question, the same convergence in law as in 1°) holds.

3°) Let $B^1, \dots, B^n$, be n independent planar BM's such that $B_0^i \neq B_0^j$ a.s. Call $\theta_t^{i,j}$ the winding number of $B^i - B^j$ around 0. Show that

$$\frac{2}{\log t}\left(\theta_t^{i,j}; 1 \leq i < j \leq n\right) \xrightarrow[t \to \infty]{(d)} \left(C^{i,j}; 1 \leq i < j \leq n\right)$$

where the $C^{i,j}$'s are independent Cauchy r.v.'s with parameter 1.

Notes and Comments

Sect. 1. For the basic definitions and results, as well as for those in Sect. 5 Chap. 0, we refer to the books of Billingsley [1] and Parthasarathy [1]. A more recent exposition is found in the book of Jacod and Shiryaev [1]. Our proof of Donsker's theorem (Donsker [1]) as well as Exercise (1.14) are borrowed from the former. This theorem is constantly being used as a tool to obtain properties of Brownian motion which have first been remarked on its random walk skeletons. This method, which we have refrained from using in this book, is, for instance, found in Pitman [1] and Le Gall [5]. It is interesting to note that conversely some original limit laws on random walks can only be understood in terms of Brownian motion as is seen in the work of Le Gall [7] completing former work of Jain and Pruitt.

Proposition (1.10) is taken from Stroock-Varadhan [1]; although it is of marginal importance in our development it is fundamental in theirs and, as is more generally the case with Martingale problems, has been extended to many situations.

For Exercise (1.13), see Nualart [1] and Yor [14]. Exercise (1.15) is due to Pagès [1] and Exercise (1.16) is inspired by Rebolledo ([1] and 2]).

Exercises (1.17) and (1.18) were suggested respectively by J. Pitman and L. Dubins. Exercise (1.17) allows to simplify the proofs of some limit theorems found in Getoor-Sharpe [4] (see also Jeulin [2] page 128).

Sect. 2. The main result of this section is due to Papanicolaou et al. [1], but their proof was different. The asymptotic version of Knight's theorem comes from Pitman-Yor [5]; another proof is found in Le Gall-Yor [2] and Exercise (2.9) is a variation on the same theme.

Kasahara and Kotani [1] have studied the same problem as Papanicolaou et al. [1] in the case of BM^2. We also refer to Kasahara [4] to whom Exercise (2.13) is due. Biane [3] unifies the asymptotic limit theorems for (multiple) additive functionals of several Brownian motions in $\mathbb{R}^d$.

A number of extensions of Exercises (2.11) through (2.15) have been obtained in recent years by Berman, Borodin [1] and in particular Rosen in the case of stable Lévy processes.

The SDE presentation of the Brownian motion on the sphere found in Exercise (2.16), is a very particular case of that given in Lewis [1] and Van den Berg-Lewis [1]; more generally, see Rogers-Williams [1] and Elworthy [1] for constructions of Brownian motions on surfaces.

Sect. 3. This section is entirely taken from Pitman and Yor ([4], [5] and [7]). Exercise (3.13) is taken from Yor [22] who answers a question of Mitchell Berger. The result in question 2°) of this exercise was originally obtained in a different manner in Le Gall-Yor [3]. More general asymptotic studies for the windings of BM^3 around curves in $\mathbb{R}^3$ are obtained in Le Gall-Yor [4]; the computation of the characteristic functions of the limit laws led the authors to some extension of the Ray-Knight theorems for Brownian local times, presented in Le Gall-Yor [5]; see also the Notes and Comments on Sect. 4 of Chap. XII.

Knight [11] and Yamazaki [1] give convergence results in the sense of fdd's which are closely related to what is called "log-scaling laws", namely limit theorems such as

$$\theta(\exp \lambda u)/u \xrightarrow[u\to\infty]{f\cdot d} F(\lambda)$$

found in Pitman-Yor [7].

