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Limit Theorems for Randomly Stopped Stochastic Processes

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Abstract. The book contains a survey of basic results related to weak convergence of random variables and **J**-convergence of càdlàg stochastic processes. General conditions of weak convergence of randomly stopped stochastic processes and weak convergence of compositions of stochastic processes are presented as well as functional limit on **J**-convergence of compositions of càdlàg stochastic processes. Applications to random sums, extremes with random sample size, generalised exceeding processes, sum-processes with renewal stopping, accumulation processes, max-processes with renewal stopping and shock processes are given. The book also contains an extended bibliography of works in the area.

Key words and phrases. Càdlàg stochastic process, random stopping, composition of càdlàg processes, weak convergence, functional limit theorem, **J**-topology, **U**-topology, **M**-topology, random sum, extreme with random sample size, generalised exceeding process, exceeding time process, renewal process, accumulation process, sum-process with renewal stopping, max-process with renewal stopping, sum-max process, shock process.

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Contents

Preface[1]

Limit theorems for stochastic processes are an important part of probability theory and mathematical statistics.

One model that has attracted the attention of many researchers in this area is that of limit theorems for randomly stopped stochastic processes and for compositions of stochastic processes.

This model can appear in a natural way: for example, when studying limit theorems for additive or extremal functionals of stochastic processes; in models connected with a random change of time, change point problems and problems related to optimal stopping of stochastic processes; and in different renewal models, particularly those which appear in applications for risk processes, queuing systems, etc.

The model also appears in statistical applications connected with studies of samples with a random sample size. Such sample models play an important role in sequential analysis. They also appear in sample survey models, or in statistical models where sample variables are associated with stochastic flows. The latter models are typical for insurance, queueing and reliability applications, as well as many others.

A large number of works in the area is devoted to studies of limit theorems for randomly stopped stochastic processes and compositions of stochastic processes under assumptions, which imply independence or asymptotical independence of external processes and internal stopping moments. In general limit theorems, the assumption of asymptotical independence is replaced by the condition of joint weak convergence of external processes and internal stopping moments. These theorems are oriented to be applied to models with dependent external processes and internal stopping moments.

The first book on this subject was published by the author in 1974. Since that time many new results and applications have been developed by the author and other researchers. At the moment there is no book that would provide a 'state of the art' reflection of general limit theorems for randomly stopped stochastic processes. These realities have stimulated me to begin work on a new book on this subject that should fill the gap in the existing literature.

The aim of this book is to present general limit theorems about weak convergence of randomly stopped stochastic processes and compositions of stochastic processes as well

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as functional limit theorems about convergence of compositions of càdlàg stochastic processes in topologies **U** and **J**.

This book contains four chapters. Chapter 1 is a survey of basic results related to weak convergence of random variables and stochastic processes, including basic facts concerning the convergence of càdlàg processes in topologies **J** and **U**. In Chapter 2, general conditions of weak convergence of randomly stopped stochastic processes and compositions of càdlàg processes are presented. In Chapter 3, functional limit theorems about convergence of compositions of càdlàg processes in topologies **J** and **U** are given. Chapter 4 presents a summary of applications to random sums, extremes with random sample size, generalised exceeding processes, sum-processes with renewal stopping, accumulation processes, max-processes with renewal stopping, and shock processes.

Many results included in the book are published for the first time. In particular, these include limit theorems for randomly stopped processes and compositions of stochastic processes based on new weaken continuity conditions as well as their applications to generalised exceeding processes. Other new results are indicated in the reference remarks at the end of each chapter.

The bibliography, which contains more than 750 references, is also supplemented with short bibliographic remarks.

The presentation of material in the chapters is organised in a way that I hope will be appreciated by readers. Each chapter has a preamble in which the main results are outlined and the chapter content (by sections) is presented. The first section of each chapter contains introductory remarks. Here models, basic conditions and results are introduced in an informal way. In addition, examples and counter-examples, illustrated by figures if possible, are given along with comments. Each section is broken up into titled subsections. Subsections containing formulations and proofs of main theorems are given first. These are followed by subsections that present various modifications to the main theorems and their conditions. The reference remarks, at the end of each chapter, highlight the origins of main results as well as indicate new results.

I would like also to comment on the notation system used in the book. Throughout the text I make use of several basic classes of conditions. Conditions that belong to a specific class are denoted by the same letter. For example, the letter $\mathcal{A}$ is used for all weak convergence conditions, the letter $\mathfrak{B}$ for continuity conditions, and so forth. Conditions belonging to a specific class have subscripts numbering conditions in the class. A list of all conditions is given in a special index. Local conditions used in theorems, lemmas, definitions or remarks are indicated by small Greek letters in brackets as **(α)**, **(β)**, etc. Local conditions in the text can be indicated as **(a)**, **(b)**, etc. This indication always acts within the limits of a *subsection* where these conditions are introduced. Subsections, theorems, lemmas, definitions and remarks have a triple numeration. For example, Theorem 1.2.3 means Theorem 3 of Section 1.2. Formulas also have a triple numeration. For example, label (1.2.3) refers to formula 3 in Section 1.2.

I hope that the publication of this new book and the comprehensive bibliography of works related to limit theorems for randomly stopped stochastic processes will be a useful contribution to the continuing intensive studies in this area. In addition to research and reference purposes, the book can be used in special courses on the subject and as a complementary reading in general courses on stochastic processes. In this respect, the book may be useful for specialists as well as doctoral and advanced undergraduate students.

I would like to thank Dr Evelina Silvestrova for her continuous encouragement and support of various aspects of my work on this book.

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Dmitrii Silvestrov

Chapter 1

Weak convergence of stochastic processes

In this chapter we present a survey of results concerning weak convergence of random variables in metric spaces and functional limit theorems for càdlàg processes. These results form a basis for the study of limit theorems for randomly stopped stochastic processes and compositions of stochastic processes.

As usual, we first present general results concerning weak convergence of random variables in the space $\mathbb{R}_m$ and in a Polish space. Then we give basic results concerning functional limit theorems for càdlàg processes. As is known, the space **D** of càdlàg functions can be equipped with a metric in such a way that convergence in this metric is equivalent to convergence in the topology **J**. Correspondingly, càdlàg processes can be considered as random variables taking values in **D**, and their convergence in the topology **J** can be regarded as weak convergence of these random variables. We also describe an alternative approach to functional theorems for càdlàg processes, which is based on the Skorokhod representation theorem. According to this theorem, if random variables that take values in a Polish space weakly converge, then one can construct new random variables with the same distributions that converge with probability 1. Using this theorem one can often reduce the corresponding functional limit theorems to simpler analogues of these theorems for non-random càdlàg functions.

Section 1.1 contains examples and introductory comments. In Sections 1.2 and 1.3, general results concerning weak convergence of random variables that take values in the space $\mathbb{R}_m$ and in a Polish space are formulated. Section 1.4 gives general facts concerning the space **D** of càdlàg functions. Section 1.5 describes the main classes of **J**-continuous functionals. In Section 1.6, the main limit theorems concerning **J**-convergence of càdlàg stochastic processes are formulated. The last subsection also contains bibliographical remarks.

It is necessary to note that Chapter 1 contains only a survey of the corresponding results. The proofs are omitted in most cases. I refer to the well known books by Billingsley (1968, 1999), Gikhman and Skorokhod (1965, 1971), Pollard (1984), Ethier and Kurtz (1986), and Jacod and Shiryaev (1987), which give a full presentation of the theory. These books also contain bibliographies on works in the area.

1.1 Introductory remarks

In this section, some examples that clarify the concept of weak convergence for random variables are considered. The concept of weak convergence is introduced for the simplest case of real-valued random variables. Possible ways to generalise this concept to the case of general metric spaces, in particular, to the spaces **C** of continuous functions, and **D** of càdlàg functions are also discussed.

1.1.1. Weak convergence of random variables. Let $\xi_\varepsilon, \varepsilon \geq 0$ be a family of real-valued random variables depending on a parameter $\varepsilon \geq 0$. We denote by $F_\varepsilon(x) = \mathsf{P}\{\xi_\varepsilon \leq x\}$, $x \in \mathbb{R}_1$, the *distribution function* of a random variable ξ_ε.

The concept of weak convergence plays a central role in probability theory and its applications. It is enough to recall that the fundamental limit theorems such as the weak law of large numbers and the central limit theorem are, actually, statements about weak convergence of random variables.

We say that random variables ξ_ε *weakly converge* to a random variable ξ_0 as $\varepsilon \to 0$ if $F_\varepsilon(x) \to F_0(x)$ as $\varepsilon \to 0$ for all points x which are points of continuity for the limiting distribution function. This is denoted by $\xi_\varepsilon \Rightarrow \xi_0$ as $\varepsilon \to 0$.

Weak convergence of random variables is, actually, a convergence of their distribution functions. That is why we can also talk about weak convergence of distribution functions $F_\varepsilon(\cdot)$, instead of random variables ξ_ε, and to use the notation $F_\varepsilon(\cdot) \Rightarrow F_0(\cdot)$ as $\varepsilon \to 0$.

To distinguish the weak convergence of random variables from the weak convergence of their distributions, the term *convergence in distribution* could be used instead of the term *weak convergence*, when one talks about weak convergence of random variables. In such a case, the notation $\xi_\varepsilon \xrightarrow{\mathrm{d}} \xi_0$ as $\varepsilon \to 0$ would be more appropriate. However, we prefer to use the term weak convergence and the symbol $\Rightarrow$ in both cases. Usually, it is obvious what objects (random variables or distribution functions) are considered in the corresponding relation of weak convergence.

It is also useful to note that random variables can be indexed in different ways. For example, a sequence of random variables ξ_n that depends on the index $n = 1, 2, \ldots$ can be an object of consideration. The notation of weak convergence is modified in an obvious way: $\xi_n \Rightarrow \xi_0$ as $n \to \infty$.

The definition of weak convergence gives rise to the following question. Why is the pointwise convergence of distribution functions required only in points of continuity of the corresponding limiting distribution function?

The following standard example explains why should points of discontinuity be excluded from the set of pointwise convergence. Let us consider a sequence of numbers a_n, $n = 1, 2, \ldots$, such that $a_n \to a_0$ as $n \to \infty$, where a_0 is a finite real constant. The constant a_n can be considered as a random variable. It is natural to expect that weak convergence $a_n \Rightarrow a_0$ as $n \to \infty$ would be equivalent to the usual convergence $a_n \to a_0$

as $n \to \infty$. The distribution function of a_n, considered as a random variable, is the indicator function $F_n(x) = \chi_{[a_n,\infty)}(x)$. It is easy to check that $a_n \to a_0$ as $n \to \infty$ if and only if $F_n(x) \to F_0(x)$ as $n \to \infty$ for all $x \neq a_0$, i.e., $F_n(x) \to 0$ as $n \to \infty$ for $x < a_0$ and $F_n(x) \to 1$ as $n \to \infty$ for $x > a_0$. Note that convergence of $F_n(x)$ to $F_0(x)$ in the point a_0, which is the only point of discontinuity of the limiting distribution function $F_0(x)$, is not required to provide convergence of a_n to a_0. If also, for example, a_n is a decreasing sequence, then $F_n(a_0) = 0$ for all $n = 1, 2, \ldots$, but $F_0(a_0) = 1$. Therefore, convergence of a_n to a_0 does not imply convergence of $F_n(a_0)$ to $F_0(a_0)$.

It should also be noted that weak convergence of random variables is equivalent to the usual pointwise convergence of their distribution functions in all points $x \in \mathbb{R}_1$, if the limiting distribution function is continuous.

The definition of weak convergence given above can easily be extended from random variables to random vectors, i.e., random variables that take values in the space $\mathbb{R}_m$. In this case, one-dimensional distribution functions should be replaced by the corresponding multi-dimensional distribution functions. One can use the definition of week convergence as pointwise convergence in points of continuity of the corresponding limiting multi-dimensional distribution function.

However, if the random variables take values in a metric space, the definition of weak convergence should be modified. It can happen that direct analogues of the distribution functions do not exist. In this case, the definition can be given with the use of convergence of values of the probability measures generated by the random variables. Convergence should be required for values of these measures on sets of continuity for the corresponding limiting measure.

1.1.2. Extension of convergence to Borel sets. As well known, any distribution function $F_\varepsilon(x) = \mathsf{P}\{\xi_\varepsilon \leq x\}$ uniquely determines a measure $F_\varepsilon(A)$ on the σ-algebra $\mathfrak{B}_1$ of Borel subsets of $\mathbb{R}_1$. By the definition, $F_\varepsilon(A) = \mathsf{P}\{\xi_\varepsilon \in A\}$. In particular, $F_\varepsilon(x) = F_\varepsilon((-\infty, x])$.

The following natural question arises. Does the weak convergence $F_\varepsilon(\cdot) \Rightarrow F_0(\cdot)$ as $\varepsilon \to 0$ implies convergence of $F_\varepsilon(A)$ to $F_0(A)$ as $\varepsilon \to 0$ for all $A \in \mathfrak{B}_1$?

In some special cases, the answer is affirmative. For example, let ξ_n, $n = 0, 1, 2, \ldots$ be a sequence of discrete random variables which take values k with probabilities $p_n(k)$ for $k = 0, 1, \ldots$. It is easy to show that the random variables $\xi_n \Rightarrow \xi_0$ as $n \to \infty$ if and only if $p_n(k) \to p_0(k)$ as $n \to \infty$ for every $k = 0, 1, \ldots$. In this case, we also have $\mathsf{P}\{\xi_n \in A\} \to \mathsf{P}\{\xi_0 \in A\}$ as $n \to \infty$ for any Borel set A.

In the general case, the answer is negative. Let us consider the following example. Let η_n, $n = 1, 2, \ldots$, be a sequence of random variables which have geometrical distributions with parameters $p_n = 1/n$. So, the random variable η_n takes a value k with probability $p_n(1 - p_n)^{k-1}$ for $k = 1, 2, \ldots$. It is easy to show that the random variables $\xi_n = p_n\eta_n \Rightarrow \xi_0$ as $n \to \infty$, where ξ_0 is a random variable that has exponential distribution with parameter 1. Let us define the set $A_0 = \{k/n\colon k, n = 1, 2, \ldots\}$. The set

A_0 is a countable Borel set. Obviously, $\mathsf{P}\{\xi_n \in A_0\} = 1$ for every $n = 1, 2, \ldots$. But $\mathsf{P}\{\xi_0 \in A_0\} = 0$, since the limiting exponential distribution is continuous. That is why the probabilities $\mathsf{P}\{\xi_n \in A_0\}$ do not converge to $\mathsf{P}\{\xi_0 \in A_0\}$ as $n \to \infty$.

Let us denote by ∂A the *boundary* of a Borel set A, i.e., the set of all points x such that any interval $(x - \delta, x + \delta)$ contains points which belong to both sets A and $\overline{A}$. Obviously, the class of all Borel sets A with $F_0(\partial A) = 0$ (the sets of continuity for the measure $F_0(A)$) is a σ-algebra.

It can be shown that weak convergence $F_\varepsilon(\cdot) \Rightarrow F_0(\cdot)$ as $\varepsilon \to 0$ implies that $F_\varepsilon(A) \to F_0(A)$ as $\varepsilon \to 0$ for all sets of continuity of the measure $F_0(A)$. Obviously, $\partial(-\infty, x] = \{x\}$. An interval $(-\infty, x]$ is a set of continuity for the measure $F_0(A)$ if and only if x is a continuity point for the distribution function $F_0(x)$.

That is why $F_\varepsilon(A) \to F_0(A)$ as $\varepsilon \to 0$ for all sets of continuity of the measure $F_0(A)$ if and only if $F_\varepsilon(\cdot) \Rightarrow F_0(\cdot)$ as $\varepsilon \to 0$.

This is one of the key general statements concerning weak convergence. It can be used to define weak convergence of random variables that take values in a metric space.

1.1.3. Subsequence approach to weak convergence. Let $F_\varepsilon(x), \varepsilon \geq 0$ be a family of distribution functions depending on parameter $\varepsilon \geq 0$.

It follows from the definition of weak convergence that, in order to prove that distribution functions $F_\varepsilon(x)$ weakly converge as $\varepsilon \to 0$, one can use the following subsequence approach.

First, an arbitrary subsequence $\varepsilon_n \to 0$ as $n \to \infty$ should be selected. Second, it should be shown that a subsequence $\varepsilon'_k = \varepsilon_{n_k}$ can be selected from the first subsequence such that $F_{\varepsilon'_k}(\cdot) \Rightarrow F(\cdot)$, where $F(x)$ is a distribution function. Third, it should be shown that the distribution function $F(x) \equiv F_0(x)$ does not depend on the choice of subsequences ε_n and ε'_k. Then $F_\varepsilon(\cdot) \Rightarrow F_0(\cdot)$ as $\varepsilon \to 0$.

Let R be a subset of $\mathbb{R}_1$. A set $S \subseteq R$ is *dense* in R, if $\inf_{y \in S} |x - y| = 0$ for every $x \in R$.

Let us choose a set S dense in $\mathbb{R}_1$. Note that S can be a countable subset of $\mathbb{R}_1$. Due to continuity from the right, any distribution function $F(x)$ is completely determined by its values in points of the set S. In this sense, S can be referred to as a *defining set*. Let now $\varepsilon_n \geq 0$ be an arbitrary sequence such that $\varepsilon_n \to 0$ as $n \to \infty$. Using Cantor's diagonal method it is always possible to find a subsequence $\varepsilon'_k = \varepsilon_{n_k}$ such that $F_{\varepsilon'_k}(x) \to F(x)$ as $k \to \infty$ for all $x \in S$, where the limits $F(x) \in [0, 1]$. The function $F(x)$, defined on the set S, is non-decreasing. Using this fact one can always define this function at every point $x \in \mathbb{R}_1 \setminus S$ as the right limit of the values $F(x_k)$ for some sequence of points $x_k \in S$, $x_k > x$, $x_k \to x$. The function $F(x)$, defined on $\mathbb{R}_1$ in this way, is non-decreasing, continuous from the right, and $F(x) \in [0, 1]$ for every $x \in \mathbb{R}_1$. So, it is a distribution function. However, it can be an improper distribution function, i.e., it can be such that $F(+\infty) - F(-\infty) < 1$, where $F(\pm\infty) = \lim_{x \to \pm\infty} F(x)$. For example, let $F_\varepsilon(x) = \chi_{[a_\varepsilon, \infty)}(x)$, where $a_\varepsilon \to a_0 = \infty$ as $\varepsilon \to 0$. In this case, $F_\varepsilon(x) \to F_0(x) \equiv 0$ as $\varepsilon \to 0$.

In order to prove that $F(x)$ is a proper distribution function (for any subsequences ε_n and ε'_k chosen as is described above), one should require that the initial family of distribution functions $F_\varepsilon(x)$, $\varepsilon \geq 0$ be stochastically bounded as $\varepsilon \to 0$:

$\mathcal{K}_1$: $\lim_{x\to\infty} \overline{\lim}_{\varepsilon\to 0}(F_\varepsilon(-x) + 1 - F_\varepsilon(x)) = 0.$

Condition $\mathcal{K}_1$ implies that, for any subsequence $\varepsilon_n \to 0$ as $n \to \infty$, a subsequence $\varepsilon'_k = \varepsilon_{n_k}$ can be selected from the first subsequence in such a way that the distribution functions $F_{\varepsilon'_k}(\cdot) \Rightarrow F(\cdot)$ as $k \to \infty$, where $F(x)$ is a proper distribution function.

Now, let us also require convergence of distribution functions $F_\varepsilon(x)$ in points of the defining set S:

$\mathcal{A}_1$: $F_\varepsilon(x) \to F_0(x)$ as $\varepsilon \to 0$ for $x \in S$.

Note that limits in $\mathcal{A}_1$ are some numbers from the interval $[0, 1]$. The function $F_0(x)$, defined in $\mathcal{A}_1$, is automatically non-decreasing. But it is not required that the corresponding limits of $F_0(x)$, as x tends to $-\infty$ or $+\infty$, be equal to 0 and 1, respectively. The function $F_0(x)$ can be continued to the whole real line, as it was described above, by using right limits. It is a proper or improper distribution function.

Condition $\mathcal{A}_1$ implies, obviously, that $F(x) = F_0(x)$, $x \in S$. Since S is a defining set, $F(x) = F_0(x)$, $x \in \mathbb{R}_1$. So, the distribution function $F(x)$ does not depend on the choice of the subsequences ε_n and ε'_k.

Summarising the remarks made above one can conclude that, in order to prove weak convergence of distribution functions $F_\varepsilon(\cdot)$ as $\varepsilon \to 0$ it is sufficient to assume that both conditions $\mathcal{K}_1$ and $\mathcal{A}_1$ hold.

Moreover, it can be easily shown that conditions $\mathcal{K}_1$ and $\mathcal{A}_1$ are not only sufficient but also necessary for weak convergence.

All the remarks made above can be repeated in the case where random variables take values in $\mathbb{R}_m$.

Moreover, the method of proof of weak convergence described above can be generalised and effectively used when dealing with weak convergence in metric spaces. The corresponding theory was developed by Prokhorov (1956).

1.1.4. Skorokhod representation theorem. Let us consider a family of random variables $\xi_\varepsilon, \varepsilon \geq 0$ depending on a parameter $\varepsilon \geq 0$. Let us assume that the random variables ξ_ε are defined on the same probability space $(\Omega, \mathfrak{F}, \mathsf{P})$ for all $\varepsilon \geq 0$ and ξ_ε *converge a.s.* (almost sure) to ξ_0 as $\varepsilon \to 0$, i.e., $\mathsf{P}\{\,\omega\colon \lim_{\varepsilon\to 0} \xi_\varepsilon(\omega) = \xi_0(\omega)\,\} = 1$. It can easily be shown that, in this case, the random variables ξ_ε weakly converge to ξ_0 as $\varepsilon \to 0$.

The inverse implication does not need to hold. For example, let $\xi_n = \xi_0$ if $n = 1, 3, \ldots$, and $\xi_n = 1 - \xi_0$ if $n = 2, 4, \ldots$, where ξ_0 is a random variable uniformly distributed on $[0, 1]$. In this case, the random variables ξ_n converge to ξ_0 weakly but not a.s., as $n \to \infty$.

Note also that weak convergence of random variables is actually a convergence of their distribution functions. For this reason, random variables ξ_ε can be defined on different probability spaces for different ε. This is a typical situation, when one considers the so-called triangular array models. In such cases, the random variables can converge weakly but not almost sure.

However, if $\xi_\varepsilon \Rightarrow \xi_0$ as $\varepsilon \to 0$, then it is possible to construct a probability space $(\Omega, \mathfrak{F}, \mathsf{P})$ and random variables $\tilde{\xi}_\varepsilon$, $\varepsilon \geq 0$, defined on this space, such that **(a)** for every $\varepsilon \geq 0$ the random variables $\tilde{\xi}_\varepsilon$ and ξ_ε have the same distribution, and **(b)** the random variables $\tilde{\xi}_\varepsilon$ a.s. converge to $\tilde{\xi}_0$ as $\varepsilon \to 0$.

Let, for example, a random variable ξ_ε have an exponential distribution with parameter $\lambda_\varepsilon > 0$, i.e., $\mathsf{P}\{\xi_\varepsilon \leq x\} = F_\varepsilon(x) = 1 - \exp\{-\lambda_\varepsilon x\}$ for $x \geq 0$. Let also $\lambda_\varepsilon \to \lambda_0 > 0$ as $\varepsilon \to 0$. In this case, it is obvious that $\xi_\varepsilon \Rightarrow \xi_0$ as $\varepsilon \to 0$. Let us consider the function $F_\varepsilon^{-1}(y) = (-1/\lambda_\varepsilon)\log(1-y)$, which is the inverse of the exponential distribution function $F_\varepsilon(x)$ introduced above. Let also ρ be a random variable uniformly distributed on $[0,1]$. Let us now consider the random variables $\tilde{\xi}_\varepsilon = -(1/\lambda_\varepsilon)\log(1-\rho)$. It is easy to check that the random variable $\tilde{\xi}_\varepsilon$ has the exponential distribution with parameter $\lambda_\varepsilon > 0$. So, for every $\varepsilon \geq 0$, the random variables ξ_ε and $\tilde{\xi}_\varepsilon$ have the same distribution. Also, the random variables $\tilde{\xi}_\varepsilon$ a.s. converge to $\tilde{\xi}_0$ as $\varepsilon \to 0$. This is so, because $(-1/\lambda_\varepsilon)\log(1-y) \to (-1/\lambda_0)\log(1-y)$ as $\varepsilon \to 0$ for every $y \in [0,1)$ and $\mathsf{P}\{\rho \in [0,1)\} = 1$.

In the case of real-valued random variables, the above construction can be realised in a similar way. Let $F_\varepsilon(x)$ be a distribution function of a random variable ξ_ε, and $F_\varepsilon^{-1}(y) = \inf(x\colon F_\varepsilon(x) > y)$ for $y \in [0,1]$. Let also ρ be a random variable uniformly distributed on $[0,1]$. For example, we can use the probability space with the space of outcomes $[0,1]$, the Borel σ-algebra of random events, and the Lebesgue measure as the corresponding probability measure. Then we can define $\rho(\omega) = \omega$. As is known, the random variable $\tilde{\xi}_\varepsilon = F_\varepsilon^{-1}(\rho)$ has the distribution function $F_\varepsilon(x)$. It is not difficult to show that the pointwise convergence of $F_\varepsilon(x)$ to $F_0(x)$ as $\varepsilon \to 0$ (in all points of continuity of the limiting distribution function $F_0(x)$) implies that their inverses, $F_\varepsilon^{-1}(y)$, pointwise converge to $F_0^{-1}(y)$ as $\varepsilon \to 0$ (in all points of continuity of the limiting function $F_0^{-1}(y)$). Since this function is monotone, it has at most a countable set of discontinuity points. So, the set C_0' of continuity points of this function has Lebesgue measure 1, i.e., $\mathsf{P}\{\rho \in C_0'\} = 1$. Obviously, $\{\rho \in C_0'\} \subseteq \{\lim_{\varepsilon\to 0} \tilde{\xi}_\varepsilon = \tilde{\xi}_0\}$. Hence, $\mathsf{P}\{\lim_{\varepsilon\to 0} \tilde{\xi}_\varepsilon = \tilde{\xi}_0\} = 1$.

The Skorokhod representation theorem generalises this result to random variables that take values in a complete separable metric space. This generalisation is not trivial. The theorem allows to simplify proofs of some important limit theorems.

1.1.5. Weak convergence of transformed random variables. One of the important statements connected with weak convergence deals with weak convergence of transformed random variables.

Let real-valued random variables $\xi_\varepsilon \Rightarrow \xi_0$ as $\varepsilon \to 0$. Let also $f(x)$ be a measurable real-valued function (the inverse image of any Borel set is a Borel set) defined on the real line. In this case, $f(\xi_\varepsilon)$ is also a real-valued random variable. The question arises if the random variables $f(\xi_\varepsilon) \Rightarrow f(\xi_0)$ as $\varepsilon \to 0$?

It can be shown that this is true for all measurable functions $f(x)$ that are a.s. continuous with respect to the distribution of the limiting random variable ξ_0.

This statement can be easily proved by using the Skorokhod representation theorem. Indeed, let $f(x)$ be a function that is a.s. continuous with respect to the distribution of the limiting random variable ξ_0. Then the random variables $f(\tilde{\xi}_\varepsilon)$ a.s. converge to $f(\tilde{\xi}_0)$ as $\varepsilon \to 0$. Since a.s. convergence implies weak convergence, the random variables $f(\tilde{\xi}_\varepsilon)$ converge weakly to $f(\tilde{\xi}_0)$ as $\varepsilon \to 0$. But the random variables $f(\tilde{\xi}_\varepsilon)$ and $f(\xi_\varepsilon)$ have the same distribution, since this is so for the random variables $\tilde{\xi}_\varepsilon$ and ξ_ε. Therefore, the random variables $f(\xi_\varepsilon)$ converge weakly to $f(\xi_0)$ as $\varepsilon \to 0$.

This statement plays a very important role in the general theory of weak convergence of random variables in metric spaces. In the case of functional metric spaces, random variables that take values in such spaces are, actually, stochastic processes. While the corresponding transformed random variables are functionals defined on trajectories of these processes.

1.1.6. Weak convergence in the spaces of continuous and càdlàg functions. Let us consider the space $\mathbf{C}_{[0,1]}$ of real-valued continuous functions defined on the interval $[0, 1]$. This space can be equipped with a uniform metric that transforms the space $\mathbf{C}_{[0,1]}$ in a metric space,

$$d_U(x(\cdot), y(\cdot)) = \sup_{0 \le t \le 1} |x(t) - y(t)|.$$

Let now $\boldsymbol{\xi}_n = \{\xi_n(t), t \in [0, 1]\}$ be a continuous stochastic process for every $n = 0, 1, \ldots$. One can consider $\boldsymbol{\xi}_n$ as a random variable taking values in the functional metric space $\mathbf{C}_{[0,1]}$. Weak convergence of such random variables (stochastic processes) is a subject of the so-called *functional limit theorems* for continuous stochastic processes.

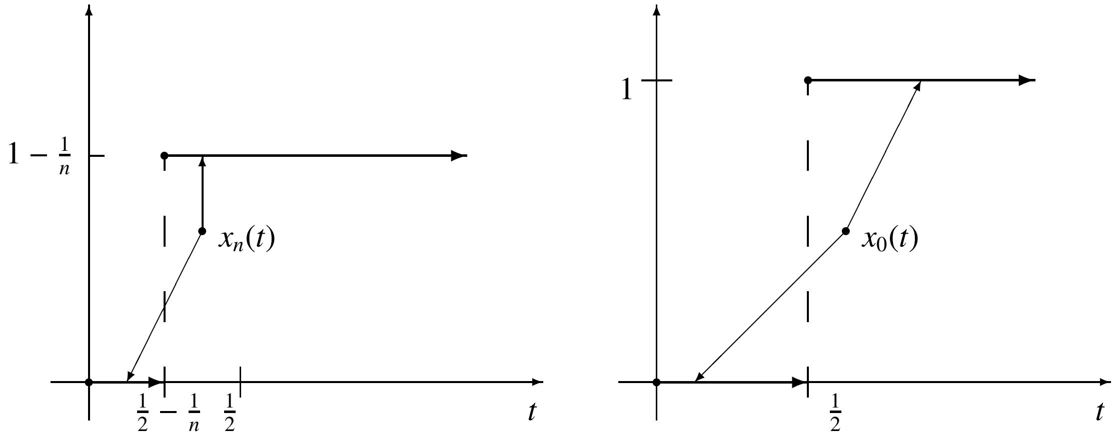
An approach to functional limit theorems for continuous stochastic processes, which is based on their reduction to weak limit theorems for random variables taking values in the functional metric space $\mathbf{C}_{[0,1]}$, was developed by Prokhorov (1956).

However, the class of continuous stochastic processes does not include many important stochastic processes. For example, general processes with independent increments have discontinuous trajectories.

An appropriate space for discontinuous stochastic processes is the space $\mathbf{D}_{[0,1]}$ of real-valued càdlàg functions defined on the interval $[0, 1]$, i.e., functions that are continuous from the right and possessing finite left limits at all points of the interval $(0, 1]$.

The uniform metric $d_U(x(\cdot), y(\cdot))$ is not an appropriate metric for the space $\mathbf{D}_{[0,1]}$, since some sequences of càdlàg functions, which would be expected to converge, do not converge.

For example, let us consider the functions $x_n(t) = (1 - \frac{1}{n})\chi(\frac{1}{2} - \frac{1}{n} \le t), t \in [0, 1]$. These functions converge pointwise to the limiting function $x_0(t) = \chi(\frac{1}{2} \le t), t \in [0, 1]$. Figure 1.1 illustrates this example.



(a) $x_n(t), t \geq 0$. (b) $x_0(t), t \geq 0$.

Figure 1.1: Functions, which **J**-converge.

These functions obviously do not converge in the uniform metric, since $d_U(x_n(\cdot), x_0(\cdot)) = 1 - \frac{1}{n}$ for $n \geq 1$. Let us define a function $\lambda_n(t)$ to be equal to $(1 - \frac{2}{n})t$ for $0 \leq t < \frac{1}{2}$, and $(1 + \frac{2}{n})t - \frac{2}{n}$ for $\frac{1}{2} \leq t \leq 1$. This function is continuous, strictly monotone, and $\lambda_n(0) = 0, \lambda_n(1) = 1$. Moreover, $\sup_{0 \leq t \leq 1} |\lambda_n(t) - t| \to 0$ as $n \to \infty$.

Now, $x_n(\lambda_n(t)) = (1 - \frac{1}{n})x_0(t)$ and, hence, $d_U(x_n(\lambda_n(\cdot)), x_0(\cdot)) = \frac{1}{n} \to 0$ as $n \to \infty$. This shows that small deformations of time applied to the functions $x_n(t)$ can transform these functions to new ones that converge uniformly to $x_0(t)$.

Skorokhod (1956) invented the so-called **J**-topology of convergence in the space $\mathbf{D}_{[0,1]}$. This topology is based on the use of small time deformations that transform càdlàg functions in uniformly convergent functions. More precisely, càdlàg functions $x_n(t)$ **J**-converge to a càdlàg function $x_0(t)$ as $n \to \infty$ if there exists a sequence of continuous strictly monotone mappings $\lambda_n(t)$ of the interval $[0, 1]$ onto itself such that $\lambda_n(0) = 0$, $\lambda_n(1) = 1$, and $\sup_{0 \leq t \leq 1}(|\lambda_n(t) - t| + d_U(x_n(\lambda_n(\cdot)), x_0(\cdot))) \to 0$ as $n \to \infty$.

To better understand the meaning of **J**-convergence, let us give two examples of sequences of càdlàg functions that converge pointwise but do not **J**-converge.

In the first example, consider the functions $x_n(t) = \frac{1}{2}\chi(\frac{1}{2} \leq t) + \frac{1}{2}\chi(\frac{1}{2} - \frac{1}{n} \leq t), t \in [0, 1]$. These functions converge pointwise to the limiting function $x_0(t) = \chi(\frac{1}{2} \leq t), t \in [0, 1]$. But these functions do not **J**-converge. There always exists a point in which the function $x_n(\lambda_n(\cdot))$ takes the value $\frac{1}{2}$ for any mapping $\lambda_n(t)$ with the properties described above. Figure 1.2 illustrates this example.

As the second example, consider the functions $x_n(t)$ which take the values 0 for $t < \frac{1}{2} - \frac{1}{n}$, $n(t - \frac{1}{2}) + 1$ for $\frac{1}{2} - \frac{1}{n} \leq t < \frac{1}{2}$, and 1 for $\frac{1}{2} \leq t \leq 1$. Again, the functions $x_n(t)$ converge pointwise to the limiting function $x_0(t) = \chi(\frac{1}{2} \leq t), t \in [0, 1]$. But these functions do not **J**-converge. As in the first example, there always exists a point in

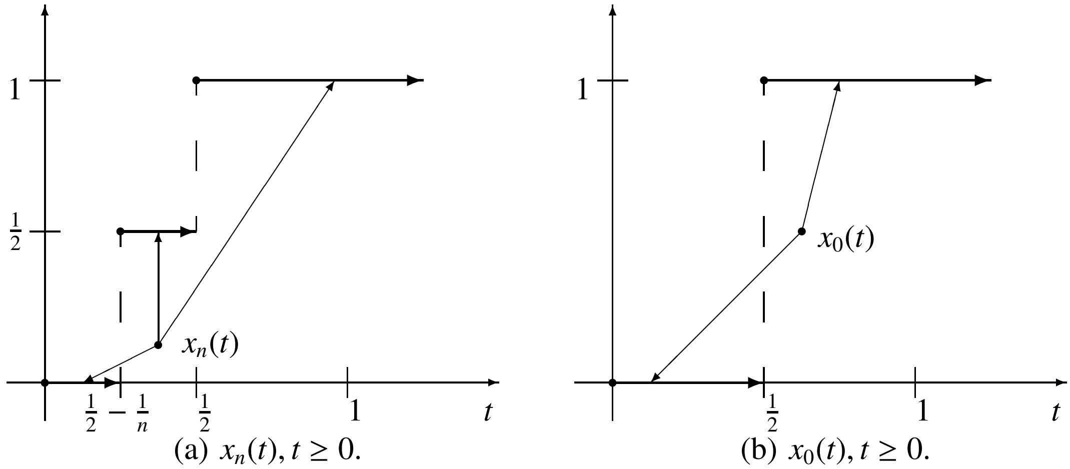


Figure 1.2: Functions, which do not **J**-converge.

which the function $x_n(\lambda_n(\cdot))$ takes the value $\frac{1}{2}$ for any mapping $\lambda_n(t)$ with the properties described above. Figure 1.3 illustrates this example.

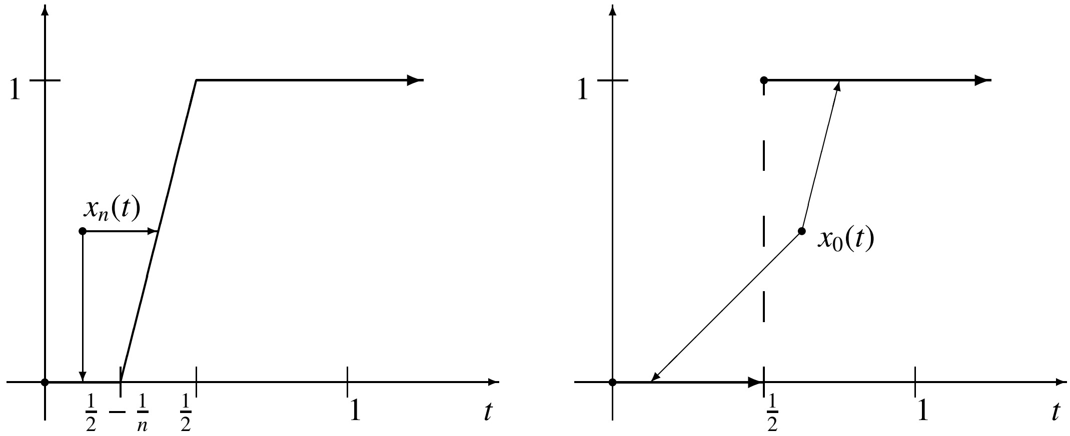
In both examples, the functions do not **J**-converge, because there are close points $t' < t < t''$ such that both increments $x_n(t) - x_n(t')$ and $x_n(t'') - x_n(t)$ are large (separated from zero) uniformly for all n large enough. Let us formulate this more precisely.

The modulus of **J**-compactness $\Delta_J(x_n(\cdot), c, 1)$ can be introduced to be the maximum of the quantities $|x_n(t)-x_n(t')| \wedge |x_n(t'')-x_n(t)|$ taken over all points $0 \leq t', t, t'' \leq 1, t-c \leq t' < t < t'' \leq t+c$. Using this, the condition of **J**-compactness can be formulated. It requires that $\Delta(x_n(\cdot), c, 1)$ tend to 0 as, first, $n \to \infty$ (here the upper limit must be used) and then $c \to 0$. As was shown by Skorokhod, the pointwise convergence of functions $x_n(t)$ and their **J**-compactness do imply **J**-convergence of the functions $x_n(t)$.

In both examples given above, $\Delta(x_n(\cdot), c, 1) = \frac{1}{2}$ for $n \geq c^{-1}$. Therefore, the iterated limit of $\Delta(x_n(\cdot), c, 1)$ equals $\frac{1}{2}$. This means that the condition of **J**-compactness does not hold.

What is very important that the space $\mathbf{D}_{[0,1]}$ can be equipped with a metric $d_J(x(\cdot), y(\cdot))$ such that convergence of càdlàg functions in this metric is equivalent to their convergence in the **J**-topology. An explicit formula that defines this metric is not simple. It will be given in Section 1.4.

Let now $\xi_n = \{\xi_n(t), t \in [0, T]\}$ be a càdlàg stochastic process for every $n = 0, 1, \ldots$. One can consider ξ_n as a random variable that takes values in the functional metric space $\mathbf{D}_{[0,1]}$. Weak convergence of such random variables (**J**-convergence of stochastic processes) is a subject of the so-called *functional limit theorems* for càdlàg stochastic processes. Weak limit theorems for the random variables $f(\xi_n)$ play an important role in the theory. Actually, $f(\xi_n)$ are random functionals defined on trajectories of the process



(a) $x_n(t), t \geq 0$. (b) $x_0(t), t \geq 0$.

Figure 1.3: Functions, which do not **J**-converge.

$\xi_n = \{\, \xi_n(t), t \in [0, T]\,\}$. In particular, functional limit theorems give effective conditions of weak convergence for important functionals (defined on càdlàg processes) such as maxima, exceeding times, integral functionals, etc.

In the case of stochastic processes, the condition of weak convergence of the so-called finite dimensional distributions, which replaces the condition of pointwise convergence, plus the condition of **J**-compactness of these processes in probability constitute conditions for **J**-convergence of stochastic processes. Fortunately, realisations of càdlàg stochastic processes usually possess the property of **J**-compactness in probability under some natural minor conditions that should be added to those conditions that provide weak convergence of finite dimensional distributions. This makes **J**-topology a very natural instrument in limit theorems for càdlàg processes.

We will give a detailed description of basic results concerning functional limit theorems in Sections 1.2–1.6.

1.2 Weak convergence in $\mathbb{R}_m$

1.2.1. Weak convergence in $\mathbb{R}_m$. Let $\xi_\varepsilon = (\xi_{\varepsilon 1}, \dots, \xi_{\varepsilon m}), \varepsilon \geq 0$ be a family of random variables (vectors) taking values in the *Euclidian space* $\mathbb{R}_m$ and depending on a parameter $\varepsilon \geq 0$. We denote by $F_\varepsilon(\mathbf{x}) = \mathsf{P}\{\xi_{\varepsilon 1} \leq x_1, \dots, \xi_{\varepsilon m} \leq x_m\}$, $\mathbf{x} = (x_1, \dots, x_m) \in \mathbb{R}_m$, the *distribution function* of the random variable ξ_ε.

We start with the following traditional definition of *weak convergence* of random variables.

Definition 1.2.1. Random variables ξ_ε weakly converge to ξ_0 as $\varepsilon \to 0$ ($\xi_\varepsilon \Rightarrow \xi_0$ as

$\varepsilon \to 0$) if $F_\varepsilon(\mathbf{x}) \to F_0(\mathbf{x})$ as $\varepsilon \to 0$ for all points $\mathbf{x} \in \mathbb{R}_m$ in which the limiting distribution function is continuous.

We denote the *set of continuity* of the distribution function F_0 by C_0. In the one-dimensional case, the set $\overline{C}_0$ is at most countable. In multi-dimensional case, this is not true. However, the set $\overline{C}_0 \subseteq U_0$, where U_0 is the union of at most countable number of hyper-planes parallel to one of the coordinate hyper-planes. Namely, U_0 is the set of points $\mathbf{x} \in \mathbb{R}_m$ that have at least one of the coordinates belonging to V_0. Here V_0 is the set of points $x \in \mathbb{R}_1$ such that $\sum_{i=1}^m \mathsf{P}\{\xi_{0i} = x\} > 0$. Obviously, the set C_0 is dense in $\mathbb{R}_m$.

Let $\mathfrak{B}_m$ be the *Borel* σ-*algebra* of subsets of $\mathbb{R}_m$ (the minimal σ-algebra containing all balls in $\mathbb{R}_m$) and $F_\varepsilon(A) = \mathsf{P}\{\xi_\varepsilon \in A\}$ be the probability measure on $\mathfrak{B}_m$ generated by the random variable ξ_ε. This measure is called a distribution of the random variable ξ_ε. It is connected with the distribution function $F_\varepsilon(\mathbf{x})$ by the formula $F_\varepsilon(A(\mathbf{x})) = F_\varepsilon(\mathbf{x})$, where $A(\mathbf{x}) = (-\infty, x_1] \times \cdots \times (-\infty, x_m]$, $\mathbf{x} \in \mathbb{R}_m$, and is uniquely defined by this distribution function via the corresponding extension theorem of measure theory.

Let ∂A denote the *b*oundary of a set A, i.e., the set of points $\mathbf{x}$ such that every *ball* $B_r(\mathbf{x}) = \{\mathbf{y} : |\mathbf{x} - \mathbf{y}| \le r\}$, with centre in $\mathbf{x}$ and radius $r > 0$, has non-empty intersections with both sets A and $\overline{A}$. If $F_0(\partial A) = 0$, then A is called a *set of continuity* for the distribution F_0. The class of such sets, $\mathfrak{B}_m(F_0)$, is a σ-algebra of subsets of $\mathfrak{B}_m$.

The following statement shows that weak convergence of random variables ξ_ε, which is actually the convergence of the probabilities $F_\varepsilon(A(\mathbf{x}))$ for $\mathbf{x} \in C_0$, can be extended to all sets of continuity for the distribution F_0.

Theorem 1.2.1. *Weak convergence $\xi_\varepsilon \Rightarrow \xi_0$ as $\varepsilon \to 0$ is a necessary and sufficient condition for the following relation to hold*:

$$F_\varepsilon(A) \to F_0(A) \text{ as } \varepsilon \to 0, \quad A \in \mathfrak{B}_m(F_0). \tag{1.2.1}$$

Theorem 1.2.1 shows a way to introduce weak convergence of random variables that take values in a general metric space. It can happen that this space does not possess a partial order similar to the one defined by the relation $\mathbf{x} \le \mathbf{y}$ in $\mathbb{R}_m$. In the sequel, random variables do not possess distribution functions similar to those defined for random variables taking values in $\mathbb{R}_m$. In this case, one can use relation (1.2.1) to define weak convergence of random variables.

The next natural step is to characterise weak convergence via convergence of expectations for the transformed random variables. In particular, it is possible to show that weak convergence of random variables ξ_ε is equivalent to weak convergence of the transformed random variables $f(\xi_\varepsilon)$ for all continuous functions f, as well as to convergence of expectations $\mathsf{E} f(\xi_\varepsilon)$ for all bounded continuous functions f. These results are absolutely analogous for the space $\mathbb{R}_m$ and for general complete separable metric spaces. The latter case is considered in Section 1.3.

1.2.2. Characteristic functions. Characteristic functions provide a very powerful tool for weak limit theorems. Let us consider a parametric class of exponential functions, $f_{\mathbf{t}}(\mathbf{x}) = \exp\{i(\mathbf{t}, \mathbf{x})\}$, $\mathbf{t} \in \mathbb{R}_m$, where $(\mathbf{t}, \mathbf{x}) = t_1x_1 + \ldots + t_mx_m$. The *characteristic function* of a random variable ξ_ε is the expectation $\varphi_\varepsilon(\mathbf{t}) = \mathsf{E}f_{\mathbf{t}}(\xi_\varepsilon)$ considered as a function of $\mathbf{t} \in \mathbb{R}_m$. Let us formulate the condition:

$\mathcal{A}_2$: $\varphi_\varepsilon(\mathbf{t}) = \mathsf{E}f_{\mathbf{t}}(\xi_\varepsilon) \to \varphi_0(\mathbf{t}) = \mathsf{E}f_{\mathbf{t}}(\xi_0)$ as $\varepsilon \to 0$ for $\mathbf{t} \in \mathbb{R}_m$.

Theorem 1.2.2. *Condition $\mathcal{A}_2$ is necessary and sufficient for the weak convergence $\xi_\varepsilon \Rightarrow \xi_0$ as $\varepsilon \to 0$.*

It is useful to note that the assumption of existence of a limiting random variable ξ_0 can be omitted in condition $\mathcal{A}_2$. It is enough to assume that the following holds: **(a)** the characteristic functions $\mathsf{E}f_{\mathbf{t}}(\xi_\varepsilon)$ converge pointwise to some limiting function $\varphi_0(\mathbf{t})$ as $\varepsilon \to 0$ for $\mathbf{t} \in \mathbb{R}_m$, and **(b)** the limiting function $\varphi_0(\mathbf{t})$ is continuous at the point 0. In this case, it is possible to show that **(c)** $\varphi_0(\mathbf{t})$ is a characteristic function of some random variable, i.e., it can be represented in the form $\varphi_0(\mathbf{t}) = \mathsf{E}f_{\mathbf{t}}(\xi_0)$. If **(c)** is proved, then weak convergence of random variables ξ_ε to ξ_0 as $\varepsilon \to 0$, follows from Theorem 1.2.2.

The following useful lemma, known as the *Wold–Cramér device* allows to prove weak convergence of m-dimensional random variables by proving weak convergence of one-dimensional random variables from some related parametric family. The proof of the latter could be simpler. Let us consider a parametric class of linear functions, $g_{\mathbf{t}}(\mathbf{x}) = (\mathbf{t}, \mathbf{x})$, $\mathbf{t} \in \mathbb{R}_m$. Let us introduce the following condition:

$\mathcal{A}_3$: $g_{\mathbf{t}}(\xi_\varepsilon) \Rightarrow g_{\mathbf{t}}(\xi_0)$ as $\varepsilon \to 0$ for $\mathbf{t} \in \mathbb{R}_m$.

Lemma 1.2.1. *Condition $\mathcal{A}_3$ is necessary and sufficient for the weak convergence $\xi_\varepsilon \Rightarrow \xi_0$ as $\varepsilon \to 0$.*

This lemma is a corollary of Theorem 1.2.2. Indeed, the expectation $\mathsf{E}\exp\{isg_{\mathbf{t}}(\xi_\varepsilon)\}$ is characteristic function of the one-dimensional random variable $g_{\mathbf{t}}(\xi_\varepsilon)$ taken at the point $s \in \mathbb{R}_1$. At the same time, it is a characteristic function of the m-dimensional random variable ξ_ε taken at the point $s\mathbf{t} \in \mathbb{R}_m$.

1.2.3. Reduction of the set determining weak convergence. According to the initial definition, in order to prove that random variables ξ_ε converge weakly to ξ_0 as $\varepsilon \to 0$, one must check the pointwise convergence of their distribution functions, $F_\varepsilon(\mathbf{x})$, for all continuity points of the limiting distribution function $F_0(\mathbf{x})$. Due to monotonicity of the distribution functions $F_\varepsilon(\mathbf{x})$ (if $\mathbf{x} \leq \mathbf{y}$, then $F_\varepsilon(\mathbf{x}) \leq F_\varepsilon(\mathbf{y})$), it is possible to limit this verification to some set S dense in $\mathbb{R}_m$. This set can be a countable set and it is not required to contain only points of continuity of the limiting distribution function F_0. Let us formulate the following condition:

$\mathcal{A}_4$: $F_\varepsilon(\mathbf{x}) \to F_0(\mathbf{x})$ as $\varepsilon \to 0$ for $\mathbf{x} \in S$, where S is a set dense in $\mathbb{R}_m$.

Lemma 1.2.2. *Condition* $\mathcal{A}_4$ *is necessary and sufficient for the weak convergence* $\xi_\varepsilon \Rightarrow \xi_0$ *as* $\varepsilon \to 0$.

The statement of necessity of this lemma can be improved. In condition $\mathcal{A}_4$, existence of the limiting distribution is assumed. However, instead of $\mathcal{A}_4$, one can only assume that: **(a)** the limits $\lim_{\varepsilon\to 0} F_\varepsilon(\mathbf{x}) = F_0(\mathbf{x})$ exist for $\mathbf{x} \in S$, where S is a set dense in $\mathbb{R}_m$, and **(b)** $F_0(\mathbf{x}) \to 0$ as $x_{\min} = \min_{1\le i\le m} x_i \to -\infty$ and $F_0(\mathbf{x}) \to 1$ as $x_{\min} \to \infty$. The limiting function $F_0(\mathbf{x})$, defined in **(a)** for $\mathbf{x} \in S$, can also be defined as $F_0(\mathbf{x}) = \lim_{\mathbf{x}<\mathbf{y}\in S, \mathbf{y}\to\mathbf{x}} F_0(\mathbf{y})$ for $\mathbf{x} \in \overline{S}$. Under conditions **(a)** and **(b)**, the function $F_0(\mathbf{x})$ is a distribution function and, according to **(a)**, condition $\mathcal{A}_4$ holds.

Note that assumption **(b)** plays here an essential role. Without this condition there is no guarantee that the function $F_0(\mathbf{x})$ is a distribution function. It is easy to give an example where **(a)** is satisfied but the limits $F_0(\mathbf{x}) = 0$ for all $\mathbf{x} \in S$ and, consequently, for all $\mathbf{x} \in \mathbb{R}_m$.

1.2.4. Slutsky theorem and related results. Weak convergence of random variables is a convergence of their distributions. For this reason, it is possible for random variables ξ_ε, which weakly converge to ξ_0 as $\varepsilon \to 0$, to be defined on different probability spaces for different $\varepsilon \ge 0$.

A special and important case is where the limiting random variable $\xi_0 = \text{const}$ with probability 1. The following simple lemma shows that, in this case, weak convergence can be interpreted as convergence in probability, despite the possibility that the random variables ξ_ε can be defined on different probability spaces.

Lemma 1.2.3. *Random variables* $\xi_\varepsilon \Rightarrow \xi_0$ *as* $\varepsilon \to 0$, *where* $\xi_0 = \text{const}$ *with probability* 1, *if and only if* **(α)** $\mathsf{P}\{|\xi_\varepsilon - \xi_0| > \delta\} \to 0$ *as* $\varepsilon \to 0$ *for* $\delta > 0$.

Let random variables $\xi'_\varepsilon = (\xi'_{\varepsilon 1}, \dots, \xi'_{\varepsilon m})$ and $\xi''_\varepsilon = (\xi''_{\varepsilon 1}, \dots, \xi''_{\varepsilon l})$ be defined on the same probability space for every $\varepsilon \ge 0$ (possibly different for different ε) and take values in the spaces $\mathbb{R}_m$ and $\mathbb{R}_l$, respectively. In this case, the vector $\xi_\varepsilon = (\xi'_{\varepsilon 1}, \dots, \xi'_{\varepsilon m}, \xi''_{\varepsilon 1}, \dots, \xi''_{\varepsilon l})$ is a random variable that takes values in the space $\mathbb{R}_{m+l}$.

Suppose that the random variables ξ'_ε and ξ''_ε converge weakly to ξ'_0 and ξ''_0 as $\varepsilon \to 0$, respectively. This does not imply that the random variables ξ_ε weakly converge to ξ_0 as $\varepsilon \to 0$. However, this is true if at least one of the limiting variables ξ'_0 or ξ''_0 is a constant.

Theorem 1.2.3. *Let* **(α)** $\xi'_\varepsilon \Rightarrow \xi'_0$ *as* $\varepsilon \to 0$, *and* **(β)** $\xi''_\varepsilon \Rightarrow \xi''_0$ *as* $\varepsilon \to 0$, *where* $\xi''_0 = \text{const}$ *with probability* 1. *Then* $\xi_\varepsilon \Rightarrow \xi_0$ *as* $\varepsilon \to 0$.

As a corollary we have that, under conditions of Theorem 1.2.3, the random variables $f(\xi_\varepsilon) \Rightarrow f(\xi_0)$ as $\varepsilon \to 0$ for any measurable function f acting from $\mathbb{R}_{m+l}$ to $\mathbb{R}_k$ and a.s. continuous with respect to the distribution F_0 of the random variable ξ_0.

In particular, we have (for the case where $m = l$) that the sums $\xi'_\varepsilon + \xi''_\varepsilon \Rightarrow \xi'_0 + \xi''_0$ as $\varepsilon \to 0$. In the case where $\xi''_0 = 0$, we obtain the following very useful result.

Lemma 1.2.4. *Let* **(α)** $\xi'_\varepsilon \Rightarrow \xi'_0$ *as* $\varepsilon \to 0$, *and* **(β)** $\mathsf{P}\{|\xi''_\varepsilon| > \delta\} \to 0$ *as* $\varepsilon \to 0$ *for* $\delta > 0$. *Then* $\xi'_\varepsilon + \xi''_\varepsilon \Rightarrow \xi'_0$ *as* $\varepsilon \to 0$.

This lemma can be generalised in the following way. Suppose that the random variables ξ_ε can be represented, for every $\varepsilon \geq 0$ and $n = 0, 1, \ldots$, as a sum of two random variables,

$$\xi_\varepsilon = \xi'_{\varepsilon n} + \xi''_{\varepsilon n}. \tag{1.2.2}$$

Lemma 1.2.5. *Let* **(α)** $\xi'_{\varepsilon n} \Rightarrow \xi_{0n}$ *as* $\varepsilon \to 0$ *for* $n = 0, 1, \ldots$, **(β)** $\xi'_{0n} \Rightarrow \xi_0$ *as* $n \to \infty$, *and* **(γ)** $\lim_{n\to\infty} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{|\xi''_{\varepsilon n}| \geq \delta\} \to 0$ *for* $\delta > 0$. *Then* $\xi_\varepsilon \Rightarrow \xi_0$ *as* $\varepsilon \to 0$.

The following lemma deals with the case where random variables ξ_ε possess upper and lower approximations $\xi^\pm_{\varepsilon n}$. Here $\xi^\pm_{\varepsilon n}$ are random variables such that, for every $\varepsilon \geq 0$ and $n = 0, 1, \ldots$, the following inequalities hold (for every component):

$$\xi^-_{\varepsilon n} \leq \xi_\varepsilon \leq \xi^+_{\varepsilon n}. \tag{1.2.3}$$

Lemma 1.2.6. *Let* **(α)** $\xi^\pm_{\varepsilon n} \Rightarrow \xi^\pm_{0n}$ *as* $\varepsilon \to 0$ *for* $n = 0, 1, \ldots$, *and* **(β)** $\xi^\pm_{0n} \Rightarrow \xi_0$ *as* $n \to \infty$. *Then* $\xi_\varepsilon \Rightarrow \xi_0$ *as* $\varepsilon \to 0$.