Another extension of these results is provided by Watanabe [5] who studies asymptotics of Abelian differentials along Brownian paths on a Riemann surface.

Supplementing these multidimensional limits in law, there are also deep investigations of the pathwise behavior of multiwindings, such as for example of their speed of transience originating with Lyons-McKean [1] and continuing with Gruet [1], Gruet-Mountford [1] and Mountford [1].

We also mention that limit theorems for a large class of diffusions, including the Jacobi processes (see, e.g., Warren-Yor [1]) are developed in Hu-Shi-Yor [1]. These limit theorems are closely related to the asymptotics of diffusion processes in random environments (Kawazu-Tanaka [1], Tanaka [4]).

Intensive discussions of recent studies on the geometry of the planar Brownian curve are found in Le Gall [9] and Duplantier et al. [1].

Appendix

§1. Gronwall's Lemma

Theorem. *If ϕ is a positive locally bounded Borel function on $\mathbb{R}_+$ such that*

$$\phi(t) \leq a + b\int_0^t \phi(s)ds$$

for every t and two constants a and b, then $\phi(t) \leq ae^{bt}$. If in particular $a = 0$ then $\phi \equiv 0$.

Proof. Plainly,

$$\begin{aligned}\phi(t) &\leq a + b\left(\int_0^t \left(a + b\int_0^s \phi(u)du\right)ds\right)\\ &= a + abt + b^2\int_0^t (t-u)\phi(u)du \leq a + abt + b^2 t\int_0^t \phi(u)du.\end{aligned}$$

Proceeding inductively one gets in this fashion

$$\phi(t) \leq a + abt + \ldots + ab^n\frac{t^n}{n!} + \frac{b^{n+1}t^n}{n!}\int_0^t \phi(u)du.$$

Since ϕ is locally bounded, the last term on the right converges to zero as n tends to infinity and the result follows.

§2. Distributions

Let U be a fixed open set in $\mathbb{R}^d$. We denote by C_K^∞ the space of infinitely differentiable functions on U which have a compact support contained in U.

(2.1) Definition. *A sequence (ϕ_n) in C_K^∞ is said to converge to an element ϕ of C_K^∞ if the supports of the ϕ_n's are contained in a fixed compact subset of U and if the k-th derivatives of $\phi_n - \phi$ converge uniformly to zero for every $k \geq 0$.*

(2.2) Definition. *A distribution T on U is a linear form on C_K^∞ such that $T(\phi_n)$ converges to 0 whenever (ϕ_n) is a sequence in C_K^∞ which converges to zero as n tends to infinity.*

We will also write $\langle T, \phi \rangle$ for the value taken by the distribution T on the function ϕ of C_K^∞. With every Radon measure μ on U, we associate a distribution T_μ by setting

$$\langle T_\mu, \phi \rangle = \int \phi d\mu.$$

Likewise, if f is a locally integrable Borel function we write T_f for T_μ where $\mu(dx) = f(x)dx$; in other words

$$\langle T_f, \phi \rangle = \int \phi(x) f(x) dx.$$

(2.3) Definition. *If T is a distribution and $\partial^\alpha / \partial x_1^{\alpha_1} \ldots \partial x_d^{\alpha_d}$ a partial derivation operator, we define the corresponding* partial derivative *of T by setting for $\phi \in C_K^\infty$*

$$\left\langle \frac{\partial^\alpha T}{\partial x_1^{\alpha_1} \ldots \partial x_d^{\alpha_d}}, \phi \right\rangle = (-1)^{|\alpha|} \left\langle T, \frac{\partial^\alpha \phi}{\partial x_1^{\alpha_1} \ldots \partial x_d^{\alpha_d}} \right\rangle$$

where $|\alpha| = \alpha_1 + \ldots \alpha_d$.