Let $F_\varepsilon(\mathbf{x})$ and $F^\pm_{\varepsilon n}(\mathbf{x})$ be distribution functions for the random variables ξ_ε and $\xi^\pm_{\varepsilon n}$, respectively. Approximation inequalities (1.2.3) can be replaced in Lemma 1.2.6 by the family of stochastic inequalities: **(a)** $F^+_{\varepsilon n}(\mathbf{x}) \leq F_\varepsilon(\mathbf{x}) \leq F^-_{\varepsilon n}(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}_m$, which obviously hold if (1.2.3) holds. Under **(a)**, conditions of Lemma 1.2.6 imply that **(b)** $\underline{\lim}_{n\to\infty} \underline{\lim}_{\varepsilon\to 0} F^+_{\varepsilon n}(\mathbf{x}) \geq F_0(\mathbf{x})$ and $\overline{\lim}_{n\to\infty} \overline{\lim}_{\varepsilon\to 0} F^-_{\varepsilon n}(\mathbf{x}) \leq F_0(\mathbf{x})$ for $\mathbf{x} \in C_0$, where C_0 is the set of continuity points for the distribution function $F_0(\mathbf{x})$. Conditions **(a)** and **(b)** form a combination, minimal in some sense, of conditions that are based on the upper and the lower approximations and provide weak convergence of the random variables ξ_ε to ξ_0 as $\varepsilon \to 0$.

If the stronger approximation inequalities (1.2.3) hold, then condition **(β)** in Lemma 1.2.6 can be replaced with the condition **(c)** $\lim_{n\to\infty} \mathsf{P}\{|\xi^+_{0n} - \xi^-_{0n}| > \delta\} = 0$ for $\delta > 0$. Obviously, **(c)** implies **(β)**, due to inequality (1.2.3) for $\varepsilon = 0$.

1.3 Weak convergence in metric spaces

1.3.1. Weak convergence in metric spaces. Let X be a metric space with a metric $d(x, y)$.

The space X is *complete* if for any fundamental sequence of points $x_n \in X$, i.e., a sequence such that $d(x_n, x_m) \to 0$ as $n, m \to \infty$, there exists a point $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$.

The space X is *separable* if there exists a countable subset $Y = \{y_1, y_2, \ldots\} \subseteq X$ such that $\min_{k\leq n} d(y_k, x) \to 0$ as $n \to \infty$ for any point $x \in X$.

The term *Polish space* is used to indicate that X is a complete separable metric space. Below, X is always a Polish space.

A set K is a *compact* (set) in a Polish space X if there exists a countable set $Y = \{y_1, y_2, \ldots\} \subseteq X$ such that $\min_{k \le n} \sup_{x \in K} d(y_k, x) \to 0$ as $n \to \infty$.

Let $\mathfrak{B}$ be the *Borel σ-algebra* of subsets of X (the minimal σ-algebra containing any *ball* $B_r(x) = \{\, y \colon d(x, y) \le r \,\}$ in the space X).

The space $\mathbb{R}_m$ is a particular example of a Polish space. Other examples that we will be interested in are the functional spaces of continuous functions, **C**, and càdlàg functions, **D**. These spaces become Polish spaces if appropriate metrics are introduced in these spaces.

Random variables that take values in a Polish space may not possess distribution functions defined in the same way as for random variables with values in the space $\mathbb{R}_m$. For this reason, the definition of weak convergence in $\mathbb{R}_m$ can not be directly extended to Polish spaces. However, as it was mentioned above, such a definition can be made by using a condition analogous to relation (1.2.1).

Let $\xi_\varepsilon, \varepsilon \ge 0$ be a family of random variables that take values in X and depend on a parameter $\varepsilon \ge 0$. We denote by $F_\varepsilon(A) = \mathsf{P}\{\xi_\varepsilon \in A\}, A \in \mathfrak{B}$, the distribution of the random variable ξ_ε.

Let ∂A denote the *boundary* of the set A, i.e., the set of points x such that every ball $B_r(x)$, with centre in x and a radius $r > 0$, has non-empty intersections with both sets A and $\overline{A}$. If $F_0(\partial A) = 0$, then A is called a *set of continuity* for the distribution F_0. The class of such sets, $\mathfrak{B}(F_0)$, is a σ-algebra of subsets of $\mathfrak{B}$.

Definition 1.3.1. Random variables ξ_ε weakly converge to ξ_0 as $\varepsilon \to 0$ ($\xi_\varepsilon \Rightarrow \xi_0$ as $\varepsilon \to 0$) if $F_\varepsilon(A) \to F_0(A)$ as $\varepsilon \to 0$ for all sets $A \in \mathfrak{B}(F_0)$.

1.3.2. Convergence of expectations for transformed random variables. It is obvious that if $f(x)$ is a measurable real-valued function defined on a space X (the inverse image of any Borel set in $\mathbb{R}_1$ is a Borel set in X), then $f(\xi_\varepsilon)$ is a real-valued random variable.

The indicator function $\chi_A(x)$ of a Borel set A is a measurable function and it has the set of discontinuity points, ∂A. The condition $F_0(\partial A) = 0$ means that $\chi_A(x)$ is an a.s. continuous function with respect to the measure F_0. The definition of weak convergence given above requires that $\mathsf{E}\chi_A(\xi_\varepsilon) = F_\varepsilon(A) \to \mathsf{E}\chi_A(\xi_0) = F_0(A)$ as $\varepsilon \to 0$ for all sets of continuity for the limiting distribution F_0.

Let us denote by $\mathfrak{C}_b(F_0)$ the class of all real-valued measurable bounded functions f that are a.s. continuous with respect to the limiting distribution F_0.

The following theorem connects weak convergence with convergence of expectations of the transformed random variables for the class of bounded a.s. continuous functions. This class is wider than the class of indicator functions.

Theorem 1.3.1. *Weak convergence $\xi_\varepsilon \Rightarrow \xi_0$ as $\varepsilon \to 0$ is a necessary and sufficient condition for the following relation to hold*:

$$\mathsf{E} f(\xi_\varepsilon) \to \mathsf{E} f(\xi_0) \text{ as } \varepsilon \to 0, \quad f \in \mathfrak{C}_b(F_0). \tag{1.3.1}$$

The statement of sufficiency in Theorem 1.3.1 is substantial. A standard proof of this statement is based on approximating a function f from the class $\mathfrak{C}_b(F_0)$, appropriately, by linear combinations of indicator functions. As far as the statement of necessity is concerned, it is useful to note that the requirement of a.s. continuity of the functions f can be replaced with the requirement of them being continuous.

It is appropriate to note that Theorem 1.3.1 allows to give another definition of weak convergence. This definition is based on the use of relation (1.3.1), and equivalent to Definition 1.3.1.

1.3.3. Weak convergence of transformed random variables. Let us denote by $\mathfrak{C}(F_0)$ the class of all real-valued measurable functions f that are a.s. continuous with respect to the limiting distribution F_0. Note that boundedness of the functions f is not required.

The following statement shows a connection between weak convergence of random variables and their transformations.

Theorem 1.3.2. *Weak convergence $\xi_\varepsilon \Rightarrow \xi_0$ as $\varepsilon \to 0$ is a necessary and sufficient condition for the following relation to hold*:

$$f(\xi_\varepsilon) \Rightarrow f(\xi_0) \text{ as } \varepsilon \to 0, \quad f \in \mathfrak{C}(F_0). \tag{1.3.2}$$

As in Theorem 1.3.1, the statement of sufficiency in Theorem 1.3.2 is substantial. The proof can be based on the use of the characteristic functions $\mathsf{E}\exp\{\mathrm{i}tf(\xi_\varepsilon)\}$. Their pointwise convergence follows from Theorem 1.3.1. As far as the statement of necessity is concerned, the requirement of a.s. continuity of functions f can be replaced with the requirement of their continuity.

It is appropriate to note that Theorem 1.3.2 allows to give the third definition of weak convergence of random variables ξ_ε. This definition is based on the relation (1.3.2), and is also equivalent to Definition 1.3.1.

Theorem 1.3.2 plays an essential role in the theory. In the case of the functional spaces **C** and **D**, this theorem is the main tool in studies of weak convergence of functionals defined on trajectories of stochastic processes.

Sometimes one can be interested in proving the joint weak convergence of several functions of random variables which weakly converge. In this context, the following remark is useful.

Let $f_1(x), \dots, f_k(x)$ be functions a.s. continuous with respect to the measure F_0. Then their linear combination $g_\mathbf{t}(x) = t_1 f_1(x) + \cdots + t_k f_k(x)$ is also a.s. continuous with respect to the measure F_0 for every $\mathbf{t} = (t_1, \dots, t_k) \in \mathbb{R}_k$. This shows that, if $\xi_\varepsilon \Rightarrow \xi_0$

as $\varepsilon \to 0$, then, as follows from Theorem 1.3.2, $g_{\mathbf{t}}(\xi_\varepsilon) \Rightarrow g_{\mathbf{t}}(\xi_0)$ as $\varepsilon \to 0$ for every $\mathbf{t} \in \mathbb{R}_k$. Due to Lemma 1.2.1, this implies joint weak convergence of the random vectors $(f_1(\xi_\varepsilon), \ldots, f_k(\xi_\varepsilon)) \Rightarrow (f_1(\xi_0), \ldots, f_k(\xi_0))$ as $\varepsilon \to 0$.

1.3.4. A subsequence approach and Prokhorov's theorems. In the case of a general Polish space, some effective tools related to the weak convergence do not exist (for example, characteristic functions) or do not work so effectively. The most effective approach in the case of a metric space is based on a subsequence approach and notions of relative compactness and tightness of a family of distributions. The corresponding general theory was developed by Prokhorov (1956).

First of all note that weak convergence is, actually, a convergence of distributions. So, one can consider weak convergence of distributions (probability measures) F_ε, instead of weak convergence of the corresponding random variables ξ_ε; we will use the symbol $F_\varepsilon \Rightarrow F_0$ as $\varepsilon \to 0$ instead of $\xi_\varepsilon \Rightarrow \xi_0$ as $\varepsilon \to 0$.

The following theorem is an analogue of the corresponding statement concerning numerical limits: $a_\varepsilon \to a_0$ as $\varepsilon \to 0$ if and only if any subsequence $0 \le \varepsilon_n \to 0$ as $n \to \infty$ contains a subsequence $\varepsilon'_k = \varepsilon_{n_k}$, where $n_k \to \infty$ as $k \to \infty$, such that $a_{\varepsilon'_k} \to a_0$ as $k \to \infty$.

Theorem 1.3.3. *Distributions $F_\varepsilon \Rightarrow F_0$ as $\varepsilon \to 0$ if and only if* **(α)** *any subsequence $\varepsilon_n \to 0$ as $n \to \infty$ contains a subsequence $\varepsilon'_k = \varepsilon_{n_k}$, where $n_k \to \infty$ as $k \to \infty$, such that $F_{\varepsilon'_k} \Rightarrow F_0$ as $k \to \infty$.*

The notions of tightness and relative compactness for a family of distributions play a principle role in the theory. Let us introduce the following condition:

$\mathcal{K}_2$: There exists a sequence of compact sets $K_n \subseteq X$, $n = 1, 2, \ldots$, such that $\lim_{n\to\infty} \overline{\lim}_{\varepsilon\to 0} F_\varepsilon(\overline{K}_n) = 0$.

Definition 1.3.2. A family of distributions F_ε, $\varepsilon \ge 0$, is *tight* as $\varepsilon \to 0$, if condition $\mathcal{K}_2$ holds.

Definition 1.3.3. A family of distributions F_ε, $\varepsilon \ge 0$, is *relatively compact* as $\varepsilon \to 0$, if any subsequence $\varepsilon_n \to 0$ as $n \to \infty$ contains a subsequence $\varepsilon'_k = \varepsilon_{n_k}$, where $n_k \to \infty$ as $k \to \infty$, such that distributions $F_{\varepsilon'_k}$ weakly converge to some probability measure F'_0 as $k \to \infty$ (possibly depending of the subsequence ε'_k).

The definition of tightness and relative compactness of a family of distributions given above slightly differs from the standard ones, since we are interested in weak convergence of the corresponding probability measures F_ε only for $\varepsilon \to 0$. Thus we only consider subsequences ε_n that converge to 0, instead of arbitrary subsequences in the set of the parameters $\{\varepsilon \ge 0\}$.

The following Prokhorov theorem plays a fundamental role in the theory.

Theorem 1.3.4. *A family of probability measures F_ε, $\varepsilon \geq 0$ is relatively compact as $\varepsilon \to 0$ if and only if it is tight as $\varepsilon \to 0$.*

Another notion of defining class for a distribution is also important.

Definition 1.3.4. A class of sets $\mathfrak{D}_F$ from the σ-algebra $\mathfrak{B}$ is a *defining class* for a probability measure F, if any probability measure F' that takes the same values as F on sets from the class $\mathfrak{D}_F$ coincides with F.

Let us introduce the following condition:

$\mathcal{A}_5$: $F_\varepsilon(A) \to F_0(A)$ as $\varepsilon \to 0$ for $A \in \mathfrak{D}_{F_0}$, where $\mathfrak{D}_{F_0}$ is some defining class for the distribution F_0.

Now we can formulate the main Prokhorov theorem that gives effective conditions for weak convergence of distributions in Polish spaces.

Theorem 1.3.5. *Conditions $\mathcal{K}_2$ and $\mathcal{A}_5$ are necessary and sufficient for the weak convergence $F_\varepsilon \Rightarrow F_0$ as $\varepsilon \to 0$.*

It follows from Theorem 1.3.3 that, in order to prove that $F_\varepsilon \Rightarrow F_0$ as $\varepsilon \to 0$, it is sufficient **(a)** to show that the family of distributions F_ε is relatively compact as $\varepsilon \to 0$, and **(b)** to prove that all weakly converging subsequences $F_{\varepsilon'_k}$ have the same limiting distribution F_0. Claim **(a)** follows from Theorem 1.3.4. Claim **(b)** is also true. Indeed, for any converging subsequence $F_{\varepsilon'_k}$, the corresponding limiting distribution F'_0 takes the same values as F_0 for sets in $\mathfrak{D}_{F_0}$ and, therefore, coincides with F_0.

Obviously, if $F_\varepsilon \Rightarrow F_0$ as $\varepsilon \to 0$, then the family of distributions F_ε, $\varepsilon \geq 0$, is relatively compact as $\varepsilon \to 0$. Therefore, due to Theorem 1.3.4, this family of distributions is also tight as $\varepsilon \to 0$. Also, the class of sets of continuity for the distribution F_0 is a defining class for this distribution. So, conditions $\mathcal{K}_2$ and $\mathcal{A}_5$ are also necessary for weak convergence $F_\varepsilon \Rightarrow F_0$ as $\varepsilon \to 0$.

Let us go back to the case of the space $\mathbb{R}_m$. Here, Theorem 1.3.5 can be considered as a generalisation of Lemma 1.2.2. Indeed, the class of sets $A(\mathbf{x}) = (-\infty, x_1] \times \cdots \times (-\infty, x_m]$, $\mathbf{x} \in S$, is a defining class for any probability measure F, if S is a set dense in $\mathbb{R}_m$. Thus $\mathcal{A}_4$ implies condition $\mathcal{A}_5$. Condition $\mathcal{A}_4$ also implies that the family F_ε, $\varepsilon \geq 0$, is tight, since $\mathcal{A}_4$ includes the assumption of existence of a limiting distribution function. Indeed, due to monotonicity of the distribution functions $F_\varepsilon(A(\mathbf{x}))$, the set S, in condition $\mathcal{A}_4$, can be extended to the set $S \cup S_0$. Here S_0 is the set of continuity of the limiting distribution function $F_0(A(\mathbf{x}))$. Note that the set S_0 is also dense in $\mathbb{R}_m$. Let us define a sequence of the compacts $K_n = \{\, \mathbf{y} \colon \mathbf{x}'_n \leq \mathbf{y} \leq \mathbf{x}''_n \,\}$, where $\mathbf{x}'_n = (x'_{1n}, \dots, x'_{mn})$ and $\mathbf{x}''_n = (x''_{1n}, \dots, x''_{mn})$ are chosen in such a way that **(c)** $\mathbf{x}'_n, \mathbf{x}''_n \in S_0$, and **(d)** $\min_{1\leq i\leq m} x'_{in} \to -\infty$ and $\min_{1\leq i\leq m} x''_{in} \to \infty$ as $n \to \infty$. Obviously, **(e)** $F_\varepsilon(\overline{K}_n) \to F_0(\overline{K}_n)$ as $\varepsilon \to 0$, and the sequence K_n, $n \geq 1$ satisfies the condition **(f)** $F_0(\overline{K}_n) \to 0$ as $n \to \infty$.

1.3.5. Convergence in probability and convergence with probability 1. These two types of convergence relate to a model where all random variables ξ_ε, $\varepsilon \geq 0$ are defined on the same probability space. So, let us suppose that $(\Omega, \mathfrak{F}, \mathsf{P})$ is a probability space and $\xi_\varepsilon = \xi_\varepsilon(\omega)$ is a random variable for every $\varepsilon \geq 0$, that is, a measurable function acting from Ω to X.

Let us first give a definition of *convergence in probability*.

Definition 1.3.5. Random variables ξ_ε converge in probability to ξ_0 as $\varepsilon \to 0$ ($\xi_\varepsilon \xrightarrow{\mathsf{P}} \xi_0$ as $\varepsilon \to 0$) if $\mathsf{P}\{d(\xi_\varepsilon, \xi_0) > \delta\} \to 0$ as $\varepsilon \to 0$ for $\delta > 0$.

It is easy to construct an example in which random variables are defined on the same probability space and weakly converge but do not converge in probability. However random variables, which converge in probability, always converge weakly.

Lemma 1.3.1. *If* $\xi_\varepsilon \xrightarrow{\mathsf{P}} \xi_0$ *as* $\varepsilon \to 0$*, then* $\xi_\varepsilon \Rightarrow \xi_0$ *as* $\varepsilon \to 0$.

Let us now give a definition of *convergence with probability* 1.

Definition 1.3.6. Random variables ξ_ε converge with probability 1 to ξ_0 as $\varepsilon \to 0$ ($\xi_\varepsilon \xrightarrow{\mathsf{P}1} \xi_0$ as $\varepsilon \to 0$) if there exists a random event $A_0 \in \mathfrak{F}$ such that **(α)** $\xi_\varepsilon(\omega) \to \xi_0(\omega)$ as $\varepsilon \to 0$ for every $\omega \in A_0$, and **(β)** $\mathsf{P}(A_0) = 1$.

Convergence with probability 1 is also known as a.s. (almost sure) convergence. The symbol $\xrightarrow{\text{a.s.}}$ can be used instead of $\xrightarrow{\mathsf{P}1}$.

Definition 1.3.6 requires some comments. In the case of weak convergence and convergence in probability, random variables ξ_ε converge to ξ_0 as $\varepsilon \to 0$ if and only if, for any subsequence $0 \leq \varepsilon_n \to 0$ as $n \to \infty$, the random variables ξ_{ε_n} converge to ξ_0 as $n \to \infty$. This follows from the corresponding property of limits of non-random functions.

In the case of a.s. convergence, the situation is slightly different.

If there exists a monotone sequence $\varepsilon_n \downarrow 0$ as $n \to \infty$ and $\xi_\varepsilon \equiv \xi_{\varepsilon_n}$ for $\varepsilon_n \leq \varepsilon < \varepsilon_{n+1}$, $n \geq 1$, we actually have a countable family of random variables. In this situation, the a.s. convergence of random variables ξ_{ε_n} to ξ_0 as $\varepsilon_n \to 0$ is obviously equivalent to the a.s. convergence of $\xi_{\varepsilon_{n_k}}$ to ξ_0 as $\varepsilon_{n_k} \to 0$ for all subsequences $n_k \to \infty$ as $k \to \infty$. The situation is different in the general case where a continuum of random variables is considered. In such a case, the a.s. convergence of ξ_ε to ξ_0 as $\varepsilon \to 0$, in the sense of the definition given above, obviously implies that ξ_{ε_n} a.s. converge to ξ_0 as $n \to \infty$ for any subsequence $0 \leq \varepsilon_n \to 0$ as $n \to \infty$. But the opposite implication is not always true.

For example, let us consider the probability space with the space of outcomes $[0, 1]$, the Borel σ-algebra of random events, and the Lebesgue measure as the corresponding probability measure. Let us define, on this probability space, a random variable $\xi_0(\omega) = \omega$ that is uniformly distributed on $[0, 1]$, and then the random variables $\xi_\varepsilon = \xi_0 \chi(\xi_0 \neq \varepsilon 2^k)$

for $2^{-k-1} \leq \varepsilon < 2^{-k}$, $k = 0, 1, \ldots$. The random variables ξ_{ε_n} a.s. converge to ξ_0 for any subsequence $\varepsilon_n \to 0$ as $n \to \infty$, but $\xi_\varepsilon(\omega)$ do not converges as $\varepsilon \to 0$ for $1/2 \leq \omega < 1$. Therefore, the random variables ξ_ε do not a.s. converge to ξ_0 as $\varepsilon \to 0$ in the sense of the definition given above.

This remark leads us to a slightly different definition of convergence which one can call *sub-sequential convergence with probability* 1 or *sub-sequential a.s. convergence*.

Definition 1.3.7. Random variables ξ_ε sub-sequentially converge with probability 1 to ξ_0 as $\varepsilon \to 0$ ($\xi_\varepsilon \xrightarrow{\text{s-P1}} \xi_0$ as $\varepsilon \to 0$) if the random variables $\xi_{\varepsilon_n} \xrightarrow{\text{P1}} \xi_0$ as $n \to \infty$ for any subsequence $\varepsilon_n \to 0$ as $n \to \infty$.

According to the remarks made above both definitions coincide in the case of a countable family of random variables. In the case of a continuum family, the sub-sequential convergence with probability 1 is weaker than the convergence with probability 1 in the sense of the first definition.

It is easy to construct an example in which random variables, which are defined on the same probability space, converge in probability but do not converge with probability 1. However, random variables that converge with probability 1 always converge in probability.

Lemma 1.3.2. *If* $\xi_\varepsilon \xrightarrow{\text{P1}} \xi_0$ *as* $\varepsilon \to 0$*, then* $\xi_\varepsilon \xrightarrow{\text{P}} \xi_0$ *as* $\varepsilon \to 0$.

Note that the assumption $\xi_\varepsilon \xrightarrow{\text{P1}} \xi_0$ as $\varepsilon \to 0$ can be replaced in this lemma by a weaker assumption, $\xi_\varepsilon \xrightarrow{\text{s-P1}} \xi_0$ as $\varepsilon \to 0$. This follows from the following useful lemma.

Lemma 1.3.3. *For any subsequence* $0 \leq \varepsilon_n \to 0$ *as* $n \to \infty$*, random variables* $\xi_{\varepsilon_n} \xrightarrow{\text{P1}} \xi_0$ *as* $n \to \infty$ *if and only if* **(α)** $\max_{k \geq n} d(\xi_{\varepsilon_k}, \xi_0) \xrightarrow{\text{P}} 0$ *as* $n \to \infty$.

Remark 1.3.1. It follows from Lemma 1.3.3 that the relations $d(\xi_{\varepsilon_n}, \xi_0) \xrightarrow{\text{P}} 0$ as $n \to \infty$ and $d(\xi_{\varepsilon_n}, \xi_0) \xrightarrow{\text{P1}} 0$ as $n \to \infty$ are equivalent if the sequence of random variables $d(\xi_{\varepsilon_n}, \xi_0)$, $n = 0, 1, \ldots$ is monotonically non-increasing with probability 1.

Remark 1.3.2. In the case of random variables with values in $\mathbb{R}_m$, if $\boldsymbol{\xi}_{\varepsilon_{n+1}} \geq \boldsymbol{\xi}_{\varepsilon_n}$ for all $n = 0, 1, \ldots$ or $\boldsymbol{\xi}_{\varepsilon_{n+1}} \leq \boldsymbol{\xi}_{\varepsilon_n}$ for all $n = 0, 1, \ldots$, then the relations $\boldsymbol{\xi}_{\varepsilon_n} \xrightarrow{\text{P}} \boldsymbol{\xi}_0$ as $n \to \infty$ and $\boldsymbol{\xi}_{\varepsilon_n} \xrightarrow{\text{P1}} \boldsymbol{\xi}_0$ as $n \to \infty$ are equivalent.

The following lemma shows in which way the convergence in probability can be characterised via the convergence with probability 1 for subsequences.

Lemma 1.3.4. *Random variables* $\xi_\varepsilon \xrightarrow{\text{P}} \xi_0$ *as* $\varepsilon \to 0$ *if and only if any subsequence* $0 \leq \varepsilon_n \to 0$ *as* $n \to \infty$ *contains a subsequence* $\varepsilon'_k = \varepsilon_{n_k}$*, where* $n_k \to \infty$ *as* $k \to \infty$*, such that* $\xi_{\varepsilon'_k} \xrightarrow{\text{P1}} \xi_0$ *as* $k \to \infty$.

Let also formulate the following useful lemma.

Lemma 1.3.5. *If, for some subsequence* $\varepsilon_n \to 0$ *as* $n \to \infty$, **(α)** *the random variables* $\xi_{\varepsilon_n} \xrightarrow{\mathsf{P1}} \xi_0$ *as* $n \to \infty$, *and* **(β)** *the non-negative integer random variables* $\mu_n \xrightarrow{\mathsf{P}} \infty$ *as* $n \to \infty$, *then* $\xi_{\varepsilon_{\mu_n}} \xrightarrow{\mathsf{P}} \xi_0$ *as* $n \to \infty$.

1.3.6. Skorokhod representation theorem. This theorem helps to simplify proofs of many results on weak convergence by replacing weakly converging random variables with random variables that have the same distributions and converge almost sure.

We use the symbol $\tilde{\xi}_\varepsilon \stackrel{d}{=} \xi_\varepsilon$ to indicate that the random variables $\tilde{\xi}_\varepsilon$ and ξ_ε, which take values in X, have the same distribution, i.e., $\mathsf{P}\{\tilde{\xi}_\varepsilon \in A\} = \mathsf{P}\{\xi_\varepsilon \in A\}$ for $A \in \mathfrak{B}$.

Theorem 1.3.6. *If* $\xi_\varepsilon \Rightarrow \xi_0$ *as* $\varepsilon \to 0$, *it is possible to construct a probability space* $(\Omega, \mathfrak{F}, \mathsf{P})$ *and random variables* $\tilde{\xi}_\varepsilon$, $\varepsilon \geq 0$, *defined on this probability space such that* **(α)** $\tilde{\xi}_\varepsilon \stackrel{d}{=} \xi_\varepsilon$ *for every* $\varepsilon \geq 0$, *and* **(β)** $\tilde{\xi}_\varepsilon \xrightarrow{\mathsf{P1}} \tilde{\xi}_0$ *as* $\varepsilon \to 0$.

In the case of real-valued random variables, this construction was described in Section 1.1. In the case of a Polish space, the proof is based on a much more sophisticated construction which, however, resembles the one described above. Theorem 1.3.6 belongs to Skorokhod (1956). Let us briefly describe the original procedure from this work, since it is not very easy to find it in the literature.

The first step is to construct a hierarchical system of shrinking Borel sets $S_{i_1,\dots,i_k} \subset X$, $i_1, \dots, i_k$, $k \geq 1$, such that: **(a)** for every $k \geq 1$, the sets $S_{i_1,\dots,i_k} \cap S_{i'_1,\dots,i'_k} = \varnothing$ if $i_k \neq i'_k$, **(b)** $\cup_{i_k \geq 1} S_{i_1,\dots,i_k} = S_{i_1,\dots,i_{k-1}}$, $k > 1$, and $\cup_{i_1 \geq 1} S_{i_1} = X$, **(c)** $\sup_{x,y \in S_{i_1,\dots,i_k}} d(x, y) \leq 2^{-k}$ for all $i_1, \dots, i_k$, $k \geq 1$, and **(d)** $F_0(\partial S_{i_1,\dots,i_k}) = 0$ for all $i_1, \dots, i_k$, $k \geq 1$.

Such a hierarchical system can be constructed in the following way. Since the space X is separable, it is possible to find, for every $k \geq 1$, a sequence of points $x_{i,k}$, $i = 1, 2, \dots$, such that every point of X lies at a distance not greater than $2^{-(k+1)}$ to at least one point from this sequence. It is possible to find $2^{-(k+1)} < r_k < 2^{-k}$ such that $F_0(B_{r_k}(x_{i,k})) = 0$ for $i \geq 1$ and every $k \geq 1$ (there exists at most a countable number of $2^{-(k+1)} < r < 2^{-k}$ for which this probability is positive). Then one can define $S_{i_1,\dots,i_k} = W_{i_1,1} \cap \dots \cap W_{i_k,k}$, where $W_{i,k} = B_{r_k}(x_{i,k}) \setminus B_{r_k}(x_{1,k}) \cap \dots \cap B_{r_k}(x_{i-1,k})$.

The second step is to construct a similar hierarchical system of sub-intervals of the interval $[0, 1]$. We define $I_{\varepsilon,i_1,\dots,i_k}$, $i_1, \dots, i_k$, $k \geq 1$, to be sub-intervals of $[0, 1]$ such that **(e)** for $k \geq 1$, the intervals $I_{\varepsilon,i_1,\dots,i_k} \cap I_{\varepsilon,i'_1,\dots,i'_k} = \varnothing$ if $i_k \neq i'_k$, **(f)** the interval $I_{\varepsilon,i_1,\dots,i_k}$ lies to the left of $I_{\varepsilon,i'_1,\dots,i''_k}$ if there exists r such that $i_1 = i'_1, \dots, i_{r-1} = i'_{r-1}$, $i_r < i'_r$, **(g)** the length of $I_{\varepsilon,i_1,\dots,i_k} = F_\varepsilon(S_{i_1,\dots,i_k})$ for $i_1, \dots, i_k$, $k \geq 1$.

The third step is to define appropriate measurable functions $f_\varepsilon(y)$ acting from $[0, 1]$ into X. Let us choose a point $x_{i_1,\dots,i_k} \in S_{i_1,\dots,i_k}$, $i_1, \dots, i_k$, $k \geq 1$, and then, for every $\varepsilon \geq 0$ and $k \geq 1$, define functions $f_{\varepsilon,k}(y) = x_{i_1,\dots,i_k}$ for $y \in I_{\varepsilon,i_1,\dots,i_k}$, $i_1, \dots, i_k \geq 1$. Since $d(f_{\varepsilon,k}(y), f_{\varepsilon,k+m}(y)) \leq 2^{-k}$, there exist limits $f_\varepsilon(y) = \lim_{k\to\infty} f_{\varepsilon,k}(y)$ for every $y \in [0, 1]$

and $\varepsilon \geq 0$. The length of the intervals $I_{\varepsilon,i_1,\dots,i_k}$ converge to that of $I_{0,i_1,\dots,i_k}$ as $\varepsilon \to 0$ and, therefore, for internal points y of the intervals $I_{0,i_1,\dots,i_k}$, one has that $\lim_{\varepsilon\to 0} d(f_\varepsilon(y), f_0(y)) \leq 2^{-(k+1)}$. This shows that **(h)** $\lim_{\varepsilon\to 0} f_\varepsilon(y) = f_0(y)$ for all $y \in [0, 1]$ except for at most a countable set of points.

Let now ρ be a random variable uniformly distributed in $[0, 1]$. Due to **(h)**, the random variables $\tilde{\xi}_\varepsilon = f_\varepsilon(\rho) \xrightarrow{\mathsf{P1}} \tilde{\xi}_0 = f_0(\rho)$ as $\varepsilon \to 0$. It is also not difficult to check that the random variable $\tilde{\xi}_\varepsilon$ has the distribution $F_\varepsilon(A)$ for every $\varepsilon \geq 0$.

A typical application of Theorem 1.3.6 relates to proofs of Theorems 1.3.1 and 1.3.2. Let $\xi_\varepsilon \Rightarrow \xi_0$ as $\varepsilon \to 0$, and $\tilde{\xi}_\varepsilon$ be the random variables constructed according to Theorem 1.3.6. Then, for any function f a.s. continuous with respect to the distribution of the limiting random variable ξ_0, the random variables $f(\tilde{\xi}_\varepsilon)$ converge with probability 1 to $f(\tilde{\xi}_0)$. But $f(\tilde{\xi}_\varepsilon) \stackrel{\mathsf{d}}{=} f(\tilde{\xi}_\varepsilon)$. Since the a.s. convergence implies the weak convergence, the random variables $f(\xi_\varepsilon)$ converge weakly to $f(\xi_0)$. Also, in the case where the function f is bounded, $\mathsf{E}f(\tilde{\xi}_\varepsilon)$ converge to $\mathsf{E}f(\tilde{\xi}_0)$ via the Lebesgue theorem. Hence, $\mathsf{E}f(\xi_\varepsilon)$ converge to $\mathsf{E}f(\xi_0)$, since $\mathsf{E}f(\tilde{\xi}_\varepsilon) = \mathsf{E}f(\xi_\varepsilon)$.

1.4 The space D of càdlàg functions

In this section, we give a brief survey of facts related to the geometry of the space **D** of càdlàg functions. The special **J**-metric makes this space a Polish space. This allows to consider càdlàg processes as random variables that take values in the Polish space **D** and, therefore, to study limit theorems for càdlàg processes applying general results on weak convergence in metric spaces.

First, we consider the basic case of the space of càdlàg functions defined on a finite interval $[0, T]$. Then we show in which way the results can be extended to the space of càdlàg functions defined on the semi-finite interval $[0, \infty)$ and other types of intervals.

1.4.1. The space of càdlàg functions. Let $\boldsymbol{I} \subseteq [0, \infty)$ be a finite or semi-finite and closed, semi-closed, or open sub-interval of $[0, \infty)$.

In the two main cases, we will be dealing with $[0, T]$ and $[0, \infty)$. However, we will also consider other intervals, e.g., the interval $(0, \infty)$. The intervals $[0, T]$ and $[0, \infty)$ contain the left endpoint 0. The first interval also contains the right endpoint T, while the second one has no right endpoint.

We now introduce $\mathbf{D}_{\boldsymbol{I}}$, a space of m-dimensional *càdlàg functions* defined on the interval $\boldsymbol{I}$.

Definition 1.4.1. $\mathbf{D}_{\boldsymbol{I}}$ is a space of functions $\mathbf{x}(t) = (x_1(t), \dots, x_m(t))$, $t \in \boldsymbol{I}$, which are defined on the interval $\boldsymbol{I}$, take values in $\mathbb{R}_m$, and are continuous from the right, that is, possess finite right limits $\lim_{s>t, s\to t} \mathbf{x}(s) = \mathbf{x}(t)$ at every point $t \in \boldsymbol{I}$ which is not the right endpoint of $\boldsymbol{I}$, and finite left limits $\lim_{s<t, s\to t} \mathbf{x}(s) = \mathbf{x}(t - 0) \in \mathbb{R}_m$ at every point $t \in \boldsymbol{I}$ which is not the left endpoint of $\boldsymbol{I}$.

To stress that a specific interval $\boldsymbol{I}$ is used, we employ the notations $\mathbf{D}_{[0,T]}$, $\mathbf{D}_{[0,\infty)}$, etc. The dimension m is indicated, if confusion can arise, with the notations $\mathbf{D}_{\boldsymbol{I}}^{(m)}$, $\mathbf{D}_{[0,T]}^{(m)}$, etc. The notation $\mathbf{D}$ is used for the *space of càdlàg functions* with no explicit reference to the interval or the dimension of the space. Let us also list some important special subspaces of the space $\mathbf{D}_{\boldsymbol{I}}$.

We denote by $\mathbf{C}_{\boldsymbol{I}}$ the space of m-dimensional continuous functions defined on the interval $\boldsymbol{I}$. To indicate a specific interval $\boldsymbol{I}$, the notations $\mathbf{C}_{[0,T]}$, etc. are used. To specify the dimension m, if needed, the notations $\mathbf{C}_{\boldsymbol{I}}^{(m)}$, $\mathbf{C}_{[0,T]}^{(m)}$, etc., are also utilised. The notation $\mathbf{C}$ refers to the *space of continuous functions* without specifying the interval and the dimension of the space.

We also denote by $\mathbf{D}_{\boldsymbol{I}+}$ the space of m-dimensional functions $\mathbf{x}(t) = (x_1(t), \dots, x_m(t))$, $t \in \boldsymbol{I}$, with components $x_i(t)$, $t \in \boldsymbol{I}$, that are non-negative and non-decreasing càdlàg functions for every $i = 1, \dots, m$. To indicate a specific interval $\boldsymbol{I}$, the notations $\mathbf{D}_{[0,T]+}$, etc. are used. The dimension m is indicated with the notations $\mathbf{D}_{\boldsymbol{I}+}^{(m)}$, $\mathbf{D}_{[0,T]+}^{(m)}$, etc.

The following lemma permits to clarify the structure of the set of discontinuity (jump) points for a càdlàg function.

Lemma 1.4.1. *If a function* $\mathbf{x}(t)$ *belongs to the space* $\mathbf{D}_{[T_1,T_2]}$, *then for every* $\delta > 0$ *there exist points* $T_1 = t_{\delta,0} < \cdots < t_{\delta,n_\delta+1} = T_2$ *such that* $|\mathbf{x}(t')-\mathbf{x}(t'')| \leq \delta$ *for* $t', t'' \in [t_{\delta,i}, t_{\delta,i+1})$, $i = 0, 1, \dots, n_\delta$.

This lemma implies that a càdlàg function has at most a finite set of discontinuity points in which the absolute values of jumps are greater or equal to any $\delta > 0$, if $\boldsymbol{I}$ is a closed finite interval, and at most a countable set of such discontinuity points, if $\boldsymbol{I}$ is a semi-open or open interval. The total number of discontinuity points for a càdlàg function defined on an interval of any type is at most countable.

The following lemma supplements Lemma 1.4.1.

Lemma 1.4.2. *If a function* $\mathbf{x}(t)$ *belongs to the space* $\mathbf{D}_{[T_1,T_2]}$ *and* $T_1 \leq z_{\delta,1} < \cdots < z_{\delta,n_\delta} \leq T_2$ *are points of discontinuity for the function* $\mathbf{x}(t)$ *with the absolute values of jumps not less than* δ, *then there exists* $h_\delta > 0$ *such that* $|\mathbf{x}(t') - \mathbf{x}(t'')| \leq \delta$ *for* $|t' - t''| \leq h_\delta$, t', $t'' \in [z_{\delta,i}, z_{\delta,i+1})$, $i = 0, 1, \dots, n_\delta$ (*here* $z_{\delta,0} = T_1$, $z_{\delta,n_\delta+1} = T_2$).

Let us introduce the *modulus of* **J**-*compactness* which plays the same role for càdlàg functions as the modulus of continuity for continuous functions. We define, for $0 \leq T_1 < T_2$ and $c > 0$,

$$\begin{aligned} &\Delta_J(\mathbf{x}(\cdot), c, T_1, T_2) = \\ &\quad = \sup_{T_1 \vee (t-c) \leq t' \leq t \leq t'' \leq (t+c) \wedge T_2} \min(|\mathbf{x}(t') - \mathbf{x}(t)|, |\mathbf{x}(t) - \mathbf{x}(t'')|). \end{aligned} \tag{1.4.1}$$

The simplified notation $\Delta_J(\mathbf{x}(\cdot), c, T)$ is usually used instead of $\Delta_J(\mathbf{x}(\cdot), c, 0, T)$ in the case of an interval $[0, T]$.

Lemma 1.4.3. *A function* $\mathbf{x}(t)$ *defined on an interval* $\boldsymbol{I}$ *with values in* $\mathbb{R}_m$ *belongs to the space* $\mathbf{D}_{\boldsymbol{I}}$ *if and only if:* **(α)** $\mathbf{x}(t)$ *is a function continuous from the right at points* $t \in \boldsymbol{I}$ *excluding, possibly, the right endpoint, if this interval has such a point, and* **(β)** $\lim_{c\to 0} \Delta_J(\mathbf{x}(\cdot), c, T_1, T_2) = 0$, $[T_1, T_2] \subseteq \boldsymbol{I}$.

Let $[T_1, T_2] \subseteq [T_1', T_2'']$. Then $\Delta_J(\mathbf{x}(\cdot), c, T_1, T_2) \leq \Delta_J(\mathbf{x}(\cdot), c, T_1', T_2'')$. This means that, in the case of a closed finite interval $\boldsymbol{I} = [T', T'']$, it is sufficient to require that condition **(β)** in Lemma 1.4.1 holds only for this interval.

1.4.2. J-topology in the space $\mathbf{D}_{[0,T]}$. It is obvious how the results concerning càdlàg functions defined on an interval $[0, T]$ can be carried over to the case of any interval $[T_1, T_2]$. So, the following consideration is reduced to the case of an interval $[0, T]$.

Let us introduce, in the space $\mathbf{D}_{[0,T]}$, a natural topology of convergence.

The first candidate is the *uniform topology* **U** of convergence. It is generated by the *uniform metric*

$$d_{U,T}(\mathbf{x}(\cdot), \mathbf{y}(\cdot)) = \sup_{0\leq t\leq T} |\mathbf{x}(t) - \mathbf{y}(t)|. \tag{1.4.2}$$

Definition 1.4.2. Functions $\mathbf{x}_\varepsilon(t), t \in [0, T]$ converge in the topology **U** to a function $\mathbf{x}_0(t), t \in [0, T]$ as $\varepsilon \to 0$ ($\mathbf{x}_\varepsilon(t), t \in [0, T] \xrightarrow{\mathbf{U}} \mathbf{x}_0(t), t \in [0, T]$ as $\varepsilon \to 0$) if $d_{U,T}(\mathbf{x}_\varepsilon(\cdot), \mathbf{x}_0(\cdot)) \to 0$ as $\varepsilon \to 0$.

U-topology is very natural for the space of continuous functions $\mathbf{C}_{[0,T]}$. Unfortunately, this topology is too strong for the space $\mathbf{D}_{[0,T]}$.

Let, for example, $x_\varepsilon(t) = \chi(a_\varepsilon \leq t)$, where $a_\varepsilon \neq a$ for $\varepsilon \neq 0$ but $a_\varepsilon \to a_0$ as $\varepsilon \to 0$. For ε small enough, the function $x_\varepsilon(t)$ differs from the function $x_0(t)$ only by a small shift in the time of the jump. However, $d_{U,T}(x_\varepsilon(\cdot), x_0(\cdot)) = 1$ for all $\varepsilon \neq 0$ and, therefore, the functions $x_\varepsilon(t)$ do not converge in the uniform topology to $x_0(t)$ as $\varepsilon \to 0$.

A natural so-called **J**-*topology* of convergence in the space $\mathbf{D}_{[0,T]}$ was introduced by Skorokhod (1955a, 1956). Some times it is also referred to as the Skorokhod topology.

Let $\boldsymbol{\Lambda}_{[0,T]}$ be the space of all continuous strictly monotone mappings $\lambda(t)$ of the interval $[0, T]$ onto itself such that $\lambda(0) = 0$ and $\lambda(T) = T$.

Definition 1.4.3. Functions $\mathbf{x}_\varepsilon(t), t \in [0, T]$ converge in the topology **J** to a function $\mathbf{x}_0(t)$, $t \in [0, T]$ as $\varepsilon \to 0$ ($\mathbf{x}_\varepsilon(t), t \in [0, T] \xrightarrow{\mathbf{J}} \mathbf{x}_0(t), t \in [0, T]$ as $\varepsilon \to 0$) if there exist mappings $\lambda_\varepsilon \in \boldsymbol{\Lambda}_{[0,T]}$ such that: **(α)** $\sup_{0\leq t\leq T} |\lambda_\varepsilon(t) - t| \to 0$ as $\varepsilon \to 0$, and **(β)** $d_{U,T}(\mathbf{x}_\varepsilon(\lambda_\varepsilon(\cdot)), \mathbf{x}_0(\cdot)) \to 0$ as $\varepsilon \to 0$.

Remark 1.4.1. Obviously, a mapping λ belongs to the space $\boldsymbol{\Lambda}_{[0,T]}$ if and only if the corresponding inverse mapping $\lambda^{-1}(t) = \sup(s \geq 0 : \lambda(s) \leq t), t \in [0, T]$ belongs to this space. Also,

$$\begin{aligned} d_{U,T}(\mathbf{x}_\varepsilon(\lambda_\varepsilon(\cdot)), \mathbf{x}_0(\cdot)) &= \sup_{0\le t\le T} |\mathbf{x}_\varepsilon(\lambda_\varepsilon(t)) - \mathbf{x}_0(t)| \\ &= \sup_{0\le t\le T} |\mathbf{x}_\varepsilon(t) - \mathbf{x}_0(\lambda_\varepsilon^{-1}(t))| = d_{U,T}(\mathbf{x}_\varepsilon(\cdot), \mathbf{x}_0(\lambda_\varepsilon^{-1}(\cdot))). \end{aligned} \tag{1.4.3}$$

Thus, the condition **(β)** in Definition 1.4.3 can be replaced by the condition **(β′)** $d_{U,T}(\mathbf{x}_\varepsilon(t), \mathbf{x}_0(\lambda_\varepsilon(\cdot))) \to 0$ as $\varepsilon \to 0$.

It is easy to show that **J**-topology is weaker than **U**-topology.

Lemma 1.4.4. *Let* $\mathbf{x}_\varepsilon(t), t \in [0,T] \xrightarrow{\mathbf{U}} \mathbf{x}_0(t), t \in [0,T]$ *as* $\varepsilon \to 0$. *Then* $\mathbf{x}_\varepsilon(t), t \in [0,T] \xrightarrow{\mathbf{J}} \mathbf{x}_0(t), t \in [0,T]$ *as* $\varepsilon \to 0$.

The proof follows immediately from the definitions of **U** and **J** topologies, since the identical mapping $\lambda(t) = t$ belongs to the space $\mathbf{\Lambda}_{[0,T]}$.

Note also that the functions $x_\varepsilon(t) = \chi(a_\varepsilon \le t)$, used in the example above, do **J**-converge to $x_0(t)$ as $\varepsilon \to 0$. This shows that the implication inverse to the one in Lemma 1.4.4 is not true.

1.4.3. The J-metrics in the space $\mathbf{D}^{(m)}_{[0,T]}$. As is known, the space of continuous functions $\mathbf{C}^{(m)}_{[0,T]}$ with the uniform metric is a Polish space.

There is a question whether it is possible to construct an appropriate metric in space $\mathbf{D}^{(m)}_{[0,T]}$ that would make this space a Polish space.

The metric in the space $\mathbf{D}^{(m)}_{[0,T]}$ that induces a topology of convergence equivalent to the **J**-convergence was constructed by Kolmogorov (1956). It was simplified by Prokhorov (1956). The simplest modification was given in Gikhman and Skorokhod (1965). This metric can defined in the following way:

$$d'_{J,T}(\mathbf{x}(\cdot), \mathbf{y}(\cdot)) = \inf_{\lambda\in\mathbf{\Lambda}_{[0,T]}} (d_{U,T}(\lambda(\cdot), \lambda_0(\cdot)) + d_{U,T}(\mathbf{x}(\lambda(\cdot)), \mathbf{y}(\cdot))), \tag{1.4.4}$$

where $\lambda_0(t) = t, t \in [0,T]$.

Theorem 1.4.1. *Formula* (1.4.4) *introduces a metric in the space* $\mathbf{D}^{(m)}_{[0,T]}$. *Convergence in this metric is equivalent to the* **J***-convergence, i.e., functions* $\mathbf{x}_\varepsilon(t), t \in [0,T] \xrightarrow{\mathbf{J}} \mathbf{x}_0(t), t \in [0,T]$ *as* $\varepsilon \to 0$ *if and only if* $d'_{J,T}(\mathbf{x}_\varepsilon(\cdot), \mathbf{x}_0(\cdot)) \to 0$ *as* $\varepsilon \to 0$.

The space $\mathbf{D}^{(m)}_{[0,T]}$ equipped with the metric $d'_{J,T}$ is a separable. Unfortunately, it is not a complete metric space. For example, the sequence of functions $x_n(t) = \chi(\frac{1}{2} \le t < \frac{1}{2} + \frac{1}{n})$, $t \in [0,T]$ is fundamental in the metric $d'_{J,T}$. But it does not converge in this metric to a function in the space $\mathbf{D}_{[0,T]}$.

The metric in the space $\mathbf{D}^{(m)}_{[0,T]}$ that induces the topology of convergence equivalent to the $\mathbf{J}$-convergence and makes this space a Polish space was constructed by Billingsley (1968).

Let us define, for a function $\lambda \in \mathbf{\Lambda}_{[0,T]}$,

$$\|\lambda\|_T = \sup_{0\leq t,s\leq T, t\neq s} \left| \ln \frac{\lambda(t) - \lambda(s)}{t - s} \right| \leq \infty. \tag{1.4.5}$$

The metric constructed by Billingsley (1968) is given by the following formula:

$$d_{J,T}(\mathbf{x}(\cdot), \mathbf{y}(\cdot)) = \inf_{\lambda \in \mathbf{\Lambda}_{[0,T]}} (\|\lambda\|_T + d_{U,T}(\mathbf{x}(\lambda(\cdot)), \mathbf{y}(\cdot))). \tag{1.4.6}$$

Theorem 1.4.2. *Formula* (1.4.6) *introduces a metric in the space* $\mathbf{D}^{(m)}_{[0,T]}$. *This space, equipped with the metric* $d_{J,T}$, *is a Polish space. The convergence in this metric is equivalent to the* $\mathbf{J}$*-convergence, i.e., functions* $\mathbf{x}_\varepsilon(t), t \in [0, T] \xrightarrow{\mathbf{J}} \mathbf{x}_0(t), t \in [0, T]$ *as* $\varepsilon \to 0$ *if and only if* $d_{J,T}(\mathbf{x}_\varepsilon(\cdot), \mathbf{x}_0(\cdot)) \to 0$ *as* $\varepsilon \to 0$.

1.4.4. Necessary and sufficient conditions for J-convergence. By using the modulus Δ_J, it is possible to give necessary and sufficient conditions for the $\mathbf{J}$-convergence that would not include the mappings $\lambda \in \mathbf{\Lambda}_{[0,T]}$ in an explicit form.

First, let us assume that the following condition holds:

$\mathfrak{O}^{(\mathbf{T})}_{\mathbf{1}}$: $\mathbf{x}_0(T - 0) = \mathbf{x}_0(T)$.

Let us also introduce the following condition of *pointwise convergence*:

$\mathcal{A}_{\mathbf{6}}$: $\mathbf{x}_\varepsilon(t) \to \mathbf{x}_0(t)$ as $\varepsilon \to 0$ for $t \in S$, where S is a subset of $[0, T]$ that is dense in this interval and contains the points 0 and T.

We use also the following $\mathbf{J}$*-compactness* condition:

$\mathfrak{J}_{\mathbf{1}}$: $\lim_{c\to 0} \overline{\lim}_{\varepsilon\to 0} \Delta_J(\mathbf{x}_\varepsilon(\cdot), c, T) = 0$.

Now we can formulate the corresponding *Skorokhod theorem*.

Theorem 1.4.3. *Let condition* $\mathfrak{O}^{(\mathbf{T})}_{\mathbf{1}}$ *hold. In this case, conditions* $\mathcal{A}_{\mathbf{6}}$ *and* $\mathfrak{J}_{\mathbf{1}}$ *are necessary and sufficient for* $\mathbf{J}$*-convergence of càdlàg functions* $\mathbf{x}_\varepsilon(t), t \in [0, T] \xrightarrow{\mathbf{J}} \mathbf{x}_0(t), t \in [0, T]$ *as* $\varepsilon \to 0$.

In the general case, where it is not known whether $\mathfrak{O}^{(\mathbf{T})}_{\mathbf{1}}$ holds or not, condition $\mathcal{A}_{\mathbf{6}}$ must be strengthen in the following way:

$\mathcal{A}_7$: (a) $\mathbf{x}_\varepsilon(t) \to \mathbf{x}_0(t)$ as $\varepsilon \to 0$ for $t \in S$, where S is a subset of $[0, T]$ that is dense in this interval and contains the points 0 and T;

(b) $\mathbf{x}_\varepsilon(T - 0) \to \mathbf{x}_0(T - 0)$ as $\varepsilon \to 0$.

The conditions of **J**-convergence take, in this case, the following form.

Theorem 1.4.4. *Conditions $\mathcal{A}_7$ and $\mathcal{J}_1$ are necessary and sufficient for* **J***-convergence of càdlàg functions* $\mathbf{x}_\varepsilon(t), t \in [0, T] \xrightarrow{\mathbf{J}} \mathbf{x}_0(t), t \in [0, T]$ *as* $\varepsilon \to 0$.

Under conditions $\mathcal{A}_6$ and $\mathcal{J}_1$, condition $\mathcal{A}_7$ (b) is equivalent to the following condition:

$\mathcal{O}_2^{(\mathbf{T})}$: $\lim_{0<c\to 0} \overline{\lim}_{\varepsilon\to 0} |\mathbf{x}_\varepsilon(T - c) - \mathbf{x}_\varepsilon(T - 0)| = 0.$

This remark permits to replace conditions $\mathcal{A}_7$ and $\mathcal{J}_1$ in Theorem 1.4.4 with conditions $\mathcal{O}_2^{(\mathbf{T})}$, $\mathcal{A}_6$, and $\mathcal{J}_1$.

If condition $\mathcal{O}_1^{(\mathbf{T})}$ holds, conditions $\mathcal{A}_6$ and $\mathcal{J}_1$ imply that conditions $\mathcal{A}_7$ (b) and $\mathcal{O}_2^{(\mathbf{T})}$ hold. Thus the conditions of Theorem 1.4.4 reduce to the conditions of Theorem 1.4.3.

1.4.5. Compact sets in $\mathbf{D}_{[0,T]}$. Let $\alpha(t)$ be a continuous increasing function defined for $t > 0$ such that $\alpha(0 + 0) = 0$, and a constant $\beta > 0$. Let us denote by $K[\alpha(\cdot), \beta, T]$ the set of càdlàg functions $\mathbf{x}(t)$, $t \in [0, T]$, such that $\sup_{0\le t\le T} |\mathbf{x}(t)| \le \beta$ and $\Delta_J(\mathbf{x}(\cdot), c, T) \le \alpha(c)$ for $c > 0$.

The following lemma characterises compact sets in $\mathbf{D}_{[0,T]}$.

Lemma 1.4.5. *The set $K[\alpha(\cdot), \beta, T]$ is the space $\mathbf{D}_{[0,T]}$, and for any compact K in the space $\mathbf{D}_{[0,T]}$ there exists a compact $K[\alpha(\cdot), \beta, T] \supseteq K$.*

Let condition $\mathcal{A}_7$ hold. Using Lemma 1.4.5 it is possible to show that, in this case, condition $\mathcal{J}_1$ guarantees that for any $\varepsilon_n \to 0$ the sequence of functions $\mathbf{x}_{\varepsilon_n}(t)$ contains a **J**-convergent subsequence.

For this reason, the quantity $\Delta_J(\mathbf{x}_\varepsilon(\cdot), c, T)$ can be referred to as a *modulus of* **J***-compactness*, and conditions of type $\mathcal{J}_1$ as **J**-compactness conditions.

1.4.6. J-convergence on subintervals. Let $0 < T' < T$. If **(a)** $\mathbf{x}_\varepsilon(t), t \in [0, T] \xrightarrow{\mathbf{J}} \mathbf{x}_0(t), t \in [0, T]$ as $\varepsilon \to 0$, then **(b)** $\mathbf{x}_\varepsilon(T') \to \mathbf{x}_0(T')$ for any point $T' \in [0, T]$ that is a point of continuity for the limiting function. Also, the corresponding modula of **J**-compactness are connected by the inequality **(c)** $\Delta_J(\mathbf{x}_\varepsilon(\cdot), c, T') \le \Delta_J(\mathbf{x}_\varepsilon(\cdot), c, T)$.

So, relation **(a)** implies that **(d)** $\mathbf{x}_\varepsilon(t), t \in [0, T'] \xrightarrow{\mathbf{J}} \mathbf{x}_0(t), t \in [0, T']$ as $\varepsilon \to 0$ if T' is a point of continuity for the limiting function $\mathbf{x}_0(t)$.

Assumption **(a)** does not automatically imply **(b)** without the assumption of continuity of the function $\mathbf{x}_0(t)$ at the point T'. However, **(a)** does imply **(b)** without this assumption, if it is assumed that **(e)** $\mathbf{x}_\varepsilon(T' \pm 0) \to \mathbf{x}_0(T' \pm 0)$ as $\varepsilon \to 0$.

It is useful to note that one can replace **(a)** with the following weaker assumption: **(f)** conditions $\mathcal{A}_6$ and $\mathfrak{J}_1$ hold (as it was pointed out in Subsection 1.4.4, condition $\mathcal{A}_7$ (b) or $\mathfrak{O}_2^{(T)}$ would be required in this case to provide **(a)**). Assumption **(f)** also implies **(d)**, if T' is a continuity point for the limiting function $\mathbf{x}_0(t)$. Indeed, one can always choose a point $T' < T'' < T$ such that T'' is a continuity point for the function $\mathbf{x}_0(t)$. Obviously, **(f)** is satisfied for the point T'' and, therefore, **(a)** holds for the functions $\mathbf{x}_\varepsilon(t), t \in [0, T'']$. Consequently, **(d)** holds. By the same arguments, **(f)** implies **(d)** without the assumption of continuity of the function $\mathbf{x}_0(t)$ at the point T' if **(e)** holds.

1.4.7. J-convergence of transformed càdlàg functions. Let $g(t, \mathbf{x})$ be a continuous function defined on the space $[0, T] \times \mathbb{R}_m$ with values in the space $\mathbb{R}_l$. Let us also $\mathbf{x}_\varepsilon(t), t \in [0, T]$ be càdlàg function from the space $\mathbf{D}^{(m)}_{[0,T]}$. Then functions $g(t, \mathbf{x}_\varepsilon(t)), t \in [0, T]$ belong to the space $\mathbf{D}^{(l)}_{[0,T]}$. the following simple statement readily follows from Definition 1.4.3.

Lemma 1.4.6. *If the functions* $\mathbf{x}_\varepsilon(t), t \in [0, T] \stackrel{\mathbf{J}}{\longrightarrow} \mathbf{x}_0(t), t \in [0, T]$ *as* $\varepsilon \to 0$*, then the functions* $g(t, \mathbf{x}_\varepsilon(t)), t \in [0, T] \stackrel{\mathbf{J}}{\longrightarrow} g(t, \mathbf{x}_0(t)), t \in [0, T]$ *as* $\varepsilon \to 0$.

1.4.8. J-convergence of vector-valued càdlàg functions. Let $\mathbf{x}_{\varepsilon j}(t), t \in [0, T]$ be a m-dimensional càdlàg function for every $j = 1, \dots, r$ and $\varepsilon \geq 0$. Let us also consider the rm-dimensional càdlàg functions $\tilde{\mathbf{x}}_\varepsilon(t) = (\mathbf{x}_{\varepsilon j}(t), j = 1, \dots, r), t \in [0, T]$.

The following useful result belongs to Whitt (1973, 1980).

Lemma 1.4.7. *Let* **(α)** *functions* $\mathbf{x}_{\varepsilon j}(t), t \in [0, T] \stackrel{\mathbf{J}}{\longrightarrow} \mathbf{x}_{0j}(t), t \in [0, T]$ *as* $\varepsilon \to 0$ *for every* $j = 1, \dots, r$*, and* **(β)** $\mathbf{x}_{0j}(t), j = 1, \dots, r$ *have no joint jump points in the interval* $[0, T]$*, then the functions* $\tilde{\mathbf{x}}_\varepsilon(t), t \in [0, T] \stackrel{\mathbf{J}}{\longrightarrow} \tilde{\mathbf{x}}_0(t), t \in [0, T]$ *as* $\varepsilon \to 0$.

1.4.9. J-convergence to continuous functions. The case of **J**-convergence to a continuous function deserves a special consideration.

Let us define the *modulus of* **U**-*compactness* for all $c, T > 0$,

$$\Delta_U(\mathbf{x}(\cdot), c, T) = \sup_{0 \leq t', t'' \leq T, |t' - t''| \leq c} |\mathbf{x}(t') - \mathbf{x}(t'')|. \tag{1.4.7}$$

Lemma 1.4.8. *A function* $\mathbf{x}(t)$ *defined on an interval* $[0, T]$ *is continuous if and only if* $\lim_{c \to 0} \Delta_U(\mathbf{x}(\cdot), c, T) = 0$.

In what follows we assume that condition $\mathcal{A}_6$ holds. Let us introduce the following condition of **U**-compactness:

$\mathcal{U}_1$: $\lim_{c\to 0} \overline{\lim}_{\varepsilon \to 0} \Delta_U(\mathbf{x}_\varepsilon(\cdot), c, T) = 0$.

Note that condition $\mathcal{U}_1$ also includes the relation $\lim_{c\to 0} \Delta_U(\mathbf{x}_0(\cdot), c, T) = 0$. Therefore, under condition $\mathcal{U}_1$, the limiting function $\mathbf{x}_0(t)$, $t \in [0, T]$ in condition $\mathcal{A}_6$ is continuous.

The following theorem is an extension of the *Ascoli-Arzelá theorem* for continuous functions to the case where the pre-limiting functions are càdlàg functions.

Theorem 1.4.5. *Conditions* $\mathcal{A}_6$ *and* $\mathcal{U}_1$ *are necessary and sufficient for* **U***-convergence of càdlàg functions* $\mathbf{x}_\varepsilon(t), t \in [0, T] \xrightarrow{\mathbf{U}} \mathbf{x}_0(t), t \in [0, T]$ *as* $\varepsilon \to 0$*, where* $\mathbf{x}_0(t), t \in [0, T]$ *is a continuous function.*

The following simple inequality connects the modula Δ_J and Δ_U.

Lemma 1.4.9. *Let* $\mathbf{x}(t)$ *and* $\mathbf{y}(t)$ *be two functions from* $\mathbf{D}_{[0,T]}$*. Then, for every* $c, T > 0$,

$$\Delta_J(\mathbf{x}(\cdot) + \mathbf{y}(\cdot), c, T) \leq \Delta_U(\mathbf{x}(\cdot), c, T) + \Delta_J(\mathbf{y}(\cdot), c, T). \tag{1.4.8}$$

Inequality (1.4.8) implies that for all $c, T > 0$,

$$\Delta_J(\mathbf{x}(\cdot), c, T) \leq \Delta_U(\mathbf{x}(\cdot), c, T). \tag{1.4.9}$$

As it was mentioned above, if càdlàg functions $\mathbf{x}_\varepsilon(t)$ **U**-converge to a càdlàg function $\mathbf{x}_0(t)$ as $\varepsilon \to 0$, then they also **J**-converge to this function.

In general, the inverse statement is not true if the limiting function is not continuous. However, it is possible to prove that the topologies of convergence **J** and **U** are equivalent if the limiting function $\mathbf{x}_0(t)$ is continuous.

Theorem 1.4.6. *Càdlàg functions* $\mathbf{x}_\varepsilon(t), t \in [0, T] \xrightarrow{\mathbf{J}} \mathbf{x}_0(t), t \in [0, T]$ *as* $\varepsilon \to 0$*, where* $\mathbf{x}_0(t), t \in [0, T]$ *is a continuous function, if and only if* $\mathbf{x}_\varepsilon(t), t \in [0, T] \xrightarrow{\mathbf{U}} \mathbf{x}_0(t), t \in [0, T]$ *as* $\varepsilon \to 0$.

1.4.10. A decomposition of càdlàg functions based on separation of large jumps, and an alternative definition of J-convergence. Let us denote by $\Delta_t(\mathbf{x}(\cdot)) = \mathbf{x}(t) - \mathbf{x}(t-0)$ the value of the jump of a càdlàg function $\mathbf{x}(\cdot)$ at a point t.

Let us take $\delta > 0$ and decompose $\mathbf{x}(t)$ into a sum of two components, $\mathbf{x}(t) = \mathbf{x}_+^{(\delta)}(t) + \mathbf{x}_-^{(\delta)}(t), t \in [0, T]$. Here $\mathbf{x}_+^{(\delta)}(t) = \sum_{s\leq t} \Delta_s(\mathbf{x}(\cdot))\chi(|\Delta_s(\mathbf{x}(\cdot))| \geq \delta), t \in [0, T]$, and $\mathbf{x}_-^{(\delta)}(t) = \mathbf{x}(t) - \mathbf{x}_+^{(\delta)}(t), t \in [0, T]$. By the definition, $\mathbf{x}_+^{(\delta)}(t)$ is the sum of jumps, of the function $\mathbf{x}(\cdot)$ in the interval $[0, t]$, whose absolute values are greater than or equal to δ. According Lemma 1.4.1, the function $\mathbf{x}(t)$ has at most a finite number of such jumps in the interval $[0, T]$. So, $\mathbf{x}_+^{(\delta)}(t)$ is a step càdlàg function with absolute values of jumps greater than or equal to δ, whereas $\mathbf{x}_-^{(\delta)}(t)$ is a càdlàg function that has no jumps with absolute value greater than or equal to δ.

Let us also denote $\mathbf{x}_\#^{(\delta)}(t) = \sum_{s\leq t} \chi(|\Delta_s(\mathbf{x}(\cdot))| \geq \delta), t \in [0, T]$. By the definition, $\mathbf{x}_\#^{(\delta)}(t)$ is the number of jumps of the function $\mathbf{x}(\cdot)$ in the interval $[0, t]$, whose absolute values are greater than or equal to δ.

In Skorokhod (1955a, 1964), the **J**-topology of convergence was defined in a form equivalent to the one given in Definition 1.4.3. We formulate this alternative definition in the form of a lemma. It is slightly modified to include the case where the right endpoint T may be not a point of continuity of the limiting function.

Let us denote by $Z[\mathbf{x}_0(\cdot)]$ the set of all $\delta > 0$ such that the càdlàg function $\mathbf{x}_0(t), t \in [0, T]$ has no jumps with absolute values equal to δ. The set $Z[\mathbf{x}_0(\cdot)]$ is $(0, \infty)$ except for at most a countable set of points. Let also $S[\mathbf{x}_0(\cdot)]$ denote the set of all points of continuity for the function $\mathbf{x}_0(t), t \in [0, T]$. Note that $0 \in S[\mathbf{x}_0(\cdot)]$.

Lemma 1.4.10. *Functions* $\mathbf{x}_\varepsilon(t), t \in [0, T] \xrightarrow{\mathbf{J}} \mathbf{x}_0(t), t \in [0, T]$ *as* $\varepsilon \to 0$ *if and only if*: **(α)** $\mathbf{x}^{(\delta)}_{\varepsilon,+}(t) \to \mathbf{x}^{(\delta)}_{0,+}(t)$ *as* $\varepsilon \to 0$ *and* $\mathbf{x}^{(\delta)}_{\varepsilon,\#}(t) \to \mathbf{x}^{(\delta)}_{0,\#}(t)$ *as* $\varepsilon \to 0$ *for every* $\delta \in Z[\mathbf{x}_0(\cdot)]$, $t \in S[\mathbf{x}_0(\cdot)] \cup \{T\}$, **(β)** $\lim_{\delta\to 0} \overline{\lim}_{\varepsilon\to 0} d_{U,T}(\mathbf{x}^{(\delta)}_{\varepsilon,-}(\cdot), \mathbf{x}^{(\delta)}_{0,-}(\cdot)) = 0$.

Using this lemma one can easily prove the following useful statement.

Lemma 1.4.11. *Let functions* $\mathbf{x}_\varepsilon(t), t \in [0, T] \xrightarrow{\mathbf{J}} \mathbf{x}_0(t), t \in [0, T]$ *as* $\varepsilon \to 0$. *Then the functions* $(\mathbf{x}_\varepsilon(t), \mathbf{x}^{(\delta)}_{\varepsilon,+}(t), \mathbf{x}^{(\delta)}_{\varepsilon,-}(t)), t \in [0, T] \xrightarrow{\mathbf{J}} (\mathbf{x}_0(t), \mathbf{x}^{(\delta)}_{0,+}(t), \mathbf{x}^{(\delta)}_{0,-}(t)), t \in [0, T]$ *as* $\varepsilon \to 0$ *for every* $\delta \in Z[\mathbf{x}_0(\cdot)]$.

In order to prove this lemma, one can apply Lemmas 1.4.6, 1.4.7 and 1.4.10 to the càdlàg functions $\tilde{\mathbf{x}}_\varepsilon(t) = \mathbf{x}^{(\delta)}_{\varepsilon,+}(t)$ and $\hat{\mathbf{x}}_\varepsilon(t) = \mathbf{x}^{(\delta)}_{\varepsilon,-}(t)$, where $\delta \in Z[\mathbf{x}_0(\cdot)]$.

By the definition, **(a)** $\tilde{\mathbf{x}}^{(\delta')}_{\varepsilon,+}(t) = \mathbf{x}^{(\delta'\vee\delta)}_{\varepsilon,+}(t)$, $\tilde{\mathbf{x}}^{(\delta')}_{\varepsilon,\#}(t) = \mathbf{x}^{(\delta'\vee\delta)}_{\varepsilon,\#}(t)$ and $\tilde{\mathbf{x}}^{(\delta')}_{\varepsilon,-}(t) = \mathbf{x}^{(\delta)}_{\varepsilon,+}(t) - \mathbf{x}^{(\delta'\vee\delta)}_{\varepsilon,+}(t)$. It is obvious that **(b)** $Z[\tilde{\mathbf{x}}_0(\cdot)] = Z[\mathbf{x}_0(\cdot)] \cup (0, \delta)$ and $S[\mathbf{x}_0(\cdot)] \subseteq S[\tilde{\mathbf{x}}_0(\cdot)] \subseteq S[\tilde{\mathbf{x}}^{(\delta')}_{0,+}(\cdot)]$. By the definition, $\tilde{\mathbf{x}}^{(\delta')}_{\varepsilon,+}(t)$ is a step càdlàg function. Also the set $S[\mathbf{x}_0(\cdot)]$ is dense in $[0, T]$. That is why, **(c)** if $t \in S[\tilde{\mathbf{x}}_0(\cdot)]$ then there exist points $t' \leq t \leq t'', t', t'' \in S[\mathbf{x}_0(\cdot)]$ such that $\tilde{\mathbf{x}}^{(\delta')}_{0,+}(t') = \tilde{\mathbf{x}}^{(\delta')}_{0,+}(t) = \tilde{\mathbf{x}}^{(\delta')}_{0,+}(t'')$ and $\tilde{\mathbf{x}}^{(\delta')}_{0,\#}(t') = \tilde{\mathbf{x}}^{(\delta')}_{0,\#}(t) = \tilde{\mathbf{x}}^{(\delta')}_{0,\#}(t'')$. It follows from **(a)** – **(c)** that conditions **(α)** and **(β)** of Lemma 1.4.10 hold for the functions $\tilde{\mathbf{x}}_\varepsilon(t)$. Thus, **(d)** $\tilde{\mathbf{x}}_\varepsilon(t), t \in [0, T] \xrightarrow{\mathbf{J}} \tilde{\mathbf{x}}_0(t), t \in [0, T]$ as $\varepsilon \to 0$.

By the definition, **(e)** $\hat{\mathbf{x}}^{(\delta')}_{\varepsilon,+}(t) = \mathbf{x}^{(\delta'\wedge\delta)}_{\varepsilon,+}(t) - \mathbf{x}^{(\delta)}_{\varepsilon,+}(t)$, $\hat{\mathbf{x}}^{(\delta')}_{\varepsilon,\#}(t) = \mathbf{x}^{(\delta'\wedge\delta)}_{\varepsilon,\#}(t) - \mathbf{x}^{(\delta)}_{\varepsilon,\#}(t)$ and $\hat{\mathbf{x}}^{(\delta')}_{\varepsilon,-}(t) = \mathbf{x}_\varepsilon(t) - \mathbf{x}^{(\delta'\wedge\delta)}_{\varepsilon,+}(t)$. It is obvious that **(f)** $Z[\hat{\mathbf{x}}_0(\cdot)] = Z[\mathbf{x}_0(\cdot)] \cup [\delta, \infty)$ and $S[\mathbf{x}_0(\cdot)] \subseteq S[\hat{\mathbf{x}}_0(\cdot)] \subseteq S[\hat{\mathbf{x}}^{(\delta')}_{0,+}(\cdot)]$. By the definition, $\hat{\mathbf{x}}^{(\delta')}_{\varepsilon,+}(\cdot)$ is a step function. Also the set $S[\mathbf{x}_0(\cdot)]$ is dense in $[0, T]$. That is why, **(g)** if $t \in S[\hat{\mathbf{x}}_0(\cdot)]$ then there exist points $t' \leq t \leq t''$, $t', t'' \in S[\mathbf{x}_0(\cdot)]$ such that $\hat{\mathbf{x}}^{(\delta')}_{0,+}(t') = \hat{\mathbf{x}}^{(\delta')}_{0,+}(t) = \hat{\mathbf{x}}^{(\delta')}_{0,+}(t'')$ and $\hat{\mathbf{x}}^{(\delta')}_{0,\#}(t') = \hat{\mathbf{x}}^{(\delta')}_{0,\#}(t) = \hat{\mathbf{x}}^{(\delta')}_{0,\#}(t'')$. It follows from **(e)** – **(g)** that conditions **(α)** and **(β)** of Lemma 1.4.10 hold for the functions $\hat{\mathbf{x}}_\varepsilon(t)$. Thus, **(h)** $\hat{\mathbf{x}}_\varepsilon(t), t \in [0, T] \xrightarrow{\mathbf{J}} \hat{\mathbf{x}}_0(t), t \in [0, T]$ as $\varepsilon \to 0$.

Obviously, **(i)** the functions $\tilde{\mathbf{x}}_0(t)$ and $\hat{\mathbf{x}}_0(t)$ have no joint jump points in the interval $[0, T]$. Using **(d)**, **(h)**, **(i)**, and Lemma 1.4.7 we get that **(j)** the functions $(\tilde{\mathbf{x}}_\varepsilon(t), \hat{\mathbf{x}}_\varepsilon(t)), t \in [0, T] \xrightarrow{\mathbf{J}} (\tilde{\mathbf{x}}_0(t), \hat{\mathbf{x}}_0(t)), t \in [0, T]$ as $\varepsilon \to 0$. To complete the proof one can apply Lemma 1.4.6. Indeed, the function $(\mathbf{x}_\varepsilon(t), \mathbf{x}^{(\delta)}_{\varepsilon,+}(t), \mathbf{x}^{(\delta)}_{\varepsilon,-}(t)) = (\mathbf{x}^{(\delta)}_{\varepsilon,+}(t) + \mathbf{x}^{(\delta)}_{\varepsilon,-}(t), \mathbf{x}^{(\delta)}_{\varepsilon,+}(t), \mathbf{x}^{(\delta)}_{\varepsilon,-}(t))$ is a continuous transformation of the function $(\mathbf{x}^{(\delta)}_{\varepsilon,+}(t), \mathbf{x}^{(\delta)}_{\varepsilon,-}(t))$.

1.4.11. The M-topology in the space $\mathbf{D}_{[0,T]}$. The **U**-topology is stronger than the **J**-topology of convergence. Let us also introduce an **M**-topology in this space. This topology, introduced by Skorokhod (1956), is weaker than the **J**-topology. The **M**-topology is connected with an important class of maximum and minimum functionals.

Let us introduce a notion of the graph of a function $\mathbf{x}(t), t \in [0, T]$ from the space $\mathbf{D}^{(m)}_{[0,T]}$.

Definition 1.4.4. The graph $\Gamma[\mathbf{x}(\cdot)]$ is a closed set in $\mathbb{R}_m \times [0, T]$ that contains all pairs $(\mathbf{x}, t)$ such that the point $\mathbf{x}$ belongs to the segment $[\mathbf{x}(t-0), \mathbf{x}(t)]$ (the set $\{\mathbf{x}(t-0)+s(\mathbf{x}(t)-\mathbf{x}(t-0)), 0 \le s \le 1\}$).

Let us define, for functions $\mathbf{x}(\cdot), \mathbf{y}(\cdot) \in \mathbf{D}^{(m)}_{[0,T]}$,

$$d_{M,T}(\mathbf{x}(\cdot), \mathbf{y}(\cdot)) = \sup_{(\mathbf{x},t)\in\Gamma[\mathbf{x}(\cdot)]} \inf_{(\mathbf{y},s)\in\Gamma[\mathbf{y}(\cdot)]} (|t-s| + |\mathbf{x}-\mathbf{y}|). \tag{1.4.10}$$

Definition 1.4.5. Functions $\mathbf{x}_\varepsilon(t), t \in [0, T]$ converge in the **M**-topology to a function $\mathbf{x}_0(t), t \in [0, T]$ as $\varepsilon \to 0$ ($\mathbf{x}_\varepsilon(t), t \in [0, T] \xrightarrow{\mathbf{M}} \mathbf{x}_0(t), t \in [0, T]$ as $\varepsilon \to 0$) if $d_{M,T}(\mathbf{x}_\varepsilon(\cdot), \mathbf{x}_0(\cdot)) \to 0$ as $\varepsilon \to 0$.

As was mentioned above, the **J**-topology is stronger than the **M**-topology.

Lemma 1.4.12. *If càdlàg functions* $\mathbf{x}_\varepsilon(t), t \in [0, T] \xrightarrow{\mathbf{J}} \mathbf{x}_0(t), t \in [0, T]$ *as* $\varepsilon \to 0$*, then* $\mathbf{x}_\varepsilon(t), t \in [0, T] \xrightarrow{\mathbf{M}} \mathbf{x}_0(t), t \in [0, T]$ *as* $\varepsilon \to 0$.

It is possible to give an example of càdlàg functions that converge in the **M**-topology but do not converge in the **J**-topology.

In what follows, we assume that condition $\mathcal{A}_6$ holds.

Let us introduce the *modulus of* **M***-compactness*,

$$\Delta_M(\mathbf{x}(\cdot), c, T) = \sup_{t\in[0,T], t'\in[t_c^-, t_c^- + \frac{c}{2}], t''\in[t_c^+ - \frac{c}{2}, t_c^+]} H(\mathbf{x}(t'), \mathbf{x}(t), \mathbf{x}(t'')), \tag{1.4.11}$$

where $t_c^- = 0 \vee t - c$, $t_c^+ = t + c \wedge T$, and $H(\mathbf{x}, [\mathbf{x}', \mathbf{x}''])$ is the distance from the point $\mathbf{x}$ to the segment $[\mathbf{x}', \mathbf{x}'']$, $c, T > 0$.

Let us introduce the following condition of **M**-compactness:

$\mathcal{M}_1$: $\lim_{c\to 0} \overline{\lim}_{\varepsilon\to 0} \Delta_M(\mathbf{x}_\varepsilon(\cdot), c, T) = 0$.

As in the case of the **J**-topology, we formulate the corresponding condition for **M**-convergence in the case where the right endpoint T is a point of continuity for the corresponding limiting function, i.e., condition $\mathcal{O}_1^{(\mathbf{T})}$ holds.

Theorem 1.4.7. *Let condition $\mathcal{O}_1^{(\mathbf{T})}$ hold. In this case, conditions $\mathcal{A}_6$ and $\mathcal{M}_1$ are necessary and sufficient for* **M***-convergence of càdlàg functions* $\mathbf{x}_\varepsilon(t), t \in [0, T] \xrightarrow{\mathbf{M}} \mathbf{x}_0(t), t \in [0, T]$ *as* $\varepsilon \to 0$.

In the general case, if it is not known whether $\mathcal{O}_1^{(\mathbf{T})}$ holds or not, $\mathcal{A}_7$ and $\mathcal{M}_1$ are necessary and sufficient conditions for the **M**-convergence $\mathbf{x}_\varepsilon(t), t \in [0, T] \xrightarrow{\mathbf{M}} \mathbf{x}_0(t), t \in [0, T]$ as $\varepsilon \to 0$.

1.4.12. Geometry of the space $\mathbf{D}_{[0,\infty)}$. The **J**-topology was introduced so far for càdlàg functions defined on a finite interval $[0, T]$. However, all results can be carried over to càdlàg functions defined on intervals of other types. The most interesting is the case of the semi-infinite interval $[0, \infty)$.

We will follow the approach of Stone (1963), whereby the **J**-convergence of càdlàg functions on the interval $[0, \infty)$ is equivalent to the **J**-convergence of these functions on finite intervals $[0, T_n]$ for some sequence $T_n \to \infty$ as $n \to \infty$.

Let us consider the space $\mathbf{D}_{[0,\infty)}^{(m)}$ of m-dimensional càdlàg functions defined on the semi-infinite interval $[0, \infty)$. In this case, the notation $\mathbf{x}(t), t \in [0, \infty)$ is usually replaced by a simpler notation $\mathbf{x}(t), t \geq 0$.

Definition 1.4.6. Functions $\mathbf{x}_\varepsilon(t), t \geq 0$ from the space $\mathbf{D}_{[0,\infty)}^{(m)}$ converge in the topology **J** to a function $\mathbf{x}_0(t), t \geq 0$ as $\varepsilon \to 0$ ($\mathbf{x}_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{J}} \mathbf{x}_0(t), t \geq 0$ as $\varepsilon \to 0$) if there exists a sequence $0 < T_n \to \infty$ as $n \to \infty$ such that **(α)** $\mathbf{x}_\varepsilon(t), t \in [0, T_n] \xrightarrow{\mathbf{J}} \mathbf{x}_0(t), t \in [0, T_n]$ as $\varepsilon \to 0$ for every $n = 1, 2, \ldots$.

As it follows from the remarks made in Subsection 1.4.7, existence of a sequence T_n in Definition 1.4.6 implies that **(α)** is satisfied for any other sequence $0 < T'_n \to \infty$ as $n \to \infty$ if $T'_n, n \geq 1$ are points of continuity for the limiting function.

There was a question whether the definition given above can be replaced with a definition similar to the one used by Skorokhod for finite intervals. Another question was whether there exists a metric that turns the space $\mathbf{D}_{[0,\infty)}^{(m)}$ into a Polish space. Both questions were answered in the affirmative by Lindvall (1973).

Let $\mathbf{\Lambda}$ be the space of all continuous strongly monotone functions $\lambda(t)$ defined on $[0, \infty)$ such that $\lambda(0) = 0$ and $\lambda(t) \to \infty$ as $t \to \infty$.

Lemma 1.4.13. *Functions* $\mathbf{x}_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{J}} \mathbf{x}_0(t), t \geq 0$ *as* $\varepsilon \to 0$ *if and only if there exist functions* $\lambda_\varepsilon \in \mathbf{\Lambda}$ *such that* **(α)** $\sup_{0 \leq t < T} |\lambda_\varepsilon(t) - t| \to 0$ *as* $\varepsilon \to 0$, $T > 0$, *and* **(β)** $\sup_{0 \leq t < T} |\mathbf{x}_\varepsilon(\lambda_\varepsilon(t)) - \mathbf{x}_0(t)| \to 0$ *as* $\varepsilon \to 0$, $T > 0$.

Let us define, for a function $\lambda \in \mathbf{\Lambda}$,

$$||\lambda|| = \sup_{0 \leq t, s < \infty, t \neq s} \left| \ln \frac{\lambda(t) - \lambda(s)}{t - s} \right| \leq \infty. \tag{1.4.12}$$

Let us also define the functions $g_N(t) = 1\chi(t \leq N) + (N+1-t)\chi(N < t \leq N+1)$, $t \in [0,\infty)$, for $= 1, 2, \ldots$. By the definition, for any function $\mathbf{x}(t)$, $t \geq 0$ from the space $\mathbf{D}^{(m)}_{[0,\infty)}$, the product $g_N(t)\mathbf{x}(t)$, $t \geq 0$ is a function from $\mathbf{D}^{(m)}_{[0,\infty)}$, continuous at the point $N+1$, and equal to 0 for $t \geq N+1$.

The following formula defines the desirable metric:

$$d_J(\mathbf{x}(\cdot), \mathbf{y}(\cdot)) = \sum_{N \geq 1} 2^{-N}(1 \wedge \inf_{\lambda \in \mathbf{\Lambda}}(\|\lambda\| + d_{U,N+1}(g_N(\lambda(\cdot))\mathbf{x}(\lambda(\cdot)), g_N(\cdot)\mathbf{y}(\cdot))). \tag{1.4.13}$$

Theorem 1.4.8. *Formula* (1.4.13) *introduces a metric in the space* $\mathbf{D}^{(m)}_{[0,\infty)}$. *This space, equipped with the metric* d_J, *is a Polish space. The convergence in this metric is equivalent to the* **J***-convergence, i.e., functions* $\mathbf{x}_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{J}} \mathbf{x}_0(t), t \geq 0$ *as* $\varepsilon \to 0$ *if and only if* $d_J(\mathbf{x}_\varepsilon(\cdot), \mathbf{x}_0(\cdot)) \to 0$ *as* $\varepsilon \to 0$.

The following condition is an analogue of the weak convergence condition $\mathcal{A}_6$, but relate to the semi-infinite interval $[0, \infty)$:

$\mathcal{A}_8$: $\mathbf{x}_\varepsilon(t) \to \mathbf{x}_0(t)$ as $\varepsilon \to 0$ for $t \in S$, where S is a subset of $[0, \infty)$ that is dense in this interval and contains the point 0.

Also, the following condition is an analogue of the **J**-compactness condition $\mathfrak{J}_1$:

$\mathfrak{J}_2$: $\lim_{c\to 0} \overline{\lim}_{\varepsilon\to 0} \Delta_J(\mathbf{x}_\varepsilon(\cdot), c, T) = 0, T > 0$.

The following theorem is a direct corollary of Theorem 1.4.3 and the definition of **J**-convergence on the semi-infinite interval $[0, \infty)$.

Theorem 1.4.9. *Conditions* $\mathcal{A}_8$ *and* $\mathfrak{J}_2$ *are necessary and sufficient for* **J***-convergence of càdlàg functions,* $\mathbf{x}_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{J}} \mathbf{x}_0(t), t \geq 0$ *as* $\varepsilon \to 0$.

It is useful to note that the **J**-compactness relation in condition $\mathfrak{J}_2$ holds for all $T > 0$ if and only if it holds for some sequence $T_n \to \infty$ as $n \to \infty$.

Condition $\mathfrak{J}_2$ can also be replaced in Theorem 1.4.9 with a similar condition in which the modulus $\Delta_J(\mathbf{x}_\varepsilon(\cdot), c, T)$ is replaced with the modulus $\Delta'_J(\mathbf{x}_\varepsilon(\cdot), c, T)$.

1.4.13. The U and M-topologies in the space $\mathbf{D}_{[0,\infty)}$. Let us now carry over, to the case of semi-infinite interval $[0, \infty)$, the corresponding definitions and results concerning the **U** and **M**-topologies.

Definition 1.4.7. Functions $\mathbf{x}_\varepsilon(t), t \geq 0$ from the space $\mathbf{D}^{(m)}_{[0,\infty)}$ converge in the **U**-topology to a function $\mathbf{x}_0(t), t \geq 0$ as $\varepsilon \to 0$ ($\mathbf{x}_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{U}} \mathbf{x}_0(t), t \geq 0$ as $\varepsilon \to 0$) if there exists a sequence $0 < T_n \to \infty$ as $n \to \infty$ such that **(α)** $\mathbf{x}_\varepsilon(t), t \in [0, T_n] \xrightarrow{\mathbf{J}} \mathbf{x}_0(t), t \in [0, T_n]$ as $\varepsilon \to 0$ for every $n = 1, 2, \ldots$.

The following **U**-compactness condition is an analogue of conditions $\mathcal{U}_1$, but it relates to the semi-infinite interval $[0, \infty)$:

$\mathcal{U}_2$: $\lim_{c\to 0} \overline{\lim}_{\varepsilon\to 0} \Delta_U(\mathbf{x}_\varepsilon(\cdot), c, T) = 0, T > 0$.

Note that condition $\mathcal{U}_2$ also includes the relation $\lim_{c\to 0} \Delta_U(\mathbf{x}_0(\cdot), c, T) = 0, T > 0$. Therefore, under condition $\mathcal{U}_2$, the limiting function $\mathbf{x}_0(t)$, $t \geq 0$ in condition $\mathcal{A}_8$ is continuous.

Theorem 1.4.10. *Conditions $\mathcal{A}_8$ and $\mathcal{U}_2$ are necessary and sufficient for* **U***-convergence of càdlàg functions,* $\mathbf{x}_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{U}} \mathbf{x}_0(t), t \geq 0$ *as* $\varepsilon \to 0$.

The case of the **M**-topology can be considered analogously.

Definition 1.4.8. Functions $\mathbf{x}_\varepsilon(t), t \geq 0$ from the space $\mathbf{D}^{(m)}_{[0,\infty)}$ converge in the **M**-topology to a function $\mathbf{x}_0(t), t \geq 0$ as $\varepsilon \to 0$ ($\mathbf{x}_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{M}} \mathbf{x}_0(t), t \geq 0$ as $\varepsilon \to 0$) if there exists a sequence $0 < T_n \to \infty$ as $n \to \infty$ such that **(α)** $\mathbf{x}_\varepsilon(t), t \in [0, T_n] \xrightarrow{\mathbf{M}} \mathbf{x}_0(t), t \in [0, T_n]$ as $\varepsilon \to 0$ for every $n = 1, 2, \ldots$.

The following **M**-compactness condition is an analogue of condition $\mathcal{M}_1$ but relates to the semi-infinite interval $[0, \infty)$:

$\mathcal{M}_2$: $\lim_{c\to 0} \overline{\lim}_{\varepsilon\to 0} \Delta_M(\mathbf{x}_\varepsilon(\cdot), c, T) = 0, T > 0$.

Theorem 1.4.11. *Conditions $\mathcal{A}_8$ and $\mathcal{M}_2$ are necessary and sufficient for* **M***-convergence of càdlàg functions,* $\mathbf{x}_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{M}} \mathbf{x}_0(t), t \geq 0$ *as* $\varepsilon \to 0$.

1.4.14. J-topology for other types of intervals. The cases of other types of open or semi-open intervals can be treated in a similar way.

In the case of an interval $\boldsymbol{I} = (a, b)$, where $0 \leq a < b \leq \infty$, a nested sequence of intervals is constructed, $a < T_{1n} < T_{2n} < b$, $T_{1n} \to a$, $T_{2n} \to b$ as $n \to \infty$. For a semi-closed interval $\boldsymbol{I} = [a, b)$, we take $a = T_{1n} < T_{2n} < b$, $T_{2n} \to b$, and in the case $I = (a, b]$, $a < T_{1n} < T_{2n} = b$, $T_{1n} \to a$.

One says that càdlàg functions $\mathbf{x}_\varepsilon(t), t \in \boldsymbol{I} \xrightarrow{\mathbf{J}} \mathbf{x}_0(t), t \in \boldsymbol{I}$ as $\varepsilon \to 0$ if $\mathbf{x}_\varepsilon(t), t \in [T_{1n}, T_{2n}] \xrightarrow{\mathbf{J}} \mathbf{x}_0(t), t \in [T_{1n}, T_{2n}]$ as $\varepsilon \to 0$ for some sequence of intervals $[T_{1n}, T_{2n}], n = 1, 2, \ldots$ that satisfy that requirements described above.

In a way analogous to that used for the intervals $[0, T]$ and $[0, \infty)$, it is possible to define the **J**-topology in the space $\mathbf{D}_{\boldsymbol{I}}$ and then to construct a metric $d_{J,\boldsymbol{I}}$ that makes this space a Polish space and that generates a topology of convergence in $\mathbf{D}_{\boldsymbol{I}}$ equivalent to the **J**-topology.

The **U** and **M** topologies of convergence on intervals of other types are introduced in a similar way.

1.5 J-continuous functionals

Conditions of **J**-continuity for various functionals defined on the space **D** of càdlàg functions play an essential role in the theory. They lead to conditions of weak convergence for random functionals defined on càdlàg processes. In this section, we formulate conditions of **J**-continuity for some important functionals.

1.5.1. A.s. J-continuous functionals in space $\mathbf{D}_{[0,T]}$. Let $\mathfrak{B}^{(m)}_{[0,T]}$ be the Borel σ-algebra of subsets of the space $\mathbf{D}^{(m)}_{[0,T]}$ equipped with the metric $d_{J,T}$ (the minimal σ-algebra containing all balls in this space). Let also $f(\mathbf{x}(\cdot))$ be a *measurable functional* (function) acting from the space $\mathbf{D}^{(m)}_{[0,T]}$ into $\mathbb{R}_l$ (the inverse image $f^{-1}(B) = \{\mathbf{x}(\cdot) \colon f(\mathbf{x}(\cdot)) \in B\}$ belongs to the σ-algebra $\mathfrak{B}^{(m)}_{[0,T]}$ for every $B \in \mathfrak{B}_l$, where $\mathfrak{B}_l$ is the Borel σ-algebra of subsets of $\mathbb{R}_l$).

Definition 1.5.1. A functional f is **J**-continuous at a càdlàg function $\mathbf{x}_0(t)$, $t \in [0, T]$ if $f(\mathbf{x}_\varepsilon(\cdot)) \to f(\mathbf{x}_0(\cdot))$ as $\varepsilon \to 0$ for any càdlàg functions $\mathbf{x}_\varepsilon(t), t \in [0, T] \xrightarrow{\mathbf{J}} \mathbf{x}_0(t), t \in [0, T]$ as $\varepsilon \to 0$.

Definition 1.5.2. A measurable set $\mathfrak{C}_J[f]$ of càdlàg functions $\mathbf{x}_0(t), t \in [0, T]$ at which the functional f is **J**-continuous is called the set of **J**-continuity of this functional.

Definition 1.5.3. A functional f is called **J**-continuous if $\mathfrak{C}_J[f] = \mathbf{D}^{(m)}_{[0,T]}$ and it is called a.s. **J**-continuous with respect to a probability measure F on $\mathfrak{B}^{(m)}_{[0,T]}$ if $F(\mathfrak{C}_J[f]) = 1$. The class of all functionals a.s. **J**-continuous with respect to the measure F is denoted by $\mathfrak{H}_J[F]$.

To indicate the interval, the notations $\mathfrak{C}_{J,T}[f]$ and $\mathfrak{H}_{J,T}[F]$ are used instead of $\mathfrak{C}_J[f]$ and $\mathfrak{H}_J[F]$.

Note that in order to show a.s. continuity of a functional f with respect to a measure F it is enough to show that $F(\mathfrak{C}'_J[f]) = 1$ for some Borel set $\mathfrak{C}'_J[f] \subseteq \mathfrak{C}_J[f]$.

The problem concerning conditions under which a functional f is a.s. **J**-continuous with respect to a measure F can be split into two subproblems. The first one is to describe the structure of the set of **J**-continuity $\mathfrak{C}_J[f]$ or some appropriate set $\mathfrak{C}'_J[f]$. Below we give an answer to this question for some important classes of functionals. The second problem is to give conditions which would imply that $F(\mathfrak{C}_J[f]) = 1$ or $F(\mathfrak{C}'_J[f]) = 1$ for the measure F generated by a càdlàg process from a given class.

1.5.2. The value of a càdlàg function and the value of jump at a point. The simplest measurable functional is $f_t^{\pm}(\mathbf{x}(\cdot)) = \mathbf{x}(t \pm 0)$. Here $0 \le t \le T$.

Lemma 1.5.1. *If $0 < t < T$, then $\mathfrak{C}_J[f_t^{\pm}]$ is the set of càdlàg functions that are* **(α)** *continuous at point t. If $t = 0$ or $t = T$, then $\mathfrak{C}_J[f_t^{\pm}] = \mathbf{D}^{(m)}_{[0,T]}$.*

Note also that any càdlàg function $\mathbf{x}(t)$ is continuous at 0. That is why, the assumption of continuity can be omitted in **(α)** if $t = 0$.

As it follows from Theorem 1.4.4, if càdlàg functions $\mathbf{x}_\varepsilon(t), t \in [0, T] \xrightarrow{\mathbf{J}} \mathbf{x}_0(t), t \in [0, T]$ as $\varepsilon \to 0$ then $\mathbf{x}_\varepsilon(T \pm 0) \to \mathbf{x}_0(T \pm 0)$ as $\varepsilon \to 0$. That is why the assumption of continuity can also be omitted in **(α)** if $t = T$.

Let us denote by $\Delta_t(\mathbf{x}(\cdot)) = \mathbf{x}(t) - \mathbf{x}(t - 0)$ the value of the jump of a càdlàg function $\mathbf{x}(\cdot)$ at a point t. Here $0 \le t \le T$. Note that, for any càdlàg function, $\Delta_0(\mathbf{x}(\cdot)) = 0$. The following lemma is a corollary of Lemma 1.5.1.

Lemma 1.5.2. *If* $0 < t < T$, *then* $\mathfrak{C}_J[\Delta_t]$ *is a set of càdlàg functions that are* **(α)** *continuous at the point t. If* $t = 0$ *or* $t = T$, *then* $\mathfrak{C}_J[\Delta_t] = \mathbf{D}^{(m)}_{[0,T]}$.

1.5.3. The sum of large jumps, the number of large jumps, and the maximal jump. Let us denote by $\Sigma^{(\delta)}_{t_1,t_2}(\mathbf{x}(\cdot)) = \sum_{t_1 \le t \le t_2} \Delta_s(\mathbf{x}(\cdot))\chi(|\Delta_s(\mathbf{x}(\cdot))| \ge \delta)$ the sum of all jumps, in the interval $[t_1, t_2]$, the absolute values of which is greater than or equal to δ. Here $0 \le t_1 \le t_2 \le T$, $\delta > 0$.

Lemma 1.5.3. *If* $0 < t_1 \le t_2 < T$, *then* $\mathfrak{C}_J[\Sigma^{(\delta)}_{t_1,t_2}]$ *is a set of càdlàg functions that are* **(α)** *continuous at the points* t_1 *and* t_2, *and* **(β)** $\Delta_t(\mathbf{x}(\cdot)) \ne \delta$ *for all* $t \in [t_1, t_2]$. *If at least one of the points* t_1, t_2 *coincides with* 0 *or T, then the condition of continuity in the definition of the set* $\mathfrak{C}_J[\Sigma^{(\delta)}_{t_1,t_2}]$ *should be omitted for the corresponding endpoint.*

We will also denote by $N^{(\delta)}_{t_1,t_2}(\mathbf{x}(\cdot)) = \sum_{t_1 \le t \le t_2} \chi(|\Delta_s(\mathbf{x}(\cdot))| \ge \delta)$ the number of jumps, in the interval $[t_1, t_2]$, that have absolute values greater than or equal to δ. Here $0 \le t_1 < t_2 \le T$, $\delta > 0$.

Lemma 1.5.4. *The set* $\mathfrak{C}_J[N^{(\delta)}_{t_1,t_2}] = \mathfrak{C}_J[\Sigma^{(\delta)}_{t_1,t_2}]$ *for every* $0 \le t_1 \le t_2 \le T$.

Let us now define the functional $M_{t_1,t_2}(\mathbf{x}_\varepsilon(\cdot)) = \sup_{t_1 \le t \le t_2} |\Delta_t(\mathbf{x}(\cdot))|$ to be the maximal (by absolute value) jump in the interval $[t_1, t_2]$. Here $0 \le t_1 \le t_2 \le T$.

Lemma 1.5.5. *If* $0 < t_1 \le t_2 < T$, *then* $\mathfrak{C}_J[M_{t_1,t_2}]$ *is the set of càdlàg functions that are* **(α)** *continuous at the points* t_1 *and* t_2. *If at least one of the points* t_1, t_2 *coincides with* 0 *or T, then the condition of continuity in the definition of the set* $\mathfrak{C}_J[M_{t_1,t_2}]$ *should be omitted for the corresponding endpoint.*

1.5.4. The moments of large jumps, the values of large jumps and the sums of large jumps. Let us denote $\alpha^{(\delta)}_{kT}(\mathbf{x}(\cdot)) = \inf(s > \alpha^{(\delta)}_{k-1T}(\mathbf{x}(\cdot)) : |\Delta_s(\mathbf{x}(\cdot))| \ge \delta) \wedge T$, $k = 1, 2, \ldots$, where $\alpha^{(\delta)}_{0T}(\mathbf{x}(\cdot)) = 0$. These functionals are the successive moments of large jumps for a càdlàg function $\mathbf{x}(t)$ truncated in time by T.

To simplify the formulation, we describe the corresponding subsets, instead of the sets of **J**-continuity of the functionals introduced in this subsection.

Lemma 1.5.6. *The set* $\mathfrak{C}'_J[\alpha^{(\delta)}_{kT}]$ *of càdlàg functions such that* **(α)** $|\Delta_s(\mathbf{x}(\cdot))| \neq \delta$ *for* $t \in [0, T]$ *is a subset of* $\mathfrak{C}_J[\alpha^{(\delta)}_{kT}]$ *for every* $k = 1, 2, \ldots$.

Also, let us denote $\beta^{(\delta)}_{kT}(\mathbf{x}(\cdot)) = \Delta_{\alpha^{(\delta)}_{kT}(\mathbf{x}(\cdot))}(\mathbf{x}(\cdot))$, $k = 1, 2, \ldots$. These functionals constitute values of successive large jumps for a càdlàg function $\mathbf{x}(t)$. More precisely, $\beta^{(\delta)}_{kT}(\mathbf{x}(\cdot))$ is the value of k-th large jump if $\alpha^{(\delta)}_{kT}(\mathbf{x}(\cdot)) < T$. But $\beta^{(\delta)}_{kT}(\mathbf{x}(\cdot)) = \Delta_T(\mathbf{x}(\cdot))$ if $\alpha^{(\delta)}_{kT}(\mathbf{x}(\cdot)) = T$.

Lemma 1.5.7. *The set* $\mathfrak{C}'_J[\beta^{(\delta)}_{kT}] = \mathfrak{C}'_J[\alpha^{(\delta)}_{kT}]$ *is a subset of* $\mathfrak{C}_J[\beta^{(\delta)}_{kT}]$ *for every* $k = 1, 2, \ldots$.

Let us also introduce, for $k = 0, 1, \ldots$, the functional $\rho^{(\delta)}_{kT}(\mathbf{x}(\cdot)) = \Sigma^{(\delta)}_{0,\alpha_{kT}}(\mathbf{x}(\cdot))$, where $\alpha_{kT} = \alpha^{(\delta)}_{kT}(\mathbf{x}(\cdot))$. This functional is the sum of large jumps for the càdlàg function $\mathbf{x}(t)$ if $\alpha^{(\delta)}_{kT}(\mathbf{x}(\cdot)) < T$. But $\rho^{(\delta)}_{kT}(\mathbf{x}(\cdot)) = \Sigma^{(\delta)}_{0,T}(\mathbf{x}(\cdot))$ if $\alpha^{(\delta)}_{kT}(\mathbf{x}(\cdot)) = T$.

Lemma 1.5.8. *The set* $\mathfrak{C}'_J[\rho^{(\delta)}_{kT}] = \mathfrak{C}'_J[\alpha^{(\delta)}_{kT}]$ *is a subset of* $\mathfrak{C}_J[\rho^{(\delta)}_{kT}]$ *for every* $k = 1, 2, \ldots$.

1.5.5. The maximum and the minimum. Let us define $\mathbf{m}^{\pm}_{t_1,t_2}(\mathbf{x}(\cdot)) = (m^{\pm}_{t_1,t_2}(x_i(\cdot)), i = 1, \ldots, m)$, where $m^{+}_{t_1,t_2}(x_i(\cdot)) = \sup_{t_1 \leq t \leq t_2} x_i(t)$ and $m^{-}_{t_1,t_2}(x_i(\cdot)) = \inf_{t_1 \leq t \leq t_2} x_i(t)$ for $i = 1, \ldots, m$. Here $0 \leq t_1 \leq t_2 \leq T$.

Lemma 1.5.9. *If* $0 < t_1 \leq t_2 < T$, *then* $\mathfrak{C}_J[\mathbf{m}^{\pm}_{t_1,t_2}]$ *is the set of càdlàg functions, that are* **(α)** *continuous at the points* t_1 *and* t_2. *If at least one of the points* t_1, t_2 *coincides with* 0 *or* T, *then the condition of continuity in the definition of the set* $\mathfrak{C}_J[\mathbf{m}^{\pm}_{t_1,t_2}]$ *should be omitted for the corresponding endpoint.*

Note that if **(a)** the functions $\mathbf{x}_\varepsilon(t), t \in [0, T] \stackrel{\mathbf{J}}{\longrightarrow} \mathbf{x}_0(t), t \in [0, T]$ as $\varepsilon \to 0$ and also **(b)** $\mathbf{x}_\varepsilon(t \pm 0) \to \mathbf{x}_0(t \pm 0)$ as $\varepsilon \to 0$ for $t \in V \subseteq [0, T]$, then $\mathbf{m}^{\pm}_{t_1,t_2}(\mathbf{x}_\varepsilon(\cdot)) \to \mathbf{m}^{\pm}_{t_1,t_2}(\mathbf{x}_0(\cdot))$ as $\varepsilon \to 0$ without requiring that the limiting function be continuous at those endpoints t_1, t_2 that belong to the set V.

1.5.6. The exceeding time and the over-jump. Exceeding times are functionals dual to the maximum and the minimum functionals. Let us define $\boldsymbol{\tau}^{\pm}_{\mathbf{a},T}(\mathbf{x}(\cdot)) = (\tau^{\pm}_{a_i,T}(x_i(\cdot)), i = 1, \ldots, m)$, where $\tau^{\pm}_{a_i,T}(x_i(\cdot)) = \inf(t \geq 0\colon \pm x_i(t) > \pm a_i) \wedge T$ for $i = 1, \ldots, m$. Here $\mathbf{a} = (a_1, \ldots, a_m) \in \mathbb{R}_m$.

Lemma 1.5.10. $\mathfrak{C}_J[\boldsymbol{\tau}^{\pm}_{\mathbf{a},T}]$ *is the set of càdlàg functions such that* **(α)** *for every* $i = 1, \ldots, m$ *there do not exist points* $0 \leq t_1 < t_2 \leq T$ *such that* $m^{\pm}_{0,t_1}(x_i(\cdot)) = m^{\pm}_{0,t_2}(x_i(\cdot)) = a_i$.

The formulation of Lemma 1.5.10 will not change if the inequalities $> a_i$ are replaced with the inequalities $\geq a_i$. in the definition of the functionals $\tau^{\pm}_{a_i,T}(x_i(\cdot))$.

An over-jump is a functional defined as $\boldsymbol{\gamma}^{\pm}_{\mathbf{a},T}(\mathbf{x}(\cdot)) = (\gamma^{\pm}_{a_i,T}(x_i(\cdot)), i = 1, \ldots, m)$. Here $\gamma^{\pm}_{a_i,T}(x_i(\cdot)) = x_i(\tau^{\pm}_{a_i,T}(x_i(\cdot))) - a_i$ for $i = 1, \ldots, m$.

Lemma 1.5.11. *The set* $\mathfrak{C}_J[\boldsymbol{\gamma}^{\pm}_{\mathbf{a},T}]$ *is a set of càdlàg functions that satisfy the condition* **(α)** *given in Lemma* 1.5.10, *and* **(β)** *for every* $i = 1, \ldots, m$, *the functional* $m^{\pm}_{0,\tau_i-0}(x_i(\cdot)) \neq a_i$ *if* $m^{\pm}_{0,\tau_i}(x_i(\cdot)) \neq a_i$, *where* $\tau_i = \tau^{\pm}_{a_i,T}(x_i(\cdot))$.

1.5.7. The number of intersects of a strip. This functional can be defined as $\mathbf{v}^+_{T,\mathbf{a}^-,\mathbf{a}^+}(\mathbf{x}(\cdot)) = (\nu^+_{T,a_i^-,a_i^+}(x_i(\cdot)),\ i = 1,\dots,m)$. Here $\nu^+_{T,a_i^-,a_i^+}(x_i(\cdot))$ is the number of times the càdlàg function $x_i(s)$ intersects the strip $[a_i^-, a_i^+]$ from below in the interval $[0,T]$. This functional is set to be equal to k if there are $k+1$ points $0 \le t_0 < t_1 < \cdots < t_k \le T$ such that $x_i(t_0) < a_i^-$, $x_i(t_1) > a_i^+$, $x_i(t_2) < a_i^-, \dots$ and there are no $k+2$ points that have this property. The functionals $\mathbf{v}^-_{T,\mathbf{a}^-,\mathbf{a}^+}(\mathbf{x}(\cdot))$ and $\nu^-_{T,a_i^-,a_i^+}(x_i(\cdot))$, which are the number of intersects of the same strip from above, are defined analogously by replacing the determining inequalities given above by the inequalities $x_i(t_0) > a_i^+$, $x_i(t_1) < a_i^-$, $x_i(t_2) > a_i^+, \dots$.

Lemma 1.5.12. *The set* $\mathfrak{C}_J[\mathbf{v}^\pm_{T,\mathbf{a}^-,\mathbf{a}^+}]$ *is the set of càdlàg functions that satisfy the conditions* **(α)** *and* **(β)** *given, respectively, in Lemmas* 1.5.10 *and* 1.5.11 *for both points* $\mathbf{a}^-$ *and* $\mathbf{a}^+$.

1.5.8. Integral functionals. This is an important class of functionals. Let us consider the case where an integral functional is defined as $I_h(\mathbf{x}(\cdot)) = \int_{[0,T]} h(t,\mathbf{x}(t))\,dt$ (the Lebesgue integration), where h is a real-valued measurable function defined on $[0,\infty) \times \mathbb{R}_m$ and bounded in every finite cube of this set.

Lemma 1.5.13. *Let h be a continuous function. Then* $\mathfrak{C}_J[I_h] = \mathbf{D}^{(m)}_{[0,T]}$.

Let now consider the case when the function h can be discontinuous. Below, $\mathbf{C}_h$ is the set of continuity points for the function h.

In order to simplify the formulation, we prefer to describe an appropriate subset rather than the corresponding set of **J**-continuity.

Lemma 1.5.14. *The set* $\mathfrak{C}'_J[I_h]$ *of càdlàg functions that satisfy the condition*
(α) $\int_{[0,T]} \chi((t,\mathbf{x}(t)) \in \overline{\mathbf{C}}_h)\,dt = 0$ *is a subset of* $\mathfrak{C}_J[I_h]$.

Note that condition **(α)** implies that the function $h(t,\mathbf{x}(t))$ is Riemann integrable in the interval $[0,T]$.

1.5.9. The modulus of J-compactness. The modulus of **J**-compactness, $\Delta_{J,c}(\mathbf{x}(\cdot)) = \Delta_J(\mathbf{x}(\cdot), c, T)$, is also a measurable functional. Here $c > 0$.

Lemma 1.5.15. $\mathfrak{C}_J[\Delta_{J,c}]$ *is the set of càdlàg functions that* **(α)** *have no jumps with the absolute value equal to c.*

1.5.10. A.s. J-continuous functionals for transformed càdlàg functions. The following construction allows to extend the results formulated above.

Let $g(t,\mathbf{x})$ be a continuous function acting from $[0,\infty) \times \mathbb{R}_m$ into $\mathbb{R}_l$. Obviously, the function $y(t) = g(t,\mathbf{x}(t))$ belongs to the space $\mathbf{D}^{(l)}_{[0,T]}$ if the function $\mathbf{x}(t)$ belongs to the space $\mathbf{D}^{(m)}_{[0,T]}$.

According to Lemma 1.4.6, if functions $\mathbf{x}_\varepsilon(t), t \in [0,T] \xrightarrow{\mathbf{J}} \mathbf{x}_0(t), t \in [0,T]$ as $\varepsilon \to 0$, then the functions $\mathbf{y}_\varepsilon(t) = g(t, \mathbf{x}_\varepsilon(t)), t \in [0,T] \xrightarrow{\mathbf{J}} \mathbf{y}_0(t) = g(t, \mathbf{x}_0(t)), t \in [0,T]$ as $\varepsilon \to 0$.

The conditions of **J**-continuity formulated in Lemmas 1.5.1 – 1.5.15 can be applied to transformed càdlàg functions. Let f be a measurable functional acting from $\mathbf{D}^{(l)}_{[0,T]}$ to $\mathbb{R}_k$. Then $f_g(\mathbf{x}(\cdot)) = f(g(\cdot, \mathbf{x}(\cdot)))$ is a measurable functional acting from $\mathbf{D}^{(m)}_{[0,T]}$ to $\mathbb{R}_k$.

Let us denote by $\mathfrak{C}^{(l)}_J[f]$ the set of **J**-continuity of the functional f in the space $\mathbf{D}^{(l)}_{[0,T]}$ and by $\hat{\mathfrak{C}}^{(m)}_J[f_g]$ the set of functions $\mathbf{x}(t)$ from the space $\mathbf{D}^{(m)}_{[0,T]}$ such that the corresponding transformed function $\mathbf{y}(t) = g(t, \mathbf{x}(t))$ belongs to $\mathfrak{C}^{(l)}_J[f]$. By the definition, the set $\hat{\mathfrak{C}}^{(m)}_J[f_g] \subseteq \mathfrak{C}^{(m)}_J[f_g]$. Here $\mathfrak{C}^{(m)}_J[f_g]$ is the set of **J**-continuity of the functional f_g.

Therefore, condition $F(\hat{\mathfrak{C}}^{(m)}_J[f_g]) = 1$ is a sufficient condition for a.s. **J**-continuity of the functional f_g with respect to the measure F.

1.5.11. A.s. J-continuous functionals on the space $\mathbf{D}_{[0,\infty)}$. The definitions that follow repeat those given above for the space $\mathbf{D}_{[0,T]}$.

Let $\mathfrak{B}^{(m)}_{[0,\infty)}$ be the Borel σ-algebra of subsets of the space $\mathbf{D}^{(m)}_{[0,\infty)}$ equipped with the metric d_J and $f(\mathbf{x}(\cdot))$ be a measurable functional acting from the space $\mathbf{D}^{(m)}_{[0,\infty)}$ into $\mathbb{R}_l$.

A functional f is **J**-continuous at a càdlàg function $\mathbf{x}_0(t), t \geq 0$, if $f(\mathbf{x}_\varepsilon(\cdot)) \to f(\mathbf{x}_0(\cdot))$ as $\varepsilon \to 0$ for any càdlàg functions $\mathbf{x}_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{J}} \mathbf{x}_0(t), t \geq 0$ as $\varepsilon \to 0$.

The measurable set $\mathfrak{C}_J[f]$ of càdlàg functions $\mathbf{x}_0(\cdot)$ at which the functional f is **J**-continuous is called the set of **J**-continuity of this functional.

A functional f is called **J**-continuous if $\mathfrak{C}_J[f] = \mathbf{D}^{(m)}_{[0,\infty)}$ and it is called a.s. **J**-continuous with respect to a probability measure F on $\mathfrak{B}^{(m)}_{[0,\infty)}$ if $F(\mathfrak{C}_J[f]) = 1$. The class of all functionals that are a.s. **J**-continuous with respect to the measure F is denoted by $\mathfrak{H}_J[F]$.

To exhibit the interval, the notations $\mathfrak{C}_{J,\infty}[f]$ and $\mathfrak{H}_{J,\infty}[F]$ are used instead of $\mathfrak{C}_J[f]$ and $\mathfrak{H}_J[F]$, respectively.

J-convergence for càdlàg functions that are defined on the interval $[0,\infty)$ is defined via **J**-convergence of their time-truncations on finite intervals. This makes it possible to connect **J**-continuity of functionals defined on $\mathbf{D}^{(m)}_{[0,\infty)}$ with **J**-continuity of time-truncated versions of these functionals.