This obviously defines another distribution and in the case of T_f above, if f is $|\alpha|$ times continuously differentiable, then

$$\frac{\partial^\alpha T_f}{\partial x_1^{\alpha_1} \ldots \partial x_d^{\alpha_d}} = T_g,$$

where

$$g = \frac{\partial^\alpha f}{\partial x_1^{\alpha_1} \ldots \partial x_d^{\alpha_d}} .$$

§3. Convex Functions

We recall that a real-valued function f defined on an open interval I of $\mathbb{R}$ is *convex* if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y),$$

for every $0 \le t \le 1$ and $x, y \in I$. It follows from this definition that for fixed x, the ratio $(f(y) - f(x))/(y - x)$ increases with y. This, in turn, entails immediately that in each point x the function f has a left-hand derivative $f'_-(x)$ and a right-hand derivative $f'_+(x)$ and that, for $y > x$

$$f'_+(x) \le \frac{f(y) - f(x)}{y - x} \le f'_-(y).$$

We moreover have the

(3.1) Proposition. *The functions f'_- and f'_+ are increasing, respectively left and right-continuous and the set $\{x : f'_-(x) \ne f'_+(x)\}$ is at most countable.*

Proof. Since $f'_-(x) \le f'_+(x)$, the first property follows at once from the above inequality. To prove that f'_+ is right-continuous, we interchange increasing limits to the effect that if $a_n \downarrow 0$ and $b_m \downarrow 0$,

$$\begin{aligned}\lim_{n\to\infty} f'_+(x+a_n) &= \lim_{n\to\infty}\left(\lim_{m\to\infty}\frac{f(x+a_n+b_m)-f(x+a_n)}{b_m}\right)\\ &= \lim_{m\to\infty}\frac{f(x+b_m)-f(x)}{b_m} = f'_+(x).\end{aligned}$$

Finally, f'_- and f'_+ have only countably many discontinuities and where f'_- is continuous, we have $f'_- = f'_+$ thanks to the above inequalities. □

We now study the second derivative of f. If f is C^2, then f'' is positive, as is easily seen. More generally, we have the

(3.2) Proposition. *The second derivative f'' of f in the sense of distributions is a positive Radon measure; conversely, for any Radon measure μ on $\mathbb{R}$, there is a convex function f such that $f'' = \mu$ and for any interval I and $x \in \mathring{I}$,*

$$\begin{aligned}f(x) &= \frac{1}{2}\int_I |x-a|\mu(da) + \alpha_I x + \beta_I\\ f'_-(x) &= \frac{1}{2}\int_I \operatorname{sgn}(x-a)\mu(da) + \alpha_I\end{aligned}$$

where α_I and β_I are constants and sgn $x = 1$ *if $x > 0$ and -1 if $x \le 0$.*

Proof. Let $\phi \in C_K^\infty$; the derivative Df of f in the sense of distributions is given by

$$\begin{aligned}\langle Df, \phi\rangle &= -\int f(x)\phi'(x)dx = -\int\left(\lim_{\varepsilon\to 0}\frac{\phi(x+\varepsilon)-\phi(x)}{\varepsilon}\right)f(x)dx\\ &= -\lim_{\varepsilon\to\infty}\int \phi(x)\left(\frac{f(x-\varepsilon)-f(x)}{\varepsilon}\right)dx = \int \phi(x)f'_-(x)dx.\end{aligned}$$

By the integration by parts formula for Stieltjes integrals, the second derivative is the measure associated with the increasing function f'_-. Of course by the above results, we could have used f'_+ instead of f'_- without altering the result.

Conversely, if $I \subset J$ the integrals

$$\frac{1}{2}\int_I |x-a|\mu(da) \qquad \text{and} \qquad \frac{1}{2}\int_J |x-a|\mu(da)$$

are convex on $\mathring{I}$ and differ by an affine function. As a result one can define a convex function f on the whole line such that on $\mathring{I}$

$$f(x) = \frac{1}{2}\int_I |x-a|\mu(da) + \alpha_I x + \beta_I.$$

An application of Lebesgue's theorem yields, for $x \in \overset{\circ}{I}$,

$$f'_-(x) = \frac{1}{2} \int_I \operatorname{sgn}(x - a)\mu(da) + \alpha_I$$

and if ϕ is a test function with support in $\overset{\circ}{I}$, then

$$\int f'_-(x)\phi'(x)dx = \int_I \mu(da) \left(\frac{1}{2} \int_{-\infty}^{+\infty} \phi'(x)\operatorname{sgn}(x - a)dx \right) = - \int \phi(a)\mu(da)$$

which proves that the second derivative of f is μ.