Let us define, for a measurable functional $f(\mathbf{x}(\cdot))$ defined on $\mathbf{D}^{(m)}_{[0,\infty)}$, the time-truncated version of this functional, f_T, defined on $\mathbf{D}^{(m)}_{[0,T]}$ to be $f_T(\mathbf{x}_T(\cdot)) = f(\mathbf{x}(\cdot \wedge T))$. Here, $\mathbf{x}_T(t) = \mathbf{x}(t)$, $t \in [0,T]$, is the truncation of the càdlàg function $\mathbf{x}(t)$, $t \in [0,\infty)$, to the interval $[0,T]$ and $\mathbf{x}(t \wedge T)$, $t \in [0,\infty)$, is the càdlàg function that coincides with the function $\mathbf{x}(t)$ for $t \in [0,T]$ and takes the constant value $\mathbf{x}(T)$ for $t > T$.

In many cases, the functional $f(\mathbf{x}(\cdot))$ depends on values that the functions $\mathbf{x}(t)$ take on some finite interval $[0,T]$. In particular, all the functionals considered above have this property. The truncation operation described above allows to reduce consideration to the space $\mathbf{D}^{(m)}_{[0,T]}$.

However, some important functionals can also be defined in an essentially non-truncated version. In this case, the reduction to the case of a finite interval still can be accomplished in the following way.

Let us assume that, for a measurable functional f and a càdlàg function $\mathbf{x}_0(t)$, $t \in [0, \infty)$, there exists a sequence $0 < T_n \to T \leq \infty$ as $n \to \infty$ of continuity points for the function $\mathbf{x}_0(t)$ such that: **(a)** $\lim_{n\to\infty} \overline{\lim}_{\varepsilon\to 0} |f(\mathbf{x}_\varepsilon(\cdot \wedge T_n)) - f(\mathbf{x}_\varepsilon(\cdot))| = 0$ for any càdlàg functions $\mathbf{x}_\varepsilon(t), t \in [0, \infty) \stackrel{\mathbf{J}}{\longrightarrow} \mathbf{x}_0(t), t \in [0, \infty)$ as $\varepsilon \to 0$, and **(b)** the time-truncated version of this functional, f_{T_n}, is **J**-continuous at the truncated function $\mathbf{x}_0(t)$, $t \in [0, T_n]$, for every $n = 1, 2, \ldots$. Then the functional f is **J**-continuous at the function $\mathbf{x}_0(t)$, $t \in [0, \infty)$.

For example, exceeding times give an example of functionals that can be defined in a non-truncated form as $\boldsymbol{\tau}^{\pm}_{\mathbf{a}}(\mathbf{x}(\cdot)) = (\tau^{\pm}_{a_i}(x_i(\cdot)), i = 1, \ldots, m)$, where $\tau^{\pm}_{a_i}(x_i(\cdot)) = \inf(t \geq 0\colon \pm x_i(t) > \pm a_i)$ for $i = 1, \ldots, m$.

Assume that **(c)** $\lim_{t\to\infty} \pm m^{\pm}_{0,t}(x_{0i}(\cdot)) > \pm a_i$, $i = 1, \ldots, m$, and **(d)** $m^{\pm}_{0,t}(x_{0i}(\cdot))$ is a strictly monotone function in t for every $i = 1, \ldots, m$.

Let $0 < T_n \to \infty$ as $n \to \infty$ be a sequence of continuity points of a càdlàg function $\mathbf{x}_0(t)$, $t \in [0, \infty)$. It follows from Lemma 1.5.10 that **(c)** and **(d)** imply **(b)**. Obviously, **(e)** $\tau^{\pm}_a(x(\cdot)) - \tau^{\pm}_{a,T_n}(x(\cdot)) = 0$ if $\pm m^{\pm}_{0,T_n}(x_i(\cdot)) > \pm a_i$. Thus **(b)**, **(c)** and **(e)** yield that **(f)** $\tau^{\pm}_{a_i}(x_{\varepsilon i}(\cdot)) < \infty, i = 1, \ldots, m$, for all ε sufficiently small, and **(a)** holds.

The following lemma follows from the remarks made above.

Lemma 1.5.16. *The set* $\mathfrak{C}'_J[\boldsymbol{\tau}^{\pm}_{\mathbf{a}}]$ *of càdlàg functions that satisfy conditions* **(c)** *and* **(d)** *is a subset of* $\mathfrak{C}_J[\boldsymbol{\tau}^{\pm}_{\mathbf{a}}]$.

In a way similar to the above, the definitions of **J**-continuity can be extended to the spaces $\mathbf{D}_{\boldsymbol{I}}$ for other types of intervals.

1.5.12. A.s. J-continuous mappings. The definitions and some results concerned a.s. **J**-continuous functionals can be generalised to the case of a.s. **J**-continuous mappings. Let $\boldsymbol{I}$ and $\boldsymbol{I}'$ be subintervals of $\mathbb{R}_1$ and $\mathbf{g}$ be a measurable mapping that acts from the space $\mathbf{D}^{(m)}_{\boldsymbol{I}}$ to the space $\mathbf{D}^{(l)}_{\boldsymbol{I}'}$. Let $\mathbf{x}(t), t \in \boldsymbol{I}$ be a function from the space $\mathbf{D}^{(m)}_{\boldsymbol{I}}$. Let us consider a function $\mathbf{x}^{(\mathbf{g})}(\cdot) = \mathbf{g}(\mathbf{x}(\cdot))$. By the definition, the function $\mathbf{x}^{(\mathbf{g})}(t), t \in \boldsymbol{I}'$ belongs to the space $\mathbf{D}^{(l)}_{\boldsymbol{I}'}$.

Definition 1.5.4. A mapping $\mathbf{g}$ is **J**-continuous at a càdlàg function $\mathbf{x}_0(t)$, $t \in \boldsymbol{I}$ if $\mathbf{x}^{(\mathbf{g})}_\varepsilon(t), t \in \boldsymbol{I}' \stackrel{\mathbf{J}}{\longrightarrow} \mathbf{x}^{(\mathbf{g})}_0(t), t \in \boldsymbol{I}'$ as $\varepsilon \to 0$ for any càdlàg functions $\mathbf{x}_\varepsilon(t), t \in \boldsymbol{I} \stackrel{\mathbf{J}}{\longrightarrow} \mathbf{x}_0(t), t \in \boldsymbol{I}$ as $\varepsilon \to 0$.

Definition 1.5.5. A measurable set $\tilde{\mathfrak{C}}_J[\mathbf{g}]$ of càdlàg functions $\mathbf{x}_0(t), t \in \boldsymbol{I}$ at which the mapping $\mathbf{g}$ is **J**-continuous is called the set of **J**-continuity of this mapping.

Definition 1.5.6. A mapping $\mathbf{g}$ is called **J**-continuous if $\tilde{\mathfrak{C}}_J[\mathbf{g}] = \mathbf{D}^{(m)}_{\boldsymbol{I}}$ and it is called a.s. **J**-continuous with respect to a probability measure F on $\mathfrak{B}^{(m)}_{\boldsymbol{I}}$ if $F(\tilde{\mathfrak{C}}_J[\mathbf{g}]) = 1$. The class of all mappings a.s. **J**-continuous with respect to the measure F is denoted by $\tilde{\mathfrak{H}}_J[F]$.

A typical example is a *transformation mapping* $\mathbf{g}: \mathbf{x}^{(\mathbf{g})}(t) = g(t, \mathbf{x}(t)), t \in [0, T]$. Hear $g(t, \mathbf{x})$ is a continuous function defined on the space $[0, T] \times \mathbb{R}_m$ with values in the space $\mathbb{R}_l$. This mapping acts from the space $\mathbf{D}^{(m)}_{[0,T]}$ to the space $\mathbf{D}^{(l)}_{[0,T]}$. The mapping $\mathbf{g}$ is **J**-continuous that follows from Lemma 1.4.6.

Another typical example is a *decomposition mapping* $\mathbf{d}: \mathbf{x}^{(\mathbf{d})}(t) = (\mathbf{x}(t), \mathbf{x}^{(\delta)}_+(t), \mathbf{x}^{(\delta)}_-(t))$, $t \in [0, T]$. Here $\mathbf{x}^{(\delta)}_+(t) = \sum_{s \leq t} \Delta_s(\mathbf{x}(\cdot))\chi(|\Delta_s(\mathbf{x}(\cdot))| \geq \delta), t \in [0, T]$, and $\mathbf{x}^{(\delta)}_-(t) = \mathbf{x}(t) - \mathbf{x}^{(\delta)}_+(t), t \in [0, T]$. This mapping acts from the space $\mathbf{D}^{(m)}_{[0,T]}$ to the space $\mathbf{D}^{(3m)}_{[0,T]}$. The following lemma supplements the result of Lemma 1.4.11.

Lemma 1.5.17. *The set of* **J***-continuity for the mapping* **d** *coincides with the set càdlàg functions that* **(α)** *have not jumps with absolute values equal to* δ.

Let us also consider a *max-mapping* $\mathbf{m}: \mathbf{x}^{(\mathbf{m})}(t) = (\mathbf{x}(t), \mathbf{x}^+(t)), t \in [0, T]$, where $\mathbf{x}(t) = (x_i(t), i = 1, \dots, m) \in \mathbf{D}^{(m)}_{[0,T]}$, $\mathbf{x}^+(t) = (x^+_i(t), i = 1, \dots, m)$ and $x^+_i(t) = \sup_{s \in [0,t]} x_i(s)$, $i = 1, \dots, m$. This mapping acts from the space $\mathbf{D}^{(m)}_{[0,T]}$ to the space $\mathbf{D}^{(2m)}_{[0,T]}$.

The following simple lemma supplement the result of Lemma 1.5.9.

Lemma 1.5.18. *The mapping* **m** *is* **J***-continuous.*

In order to prove this lemma one can apply Definition 1.4.3. According to this definition, if càdlàg functions $\mathbf{x}_\varepsilon(t), t \in [0, T] \xrightarrow{\mathbf{J}} \mathbf{x}_0(t), t \in [0, T]$ as $\varepsilon \to 0$, then there exist mappings $\lambda_\varepsilon \in \mathbf{\Lambda}_{[0,T]}$ such that **(a)** $\sup_{0 \leq t \leq T} |\lambda_\varepsilon(t) - t| \to 0$ as $\varepsilon \to 0$, and **(b)** $d_{U,T}(\mathbf{x}_\varepsilon(\lambda_\varepsilon(\cdot)), \mathbf{x}_0(\cdot)) \to 0$ as $\varepsilon \to 0$. Obviously, **(b)** holds if and only if **(c)** $d_{U,T}(x_{\varepsilon i}(\lambda_\varepsilon(\cdot)), x_{0i}(\cdot)) \to 0$ as $\varepsilon \to 0$, for every $i = 1, \dots, m$. Using **(c)** one get **(d)** $d_{U,T}(x^+_{\varepsilon i}(\lambda_\varepsilon(\cdot)), x^+_{0i}(\cdot)) \leq \sup_{0 \leq t \leq T} \sup_{0 \leq s \leq t} |x_{\varepsilon i}(\lambda_\varepsilon(s)) - x_{0i}(s)| = d_{U,T}(x_{\varepsilon i}(\lambda_\varepsilon(\cdot)), x_{0i}(\cdot)) \to 0$ as $\varepsilon \to 0$, for $i = 1, \dots, m$. But, **(c)** and **(d)** hold if and only if **(e)** $d_{U,T}(\mathbf{x}^{(\mathbf{m})}_\varepsilon(\lambda_\varepsilon(\cdot)), \mathbf{x}^{(\mathbf{m})}_0(\cdot)) \to 0$ as $\varepsilon \to 0$. By Definition 1.4.3, **(e)** implies **J**-convergence of the functions $\mathbf{x}^{(\mathbf{m})}_\varepsilon(t), t \in [0, T]$ to the function $\mathbf{x}^{(\mathbf{m})}_0(t), t \in [0, T]$ as $\varepsilon \to 0$.

In an obvious way the definitions given above can be generalised to the case where **g** be a measurable mapping that acts from some closed with respect to **J**-convergence sub-space $\tilde{\mathbf{D}} \subseteq \mathbf{D}^{(m)}_{\mathbf{I}}$ to the space $\mathbf{D}^{(l)}_{\mathbf{I}'}$.

In context of this book, an important *composition mapping* is an object of special interest. It is defined as $\mathbf{c}: \tilde{\mathbf{x}}^{(\mathbf{c})}(t) = (x_1(y_1(t)), \dots, x_m(y_m(t))), t \in [0, \infty)$ where $\tilde{\mathbf{x}}(t) = (y_1(t), \dots, y_m(t), x_1(t), \dots, x_m(t)), t \in [0, \infty)$ belongs to the space $\mathbf{D}^{(m)}_{[0,\infty)+} \times \mathbf{D}^{(m)}_{[0,\infty)}$. This mapping acts from the space $\mathbf{D}^{(m)}_{[0,\infty)+} \times \mathbf{D}^{(m)}_{[0,\infty)}$ to the space $\mathbf{D}^{(m)}_{[0,\infty)}$.

The notation $\mathbf{x} \circ \mathbf{y}(t)$ can be used in order to show explicitly functions that are composed.

Conditions of a.s. **J**-continuity for the composition mapping are studied in Chapter 3. What is interesting that the compositions $\tilde{\mathbf{x}}^{(\mathbf{c})}_\varepsilon(t) = (x_{\varepsilon 1}(y_{\varepsilon 1}(t)), \dots, x_{\varepsilon m}(y_{\varepsilon m}(t))), t \in [0, \infty)$ can **J**-converge even if the functions $\tilde{\mathbf{x}}_\varepsilon(t) = (y_{\varepsilon 1}(t), \dots, y_{\varepsilon m}(t), x_{\varepsilon 1}(t), \dots, x_{\varepsilon m}(t)), t \in [0, \infty)$ do not **J**-converge. Such statements require to extend the concept of a.s. **J**-continuous mappings.

1.5.13. A.s. U-continuous and a.s. M-continuous functionals. Analogous definitions can be given for the topologies **U** and **M**. The definitions are given in a unified form for both cases of the intervals $\boldsymbol{I} = [0, T]$ and $\boldsymbol{I} = [0, \infty)$, as to avoid repetitions.

In the case of the **U**-topology, the only case of convergence to continuous functions is considered.

A measurable functional f is called **U**-*continuous* at a continuous function $\mathbf{x}_0(t)$ if $f(\mathbf{x}_\varepsilon(\cdot)) \to f(\mathbf{x}_0(\cdot))$ as $\varepsilon \to 0$ for any càdlàg functions $\mathbf{x}_\varepsilon(\cdot) \xrightarrow{\mathbf{U}} \mathbf{x}_0(\cdot)$ as $\varepsilon \to 0$.

The measurable set $\mathfrak{C}_U[f]$ of continuous functions $\mathbf{x}_0(t)$ at which the functional f is **U**-continuous is called the *set of* **U**-*continuity* of this functional.

A functional f is called **U**-*continuous* if $\mathfrak{C}_U[f] = \mathbf{D}_{\boldsymbol{I}}^{(m)}$ and it is called a.s. **U**-continuous with respect to a probability measure F, defined on the Borel σ-algebra of the corresponding space of càdlàg functions, if $F(\mathfrak{C}_U[f]) = 1$. The class of all functionals that are a.s. **U**-continuous with respect to a measure F is denoted by $\mathfrak{H}_U[F]$.

To indicate the interval, the notations $\mathfrak{C}_{U,T}[f]$ and $\mathfrak{H}_{U,T}[F]$ or $\mathfrak{C}_{U,\infty}[f]$ and $\mathfrak{H}_{U,\infty}[F]$ replace $\mathfrak{C}_U[f]$ and $\mathfrak{H}_U[F]$, respectively.

By the definition, the set $\mathfrak{C}_U[f]$ contains only continuous functions. Let us also recall that **J**-convergence and **U**-convergence are equivalent if the limit functions are continuous. So, for any measurable functional f, the set of **U**-continuity coincides with the intersection of the set of **J**-continuity of this functional and the corresponding space **C** of continuous functions.

Lemma 1.5.19. *The set* $\mathfrak{C}_U[f] = \mathfrak{C}_J[f] \cap \mathbf{C}$.

This lemma makes it simple to describe the set $\mathfrak{C}_U[f]$ for most of the functional described in Subsections 1.5.2 – 1.5.9. For example, the set $\mathfrak{C}_{U,T}[f]$ coincides with the the space $\mathbf{C}_{[0,T]}^{(m)}$ for all functionals introduced in Subsections 1.5.2 – 1.5.5.

It is also useful to note also that, due to Lemma 1.5.19, a measurable functional f can be a.s. **U**-continuous with respect to a probability measure F only if the measure F is concentrated on the corresponding space **C** of continuous functions.

In the case of the **M**-topology, all the definitions are analogous.

One says that a measurable functional f is **M**-*continuous* at a càdlàg function $\mathbf{x}_0(t)$ if $f(\mathbf{x}_\varepsilon(\cdot)) \to f(\mathbf{x}_0(\cdot))$ as $\varepsilon \to 0$ for any càdlàg functions $\mathbf{x}_\varepsilon(\cdot) \xrightarrow{\mathbf{M}} \mathbf{x}_0(\cdot)$ as $\varepsilon \to 0$.

The measurable set $\mathfrak{C}_M[f]$ of càdlàg functions $\mathbf{x}_0(t)$ at which the functional f is **M**-continuous is called the *set of* **M**-*continuity* of this functional.

A functional f is called a.s. **M**-*continuous* if $\mathfrak{C}_M[f] = \mathbf{D}_{\boldsymbol{I}}^{(m)}$ and it is called *a.s.* **M**-*continuous* with respect to a probability measure F, defined on the Borel σ-algebra of the corresponding space of càdlàg functions, if $F(\mathfrak{C}_M[f]) = 1$. The class of all functionals that are a.s. **M**-continuous with respect to a measure F is denoted by $\mathfrak{H}_M[F]$.

The cases of intervals $[0, T]$ or $[0, \infty)$ are specified, respectively, by the notations $\mathfrak{C}_{M,T}[f]$ and $\mathfrak{H}_{M,T}[F]$ or $\mathfrak{C}_{M,\infty}[f]$ and $\mathfrak{H}_{M,\infty}[F]$.

A study of conditions for **M**-continuity of concrete functionals is beyond the scope of this book. We will only formulate the related result for maximum and minimum functionals that play an important role in limit theorems for randomly stopped càdlàg processes.

Lemma 1.5.20. $\mathfrak{C}_{M,\infty}[\mathbf{m}^{\pm}_{t_1,t_2}]$ *is the set of càdlàg functions that are* **(α)** *continuous at the points* t_1 *and* t_2.

As it was pointed out by Skorokhod (1956), **M**-convergence can actually be characterised in terms of convergence of maximum and minimum functionals. Namely, $\mathbf{x}_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{M}} \mathbf{x}_0(t), t \geq 0$ as $\varepsilon \to 0$ if and only if $\mathbf{m}^{(\pm)}_{t',t''}(x_\varepsilon(\cdot)) \to \mathbf{m}^{(\pm)}_{t',t''}(x_0(\cdot))$ as $\varepsilon \to 0$ for all points $0 \leq t_1 \leq t_2 < \infty$ that are continuity points for the function $x_0(t)$.

1.6 J-convergence of càdlàg processes

In this section, we give a survey of general results related to functional limit theorems for càdlàg stochastic processes. These processes can be considered as random variables taking values in the corresponding space of càdlàg functions **D**. This space becomes a Polish space if an appropriate metric is introduced. This allows to apply the general theory of weak convergence in metric spaces for obtaining functional limit theorems for càdlàg processes.

1.6.1. Càdlàg stochastic processes. Let us consider a stochastic process $\xi(t) = (\xi_1(t), \ldots, \xi_m(t)), t \in \boldsymbol{I}$, defined on an interval $\boldsymbol{I}$ and taking values in the space $\mathbb{R}_m$. Actually, the process $\xi(t)$ is a family of m-dimensional random variables $\xi(t)$. These random variables are defined on some probability space $(\Omega, \mathfrak{F}, \mathsf{P})$ and depend on a parameter $t \in \boldsymbol{I}$ which should be interpreted as time.

A process $\xi(t), t \in \boldsymbol{I}$ is a *càdlàg process* if, for every outcome $\omega \in \Omega$, the realisation of this process $\{\xi(t, \omega), t \in \boldsymbol{I}\}$ belongs to the space $\mathbf{D}^{(m)}_{\boldsymbol{I}}$.

1.6.2. Finite dimensional distributions and measures generated by càdlàg processes. Let $t_1 < \cdots < t_n$ be a finite sequence of times in the interval $\boldsymbol{I}$. Obviously, $(\xi(t_1), \ldots, \xi(t_n))$ is a random variable (vector) taking values in the space $\mathbb{R}_{mn}$. Let us consider the distribution function of this random vector,

$$\begin{aligned} F_{t_1,\ldots,t_m}(\bar{\mathbf{x}}) &= F_{t_1,\ldots,t_m}(\mathbf{x}_1, \ldots, \mathbf{x}_n) \\ &= \mathsf{P}\{\xi(t_1) \leq \mathbf{x}_1, \ldots, \xi(t_n) \leq \mathbf{x}_n\},\ \bar{\mathbf{x}} = (\mathbf{x}_1, \ldots, \mathbf{x}_n) \in \mathbb{R}_{mn}. \end{aligned}$$

The distribution function given above is in the form of a joint distribution function of the random variables $\xi(t_1), \ldots, \xi(t_n)$. It is called a *finite-dimensional distribution* of the process $\xi(t), t \in \boldsymbol{I}$ at the points $t_1, \ldots, t_n$.

The family of all finite-dimensional distributions for arbitrary varying sequences of times $t_1 < \cdots < t_n$, $t_1, \ldots, t_n \in \boldsymbol{I}$, $n = 1, 2, \ldots$, determines probability properties of the process $\xi(t), t \in \boldsymbol{I}$ in the following sense.

Denote by $\mathfrak{Z}_{\boldsymbol{I}}$ the class of all *cylindric* subsets $\mathfrak{Z}_{t_1,\dots,t_m,\mathbf{x}_1,\dots,\mathbf{x}_n} = \{\mathbf{x}(\cdot) \in \mathbf{D}_I \colon \mathbf{x}(t_1) \leq \mathbf{x}_1, \dots, \mathbf{x}(t_n) \leq \mathbf{x}_n\}$, $\mathbf{x}_1, \dots, \mathbf{x}_n \in \mathbb{R}_m$, $t_1 < \dots < t_n$, $t_1, \dots t_n \in \boldsymbol{I}$, $n = 1, 2, \dots$.

Let us also use the symbol $\mathfrak{B}_{\boldsymbol{I}}^{(m)}$ to denote the minimal σ-algebra which contains all cylindric subsets from the class $\mathfrak{Z}_{\boldsymbol{I}}$.

Via the extension measure theorem, the family of finite-dimensional distributions of the process $\xi(t), t \in \boldsymbol{I}$ uniquely determines a probability measure $F(A)$ on the σ-algebra $\mathfrak{B}_{\boldsymbol{I}}^{(m)}$. This measure takes the values $F(\mathfrak{Z}_{t_1,\dots,t_m,\mathbf{x}_1,\dots,\mathbf{x}_n}) = F_{t_1,\dots,t_m}(\mathbf{x}_1, \dots, \mathbf{x}_n)$ on cylindric sets from the class $\mathfrak{Z}_{\boldsymbol{I}}$. By the definition, $F(A) = \mathsf{P}\{\omega \colon \xi(\omega) = \{\xi(\omega, t), t \in \boldsymbol{I}\} \in A\}$ is a probability that the realisation of the process $\xi(t), t \in \boldsymbol{I}$ belongs to a set A from the σ-algebra $\mathfrak{B}_{\boldsymbol{I}}^{(m)}$. This measure is called a *measure generated by the càdlàg process* $\xi(t), t \in \boldsymbol{I}$.

As was pointed out in Section 1.4, the space $\mathbf{D}_{\boldsymbol{I}}^{(m)}$ can be equipped with a metric $d_{J,\boldsymbol{I}}$ that makes this space a Polish space and such that convergence in this metric is equivalent to **J**-convergence. These metrics were explicitly introduced in Subsections 1.4.3 and 1.4.11 for two most important types of intervals, $[0, T]$ and $[0, \infty)$, respectively. The corresponding procedure was also described in Subsection 1.4.13 for other types of intervals.

We will use the same symbol $\mathfrak{B}_{\boldsymbol{I}}^{(m)}$ to denote the Borel σ-algebra of subsets of $\mathbf{D}_{\boldsymbol{I}}^{(m)}$ equipped with the metric $d_{J,\boldsymbol{I}}$ (the minimal σ-algebra containing all balls in $\mathbf{D}_{\boldsymbol{I}}^{(m)}$). The following theorem explains why the double use of the symbol $\mathfrak{B}_{\boldsymbol{I}}^{(m)}$ is justified.

Theorem 1.6.1. *The minimal* σ*-algebra that contains all cylindric subsets from the class* $\mathfrak{Z}_{\boldsymbol{I}}$ *coincides with the Borel* σ*-algebra of subsets of* $\mathbf{D}_{\boldsymbol{I}}^{(m)}$ *equipped with the metric* $d_{J,\boldsymbol{I}}$.

This theorem allows to consider a càdlàg process $\xi = \{\xi(t), t \in \boldsymbol{I}\}$ defined on a probability space $(\Omega, \mathfrak{F}, \mathsf{P})$ as a random variable defined on this probability space and taking values in the Polish space $\mathbf{D}_{\boldsymbol{I}}^{(m)}$ endowed with the metric $d_{J,\boldsymbol{I}}$. The distribution of the random variable ξ is the measure $F(A)$.

1.6.3. A.s. càdlàg stochastic processes. Let us also give a definition of an a.s. càdlàg process. A process $\xi(t), t \in \boldsymbol{I}$ is an *a.s. càdlàg process* if there exists a set $A_0 \in \mathfrak{F}$ such that: **(a)** for every $\omega \in A_0$, the realisation $\{\xi(t, \omega), t \in I\}$ belongs to the space $\mathbf{D}_{\boldsymbol{I}}^{(m)}$, **(b)** $\mathsf{P}(A_0) = 1$.

It is always possible to replace the realisation of an a.s. càdlàg process $\xi(t), t \in \boldsymbol{I}$ by some fixed càdlàg functions for $\omega \notin A_0$. For example, one can define $\xi'(t, \omega) = \xi(t, \omega)\chi_{A_0}(\omega)$, $\omega \in \Omega$, $t \in \boldsymbol{I}$. The new càdlàg process $\xi'(t)$, $t \in \boldsymbol{I}$ will be *stochastically equivalent* to the old one, i.e., $\mathsf{P}\{\,\omega \colon \xi(t, \omega) = \xi'(t, \omega)\,\} = 1$ for $t \in \boldsymbol{I}$. Moreover, since $\xi(t, \omega) = \xi'(t, \omega), t \in I$ for $\omega \in A_0$, we have that $\mathsf{P}\{\,\omega \colon \xi(t, \omega) = \xi'(t, \omega), t \in \boldsymbol{I}\} = 1$.

This shows that both processes have the same finite-dimensional distributions and, therefore, they generate the same measure $F(A)$ on the σ-algebra $\mathfrak{B}_{\boldsymbol{I}}^{(m)}$.

Let f be a measurable functional that is defined on $\mathbf{D}_{\boldsymbol{I}}^{(m)}$ and takes values in $\mathbb{R}_l$. If $\xi(t), t \in \boldsymbol{I}$ is a càdlàg process, then $f(\xi(\cdot))$ is a random variable that takes values in $\mathbb{R}_l$. It is called a random functional defined on the càdlàg process $\xi(t)$.

If $\xi(t), t \in \boldsymbol{I}$, is an a.s. càdlàg process, then, formally, $f(\xi(\cdot, \omega))$ is possibly not be defined for $\omega \notin A_0$. To avoid this problem, one can define $f(\mathbf{x}(\cdot)) = 0$ for $\mathbf{x}(\cdot) \notin \mathbf{D}_{\boldsymbol{I}}^{(m)}$ and then use the càdlàg modification $\xi'(t), t \in \boldsymbol{I}$, defined above. Since $f(\xi(\cdot, \omega)) = f(\xi'(\cdot, \omega))$ for $\omega \in A_0$, we have that $\mathsf{P}\{\omega : f(\xi(\cdot, \omega)) = f(\xi'(\cdot, \omega))\} = 1$. So, the random variables $f(\xi'(\cdot))$ and $f(\xi(\cdot))$ have the same distribution.

All definitions, limit theorems, and other statements concerning càdlàg stochastic processes can be immediately translated to a.s. càdlàg stochastic processes and, hence, one can always reduce the consideration to càdlàg processes.

Absolutely similar remarks can be made about continuous and a.s. continuous stochastic processes.

A process $\xi(t)$ is a *continuous process* if, for every outcome $\omega \in \Omega$, the realisation of this process $\{\xi(t, \omega), t \in \boldsymbol{I}\}$ belongs to the space $\mathbf{C}_{\boldsymbol{I}}^{(m)}$.

A process $\xi(t)$ is an *a.s. continuous process* if there exists a set $A_0 \in \mathfrak{F}$ such that: **(a)** for every $\omega \in A_0$, the realisation $\{\xi(t, \omega), t \in \boldsymbol{I}\}$ belongs to the space $\mathbf{C}_{\boldsymbol{I}}^{(m)}$, **(b)** $\mathsf{P}(A_0) = 1$.

Any continuous process is a càdlàg process, and any a.s. continuous process is an a.s. càdlàg process. Thus one can consider the measure $F(A)$ generated by the process $\{\xi(t, \omega), t \in \boldsymbol{I}\}$ on the σ-algebra $\mathfrak{B}_{\boldsymbol{I}}^{(m)}$. Actually, this measure is concentrated on the space of continuous functions, $\mathbf{C}_{\boldsymbol{I}}^{(m)}$, i.e., $F(\mathbf{C}_{\boldsymbol{I}}^{(m)}) = 1$.

It is possible to replace realisations of the process $\{\xi(t, \omega), t \in \boldsymbol{I}\}$ with some fixed continuous functions for $\omega \in A_0$. For example, one can define $\xi'(t, \omega) = \xi(t, \omega)\chi_{A_0}(\omega)$, $\omega \in \Omega$, $t \in \boldsymbol{I}$. The new continuous process $\xi'(t)$, $t \in \boldsymbol{I}$, will be stochastically equivalent to the old one, moreover, $\mathsf{P}\{\omega : \xi(t, \omega) = \xi'(t, \omega), t \in \boldsymbol{I}\} = 1$. So, both processes have the same finite-dimensional distributions and, therefore, they generate the same measure $F(A)$.

Again, all definitions, limit theorems, and other statements concerning continuous stochastic processes can immediately be rephrased for a.s. continuous stochastic processes. For this reason, the consideration can always be reduced to the case of continuous processes.

1.6.4. Defining classes for measures generated by càdlàg processes. Let $\xi(t), t \in \boldsymbol{I}$ be a càdlàg process defined on an interval $\boldsymbol{I}$. Let also S be a subset of the interval $\boldsymbol{I}$, which is dense in this interval and contains its endpoints. Because the càdlàg process $\xi(t)$ is continuous from the right, the measure $F(B)$ generated by this process is uniquely determined by the family of finite-dimensional distributions, $F_{t_1,\dots,t_m}(\bar{\mathbf{x}}) = F_{t_1,\dots,t_m}(\mathbf{x}_1, \dots, \mathbf{x}_n)$, taken at the points $\bar{\mathbf{x}} = (\mathbf{x}_1, \dots, \mathbf{x}_m) \in C_{t_1,\dots,t_m}$, $t_1 < \cdots < t_m$, $t_1, \dots, t_m \in S$, $m = 1, 2, \dots$, where $C_{t_1,\dots,t_m}$ is the set of continuity points for the distribution function $F_{t_1,\dots,t_m}(\bar{\mathbf{x}})$.

Denote by $\mathfrak{Z}_{\boldsymbol{I}}[F, S]$ the class of cylindric sets $\mathfrak{Z}_{t_1,\dots,t_m,\mathbf{x}_1,\dots,\mathbf{x}_n}$, $\bar{\mathbf{x}} = (\mathbf{x}_1, \dots, \mathbf{x}_n) \in C_{t_1,\dots,t_m}$, $t_1 < \cdots < t_m$, $t_1, \dots, t_m \in S$, $m = 1, 2, \dots$. Then the fact described above can be formulated in the following way.

Lemma 1.6.1. *The class* $\mathcal{Z}_{\boldsymbol{I}}[F, S]$ *is a defining class for the measure* $F(A)$ *generated by a càdlàg process* $\xi(t)$, $t \in \boldsymbol{I}$.

1.6.5. The set of stochastic continuity of a càdlàg process. It follows from Lemma 1.4.1 that a càdlàg process has, with probability 1, at most a finite number of discontinuity points with the absolute values of jumps greater than a positive number $\delta > 0$ in any finite closed interval. Therefore, a càdlàg process has, with probability 1, at most a countable set of discontinuity points.

Definition 1.6.1. A process $\xi(t)$, $t \in \boldsymbol{I}$ is stochastically continuous at a point $t \in \boldsymbol{I}$ if **(α)** $\xi(t + s) - \xi(t) \xrightarrow{\mathsf{P}} 0$ as $s \to 0$.

If **(α)** holds as $0 < s \to 0$ or $0 > s \to 0$, the process $\xi(t)$, $t \in \boldsymbol{I}$ is called stochastically continuous from the right or from the left, respectively.

By the definition, a càdlàg process is continuous and, therefore, it is also stochastically continuous from the right at any point $t \in \boldsymbol{I}$ that is not the right endpoint of this interval.

Lemma 1.6.2. *The set* S *of stochastic continuity of a càdlàg process* $\xi(t)$, $t \in \boldsymbol{I}$ *is the whole interval* $\boldsymbol{I}$ *excluding at most a countable set of points. The process* $\xi(t)$ *is continuous with probability* 1 *at every point of stochastic continuity.*

It follows from Lemma 1.6.2 that a stochastically continuous càdlàg process can possess only random points of discontinuity. However, a stochastically continuous càdlàg process may be not an a.s continuous process. The following lemma gives conditions for a.s. continuity of a càdlàg process.

Lemma 1.6.3. *A càdlàg process* $\xi(t)$, $t \in \boldsymbol{I}$ *is a.s. continuous if and only if*
(α) $\lim_{c\to 0} \mathsf{P}\{\Delta_U(\xi(\cdot), c, T_1, T_2) > \delta\} = 0, \delta > 0,$ *for every interval* $[T_1, T_2] \subseteq I$.

1.6.6. Weak convergence of càdlàg processes. Let $\xi_\varepsilon(t)$, $t \in \boldsymbol{I}$ be a càdlàg stochastic process for every $\varepsilon \geq 0$. Note that the processes $\xi_\varepsilon(t)$, $t \in \boldsymbol{I}$ can be defined on different probability spaces for different $\varepsilon \geq 0$.

Definition 1.6.2. We say that càdlàg processes $\xi_\varepsilon(t)$ weakly converge to a càdlàg process $\xi_0(t)$ on a set $V \subseteq \boldsymbol{I}$ as $\varepsilon \to 0$ ($\xi_\varepsilon(t), t \in V \Rightarrow \xi_0(t), t \in V$ as $\varepsilon \to 0$) if for any finite sequence of times $t_1 < \cdots < t_n$, $t_1, \ldots, t_n \in V$, $n = 1, 2, \ldots$, the random vectors $(\xi_\varepsilon(t_1), \ldots, \xi_\varepsilon(t_n)) \Rightarrow (\xi_0(t_1), \ldots, \xi_0(t_n))$ as $\varepsilon \to 0$. The set V is called a set of weak convergence.

1.6.7. J-convergence of càdlàg processes. There exist several equivalent ways to define **J**-convergence of càdlàg processes on various intervals $\boldsymbol{I}$.

One universal way is to consider the càdlàg process as a random variable $\xi_\varepsilon = \{\xi_\varepsilon(t), t \in \boldsymbol{I}\}$ taking values in the Polish space $\mathbf{D}_{\boldsymbol{I}}^{(m)}$ with the metric generating the **J**-topology, and then to reduce the definition of **J**-convergence to the definition of weak convergence of the random variables ξ_ε. This method can be used for any kind of intervals $\boldsymbol{I}$.

Such an approach was used by Prokhorov (1956) to introduce **U**-convergence of continuous processes defined on a finite interval. Due to Kolmogorov (1956) and Billingsley (1968), this approach was also extended to the case of **J**-convergence of càdlàg processes.

Theorem 1.3.2, which states that weak convergence of random variables ξ_ε that take values in a Polish space is equivalent to weak convergence of the transformed random variables $f(\xi_\varepsilon)$ for all functions f that are a.s. continuous with respect to the distribution of the limiting random variable ξ_0, gives another universal way to define **J**-convergence of càdlàg processes. **J**-convergence of càdlàg processes can be defined via weak convergence of random functionals $f(\xi_\varepsilon(\cdot))$ for all functionals that are a.s. **J**-continuous with respect to the measure generated by the corresponding limiting càdlàg process.

This way was used by Skorokhod (1956) in his originating paper, where the topology **J** was invented. The advantage of this approach is that it permits to avoid explicit metric considerations and use the same functional approach for other topologies of convergence, for example, **U** and **M**.

Both methods are described below in the basic case of a closed finite interval $[0, T]$ and the semi-infinite interval $[0, \infty)$. There also exists the third equivalent method to define **J** convergence of càdlàg processes on the interval $[0, \infty)$ and other types of semi-open and open intervals. This method was proposed by Stone (1963). Similarly to the case of non-random càdlàg functions, one can define **J**-convergence of càdlàg processes $\xi_\varepsilon(t), t \in \boldsymbol{I}$ via **J**-convergence of the time-truncations of these processes $\xi_\varepsilon(t), t \in \boldsymbol{I}_n$ for some sequence of embedded closed finite intervals $\boldsymbol{I}_n \subseteq \boldsymbol{I}$ such that $\cup_{n \geq 1} \boldsymbol{I}_n = \boldsymbol{I}$.

This method yields an equivalent definition of **J**-convergence and, at the same time, it has a certain advantage. It permits to avoid the explicit consideration of **J**-metrics for the interval $[0, \infty)$ and other semi-open or open intervals. These metrics have structures too complicated to apply them effectively in calculations related to **J**-convergence.

It is appropriate to note that **J**-convergence of càdlàg processes and their weak convergence is not the same. In general, weak convergence of càdlàg processes $\xi_\varepsilon(t)$ on a interval $\boldsymbol{I}$ does not imply their **J**-convergence. Also, **J**-convergence of $\xi_\varepsilon(t)$ processes does not imply weak convergence of these processes on the whole interval $\boldsymbol{I}$. But it does imply that they weakly converge on the set of all points of stochastic continuity of the limiting process.

Actually, the meaning of **J**-convergence for càdlàg processes is that it is equivalent to weak convergence of random functionals that are a.s. **J**-continuous with respect to the measure generated by the corresponding limiting process.

The structures of the sets of **J**-continuity for various functionals have been described in Section 1.5 in the case where the càdlàg functions are defined on the intervals $[0, T]$ or $[0, \infty)$. Of course, the conditions that provide a.s. **J**-continuity of some specific functional with respect to the measure generated by the limiting càdlàg process from a specific class require a special investigation. As a rule, the purpose of such an investigation would be to express **J**-continuity conditions in terms of some natural characteristics of the corresponding limiting process.

For example, this can be effectively done for processes with independent increments and Markov càdlàg processes for the functionals listed in Section 1.5. An exposition of the corresponding results is, however, beyond the scope of this work.

1.6.8. J-convergence of càdlàg processes defined on the interval $[0, T]$. Let, for every $\varepsilon \geq 0$, $\xi_\varepsilon(t)$, $t \in [0, T]$ be a m-dimensional càdlàg stochastic process. As above, the processes $\xi_\varepsilon(t)$, $t \in \boldsymbol{I}$, can be defined on different probability spaces for different ε.

The process $\xi_\varepsilon = \{\xi_\varepsilon(t), t \in [0, T]\}$ can be considered as a random variable taking values in the Polish space $\mathbf{D}^{(m)}_{[0,T]}$ with the **J**-metric $d_{J,T}$ introduced in Subsection 1.4.3. The measure $F_\varepsilon(A)$ generated by the process $\xi_\varepsilon(t), t \in [0, T]$ on the σ-algebra $\mathfrak{B}^{(m)}_{[0,T]}$ can be regarded as distribution of the random variable ξ_ε.

As follows from Theorem 1.3.2, the next two definitions of **J**-convergence of càdlàg processes are equivalent.

Definition 1.6.3. Càdlàg processes $\xi_\varepsilon(t)$, $t \in [0, T]$ converge in the topology **J** to a càdlàg process $\xi_0(t), t \in [0, T]$ as $\varepsilon \to 0$ ($\xi_\varepsilon(t), t \in [0, T] \xrightarrow{\mathbf{J}} \xi_0(t), t \in [0, T]$ as $\varepsilon \to 0$) if **(α)** the random variables $\xi_\varepsilon \Rightarrow \xi_\varepsilon$ as $\varepsilon \to 0$.

Definition 1.6.4. Càdlàg processes $\xi_\varepsilon(t), t \in [0, T] \xrightarrow{\mathbf{J}} \xi_0(t), t \in [0, T]$ as $\varepsilon \to 0$ if **(α)** the random variables $f(\xi_\varepsilon(\cdot)) \Rightarrow f(\xi_0(\cdot))$ as $\varepsilon \to 0$ for every functional $f \in \mathfrak{H}_{J,T}[F_0]$.

Let us introduce the following condition for weak convergence:

$\mathcal{A}_9$: $\xi_\varepsilon(t), t \in S \Rightarrow \xi_0(t), t \in S$ as $\varepsilon \to 0$, where S is a subset of $[0, T]$ that is dense in this interval and contains the points 0 and T.

We also use the following **J**-*compactness* condition:

$\mathfrak{J}_3$: $\lim_{c \to 0} \overline{\lim}_{\varepsilon \to 0} \mathsf{P}\{\Delta_J(\xi_\varepsilon(\cdot), c, T) > \delta\} = 0, \ \delta > 0.$

The next condition is an analogue of condition $\mathfrak{O}^{(\mathbf{T})}_{\mathbf{1}}$:

$\mathfrak{O}^{(\mathbf{T})}_{\mathbf{3}}$: $\mathsf{P}\{\xi_0(T - 0) = \xi_0(T)\} = 1.$

The following *functional limit theorem* belongs to Skorokhod (1956). It gives conditions for **J**-convergence of càdlàg stochastic processes in the case where the corresponding limiting process is stochastically continuous at the right endpoint T.

Theorem 1.6.2. *Let condition $\mathcal{O}_3^{(\mathbf{T})}$ hold. In this case, conditions $\mathcal{A}_9$ and $\mathcal{J}_3$ are necessary and sufficient for* **J***-convergence* $\xi_\varepsilon(t), t \in [0, T] \xrightarrow{\mathbf{J}} \xi_0(t), t \in [0, T]$ *as* $\varepsilon \to 0$.

The original proof of this theorem was given by Skorokhod with the use of his method of one probability space. Later, Billingsley constructed a metric making the space $\mathbf{D}_{[0,T]}^{(m)}$ a Polish space. This permitted to give a proof based on the use of general Prokhorov's theorems about weak convergence in a metric space. A detailed presentation of this proof can be found, for example, in Billingsley (1968, 1999). Let us briefly describe the main steps of the proof based on applying Prokhorov's Theorem 1.3.5.

Let $F_\varepsilon(A)$ be measures generated by the càdlàg processes $\xi_\varepsilon(t), t \in [0, T]$ on the σ-algebra $\mathfrak{B}_{[0,T]}^{(m)}$. Due to Lemma 1.6.1, condition $\mathcal{A}_9$ implies convergence of values of the measures $F_\varepsilon(A)$ for sets A from the class $\mathfrak{Z}_{[0,T]}[F_0, S]$. This is a defining class for the limiting measure $F_0(A)$. Therefore, condition $\mathcal{A}_5$ of Theorem 1.3.5 holds.

According to this theorem, one should also prove the tightness of the family of measures $F_\varepsilon(A)$ as $\varepsilon \to 0$, that is to show that condition $\mathcal{K}_2$ holds for this family.

Here Lemma 1.4.5, which gives the form of compact sets in the space $\mathbf{D}_{[0,T]}^{(m)}$, can be employed. Using this lemma it is possible to show that the tightness of the measures $F_\varepsilon(A)$ as $\varepsilon \to 0$ follows from $\mathcal{J}_3$ and the following two additional conditions: **(a)** $\lim_{c\to 0} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\sup_{0\le t\le T} |\xi_\varepsilon(t)| > \delta\} = 0,\ \delta > 0$, **(b)** $\lim_{c\to 0} \overline{\lim}_{\varepsilon\to 0}(\mathsf{P}\{\Delta_U(\xi_\varepsilon(\cdot), 0, c) > \delta\} + \mathsf{P}\{\Delta_U(\xi_\varepsilon(\cdot), T - c, T) > \delta\}) = 0,\ \delta > 0$. However, it is not difficult to show that conditions $\mathcal{J}_3$, $\mathcal{A}_9$, and $\mathcal{O}_3^{(\mathbf{T})}$ imply **(a)** and **(b)**. Therefore, the tightness condition $\mathcal{K}_2$ holds.

By applying Theorem 1.3.5 to the measures $F_\varepsilon(A)$, one obtains a proof of the statement of sufficiency in Theorem 1.6.2.

The necessity of condition $\mathcal{J}_3$ follows from Lemma 1.5.15 that states that the functionals $\Delta_{J,c}$ are a.s. **J**-continuous with respect to the measure $F_0(A)$ for all $c > 0$ save for at most a countable number of c-values. The necessity of condition $\mathcal{A}_9$ follows from Lemma 1.5.1 that provides a.s. **J**-continuity of the corresponding functionals f_0^+, $f_T^\pm$, and f_t^+ ($f_t^\pm(\mathbf{x}(\cdot)) = \mathbf{x}(t \pm 0)$) for all points t of stochastic continuity of the process $\xi_0(t), t \in [0, T]$. According to Lemma 1.6.2, this set is $[0, T]$ excluding at most a countable set of points. This set is dense in $[0, T]$. It also contains the point 0 and, due to condition $\mathcal{O}_3^{(\mathbf{T})}$, the point T.

In the general case, where it is not known whether $\mathcal{O}_3^{(\mathbf{T})}$ holds or not, condition $\mathcal{A}_9$ must be strengthen in the following way:

$\mathcal{A}_{10}$: $(\xi_\varepsilon(t), \xi_\varepsilon(T - 0)), t \in S \Rightarrow (\xi_0(t), \xi_0(T - 0)), t \in S$ as $\varepsilon \to 0$, where S is a subset of $[0, T]$ that is dense in this interval and contains the points 0 and T.

Conditions for **J**-convergence take, in this case, the following form.

Theorem 1.6.3. *Conditions $\mathcal{A}_{10}$ and $\mathcal{J}_3$ are necessary and sufficient for* **J***-convergence* $\xi_\varepsilon(t), t \in [0, T] \xrightarrow{\mathbf{J}} \xi_0(t), t \in [0, T]$ *as* $\varepsilon \to 0$.

The following condition is an analogue of condition $\mathfrak{O}_2^{(\mathbf{T})}$:

$\mathfrak{O}_4^{(\mathbf{T})}$: $\lim_{0<s\to 0}\overline{\lim}_{\varepsilon\to 0}\mathsf{P}\{|\xi_\varepsilon(T-s)-\xi_\varepsilon(T-0)|>\delta\}=0,\ \delta>0.$

Note that, if $\mathcal{J}_3$ and $\mathfrak{O}_4^{(\mathbf{T})}$ hold, conditions $\mathcal{A}_9$ and $\mathcal{A}_{10}$ are equivalent. Using this, it possible to replace conditions $\mathcal{A}_{10}$ and $\mathcal{J}_3$ in Theorem 1.6.3 with conditions $\mathfrak{O}_4^{(\mathbf{T})}$, $\mathcal{A}_9$, and $\mathcal{J}_3$.

If condition $\mathfrak{O}_3^{(\mathbf{T})}$ holds, then conditions $\mathcal{A}_9$ and $\mathcal{J}_3$ imply conditions $\mathcal{A}_{10}$ and $\mathfrak{O}_4^{(\mathbf{T})}$. So, the conditions of Theorem 1.6.3 reduce to the conditions of Theorem 1.6.2 in this case.

1.6.9. U-convergence of càdlàg processes defined on the interval $[0,T]$. Let us formulate conditions for **U**-convergence of càdlàg processes for the case of an a.s. continuous limiting process. In this case, it is more suitable to use the definition that is based on weak convergence of random a.s. **J**-continuous functionals.

Definition 1.6.5. Càdlàg processes $\xi_\varepsilon(t)$, $t\in[0,T]$ converge in the topology **U** to an a.s. continuous process $\xi_0(t)$, $t\in[0,T]$ as $\varepsilon\to 0$ ($\xi_\varepsilon(t), t\in[0,T]\xrightarrow{\mathbf{U}}\xi_0(t), t\in[0,T]$ as $\varepsilon\to 0$) if the random variables $f(\xi_\varepsilon(\cdot))\Rightarrow f(\xi_0(\cdot))$ as $\varepsilon\to 0$ for every functional $f\in\mathfrak{H}_{U,T}[F_0]$.

Let introduce the following continuity condition:

$\mathcal{B}_1$: $\xi_0(t), t\in[0,T]$ is an a.s. continuous process.

As was mentioned in the Subsection 1.5.12, the equivalence of **J**-convergence and **U**-convergence, if the limiting function is continuous, implies that the set $\mathfrak{C}_{U,T}[f]=\mathfrak{C}_{J,T}[f]\cap\mathbf{C}^{(m)}_{[0,T]}$. Condition $\mathcal{B}_1$ implies, obviously, that $F_0(\mathbf{C}^{(m)}_{[0,T]})=1$. So, $F_0(\mathfrak{C}_{U,T}[f])=1$ if and only if $F_0(\mathfrak{C}_{J,T}[f])=1$. The following lemma follows from these remarks.

Lemma 1.6.4. *Let condition $\mathcal{B}_1$ hold. Then $\xi_\varepsilon(t), t\in[0,T]\xrightarrow{\mathbf{U}}\xi_0(t), t\in[0,T]$ as $\varepsilon\to 0$ if and only if $\xi_\varepsilon(t), t\in[0,T]\xrightarrow{\mathbf{J}}\xi_0(t), t\in[0,T]$ as $\varepsilon\to 0$.*

In the sequel, $\mathcal{A}_9$ is assumed to hold. Let us introduce the following **U**-*compactness* condition:

$\mathcal{U}_3$: $\lim_{c\to 0}\overline{\lim}_{\varepsilon\to 0}\mathsf{P}\{\Delta_U(\xi_\varepsilon(\cdot),c,T)>\delta\}=0,\ \delta>0.$

Note that condition $\mathcal{U}_3$ includes the relation $\lim_{c\to 0}\mathsf{P}\{\Delta_U(\xi_0(\cdot),c,T)>\delta\}=0, \delta>0$. Therefore, under condition $\mathcal{U}_3$, the limiting process $\xi_0(\cdot)$, $t\in[0,T]$ (which appears in condition $\mathcal{A}_9$) is a.s. continuous.

The following *functional limit theorem* belongs to Prokhorov (1956). It gives conditions for **U**-convergence of càdlàg stochastic processes to an a.s. continuous process.

Theorem 1.6.4. *Conditions $\mathcal{A}_9$ and $\mathfrak{U}_3$ are necessary and sufficient for* **U**-*convergence* $\xi_\varepsilon(t), t \in [0, T] \xrightarrow{\mathbf{U}} \xi_0(t), t \in [0, T]$ *as* $\varepsilon \to 0$, *where* $\xi_0(t)$, $t \in [0, T]$ *is an a.s. continuous process.*

Remark 1.6.1. Actually, the original Prokhorov's formulation of Theorem 1.6.4 is concerned the case, when the corresponding pre-limiting processes are also a.s. continuous. Above, the theorem is formulated in the extended form given by Skorokhod (1956), that is, when the limiting process is continuous but the pre-limiting processes can be càdlàg processes.

Due to inequality (1.4.9), condition $\mathfrak{U}_3$ implies $\mathfrak{J}_3$. Also, as was mentioned above, $\mathfrak{U}_3$ implies $\mathcal{B}_1$.

Recall that $\mathcal{A}_9$ and $\mathfrak{J}_3$ are necessary and sufficient conditions for **J**-convergence of the processes $\xi_\varepsilon(t)$, if condition $\mathfrak{O}_3^{(\mathbf{T})}$ holds. Obviously, $\mathcal{B}_1$ implies $\mathfrak{O}_3^{(\mathbf{T})}$. Hence, the processes $\xi_\varepsilon(t)$ **J**-converge and, therefore, they also **U**-converge.

So, under condition $\mathcal{A}_9$, conditions $\mathfrak{J}_3$ and $\mathcal{B}_1$ are equivalent to condition $\mathfrak{U}_3$.

1.6.10. M-convergence of càdlàg processes defined on the interval [0, *T*]. Let us formulate conditions for **M**-convergence of càdlàg processes.

Definition 1.6.6. Càdlàg processes $\xi_\varepsilon(t)$, $t \in [0, T]$ converge in the topology **M** to a càdlàg process $\xi_0(t), t \in [0, T]$ as $\varepsilon \to 0$ ($\xi_\varepsilon(t), t \in [0, T] \xrightarrow{\mathbf{M}} \xi_0(t), t \in [0, T]$ as $\varepsilon \to 0$) if the random variables $f(\xi_\varepsilon(\cdot)) \Rightarrow f(\xi_0(\cdot))$ as $\varepsilon \to 0$ for every functional $f \in \mathfrak{H}_{M,T}[F_0]$.

We assume in what follows that condition $\mathcal{A}_{10}$ holds. Let us introduce the condition:

$\mathfrak{M}_3$: $\lim_{c\to 0} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\Delta_M(\xi_\varepsilon(\cdot), c, T) > \delta\} = 0, \ \delta > 0.$

The *functional limit theorem* gives conditions for **M**-convergence of càdlàg processes.

Theorem 1.6.5. *Conditions $\mathcal{A}_{10}$ and $\mathfrak{M}_3$ are necessary and sufficient for* **M**-*convergence* $\xi_\varepsilon(t), t \in [0, T] \xrightarrow{\mathbf{M}} \xi_0(t), t \in [0, T]$ *as* $\varepsilon \to 0$.

1.6.11. J-convergence of càdlàg processes defined on interval the [0, ∞). Let, for every $\varepsilon \geq 0$, $\xi_\varepsilon(t)$, $t \geq 0$ be a m-dimensional càdlàg stochastic process. As above, the processes $\xi_\varepsilon(t), t \geq 0$ can be defined on different probability spaces for different ε.

The process $\xi_\varepsilon = \{\xi_\varepsilon(t), t \geq 0\}$ can be considered as a random variable taking values in the Polish space $\mathbf{D}_{[0,\infty)}^{(m)}$ with the **J**-metric d_J that was introduced in Subsection 1.4.11. The measure $F_\varepsilon(A)$ generated by the process $\xi_\varepsilon(t), t \geq 0$ on the σ-algebra $\mathfrak{B}_{[0,\infty)}^{(m)}$ serves as the distribution of the random variable ξ_ε.

As was mentioned above, in this case there exist three equivalent ways to define **J**-convergence of càdlàg processes.

Definition 1.6.7. Càdlàg processes $\xi_\varepsilon(t)$, $t \geq 0$ converge in the topology **J** to a càdlàg process $\xi_0(t)$, $t \geq 0$ as $\varepsilon \to 0$ ($\xi_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{J}} \xi_0(t), t \geq 0$ as $\varepsilon \to 0$) if the random variables $\xi_\varepsilon \Rightarrow \xi_\varepsilon$ as $\varepsilon \to 0$.

Definition 1.6.8. Càdlàg processes $\xi_\varepsilon(t) \xrightarrow{\mathbf{J}} \xi_0(t), t \geq 0$ as $\varepsilon \to 0$ if the random variables $f(\xi_\varepsilon(\cdot)) \Rightarrow f(\xi_0(\cdot))$ as $\varepsilon \to 0$ for every functional $f \in \mathfrak{H}_{J,\infty}[F_0]$.

Definition 1.6.9. Càdlàg processes $\xi_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{J}} \xi_0(t), t \geq 0$ as $\varepsilon \to 0$ if there exists a sequence $0 < T_n \to \infty$ as $n \to \infty$ such that the time truncated processes $\xi_\varepsilon(t), t \in [0, T_n] \xrightarrow{\mathbf{J}} \xi_0(t), t \in [0, T_n]$ as $\varepsilon \to 0$ for every $n = 1, 2, \ldots$.

The equivalence of the first two definitions follows from Theorem 1.3.2, while their equivalence to the third one follows from Theorem 1.6.6 formulated below.

Let us introduce the weak convergence and **J**-compactness conditions that are analogues of conditions $\mathcal{A}_9$ and $\mathcal{J}_3$ in the case of the interval $[0, \infty)$:

$\mathcal{A}_{11}$: $\xi_\varepsilon(t), t \in S \Rightarrow \xi_0(t), t \in S$ as $\varepsilon \to 0$, where S is a subset of $[0, \infty)$ that is dense in this interval and contains the point 0.

$\mathcal{J}_4$: $\lim_{c \to 0} \overline{\lim}_{\varepsilon \to 0} \mathsf{P}\{\Delta_J(\xi_\varepsilon(\cdot), c, T) > \delta\} = 0, \ \delta, T > 0.$

Note that the asymptotic relation in $\mathcal{J}_4$ holds for all $T > 0$ if it holds for some sequence of T-values, $0 < T_n \to \infty$ as $n \to \infty$.

The functional limit theorem gives conditions for **J**-convergence of càdlàg stochastic processes defined on the interval $[0, \infty)$.

Theorem 1.6.6. *Conditions $\mathcal{A}_{11}$ and $\mathcal{J}_4$ are necessary and sufficient for* **J**-*convergence* $\xi_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{J}} \xi_0(t), t \geq 0$ *as* $\varepsilon \to 0$.

1.6.12. Weak convergence of random a.s. J-continuous functionals. As was already mentioned in Subsection 1.6.7, the meaning of **J**-convergence of càdlàg processes is that it is equivalent to the weak convergence of random a.s. **J**-continuous functionals defined on these processes.

In what follows, we use also the notion of joint weak convergence of random variables which belong to some parametric family. Let, for every $\varepsilon \geq 0$, $\{\xi_\varepsilon^{(\theta)}, \theta \in \Theta\}$ be a parametric family of m-dimensional random variables (vectors) defined on the same probability space.

Definition 1.6.10. Random variables $\xi_\varepsilon^{(\theta)}, \theta \in \Theta$ jointly weakly converge to random vectors $\xi_0^{(\theta)}, \theta \in \Theta$ as $\varepsilon \to 0$ ($\xi_\varepsilon^{(\theta)}, \theta \in \Theta \Rightarrow \xi_0^{(\theta)}, \theta \in \Theta$ as $\varepsilon \to 0$) if for any finite sequence of parameters $\theta_1, \ldots, \theta_n \in \Theta$, $n = 1, 2, \ldots$, the random vectors $(\xi_\varepsilon^{(\theta_1)}, \ldots, \xi_\varepsilon^{(\theta_n)}) \Rightarrow (\xi_0^{(\theta_1)}, \ldots, \xi_0^{(\theta_n)})$ as $\varepsilon \to 0$.

Any linear combination of functional a.s. **J**-continuous with respect to some measure is also an functional a.s. **J**-continuous with respect to this measure. Taking in account this fact and Lemma 1.2.1, one can easily prove that **J**-convergence $\xi_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{J}} \xi_0(t), t \geq 0$ as $\varepsilon \to 0$ implies the following relation of *joint weak convergence* for random a.s. **J**-continuous functionals

$$f(\xi_\varepsilon(\cdot)),\ f \in \mathfrak{H}_{J,\infty}[F_0] \Rightarrow f(\xi_0(\cdot)),\ f \in \mathfrak{H}_{J,\infty}[F_0] \text{ as } \varepsilon \to 0. \tag{1.6.1}$$

Moreover, let us choose some points $t_1, \dots, t_n \in S$, where S is the set of weak convergence in condition $\mathcal{A}_{11}$. Consider the process $\tilde{\xi}_\varepsilon(t) = (\xi_\varepsilon(t), \xi_\varepsilon(t_i), i = 1, \dots, n), t \geq 0$. If the conditions of Theorem 1.6.6 hold for the processes $\xi_\varepsilon(t), t \geq 0$, then these conditions also hold for the processes $\tilde{\xi}_\varepsilon(t), t \geq 0$. Thus, the following relation of **J**-convergence holds: **(a)** $\tilde{\xi}_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{J}} \tilde{\xi}_0(t), t \geq 0$ as $\varepsilon \to 0$. Taking into account **(a)**, arbitrariness in the choice of the points $t_1, \dots, t_n \in S$, and applying (1.6.1) to the processes $\tilde{\xi}_\varepsilon(t)$, one can write a relation generalising (1.6.1).

Theorem 1.6.7. *Let conditions $\mathcal{A}_{11}$ and $\mathfrak{J}_4$ hold. Then*

$$\begin{aligned}&(\xi_\varepsilon(t), f(\xi_\varepsilon(\cdot))),\ (t, f) \in S \times \mathfrak{H}_{J,\infty}[F_0] \\ &\qquad \Rightarrow (\xi_0(t), f(\xi_0(\cdot))),\ (t, f) \in S \times \mathfrak{H}_{J,\infty}[F_0] \text{ as } \varepsilon \to 0.\end{aligned} \tag{1.6.2}$$

Let us use relation (1.6.2) to enlarge the set of weak convergence S, which appears in condition $\mathcal{A}_{11}$, by adding to the set S all points of stochastic continuity of the corresponding limiting process. Note that the set S can also include points of stochastic discontinuity of the process $\xi_0(t)$. In principle, all points in this set can be points of stochastic discontinuity of the process $\xi_0(t)$.

Let S_0 be the set of points of stochastic continuity of the process $\xi_0(t), t \geq 0$.

Lemma 1.6.5. *Let conditions $\mathcal{A}_{11}$ (with the set S) and $\mathfrak{J}_4$ hold and, therefore, $\xi_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{J}} \xi_0(t), t \geq 0$ as $\varepsilon \to 0$. Then $\xi_\varepsilon(t), t \in S \cup S_0 \Rightarrow \xi_0(t), t \in S \cup S_0$ as $\varepsilon \to 0$.*

This statement follows from the fact that the functional $f_t^+(\mathbf{x}(\cdot)) = \mathbf{x}(t)$ is a.s. **J**-continuous with respect to the measure generated by the limiting process $\xi_0(t), t \in [0, T]$, for every point $t \in S_0$.

So, **J**-convergence of càdlàg processes implies their weak convergence on the set $S \cup S_0$.

If the process $\xi_0(t), t \geq 0$, is stochastically continuous, the set of weak convergence is $[0, \infty)$. However, the inverse implication does not hold, since weak convergence of càdlàg processes and their **J**-convergence is not the same.

For example, let us consider the process $\xi_n(t) = \chi_{[\tau-1/n,\infty)}(t) + \chi_{[\tau,\infty)}(t), t \geq 0$, where τ is a random variable exponentially distributed with parameter 1. Obviously, the processes $\xi_n(t), t \geq 0 \Rightarrow \xi_0(t), t \geq 0$ as $n \to \infty$, where the process $\xi_0(t) = 2\chi_{[\tau,\infty)}(t), t \geq 0$. At the same time, these processes do not **J**-converge, since the process $\xi_n(t), t \in [0, T]$ has two unit jumps in close points $\tau - 1/n$ and τ if $1/n < \tau \leq T$. So, $\Delta_J(\xi_n(\cdot), c, T) = \chi(1/n < \tau \leq T)$.

Let us formulate also conditions for weak convergence for some other **J**-continuous functionals considered in Section 1.5.

Lemma 1.6.6. *Let conditions* $\mathcal{A}_{11}$ *and* $\mathfrak{J}_4$ *hold. The functional* $\mathbf{m}^{\pm}_{t_1,t_2} \in \mathfrak{H}_{J,\infty}[F_0]$ *for* $0 \leq t_1 \leq t_2 < \infty, t_1, t_2 \in S_0$ *and, therefore,* $\mathbf{m}^{\pm}_{t_1,t_2}(\xi_\varepsilon(\cdot)), t_1 \leq t_2, t_1, t_2 \in S_0 \Rightarrow \mathbf{m}^{\pm}_{t_1,t_2}(\xi_0(\cdot)), t_1 \leq t_2, t_1, t_2 \in S_0$ *as* $\varepsilon \to 0$.

Let Y_0 be the set of points $\mathbf{a} = (a_1, \ldots, a_m) \in \mathbb{R}_m$ such that $\mathsf{P}\{m^{\pm}_{0,t_1}(\xi_{0i}(\cdot)) = m^{\pm}_{0,t_2}(\xi_{0i}(\cdot)) = a_i\} = 0$ for all $0 \leq t_1 < t_2 \leq T, i = 1, \ldots, m$.

Lemma 1.6.7. *Let conditions* $\mathcal{A}_{11}$ *and* $\mathfrak{J}_4$ *hold. The functional* $\boldsymbol{\tau}^{\pm}_{\mathbf{a},T} \in \mathfrak{H}_{J,\infty}[F_0]$ *for* $\mathbf{a} \in Y_0$ *and, therefore,* $\boldsymbol{\tau}^{\pm}_{\mathbf{a},T}(\xi_\varepsilon(\cdot)), \mathbf{a} \in Y_0 \Rightarrow \boldsymbol{\tau}^{\pm}_{\mathbf{a},T}(\xi_0(\cdot)), \mathbf{a} \in Y_0$ *as* $\varepsilon \to 0$.

Let Z_0 be a set of all $\delta > 0$ such that $\mathsf{P}\{|\Delta_s(\xi_0(\cdot))| \neq \delta, s \in [0, T]\} = \mathsf{P}\{\beta^{(\delta)}_{kT}(\xi_0(\cdot)) \neq \delta, k \geq 1\} = 0$. The set Z_0 coincides with $(0, \infty)$ except for at most a countable set of points.

Lemma 1.6.8. *Let conditions* $\mathcal{A}_{11}$ *and* $\mathfrak{J}_4$ *hold. The functionals* $\alpha^{(\delta)}_{kT}, \beta^{(\delta)}_{kT} \in \mathfrak{H}_{J,\infty}[F_0]$ *for* $k \geq 1$, $\delta \in Z_0, T \in S_0$ *and, therefore,* $(\alpha^{(\delta)}_{kT}(\xi_\varepsilon(\cdot)), \beta^{(\delta)}_{kT}(\xi_\varepsilon(\cdot))), k \geq 1 \Rightarrow (\alpha^{(\delta)}_{kT}(\xi_0(\cdot)), \beta^{(\delta)}_{kT}(\xi_0(\cdot))), k \geq 1$ *as* $\varepsilon \to 0$.

Lemma 1.6.9. *Let conditions* $\mathcal{A}_{11}$ *and* $\mathfrak{J}_4$ *hold. The functional* $\Sigma^{(\delta)}_{t_1,t_2} \in \mathfrak{H}_{J,\infty}[F_0]$ *for all* $0 \leq t_1 \leq t_2 < \infty, t_1, t_2 \in S_0$, $\delta \in Z_0$ *and, therefore,* $\Sigma^{(\delta)}_{t_1,t_2}(\xi_\varepsilon(\cdot)), t_1 \leq t_2, t_1, t_2 \in S_0 \Rightarrow \Sigma^{(\delta)}_{t_1,t_2}(\xi_0(\cdot)), t_1 \leq t_2, t_1, t_2 \in S_0$ *as* $\varepsilon \to 0$.

Lemma 1.6.10. *Let conditions* $\mathcal{A}_{11}$ *and* $\mathfrak{J}_4$ *hold. The functional* $N^{(\delta)}_{t_1,t_2} \in \mathfrak{H}_{J,\infty}[F_0]$ *for all* $0 \leq t_1 \leq t_2 < \infty, t_1, t_2 \in S_0$, $\delta \in Z_0$ *and, therefore,* $N^{(\delta)}_{t_1,t_2}(\xi_\varepsilon(\cdot)), t_1 \leq t_2, t_1, t_2 \in S_0 \Rightarrow N^{(\delta)}_{t_1,t_2}(\xi_0(\cdot)), t_1 \leq t_2, t_1, t_2 \in S_0$ *as* $\varepsilon \to 0$.

Let us note that, according to (1.6.2), one can also write relations of joint weak convergence for random functionals considered above in Lammas 1.6.6 – 1.6.10.

In similar way, one can formulate conditions for weak convergence for other **J**-continuous functionals introduced in Section 1.5.

1.6.13. Joint J-convergence of several càdlàg processes. In this subsection, a useful "vector" extension of Theorem 1.6.7 is given. Let $\xi_{\varepsilon j}(t)$, $t \geq 0$ be a m-dimensional càdlàg stochastic process for every $j = 1, \ldots, r$ and $\varepsilon \geq 0$. It is assumed that the processes $\xi_{\varepsilon j}(t), t \geq 0$ for $j = 1, \ldots, r$ are defined on the same probability space for a fixed ε, but these spaces can be different for different ε.

Further, it is assumed that the following "vector" versions of conditions $\mathcal{A}_{11}$ and $\mathfrak{J}_4$ are satisfied:

$\mathcal{A}_{12}$: $(\xi_{\varepsilon 1}(t_1), \dots, \xi_{\varepsilon r}(t_r)), (t_1, \dots, t_r) \in S_1 \times \cdots \times S_r \Rightarrow (\xi_{01}(t_1), \dots, \xi_{0r}(t_r)), (t_1, \dots, t_r) \in S_1 \times \cdots \times S_r$ as $\varepsilon \to 0$, where $S_j, j = 1, \dots, r$ are subsets of $[0, \infty)$ that are dense in this interval and contain the point 0.

$\mathcal{J}_5$: $\lim_{c \to 0} \overline{\lim}_{\varepsilon \to 0} \mathsf{P}\{\Delta_J(\xi_{\varepsilon j}(\cdot), c, T) > \delta\} = 0,\ \delta, T > 0, j = 1, \dots, r.$

According Theorem 1.6.6, conditions $\mathcal{A}_{12}$ and $\mathcal{J}_5$ guarantee that **(a)** the processes $\xi_{\varepsilon j}(t), t \geq 0 \xrightarrow{\mathbf{J}} \xi_{0j}(t), t \geq 0$ as $\varepsilon \to 0$ for every $j = 1, \dots, r$. But these conditions do not imply **(b) J**-convergence of the vector processes $(\xi_{\varepsilon j}(t), j = 1, \dots, r), t \geq 0$. However, these processes jointly weakly converge in the sense of condition $\mathcal{A}_{12}$. This makes it possible to write an analogue of relation (1.6.2).

Let us denote by F_{0j} the measure generated by the process $\xi_{0j}(t), t \geq 0$, on the σ-algebra$\mathfrak{B}^{(m)}_{[0,\infty)}$ for $j = 1, \dots, r$.

Theorem 1.6.8. *Let conditions $\mathcal{A}_{12}$ and $\mathcal{J}_5$ hold. Then*

$$\begin{aligned} &(\xi_{\varepsilon j}(t_j), f_j(\xi_{\varepsilon j}(\cdot))),\ (t_j, f_j) \in S_j \times \mathfrak{H}_{J,\infty}[F_{0j}], j = 1, \dots, r \\ &\quad \Rightarrow (\xi_{0j}(t_j), f_j(\xi_0(\cdot))),\ (t_j, f_j) \in S_j \times \mathfrak{H}_{J,\infty}[F_{0j}], j = 1, \dots, r \text{ as } \varepsilon \to 0. \end{aligned} \tag{1.6.3}$$

The proof can be accomplished in the following way. First, one can take an arbitrary $n \geq 1$ and arbitrary points $t_{1j}, \dots, t_{nj} \in S_j, j = 1, \dots, r$, and consider the process $\tilde{\xi}_{\varepsilon 1}(t) = (\xi_{\varepsilon 1}(t), \xi_{\varepsilon j}(t_{kj}), k = 1, \dots, n, j = 1, \dots, r), t \geq 0$. Obviously, condition $\mathcal{A}_{12}$ implies that condition $\mathcal{A}_{11}$ holds for these processes. The processes $\tilde{\xi}_{\varepsilon 1}(t), t \geq 0$ and $\xi_{\varepsilon 1}(t), t \geq 0$ have the same **J**-compactness modulus on every finite interval. That is why condition $\mathcal{J}_5$ implies that condition $\mathcal{J}_4$ holds for the processes $\tilde{\xi}_{\varepsilon 1}(t), t \geq 0$. So, $\tilde{\xi}_{\varepsilon 1}(t), t \geq 0 \xrightarrow{\mathbf{J}} \tilde{\xi}_{01}(t), t \geq 0$ as $\varepsilon \to 0$. By applying (1.6.2) to the processes $\tilde{\xi}_{\varepsilon 1}(t)$ and taking into account arbitrariness of the choice of the points $t_{1j}, \dots, t_{nj} \in S_j, j = 1, \dots, r$, we can write the following relation:

$$\begin{aligned} &(\xi_{\varepsilon j}(t_j), f_1(\xi_{\varepsilon 1}(\cdot))),\ t_j \in S_j, j = 1, \dots, r, f_1 \in \mathfrak{H}_{J,\infty}[F_{01}] \\ &\quad \Rightarrow (\xi_{0j}(t_j), f_1(\xi_{01}(\cdot))),\ t_j \in S_j, j = 1, \dots, r, f_1 \in \mathfrak{H}_{J,\infty}[F_{01}] \text{ as } \varepsilon \to 0. \end{aligned} \tag{1.6.4}$$

Second, let us take arbitrary $n \geq 1$ points $t_{1j}, \dots, t_{nj} \in S_j, j = 1, \dots, r$, and functionals $f_{11}, \dots, f_{1n} \in \mathfrak{H}_{J,\infty}[F_{01}]$. Let us consider the process $\tilde{\xi}_{\varepsilon 2}(t) = (\xi_{\varepsilon 2}(t), \xi_{\varepsilon j}(t_{kj}), f_{1k}(\xi_{\varepsilon 1}(\cdot)), k = 1, \dots, n, j = 1, \dots, r), t \geq 0$. Obviously, condition $\mathcal{A}_{12}$ implies that condition $\mathcal{A}_{11}$ holds for these processes. The processes $\tilde{\xi}_{\varepsilon 2}(t), t \geq 0$ and $\xi_{\varepsilon 2}(t), t \geq 0$ have the same **J**-compactness modulus on every finite interval. Thus, condition $\mathcal{J}_5$ implies that condition $\mathcal{J}_4$ holds for the processes $\tilde{\xi}_{\varepsilon 2}(t), t \geq 0$. That is why $\tilde{\xi}_{\varepsilon 2}(t), t \geq 0 \xrightarrow{\mathbf{J}} \tilde{\xi}_{02}(t), t \geq 0$ as $\varepsilon \to 0$. By applying (1.6.2) to the processes $\tilde{\xi}_{\varepsilon 2}(t)$ and taking into account arbitrariness of the choice of the points $t_{1j}, \dots, t_{nj} \in S_j, j = 1, \dots, r$, and the functionals

$f_{11}, \ldots, f_{1n} \in \mathfrak{H}_{J,\infty}[F_{01}]$, we can write the following relation:

$$\begin{aligned}&(\xi_{\varepsilon j}(t_j), f_l(\xi_{\varepsilon l}(\cdot))),\ t_j \in S_j, j = 1, \ldots, r, f_l \in \mathfrak{H}_{J,\infty}[F_{0l}], l = 1, 2 \\ &\quad \Rightarrow (\xi_{0j}(t_j), f_l(\xi_{0l}(\cdot))),\ t_j \in S_j, j = 1, \ldots, r, f_l \in \mathfrak{H}_{J,\infty}[F_{0l}], l = 1, 2 \text{ as } \varepsilon \to 0.\end{aligned} \tag{1.6.5}$$

By proceeding in the same way, we get, after r steps, relation (1.6.3).

In conclusion, let us formulate one useful statement that belongs to Whitt (1973, 1980). Assume that the following condition holds:

$\mathcal{H}_1$: $\mathsf{P}\{\sum_{j=1}^{r} \chi(|\Delta_t(\xi_{0j}(\cdot)| > 0) \leq 1 \text{ for } t \geq 0\} = 1$.

Condition $\mathcal{H}_1$ means that the processes $\xi_{0j}(t), t \geq 0,\ j = 1, \ldots, r$ have no joint jump points with probability 1.

For example, this condition satisfies if the corresponding jump components of the processes $\xi_{0i}(t), t \geq 0$ and $\xi_{0j}(t), t \geq 0$ are independent for every $i, j = 1, \ldots, r, i \neq j$.

Consider the vector processes $\tilde{\xi}_\varepsilon(t) = (\xi_{\varepsilon j}(t), j = 1, \ldots, r), t \geq 0$.

Lemma 1.6.11. *Let conditions* $\mathcal{A}_{12}$, $\mathcal{J}_5$, *and* $\mathcal{H}_1$ *hold. Then*

$$\tilde{\xi}_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{J}} \tilde{\xi}_0(t), t \geq 0 \text{ as } \varepsilon \to 0. \tag{1.6.6}$$

1.6.14. J-convergence of transformed càdlàg processes. The following useful lemma permits to extend **J**-convergence of càdlàg processes to transformed càdlàg processes.

Lemma 1.6.12. *Let the following conditions hold*: **(α)** $\xi_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{J}} \xi_0(t), t \geq 0$ *as* $\varepsilon \to 0$, *and* **(β)** $g(t, \mathbf{x})$ *is a continuous function that acts from* $[0, \infty) \times \mathbb{R}_m$ *to* $\mathbb{R}_l$. *Then the càdlàg processes* $g(t, \xi_\varepsilon(t)), t \geq 0 \xrightarrow{\mathbf{J}} g(t, \xi_0(t)), t \geq 0$ *as* $\varepsilon \to 0$.

1.6.15. The continuous mapping theorem. Let, for every $\varepsilon \geq 0$, $\xi_\varepsilon(t), t \geq 0$ be a m-dimensional càdlàg process with real-valued components and $\mathbf{g}$ be a measurable mapping that acts from the space $\mathbf{D}^{(m)}_{[0,\infty)}$ to the space $\mathbf{D}^{(l)}_{[0,\infty)}$. Let also $\xi^{(\mathbf{g})}_\varepsilon(\cdot) = \mathbf{g}(\xi_\varepsilon(\cdot))$. By the definition, $\xi^{(\mathbf{g})}_\varepsilon(t), t \geq 0$ is an l-dimensional càdlàg process with real-valued components. Denote also by F_0 the measure generated by the process $\xi_0(t), t \geq 0$ on the σ-algebra $\mathfrak{B}^{(m)}_{[0,\infty)}$.

The following statement is known as the *continuous mapping theorem*. It can be found, for example, in Billingsley (1968).

Theorem 1.6.9. *Let the following conditions hold*: **(α)** $\xi_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{J}} \xi_0(t), t \geq 0$ *as* $\varepsilon \to 0$, *and* **(β)** $\mathbf{g}$ *is an a.s.* **J***-continuous mapping with respect to the measure* F_0. *Then the càdlàg processes* $\xi^{(\mathbf{g})}_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{J}} \xi^{(\mathbf{g})}_0(t), t \geq 0$ *as* $\varepsilon \to 0$.

Recall that the processes $\xi_\varepsilon = \{\xi_\varepsilon(t), t \geq 0\}$ can be considered as random variables taking values in the Polish space $\mathbf{D}^{(m)}_{[0,\infty)}$ equipped with the metric defined in Subsection 1.4.12. In this context, continuous mapping theorem can be considered as a generalisation of the sufficiency statement of Theorem 1.3.2 about weak convergence of transformed random variables. The only difference is that transformation functions take values in the functional Polish space $\mathbf{D}^{(l)}_{[0,\infty)}$ instead of $\mathbb{R}_1$.

An elegant proof of Theorem 1.6.9 can be given with the use of Skorokhod representation Theorem 1.3.6. According to this theorem, condition **(α)** implies that random variables $\tilde{\xi}_\varepsilon, \varepsilon \geq 0$ can be constructed on some probability space such that **(a)** $\tilde{\xi}_\varepsilon \overset{d}{=} \xi_\varepsilon$ for every $\varepsilon \geq 0$, and **(b)** $\tilde{\xi}_\varepsilon$ a.s. converge to $\tilde{\xi}_0$ as $\varepsilon \to 0$. Relation **(a)** yields that the random variable $\tilde{\xi}_0$ has the distribution F_0. Relation **(b)** and condition **(β)** imply, in an obvious way, that **(c)** the random variables $\mathbf{g}(\tilde{\xi}_\varepsilon)$ a.s. converge to $\mathbf{g}(\tilde{\xi}_0)$ as $\varepsilon \to 0$. Since a.s. convergence implies weak convergence, **(d)** the random variables $\mathbf{g}(\tilde{\xi}_\varepsilon)$ weakly converge to $\mathbf{g}(\tilde{\xi}_0)$ as $\varepsilon \to 0$. But, **(e)** $\mathbf{g}(\tilde{\xi}_\varepsilon) \overset{d}{=} \mathbf{g}(\xi_\varepsilon)$ that follows from **(a)**. It remains to note that random variables $\mathbf{g}(\tilde{\xi}_\varepsilon)$ take values in the Polish space $\mathbf{D}^{(l)}_{[0,\infty)}$ and their weak convergence means, actually, **J**-convergence of transformed càdlàg processes $\xi^{(\mathbf{g})}_\varepsilon(t), t \geq 0$.

It should be noted that the actual value of Theorem 1.6.9 must not be overestimated. This theorem is just a convenient way to split the proof of **J**-convergence of càdlàg processes $\xi^{(\mathbf{g})}_\varepsilon(t), t \geq 0$ in two steps, namely, the proof of **J**-convergence of the initial processes $\xi_\varepsilon(t), t \geq 0$ and the proof of a.s. **J**-continuity of the mapping **g**.

A gain can be usually achieved, when the mapping **g** has a comparatively simple structure.

Lemma 1.6.12 gives the first example concerned transformed càdlàg processes.

Let also give two examples that are used in our further considerations. The following two statements are direct corollaries of Theorem 1.6.9, and Lemmas 1.5.17 and 1.5.18 that provide conditions of **J**-continuity for the corresponding mappings.

First, let us consider the process $\xi^{(\mathbf{d})}_\varepsilon(t) = (\xi_\varepsilon(t), \xi^{(\delta)}_{\varepsilon+}(t), \xi^{(\delta)}_{\varepsilon-}(t)), t \geq 0$ defined with the use of the decomposition mapping.

Lemma 1.6.13. *Let the following conditions hold*: **(α)** $\xi_\varepsilon(t), t \geq 0 \overset{\mathbf{J}}{\longrightarrow} \xi_0(t), t \geq 0$ *as* $\varepsilon \to 0$, **(β)** $\xi_0(t), t \geq 0$ *has not jumps with the absolute values equal* δ *with probability 1. Then the processes* $\xi^{(\mathbf{d})}_\varepsilon(t), t \geq 0 \overset{\mathbf{J}}{\longrightarrow} \xi^{(\mathbf{d})}_0(t), t \geq 0$ *as* $\varepsilon \to 0$.

Second, let us consider the *max-process* $\xi^{(\mathbf{m})}_\varepsilon(t) = (\xi_\varepsilon(t), \xi^{+}_\varepsilon(t)), t \geq 0$ defined with the use of the max-mapping.

Lemma 1.6.14. *Let the following condition holds*: **(α)** $\xi_\varepsilon(t), t \geq 0 \overset{\mathbf{J}}{\longrightarrow} \xi_0(t), t \geq 0$ *as* $\varepsilon \to 0$. *Then the processes* $\xi^{(\mathbf{m})}_\varepsilon(t), t \geq 0 \overset{\mathbf{J}}{\longrightarrow} \xi^{(\mathbf{m})}_0(t), t \geq 0$ *as* $\varepsilon \to 0$.

The main object of studies in this book is *compositions of càdlàg processes*. Such processes are defined with the use of the composition mapping as $\tilde{\xi}^{(\mathbf{c})}_\varepsilon(t) = (\xi_{\varepsilon 1}(\nu_{\varepsilon 1}(t)), \ldots,$

$\xi_{\varepsilon m}(\nu_{\varepsilon m}(t)), t \geq 0$. Here $\tilde{\xi}_\varepsilon(t) = (\nu_{\varepsilon 1}(t), \dots, \nu_{\varepsilon m}(t), \xi_{\varepsilon 1}(t), \dots, \xi_{\varepsilon m}(t)), t \geq 0$ is a $2m$-dimensional càdlàg process with non-negative and non-decreasing first m components and real-valued last m components.

The conditions of **J**-convergence for compositions of càdlàg processes are thoroughly studied in Chapter 3. As was mentioned in Subsection 1.5.11, càdlàg functions defined with the use of composition mapping can **J**-converge when the corresponding initial processes do not **J**-converge. Analogously, the compositions $\tilde{\xi}_\varepsilon^{(\mathbf{c})}(t), t \geq 0$ can **J**-converge when the initial processes $\tilde{\xi}_\varepsilon(t), t \geq 0$ do not **J**-converge. The corresponding results are not covered by the continuous mapping theorem.

1.6.16. U-convergence of càdlàg processes defined on the interval $[0, \infty)$. Let us define and formulate conditions for **U**-convergence of càdlàg processes in the case where the limiting process is continuous. We think that, in this case, the definition based on **U**-convergence on embedded intervals is preferable.

Definition 1.6.11. Càdlàg processes $\xi_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{U}} \xi_0(t), t \geq 0$ as $\varepsilon \to 0$ if **(α)** there exists a sequence $0 < T_n \to \infty$ as $n \to \infty$ such that the time truncated processes $\xi_\varepsilon(t), t \in [0, T_n] \xrightarrow{\mathbf{U}} \xi_0(t), t \in [0, T_n]$ as $\varepsilon \to 0$ for every $n = 1, 2, \dots$.

Let us introduce the following **U**-compactness condition:

$\mathcal{U}_4$: $\lim_{c\to 0} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\Delta_U(\xi_\varepsilon(\cdot), c, T) > \delta\} = 0, \ \delta, T > 0.$

The functional limit theorem gives conditions for **U**-convergence.

Theorem 1.6.10. *Conditions $\mathcal{A}_{11}$ and $\mathcal{U}_4$ are necessary and sufficient for* **U***-convergence* $\xi_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{U}} \xi_0(t), t \geq 0$ *as* $\varepsilon \to 0$*, where* $\xi_0(t)$*,* $t \geq 0$ *is an a.s. continuous process.*

Let introduce the following continuity condition:

$\mathcal{B}_2$: $\xi_0(t), t \geq 0$ is an a.s. continuous process.

Note that condition $\mathcal{U}_4$ implies $\mathcal{J}_4$ and $\mathcal{B}_2$. Moreover, under condition $\mathcal{A}_{11}$, conditions $\mathcal{J}_4$ and $\mathcal{B}_2$ are equivalent to condition $\mathcal{U}_4$.

Lemma 1.6.15. *If condition $\mathcal{B}_2$ holds, the processes* $\xi_\varepsilon(t), t \in [0, T] \xrightarrow{\mathbf{U}} \xi_0(t), t \in [0, T]$ *as* $\varepsilon \to 0$ *if and only if* $\xi_\varepsilon(t), t \in [0, T] \xrightarrow{\mathbf{J}} \xi_0(t), t \in [0, T]$ *as* $\varepsilon \to 0$.

Let us also formulate a theorem which is an analogue of Theorem 1.6.7.

Theorem 1.6.11. *Let $\mathcal{A}_{11}$ and $\mathcal{U}_4$ hold. Then*

$$
\begin{aligned}
&(\xi_\varepsilon(t), f(\xi_\varepsilon(\cdot))), \ (t, f) \in [0, \infty) \times \mathfrak{H}_{U,\infty}[F_0] \\
&\qquad \Rightarrow (\xi_0(t), f(\xi_0(\cdot))), \ (t, f) \in [0, \infty) \times \mathfrak{H}_{U,\infty}[F_0] \text{ as } \varepsilon \to 0.
\end{aligned}
\tag{1.6.7}
$$

In the case of the **U**-topology, condition $\mathfrak{U}_4$ is equivalent to the following condition:

$\mathfrak{U}'_4$: $\lim_{c\to 0}\overline{\lim}_{\varepsilon\to 0}\mathsf{P}\{\Delta_U(\xi_{\varepsilon i}(\cdot), c, T) > \delta\} = 0,\ \delta, T > 0, i = 1,\dots,m.$

This shows that there is no need to formulate an analogue of Theorem 1.6.8 for the case of **U**-topology. Such a theorem would be equivalent to Theorem 1.6.11.

Let us also formulate the following useful lemma. Assume that a càdlàg process $\xi_\varepsilon(t), t \geq 0$ can be represented, for every $\varepsilon \geq 0$, as a sum of two càdlàg processes,

$$\xi_\varepsilon(t) = \xi'_\varepsilon(t) + \xi''_\varepsilon(t),\ t \geq 0.$$

Lemma 1.6.16. *Let the conditions* **(α)** $\xi'_\varepsilon(t), t \geq 0 \stackrel{\mathbf{J}}{\longrightarrow} \xi'_0(t), t \geq 0$ *as* $\varepsilon \to 0$ *and* **(β)** $\xi''_\varepsilon(t), t \geq 0 \stackrel{\mathbf{U}}{\longrightarrow} \xi''_0(t) \equiv 0, t \geq 0$ *as* $\varepsilon \to 0$ *hold. Then the processes* $\xi_\varepsilon(t), t \geq 0 \stackrel{\mathbf{J}}{\longrightarrow} \xi'_0(t), t \geq 0$ *as* $\varepsilon \to 0$.

1.6.17. M-convergence of càdlàg processes defined on the interval [0, ∞). Let us now formulate conditions for **M**-convergence of càdlàg processes.

Definition 1.6.12. Càdlàg processes $\xi_\varepsilon(t) \stackrel{\mathbf{M}}{\longrightarrow} \xi_0(t), t \geq 0$ as $\varepsilon \to 0$ if there exists a sequence $0 < T_n \to \infty$ as $n \to \infty$ such that the time truncated processes $\xi_\varepsilon(t), t \in [0, T_n]$ $\stackrel{\mathbf{M}}{\longrightarrow} \xi_0(t), t \in [0, T_n]$ as $\varepsilon \to 0$ for every $n = 1, 2, \dots$.

Let us introduce the following **M**-*compactness* condition:

$\mathfrak{M}_4$: $\lim_{c\to 0}\overline{\lim}_{\varepsilon\to 0}\mathsf{P}\{\Delta_M(\xi_\varepsilon(\cdot), c, T) > \delta\} = 0,\ \delta, T > 0.$

The gives conditions for **M**-convergence.

Theorem 1.6.12. *Conditions* $\mathcal{A}_{11}$ *and* $\mathfrak{M}_4$ *are necessary and sufficient for* **M**-*convergence* $\xi_\varepsilon(t), t \geq 0 \stackrel{\mathbf{M}}{\longrightarrow} \xi_0(t), t \geq 0$ *as* $\varepsilon \to 0$.

Let us also introduce the following **M**-compactness condition:

$\mathfrak{M}_5$: $\lim_{c\to 0}\overline{\lim}_{\varepsilon\to 0}\mathsf{P}\{\Delta_M(\xi_{\varepsilon i}(\cdot), c, T) > \delta\} = 0,\ \delta, T > 0,\ i = 1,\dots,m.$

In the case of the **M**-topology, it is possible to formulate an analogue of Theorem 1.6.8.

Theorem 1.6.13. *Let conditions* $\mathcal{A}_{12}$ *and* $\mathfrak{M}_5$ *hold. Then*

$$\begin{aligned}&(\xi_{\varepsilon j}(t_j), f_j(\xi_{\varepsilon j}(\cdot))),\ (t_j, f_j) \in S_j \times \mathfrak{H}_{M,\infty}[F_{0j}], j = 1,\dots,r\\ &\quad \Rightarrow (\xi_{0j}(t_j), f_j(\xi_0(\cdot))),\ (t_j, f_j) \in S_j \times \mathfrak{H}_{M,\infty}[F_{0j}], j = 1,\dots,r \text{ as } \varepsilon \to 0.\end{aligned} \tag{1.6.8}$$

The following lemma, which is due to Skorokhod (1956), shows the connection between **M**-convergence of càdlàg processes and the weak convergence of the maximum and the minimum functionals.

Denote by S_0 the set of points of stochastic continuity of the process $\xi_0(t), t \geq 0$.

Lemma 1.6.17. *Càdlàg processes* $\xi_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{M}} \xi_0(t), t \geq 0$ *as* $\varepsilon \to 0$ *if and only if* **(α)** $\mathbf{m}^{\pm}_{t_1,t_2}(\xi_\varepsilon(\cdot)), t_1 \leq t_2, t_1, t_2 \in S_0 \Rightarrow \mathbf{m}^{\pm}_{t_1,t_2}(\xi_0(\cdot)), t_1 \leq t_2, t_1, t_2 \in S_0$ *as* $\varepsilon \to 0$.

1.6.18. Skorokhod representation theorems for càdlàg processes. Let, for every $\varepsilon \geq 0$, $\xi_\varepsilon(t), t \in \boldsymbol{I}$ be a m-dimensional càdlàg process that is defined on an interval $\boldsymbol{I}$. Note that the processes $\xi_\varepsilon(t)$, $t \in \boldsymbol{I}$ can be defined on different probability spaces for different ε.

Let us introduce the following weak convergence condition:

$\mathcal{A}_{\mathbf{13}}$: $\xi_\varepsilon(t), t \in \tilde{S} \Rightarrow \xi_0(t), t \in \tilde{S}$ as $\varepsilon \to 0$, where $\tilde{S}$ is a countable subset of $\boldsymbol{I}$ that is dense in this interval and contains the endpoints of $\boldsymbol{I}$.

Let us also use the symbol $\tilde{\xi}_\varepsilon(t), t \in \boldsymbol{I} \overset{\mathrm{d}}{=} \xi_\varepsilon(t), t \in \boldsymbol{I}$ to indicate that the processes $\tilde{\xi}_\varepsilon(t)$, $t \in \boldsymbol{I}$ and $\xi_\varepsilon(t), t \in \boldsymbol{I}$ have the same finite-dimensional distributions.

The following results belong to Skorokhod (1956).

Theorem 1.6.14. *Let condition* $\mathcal{A}_{\mathbf{13}}$ *hold. Then it is possible to construct a probability space* $(\Omega, \mathfrak{F}, P)$ *and a.s. càdlàg processes* $\tilde{\xi}_\varepsilon(t), t \in \boldsymbol{I}$ *defined on this probability space for every* $\varepsilon \geq 0$ *such that*: **(α)** $\tilde{\xi}_\varepsilon(t), t \in \boldsymbol{I} \overset{\mathrm{d}}{=} \xi_\varepsilon(t), t \in \boldsymbol{I}$ *for every* $\varepsilon \geq 0$, **(β)** $\tilde{\xi}_{\varepsilon_n}(s) \xrightarrow{a.s.} \tilde{\xi}_0(s)$ *as* $n \to \infty, s \in \tilde{S}$ *for any subsequence* $\varepsilon_n \to 0$ *as* $n \to \infty$.

Theorem 1.6.15. *Let the interval be* $\boldsymbol{I} = [0, T]$, *conditions* $\mathcal{A}_{\mathbf{9}}$ (*with the set* S), $\mathfrak{J}_{\mathbf{3}}$, *and* $\mathfrak{O}^{(\mathbf{T})}_{\mathbf{3}}$ *be satisfied for the càdlàg processes* $\xi_\varepsilon(t), t \in [0, T]$ *and* $\tilde{S}$ *be a countable set that is dense in* $[0, T]$, *contains the points* $0, T$, *and is a subset of* S. *Let also* $\tilde{\xi}_\varepsilon(t), t \in [0, T]$ *be a.s. càdlàg processes constructed according to Theorem* 1.6.14 *with the use of the set* $\tilde{S}$ *and, therefore, defined on the same probability space for all* $\varepsilon \geq 0$ *and such that*: **(α′)** $\tilde{\xi}_\varepsilon(t), t \in [0, T] \overset{\mathrm{d}}{=} \xi_\varepsilon(t), t \in [0, T]$ *for every* $\varepsilon \geq 0$, **(β′)** $\tilde{\xi}_{\varepsilon_n}(s) \xrightarrow{a.s.} \tilde{\xi}_0(s)$ *as* $n \to \infty, s \in \tilde{S}$ *for any sequence* $0 \leq \varepsilon_n \to 0$ *as* $n \to \infty$. *Then* **(γ′)** *any sequence* $\varepsilon_n \to 0$ *contains a subsequence* $\varepsilon'_k = \varepsilon_{n_k} \to 0$ *as* $k \to \infty$ *such that* $\mathsf{P}\{\omega : \tilde{\xi}_{\varepsilon'_k}(t, \omega), t \in [0, T] \xrightarrow{\mathbf{J}} \tilde{\xi}_0(t, \omega), t \in [0, T]$ as $k \to \infty\} = 1$.

Let give a sketch of Skorokhod's proofs. The proof of Theorem 1.6.14 is based on the use of Theorem 1.3.6. Denote by X_∞ the space of sequences $\boldsymbol{x} = (\mathbf{x}_1, \mathbf{x}_2, \ldots)$, where $\mathbf{x}_i \in \mathbb{R}_m$. Define a metric in X_∞ by the formula $d(\boldsymbol{x}', \boldsymbol{x}'') = \sum_{n \geq 1} 1/n!(1 - \exp\{-|\mathbf{x}'_n - \mathbf{x}''_n|\})$. With this metric, X_∞ is a Polish space. Choose some countable set $\tilde{S} = \{t_1, t_2, \ldots\}$, which is dense in $[0, T]$, contains points $0, T$ and is a subset of S. The sequence $\xi_\varepsilon = (\xi_\varepsilon(t_1), \xi_\varepsilon(t_2), \ldots)$ can be considered as a random variable taking values in the space $\mathbb{R}_\infty$.

It follows from the weak convergence condition $\mathcal{A}_{13}$ that the random variables $\xi_\varepsilon \Rightarrow \xi_0$ as $\varepsilon \to 0$. So, one can construct, using Theorem 1.3.6, a probability space $(\Omega, \mathfrak{F}, P)$ and random variables $\xi'_\varepsilon = (\xi'_\varepsilon(t_1), \xi'_\varepsilon(t_2), \dots)$ defined on this probability space such that **(a)** $\xi'_\varepsilon \overset{\mathrm{d}}{=} \xi_\varepsilon$ for every $\varepsilon \geq 0$, **(b)** $\xi'_\varepsilon \overset{\text{a.s.}}{\longrightarrow} \xi'_0$ as $\varepsilon \to 0$. The condition **(a)** is equivalent to the relation **(c)** $\xi'_\varepsilon(s), s \in \tilde{S} \overset{\mathrm{d}}{=} \xi_\varepsilon(s), s \in \tilde{S}$, and the condition **(b)** is equivalent to the relation **(d)** $\xi'_\varepsilon(s) \overset{\text{a.s.}}{\longrightarrow} \xi'_0(s)$ as $\varepsilon \to 0,\ s \in \tilde{S}$.

Since $\xi_\varepsilon(t), t \in \boldsymbol{I}$ is an a.s. càdlàg process, **(b)** implies that **(e)** the random variables $\xi'_\varepsilon(s)$ a.s. converge to some limiting random variables $\xi'_\varepsilon(t)$ as $t < s, s \in \tilde{S}, s \to t$ for every $t \in \boldsymbol{I}$ which is not the right endpoint of this interval. The process $\xi'_\varepsilon(t), t \in \boldsymbol{I}$, defined in this way, has the same finite-dimensional distributions as the process $\xi_\varepsilon(t), t \in \boldsymbol{I}$.

The process $\xi'_\varepsilon(t), t \in \boldsymbol{I}$ possesses the following properties: **(f)** it takes values in $\mathbb{R}_m$, **(g)** it is an a.s. continuous from the right in every point $t \in \boldsymbol{I}$ that is not the right endpoint of this interval, and **(h)** it has the same finite-dimensional distribution as the a.s. càdlàg process $\xi_\varepsilon(t), t \in \boldsymbol{I}$. It follows from **(f)** – **(h)** that there exists a stochastically equivalent a.s. càdlàg modification $\tilde{\xi}_\varepsilon(t), t \in \boldsymbol{I}$ for the process $\xi'_\varepsilon(t), t \in \boldsymbol{I}$.

Since the processes $\tilde{\xi}_\varepsilon(t), t \in \boldsymbol{I}$ and $\xi'_\varepsilon(t), t \in \boldsymbol{I}$ are stochastically equivalent, **(c)** implies **(α′)** and **(d)** implies **(β′)**. Note that the a.s. convergence in **(β)** is guaranteed for subsequences but is not guaranteed in the case where $\varepsilon \to 0$ continuously (see Subsection 1.3.5).

The proof of Theorem 1.6.15 is based on some estimates for the modulus of **J**-compactness for the processes $\tilde{\xi}_{\varepsilon_n}(t), t \in \boldsymbol{I}$. The possibility to construct such processes possessing properties **(α′)** and **(β′)** is guaranteed by Theorem 1.6.14.

The conditions $\mathfrak{J}_3$ and **(α′)** imply that **(i)** for any sequence $\varepsilon_n \to 0$ there exist sequences of numbers $0 < n(k) \to \infty$, $0 < c(k) \to 0$, and $0 < \delta(k) \to 0$ as $k \to \infty$ such that $\max_{n \geq n(k)} \mathsf{P}\{\Delta_J(\tilde{\xi}_{\varepsilon_n}(\cdot), c(k), T) > \delta(k)\} \leq 1/k^2$. The proof given by Skorokhod (1956) is based on some further estimates for the modulus of **J**-compactness, and involve conditions $\mathfrak{J}_3$, $\mathfrak{O}_3^{(\mathbf{T})}$, and also **(β′)** and **(i)**. These estimates prove, with the use of Borel-Cantelli Lemma, that **(j)** there exist numbers $n_k > n(k)$ such that $\mathsf{P}\{\lim_{c\to 0} \overline{\lim}_{k\to\infty} \Delta_J(\tilde{\xi}_{\varepsilon'_k}(\cdot), c, T) = 0\} = 1$ for the subsequence $\varepsilon'_k = \varepsilon_{n_k}$.

Let A' be the set of elementary events for which the random variables in **(γ′)** converge for all $s \in \tilde{S}$. By **(β′)**, the probability $\mathsf{P}(A') = 1$. Let also A'' be the set of elementary events for which the convergence in **(j)** takes place. By **(j)**, the probability $\mathsf{P}(A'') = 1$. Obviously, $\mathsf{P}(A' \cap A'') = 1$. This implies that **(γ′)** holds.

Conditions $\mathcal{A}_9$, $\mathfrak{J}_3$, and $\mathfrak{O}_3^{(\mathbf{T})}$ can be replaced with conditions $\mathcal{A}_{10}$ and $\mathfrak{J}_3$. This can be achieved by including the random variables $\xi_\varepsilon(T-0)$ in the constructions described in Theorems 1.6.14 and 1.6.15.

The result similar to that in Theorem 1.6.15 can also be formulated for the case of the semi-infinite interval $[0, \infty)$.

Theorem 1.6.16. *Let the interval be* $\boldsymbol{I} = [0, \infty)$*, conditions* $\mathcal{A}_{11}$ *(with the set* S*), and* $\mathfrak{J}_4$

and be satisfied for the càdlàg processes $\xi_\varepsilon(t), t \in [0, \infty)$*, and* $\tilde{S}$ *be a countable set that is dense in* $[0, \infty)$*, contains the point* 0*, and is a subset of* S*. Let also* $\tilde{\xi}_\varepsilon(t), t \in [0, \infty)$ *be a.s. càdlàg processes constructed according to Theorem* 1.6.14 *with the use of the set* $\tilde{S}$ *and, therefore, defined on the same probability space for all* $\varepsilon \geq 0$ *and such that*: $(\boldsymbol{\alpha}'')$ $\tilde{\xi}_\varepsilon(t), t \in [0, \infty) \stackrel{\mathrm{d}}{=} \xi_\varepsilon(t), t \in [0, \infty)$ *for every* $\varepsilon \geq 0$, $(\boldsymbol{\beta}'')$ $\tilde{\xi}_{\varepsilon_n}(s) \xrightarrow{a.s.} \tilde{\xi}_0(s)$ *as* $n \to \infty, s \in \tilde{S}$ *for any sequence* $0 \leq \varepsilon_n \to 0$ *as* $n \to \infty$*. Then* $(\boldsymbol{\gamma}'')$ *any sequence* $\varepsilon_n \to 0$ *contains a subsequence* $\varepsilon'_k = \varepsilon_{n_k} \to 0$ *as* $k \to \infty$ *such that* $\mathsf{P}\{\omega : \tilde{\xi}_{\varepsilon'_k}(t, \omega), t \in [0, \infty) \xrightarrow{\mathbf{J}} \tilde{\xi}_0(t, \omega), t \in [0, \infty)$ as $k \to \infty\} = 1$.

The proof of Theorem 1.6.16 can be accomplished by the use of Theorems 1.6.14 – 1.6.15 and the Cantor selection procedure. A possibility to construct processes $\tilde{\xi}_\varepsilon(t), t \in [0, \infty)$, possessing properties $(\boldsymbol{\alpha}'')$ and $(\boldsymbol{\beta}'')$ follows from Theorem 1.6.14. Then a sequence of intervals $[0, T_r]$ can be chosen such that $0 < T_r \to \infty$ as $r \to \infty$ and $T_r, r \geq 1$, are points of stochastic continuity of the limiting process $\tilde{\xi}_0(t), t \in [0, \infty)$. Conditions $\mathcal{A}_{11}$ and $\mathfrak{J}_4$ allow to include the points $T_r, r \geq 1$, in the set S and then in the set $\tilde{S}$.

Let $0 \leq \varepsilon_n \to 0$ as $n \to \infty$. According to Theorem 1.6.15, there exists a subsequence $\varepsilon_{1,k}$ of the sequence ε_n such that $(\boldsymbol{\gamma}')$ holds for the processes $\tilde{\xi}_\varepsilon(t), t \in [0, T_1]$. According the same theorem, there exists a subsequence $\varepsilon_{2,k}$ of the subsequence $\varepsilon_{1,k}$ such that $(\boldsymbol{\gamma}')$ holds for the processes $\tilde{\xi}_\varepsilon(t), t \in [0, T_2]$. Continuing this selection process one can select, by induction, a subsequence $\varepsilon_{r,k}$ for every $r = 1, 2 \ldots$ such that $(\boldsymbol{\gamma}')$ holds for the processes $\tilde{\xi}_\varepsilon(t), t \in [0, T_r]$. Let now $\varepsilon_{k,k}$ be the corresponding diagonal subsequence. Obviously, $\mathsf{P}(A_r) = 1$, where $A_r = \{\omega : \tilde{\xi}_{\varepsilon_{k,k}}(t, \omega), t \in [0, T_r] \xrightarrow{\mathbf{J}} \tilde{\xi}_0(t, \omega), t \in [0, T_r]$ as $k \to \infty\}$. Therefore, $\mathsf{P}(A) = 1$, where $A = \cap_{r \geq 1} A_r$. But, it follows from the definition of $\mathbf{J}$-convergence on the semi-infinite interval $[0, \infty)$ that $\tilde{\xi}_{\varepsilon_{k,k}}(t, \omega), t \in [0, \infty) \xrightarrow{\mathbf{J}} \tilde{\xi}_0(t, \omega), t \in [0, \infty)$ as $k \to \infty$ for all $\omega \in A$.

It should be noted that the statements $(\boldsymbol{\gamma}')$ and $(\boldsymbol{\gamma}'')$ are, actually, statements about $\mathbf{J}$-convergence of the corresponding a.s. càdlàg processes in probability.

At the time when Skorokhod (1956) has formulated the results presented above, the metric $d_{J,T}$, which makes the space $\mathbf{D}^{(m)}_{[0,T]}$ a Polish space, has not been known. This metric was constructed by Billingsley (1968). In the light of this result and Prokhorov's theorems about weak convergence in metric spaces, it became possible to replace the $\mathbf{J}$-convergence in probability in the relations $(\boldsymbol{\gamma}')$ and $(\boldsymbol{\gamma}'')$ given in Theorem 1.6.15 by the a.s. $\mathbf{J}$-convergence.

A direct application of Skorokhod representation Theorem 1.3.6 to a.s. càdlàg processes $\xi_\varepsilon = \{\xi_\varepsilon(t), t \in [0, T]\}$, considered as random variables taking values in the Polish space $\mathbf{D}^{(m)}_{[0,T]}$ with the $\mathbf{J}$-metric $d_{J,T}$ (introduced in Subsection 1.4.3), yields the following result.

Theorem 1.6.17. *Let conditions* $\mathcal{A}_9$, $\mathfrak{J}_3$, *and* $\mathfrak{O}_3^{(\mathbf{T})}$ $(\mathcal{A}_{10}$ *and* $\mathfrak{J}_3)$ *be satisfied. Then it is possible to construct a probability space* $(\Omega, \mathfrak{F}, \mathsf{P})$ *and a.s. càdlàg processes* $\tilde{\xi}_\varepsilon(t)$,

$t \in [0,T]$, *defined on this probability space for every* $\varepsilon \geq 0$ *such that*: **(α′)** $\tilde{\xi}_\varepsilon(t), t \in [0,T] \stackrel{d}{=} \xi_\varepsilon(t), t \in [0,T]$ *for every* $\varepsilon \geq 0$, **(δ′)** $\mathsf{P}\{\omega : \tilde{\xi}_\varepsilon(t,\omega), t \in [0,T] \stackrel{\mathbf{J}}{\longrightarrow} \tilde{\xi}_0(t,\omega), t \in [0,T]$ as $\varepsilon \to 0\} = 1$.

Let us also formulate an analogue of Theorem 1.6.17 for the case where the processes are defined on the interval $[0,\infty)$. It is also a direct corollary of Skorokhod representation Theorem 1.3.6 applied to the càdlàg processes $\xi_\varepsilon = \{\xi_\varepsilon(t), t \geq 0\}$, considered as a random variable taking values in the Polish space $\mathbf{D}^{(m)}_{[0,\infty)}$ with the **J**-metric d_J (introduced in Subsection 1.4.11).

Theorem 1.6.18. *Let conditions* $\mathcal{A}_{11}$ *and* $\mathfrak{J}_4$ *be satisfied. Then it is possible to construct a probability space* $(\Omega, \mathfrak{F}, \mathsf{P})$ *and a.s. càdlàg processes* $\tilde{\xi}_\varepsilon(t)$, $t \geq 0$ *defined on this probability space for every* $\varepsilon \geq 0$ *such that*: **(α″)** $\tilde{\xi}_\varepsilon(t), t \geq 0 \stackrel{d}{=} \xi_\varepsilon(t), t \geq 0$ *for every* $\varepsilon \geq 0$, **(δ″)** $\mathsf{P}\{\omega : \tilde{\xi}_\varepsilon(t,\omega), t \geq 0 \stackrel{\mathbf{J}}{\longrightarrow} \tilde{\xi}_0(t,\omega), t \geq 0$ as $\varepsilon \to 0\} = 1$.

One should note that the combination of conditions **(α′)** and **(γ′)** can serve in proofs of functional limit theorems just as well as the stronger combination of conditions **(α′)** and **(δ′)**.

Let us, for example, show how Theorem 1.6.15 can be used to prove weak convergence of a.s. **J**-continuous functionals defined on **J**-convergent càdlàg processes.

Let f be an arbitrary functional from the class $\mathfrak{H}_{J,T}[F_0]$. This means that $\mathsf{P}(A') = 1$, where A' is the set of elementary events ω for which the realization $\{\tilde{\xi}_0(t,\omega), t \in [0,T]\}$ belongs to the set of **J**-continuity $\mathfrak{C}_{J,T}[f]$ of the functional f.

Let also $\varepsilon_n \geq 0, n = 0, 1, \ldots$, be an arbitrary sequence such that $\varepsilon_n \to 0$ as $n \to \infty$. According to **(γ′)**, one can select from this sequence a subsequence $\varepsilon'_k = \varepsilon_{n_k} \to 0$ as $k \to \infty$ such that $\mathsf{P}(A'') = 1$, where $A'' = \{\omega : \tilde{\xi}_{\varepsilon'_k}(t,\omega), t \in [0,T] \stackrel{\mathbf{J}}{\longrightarrow} \tilde{\xi}_0(t,\omega), t \in [0,T]$ as $k \to \infty\}$. Obviously, $\mathsf{P}(A' \cap A'') = 1$ and, for every $\omega \in A' \cap A''$, the sequence $f(\tilde{\xi}_{\varepsilon'_k}(\cdot,\omega)) \to f(\tilde{\xi}_0(\cdot,\omega))$ as $k \to \infty$. This implies, due to Lemma 1.3.4, that the random variables $f(\tilde{\xi}_\varepsilon(\cdot)) \stackrel{\mathsf{P}}{\longrightarrow} f(\tilde{\xi}_0(\cdot))$ as $\varepsilon \to 0$. As was pointed out in Lemma 1.3.1, the convergence in probability implies the weak convergence of random variables. That is why $f(\tilde{\xi}_\varepsilon(\cdot)) \Rightarrow f(\tilde{\xi}_0(\cdot))$ as $\varepsilon \to 0$. It remains to note that, due to **(α′)**, the random variable $f(\tilde{\xi}_\varepsilon(\cdot)) \stackrel{d}{=} f(\xi_\varepsilon(\cdot))$ for every $\varepsilon \geq 0$.

At the same time, the combination of Theorems 1.6.14 and 1.6.15 have some advantage in comparison with Theorem 1.6.17. In the first case, the construction of the corresponding processes defined on one probability space involves only the condition of weak convergence of finite-dimensional distributions. This makes it possible to extend the method of one probability space given in Theorems 1.6.14 and 1.6.15 to vector processes for which the corresponding **J**-compactness conditions hold for their components but do not hold for the vector processes. Similar remark can be made about Theorems 1.6.14, 1.6.16 and Theorem 1.6.18. We use this extension in Chapter 3 in theorems on **J**-convergence of compositions of càdlàg processes.

1.6.20. References. The book by Billingsley (1968, 1999) contains general results on weak convergence of random variables and the corresponding historical remarks.

The basic Theorems 1.3.4 and 1.3.5 concerning weak convergence in metric spaces belong to Prokhorov (1956), along with Theorem 1.6.4 that gives conditions for convergence of continuous stochastic processes in the topology **U**. It is formulated in the extended form given by Skorokhod (1956), that is, when the limiting process is continuous but the pre-limiting processes can be càdlàg processes.

The topology **J** in space **D** of càdlàg functions was invented by Skorokhod (1955a, 1955b). Theorems 1.4.3 and 1.6.2 that give conditions for **J**-convergence, respectively, of càdlàg functions and càdlàg stochastic processes, belong to Skorokhod (1956) as well as the representation Theorem 1.3.6 that plays a central role in Skorokhod's method of a single probability space. The metric $d'_{J,T}$ was constructed in a slightly different form by Kolmogorov (1956). The metric $d_{J,T}$, which makes the space $\mathbf{D}_{[0,T]}$ a Polish space, was constructed by Billingsley (1968) who is also credited for Theorem 1.4.2. The extension of **J**-topology to the semi-infinite interval $[0, \infty)$ via embedded sequences of close finite intervals was introduced by Stone (1963). The extension of the metric $d_{J,T}$ to the case of the semi-open interval $[0, \infty)$ and Theorem 1.4.8 are due to Lindvall (1973).