The convex function determined by μ is of course unique only up to addition of an affine function. If the measure μ is such that $\int |x - a|\mu(da)$ is finite for every x, which will in particular be the case if μ has compact support, then one can globally state that

$$f(x) = \frac{1}{2} \int |x - a|\mu(da) + \alpha x + \beta.$$

The constants α and β can be fixed by specifying special values for f in two points. If in particular for $a < b$ we demand that $f(a) = f(b) = 0$ then one can give for f a more compact expression which we now describe in a slightly more general setting.

Let s be a continuous, strictly increasing function on $I = [a, b]$. We will say that f is s-convex if for $a < c_1 < x < c_2 < b$,

$$(s(c_2) - s(c_1)) f(x) \leq (s(c_2) - s(x)) f(c_1) + (s(x) - s(c_1)) f(c_2).$$

Exactly as above one can define the right and left s-derivatives $df_\pm/ds$ by taking the appropriate limits of the ratios $(f(y) - f(x))/(s(y) - s(x))$. At the points where they are equal we say that f has an s-derivative. The functions thus defined are increasing and determine as above a measure μ.

If for $x \leq y$ we set

$$G(x, y) = G(y, x) = (s(x) - s(a))(s(b) - s(y))/(s(b) - s(a)),$$

then if f is s-convex and if $f(a) = f(b) = 0$,

$$f(x) = - \int_a^b G(x, y)\mu(dy).$$

Indeed, using the integration by parts formula for Stieltjes integrals,

$$\begin{aligned} \int_a^b G(x, y)\mu(dy) = & \frac{s(b) - s(x)}{s(b) - s(a)} \int_{]a,x]} (s(y) - s(a))\mu(dy) \\ & + \frac{s(x) - s(a)}{s(b) - s(a)} \int_{]x,b[} s(b) - s(y))\mu(dy) \end{aligned}$$

$$= \frac{s(b)-s(x)}{s(b)-s(a)}\left[(s(x)-s(a))\frac{df_+}{ds}(x) - \int_a^x \frac{df_+}{ds}(y)ds(y)\right]$$
$$+ \frac{s(x)-s(a)}{s(b)-s(a)}\left[-(s(b)-s(x))\frac{df_+}{ds}(x) + \int_x^b \frac{df_+}{ds}(y)ds(y)\right]$$
$$= -f(x).$$

Naturally, all we have said is valid for concave functions with the obvious changes.

§4. Hausdorff Measures and Dimension

Let h be a strictly increasing continuous function on $\mathbb{R}_+$ such that $h(0) = 0$ and $h(\infty) = \infty$. Let B be a Borel subset of a metric space E. The *Hausdorff h-measure* of B is the number

$$\Lambda^h(B) = \liminf_{\varepsilon \downarrow 0}\left(\sum_n h(|I_n|)\right)$$

where the infimum is over all coverings $\bigcup I_n$ of B where I_n is a closed set in E with diameter $|I_n| \leq \varepsilon$. Of special interest is the case where $h(t) = t^\alpha$, $\alpha > 0$, in which case we will write Λ^α and speak of α-measure. If $E = \mathbb{R}^d$, Λ^d is the ordinary Lebesgue measure.

(4.1) Lemma. *If* $h(t) = \nu(t)k(t)$ *with* $\lim_{t\to 0}\nu(t) = 0$, *then* $m_h(F) > 0$ *implies* $m_k(F) = \infty$.