It is appropriate to note that Prokhorov's approach, which is based on general theorem about weak convergence in metric spaces, and Skorokhod's approach based on his method of a single probability space, yield the same conditions and the same results about **J**-convergence of càdlàg processes.

The advantage of Prokhorov's approach is its universality, in particular, the possibility to interpret functional limit theorems for continuous and càdlàg processes as weak limit theorems in the Polish spaces **C** and **D**, respectively.

The advantage of Skorokhod's approach lies in the possibility to use it in studies of convergence for other types of topologies that, in some cases, are not induced by a metric in the same way as it is for the **J**-topology. Skorokhod (1956) has invented several such topologies, in particular, the topology **M** that is useful in studies of extremal functionals. Theorem 1.6.5, which gives conditions for **M**-convergence of càdlàg stochastic processes, is cited from Skorokhod (1956).

In studies of functional limit theorems for randomly stopped càdlàg processes, it is useful to modify formulations of the functional limit theorem in the case of a finite interval $[0, T]$ to such a form that the condition for stochastic continuity of the càdlàg processes in the right endpoint T would not be involved. Conditions for **J**-convergence of càdlàg processes, which slightly differ from the standard ones, are given in Theorem 1.6.3.

Another extension that is important for limit theorems for randomly stopped càdlàg processes is functional limit theorems for càdlàg processes defined on the semi-infinite interval $[0, \infty)$. Theorems 1.6.6, 1.6.10, and 1.6.12 give conditions for **J**-, **U**-, and **M**-convergence of càdlàg stochastic processes defined on the interval $[0, \infty)$. The relevant references are Stone (1963), Whitt (1970), Borovkov (1972b), Grigelionis (1973), Lindvall (1973), Mackevicius (1974), and Pomarede (1976).

Conditions for a.s. **J**-continuity of random functionals and mappings defined on trajectories of càdlàg processes were studied by many authors. Some of these results, related to the most important functionals, are formulated in Lemmas 1.5.1 – 1.5.16. These results are attributed to the works of Skorokhod (1956, 1961), Billingsley (1968), Borovkov (1972a, 1976), Borovkov and Pecherskij (1975), Whitt (1973, 1980, 2002), Silvestrov (1974), Serfozo (1976), Resnick (1987), Liptser and Shiryaev (1986), and Jacod and Shiryaev (1987).

I refer to the books by Skorokhod (1961, 1964), Parthasarathy (1967), Billingsley (1968, 1999), Gikhman and Skorokhod (1965, 1971), Pollard (1984), Ethier and Kurtz (1986), Liptser and Shiryaev (1986), Jacod and Shiryaev (1987), Davidson (1994), Borovkov, Mogul'skij and Sakhanenko (1995), and Whitt (2002) which contain a more detailed presentation of the theory.

Chapter 2

Weak convergence of randomly stopped stochastic processes

In this chapter, general conditions for weak convergence of randomly stopped stochastic processes and compositions of stochastic processes are considered.

The main results concerning weak convergence of randomly stopped stochastic processes are given in Theorems 2.2.1, 2.2.2, and 2.4.1.

Theorem 2.2.2 gives three conditions that, together, imply weak convergence of randomly stopped càdlàg processes. These conditions are: **(a)** the condition of joint weak convergence of random stopping moments and external stochastic processes; **(b)** the condition of **J**-compactness of external stochastic processes, and **(c)** the condition of continuity, which means that the limiting external stochastic process is continuous at the limiting stopping moment with probability 1.

This combination makes a good balance between conditions imposed on the pre-limiting and limiting external processes, on the one hand, and the stopping moments, on the other hand. Pre-limiting joint distributions of stopping moments and external processes usually have a complicated structure. However, these distributions are involved only in the simplest and most natural way via a condition of their joint weak convergence. The second **J**-compactness condition involves only the external processes themselves and not the stopping moments. This condition is a standard one. It was thoroughly studied for various classes of càdlàg stochastic processes. The third continuity condition involves joint distributions of the limiting stopping moment and the limiting external process. These limiting joint distributions are usually simpler than the corresponding pre-limiting joint distributions. This permits to check the continuity condition in various practically important cases. Due to a balance between conditions imposed on the pre-limiting and limiting processes and stopping moments, Theorem 2.2.2 becomes an effective tool for use in limit theorems for randomly stopped stochastic processes.

The continuity condition **(c)** mentioned above does not cover the cases where the limiting stopping moment is a point of continuity for the corresponding limiting external process with probability less than 1. This case is covered by Theorem 2.4.1. In this theorem, condition **(c)** is replaced with the weaker condition **(d)** that ensures the right positioning of the pre-limiting stopping moments on the right-hand side of the moments where the pre-limiting external processes experience large jumps. This condition does involve the pre-limiting joint distributions of stopping moments and external processes not only via their joint distributions but also via the joint distributions of stopping moments and moments of large jumps for external processes. Also, the latter distributions are not so complicated and the corresponding conditions can effectively be verified in some important cases.

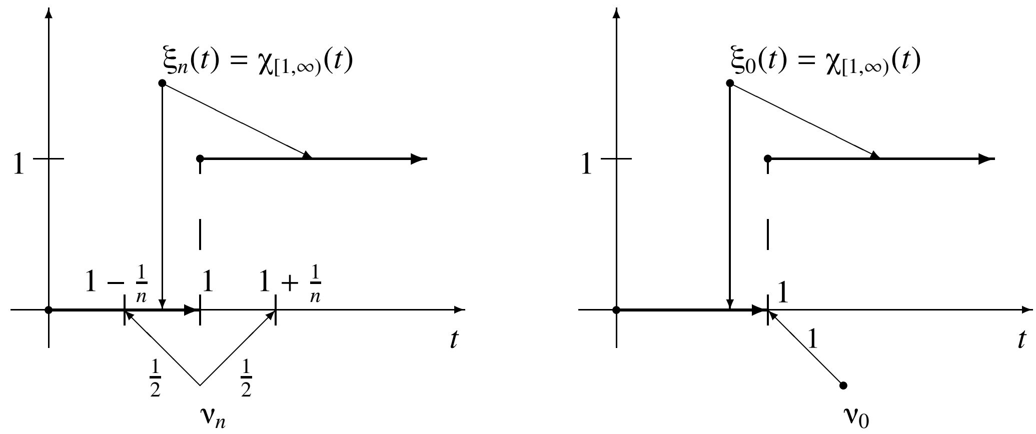
In Theorems 2.2.1, 2.2.2, and 2.4.1, a model for randomly stopped scalar (one-dimensional) càdlàg processes is considered. In Theorems 2.3.1, 2.3.4, and 2.4.3, similar results are given for randomly stopped vector càdlàg processes. In this model, each component of the external vector process is stopped in its own stopping moment. Although such a generalisation is important by itself, it also plays an essential role in theorems on weak convergence of compositions of stochastic processes. This model deals with a composition of an external càdlàg process and an internal non-decreasing càdlàg stopping process. The main results concerning weak convergence of compositions of stochastic processes are given, respectively, in Theorems 2.6.1, 2.6.3, 2.6.4, and 2.6.5 for scalar compositions, and in Theorems 2.7.1, 2.7.6, 2.7.8, and 2.7.10 for vector compositions of càdlàg processes.

Section 2.1 gives examples that clarify the formulation of the problem and conditions for weak convergence of randomly stopped random processes and compositions of stochastic processes. In Sections 2.2 and 2.3, conditions for weak convergence are given for randomly stopped scalar and vector càdlàg processes, respectively. In Section 2.4, conditions for weak convergence of randomly stopped scalar and vector càdlàg processes are given in the case where the continuity conditions imposed on external càdlàg processes and stopping moments are weakened. In Section 2.5, some results concerned iterated weak limits for randomly stopped càdlàg processes are discussed. Sections 2.6 and 2.7 give conditions for weak convergence, respectively, of scalar and vector compositions of càdlàg processes. In Section 2.8, the so-called translation theorems are formulated. These theorems give conditions for weak convergence of randomly stopped processes with random normalisation. In Section 2.9, conditions for weak convergence are given for randomly stopped stochastic processes in a model with locally compact external processes. Reference remarks are given at the end of this section.

2.1 Introductory remarks

In this section, we discuss some examples that clarify conditions for weak convergence of randomly stopped stochastic processes and compositions of càdlàg processes.

2.1.1. A condition of joint weak convergence. Let us use a natural number n as a parameter, instead of ε, to index the corresponding external càdlàg stochastic processes and stopping moments. Actually, we can always assume that $\varepsilon = n^{-1}$ for $n \geq 1$ and $\varepsilon = 0$



(a) ν_n and $\xi_n(t), t \geq 0$.
(b) ν_0 and $\xi_0(t), t \geq 0$.

Figure 2.1: $\mathfrak{C}$: first-type continuity condition.

for $n = 0$. Let, for every $n = 0, 1, \ldots, \xi_n(t), t \geq 0$ be a real-valued càdlàg process and ν_n a non-negative random variable. We call $\xi_n(t), t \geq 0$ an *external process* and ν_n a *stopping moment*.

We are interested in conditions that should be imposed on the random variables ν_n and the processes $\xi_n(t), t \geq 0$ as to imply the following relation of weak convergence:

$$\xi_n(\nu_n) \Rightarrow \xi_0(\nu_0) \text{ as } n \to \infty. \tag{2.1.1}$$

The condition that can be expected to provide relation (2.1.1) is the following condition of joint weak convergence of random stopping moments and external càdlàg processes:

$\mathcal{A}_{14}$: $(\nu_n, \xi_n(t)), t \geq 0 \Rightarrow (\nu_0, \xi_0(t)), t \geq 0$ as $n \to \infty$.

The following simple example shows that this hypothesis is not true and condition $\mathcal{A}_{14}$ is not sufficient to imply (2.1.1) without some additional assumptions.

Let ν_n be a random variable that takes values $1 - n^{-1}$ and $1 + n^{-1}$ with probability $\frac{1}{2}$ and $\xi_n(t) = \chi_{[1,\infty)}(t)$, $t \geq 0$, for $n \geq 1$. In this case, condition $\mathcal{A}_{14}$ obviously holds. The limiting stopping moment $\nu_0 = 1$ with probability 1 and the limiting process $\xi_0(t) = \chi_{[1,\infty)}(t)$, $t \geq 0$. However, $\xi_n(\nu_n)$ is a random variable that takes values 0 and 1 with probability $\frac{1}{2}$, while $\xi_0(\nu_0) = 1$ with probability 1. Therefore, (2.1.1) does not hold. Figure 2.1 illustrates this example.

In this example, the limiting stopping moment is a discontinuity point for the corresponding limiting external process. The oscillation of stopping moments in a neighbourhood of the this discontinuity point causes a violation of (2.1.1).

2.1.2. A first-type continuity condition. The example considered above leads to the following hypothesis. In order to provide (2.1.1), it is enough to add to $\mathcal{A}_{14}$ the condition that the limiting process $\xi_0(t)$, $t \geq 0$, is continuous at a random point ν_0 with probability 1,

$\mathfrak{C}_1$: $\mathsf{P}\{\lim_{t\to 0} \xi_0(\nu_0 + t) = \xi_0(\nu_0)\} = 1$.

The following more sophisticated example shows that this hypothesis is also not true. Conditions $\mathcal{A}_{14}$ and $\mathfrak{C}_1$ together are not sufficient to provide (2.1.1) without some additional assumptions.

Let ξ_k, $k = 0, 1, \ldots$ be a sequence of non-negative i.i.d. random variables with a continuous distribution function $F(x)$ and $\zeta_n = \max_{0\leq k\leq n} \xi_k$ be the maximum of the first $n + 1$ random variables of this sequence.

Let us introduce the random variables $\mu_n = \min(r : \xi_r = \zeta_n)$. By the definition, $\zeta_n = \xi_{\mu_n}$. The last representation can be rewritten in the following form: $\zeta_n/n = \xi_n(\nu_n)$, where $\xi_n(t) = \xi_{[tn]}/n$, $t \geq 0$ and $\nu_n = \mu_n/n$.

It is easy to see that the random variable μ_n takes values $0, \ldots, n$ with probability $\frac{1}{n+1}$ and, hence,

$$\nu_n \Rightarrow \nu_0 \text{ as } n \to \infty, \tag{2.1.2}$$

where ν_0 is a random variable uniformly distributed in $[0, 1]$.

Since the random variables $\xi_k, k = 0, 1, \ldots$ are i.i.d. random variables,

$$\xi_n(t) \xrightarrow{\mathsf{P}} 0 \text{ as } n \to \infty, \ t \geq 0. \tag{2.1.3}$$

From Slutsky Theorem 1.2.3 and relations (2.1.2)–(2.1.3), it follows that

$$(\nu_n, \xi_n(t)), t \geq 0 \Rightarrow (\nu_0, 0), t \geq 0 \text{ as } n \to \infty. \tag{2.1.4}$$

So, condition $\mathcal{A}_{14}$ holds. Condition $\mathfrak{C}_1$ also holds, since the limiting process $\xi_0(t) = 0$, $t \geq 0$ is continuous.

In this case, $\xi_0(\nu_0) = 0$. However, the random variables $\zeta_n/n = \xi_n(\nu_n)$ may not converge weakly to 0 as $n \to \infty$. For example, let $F(x) = \chi_{[1,\infty)}(x)(1 - 1/x)$. Then $\mathsf{P}\{\zeta_n/n < x\} = F(xn)^n \to \exp(-x^{-1})$ as $n \to \infty$, for $x > 0$. This means that the random variables $\zeta_n/n \Rightarrow \zeta$ as $n \to \infty$, where ζ is a non-negative random variable which has the distribution function $\mathsf{P}\{\zeta \leq x\} = \chi_{[0,\infty)}(x) \exp(-x^{-1})$.

An explanation of the example above is that the processes $\xi_n(t)$, $t \geq 0$ weakly converge to the zero-process $\xi_0(t) = 0$, $t \geq 0$, but these processes do not converge neither in the topology $\mathbf{J}$ nor in the weaker topology $\mathbf{M}$. They can possess too large oscillations in small intervals. In the example above, this effect causes that the max-processes

$\xi_n^+(t) = \sup_{s \le t} \xi_n(s), t \ge 0$ do not converge weakly to the corresponding zero max-process $\xi_0^+(t) = \sup_{s \le t} \xi_0(s) = 0, t \ge 0$. At the same time, the stopping moments ν_n and the external processes $\xi_n(t), t \ge 0$ are connected so that $\xi_n(\nu_n) = \xi_n^+(\nu_n)$.

2.1.3. A condition of J-compactness. The above example shows that, in order to provide (2.1.1), one should add, to the condition of join weak convergence $\mathcal{A}_{14}$ and the continuity condition $\mathcal{C}_1$, an additional compactness condition on the external processes $\xi_n(t), t \ge 0$. For example, this can be the following **J**-compactness condition:

$\mathcal{J}_6$: $\lim_{c \to 0} \overline{\lim}_{n \to \infty} \mathsf{P}\{\Delta_J(\xi_n(\cdot), c, T) > \delta\} = 0, \ \delta, T > 0.$

As is proved in Theorem 2.2.2, the combination of three conditions listed above, i.e., the condition of joint weak convergence $\mathcal{A}_{14}$, the continuity condition $\mathcal{C}_1$, and the condition of **J**-compactness $\mathcal{J}_6$ do imply the desirable asymptotic relation (2.1.1).

What is important is that this combination of three conditions is balanced, which was discussed in the preamble to the chapter. This makes the combination of conditions $\mathcal{A}_{14}$, $\mathcal{C}_1$, and $\mathcal{J}_6$ an effective instrument for use in weak limit theorems for randomly stopped stochastic processes.

2.1.4. A condition of joint weak convergence of random stopping moments and external max-processes. Let us introduce the maximum functionals

$$\xi_n^+(t', t'') = \sup_{t' \le s < t''} \xi_n(s), \quad \xi_n^-(t', t'') = \inf_{t' \le s < t''} \xi_n(s), \ 0 \le t' < t'' < \infty.$$

The pair of conditions $\mathcal{A}_{14}$ and $\mathcal{J}_6$ can be weakened and replaced with the condition of joint weak convergence of the stopping moments ν_n and the maximum functionals $\xi_n^\pm(t', t'')$, that is, with the following condition:

$\mathcal{A}_{15}$: $(\nu_n, \xi_n^\pm(t', t'')) \Rightarrow (\nu_0, \xi_0^\pm(t', t''))$ as $n \to \infty, \ 0 \le t' \le t'' < \infty$.

It is proved in Theorem 2.2.1 that conditions $\mathcal{A}_{15}$ and $\mathcal{C}_1$ do imply the desirable asymptotical relation (2.1.1).

In principle, condition $\mathcal{A}_{15}$ is weaker than a combination of conditions $\mathcal{A}_{14}$ and $\mathcal{J}_6$. This becomes obvious considering a model with monotone processes $\xi_n(t), t \ge 0$. In this case, condition $\mathcal{A}_{15}$ is reduced to the condition of joint weak convergence of random variables ν_n and $\xi_n(t)$ for every $t \ge 0$. This condition is weaker than $\mathcal{A}_{14}$. The condition of **J**-compactness $\mathcal{J}_6$ can be omitted in this case.

2.1.5. Necessity of conditions. Let us go back to the basic conditions of joint weak convergence $\mathcal{A}_{14}$, continuity $\mathcal{C}_1$, and **J**-compactness $\mathcal{J}_6$. One can talk about a certain necessity of these conditions in the sense that, taken together, these conditions are sufficient to provide (2.1.1) but any combination of two of them is not.

In the first example considered in Subsection 2.2.2, the conditions of joint weak convergence $\mathcal{A}_{14}$ and **J**-compactness $\mathcal{J}_6$ hold, but the continuity condition $\mathcal{C}_1$ does not. For this reason, asymptotic relation (2.1.1) does not hold either.

In the second example considered in Subsection 2.2.2, the conditions of joint weak convergence $\mathcal{A}_{14}$ and continuity $\mathfrak{C}_1$ hold, but the condition of **J**-compactness $\mathfrak{J}_6$ does not. This means that the asymptotic relation(2.1.1) also does not hold.

Let us now give an example in which the conditions of continuity $\mathfrak{C}_1$ and **J**-compactness $\mathfrak{J}_6$ hold. Moreover, in this example, the random stopping moments ν_n as well as the stochastic processes $\xi_n(t)$, $t \geq 0$, weakly converge. However, the condition of joint weak convergence $\mathcal{A}_{14}$ does not hold and so neither does relation(2.1.1).

Let a random variable ν take values 0 and 1 with probability $\frac{1}{2}$. Then $\nu_n = \nu$, for $n = 0, 2, \ldots$, and $\nu_n = 1 - \nu$, for $n = 1, 3, \ldots$. Let also $\xi_n(t) = \nu\chi_{[\frac{1}{2},\infty)}(t)$, $t \geq 0$, for $n = 0$, $1, \ldots$. In this case, the conditions of continuity $\mathfrak{C}_1$ and **J**-compactness $\mathfrak{J}_6$ obviously hold. Moreover, the random variables ν_n weakly converge to $\nu_0 = \nu$ as $n \to \infty$ and the processes $\xi_n(t)$, $t \geq 0$, weakly converge to $\xi_0(t)$, $t \geq 0$, as $n \to \infty$. However, the condition of joint weak convergence $\mathcal{A}_{14}$ does not hold. Indeed, in this case, $(\nu_n, \xi_n(t)), t \geq 0$, coincides with $(1, \chi_{[\frac{1}{2},\infty)}(t)), t \geq 0$ if $\nu = 1$ and $(0, 0), t \geq 0$, if $\nu = 0$, for $n = 0, 2, \ldots$. While $(\nu_n, \xi_n(t)), t \geq 0$, coincides with $(0, \chi_{[\frac{1}{2},\infty)}(t)), t \geq 0$, if $\nu = 1$ and $(1, 0), t \geq 0$, if $\nu = 0$, for $n = 1, 3, \ldots$. In this case, the random variables $\xi_n(\nu_n) = \nu$ for $n = 0, 2, \ldots$ and $\xi_n(\nu_n) = 0$, for $n = 1, 3, \ldots$. This implies that the asymptotic relation (2.1.1) does not hold.

It should be noted that the combination of conditions of joint weak convergence $\mathcal{A}_{14}$, continuity $\mathfrak{C}_1$, and **J**-compactness $\mathfrak{J}_6$ is sufficient, rather than necessary, to imply the asymptotic relation (2.1.1). In the following example all three conditions do not hold but (2.1.1) still does.

Let a random variable ν take values 0 and 1 with probability $\frac{1}{2}$. Then $\nu_n = \nu$, for $n = 0, 2, \ldots$ and $\nu_n = 1 - \nu$, for $n = 1, 3, \ldots$. Let also $\xi_n(t) = \nu(\chi_{[1-n^{-1},1)}(t) + \chi_{[\frac{1}{2},1)}(t))$, $t \geq 0$, for $n = 1, 2, \ldots$ and $\xi_0(t) = \nu\chi_{[\frac{1}{2},1)}(t)$, $t \geq 0$. In this case, the condition of joint weak convergence $\mathcal{A}_{14}$ does not hold. However, the random variables ν_n weakly converge to $\nu_0 = \nu$ as $n \to \infty$ and the processes $\xi_n(t)$, $t \geq 0$, weakly converge to $\xi_0(t)$, $t \geq 0$ as $n \to \infty$. The condition of continuity $\mathfrak{C}_1$ does not hold, because ν takes the value 1 with probability $\frac{1}{2}$ while 1 is a discontinuity point for the limiting process $\xi_0(t)$, $t \geq 0$. The condition of **J**-compactness $\mathfrak{J}_6$ also does not hold, since for every $n = 1, 2, \ldots$, the process $\xi_n(t)$, $t \geq 0$ has with probability $\frac{1}{2}$ two jumps with values 1 and 2 at points $1 - n^{-1}$ and 1, respectively. At the same time, it is obvious that $\xi_n(\nu_n) = 0$ for all $n = 0, 1, \ldots$. Therefore, the asymptotic relation (2.1.1) holds.

2.1.6. A weakened version of the continuity condition $\mathfrak{C}_1$. Let us return to the first example considered in Subsection 2.1.1. In this example, the random variables $\xi_n(\nu_n)$ do not weakly converge to the random variable $\xi_0(\nu_0)$ as $n \to \infty$. This is because the stopping moments ν_n can take values to the "wrong" left-hand side of the corresponding point where the process $\xi_n(t)$ has the unit jump. This occurs with a probability that is asymptotically separated from zero as $n \to \infty$.

Let us modify this example by considering the same process $\xi_n(t) = \chi_{[1,\infty)}(t)$, $t \geq 0$, and a modified random variable ν_n that takes the values $1 - n^{-1}$ and $1 + n^{-1}$ with the probabilities $\frac{1}{n}$ and $1 - \frac{1}{n}$, respectively.

In this case, condition $\mathcal{A}_{14}$ obviously holds. The same is true for condition $\mathfrak{J}_6$. However, the continuity condition $\mathfrak{C}_1$ does not hold. As in the first example, the limiting stopping moment $\nu_0 = 1$ with probability 1, and the limiting process $\xi_0(t) = \chi_{[1,\infty)}(t)$, $t \geq 0$ has the unit jump at the point 1. This shows that $\xi_0(\nu_0) = 1$ with probability 1. However, $\xi_n(\nu_n)$ is a random variable that takes values 0 and 1 with probabilities $\frac{1}{n}$ and $1 - \frac{1}{n}$, respectively. Therefore, the asymptotic relation (2.1.1) does hold.

This example shows that it is possible to somewhat weaken the continuity condition $\mathfrak{C}_1$. This can be achieved by replacing $\mathfrak{C}_1$ with a condition that would guarantee asymptotically (for all n large enough) the right positioning of the stopping moment ν_n with respect to discontinuity points of the process $\xi_n(t)$, $t \geq 0$.

Let us denote by $\alpha_{nk}^{(\delta)}, k = 1, 2, \ldots$, the successive moments of jumps of the process $\xi_n(t), t \geq 0$, which have the absolute values of jumps greater than or equal to $\delta > 0$. By the definition, $\alpha_{nk}^{(\delta)} = \infty$ if there exist less than k such points. The following condition can be used instead of the continuity condition $\mathfrak{C}_1$:

$\mathfrak{D}_1$: $\lim_{c \to 0} \overline{\lim}_{n \to \infty} \mathsf{P}\{\nu_n \in [\alpha_{nk}^{(\delta)} - c, \alpha_{nk}^{(\delta)})\} = 0$ for $\delta > 0$ and $k \geq 1$.

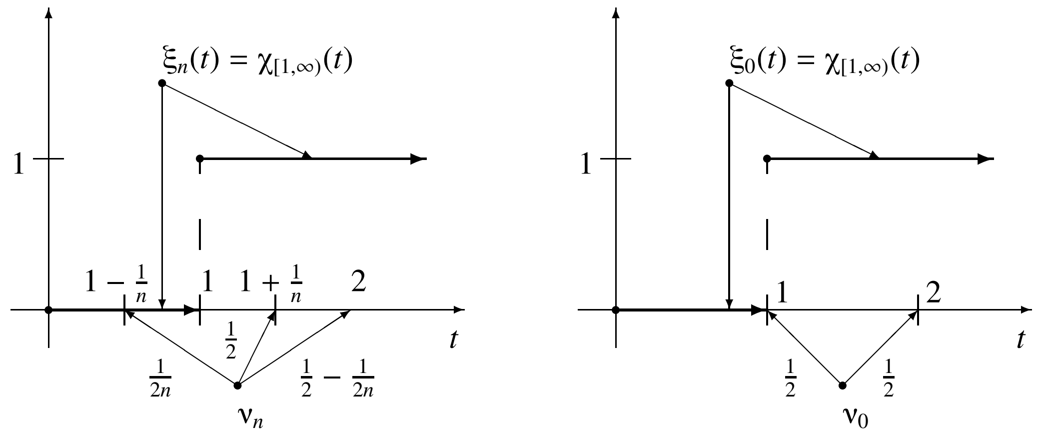
As is proved in Theorem 2.4.1, a combination of the conditions $\mathcal{A}_{14}$, $\mathfrak{J}_6$, and $\mathfrak{D}_1$ imply the desirable asymptotical relation (2.1.1).

It can be shown that, if conditions $\mathcal{A}_{14}$ and $\mathfrak{J}_6$ are verified, condition $\mathfrak{C}_1$ implies condition $\mathfrak{D}_1$. This means that it is possible to consider $\mathfrak{D}_1$ as a weakened version of condition $\mathfrak{C}_1$. It can occur that the opposite implication does not take place. In the example considered above, the conditions $\mathcal{A}_{14}$, $\mathfrak{J}_6$, and $\mathfrak{D}_1$ hold but condition $\mathfrak{C}_1$ does not.

By using condition $\mathfrak{D}_1$, instead of $\mathfrak{C}_1$, one can deal with some cases where the limiting stopping moment is a discontinuity point of the limiting external process. Some cases where the limiting stopping moment can be a point of discontinuity or continuity of the limiting external process with both positive probabilities can also be treated.

For instance, let us modify the example considered above once more. Now, let the process $\xi_n(t) = \chi_{[1,\infty)}(t)$, $t \geq 0$ be the same but the random variable ν_n take the values $1 - n^{-1}$, $1 + n^{-1}$, and 2 with the probabilities $\frac{1}{2n}$, $\frac{1}{2}$, and $\frac{1}{2} - \frac{1}{2n}$, respectively. Conditions $\mathcal{A}_{14}$, $\mathfrak{J}_6$, and $\mathfrak{D}_1$ hold. Therefore, relation (2.1.1) also holds. In this case, the random variable $\xi_n(\nu_n)$ takes the values 0 and 1 with the probabilities $\frac{1}{2n}$ and $1 - \frac{1}{2n}$, respectively. At the same time, the limiting random variable $\xi_0(\nu_0)$ takes the value 1 with probability 1. However, the limiting stopping moment ν_0 takes the values 1 and 2, with probability $\frac{1}{2}$. The point 1 is a point of discontinuity of the process $\xi_0(t) = \chi_{[1,\infty)}(t)$, $t \geq 0$, while 2 is a point of continuity of this process. Condition $\mathfrak{C}_1$ does not hold in this case. Figure 2.2 illustrates this example.

Of course, condition $\mathfrak{D}_1$ is not as simple as condition $\mathfrak{C}_1$. It involves pre-limiting external processes and stopping moments, in contrast with condition $\mathfrak{C}_1$. However, it still



(a) ν_n and $\xi_n(t), t \geq 0$. (b) ν_0 and $\xi_0(t), t \geq 0$.

Figure 2.2: $\mathfrak{D}$: weakened first-type continuity condition.

involves the stopping moments ν_n and the external processes $\xi_n(t), t \geq 0$, in an acceptable combination of joint distributions of the stopping moment ν_n and the moments of large jumps $\alpha_{nk}^{(\delta)}$ of the external process $\xi_n(t), t \geq 0$. The latter functionals were thoroughly studied for various classes of càdlàg stochastic processes. This shows that the weakened continuity condition $\mathfrak{D}_1$ can effectively be used in some practically important cases not covered by the continuity condition $\mathfrak{C}_1$.

2.1.7. Conditions of weak convergence for compositions of càdlàg processes. Let, for every $n = 0, 1, \ldots, \xi_n(t), t \geq 0$ be a real-valued càdlàg process and $\nu_n(t), t \geq 0$ be a non-negative monotone non-decreasing càdlàg process. We call $\xi_n(t), t \geq 0$ an *external process* and $\nu_n(t), t \geq 0$ an *internal stopping process*. Consider a *composition* of these processes $\xi_n(\nu_n(t)), t \geq 0$. We are interested in conditions that would provide the weak convergence of the compositions $\xi_n(\nu_n(t)), t \geq 0$ on some subset $S \subseteq [0, \infty)$,

$$\xi_n(\nu_n(t)), t \in S \Rightarrow \xi_0(\nu_0(t)), t \in S \text{ as } n \to \infty. \tag{2.1.5}$$

The simplest analogue of condition $\mathcal{A}_{14}$ is the following condition of joint weak convergence of internal stopping processes and external càdlàg processes:

$\mathcal{A}_{16}$: $(\nu_n(t), \xi_n(t)), t \geq 0 \Rightarrow (\nu_0(t), \xi_0(t)), t \geq 0$ as $n \to \infty$.

The condition of **J**-compactness, $\mathcal{J}_6$, does not need to be changed.

For a continuity condition, one can take the following analogue of condition $\mathfrak{C}_1$:

$\mathfrak{C}_2^S$: $\mathsf{P}\{\lim_{s \to 0} \xi_0(\nu_0(t) + s) = \xi_0(\nu_0(t))\} = 1$ for $t \in S$.

Conditions $\mathcal{A}_{16}$, $\mathcal{J}_6$, and $\mathcal{C}_2^S$ imply weak convergence of the processes $\xi_n(\nu_n(t))$ on the set $S \subseteq [0, \infty)$. The set S is called a set of weak convergence.

Condition $\mathcal{C}_2^S$ looks rather restrictive. But, actually, it is satisfied in many important cases. For instance, it is satisfied if the process $\xi_0(t) = \xi'_0(t) + \xi''_0(t)$, $t \geq 0$ is a sum of two càdlàg processes such that the first one is a continuous process, possibly dependent on the process $\nu_0(t)$, $t \geq 0$, while the second one is a stochastically continuous càdlàg process independent of the stopping process $\nu_0(t)$, $t \geq 0$. In this case, $\mathcal{C}_2^S$ holds with the set $S = [0, \infty)$.

The problem has an additional new aspect if one does not prescribe a set of weak convergence but would like to only guarantee the weak convergence of compositions $\xi_n(\nu_n(t))$ on some subset S dense in the interval $[0, \infty)$. This is an important problem in studies of functional limit theorems.

Let us note that, in the case where the process $\xi_0(t)$, $t \geq 0$ admits an additive decomposition described above, the set S is $[0, \infty)$.

There is another important case in which the existence of a desirable subset S is guaranteed without any decomposition assumptions. This is true if the process $\nu_0(t)$, $t \geq 0$, is an a.s. strictly monotone process. In this case, there exists at most a countable set of points $t \geq 0$ such that the random moment $\nu_0(t)$ is a point of discontinuity of the process $\xi_0(t)$, $t \geq 0$ with a positive probability.

We show in Lemma 2.6.2 that this is also true in a situation more general than the case of a strictly monotone limiting stopping process $\nu_0(t)$. Let $R[\xi_0(\cdot)]$ be a random set of all discontinuity points of the process $\xi_0(t)$, $t \geq 0$. Then condition $\mathcal{C}_2^S$ holds with some subset S dense in the interval $[0, \infty)$ if the following continuity type condition holds:

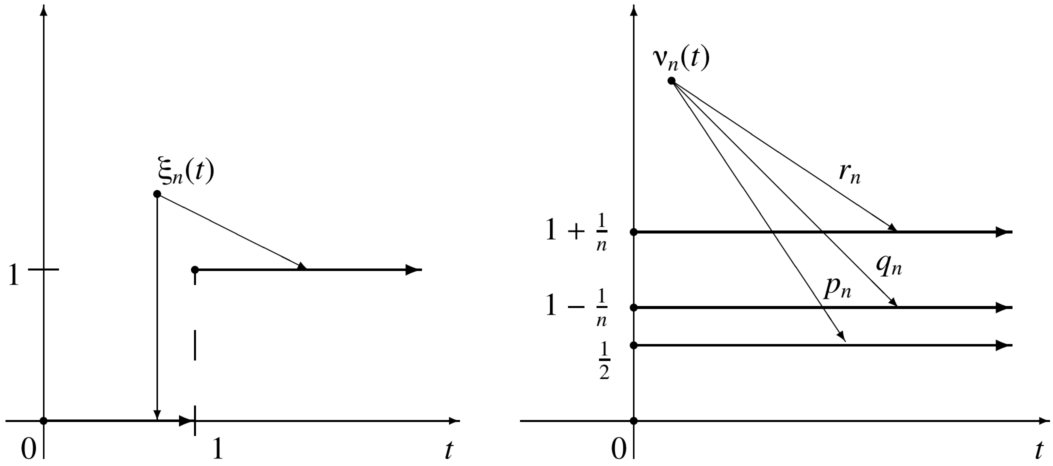
$\mathcal{E}_1$: $\mathsf{P}\{\nu_0(t') = \nu_0(t'') \in R[\xi_0(\cdot)]\} = 0$ for $0 \leq t' < t'' < \infty$.

2.1.8. Weakened continuity conditions and weak convergence for compositions of càdlàg processes. The compositions $\xi_n(\nu_n(t))$ can, however, weakly converge on some subset S dense in the interval $[0, \infty)$ in situation when the continuity condition $\mathcal{E}_1$ does not hold. As is proved in Theorem 2.6.5, it is so, if the conditions $\mathcal{A}_{16}$, $\mathcal{J}_6$ hold together with the following continuity type condition, which is weaker than condition $\mathcal{E}_1$:

$\mathcal{F}_1$: $\lim_{0<c\to 0} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\alpha_{\varepsilon k}^{(\delta)} - c \leq \nu_\varepsilon(t'), \nu_\varepsilon(t'') < \alpha_{\varepsilon k}^{(\delta)}\} = 0$ for $0 \leq t' < t'' < \infty$, $\delta > 0$ and $k \geq 1$.

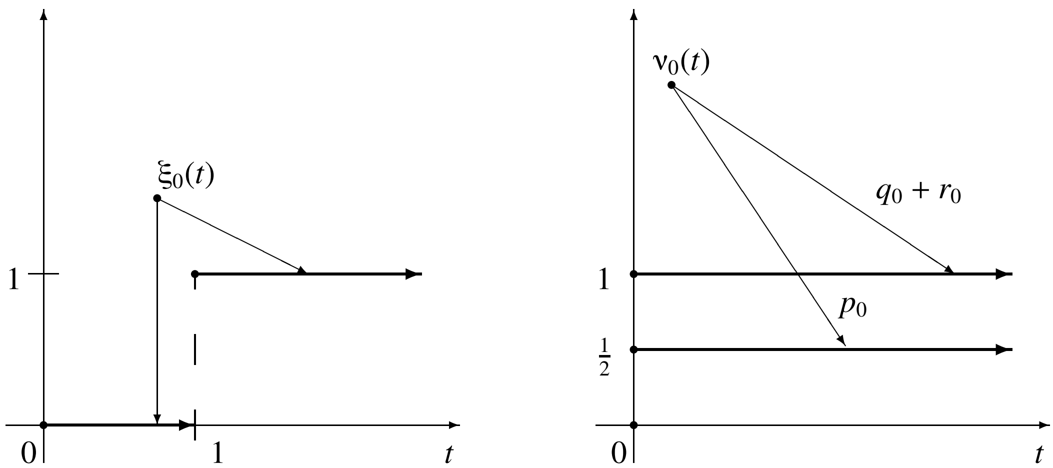
Let us consider the following example shown in Figures 2.3, 2.4, and 2.5. Let $\xi_n(t) = \chi_{[1,\infty)}(t)$, $t \geq 0$, for $n \geq 1$. Let also, for $n \geq 1$, the process $\nu_n(t)$, $t \geq 0$ have three possible realisations that occur with the probabilities p_n, q_n and r_n, where $p_n + q_n + r_n = 1$. These realisations are $\frac{1}{2}$ for $t \geq 0$; $1 - n^{-1}$ for $t \geq 0$; $1 + n^{-1}$ for $t \geq 0$;

We assume that probabilities p_n, q_n and r_n converge as $n \to \infty$ to the limiting values p_0, q_0 and r_0, respectively. In this case, condition $\mathcal{A}_{16}$ obviously holds. The limiting



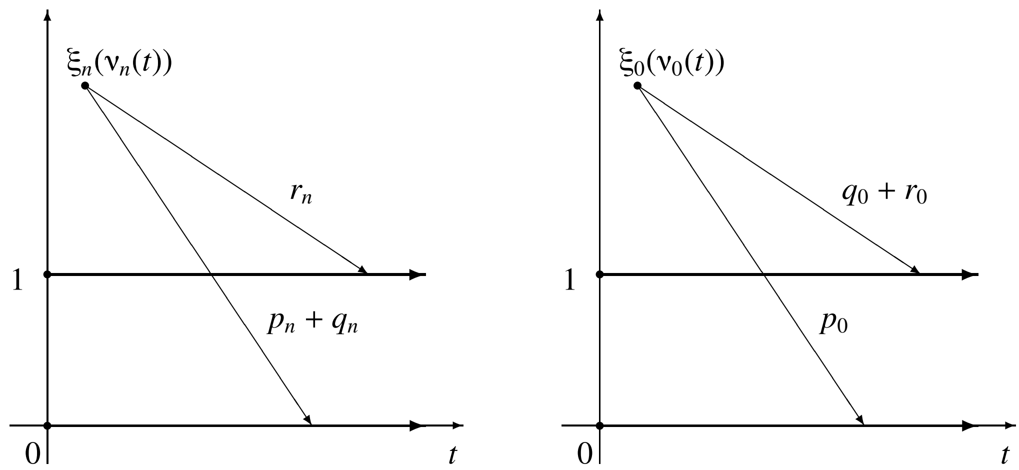
(a) $\xi_n(t), t \geq 0$.   (b) $\nu_n(t), t \geq 0$.

Figure 2.3: $\mathcal{E}$ and $\mathfrak{F}$: second-type and weakened second-type continuity conditions.



(a) $\xi_0(t), t \geq 0$.   (b) $\nu_0(t), t \geq 0$.

Figure 2.4: $\mathcal{E}$ and $\mathfrak{F}$: second-type and weakened second-type continuity conditions.



(a) $\xi_n(\nu_n(t)), t \geq 0$. (b) $\xi_0(\nu_0(t)), t \geq 0$.

Figure 2.5: $\mathcal{E}$ and $\mathcal{F}$: second-type and weakened second-type continuity conditions.

process $\xi_0(t) = \chi_{[1,\infty)}(t)$, $t \geq 0$. The limiting stopping process $\nu_0(t), t \geq 0$ has only two possible realisations that occur with the probabilities p_0 and $q_0 + r_0$, respectively. These realisations are $\frac{1}{2}$ for $t \geq 0$, and 1 for $t \geq 0$. The condition of **J**-compactness $\mathcal{J}_6$ also holds.

For $n \geq 1$, the composition $\xi_n(\nu_n(t)), t \geq 0$ has two possible realisations that occur with the probabilities $p_n + q_n$ and r_n, respectively. These realisations are 0 for $t \geq 0$, and 1 for $t \geq 0$. At the same time, the composition $\xi_0(\nu_0(t)), t \geq 0$ has the same two possible realisations that occur with the probabilities p_0 and $q_0 + r_0$, respectively.

Condition $\mathcal{E}_1$ holds if and only if **(a)** $p_0 = 1$. In this case, the limiting composition $\xi_0(\nu_0(t)), t \geq 0$ has only one realisation 0 for $t \geq 0$. The compositions $\xi_n(\nu_n(t))$ weakly converge to $\xi_0(\nu_0(t))$ on the set $S = [0, \infty)$.

Condition $\mathcal{F}_1$ holds if and only if **(b)** $q_0 = 0$. If also, $p_0 < 1$, then condition $\mathcal{F}_1$ hold but $\mathcal{E}_1$ does not. If $q_0 = 0$, the limiting composition $\xi_0(\nu_0(t)), t \geq 0$ has two possible realisations 0 for $t \geq 0$, and 1 for $t \geq 0$. They occur with the probabilities p_0 and r_0, respectively. Again, the $\xi_n(\nu_n(t))$ weakly converge to $\xi_0(\nu_0(t))$ on the set $S = [0, \infty)$.

Neither condition $\mathcal{E}_1$ nor $\mathcal{F}_1$ hold, if **(c)** $q_0 > 0$. In this case $p_n + q_n \not\to p_0$ and $r_n \not\to q_0 + r_0$ as $n \to \infty$. Therefore, the compositions $\xi_n(\nu_n(t))$ do not weakly converge to $\xi_0(\nu_0(t))$ for every $t \in [0, \infty)$.

These statements are consistent with the remarks above.

Non-trivial examples of applications of weak convergence results for compositions of càdlàg processes, based on continuity type condition $\mathcal{F}_1$, are given in Chapter 4. These results are applied there to so-called generalised exceeding processes that describe various renewal type models.

2.2 Randomly stopped scalar càdlàg processes

In this section, we formulate conditions for weak convergence of randomly stopped one-dimensional càdlàg stochastic processes. In this case, the corresponding conditions have the most clear form.

2.2.1. Main results. Let, for every $\varepsilon \geq 0$, $\xi_\varepsilon(t)$, $t \geq 0$ be a real-valued càdlàg process, and ν_ε be a non-negative random variable. We call $\xi_\varepsilon(t)$, $t \geq 0$ an *external process* and ν_ε a *stopping moment*.

We are interested in conditions that should be imposed on the random variables ν_ε and the processes $\xi_\varepsilon(t)$, $t \geq 0$ in order to provide the following relation

$$\xi_\varepsilon(\nu_\varepsilon) \Rightarrow \xi_0(\nu_0) \text{ as } \varepsilon \to 0. \tag{2.2.1}$$

The following condition can be expected to provide relation (2.2.1):

$\mathcal{A}_{17}$: $(\nu_\varepsilon, \xi_\varepsilon(t)), t \in U \Rightarrow (\nu_0, \xi_0(t)), t \in U$ as $\varepsilon \to 0$, where U is a subset of $[0, \infty)$ that is dense in this interval and contains the point 0.

Examples constructed in Section 2.1 show that this condition is not sufficient to imply relation (2.2.1).

Let us introduce the following weak convergence condition:

$\mathcal{A}_{18}$: There exists a set S dense in $[0, \infty)$, containing 0, and such that $\mathsf{P}\{\nu_0 = t\} = 0$ for $t \in S \setminus \{0\}$, and, for all $t', t'' \in S$,

$$(\nu_\varepsilon, \sup_{t\in[t',t'')} \xi_\varepsilon(t)) \Rightarrow (\nu_0, \sup_{t\in[t',t'')} \xi_0(t)) \text{ as } \varepsilon \to 0,$$
$$(\nu_\varepsilon, \inf_{t\in[t',t'')} \xi_\varepsilon(t)) \Rightarrow (\nu_0, \inf_{t\in[t',t'')} \xi_0(t)) \text{ as } \varepsilon \to 0.$$

Let us also introduce the following continuity condition:

$\mathcal{C}_3$: The process $\xi_0(t)$, $t \geq 0$ is continuous at the point ν_0 with probability 1, i.e., $\mathsf{P}\{\lim_{t\to 0} \xi_0(\nu_0 + t) = \xi_0(\nu_0)\} = 1$.

The main result of this section is the following theorem from Silvestrov (1971b, 1972a).

Theorem 2.2.1. *Let conditions* $\mathcal{A}_{18}$ *and* $\mathcal{C}_3$ *hold. Then*

$$\xi_\varepsilon(\nu_\varepsilon) \Rightarrow \xi_0(\nu_0) \textit{ as } \varepsilon \to 0.$$

Theorem 2.2.1 does not require a separate proof. This theorem is a particular case of Theorem 2.3.1 that gives a similar result for a more general model of randomly stopped vector processes.

2.2.2. The condition $\mathcal{A}_{18}$ and J-convergence of the processes $(\nu_\varepsilon, \xi_\varepsilon(t))$, $t \geq 0$. Condition $\mathcal{A}_{18}$ can be replaced with a more simple condition of joint weak convergence $\mathcal{A}_{17}$ if, to assume additionally to $\mathcal{A}_{17}$, the following **J**-compactness condition is assumed:

$\mathcal{J}_7$: $\lim_{c\to 0} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\Delta_J(\xi_\varepsilon(\cdot), c, T) > \delta\} = 0,\ \delta, T > 0.$

Note that conditions $\mathcal{A}_{17}$ and $\mathcal{J}_7$ are necessary and sufficient for **J**-convergence of the vector processes $(\nu_\varepsilon, \xi_\varepsilon(t))$, $t \geq 0$.

Let S_0 be the set of points of stochastic continuity of the process $\xi_0(t)$, $t \geq 0$. This set is the interval $[0, \infty)$ except, perhaps, for some finite or countable set. Since the process $\xi_0(t)$ is continuous from the right, the point 0 also belongs to S_0. Let also Y_0 be a set that includes all continuity points of the distribution function of the random variable ν_0 and the point 0. Then $\overline{Y}_0$ is the set of all points $t > 0$ such that $\mathsf{P}\{\nu_0 = t\} > 0$. This set contains at most a countable number of points. Therefore, the set $S = S_0 \setminus \overline{Y}_0$ is dense in $[0, \infty)$ and contains the point 0. Moreover, this set is $[0, \infty)$, except for at most a countable set.

Lemma 2.2.1. *Let conditions $\mathcal{A}_{17}$ and $\mathcal{J}_7$ hold. Then condition $\mathcal{A}_{18}$ holds with the set $S = S_0 \setminus \overline{Y}_0$.*

Lemma 2.2.1 does not need to be proved separately, too. It is a particular case of Lemma 2.3.1 that gives a similar result for vector càdlàg processes.

The following theorem from Silvestrov (1971b, 1972a) is a direct corollary of Theorem 2.2.1 and Lemma 2.2.1.

Theorem 2.2.2. *Let conditions $\mathcal{A}_{17}$, $\mathcal{J}_7$, and $\mathcal{C}_3$ hold. Then*

$$\xi_\varepsilon(\nu_\varepsilon) \Rightarrow \xi_0(\nu_0) \text{ as } \varepsilon \to 0.$$

2.2.3. The case of non-random càdlàg functions. As an example, let us consider the case where the external processes and the stopping moments are non-random. Thus, let us consider the case where a non-random càdlàg function $x_\varepsilon(t)$, $t \geq 0$ is stopped at a non-random point y_ε. In this case, condition $\mathcal{A}_{17}$ is reduced to the following conditions: **(a)** $x_\varepsilon(t) \to x_0(t)$ as $\varepsilon \to 0$ for $t \in U$, where U is some set of points everywhere dense in $[0, \infty)$ and containing 0; and **(b)** $y_\varepsilon \to y_0$ as $\varepsilon \to 0$. Condition $\mathcal{J}_7$ reduces, in this case, to the condition of **J**-compactness, **(c)** $\lim_{c\to 0} \overline{\lim}_{\varepsilon\to 0} \Delta_J(x_\varepsilon(\cdot), c, T) = 0$, $\delta, T > 0$. Note that **(a)** and **(c)** are just necessary and sufficient conditions for the following relation of **J**-convergence: **(d)** $x_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{J}} x_0(t), t \geq 0$ as $\varepsilon \to 0$. Finally, condition $\mathcal{C}_3$ takes the following form: **(e)** y_0 is a continuity point of the function $x_0(t)$, $t \geq 0$. Theorem 2.2.2 states in this case that, under conditions **(a)**, **(b)**, **(c)**, and **(e)**, the following relation holds: **(f)** $x_\varepsilon(y_\varepsilon) \to x_0(y_0)$ as $\varepsilon \to 0$.

An importance and utility of this statement for càdlàg functions was pointed out, for example, in Jacod and Shiryaev (1987). The authors did not recognise that it is a particular case of Theorem 2.2.2 from Silvestrov (1971b, 1972a).

2.2.4. Condition $\mathcal{A}_{18}$ and M-convergence of the processes $(\nu_\varepsilon, \xi_\varepsilon(t))$, t ≥ 0. Conditions $\mathcal{A}_{17}$ and $\mathcal{J}_7$ can be replaced with weaker conditions of **M**-convergence of vector processes $(\nu_\varepsilon, \xi_\varepsilon(t))$, $t \geq 0$, namely, with condition $\mathcal{A}_{17}$ and the following condition of **M**-compactness:

$\mathfrak{M}_6$: $\lim_{c\to 0} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\Delta_M(\xi_\varepsilon(\cdot), c, T) > \delta\} = 0, \ \delta, T > 0.$

Lemma 2.2.2. *Let conditions $\mathcal{A}_{17}$ and $\mathfrak{M}_6$ hold. Then condition $\mathcal{A}_{18}$ holds with the set $S = S_0 \setminus \overline{Y}_0$.*

Lemma 2.2.2 also does not require a separate proof, since it is a particular case of Lemma 2.3.3 that gives a similar result for vector càdlàg processes.

The following theorem, which was also given in Silvestrov (1971b, 1972a), is a direct corollary of Theorem 2.2.1 and Lemma 2.2.2.

Theorem 2.2.3. *Let conditions $\mathcal{A}_{17}$, $\mathfrak{M}_6$, and $\mathfrak{C}_3$ hold. Then*

$$\xi_\varepsilon(\nu_\varepsilon) \Rightarrow \xi_0(\nu_0) \text{ as } \varepsilon \to 0.$$

2.2.5. Monotone external processes. Condition $\mathcal{A}_{18}$ is weaker than $\mathcal{A}_{17}$ and $\mathfrak{J}_7$. It is also weaker than $\mathcal{A}_{17}$ and $\mathfrak{M}_6$.

To clarify this, let us consider a model with monotone processes $\xi_\varepsilon(t)$, $t \geq 0$. In this case, condition $\mathcal{A}_{18}$ is equivalent to the relation $(\nu_\varepsilon, \xi_\varepsilon(t)) \Rightarrow (\nu_0, \xi_0(t))$ as $\varepsilon \to 0$ for $t \in S$, i.e., to the weak convergence of distributions that are one-dimensional with respect to t.

In regard to condition $\mathcal{A}_{17}$, this condition demands weak convergence of distributions that are finite dimensional in time. Also in this case, $\mathfrak{J}_7$ is an additional **J**-compactness condition. Now, condition $\mathfrak{M}_6$ is implied by $\mathcal{A}_{17}$, since the processes $\xi_\varepsilon(t), t \geq 0$ are monotone.

2.2.6. Decomposition condition $\mathfrak{Q}_1$. Let us introduce a condition that is very useful in applications,

$\mathfrak{Q}_1$: $\xi_0(t) = \xi_0'(t) + \xi_0''(t)$, $t \geq 0$, where (a) $\xi_0'(t)$, $t \geq 0$ is a continuous process, (b) $\xi_0''(t)$, $t \geq 0$ is a stochastically continuous càdlàg process, (c) the process $\xi_0''(t)$, $t \geq 0$ and the random variable ν_0 are independent.

Lemma 2.2.3. *Let condition $\mathfrak{Q}_1$ hold. Then the continuity condition $\mathfrak{C}_3$ holds, i.e., the process $\xi_0(t)$, $t \geq 0$ is continuous at the point ν_0 with probability* 1.

Proof of Lemma 2.2.3. The first component $\xi_0'(t)$ is a continuous process. So, it is sufficient to show that

$$\eta_0''(h, \nu_0) \xrightarrow{\mathsf{P1}} 0 \text{ as } h \to 0, \tag{2.2.2}$$

where

$$\eta_0''(h, x) = \sup_{|t|\leq h} |\xi_0''(t + x) - \xi_0''(x)|, \ x \geq 0.$$

By the definition, the quantities $\eta_0''(h, x)$ are monotonically non-decreasing in h for every $x \geq 0$, and, consequently, such are the quantities $\eta_0''(h, \nu_0)$. This and Remark 1.3.2 imply that, in order to prove (2.2.2), it is sufficient to prove the following relation:

$$\eta_0''(h, \nu_0) \xrightarrow{\mathsf{P}} 0 \text{ as } h \to 0. \tag{2.2.3}$$

As is known, every point of stochastic continuity of a càdlàg process is also a point of continuity of this process with probability 1. Therefore,

$$\eta_0''(h, x) \xrightarrow{\mathsf{P}} 0 \text{ as } h \to 0, \ x \geq 0. \tag{2.2.4}$$

Now, using (2.2.4), independence of $\xi_0''(t)$, $t \geq 0$ and ν_0, and the Lebesgue theorem, we get

$$\mathsf{P}\{\eta_0''(h, \nu_0) > \delta\} = \int_0^\infty \mathsf{P}\{\eta_0''(h, x) > \delta\}\mathsf{P}\{\nu_0 \in dx\} \to 0 \text{ as } h \to 0. \tag{2.2.5}$$

This completes the proof. □

The requirement that the process $\xi_0''(t)$ in condition $\mathbf{\Omega_1}$ is stochastically continuous can be replaced with a weaker condition that $\mathsf{P}\{\nu_0 \in S_0''\} = 1$, where S_0'' is the set of points of stochastic continuity of the process $\xi_0''(t)$, $t \geq 0$. Indeed, in this case relation (2.2.4) holds for all points $x \in S_0''$, i.e., almost everywhere with respect to the distribution of the random variable ν_0. One can still use the Lebesgue theorem and prove that the limit in (2.2.5) equals zero.

The assumption of continuity of the process $\xi_0'(t)$ in condition $\mathbf{\Omega_1}$ can also be weakened. It can be replaced with a weaker assumption that $\mathsf{P}\{\nu_0 \in U_0'\} = 1$, where U_0' is some set of points such that the process $\xi_0'(t)$ is continuous simultaneously at all points $t \in U_0'$ with probability 1. Indeed, let $A = \{\omega : \nu_0(\omega) \in S_0''\}$ and $B = \{\omega : \nu_0(\omega) \in U_0'\}$ be the corresponding sets of elementary events. Both events A and B have probability 1. Obviously, $\nu_0(\omega)$ is a point of continuity for the realisation $\xi_0'(t, \omega)$ for every elementary event $\omega \in A \cap B$ and $\mathsf{P}(A \cap B) = 1$.

For example, let ν_0 have a discrete distribution concentrated in points of a countable set $U_0' = \{u_k\}$. Then the process $\xi_0'(t)$ is continuous simultaneously at all points $t \in U_0'$ with probability 1 if this process is continuous with probability 1 at each point of the set U_0'. To prove this, it is enough to assume that the process $\xi_0'(t)$ is stochastically continuous at each point $t \in U_0'$. This is so, since this process is a càdlàg process.

Also, the additive decomposition in $\mathbf{\Omega_1}$ can be generalised to a more general form, $\xi_0(t) = f(t, \xi_0'(t), \xi_0''(t))$, $t \geq 0$, where $\xi_0'(t)$ and $\xi_0''(t)$ are processes with the same properties as in $\mathbf{\Omega_1}$, and $f(t, x, y)$ is a continuous function.

Some simple sufficient continuity conditions can also be formulated in the general case where no decomposition assumptions are made. Since the process $\xi_0(t)$, $t \geq 0$, is a càdlàg process, it has, with probability 1, a finite number of discontinuity points at which the absolute values of jumps belongs to the interval $[\frac{1}{n}, \frac{1}{n-1})$ in any finite interval $[0, T]$. This is the case for every $n = 1, 2, \ldots$. Let us recursively define $\tau_{kn} = \inf(s > \tau_{k-1n}\colon |\xi_0(s) - \xi_0(s-0)| \in [\frac{1}{n}, \frac{1}{n-1}))$, $k = 1, 2, \ldots$, and $\tau_{0n} = 0$. By the definition, τ_{kn} are successive moments of such jumps for $k < \mu_n + 1$ and $\tau_{kn} = \infty$ for $k \geq \mu_n + 1$, where $\mu_n = \max(k \geq 0\colon \tau_{kn} < \infty)$ is the total number of jumps in the interval $[0, \infty)$ which have the absolute values in the interval $[\frac{1}{n}, \frac{1}{n-1})$. The random variables μ_n can take the values $0, 1, \ldots, \infty$. Now, let us define the random set of all points of jumps,

$$R[\xi_0(\cdot)] = \{\tau_{kn}\colon 1 \leq k < \mu_n + 1, n = 1, 2, \ldots\}. \tag{2.2.6}$$

Condition $\mathcal{C}_3$ can be rewritten in an equivalent form,

$\mathcal{C}_3'$: $\mathsf{P}\{\nu_0 \in R[\xi_0(\cdot)]\} = 0$,

or as

$\mathcal{C}_3''$: $\mathsf{P}\{\nu_0 = \tau_{kn}\} = 0$ for $k, n = 1, 2, \ldots$.

If condition $\mathcal{Q}_1$ holds, then the random variable τ_{kn} can be a discontinuity point for the process $\xi_0(t)$ if and only if it is a discontinuity point for the second component $\xi_0''(t)$. This is so, because the first component $\xi_0'(t)$ is a continuous process. Therefore, ν_0 and τ_{kn} are independent, and condition $\mathcal{C}_3''$ is equivalent to the requirement that the distribution functions of ν_0 and τ_{kn} have no common discontinuity points for every $k, n \geq 1$.

Condition $\mathcal{C}_3''$ also holds if, for every k, $n = 1, 2, \ldots$, the random variables ν_0 and τ_{kn} are independent and their distribution functions have no common discontinuity points. In this case, the process $\xi_0(t)$ and the random variable ν_0 can be dependent. In particular, ν_0 can depend on the process $\xi_0'(t), t \geq 0$, and also on values of the process $\xi_0''(t), t \geq 0$ at moments of its jumps.

Moreover, condition $\mathcal{C}_3''$ holds also if the random variables ν_0 and τ_{kn} are dependent but the distributions of the random variables $\tau_{kn} - \nu_0$ are continuous at zero.

2.3 Randomly stopped vector càdlàg processes

In this section, the results formulated in Section 2.2 are generalised to the case of vector processes. This is a necessary step to weak convergence theorems for compositions of stochastic processes.

2.3.1. Main results. Let, for every $\varepsilon \geq 0$, $\boldsymbol{\xi}_\varepsilon(t) = (\xi_{\varepsilon 1}(t), \ldots, \xi_{\varepsilon m}(t))$, $t \geq 0$, be a càdlàg random process taking values in $\mathbb{R}_m$, and $\boldsymbol{\nu}_\varepsilon = (\nu_{\varepsilon 1}, \ldots, \nu_{\varepsilon m})$ be a random vector with non-negative components. We call $\boldsymbol{\xi}_\varepsilon(t)$, $t \geq 0$ an *external process* and $\boldsymbol{\nu}_\varepsilon$ a vector *stopping moment*. Consider the random vectors $\boldsymbol{\zeta}_\varepsilon = (\xi_{\varepsilon 1}(\nu_{\varepsilon 1}), \ldots, \xi_{\varepsilon m}(\nu_{\varepsilon m}))$.

Let us introduce conditions that are vector analogues of conditions $\mathcal{A}_{18}$ and $\mathcal{C}_3$,

$\mathcal{A}_{19}$: For every $i = 1, \dots, m$, there exists a set S_i dense in $[0, \infty)$, containing 0, and such that $\mathsf{P}\{\nu_{0i} = t\} = 0$ for $t \in S_i \setminus \{0\}$, and for all $t'_i, t''_i \in S_i$, $i = 1, \dots, m$,

$$\begin{aligned}&(\nu_{\varepsilon i}, \sup_{t\in[t'_i,t''_i)} \xi_{\varepsilon i}(t), i = 1, \dots, m)\\ &\qquad \Rightarrow (\nu_{0i}, \sup_{t\in[t'_i,t''_i)} \xi_{0i}(t), i = 1, \dots, m) \text{ as } \varepsilon \to 0,\\ &(\nu_{\varepsilon i}, \inf_{t\in[t'_i,t''_i)} \xi_{\varepsilon i}(t), i = 1, \dots, m)\\ &\qquad \Rightarrow (\nu_{0i}, \inf_{t\in[t'_i,t''_i)} \xi_{0i}(t), i = 1, \dots, m) \text{ as } \varepsilon \to 0.\end{aligned}$$

$\mathcal{C}_4$: The process $\xi_{0i}(t)$, $t \geq 0$, is continuous at the point ν_{0i} with probability 1 for every $i = 1, \dots, m$, i.e., $\mathsf{P}\{\lim_{t\to 0} \xi_{0i}(\nu_{0i} + t) = \xi_{0i}(\nu_{0i})\} = 1$ for $i = 1, \dots, m$.

The following theorem from Silvestrov (1971b, 1972a) is a vector analogue of Theorem 2.2.1.

Theorem 2.3.1. *Let conditions $\mathcal{A}_{19}$ and $\mathcal{C}_4$ hold. Then*

$$(\xi_{\varepsilon i}(\nu_{\varepsilon i}), i = 1, \dots, m) \Rightarrow (\xi_{0i}(\nu_{0i}), i = 1, \dots, m) \text{ as } \varepsilon \to 0.$$

Proof of Theorem 2.3.1. For each $i = 1, \dots, m$ and $n \geq 1$, choose partitions of the interval $[0, \infty)$ such that, for every $i = 1, \dots, m$, **(a)** $0 = z_{i,0,n} < z_{i,1,n} < \cdots < z_{i,n,n} < z_{i,n+1,n} = \infty$ for $i = 1, \dots, m$, **(b)** $h_i(n) = \max_{0\leq k\leq n-1} |z_{i,k+1,n} - z_{i,k,n}| \to 0$ as $n \to \infty$ for $i = 1, \dots, m$, and **(c)** $z_{i,n,n} \to \infty$ as $n \to \infty$.

For $i = 1, \dots, m$ and $n \geq 1$, define the random variables

$$\xi^+_{\varepsilon i}(n) = \sum_{k=0}^{n} \sup_{t\in[z_{i,k,n},z_{i,k+1,n})} \xi_{\varepsilon i}(t)\chi(\nu_{\varepsilon i} \in [z_{i,k,n}, z_{i,k+1,n})), \tag{2.3.1}$$

and

$$\xi^-_{\varepsilon i}(n) = \sum_{k=0}^{n} \inf_{t\in[z_{i,k,n},z_{i,k+1,n})} \xi_{\varepsilon i}(t)\chi(\nu_{\varepsilon i} \in [z_{i,k,n}, z_{i,k+1,n})). \tag{2.3.2}$$

Let also

$$\eta_{\varepsilon i}(h, x) = \sup_{|t|\leq h} |\xi_{\varepsilon i}(x + t) - \xi_{\varepsilon i}(x)|, \ x \geq 0, \ i = 1, \dots, m. \tag{2.3.3}$$

It is clear that condition $\mathcal{C}_4$ is equivalent to the relation

$$\eta_{0i}(h, \nu_{0i}) \xrightarrow{\mathsf{P1}} 0 \text{ as } h \to 0, i = 1, \dots, m. \tag{2.3.4}$$

On the other hand, by the definition of the random variables $\xi^{\pm}_{0i}(n)$ in (2.3.1) and (2.3.2), we have, for all $i = 1, \ldots, m$ and $n \geq 1$, that

$$(\xi^+_{0i}(n) - \xi^-_{0i}(n))\chi(\nu_{0i} < z_{i,n,n}) \leq 2\eta_{0i}(h_i(n), \nu_{0i}). \tag{2.3.5}$$

Thus, if condition $\mathcal{C}_4$ and, consequently, (2.3.4) hold, then for $i = 1, \ldots, m$,

$$\begin{aligned} &\mathsf{P}\{\xi^+_{0i}(n) - \xi^-_{0i}(n) > \delta\} \\ &\qquad \leq \mathsf{P}\{\nu_{0i} \geq z_{i,n,n}\} + \mathsf{P}\{\eta_{0i}(h_i(n), \nu_{0i}) > \delta/2\} \to 0 \text{ as } n \to 0, \end{aligned} \tag{2.3.6}$$

that is, for all $i = 1, \ldots, m$,

$$\xi^+_{0i}(n) - \xi^-_{0i}(n) \xrightarrow{\mathsf{P}} 0 \text{ as } n \to \infty. \tag{2.3.7}$$

It is clear that, for all $i = 1, \ldots, m$, $n \geq 1$, and $\varepsilon \geq 0$,

$$\xi^-_{\varepsilon i}(n) \leq \xi_{\varepsilon i}(\nu_{\varepsilon i}) \leq \xi^+_{\varepsilon i}(n). \tag{2.3.8}$$

It follows from (2.3.7) and (2.3.8) that

$$(\xi^+_{0i}(n), i = 1, \ldots, m) \xrightarrow{\mathsf{P}} (\xi_{0i}(\nu_{0i}), i = 1, \ldots, m) \text{ as } n \to \infty, \tag{2.3.9}$$

and

$$(\xi^-_{0i}(n), i = 1, \ldots, m) \xrightarrow{\mathsf{P}} (\xi_{0i}(\nu_{0i}), i = 1, \ldots, m) \text{ as } n \to \infty. \tag{2.3.10}$$

The sequence of partitions described in **(a)** – **(c)** can always be chosen in such a way that $z_{i,k,n} \in S_i$ for all $k = 0, \ldots n$, $n \geq 1$, and $i = 1, \ldots, m$.

Let also U be the subset of all points $\mathbf{u} = (u_1, \ldots, u_m)$ such that $\mathsf{P}\{\xi_{0i}(\nu_{0i}) = u_i\} = \mathsf{P}\{\xi^+_{0i}(n) = u_i\} = \mathsf{P}\{\xi^-_{0i}(n) = u_i\} = 0$ for all $n \geq 1$ and $i = 1, \ldots, m$. The set U is dense in $\mathbb{R}_m$.

Relations (2.3.9) and (2.3.10) imply that, for all points $\mathbf{u} \in U$,

$$\mathsf{P}\{\xi^+_{0i}(n) \leq u_i, i = 1, \ldots, m\} \to \mathsf{P}\{\xi_{0i}(\nu_{0i}) \leq u_i, i = 1, \ldots, m\} \text{ as } n \to \infty, \tag{2.3.11}$$

and

$$\mathsf{P}\{\xi^-_{0i}(n) \leq u_i, i = 1, \ldots, m\} \to \mathsf{P}\{\xi_{0i}(\nu_{0i}) \leq u_i, i = 1, \ldots, m\} \text{ as } n \to \infty. \tag{2.3.12}$$

By the definition of $\xi^{\pm}_{\varepsilon i}(n)$, we have

$$\begin{aligned} &\mathsf{P}\{\xi^+_{\varepsilon i}(n) < u_i,\ i = 1, \ldots, m\} \\ &\quad = \sum_{i=1}^{m} \sum_{k_i=0}^{n} \mathsf{P}\{ \sup_{t \in [z_{i,k_i,n}, z_{i,k_i+1,n})} \xi_{\varepsilon i}(t) \leq u_i, \nu_{\varepsilon i} \in [z_{i,k_i,n}, z_{i,k_i+1,n}), i = 1, \ldots, m\}, \end{aligned} \tag{2.3.13}$$

and

$$\mathsf{P}\{\xi^-_{\varepsilon i}(n) < u_i, i = 1, \ldots, m\} \\ = \sum_{i=1}^{m} \sum_{k_i=0}^{n} \mathsf{P}\{\inf_{t \in [z_{i,k_i,n}, z_{i,k_i+1,n})} \xi_{\varepsilon i}(t) \le u_i, \nu_{\varepsilon i} \in [z_{i,k_i,n}, z_{i,k_i+1,n}), i = 1, \ldots, m\}. \tag{2.3.14}$$

Also, by condition $\mathcal{A}_{19}$ and (2.3.13)-(2.3.14), for all points $\mathbf{u} \in U$,

$$\overline{\lim_{\varepsilon\to 0}} |\mathsf{P}\{\xi^+_{\varepsilon i}(n) < u_i, i = 1, \ldots, m\} - \mathsf{P}\{\xi^+_{0i}(n) < u_i, i = 1, \ldots, m\}| \\ \le \overline{\lim_{\varepsilon\to 0}} \sum_{i=1}^{m} (\mathsf{P}\{\nu_{\varepsilon i} \ge z_{i,n,n}\} + \mathsf{P}\{\nu_{0i} \ge z_{i,n,n}\}) = 2 \sum_{i=1}^{m} \mathsf{P}\{\nu_{0i} \ge z_{i,n,n}\}, \tag{2.3.15}$$

and

$$\overline{\lim_{\varepsilon\to 0}} |\mathsf{P}\{\xi^-_{\varepsilon i}(n) \le u_i, i = 1, \ldots, m\} - \mathsf{P}\{\xi^-_{0i}(n) \le u_i, i = 1, \ldots, m\}| \\ \le \overline{\lim_{\varepsilon\to 0}} \sum_{i=1}^{m} (\mathsf{P}\{\nu_{\varepsilon i} \ge z_{i,n,n}\} + \mathsf{P}\{\nu_{0i} \ge z_{i,n,n}\}) = 2 \sum_{i=1}^{m} \mathsf{P}\{\nu_{0i} \ge z_{i,n,n}\}. \tag{2.3.16}$$

Now, using relations (2.3.8), (2.3.11), (2.3.12), (2.3.15), and (2.3.16) we get, for every point $\mathbf{u} \in U$, the following estimates:

$$\overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\xi_{\varepsilon i}(\nu_{\varepsilon i}) \le u_i, i = 1, \ldots, m\} \\ \le \lim_{n\to\infty} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\xi^-_{\varepsilon i}(n) \le u_i, i = 1, \ldots, m\} \\ \le \lim_{n\to\infty} (\mathsf{P}\{\xi^-_{0i}(n) \le u_i, i = 1, \ldots, m\} + 2\sum\nolimits_{i=1}^{m} \mathsf{P}\{\nu_{0i} \ge z_{i,n,n}\}) \\ = \mathsf{P}\{\xi_{0i}(\nu_{0i}) \le u_i, i = 1, \ldots, m\}, \tag{2.3.17}$$

and

$$\underline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\xi_{\varepsilon i}(\nu_{\varepsilon i}) \le u_i, i = 1, \ldots, m\} \\ \ge \lim_{n\to\infty} \underline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\xi^+_{\varepsilon i}(n) \le u_i, i = 1, \ldots, m\} \\ \ge \lim_{n\to\infty} (\mathsf{P}\{\xi^+_{0i}(n) \le u_i, i = 1, \ldots, m\} - 2\sum\nolimits_{i=1}^{m} \mathsf{P}\{\nu_{0i} \ge z_{i,n,n}\}) \\ = \mathsf{P}\{\xi_{0i}(\nu_{0i}) \le u_i, i = 1, \ldots, m\}. \tag{2.3.18}$$

Obviously, relations (2.3.17) and (2.3.18) imply that for every $\mathbf{u} \in U$,

$$\lim_{\varepsilon\to 0} \mathsf{P}\{\xi_{\varepsilon i}(\nu_{\varepsilon i}) \le u_i, i = 1, \ldots, m\} = \mathsf{P}\{\xi_{0i}(\nu_{0i}) \le u_i, i = 1, \ldots, m\}. \tag{2.3.19}$$

Relation (2.3.19) proves the theorem (recall that distribution functions of random vectors weakly converge if they converge on a dense subset in $\mathbb{R}_m$). □

2.3.2. Joint weak convergence of randomly stopped processes and stopping moments. The statement of Theorem 2.3.1 can be strengthened in the following way.

Theorem 2.3.2. *Let conditions $\mathcal{A}_{19}$ and $\mathfrak{C}_4$ hold. Then*

$$(\nu_{\varepsilon i}, \xi_{\varepsilon i}(\nu_{\varepsilon i}), i = 1, \ldots, m) \Rightarrow (\nu_{0i}, \xi_{0i}(\nu_{0i}), i = 1, \ldots, m) \text{ as } \varepsilon \to 0.$$

Proof of Theorem 2.3.2. If conditions $\mathcal{A}_{19}$ and $\mathfrak{C}_4$ are fulfilled for the random vectors $(\nu_{\varepsilon i}, i = 1, \ldots, m)$ and the processes $(\xi_{\varepsilon i}(t), i = 1, \ldots, m)$, $t \geq 0$, then these conditions are also fulfilled for the random vectors $(\nu_{\varepsilon i}, \nu_{\varepsilon i}, i = 1, \ldots, m)$ and the processes $(\xi'_{\varepsilon i}(t), \xi_{\varepsilon i}(t), i = 1, \ldots, m)$, $t \geq 0$, where $\xi'_{\varepsilon i}(t) = t$, $t \geq 0$, for $i = 1, \ldots, m$. □

2.3.3. Condition $\mathcal{A}_{19}$ and J-convergence of vector càdlàg processes $(\boldsymbol{\nu}_\varepsilon, \boldsymbol{\xi}_\varepsilon(t))$, $t \geq 0$. The following condition ia a vector analogue of condition $\mathcal{A}_{17}$:

$\mathcal{A}_{20}$: $(\boldsymbol{\nu}_\varepsilon, \boldsymbol{\xi}_\varepsilon(t)), t \in U \Rightarrow (\boldsymbol{\nu}_0, \boldsymbol{\xi}_0(t)), t \in U$ as $\varepsilon \to 0$, where U is a subset of $[0, \infty)$ that is dense in this interval and contains the point 0.

The following condition of **J**-compactness of external processes was introduced in Subsection 1.6.11:

$\mathfrak{J}_4$: $\lim_{c \to 0} \overline{\lim}_{\varepsilon \to 0} \mathsf{P}\{\Delta_J(\boldsymbol{\xi}_\varepsilon(\cdot), c, T) > \delta\} = 0, \ \delta, T > 0.$

Let S_0 be the set of points of stochastic continuity of the process $\boldsymbol{\xi}_0(t)$, $t \geq 0$. This set is the interval $[0, \infty)$, except for at most a countable set. Note also that $0 \in S_0$. Let also Y_i be the set that contains all points y that are points of continuity of the distribution functions of the random variables ν_{0i} and the point 0. By the definition, $\overline{Y}_i$ is the set of points $t > 0$ for which $\mathsf{P}\{\nu_{0i} = t\} > 0$. Each such set contains at most a countable number of points. Therefore, each set $S_i = S_0 \setminus \overline{Y}_i$ is dense in $[0, \infty)$ and contains the point 0. Moreover, this set coincides with $[0, \infty)$ except for, possibly, some finite or countable set.

Lemma 2.3.1. *Let conditions $\mathcal{A}_{20}$ and $\mathfrak{J}_4$ hold. Then condition $\mathcal{A}_{19}$ holds with the sets $S_i = S_0 \setminus \overline{Y}_i$, $i = 1, \ldots, m$.*

Proof of Lemma 2.3.1. Obviously, $\tilde{\boldsymbol{\xi}}_\varepsilon(t) = (\boldsymbol{\nu}_\varepsilon, \boldsymbol{\xi}_\varepsilon(t))$, $t \geq 0$, is a càdlàg process with the phase space $\mathbb{R}_{2m}$. The first m components of this process do not depend on time.

Condition $\mathcal{A}_{20}$ yields the weak convergence of these processes on the set U from this condition. Also, the processes $\tilde{\boldsymbol{\xi}}_\varepsilon(t)$, $t \geq 0$, and $\boldsymbol{\xi}_\varepsilon(t)$, $t \geq 0$, have the same moduli of **J**-compactness, i.e., $\Delta_J(\tilde{\boldsymbol{\xi}}_\varepsilon(\cdot), c, T) = \Delta_J(\boldsymbol{\xi}_\varepsilon(\cdot), c, T)$ for every $c, T > 0$. So, $\mathfrak{J}_4$ can serve as a **J**-compactness condition for the vector processes $\tilde{\boldsymbol{\xi}}_\varepsilon(t)$, $t \geq 0$.

Thus, $\mathcal{A}_{20}$ and $\mathfrak{J}_4$ provide **J**-convergence of the processes $\tilde{\boldsymbol{\xi}}_\varepsilon(t)$, $t \geq 0$, i.e.,

$$\tilde{\boldsymbol{\xi}}_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{J}} \tilde{\boldsymbol{\xi}}_0(t), t \geq 0 \text{ as } \varepsilon \to 0. \tag{2.3.20}$$

Let $F_0(A)$ denote the measure generated by the process $\tilde{\xi}_0(t)$, $t \geq 0$ on the Borel σ-algebra $\mathfrak{B}^{(2m)}_{[0,\infty)}$.

Let $\mathbf{x}(t) = (x_1(t), \dots x_{2m}(t)), t \geq 0$ be a càdlàg function from the space $\mathbf{D}^{(2m)}_{[0,\infty)}$. Let us consider the functionals $f_i(\mathbf{x}(\cdot)) = x_i(0)$ and $m^+_{i,t',t''}(\mathbf{x}(\cdot)) = \sup_{t\in[t',t'')} x_{m+i}(t)$, $m^-_{i,t',t''}(\mathbf{x}(\cdot)) = \inf_{t\in[t',t'')} x_{m+i}(t)$ for $0 \leq t' < t'' < \infty$, $i = 1, \dots, m$.

According to Lemmas 1.5.1 and 1.5.9, the functionals $f_i(\mathbf{x}(\cdot))$ belong to the class $\mathfrak{H}_{J,\infty}[F_0]$ as well as the functionals $m^{\pm}_{i,t'_i,t''_i}(\mathbf{x}(\cdot))$ for all $0 \leq t'_i < t''_i < \infty$, $t'_i, t''_i \in S_i, i = 1, \dots, m$. By the definition of these functionals, $f_i(\tilde{\xi}_\varepsilon(\cdot)) = \nu_{\varepsilon i}$, while $m^+_{i,t'_i,t''_i}(\tilde{\xi}_\varepsilon(\cdot)) = \sup_{t\in[t'_i,t''_i)} \xi_{\varepsilon i}(t)$ and $m^-_{i,t'_i,t''_i}(\tilde{\xi}_\varepsilon(\cdot)) = \inf_{t\in[t'_i,t''_i)} \xi_{\varepsilon i}(t)$.

Now, by Theorem 1.6.7, for all $0 \leq t'_i < t''_i < \infty, t'_i, t''_i \in S_i, i = 1, \dots, m$,

$$(\nu_{\varepsilon i}, \sup_{t\in[t'_i,t''_i)} \xi_{\varepsilon i}(t), i = 1, \dots, m) \Rightarrow (\nu_{0i}, \sup_{t\in[t'_i,t''_i)} \xi_{0i}(t), i = 1, \dots, m) \text{ as } \varepsilon \to 0, \qquad (2.3.21)$$

and

$$(\nu_{\varepsilon i}, \inf_{t\in[t'_i,t''_i)} \xi_{\varepsilon i}(t), i = 1, \dots, m) \Rightarrow (\nu_{0i}, \inf_{t\in[t'_i,t''_i)} \xi_{0i}(t), i = 1, \dots, m) \text{ as } \varepsilon \to 0. \qquad (2.3.22)$$

Relations (2.3.21) and (2.3.22) imply that condition $\mathcal{A}_{19}$ holds with the sets $S_i = S_0 \setminus \overline{Y}_i, i = 1, \dots, m$. □

Now we can formulate the following theorem from Silvestrov (1971b, 1972a), which is a vector analogue of Theorem 2.2.2.

Theorem 2.3.3. *Let conditions $\mathcal{A}_{20}$, $\mathfrak{J}_4$, and $\mathfrak{C}_4$ hold. Then*

$$(\xi_{\varepsilon i}(\nu_{\varepsilon i}), i = 1, \dots, m) \Rightarrow (\xi_{0i}(\nu_{0i}), i = 1, \dots, m) \text{ as } \varepsilon \to 0.$$

Condition $\mathfrak{J}_4$ in Theorem 2.3.3 can be weakened in the following way. Let us introduce the condition:

$\mathfrak{J}_8$: $\lim_{c\to 0} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\Delta_J(\xi_{\varepsilon i}(\cdot), c, T) > \delta\} = 0, \ \delta, T > 0, \ i = 1, \dots, m.$

It should be noted that condition $\mathfrak{J}_8$ is weaker than $\mathfrak{J}_4$. An example of vector càdlàg functions that satisfy condition $\mathfrak{J}_8$ but do not satisfy condition $\mathfrak{J}_4$ is given in Section 3.1.

Lemma 2.3.2. *Let conditions $\mathcal{A}_{20}$ and $\mathfrak{J}_8$ hold. Then condition $\mathcal{A}_{19}$ holds with the sets $S_i = S_0 \setminus \overline{Y}_i$, $i = 1, \dots, m$.*

Proof of Lemma 2.3.2. The proof is analogous to the proof of Lemma 2.3.1. The difference is that one must use Theorem 1.6.8 instead of Theorem 1.6.7.

Let us consider the processes $\tilde{\xi}_{\varepsilon i}(t) = (\nu_{\varepsilon i}, \xi_{\varepsilon i}(t))$, $t \geq 0$, for $i = 1, \dots, m$. Let $F_{0i}(A)$ be the measure generated by the process $\tilde{\xi}_{0i}(t)$, $t \geq 0$ on the Borel σ-algebra of subsets of $\mathfrak{B}^{(2)}_{[0,\infty)}$. Conditions $\mathcal{A}_{20}$ and $\mathfrak{J}_8$ imply that these processes satisfy the conditions of Theorem 1.6.8.

Let $\mathbf{x}(t) = (x_1(t), x_2(t)), t \geq 0$ be a càdlàg function from the space $\mathbf{D}^{(2)}_{[0,\infty)}$. Let us consider the functionals $f(\mathbf{x}(\cdot)) = x_1(0)$ and $m^+_{t',t''}(\mathbf{x}(\cdot)) = \sup_{t\in[t',t'')} x_2(t)$, $m^-_{t',t''}(\mathbf{x}(\cdot)) = \inf_{t\in[t',t'')} x_2(t)$ for $0 \leq t' < t'' < \infty$.

Also, according Lemmas 1.5.1 and 1.5.9, for every $i = 1, \ldots, m$, the functionals $f(\mathbf{x}(\cdot))$ and $m^\pm_{t'_i,t''_i}(\mathbf{x}(\cdot))$, for all $0 \leq t'_i < t''_i < \infty$, $t'_i, t''_i \in S_i$, belong to the class $\mathfrak{H}_{J,\infty}[F_{0i}]$.

Now, by Theorem 1.6.8, we get for all $0 \leq t'_i < t''_i < \infty, t'_i, t''_i \in S_i, i = 1, \ldots, m$,

$$(\nu_{\varepsilon i}, \sup_{t\in[t'_i,t''_i)} \xi_{\varepsilon i}(t), i = 1, \ldots, m) \Rightarrow (\nu_{0i}, \sup_{t\in[t'_i,t''_i)} \xi_{0i}(t), i = 1, \ldots, m) \text{ as } \varepsilon \to 0. \quad (2.3.23)$$

In the same way, for all $0 \leq t'_i < t''_i$, t'_i, $t''_i \in S_i$, $i = 1, \ldots, m$, we get the following relation:

$$(\nu_{\varepsilon i}, \inf_{t\in[t'_i,t''_i)} \xi_{\varepsilon i}(t), i = 1, \ldots, m) \Rightarrow (\nu_{0i}, \inf_{t\in[t'_i,t''_i)} \xi_{0i}(t), i = 1, \ldots, m) \text{ as } \varepsilon \to 0. \quad (2.3.24)$$

Relations (2.3.23) and (2.3.24) obviously imply that condition $\mathcal{A}_{19}$ holds with the sets $S_i = S_0 \setminus \overline{Y}_i, i = 1, \ldots, m$. □

We can improve Theorem 2.3.3 by replacing the condition $\mathfrak{J}_4$ with the weaker condition $\mathfrak{J}_8$.

Theorem 2.3.4. *Let conditions $\mathcal{A}_{20}$, $\mathfrak{J}_8$, and $\mathfrak{C}_4$ hold. Then*

$$(\xi_{\varepsilon i}(\nu_{\varepsilon i}), i = 1, \ldots, m) \Rightarrow (\xi_{0i}(\nu_{0i}), i = 1, \ldots, m) \textit{ as } \varepsilon \to 0.$$

Remark 2.3.1. If the limiting process $\xi_0(t), t \geq 0$, is a.s. continuous, condition $\mathfrak{C}_4$ automatically holds. In this case, the modulus of **J**-compactness in the conditions $\mathfrak{J}_4$ and $\mathfrak{J}_8$ can be replaced with the corresponding modulus of **U**-compactness. After this, the conditions $\mathfrak{J}_4$ and $\mathfrak{J}_8$ become equivalent.

2.3.3. The case of non-random functions. Let us consider the case where a non-random càdlàg vector function $\mathbf{x}_\varepsilon(t) = (x_{\varepsilon i}(t), i = 1, \ldots, m), t \geq 0$ is stopped at a non-random vector point $\mathbf{y}_\varepsilon = (y_{\varepsilon i}, i = 1, \ldots, m)$. Consider the vector $\mathbf{z}_\varepsilon = (x_{\varepsilon i}(y_{\varepsilon i}), i = 1, \ldots, m)$. In this case, condition $\mathcal{A}_{20}$ reduces to the following conditions: **(a)** $\mathbf{x}_\varepsilon(t) \to \mathbf{x}_0(t)$ as $\varepsilon \to 0$ for $t \in U$, where U is some set of points everywhere dense in $[0, \infty)$ and containing 0; and **(b)** $\mathbf{y}_\varepsilon \to \mathbf{y}_0$ as $\varepsilon \to 0$. Condition $\mathfrak{J}_8$ is a condition of **J**-compactness of the functions $x_{\varepsilon i}(t), t \geq 0$, which now becomes **(c)** $\lim_{c\to 0} \overline{\lim}_{\varepsilon\to 0} \Delta_J(x_{\varepsilon i}(\cdot), c, T) = 0$, for every $\delta, T > 0$ and $i = 1, \ldots, m$. Note that **(a)** and **(c)** are just necessary and sufficient conditions of **J**-convergence of the functions $x_{\varepsilon i}(t), t \geq 0$, for every $i = 1, \ldots, m$. They can be re-casted as **(d)** $x_{\varepsilon i}(t), t \geq 0 \stackrel{\mathbf{J}}{\longrightarrow} x_{0i}(t), t \geq 0$ as $\varepsilon \to 0$, $i = 1, \ldots, m$. Note that **J**-convergence of the vector functions $\mathbf{x}_\varepsilon(t), t \geq 0$ is not required. Finally, condition $\mathfrak{C}_4$

takes the following form: **(e)** y_{0i} is a continuity point of the function $x_{0i}(t)$, $t \geq 0$ for every $i = 1, \dots, m$. In this case, Theorem 2.3.4 states that $\mathbf{z}_\varepsilon \to \mathbf{z}_0$ as $\varepsilon \to 0$ if conditions **(a)**, **(b)**, **(c)**, and **(e)** are satisfied.

2.3.4. Condition $\mathcal{A}_{19}$ and M-convergence of processes $(\nu_\varepsilon, \xi_\varepsilon(t))$, $t \geq 0$. Similar to the one-dimensional case, conditions $\mathcal{A}_{20}$ and $\mathcal{J}_8$ can be replaced with the corresponding conditions that are based on the weaker topology **M**.

Let as recall the condition of **M**-compactness introduced in Subsection 1.6.16,

$$\mathcal{M}_5\text{:}\ \lim_{c\to 0} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\Delta_M(\xi_{\varepsilon i}(\cdot), c, T) > \delta\} = 0,\ \delta, T > 0,\ i = 1, \dots, m.$$

Lemma 2.3.3. *Let conditions $\mathcal{A}_{20}$ and $\mathcal{M}_5$ hold. Then condition $\mathcal{A}_{19}$ holds with the sets $S_i = S_0 \setminus \overline{Y}_i$, $i = 1, \dots, m$.*

Proof of Lemma 2.3.3. The condition of joint weak convergence, $\mathcal{A}_{20}$, and the condition of **M**-compactness, $\mathcal{M}_5$, imply **M**-convergence of the scalar processes $\xi_{\varepsilon i}(t)$, $t \geq 0$, for every $i = 1, \dots, m$. Moreover, for every $u_i, w_i \in R_1, i = 1, \dots m$, conditions $\mathcal{A}_{20}$ and $\mathcal{M}_5$ imply **M**-convergence of the scalar processes $u_i\nu_{\varepsilon i} + w_i\xi_{\varepsilon i}(t)$, $t \geq 0$ for every $i = 1, \dots, m$. By using Theorem 1.6.13, one gets the following relations for all $0 \leq t'_i < t''_i < \infty, t'_i, t''_i \in S_i, i = 1, \dots, m$:

$$\begin{aligned} &(u_i\nu_{\varepsilon i} + w_i \sup_{t\in[t'_i,t''_i)} \xi_{\varepsilon i}(t), i = 1, \dots, m) \\ &\qquad \Rightarrow (u_i\nu_{0i} + w_i \sup_{t\in[t'_i,t''_i)} \xi_{0i}(t), i = 1, \dots, m) \text{ as } \varepsilon \to 0, \end{aligned} \tag{2.3.25}$$

and

$$\begin{aligned} &(u_i\nu_{\varepsilon i} + w_i \inf_{t\in[t'_i,t''_i)} \xi_{\varepsilon i}(t), i = 1, \dots, m) \\ &\qquad \Rightarrow (u_i\nu_{0i} + w_i \inf_{t\in[t'_i,t''_i)} \xi_{0i}(t), i = 1, \dots, m) \text{ as } \varepsilon \to 0. \end{aligned} \tag{2.3.26}$$

Since the choice of $u_i, w_i \in R_1, i = 1, \dots, m$ is arbitrary, these relations imply that for any $0 \leq t'_i < t''_i$, t'_i, $t''_i \in S_i$, $i = 1, \dots, m$,

$$(\nu_{\varepsilon i}, \sup_{t\in[t'_i,t''_i)} \xi_{\varepsilon i}(t), i = 1, \dots, m) \Rightarrow (\nu_0, \sup_{t\in[t'_i,t''_i)} \xi_{0i}(t), i = 1, \dots, m) \text{ as } \varepsilon \to 0, \tag{2.3.27}$$

and

$$(\nu_{\varepsilon i}, \inf_{t\in[t'_i,t''_i)} \xi_{\varepsilon i}(t), i = 1, \dots, m) \Rightarrow (\nu_0, \inf_{t\in[t'_i,t''_i)} \xi_{0i}(t), i = 1, \dots, m) \text{ as } \varepsilon \to 0. \tag{2.3.28}$$

Relations (2.3.27) and (2.3.28), obviously, imply that condition $\mathcal{A}_{19}$ holds with the sets $S_i = S_0 \setminus \overline{Y}_i$, $i = 1, \dots, m$. □

The following theorem is a corollary of Theorem 2.3.1 and Lemma 2.3.3.

Theorem 2.3.5. *Let conditions $\mathcal{A}_{20}$, $\mathcal{M}_5$, and $\mathcal{C}_4$ hold. Then*

$$(\xi_{\varepsilon i}(\nu_{\varepsilon i}), i = 1, \ldots, m) \Rightarrow (\xi_{0i}(\nu_{0i}), i = 1, \ldots, m) \text{ as } \varepsilon \to 0.$$

2.3.5. The continuity condition $\mathcal{C}_4$. Condition $\mathcal{C}_4$, actually, is an assumption that condition $\mathcal{C}_3$ holds for the process $\xi_{0i}(t)$, $t \geq 0$ and the random variable ν_{0i} for every $i = 1, \ldots, m$. All remarks made in Section 2.2 can be repeated without any change.

In particular, condition $\mathcal{C}_4$ holds if condition $\mathfrak{Q}_1$ holds for the process $\xi_{0i}(t)$, $t \geq 0$, and the random variable ν_{0i} for every $i = 1, \ldots, m$.

2.3.6. Time interval $(-\infty, \infty)$. All the results formulated above can be generalised to the model where the càdlàg processes $\boldsymbol{\xi}_\varepsilon(t) = (\xi_{\varepsilon 1}(t), \ldots, \xi_{\varepsilon m}(t))$, $t \geq 0$ are defined on the time interval $(-\infty, \infty)$, and the random vectors $\boldsymbol{\nu}_\varepsilon = (\nu_{\varepsilon 1}, \ldots, \nu_{\varepsilon m})$ take values in space $\mathbb{R}_m$. In this case, the sets S_i in condition $\mathcal{A}_{19}$ and the set U in condition $\mathcal{A}_{20}$ must be everywhere dense in $(-\infty, \infty)$, while the relations of $\mathbf{J}$-compactness in the conditions $\mathfrak{J}_4$ and $\mathfrak{J}_8$ must hold for any finite interval $[T', T'']$ where $-\infty < T' < T'' < \infty$.

2.3.7. Positive limiting stopping moments. In the case where the limiting stopping moments $\nu_{0i} > 0, i = 1, \ldots, m$, with probability 1, one can slightly weaken the conditions $\mathcal{A}_{19}$, $\mathcal{A}_{20}$ and the conditions $\mathfrak{J}_4$, $\mathfrak{J}_8$. In this case, the sets S_i in condition $\mathcal{A}_{19}$ and U in the condition $\mathcal{A}_{20}$ must be dense in $(0, \infty)$, and the relations of $\mathbf{J}$-compactness in the conditions $\mathfrak{J}_4$ or $\mathfrak{J}_8$ must hold for any finite interval $[T', T'']$ with $0 < T' < T'' < \infty$.

This generalisation can be achieved by using the following standard method. Let us consider the basic case where condition $\mathcal{A}_{19}$ holds. One can always choose some sequences $0 < s_{ni} \to 0$ as $n \to \infty$ such that, for every $i = 1, \ldots, m$ and $n = 0, 1, \ldots$, the point s_{ni} belongs to the set S_i. Then one can consider the processes $\xi^{(n)}_{\varepsilon i+}(t) = \xi_{\varepsilon i}(t)\chi(t \geq s_{ni})$, $\xi^{(n)}_{\varepsilon i-}(t) = \xi_{\varepsilon i}(t)\chi(t < s_{ni})$, $t \geq 0$. Obviously, $\xi_{\varepsilon i}(t) = \xi^{(n)}_{\varepsilon i+}(t) + \xi^{(n)}_{\varepsilon i-}(t)$, and, therefore, **(a)** $\xi_{\varepsilon i}(\nu_{\varepsilon i}) = \xi^{(n)}_{\varepsilon i+}(\nu_{\varepsilon i}) + \xi^{(n)}_{\varepsilon i-}(\nu_{\varepsilon i})$.

From $\mathcal{A}_{19}$ and positivity of the random variables $\nu_{0i}, i = 1, \ldots, m$, one gets the following estimate: **(b)** $\lim_{n\to\infty} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{|\xi^{(n)}_{\varepsilon i-}(\nu_{\varepsilon i})| > \delta\} \leq \lim_{n\to\infty} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\nu_{\varepsilon i} < s_{ni}\} = \lim_{n\to\infty} \mathsf{P}\{\nu_{0i} < s_{ni}\} = 0$. It is readily seen that **(c)** conditions $\mathcal{A}_{19}$ and $\mathcal{C}_4$ hold for the processes $\boldsymbol{\xi}_\varepsilon(t) = (\xi^{(n)}_{\varepsilon i+}(t), i = 1, \ldots, m), t \geq 0$, and the random vectors $\boldsymbol{\nu}_\varepsilon = (\nu_{\varepsilon i}, i = 1, \ldots, m)$ for every $n = 0, 1, \ldots$. Hence, **(d)** $(\xi^{(n)}_{0i+}(\nu_{0i}), i = 1, \ldots, m) \Rightarrow (\xi^{(n)}_{0i+}(\nu_{0i}), i = 1, \ldots, m)$ as $\varepsilon \to 0$. The random variables $\nu_{0i}, i = 1, \ldots, m$ are positive. This implies that **(e)** $(\xi^{(n)}_{0i+}(\nu_{0i}), i = 1, \ldots, m) \Rightarrow (\xi_{0i}(\nu_{0i}), i = 1, \ldots, m)$ as $n \to \infty$. Finally, by Lemma 1.2.5 and relations **(a)** – **(e)**,

$$(\xi_{\varepsilon i}(\nu_{\varepsilon i}), i = 1, \ldots, m) \Rightarrow (\xi_{0i}(\nu_{0i}), i = 1, \ldots, m) \text{ as } \varepsilon \to 0. \tag{2.3.29}$$

2.3.8. Random vectors $(\xi_{\varepsilon i}(\nu_{\varepsilon i} - 0), i = 1, \ldots, m)$. Under the same conditions $\mathcal{A}_{19}$ and $\mathcal{C}_4$, the following relation holds:

$$(\xi_{\varepsilon i}(\nu_{\varepsilon i} - 0), i = 1, \ldots, m) \Rightarrow (\xi_{0i}(\nu_{0i} - 0), i = 1, \ldots, m) \text{ as } \varepsilon \to 0. \tag{2.3.30}$$

This relation can be proved following the proof of Theorem 2.3.1 by carrying it over to the random vectors $(\xi_{\varepsilon i}(\nu_{\varepsilon i} - 0), i = 1, \dots, m)$. What needs to be slightly changed is only the definition of the random variables $\xi^{\pm}_{\varepsilon i}(n)$. The corresponding suprema and infima should be taken over the intervals $[z_{i,k-1,n}, z_{i,k+1,n})$, instead of the intervals $[z_{i,k,n}, z_{i,k+1,n})$ (here $z_{i,-1,n} = z_{i,0,n} = 0$). In this case, the same upper and lower approximations can be used for the random variables $\xi_{\varepsilon i}(\nu_{\varepsilon i} \pm 0)$, i.e., for all $i = 1, \dots, m$, $n \geq 1$, and $\varepsilon \geq 0$,

$$\xi^{-}_{\varepsilon i}(n) \leq \xi_{\varepsilon i}(\nu_{\varepsilon i} \pm 0) \leq \xi^{+}_{\varepsilon i}(n). \tag{2.3.31}$$

Moreover, the proof of Theorem 2.3.1 can also be carried over in the same way to the random vectors $(\xi_{\varepsilon i}(\nu_{\varepsilon i}), \xi_{\varepsilon i}(\nu_{\varepsilon i} - 0), i = 1, \dots, m)$. This yields that, under the conditions $\mathcal{A}_{19}$ and $\mathcal{C}_4$, the following relation holds:

$$\begin{aligned}&(\xi_{\varepsilon i}(\nu_{\varepsilon i}), \xi_{\varepsilon i}(\nu_{\varepsilon i} - 0), i = 1, \dots, m) \\ &\qquad \Rightarrow (\xi_{0i}(\nu_{0i}), \xi_{0i}(\nu_{0i} - 0), i = 1, \dots, m) \text{ as } \varepsilon \to 0.\end{aligned} \tag{2.3.32}$$

The idea of extending the intervals $[z_{i,k,n}, z_{i,k+1,n})$ in the definition of the random variables $\xi^{\pm}_{\varepsilon i}(n)$ can be modified. The intervals $(z_{i,k-1,n}, z_{i,k+1,n})$ and $[z_{i,k-1,n}, z_{i,k+1,n}]$ can serve equally well and replace the intervals $[z_{i,k-1,n}, z_{i,k+1,n})$ in the estimate 2.3.31.

This remark leads to modified versions of condition $\mathcal{A}_{19}$ in which the corresponding suprema and infima should be taken over the intervals (t'_i, t''_i) or $[t'_i, t''_i]$, instead of the intervals $[t'_i, t''_i)$. It is useful to note that the conditions $\mathcal{A}_{20}$ and $\mathcal{J}_4$ (or $\mathcal{J}_8$) imply any modification of the condition $\mathcal{A}_{19}$ described above.

In the case where the modification of condition $\mathcal{A}_{19}$ is based on the open intervals (t'_i, t''_i), an interesting method of time reversion can be employed.

Let us take some $T > 0$ such that **(a)** it is a point of stochastic continuity of the process $\boldsymbol{\xi}_0(t), t \geq 0$; **(b)** $\mathsf{P}\{\nu_{0i} = T\} = 0, i = 1, \dots, m$. Let us also assume for the moment that **(c)** $0 \leq \nu_{\varepsilon i} \leq T, i = 1, \dots, m$ for all $\varepsilon \geq 0$.

Consider the process $\boldsymbol{\xi}_\varepsilon(t-0) = (\xi_{\varepsilon i}(t-0), i = 1, \dots, m), t \geq 0$ (here $\boldsymbol{\xi}_\varepsilon(0-0) = \boldsymbol{\xi}_\varepsilon(0)$). This process is continuous from the left, whereas the original process $\boldsymbol{\xi}_\varepsilon(t), t \geq 0$, is continuous from the right. The process with reversed time can be defined by $\boldsymbol{\xi}^{(T)}_\varepsilon(t) = \boldsymbol{\xi}_\varepsilon(T - t - 0), 0 \leq t \leq T$, and $\boldsymbol{\xi}^{(T)}_\varepsilon(t) = \boldsymbol{\xi}_\varepsilon(0)$ for $t > T$. Obviously, $\boldsymbol{\xi}^{(T)}_\varepsilon(t), t \geq 0$ is a càdlàg process. Let also $\boldsymbol{\nu}^{(T)}_\varepsilon = (T - \nu_{\varepsilon i}, i = 1, \dots, m)$.

By the definition of the processes $\boldsymbol{\xi}^{(T)}_\varepsilon(t), t \geq 0$ and the random vectors $\boldsymbol{\nu}^{(T)}_\varepsilon$,

$$(\xi^{(T)}_{\varepsilon i}(T - \nu_{\varepsilon i}), i = 1, \dots, m) = (\xi_{\varepsilon i}(\nu_{\varepsilon i} - 0), i = 1, \dots, m). \tag{2.3.33}$$

Obviously, $\sup_{t\in(t',t'')} x(t) = \sup_{t\in(t',t'')} x(t - 0)$ and $\inf_{t\in(t',t'')} x(t) = \inf_{t\in(t',t'')} x(t - 0)$ for any real-valued càdlàg function $x(t), t \geq 0$. Thus, the modification of condition $\mathcal{A}_{19}$ based on open intervals holds for the processes $\boldsymbol{\xi}^{(T)}_\varepsilon(t), t \geq 0$ and the random vectors $\boldsymbol{\nu}^{(T)}_\varepsilon$. Also, condition $\mathcal{C}_4$ holds for the processes $\boldsymbol{\xi}^{(T)}_0(t), t \geq 0$ and the random vectors $\boldsymbol{\nu}^{(T)}_0$ if it holds for the initial external processes and stopping moments.

Therefore, Theorem 2.3.1 can be applied, and this yields relation (2.3.30).

The general case with unbounded stopping moments can be reduced to the case where condition **(c)** holds. Let $0 < T_n \to \infty$ as $n \to \infty$ be a sequence of points for which conditions **(a)** and **(b)** hold. Condition **(c)** obviously holds for the truncated random variables $\nu_{\varepsilon i} \wedge T, i = 1, \dots, m$.

The following obvious estimate holds for all $i = 1, \dots, m$, $n \geq 1$ and $\sigma > 0$: **(d)** $\mathsf{P}\{|\xi_{\varepsilon i}(\nu_{\varepsilon i} \pm 0) - \xi_{\varepsilon i}((\nu_{\varepsilon i} \wedge T_n) \pm 0)| > \sigma\} \leq \mathsf{P}\{\nu_{\varepsilon i} > T_n\}$. It follows from this estimate and condition $\mathcal{A}_{19}$ that **(e)** $\lim_{n\to\infty} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{|\xi_{\varepsilon i}(\nu_{\varepsilon i} \pm 0) - \xi_{\varepsilon i}((\nu_{\varepsilon i} \wedge T_n) \pm 0)| > \sigma\} = 0$. Also, **(f)** $(\xi_{0i}((\nu_{0i} \wedge T_n) \pm 0), i = 1, \dots, m) \Rightarrow (\xi_{0i}(\nu_{0i} \pm 0), i = 1, \dots, m)$ as $n \to 0$. The modified version of condition $\mathcal{A}_{19}$ and condition $\mathfrak{C}_4$ imply, by the remarks made above, **(g)** $(\xi_{\varepsilon i}((\nu_{\varepsilon i} \wedge T_n) \pm 0), i = 1, \dots, m) \Rightarrow (\xi_{0i}((\nu_{0i} \wedge T_n) \pm 0), i = 1, \dots, m)$ as $\varepsilon \to 0$ for every $n \geq 1$.

Lemma 1.2.5 and relations **(d)** – **(g)** imply that

$$(\xi_{\varepsilon i}(\nu_{\varepsilon i} \pm 0), i = 1, \dots, m) \Rightarrow (\xi_{0i}(\nu_{0i} \pm 0), i = 1, \dots, m) \text{ as } \varepsilon \to 0. \tag{2.3.34}$$

2.3.9. A Polish phase space. The results given in Section 2.3 can also be generalised to a model with càdlàg processes $\xi_{\varepsilon i}(t)$, $t \geq 0$ that take values in a Polish space X. We will show how the consideration can be reduced to real-valued processes.

The vector process $\boldsymbol{\xi}_\varepsilon(t)$, $t \geq 0$, has the phase space $X_m = X \times \cdots \times X$. A metric in the space X_m can be defined by $d_m(\mathbf{x}, \mathbf{y}) = (\sum_{i=1}^m d^2(x_i, y_i))^{1/2}$ for points $\mathbf{x} = (x_1, \dots, x_m), \mathbf{y} = (y_1, \dots, y_m) \in X_m$, where $d(x, y)$ is the corresponding metric in the space X.

The conditions $\mathcal{A}_{20}$ and $\mathfrak{C}_4$ can be kept without any changes. In the conditions $\mathfrak{J}_4$ and $\mathfrak{J}_8$, the Euclidean distance $|x - y|$ must be replaced with the corresponding metrics $d_m(x, y)$ and $d(x, y)$ in the formulas for the moduli of **J**-compactness, $\Delta_J(\boldsymbol{\xi}_\varepsilon(\cdot), c, T)$ and $\Delta_J(\xi_{\varepsilon i}(\cdot), c, T)$. The following theorem is a new result.

Theorem 2.3.6. *Let conditions $\mathcal{A}_{20}$, $\mathfrak{J}_4$, and $\mathfrak{C}_4$ hold. Then*

$$(\xi_{\varepsilon i}(\nu_{\varepsilon i}), i = 1, \dots, m) \Rightarrow (\xi_{0i}(\nu_{0i}), i = 1, \dots, m) \text{ as } \varepsilon \to 0.$$

Proof of Theorem 2.3.6. Let us consider the processes $\tilde{\boldsymbol{\xi}}_\varepsilon(t) = (\boldsymbol{\nu}_\varepsilon, \boldsymbol{\xi}_\varepsilon(t))$, $t \geq 0$. These are càdlàg processes taking values in the space $\mathbb{R}_m \times X_m$.

Conditions $\mathcal{A}_{20}$ and $\mathfrak{J}_4$ imply **J**-convergence of the processes $\tilde{\boldsymbol{\xi}}_\varepsilon(t), t \geq 0$ to the process $\tilde{\boldsymbol{\xi}}_0(t), t \geq 0$ as $\varepsilon \to 0$.

Denote by $F_0(A)$ the measure generated by the process $\tilde{\boldsymbol{\xi}}_0(t), t \geq 0$ on the Borel σ-algebra of the space $\mathbf{D}$ of càdlàg functions $\mathbf{x}(t) = (x_1(t), \dots x_{2m}(t)), t \geq 0$ taking values in the space $\mathbb{R}_m \times X_m$.

Let us $f_i(x)$ be arbitrary continuous bounded functions defined on X, u_i, $w_i \in \mathbb{R}_1$ and points $t'_i < t''_i$, $t'_i, t'_i \in S_0$ for $i = 1, \dots, m$. Here S_0 is the set of points of stochastic continuity of the process $\tilde{\boldsymbol{\xi}}_0(t)$, $t \geq 0$. The functionals $\sum_{i=1}^m u_i x_i(0) + w_i \sup_{t\in[t'_i, t''_i)} f_i(x_{m+i}(t))$

and $\sum_{i=1}^{m} u_i x_i(0) + w_i \inf_{t\in[t'_i,t''_i)} f_i(x_{m+i}(t))$ are a.s. **J**-continuous with respect to measure F_0. Thus,

$$\begin{aligned} &\sum_{i=1}^{m} u_i \nu_{\varepsilon i} + w_i \sup_{t\in[t'_i,t''_i)} f_i(\xi_{\varepsilon i}(t)) \\ &\qquad \Rightarrow \sum_{i=1}^{m} u_i \nu_{0i} + w_i \sup_{t\in[t'_i,t''_i)} f_i(\xi_{0i}(t)) \text{ as } \varepsilon \to 0, \end{aligned} \tag{2.3.35}$$

and

$$\begin{aligned} &\sum_{i=1}^{m} u_i \nu_{\varepsilon i} + w_i \inf_{t\in[t'_i,t''_i)} f_i(\xi_{\varepsilon i}(t)) \\ &\qquad \Rightarrow \sum_{i=1}^{m} u_i \nu_{0i} + w_i \inf_{t\in[t'_i,t''_i)} f_i(\xi_{0i}(t)) \text{ as } \varepsilon \to 0. \end{aligned} \tag{2.3.36}$$

Since $u_i, w_i \in \mathbb{R}_1$, $i = 1, \ldots, m$, are chosen arbitrarily, (2.3.35) and (2.3.36) imply that

$$\begin{aligned} &(\nu_{\varepsilon i}, \sup_{t\in[t'_i,t''_i)} f_i(\xi_{\varepsilon i}(t)), i = 1, \ldots, m) \\ &\qquad \Rightarrow (\nu_{0i}, \sup_{t\in[t'_i,t''_i)} f_i(\xi_{0i}(t)), i = 1, \ldots, m) \text{ as } \varepsilon \to 0, \end{aligned} \tag{2.3.37}$$

and

$$\begin{aligned} &(\nu_{\varepsilon i}, \inf_{t\in[t'_i,t''_i)} f_i(\xi_{\varepsilon i}(t)), i = 1, \ldots, m) \\ &\qquad \Rightarrow (\nu_{0i}, \inf_{t\in[t'_i,t''_i)} f_i(\xi_{0i}(t)), i = 1, \ldots, m) \text{ as } \varepsilon \to 0. \end{aligned} \tag{2.3.38}$$

Since $t'_i < t''_i$ are arbitrary points from S_0, the relations (2.3.37) and (2.3.38) mean that condition $\mathcal{A}_{19}$ holds for the processes $(f_i(\xi_{\varepsilon i}(t)), i = 1, \ldots, m)$, $t \geq 0$, and the random vectors $(\nu_{\varepsilon i}, i = 1, \ldots, m)$ with the sets $S_i = S_0 \setminus \overline{Y}_i$, $i = 1, \ldots, m$. Here $\overline{Y}_i$ are sets of points $t > 0$ for which $\mathsf{P}\{\nu_{0i} = t\} > 0$, $i = 1, \ldots, m$.

It is obvious that the processes $(f_i(\xi_{0i}(t)), i = 1, \ldots, m)$, $t \geq 0$ and the random vectors $(\nu_{0i}, i = 1, \ldots, m)$ satisfy the continuity condition $\mathcal{C}_4$ if this condition holds for the processes $(\xi_{0i}(t), i = 1, \ldots, m)$, $t \geq 0$ and the random vectors $(\nu_{0i}, i = 1, \ldots, m)$.

Thus, Theorem 2.3.1 is applicable to the processes $(f_i(\xi_{\varepsilon i}(t)), i = 1, \ldots, m)$, $t \geq 0$ and the random vectors $(\nu_{\varepsilon i}, i = 1, \ldots, m)$, which gives the relation

$$(f_i(\xi_{\varepsilon i}(\nu_{\varepsilon i})), i = 1, \ldots, m) \Rightarrow (f_i(\xi_{0i}(\nu_{0i})), i = 1, \ldots, m) \text{ as } \varepsilon \to 0. \tag{2.3.39}$$

Since the continuous bounded functions $f_i(x)$, $i = 1, \ldots, m$, are arbitrary, (2.3.39) implies the relation

$$(\xi_{\varepsilon i}(\nu_{\varepsilon i}), i = 1, \ldots, m) \Rightarrow (\xi_{0i}(\nu_{0i}), i = 1, \ldots, m) \text{ as } \varepsilon \to 0. \tag{2.3.40}$$

The proof is completed. □

Remark 2.3.2. It is possible to weaken condition $\mathfrak{J}_4$ and replace it with condition $\mathfrak{J}_8$.

2.3.10. A more general space of trajectories. As follows from the proof of Theorem 2.3.1, the assumption that $\xi_\varepsilon(t), t \geq 0$ is a càdlàg processes is not essential. It would be sufficient to only require that the quantities $\xi_{\varepsilon i}(\nu_{\varepsilon i})$, and $\sup_{t\in[t'_i,t''_i)} \xi_{\varepsilon i}(t)$, $\inf_{t\in[t'_i,t''_i)} \xi_{\varepsilon i}(t)$, $t'_i, t''_i \in S_i$, $i = 1, \dots, m$ be random variables. The formulations of condition $\mathcal{A}_{19}$ as well as Theorem 2.3.1 can be preserved without any changes.

2.4 Weakened continuity conditions

The first-type continuity condition that the limiting stopping moment is a point of continuity of the corresponding limiting external process with probability 1 is essential in the limit theorems given in Sections 2.2 and 2.3. This condition covers a significant part of applications. Nevertheless, it is desirable to weaken this condition in order to include in the consideration the models in which the limiting stopping moment can be a point of continuity of the limiting external process with a probability less than 1. In this section, we show that in such cases one can use weaker continuity type conditions that prevent the positioning of stopping moments at the "wrong" left-hand side of points of large jumps of the external processes. The results in this section, in particular Theorems 2.4.1 and 2.4.2, are new.

2.4.1. A weakened continuity condition. Let, for every $\varepsilon \geq 0$, $\xi_\varepsilon(t)$, $t \geq 0$ be a real-valued càdlàg process, and ν_ε be a non-negative random variable.

Take $\delta, T > 0$ and define, for a real-valued càdlàg function $x(t), t \geq 0$, the functionals $\alpha^{(\delta)}_{0T}(x(\cdot)) = 0$ and then, recursively, $\alpha^{(\delta)}_{kT}(x(\cdot)) = \inf(s > \alpha^{(\alpha)}_{k-1T}(x(\cdot))\colon |\Delta_s(x(\cdot))| \geq \delta) \wedge T$ for $k = 1, 2, \dots$.

Let us also consider the random variables $\alpha^{(\delta)}_{\varepsilon kT} = \alpha^{(\delta)}_{kT}(\xi_\varepsilon(\cdot))$, $k = 1, 2, \dots$. By the definition, $\alpha^{(\delta)}_{\varepsilon kT}$ are successive moments of jumps of the process $\xi_\varepsilon(t)$, $t \geq 0$, at which the absolute values of the jumps are greater than or equal to δ and which are truncated in time by T. Since $\xi_\varepsilon(t)$, $t \geq 0$ is a càdlàg process, $\mathsf{P}\{\alpha^{(\delta)}_{\varepsilon kT} = T\} \to 1$ as $k \to \infty$.