Proof. Pick $\varepsilon > 0$; there is an $\eta > 0$ such that $v \leq \eta$ implies $\nu(v) \leq \varepsilon$. Let $\bigcup I_n$ be a covering of F with $|I_n| \leq v \leq \eta$. Then

$$\sum_n k(|I_n|) = \sum_n h(|I_n|)/\nu(|I_n|) \geq \frac{1}{\varepsilon}\sum_n h(|I_n|);$$

it follows that

$$\sum_n k(|I_n|) \geq \frac{1}{\varepsilon} m_h(F),$$

hence

$$m_k(F) \geq \frac{1}{\varepsilon} m_h(F),$$

and since ε is arbitrary, the proof is complete.

A consequence of this lemma is that there is a number α_0 such that $\Lambda^\alpha(F) = +\infty$ if $\alpha < \alpha_0$ and $\Lambda^\alpha(F) = 0$ if $\alpha > \alpha_0$ (the number $\Lambda^{\alpha_0}(F)$ itself may be zero, non zero and finite or infinite). The number α_0 is called the *Hausdorff dimension* of F. For instance, one can prove that the dimension of the Cantor "middle third" set is $\log 2/\log 3$.

§5. Ergodic Theory

Let $(E, \mathscr{E}, m)$ be a σ-finite measure space. A positive contraction T of $L^1(m)$ is a linear operator on $L^1(m)$ with norm ≤ 1 and mapping positive (classes of) functions into positive (classes of) functions. A basic example of such a contraction is the map $f \to f \circ \theta$ where θ is a measurable transformation of $(E, \mathscr{E})$ which leaves m invariant.

(5.1) Theorem. (Hopf's decomposition theorem). *There is an m-essentially unique partition $C \cup D$ of E such that for any $f \in L^1_+(m)$*

i) $\sum_{k=0}^{\infty} T^k f = 0$ *or* $+\infty$ *on* C,

ii) $\sum_{k=0}^{\infty} T^k f < \infty$ *on* D.

If $D = \emptyset$, the contraction T is said to be *conservative*. In that case, the sums $\sum_{k=0}^{\infty} T^k f$ for $f \in L^1_+(m)$ take on only the values 0 and $+\infty$. The sets $\left\{\sum_0^{\infty} T^k f = \infty\right\}$ where f runs through $L^1_+(m)$ form a σ-algebra denoted by $\mathscr{C}$ and called the *invariant* σ-algebra. If all these sets are either $\emptyset$ or E (up to equivalence) or in other words if $\mathscr{C}$ is m-a.e. trivial then T is called *ergodic*.

We now state the basic *Chacon-Ornstein theorem*.

(5.2) Theorem. *If T is conservative and g is an element of $L^1_+(m)$ such that $m(g) > 0$, then for every $f \in L^1(m)$,*

$$\lim_{n\to\infty} \left(\sum_0^n T^k f \Big/ \sum_0^n T^k g \right) = E[f \mid \mathscr{C}]/E[g \mid \mathscr{C}] \qquad m\text{-a.e.}$$

The conditional expectations on the right are taken with respect to m. If m is unbounded this means that the quotient is equal to $E[(f/h) \mid \mathscr{C}]/E[(g/h) \mid \mathscr{C}]$ where h is a strictly positive element in $L^1(m)$ and the conditional expectations are taken with respect to the bounded measure $h \cdot m$; it can be shown that the result does not depend on h. If T is ergodic the quotient on the right is simply $m(f)/m(g)$.

The reader is referred to Revuz [3] for the proof of these results.

§6. Probabilities on Function Spaces

Let E be a Polish space and set $\Omega = C(\mathbb{R}_+, E)$. Let us call X the canonical process and set $\mathscr{F}_t = \sigma(X_s, s \leq t)$ and $\mathscr{F}_\infty = \sigma(X_s, s \geq 0)$.

(6.1) Theorem. *If for every $t \geq 0$, there exists a probability measure P^t on $\mathscr{F}_t$ such that for every $s < t$, P^t coincides with P^s on $\mathscr{F}_s$, then there exists a probability measure P on $\mathscr{F}_\infty$ which for every t coincides with P^t on $\mathscr{F}_t$.*

For the proof of this result the reader can refer to the book of Stroock and Varadhan [1] p. 34; see also Azéma-Jeulin [1].

§7. Bessel Functions

The modified Bessel function I_ν is defined for $\nu \geq -1$ and $x > 0$, by

$$I_\nu(x) = \sum_{k=0}^{\infty} (x/2)^{2k+\nu}/k!\Gamma(\nu + k + 1).$$

Observe that for $\nu = -1$ and $k = 0$ the term $\Gamma(\nu + k + 1)$ is infinite, and therefore the first term in the above series vanishes. By using the relationship $\Gamma(z + 1) = z\Gamma(z)$, one thus sees that $I_1 = I_{-1}$. For some details about these functions we refer the reader to Lebedev [1], pages 108–111.

This family of functions occurs in many computations of probability laws. Call for instance $d_x^{(\nu)}$ the density of a random variable with conditional law $\gamma_{\nu+k+1}$ where k is random with a Poisson law of parameter $x > 0$ and $\nu > -1$. Then

$$\begin{aligned} d_x^{(\nu)}(y) &= \sum_{k=0}^{\infty} e^{-x} \left(x^k/k!\right) y^{\nu+k} e^{-y}/\Gamma(\nu + k + 1) \\ &= e^{-(x+y)}(y/x)^{\nu/2} I_\nu \left(2\sqrt{xy}\right). \end{aligned}$$

Replacing x and y by $x/2t$ and $y/2t$ we find that, for $\nu > -1$,

$$q_t^{(\nu)}(x, y) = (1/2t)\exp(-(x + y)/2t)(y/x)^{\nu/2} I_\nu \left(\sqrt{xy}/t\right)$$

where $t > 0$, $x > 0$, $y > 0$, is also a probability density, in fact the density of $\text{BESQ}^{(\nu)}$ as found in section 1, Chapter IX. At that point we needed to know the Laplace transform of this density which is easily found from the above. Indeed, the Laplace transform of γ_k is equal to $(\lambda + 1)^{-k}$ and therefore the Laplace transform of $d_x^{(\nu)}$ is equal to

$$\sum_{k=0}^{\infty} e^{-x}(x^k/k!)(\lambda + 1)^{-(\nu+k+1)} = (\lambda + 1)^{-(\nu+1)} \exp(-\lambda x/(\lambda + 1)).$$

From this, using the same change of variables as before, one gets that the Laplace transform of $q_t^{(\nu)}(x, \cdot)$ is equal to

$$(2\lambda t + 1)^{-(\nu+1)} \exp(-\lambda x/(2\lambda t + 1)).$$

Another formula involving Bessel functions and which was of interest in Sect.3, Chap. VIII, is the following. If $x \in \mathbb{R}^d$ we call $\xi(x)$ the angle of Ox with a fixed axis, and if μ^d is the uniform probability measure on the unit sphere S^{d-1}, then

$$\int_{S^{d-1}} \exp(\rho \cos \xi(x))\mu^d(dx) = (2/\rho)^\nu \Gamma(\nu + 1) I_\nu(\rho/2)$$

where $\nu = (d/2) - 1$. This can be proved directly from the definition of I_ν by writing the exponential as a series and computing $\int_{S^{d-1}} \cos \xi(x)^p \mu^d(dx)$; to this end, it is helpful to use the duplication formula

$$\Gamma(2z) = (2\pi)^{-1/2} 2^{2z-(1/2)} \Gamma(z)\Gamma(z + (1/2)).$$

§8. Sturm-Liouville Equation

Let μ be a positive Radon measure on $\mathbb{R}$. Then there exists a unique, positive, decreasing function ϕ_μ such that

$$(*) \qquad \phi_\mu(0) = 1, \qquad \phi''_\mu = \phi_\mu \mu,$$

where ϕ''_μ is the second derivative in the sense of distributions (Appendix 2). Observe from (*) that since ϕ_μ is positive, it is convex, and ϕ''_μ is equal to $d\phi'_\mu$ where ϕ'_μ is the right derivative of ϕ_μ (Appendix 3).

To prove this existence and uniqueness result we transform (*) into the Riccati equation

$$(+) \qquad g(x) = 1 + \mu(]a, x]) - \int_a^x g^2(y)dy,$$

where $a \in \mathbb{R}$. We claim that this equation has a unique solution g on $[a, \infty[$ which satisfies the inequality $g(x) \geq 1/(1 + x - a)$, for $x \geq a$.

Indeed, since the function $x \to x^2$ is locally Lipschitz, there is a unique maximal solution to (+) on an interval $[a, \alpha[$ with $\alpha > a$. Obviously g is of finite variation on every bounded interval. We will first prove that g is > 0 on $[a, \alpha[$. Indeed, suppose that $g(x-) \leq 0$ for some $x \in [a, \alpha[$ and set $\gamma = \inf\{x \in [a, \alpha[: g(x-) \leq 0\}$. For $a \leq x < \gamma$, we have

$$(\ddagger) \qquad g(x) = g(x-) + \mu(\{x\}) > 0.$$

On the other hand, by Proposition (4.6) Chap. 0, we may write

$$\begin{aligned} 1/g(x) &= 1 - \int_{]a,x]} (g(y)g(y-))^{-1} dg(y) \\ &= 1 + (x - a) - \int_{]a,x]} (g(y)g(y-))^{-1} d\mu(y) \leq 1 + (x - a). \end{aligned}$$

As a result, $g(\gamma) \geq 1/(1 + \gamma - a) + \mu(\{\gamma\}) > 0$, and since g is right-continuous this contradicts the definition of γ. That g is > 0 then follows from ($\ddagger$).

Now, since g is > 0 on $[a, \alpha[$, if α is finite, rewriting (+) as

$$g(x) + \int_a^x g^2(y)dy = 1 + \mu(]a, x]),$$

we see that g is bounded on $]a, \alpha[$ and by letting x increase to α we get

$$g(\alpha-) + \int_a^\alpha g^2(y)dy = 1 + \mu(]a, \alpha[).$$

If we set $g(\alpha) = g(\alpha-) + \mu(\{\alpha\})$ and solve the equation (+) for $x > \alpha$, we see that α cannot be maximal. As a result $\alpha = \infty$ and we have proved our claim.

Next, if g is the solution to (+), we set, for $x \geq a$,

$$\psi(x) = \exp\left(\int_a^x g(y)dy\right).$$

One sees rapidly that $\psi(x) \geq 1 + x - a$ on $[a, \infty[$ and that $\psi'' = \psi\mu$. We further set

$$\phi(x) = \psi(x) \int_x^\infty \psi(y)^{-2} dy.$$

The function ϕ is > 0 and is another solution to the equation $\phi'' = \phi\mu$. Moreover, because ψ' is increasing, we have

$$\begin{aligned} \phi'(x) &= \psi'(x) \int_x^\infty \psi(y)^2 dy - (1/\psi(x)) \\ &\leq \int_x^\infty \left(\psi'(y)/\psi(y)^2\right) dy - (1/\psi(x)) = 0, \end{aligned}$$

which shows that ϕ is decreasing.

The space of solutions to the equation $\phi'' = \phi\mu$ is the space of functions $u\psi + v\phi$ with $u, v \in \mathbb{R}$. Since ψ increases to $+\infty$ at infinity, the only positive bounded solutions are of the form $v\phi$ with $v \geq 0$. If for $a < 0$ we put $\phi_\mu = \phi/\phi(0)$ we get the unique solution to (*) that we were looking for.

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