In what follows, it is assumed that conditions $\mathcal{A}_{17}$ and $\mathfrak{J}_7$ hold. These conditions are necessary and sufficient for **J**-convergence of the vector processes $(\nu_\varepsilon, \xi_\varepsilon(t)), t \geq 0$,

$$(\nu_\varepsilon, \xi_\varepsilon(t)), t \geq 0 \xrightarrow{\mathbf{J}} (\nu_0, \xi_0(t)), t \geq 0 \text{ as } \varepsilon \to 0. \tag{2.4.1}$$

Let us denote by Z_0 the set of all $\delta > 0$ such that $\mathsf{P}\{|\Delta_s(\xi_0(\cdot))| \neq \delta, s \geq 0\} = 1$. By the definition, Z_0 is a set of $\delta > 0$ for which the process $\xi_0(t)$, $t \geq 0$ has no jumps with the absolute value equal to δ with probability 1. Since the càdlàg process $\xi_0(t)$ has at most a countable number of jumps, the set Z_0 is $(0, \infty)$ except for at most a countable set.

Let also S_0 be a set of points of stochastic continuity of the process $\xi_0(t)$, $t \geq 0$. This set coincides with $[0, \infty)$ except for at most a countable set, and $0 \in S_0$.

Let $F_{0,T}$ be the measure generated by the process $(\nu_0, \xi_0(t)), t \in [0,T]$ on the Borel σ-algebra $\mathfrak{B}^{(2)}_{[0,T]}$.

Take $\delta \in Z_0$ and $0 < T \in S_0$. As follows from Lemma 1.6.8, the functionals $\alpha^{(\delta)}_{kT}(\cdot)$ belong to the class $\mathfrak{H}_J[F_{0,T}]$ for all $k \geq 1$. The random variable ν_ε can be considered as the value $f_0(\cdot)$ of the first component of the vector process $(\nu_\varepsilon, \xi_\varepsilon(t)), t \geq 0$ at moment 0. This functional and the difference $\alpha^{(\delta)}_{kT}(\cdot) - f_0(\cdot)$ also belong to the class $\mathfrak{H}_J[F_{0,T}]$.

So, for every $\delta \in Z_0$, $0 < T \in S_0$, and $k \geq 1$ and every u, which is a continuity point of the corresponding limiting distribution function,

$$\mathsf{P}\{\alpha^{(\delta)}_{\varepsilon kT} - \nu_\varepsilon \leq u\} \to \mathsf{P}\{\alpha^{(\delta)}_{0kT} - \nu_0 \leq u\} \text{ as } \varepsilon \to 0. \tag{2.4.2}$$

However, it is not certain that $u = 0$ is a continuity point of the limiting distribution function in (2.4.2) and, therefore, there is no guarantee that (2.4.2) holds for $u = 0$.

Let Y_0 denote the set of all continuity points of the distribution function of the random variable ν_0. This set coincides with $[0, \infty)$ except for at most a countable set. Thus, the set $S_0 \cap Y_0$ is also $[0, \infty)$ except for at most a countable set.

Let us now assume that the following condition holds:

$\mathcal{D}_2$: There exist a sequence $\delta_l \in Z_0, \delta_l \to 0$ as $l \to \infty$ and a sequence $0 < T_r \in S_0 \cap Y_0, T_r \to \infty$ as $r \to \infty$ such that, for every $l, k, r \geq 1$, $\underline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\alpha^{(\delta_l)}_{\varepsilon kT_r} - \nu_\varepsilon \leq 0\} \geq \mathsf{P}\{\alpha^{(\delta_l)}_{0kT_r} - \nu_0 \leq 0\}$.

Take some $0 < c_n \to 0$ which are points of continuity of the distribution function of the random variable $\alpha^{(\delta_l)}_{0k} - \nu_0$. Then, for every $l, k, r \geq 1$,

$$\begin{aligned}
0 &\leq \lim_{n\to\infty} \underline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\alpha^{(\delta_l)}_{\varepsilon kT_r} - \nu_\varepsilon \in (0, c_n]\} \\
&= \lim_{n\to\infty} \underline{\lim}_{\varepsilon\to 0} (\mathsf{P}\{\alpha^{(\delta_l)}_{\varepsilon kT_r} - \nu_\varepsilon \leq c_n\} - \mathsf{P}\{\alpha^{(\delta_l)}_{\varepsilon kT_r} - \nu_\varepsilon \leq 0\}) \\
&= \lim_{n\to\infty} \mathsf{P}\{\alpha^{(\delta_l)}_{0kT_r} - \nu_0 \leq c_n\} - \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\alpha^{(\delta_l)}_{\varepsilon kT_r} - \nu_\varepsilon \leq 0\} \\
&= \mathsf{P}\{\alpha^{(\delta_l)}_{0kT_r} - \nu_0 \leq 0\} - \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\alpha^{(\delta_l)}_{\varepsilon kT_r} - \nu_\varepsilon \leq 0\}.
\end{aligned} \tag{2.4.3}$$

It follows from relation (2.4.3) that the sign of inequality in $\mathcal{D}_2$ can be replaced with the sign of equality. So, under conditions $\mathcal{A}_{17}$ and $\mathcal{J}_7$, one can use the following equivalent form of condition $\mathcal{D}_2$:

$\mathcal{D}_2'$: There exist a sequence of $\delta_l \in Z_0, \delta_l \to 0$ as $l \to \infty$ and a sequence of $0 < T_r \in S_0 \cap Y_0, T_r \to \infty$ as $r \to \infty$ such that, for every $l, k, r \geq 1$, $\lim_{\varepsilon\to 0} \mathsf{P}\{\alpha^{(\delta_l)}_{\varepsilon kT_r} - \nu_\varepsilon \leq 0\} = \mathsf{P}\{\alpha^{(\delta_l)}_{0kT_r} - \nu_0 \leq 0\}$.

Theorem 2.4.1. *Let conditions $\mathcal{A}_{17}$, $\mathfrak{J}_7$, and $\mathfrak{D}_2$ hold. Then*

$$\xi_\varepsilon(\nu_\varepsilon) \Rightarrow \xi_0(\nu_0) \text{ as } \varepsilon \to 0. \tag{2.4.4}$$

Proof of Theorem 2.4.1. Let us take some $\delta \in Z_0$ and decompose the process $\xi_\varepsilon(t), t \geq 0$ into a sum of two components,

$$\xi^{(\delta)}_{\varepsilon,+}(t) = \sum_{s \leq t} \Delta_s(\xi_\varepsilon(\cdot))\chi(|\Delta_s(\xi_\varepsilon(\cdot))| \geq \delta), \ \xi^{(\delta)}_{\varepsilon,-}(t) = \xi_\varepsilon(t) - \xi^{(\delta)}_{\varepsilon,+}(t), \ t \geq 0.$$

By the definition, $\xi^{(\delta)}_{\varepsilon,+}(t)$ is the sum of all jumps of the process $\xi_\varepsilon(t)$ in the interval $[0, t]$ such that their absolute values are greater than or equal to δ. The random variable $\xi^{(\delta)}_{\varepsilon,-}(t)$ is obtained by excluding all such jumps from $\xi_\varepsilon(t)$.

A càdlàg process has, with probability 1, at most a finite number of jumps in any finite interval. All these jumps have absolute values greater than or equal to δ. The process $\xi^{(\delta)}_{\varepsilon,+}(t)$ is a càdlàg process with step trajectories. The process $\xi^{(\delta)}_{\varepsilon,-}(t)$ is a càdlàg process that has no jumps with absolute values greater than or equal to δ with probability 1.

Conditions $\mathcal{A}_{17}$ and $\mathfrak{J}_7$ imply that, for every $\delta \in Z_0$, the vector processes

$$\begin{aligned} \boldsymbol{\xi}^{(\delta)}_\varepsilon(t) &= (\nu_\varepsilon, \xi_\varepsilon(t), \xi^{(\delta)}_{\varepsilon,+}(t), \xi^{(\delta)}_{\varepsilon,-}(t)), t \geq 0 \\ &\xrightarrow{\mathbf{J}} \boldsymbol{\xi}^{(\delta)}_0(t) = (\nu_0, \xi_0(t), \xi^{(\delta)}_{0,+}(t), \xi^{(\delta)}_{0,-}(t)), t \geq 0 \text{ as } \varepsilon \to 0. \end{aligned} \tag{2.4.5}$$

A simple way to see this is to apply Lemma 1.6.13 to the processes $(\nu_\varepsilon, \xi_\varepsilon(t), t \geq 0$.

The idea of the proof of Theorem 2.4.1 is to construct and use appropriate upper and lower approximations for the random variables $\xi^{(\delta)}_{0,+}(\nu_\varepsilon)$ and $\xi^{(\delta)}_{0,-}(\nu_\varepsilon)$, and then for their sum $\xi_\varepsilon(\nu_\varepsilon) = \xi^{(\delta)}_{0,+}(\nu_\varepsilon) + \xi^{(\delta)}_{0,-}(\nu_\varepsilon)$.

Denote by $\tilde{F}_{0,T}$ the measure generated by the process $\boldsymbol{\xi}^{(\delta)}_0(t)$, $t \in [0, T]$, on the Borel σ-algebra $\mathfrak{B}^{(4)}_{[0,T]}$. Note that the process $\boldsymbol{\xi}^{(\delta)}_0(t), t \geq 0$ has the same set of points of stochastic continuity, S_0, as the process $\xi_0(t), t \geq 0$. We will be interested in certain a.s. **J**-continuous functionals from the space $\mathfrak{H}_J[\tilde{F}_{0,T}]$. Below, it is assumed that **(a)** $\delta \in Z_0$ and $0 < T \in S_0$.

Let $\mathbf{x}(t) = (x_1(t), x_2(t), x_3(t), x_4(t))$ be a function from the space $\mathbf{D}^{(4)}_{[0,T]}$.

The first class includes the functional $f(\mathbf{x}(\cdot)) = x_1(0)$. Obviously, this functional belongs to the class $\mathfrak{H}_J[\tilde{F}_{0,T}]$. The corresponding random variable, which is the value of this functional on the process $\boldsymbol{\xi}^{(\delta)}_0(t), t \in [0, T]$, is ν_ε.

We will also be interested in the functional $\chi(t_1 \leq f(\mathbf{x}(\cdot)) < t_2) = \chi(t_1 \leq x_1(0) < t_2)$, $0 \leq t_1 < t_2 \leq T$. This functional belongs to the class $\mathfrak{H}_J[\tilde{F}_{0,T}]$ for any t_1, t_2 that are points of continuity of the distribution function of the random variable ν_0. The corresponding random variables that we are interested in are $\chi(t_1 \leq \nu_\varepsilon < t_2)$.

The second class includes the functionals $\zeta^+_{t_1,t_2}(\mathbf{x}(\cdot)) = \sup_{t_1 \leq t < t_2} x_4(t)$ and $\zeta^-_{t_1,t_2}(\mathbf{x}(\cdot)) = \inf_{t_1 \leq t < t_2} x_4(t)$, $0 \leq t_1 < t_2 \leq T$. These functionals belong to the class $\mathfrak{H}_J[\tilde{F}_{0,T}]$ if $t_1, t_2 \in$

S_0. This follows from Lemmas 1.5.9 and 1.6.6. The corresponding random variables that are of interest are $\zeta^{(\delta)}_{\varepsilon,\pm}[t_1, t_2] = \zeta^{\pm}_{t_1,t_2}(\xi^{(\delta)}_{\varepsilon,-}(\cdot))$.

The third class includes the moments of large jumps, $\alpha^{(\delta)}_{kT}(\mathbf{x}(\cdot)) = \alpha^{(\delta)}_{kT}(x_2(\cdot))$, $k = 0, 1, \ldots$. Recall that $\alpha^{(\delta)}_{0T}(x_2(\cdot)) = 0$. These functionals were already discussed above. They belong to the class $\mathfrak{H}_J[\tilde{F}_{0,T}]$ for all $k = 0, 1, \ldots$. This follows from Lemmas 1.5.6 and 1.6.8. The corresponding random variables that are of interest are $\alpha^{(\delta)}_{\varepsilon kT}$.

The fourth class includes the functionals $\kappa^{(\delta)}_{kT}(\mathbf{x}(\cdot)) = \alpha^{(\delta)}_{kT}(x_2(\cdot)) - x_1(0)$, $k = 0, 1, \ldots$. These functionals were also discussed above. They belong to the class $\mathfrak{H}_J[\tilde{F}_{0,T}]$ for all $k = 0, 1, \ldots$. This follows from Lemmas 1.5.1, 1.5.6, and 1.6.8. The corresponding random variables that we are interested in are $\alpha^{(\delta)}_{\varepsilon kT} - \nu_\varepsilon$.

We will also consider the functionals $\chi(\kappa^{(\delta)}_{kT}(\mathbf{x}(\cdot)) \leq c) = \chi(\alpha^{(\delta)}_{kT}(x_2(\cdot)) - x_1(0) \leq c)$, $c \in \mathbb{R}_1$, $k = 0, 1, \ldots$. These functionals belong to the class $\mathfrak{H}_J[\tilde{F}_{0,T}]$ for all $k = 0, 1, \ldots$ if c is a continuity point of the distribution functions of the random variables $\alpha^{(\delta)}_{0kT} - \nu_0$, $k = 0, 1, \ldots$. The corresponding random variables are $\chi(\alpha^{(\delta)}_{\varepsilon kT} - \nu_\varepsilon \leq c)$.

The fifth class includes the functionals $\Sigma^{(\delta)}_t(\mathbf{x}(\cdot)) = x_3(t)$, $t \in [0, T]$. These functionals belong to the class $\mathfrak{H}_J[\tilde{F}_{0,T}]$ if $t \in S_0$, as it follows from Lemmas 1.5.3 and 1.6.9. The random variables that we are interested in now are $\xi^{(\delta)}_{\varepsilon,+}(t)$.

Finally, the last sixth class includes the functionals $\rho^{(\delta)}_{kT}(\mathbf{x}(\cdot)) = x_3(\alpha^{(\delta)}_{kT}(x_2(\cdot)))$, $k = 0, 1, \ldots$. They belong to the class $\mathfrak{H}_J[\tilde{F}_{0,T}]$ for all $k = 0, 1, \ldots$, which follows from Lemmas 1.5.6 and 1.5.8. The random variables that we are interested in are $\rho^{(\delta)}_{\varepsilon kT} = \xi^{(\delta)}_{\varepsilon,+}(\alpha^{(\delta)}_{\varepsilon kT})$.

Now we are in a position to use condition $\mathfrak{D}'_2$. Assume, therefore, that $\delta = \delta_l \in Z_0$ and $T = T_r \in S_0$ are taken from the sequences that enter this condition.

By applying Theorem 1.6.7, we can write, for every $k = 0, 1, \ldots$, the following relation that holds for any $t_1 < t_2, t_1, t_2 \in S_0$, $u_\pm \in \mathbb{R}_1$, $c \geq 0$, $w \in \mathbb{R}_1$, such that the points $t_1, t_2, u_\pm, c, w$ are continuity points of the distribution functions of the corresponding limiting random variables,

$$\begin{aligned} &\mathsf{P}\{t_1 \leq \nu_\varepsilon < t_2, \zeta^{(\delta)}_{\varepsilon,\pm}[t_1, t_2] \leq u_\pm, \\ &\quad \alpha^{(\delta_l)}_{\varepsilon kT_r} - \nu_\varepsilon \leq c, \alpha^{(\delta_l)}_{\varepsilon k+1T_r} - \nu_\varepsilon > c, \rho^{(\delta_l)}_{\varepsilon kT_r} \leq w\} \\ &\to \mathsf{P}\{t_1 \leq \nu_0 < t_2, \zeta^{(\delta_l)}_{0,\pm}[t_1, t_2] \leq u_\pm, \\ &\quad \alpha^{(\delta_l)}_{0kT_r} - \nu_0 \leq c, \alpha^{(\delta_l)}_{0k+1T_r} - \nu_0 > c, \rho^{(\delta_l)}_{0kT_r} \leq w\} \text{ as } \varepsilon \to 0. \end{aligned} \tag{2.4.6}$$

Take now a sequence of points $0 < c_n \to 0$ as $n \to \infty$ that are points of continuity of the distribution functions of the random variables $\alpha^{(\delta_l)}_{0kT_r} - \nu_0$ for all $k = 0, 1, \ldots$.

Obviously,

$$\begin{aligned}
&\mathsf{P}\{t_1 \le \nu_0 < t_2, \zeta^{(\delta_l)}_{0,\pm}[t_1,t_2] \le u_\pm,\\
&\qquad \alpha^{(\delta_l)}_{0kT_r} - \nu_0 \le c_n, \alpha^{(\delta_l)}_{0k+1T_r} - \nu_0 > c_n, \rho^{(\delta_l)}_{0kT} \le w\}\\
&\quad \to \mathsf{P}\{t_1 \le \nu_0 < t_2, \zeta^{(\delta_l)}_{0,\pm}[t_1,t_2] \le u_\pm,\\
&\qquad \alpha^{(\delta_l)}_{0kT_r} - \nu_0 \le 0, \alpha^{(\delta_l)}_{0k+1T_r} - \nu_0 > 0, \rho^{(\delta_l)}_{0kT_r} \le w\} \text{ as } n \to \infty.
\end{aligned} \tag{2.4.7}$$

At the same time,

$$\begin{aligned}
&|\mathsf{P}\{t_1 \le \nu_\varepsilon < t_2, \zeta^{(\delta_l)}_{\varepsilon,\pm}[t_1,t_2] \le u_\pm,\\
&\qquad \alpha^{(\delta_l)}_{\varepsilon kT_r} - \nu_\varepsilon \le c_n, \alpha^{(\delta_l)}_{\varepsilon k+1T_r} - \nu_\varepsilon > c_n, \rho^{(\delta_l)}_{\varepsilon kT} \le w\}\\
&\quad - \mathsf{P}\{t_1 \le \nu_\varepsilon < t_2, \zeta^{(\delta_l)}_{\varepsilon,\pm}[t_1,t_2] \le u_\pm,\\
&\qquad \alpha^{(\delta_l)}_{\varepsilon kT_r} - \nu_\varepsilon \le 0, \alpha^{(\delta_l)}_{\varepsilon k+1T_r} - \nu_\varepsilon > 0, \rho^{(\delta_l)}_{\varepsilon kT_r} \le w\}|\\
&\quad \le \mathsf{P}\{\alpha^{(\delta_l)}_{\varepsilon kT_r} - c_n \le \nu_\varepsilon < \alpha^{(\delta_l)}_{\varepsilon kT_r}\} + \mathsf{P}\{\alpha^{(\delta_l)}_{\varepsilon k+1T_r} - c_n \le \nu_\varepsilon < \alpha^{(\delta_l)}_{\varepsilon k+1T_r}\}.
\end{aligned} \tag{2.4.8}$$

Using condition $\mathfrak{D}_2'$ and then the identity $\cap_{n\ge 1}\{\alpha^{(\delta_l)}_{0kT_r} - \nu_0 \in (0, c_n]\} = \varnothing$, one gets for all $k = 0, 1, \dots$ that

$$\begin{aligned}
&\lim_{n\to\infty} \overline{\lim_{\varepsilon\to 0}} (\mathsf{P}\{\alpha^{(\delta_l)}_{\varepsilon kT_r} - c_n \le \nu_\varepsilon < \alpha^{(\delta_l)}_{\varepsilon kT_r}\} + \mathsf{P}\{\alpha^{(\delta_l)}_{\varepsilon k+1T_r} - c_n \le \nu_\varepsilon < \alpha^{(\delta_l)}_{\varepsilon k+1T_r}\})\\
&\quad = \lim_{n\to\infty} (\mathsf{P}\{\alpha^{(\delta_l)}_{0kT_r} - c_n \le \nu_0 < \alpha^{(\delta_l)}_{0kT_r}\} + \mathsf{P}\{\alpha^{(\delta_l)}_{0k+1T_r} - c_n \le \nu_0 < \alpha^{(\delta_l)}_{0k+1T_r}\}) = 0.
\end{aligned} \tag{2.4.9}$$

Relations (2.4.6), (2.4.7), (2.4.8), and (2.4.9) imply, in an obvious way, the following relation that holds for every $k = 0, 1, \dots$ and all $t_1 < t_2, t_1, t_2 \in S_0, u_\pm \in \mathbb{R}_1, c \ge 0, w \in \mathbb{R}_1$ such that the points $t_1, t_2, u_\pm, c, w$ are continuity points of the distribution functions of the corresponding limiting random variables,

$$\begin{aligned}
&\mathsf{P}\{t_1 \le \nu_\varepsilon < t_2, \zeta^{(\delta_l)}_{\varepsilon,\pm}[t_1,t_2] \le u_\pm,\\
&\qquad \alpha^{(\delta_l)}_{\varepsilon kT_r} - \nu_\varepsilon \le 0, \alpha^{(\delta_l)}_{\varepsilon k+1T_r} - \nu_\varepsilon > 0, \rho^{(\delta_l)}_{\varepsilon kT_r} \le w\}\\
&\quad = \mathsf{P}\{t_1 \le \nu_\varepsilon < t_2, \zeta^{(\delta_l)}_{\varepsilon,\pm}[t_1,t_2] \le u_\pm,\\
&\qquad \alpha^{(\delta_l)}_{\varepsilon kT_r} \le \nu_\varepsilon < \alpha^{(\delta_l)}_{\varepsilon k+1T_r}, \rho^{(\delta)}_{\varepsilon kT} \le w\}\\
&\quad \to \mathsf{P}\{t_1 \le \nu_0 < t_2, \zeta^{(\delta_l)}_{0,\pm}[t_1,t_2] \le u_\pm,\\
&\qquad \alpha^{(\delta_l)}_{0kT_r} - \nu_0 \le 0, \alpha^{(\delta_l)}_{0k+1T_r} - \nu_0 > 0, \rho^{(\delta_l)}_{0kT_r} \le w\}\\
&\quad = \mathsf{P}\{t_1 \le \nu_0 < t_2, \zeta^{(\delta_l)}_{0,\pm}[t_1,t_2] \le u_\pm,\\
&\qquad \alpha^{(\delta_l)}_{0kT_r} \le \nu_0 < \alpha^{(\delta_l)}_{0k+1T_r}, \rho^{(\delta_l)}_{0kT_r} \le w\} \text{ as } \varepsilon \to 0.
\end{aligned} \tag{2.4.10}$$

For each $n \ge 1$, choose partitions $0 = z_{0,n} < z_{1,n} < \cdots < z_{n,n} = T_r$ of the interval $[0, T_r]$ satisfying the following assumptions: **(b)** $z_{i,n}$ is a point of continuity of the distribution function of the random variable ν_0 for all $i = 1, \ldots, n, n \geq 1$; **(c)** $h(n) = \max_{0 \leq i \leq n-1} |z_{i+1,n} - z_{i,n}| \to 0$ as $n \to \infty$.

Let us define, for $n \geq 1$, the random variables

$$\zeta^{(\delta_l)}_{\varepsilon,n,\pm} = \sum_{i=0}^{n-1} \zeta^{(\delta_l)}_{\varepsilon,\pm}[z_{i,n}, z_{i+1,n}]\chi(z_{i,n} \leq \nu_\varepsilon < z_{i+1,n}) \tag{2.4.11}$$

By the definition of these random variables, for every $\varepsilon \geq 0$ and $n = 1, 2, \ldots$,

$$\zeta^{(\delta_l)}_{\varepsilon,n,-} \leq \xi^{(\delta_l)}_{\varepsilon,-}(\nu_\varepsilon)\chi(\nu_\varepsilon < T_r) \leq \zeta^{(\delta_l)}_{\varepsilon,n,+}. \tag{2.4.12}$$

Let us also consider the random variable $\xi^{(\delta_l)}_{\varepsilon,+}(\nu_\varepsilon)$. The process $\xi^{(\delta_l)}_{\varepsilon,+}(t)$ is a step process that takes the value $\rho^{(\delta_l)}_{\varepsilon kT_r} = \xi^{(\delta_l)}_{\varepsilon,+}(\alpha^{(\delta_l)}_{\varepsilon kT_r})$ in the interval $[\alpha^{(\delta_l)}_{\varepsilon kT_r}, \alpha^{(\delta_l)}_{\varepsilon k+1T_r})$ for every $k = 0, 1, \ldots$. Since the random variables $\alpha^{(\delta_l)}_{\varepsilon kT_r} \xrightarrow{\mathsf{P1}} T_r$ as $k \to \infty$ for every $\varepsilon \geq 0$, we can write the following representation:

$$\xi^{(\delta_l)}_{\varepsilon,+}(\nu_\varepsilon)\chi(\nu_\varepsilon < T_r) \overset{\mathsf{P1}}{=} \sum_{k=0}^{\infty} \rho^{(\delta_l)}_{\varepsilon kT_r}\chi(\alpha^{(\delta_l)}_{\varepsilon kT_r} \leq \nu_\varepsilon < \alpha^{(\delta_l)}_{\varepsilon k+1T_r}). \tag{2.4.13}$$

Let us also introduce the random variables

$$\xi^{(\delta_l)}_{\varepsilon,N,+} = \sum_{k=0}^{N} \rho^{(\delta_l)}_{\varepsilon kT_r}\chi(\alpha^{(\delta_l)}_{\varepsilon kT_r} \leq \nu_\varepsilon < \alpha^{(\delta_l)}_{\varepsilon k+1T_r}),\ N \geq 1.$$

The joint distribution of the random variables $\zeta^{(\delta_l)}_{\varepsilon,n,\pm}$ and $\xi^{(\delta_l)}_{\varepsilon,N,+}(\nu_\varepsilon)$ has the following form:

$$\begin{aligned}
&\mathsf{P}\{\zeta^{(\delta_l)}_{\varepsilon,n,\pm} \leq u_\pm, \xi^{(\delta_l)}_{\varepsilon,N,+} \leq w\} \\
&\quad = \sum_{i=0}^{n-1}\sum_{k=0}^{N} \mathsf{P}\{z_{i,n} \leq \nu_\varepsilon < z_{i+1,n}, \zeta^{(\delta_l)}_{\varepsilon,\pm}[z_{i,n}, z_{i+1,n}] \leq u_\pm, \\
&\quad\quad \alpha^{(\delta_l)}_{\varepsilon kT} \leq \nu_\varepsilon < \alpha^{(\delta_l)}_{\varepsilon k+1T_r}, \rho^{(\delta_l)}_{\varepsilon kT_r} \leq w\} \\
&\quad + \sum_{i=0}^{n-1} \mathsf{P}\{z_{i,n} \leq \nu_\varepsilon < z_{i+1,n}, \zeta^{(\delta_l)}_{\varepsilon,\pm}[z_{i,n}, z_{i+1,n}] \leq u_\pm, \alpha^{(\delta_l)}_{\varepsilon N+1T_r} \leq \nu_\varepsilon\}\chi(0 \leq w) \\
&\quad + \mathsf{P}\{\nu_\varepsilon \geq T_r\}\chi(0 \leq u_\pm \wedge w).
\end{aligned} \tag{2.4.14}$$

In the way absolutely similar to those used to prove relation (2.4.10) it can be proved that for all $t_1 < t_2, t_1, t_2 \in S_0$, $u_\pm \in \mathbb{R}_1$ such that points $t_1, t_2, u_\pm$ are continuity points for the distribution functions of the corresponding limiting random variables,

$$
\begin{aligned}
&\mathsf{P}\{t_1 \le \nu_\varepsilon < t_2, \zeta^{(\delta_l)}_{\varepsilon,\pm}[t_1,t_2] \le u_\pm, \alpha^{(\delta_l)}_{\varepsilon N+1 T_r} \le \nu_\varepsilon\} \\
&\quad = \mathsf{P}\{t_1 \le \nu_\varepsilon < t_2, \zeta^{(\delta_l)}_{\varepsilon,\pm}[t_1,t_2] \le u_\pm\} \\
&\quad - \sum_{k=0}^{N} \mathsf{P}\{t_1 \le \nu_\varepsilon < t_2, \zeta^{(\delta_l)}_{\varepsilon,\pm}[t_1,t_2] \le u_\pm, \alpha^{(\delta_l)}_{\varepsilon k T} \le \nu_\varepsilon < \alpha^{(\delta_l)}_{\varepsilon k+1 T_r}\} \\
&\quad \to \mathsf{P}\{t_1 \le \nu_0 < t_2, \zeta^{(\delta_l)}_{0,\pm}[t_1,t_2] \le u_\pm\} \\
&\quad - \sum_{k=0}^{N} \mathsf{P}\{t_1 \le \nu_0 < t_2, \zeta^{(\delta_l)}_{0,\pm}[t_1,t_2] \le u_\pm, \alpha^{(\delta_l)}_{0kT} \le \nu_0 < \alpha^{(\delta_l)}_{0k+1 T_r}\} \\
&\quad = \mathsf{P}\{t_1 \le \nu_0 < t_2, \zeta^{(\delta_l)}_{0,\pm}[t_1,t_2] \le u_\pm, \alpha^{(\delta_l)}_{0N+1 T_r} \le \nu_0\} \text{ as } \varepsilon \to 0.
\end{aligned}
\tag{2.4.15}
$$

It follows from relations (2.4.10), (2.4.14), and (2.4.15) that for all $u_\pm$ and w, which are continuity points of the limiting distribution function

$$
\mathsf{P}\{\zeta^{(\delta_l)}_{\varepsilon,n,\pm} \le u_\pm, \xi^{(\delta_l)}_{\varepsilon,N,+} \le w\} \to \mathsf{P}\{\zeta^{(\delta_l)}_{0,n,\pm} \le u_\pm, \xi^{(\delta_l)}_{0,N,+} \le w\} \text{ as } \varepsilon \to 0. \tag{2.4.16}
$$

It is obvious that for any $\sigma > 0$,

$$
\mathsf{P}\{|\xi^{(\delta_l)}_{\varepsilon,+}(\nu_\varepsilon)\chi(\nu_\varepsilon < T_r) - \xi^{(\delta_l)}_{\varepsilon,N,+}| > \sigma\} \le \mathsf{P}\{\alpha^{(\delta_l)}_{\varepsilon N+1 Tr} \le \nu_\varepsilon < T_r\}. \tag{2.4.17}
$$

Since $\alpha^{(\delta_l)}_{0kT_r} \xrightarrow{\mathsf{P}1} T_r$ as $k \to \infty$, relation (2.4.17) yields

$$
(\zeta^{(\delta_l)}_{0,n,\pm}, \xi^{(\delta_l)}_{0,N,+}) \Rightarrow (\zeta^{(\delta_l)}_{0,n,\pm}, \xi^{(\delta_l)}_{0,+}(\nu_0)\chi(\nu_0 < T_r)) \text{ as } N \to \infty. \tag{2.4.18}
$$

Also, taking condition $\mathbf{D}_2'$ into consideration we get

$$
\begin{aligned}
&\lim_{N\to\infty} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{|\xi^{(\delta_l)}_{\varepsilon,+}(\nu_\varepsilon)\chi(\nu_\varepsilon < T_r) - \xi^{(\delta_l)}_{\varepsilon,N,+}| > \sigma\} \\
&\quad \le \lim_{N\to\infty} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\alpha^{(\delta_l)}_{\varepsilon N+1 T_r} \le \nu_\varepsilon < T_r\} = \lim_{N\to\infty} \mathsf{P}\{\alpha^{(\delta_l)}_{0N+1 T_r} \le \nu_0 < T_r\} = 0.
\end{aligned}
\tag{2.4.19}
$$

It follows from relations (2.4.16), (2.4.18), and (2.4.19) that for any continuity points of the limiting distribution function, $u_\pm$ and w,

$$
\begin{aligned}
&\mathsf{P}\{\zeta^{(\delta_l)}_{\varepsilon,n,\pm} \le u_\pm, \xi^{(\delta_l)}_{\varepsilon,+}(\nu_\varepsilon)\chi(\nu_\varepsilon < T_r) \le w\} \\
&\qquad \to \mathsf{P}\{\zeta^{(\delta_l)}_{0,n,\pm} \le u_\pm, \xi^{(\delta_l)}_{0,+}(\nu_0)\chi(\nu_0 < T_r) \le w\} \text{ as } \varepsilon \to 0.
\end{aligned}
\tag{2.4.20}
$$

Relation (2.4.20) implies that for any points of continuity of the limiting distribution function, $u_\pm$,

$$
\begin{aligned}
&\mathsf{P}\{\zeta^{(\delta_l)}_{\varepsilon,n,\pm} + \xi^{(\delta_l)}_{\varepsilon,+}(\nu_\varepsilon)\chi(\nu_\varepsilon < T_r) \le u_\pm\} \\
&\qquad \to \mathsf{P}\{\zeta^{(\delta_l)}_{0,n,\pm} + \xi^{(\delta_l)}_{0,+}(\nu_0)\chi(\nu_0 < T_r) \le u_\pm\} \text{ as } \varepsilon \to 0.
\end{aligned}
\tag{2.4.21}
$$

Obviously,

$$\xi^{(\delta_l)}_{\varepsilon,-}(\nu_\varepsilon)\chi(\nu_\varepsilon < T_r) + \xi^{(\delta_l)}_{\varepsilon,+}(\nu_\varepsilon)\chi(\nu_\varepsilon < T_r) = \xi_\varepsilon(\nu_\varepsilon)\chi(\nu_\varepsilon < T_r). \tag{2.4.22}$$

Now, using an inequality that follows from relations (2.4.12), (2.4.22) and holds for every $\varepsilon \geq 0$ and $n = 1, 2, \ldots$, we get

$$\zeta^{(\delta_l)}_{\varepsilon,n,-} + \xi^{(\delta_l)}_{\varepsilon,+}(\nu_\varepsilon)\chi(\nu_\varepsilon < T_r) \leq \xi_\varepsilon(\nu_\varepsilon)\chi(\nu_\varepsilon < T_r) \leq \zeta^{(\delta_l)}_{\varepsilon,n,+} + \xi^{(\delta_l)}_{\varepsilon,+}(\nu_\varepsilon)\chi(\nu_\varepsilon < T_r). \tag{2.4.23}$$

Denote by U_0 the set of continuity points of the distribution functions of the random variables $\xi_0(\nu_0)\chi(\nu_0 < T_r)$, $\max(\xi_0(\nu_0), \xi_0(\nu_0))\chi(\nu_0 < T_r)$, $\min(\xi_0(\nu_0), \xi_0(\nu_0))\chi(\nu_0 < T_r)$, and $\zeta^{(\delta_l)}_{0,n,\pm} + \xi^{(\delta_l)}_{0,+}(\nu_0)\chi(\nu_0 < T_r), n \geq 1$. The set U_0 differs from $\mathbb{R}_1$ by at most a countable set. Take an arbitrary point $u \in U_0$.

Using relation (2.4.21) and inequality (2.4.23) we get

$$\begin{aligned}
&\varliminf_{\varepsilon\to 0} \mathsf{P}\{\xi_\varepsilon(\nu_\varepsilon)\chi(\nu_\varepsilon < T_r) \leq u\} \\
&\quad \geq \varliminf_{\varepsilon\to 0} \mathsf{P}\{\zeta^{(\delta_l)}_{\varepsilon,n,+} + \xi^{(\delta_l)}_{\varepsilon,+}(\nu_\varepsilon)\chi(\nu_\varepsilon < T_r) \leq u\} \\
&\quad = \mathsf{P}\{\zeta^{(\delta_l)}_{0,n,+} + \xi^{(\delta_l)}_{0,+}(\nu_0)\chi(\nu_0 < T_r) \leq u\}.
\end{aligned} \tag{2.4.24}$$

Let $x(t), t \geq 0$, be a real-valued càdlàg function. Let us consider the functionals $m^+_{t-h,t+h}(x(\cdot)) = \sup_{t-h\leq s\leq t+h} x(s)$ and $m^-_{t-h,t+h}(x(\cdot)) = \inf_{t-h\leq s\leq t+h} x(s)$ for $t, h \geq 0$. Here, we take $x(s) = x(0)$ for $s < 0$. Obviously, $m^+_{t-h,t+h}(x(\cdot)) \to \max(x(t), x(t-0))$ and $m^-_{t-h,t+h}(x(\cdot)) \to \min(x(t), x(t-0))$ as $0 < h \to 0$ for any $t \geq 0$.

Taking in consideration this remark we get

$$\begin{aligned}
&m^+_{\nu_0-h,\nu_0+h}(\xi^{(\delta_l)}_{0,-}(\cdot))\chi(\nu_0 < T_r) \\
&\quad \xrightarrow{\mathsf{P1}} \max(\xi^{(\delta_l)}_{0,-}(\nu_0), \xi^{(\delta_l)}_{0,-}(\nu_0 - 0))\chi(\nu_0 < T_r) \text{ as } 0 < h \to 0.
\end{aligned} \tag{2.4.25}$$

Note that, by the definition of random variables $\zeta^{(\delta_l)}_{0,n,+}$,

$$\zeta^{(\delta_l)}_{0,n,+} \leq m^+_{\nu_0-h(n),\nu_0+h(n)}(\xi^{(\delta_l)}_{0,-}(\cdot))\chi(\nu_0 < T_r). \tag{2.4.26}$$

Since the process $\xi^{(\delta_l)}_{0,-}(t), t \geq 0$ has no jumps with absolute values greater than or equal to δ_l,

$$\max(\xi^{(\delta_l)}_{0,-}(\nu_0), \xi^{(\delta_l)}_{0,-}(\nu_0 - 0))\chi(\nu_0 < T_r) \leq \delta_l + \xi^{(\delta_l)}_{0,-}(\nu_0)\chi(\nu_0 < T_r). \tag{2.4.27}$$

Taking in consideration relation (2.4.25) and inequalities (2.4.26) and (2.4.27) we can continue relation (2.4.24) and get

$$\begin{aligned}
&\varliminf_{n\to\infty} \mathsf{P}\{\zeta^{(\delta_l)}_{0,n,+} + \xi^{(\delta_l)}_{0,+}(\nu_0)\chi(\nu_0 < T_r) \le u\} \\
&\qquad \ge \varliminf_{n\to\infty} \mathsf{P}\{(m^+_{\nu_0-h(n),\nu_0+h(n)}(\xi^{(\delta_l)}_{0,-}(\cdot)) + \xi^{(\delta_l)}_{0,+}(\nu_0))\chi(\nu_0 < T_r) \le u\} \\
&\qquad = \mathsf{P}\{(\max(\xi^{(\delta_l)}_{0,-}(\nu_0), \xi^{(\delta_l)}_{0,-}(\nu_0 - 0)) + \xi^{(\delta_l)}_{0,+}(\nu_0))\chi(\nu_0 < T_r) \le u\} \\
&\qquad \ge \mathsf{P}\{\delta_l + (\xi^{(\delta_l)}_{0,-}(\nu_0) + \xi^{(\delta_l)}_{0,+}(\nu_0))\chi(\nu_0 < T_r) \le u\} \\
&\qquad = \mathsf{P}\{\delta_l + \xi_0(\nu_0)\chi(\nu_0 < T_r) \le u\}.
\end{aligned} \tag{2.4.28}$$

Finally, for every $u \in U_0$, using (2.4.24) and (2.4.28) we get that

$$\begin{aligned}
&\varliminf_{\varepsilon\to 0} \mathsf{P}\{\xi_\varepsilon(\nu_\varepsilon)\chi(\nu_\varepsilon < T_r) \le u\} \\
&\qquad \ge \lim_{l\to\infty} \mathsf{P}\{\delta_l + \xi_0(\nu_0)\chi(\nu_0 < T_r) \le u\} = \mathsf{P}\{\xi_0(\nu_0)\chi(\nu_0 < T_r) \le u\}.
\end{aligned} \tag{2.4.29}$$

In a similar way, it can be proved that for every $u \in U_0$,

$$\begin{aligned}
&\varlimsup_{\varepsilon\to 0} \mathsf{P}\{\xi_\varepsilon(\nu_\varepsilon)\chi(\nu_\varepsilon < T_r) \le u\} \\
&\qquad \le \lim_{l\to\infty} \mathsf{P}\{-\delta_l + \xi_0(\nu_0)\chi(\nu_0 < T_r) \le u\} = \mathsf{P}\{\xi_0(\nu_0)\chi(\nu_0 < T_r) \le u\}.
\end{aligned} \tag{2.4.30}$$

Relations (2.4.29) and (2.4.30) imply that for every $u \in U_0$,

$$\lim_{\varepsilon\to 0} \mathsf{P}\{\xi_\varepsilon(\nu_\varepsilon)\chi(\nu_\varepsilon < T_r) \le u\} = \mathsf{P}\{\xi_0(\nu_0)\chi(\nu_0 < T_r) \le u\}. \tag{2.4.31}$$

Since the set U_0 is dense in $\mathbb{R}_1$, relation (2.4.31) gives

$$\xi_\varepsilon(\nu_\varepsilon)\chi(\nu_\varepsilon < T_r) \Rightarrow \xi_0(\nu_0)\chi(\nu_0 < T_r) \text{ as } \varepsilon \to 0. \tag{2.4.32}$$

Obviously,

$$\mathsf{P}\{|\xi_\varepsilon(\nu_\varepsilon) - \xi_\varepsilon(\nu_\varepsilon)\chi(\nu_\varepsilon < T_r)| > \sigma\} \le \mathsf{P}\{\nu_\varepsilon \ge T_r\}. \tag{2.4.33}$$

This yields

$$\xi_0(\nu_0)\chi(\nu_0 < T_r) \Rightarrow \xi_0(\nu_0) \text{ as } r \to \infty, \tag{2.4.34}$$

and also

$$\begin{aligned}
&\lim_{r\to\infty} \varlimsup_{\varepsilon\to 0} \mathsf{P}\{|\xi_\varepsilon(\nu_\varepsilon) - \xi_\varepsilon(\nu_\varepsilon)\chi(\nu_\varepsilon < T_r)| > \sigma\} \\
&\qquad \le \lim_{r\to\infty} \varlimsup_{\varepsilon\to 0} \mathsf{P}\{\nu_\varepsilon \ge T_r\} = \lim_{r\to\infty} \mathsf{P}\{\nu_0 \ge T_r\} = 0.
\end{aligned} \tag{2.4.35}$$

Relation (2.4.4) follows from Lemma 1.2.5 and relations (2.4.32), (2.4.34), and (2.4.35). The proof is completed. □

2.4.2. Alternative forms of continuity condition $\mathcal{D}_2$. Condition $\mathcal{D}_2$ can be represented in equivalent forms that, in some cases, are more convenient than those given above. Recall that we assume that the conditions $\mathcal{A}_{17}$ and $\mathcal{J}_{97}$ hold and, therefore, the relation of **J**-convergence (2.4.1) holds.

The following condition, with a modified form of the corresponding asymptotic relation, is equivalent to $\mathcal{D}_2$ and $\mathcal{D}_2'$:

$\mathcal{D}_2''$: There exist a sequence $\delta_l \in Z_0, \delta_l \to 0$ as $l \to \infty$ and a sequence $0 < T_r \in S_0 \cap Y_0, T_r \to \infty$ as $r \to \infty$ such that, for every $l, k, r \geq 1$, $\lim_{0<c\to 0} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\alpha^{(\delta_l)}_{\varepsilon k T_r} - c \leq \nu_\varepsilon < \alpha^{(\delta_l)}_{\varepsilon k T_r}\} = 0$.

In condition $\mathcal{D}_2''$, the asymptotic relation is given in the form of an asymptotic estimate. This is more convenient than the form of asymptotic equality used in condition $\mathcal{D}_2'$.

Let us also introduce non-truncated versions for the moments of large jumps of the processes $\xi_\varepsilon(t)$, $t \geq 0$. Take $\delta > 0$ and define, for the real-valued càdlàg function $x(t), t \geq 0$, the functionals $\alpha^{(\delta)}_0(x(\cdot)) = 0$ and then, recursively, $\alpha^{(\delta)}_k(x(\cdot)) = \inf(s > \alpha^{(\delta)}_{k-1}(x(\cdot))\colon |\Delta_s(x(\cdot))| \geq \delta)$ for $k = 1, 2, \ldots$.

It follows from the definitions of truncated and non-truncated versions of these functionals that $\alpha^{(\delta)}_{kT}(x(\cdot)) = \alpha^{(\delta)}_k(x(\cdot))$ if $\alpha^{(\delta)}_k(x(\cdot)) < T$, while $\alpha^{(\delta)}_{kT}(x(\cdot)) = T$ if $\alpha^{(\delta)}_k(x(\cdot)) \geq T$.

Let us also consider the random variables $\alpha^{(\delta)}_{\varepsilon k} = \alpha^{(\delta)}_k(\xi_\varepsilon(\cdot))$, $k = 1, 2, \ldots$. By the definition, $\alpha^{(\delta)}_{\varepsilon k}$ are successive moments of jumps of the process $\xi_\varepsilon(t)$, $t \geq 0$ that have absolute values of jumps greater than or equal to δ. Note that the random variable $\alpha^{(\delta)}_{\varepsilon k}$ takes values in the interval $[0, \infty]$, i.e., it can be an improper random variable.

Also, the following condition is equivalent to $\mathcal{D}_2$, $\mathcal{D}_2'$, and $\mathcal{D}_2''$:

$\mathcal{D}_2'''$: There exist a sequence $\delta_l \in Z_0, \delta_l \to 0$ as $l \to \infty$ and a sequence $0 < T_r \to \infty$ as $r \to \infty$ such that, for every $l, k, r \geq 1$, $\lim_{0<c\to 0} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\alpha^{(\delta_l)}_{\varepsilon k} - c \leq \nu_\varepsilon < \alpha^{(\delta_l)}_{\varepsilon k}, \alpha^{(\delta_l)}_{\varepsilon k} < T_r\} = 0$.

The asymptotic inequality in condition $\mathcal{D}_2'''$ differs from the one in $\mathcal{D}_2''$. It involves the non-truncated versions of the moments of large jumps. Also, it is not required that $T_r \in S_0 \cap Y_0$.

The following lemma summarises all statements about equivalence of different forms of condition $\mathcal{D}_2$ given above.

Lemma 2.4.1. *Let conditions $\mathcal{A}_{17}$ and $\mathcal{J}_7$ hold. Then conditions $\mathcal{D}_2$, $\mathcal{D}_2'$, $\mathcal{D}_2''$, and $\mathcal{D}_2'''$ are equivalent.*

Proof of Lemma 2.4.1. Equivalence of conditions $\mathcal{D}_2$ and $\mathcal{D}_2'$ was proved in (2.4.3).

Let us return to relation (2.4.3), but calculate the upper limit, instead of the lower limit. Take some $0 < c_n \to 0$, which is a point of continuity of the distribution functions of the random variables $\alpha_{0kT_r}^{(\delta_l)} - \nu_0$ for $l, k, r \geq 1$. Then, for every $l, k, r \geq 1$,

$$\begin{aligned} 0 &\leq \lim_{n\to\infty} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\alpha_{\varepsilon kT_r}^{(\delta_l)} - c_n \leq \nu_\varepsilon < \alpha_{\varepsilon kT_r}^{(\delta_l)}\} \\ &= \lim_{n\to\infty} \overline{\lim_{\varepsilon\to 0}} (\mathsf{P}\{\alpha_{\varepsilon kT_r}^{(\delta_l)} - \nu_\varepsilon \leq c_n\} - \mathsf{P}\{\alpha_{\varepsilon kT_r}^{(\delta_l)} - \nu_\varepsilon \leq 0\}) \\ &= \lim_{n\to\infty} \mathsf{P}\{\alpha_{0kT_r}^{(\delta_l)} - \nu_0 \leq c_n\} - \underline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\alpha_{\varepsilon kT_r}^{(\delta_l)} - \nu_\varepsilon \leq 0\} \\ &= \mathsf{P}\{\alpha_{0kT_r}^{(\delta_l)} - \nu_0 \leq 0\} - \underline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\alpha_{\varepsilon kT_r}^{(\delta_l)} - \nu_\varepsilon \leq 0\}. \end{aligned} \tag{2.4.36}$$

It follows from relation (2.4.36) that conditions $\mathfrak{D}_2$ and $\mathfrak{D}_2''$ are equivalent. Indeed, if $\mathfrak{D}_2$ holds, then so does $\mathfrak{D}_2'$ and, therefore, the last expression in (2.4.36) is equal to zero. Therefore, $\mathfrak{D}_2'''$ holds. If $\mathfrak{D}_2'''$ holds, then, again, the last expression in (2.4.36) is equal to zero and, therefore, condition $\mathfrak{D}_2$ holds.

Due to the connection between the truncated and non-truncated moments of large jumps, we have

$$\begin{aligned} &\mathsf{P}\{\alpha_{\varepsilon kT_r}^{(\delta_l)} - c_n \leq \nu_\varepsilon < \alpha_{\varepsilon kT_r}^{(\delta_l)}\} \\ &\quad = \mathsf{P}\{\alpha_{\varepsilon k}^{(\delta_l)} - c_n \leq \nu_\varepsilon < \alpha_{\varepsilon k}^{(\delta_l)}, \alpha_{\varepsilon k}^{(\delta_l)} < T_r\} \\ &\quad + \mathsf{P}\{T_r - c_n \leq \nu_\varepsilon < T_r, \alpha_{\varepsilon k}^{(\delta_l)} \geq T_r\}. \end{aligned} \tag{2.4.37}$$

Since $T_r \in S_0 \cap Y_0$,

$$\begin{aligned} &\lim_{n\to\infty} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{T_r - c_n \leq \nu_\varepsilon < T_r, \alpha_{\varepsilon k}^{(\delta_l)} \geq T_r\} \\ &\quad = \lim_{n\to\infty} \mathsf{P}\{T_r - c_n \leq \nu_0 < T_r, \alpha_{0k}^{(\delta_l)} \geq T_r\} = 0. \end{aligned} \tag{2.4.38}$$

So, the asymptotic relation in condition $\mathfrak{D}_2''$ can be replaced with the following equivalent relation:

$$\lim_{0<c\to 0} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\alpha_{\varepsilon k}^{(\delta_l)} - c \leq \nu_\varepsilon < \alpha_{\varepsilon k}^{(\delta_l)}, \alpha_{\varepsilon k}^{(\delta_l)} < T_r\} = 0. \tag{2.4.39}$$

Thus, an equivalent form of condition $\mathfrak{D}_2''$ would be to assume that there exist a sequence $\delta_l \in Z_0, \delta_l \to 0$ as $l \to \infty$ and a sequence $0 < T_r \in S_0 \cap Y_0, T_r \to \infty$ as $r \to \infty$ such that relation (2.4.39) holds for every $l, k, r \geq 1$.

However, the probability $\mathsf{P}\{\alpha_{\varepsilon k}^{(\delta_l)} - c \leq \nu_\varepsilon < \alpha_{\varepsilon k}^{(\delta_l)}, \alpha_{\varepsilon k}^{(\delta_l)} < T_r\}$ is a nondecreasing function in T_r. This implies, in an obvious way, that relation (2.4.39) holds simultaneously for all the sequences $0 < T_r \to \infty$ as $r \to \infty$. □

The following condition is, obviously, sufficient for condition $\mathfrak{D}_2'''$ to hold:

$\mathfrak{D}_2''''$: There exists a sequence $\delta_l \in Z_0, \delta_l \to 0$ as $l \to \infty$ such that for every $l, k \geq 1$, $\lim_{0<c\to 0} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\alpha_{\varepsilon k}^{(\delta_l)} - c \leq \nu_\varepsilon < \alpha_{\varepsilon k}^{(\delta_l)}\} = 0$.

The following lemma supplements Lemma 2.4.1.

Lemma 2.4.2. *Let conditions $\mathcal{A}_{17}$ and $\mathfrak{J}_7$ hold. Then condition $\mathfrak{D}_2''''$ implies conditions $\mathfrak{D}_2$, $\mathfrak{D}_2'$, $\mathfrak{D}_2''$, and $\mathfrak{D}_2'''$, and it is equivalent to these conditions if* **(α)** $\lim_{r\to\infty} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{T_r \leq \alpha_{\varepsilon k}^{(\delta_l)} < \infty\} = 0$.

Proof of Lemma 2.4.2. It is obvious that for any $T_r > 0$ and $l, k \geq 1$, the following relation implies (2.4.39):

$$\lim_{0<c\to 0} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\alpha_{\varepsilon k}^{(\delta_l)} - c \leq \nu_\varepsilon < \alpha_{\varepsilon k}^{(\delta_l)}\} = 0. \tag{2.4.40}$$

Therefore, condition $\mathfrak{D}_2''''$ implies condition $\mathfrak{D}_2'''$.

Note that $\{\alpha_{\varepsilon k}^{(\delta_l)} - c \leq \nu_\varepsilon < \alpha_{\varepsilon k}^{(\delta_l)}\} \subseteq \{\alpha_{\varepsilon k}^{(\delta_l)} < \infty\}$. Assume that condition **(α)** holds. Conditions $\mathfrak{D}_2'''$ and **(α)** imply that, for any sequence $0 < T_r \to \infty$ as $r \to \infty$,

$$\begin{aligned} &\lim_{0<c\to 0} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\alpha_{\varepsilon k}^{(\delta_l)} - c \leq \nu_\varepsilon < \alpha_{\varepsilon k}^{(\delta_l)}\} \\ &\quad \leq \lim_{0<c\to 0} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\alpha_{\varepsilon k}^{(\delta_l)} - c \leq \nu_\varepsilon < \alpha_{\varepsilon k}^{(\delta_l)}, \alpha_{\varepsilon k}^{(\delta_l)} < T_r\} \\ &\quad + \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{T_r \leq \alpha_{\varepsilon k}^{(\delta_l)} < \infty\} = \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{T_r \leq \alpha_{\varepsilon k}^{(\delta_l)} < \infty\} \to 0 \text{ as } r \to \infty. \end{aligned} \tag{2.4.41}$$

Relation (2.4.41) implies that the conditions $\mathfrak{D}_2''''$ and $\mathfrak{D}_2$ are equivalent if condition **(α)** holds. □

The following lemma states that, in some sense, conditions $\mathfrak{D}_2$ and $\mathfrak{D}_2'$ – $\mathfrak{D}_2''''$ are invariant with respect to the choice of the sequences δ_l and T_r.

Lemma 2.4.3. *Let conditions $\mathcal{A}_{17}$ and $\mathfrak{J}_7$ hold. Then* **(α)** *condition $\mathfrak{D}_2$ as well as $\mathfrak{D}_2'$ and $\mathfrak{D}_2''$ can hold only for all sequences $\delta_l \in Z_0, \delta_l \to 0$ as $l \to \infty$ and $0 < T_r \in S_0 \cap Y_0, T_r \to \infty$ as $r \to \infty$ simultaneously;* **(β)** *condition $\mathfrak{D}_2'''$ can hold only for all sequences $\delta_l \in Z_0, \delta_l \to 0$ as $l \to \infty$ and $T_r \to \infty$ as $r \to \infty$ simultaneously;* **(γ)** *condition $\mathfrak{D}_2''''$ can hold only for all sequences $\delta_l \in Z_0, \delta_l \to 0$ as $l \to \infty$ simultaneously.*

Proof of Lemma 2.4.3. Let us prove statement **(α)**. The proofs of the statements **(β)** and **(γ)** are similar.

Let us go back to the proof of Lemma 2.4.1. It was actually proved that any sequence $\delta_l \in Z_0, \delta_l \to 0$ as $l \to \infty$ and any sequence $0 < T_r \in S_0 \cap Y_0, T_r \to \infty$ as $r \to \infty$, which is used in one of the conditions $\mathfrak{D}_2$, $\mathfrak{D}_2'$, or $\mathfrak{D}_2''$, can also be used in the other two conditions.

Moreover, it was shown that an equivalent form of condition $\mathfrak{D}_2''$ is to assume that there exist **(a)** a sequence $\delta_l \in Z_0, \delta_l \to 0$ as $l \to \infty$ and **(b)** a sequence $0 < T_r \in S_0 \cap Y_0, T_r \to \infty$ as $r \to \infty$ such that **(c)** relation (2.4.39) holds for every $l, k, r \geq 1$.

Thus, it is sufficient to prove **(α)** for condition $\mathfrak{D}_2''$. To do this, it is enough to show that the asymptotic relation used in condition $\mathfrak{D}_2''$ possesses the following property: **(d)** if the relation holds for given $\delta_l \in Z_0$ and $0 < T_r \in S_0 \cap Y_0$ for all $k \geq 1$, then it also holds for all $\delta'_l \in Z_0, \delta'_l > \delta_l$ and $0 < T'_r \in S_0 \cap Y_0, T'_r < T_r$ for all $k \geq 1$.

Let us first consider two random sequences $\alpha^{(\delta'_l)}_{\varepsilon k T_r}, k \geq 1$ and $\alpha^{(\delta_l)}_{\varepsilon k T_r}, k \geq 1$. Obviously, the sequence of moments, $\alpha^{(\delta'_l)}_{\varepsilon k T_r}, k \geq 1$, is a subsequence of the random sequence $\alpha^{(\delta_l)}_{\varepsilon k T_r}, k \geq 1$, i.e., $\alpha^{(\delta'_l)}_{\varepsilon k T_r} = \alpha^{(\delta_l)}_{\varepsilon \mu_{\varepsilon k} T_r}$. Here, $\mu_{\varepsilon k}$ are the random indices that take the values $1, 2, \ldots$ and define the corresponding subsequence. Let us show that

$$\lim_{n\to\infty} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\mu_{\varepsilon k} > n\} = 0. \tag{2.4.42}$$

Obviously, **(e)** $\mu_{\varepsilon k} \leq k + \beta_{\varepsilon, T_r}$, where β_{ε, T_r} is the number of jumps of the process $\xi_\varepsilon(t), t \geq 0$ in the interval $[0, T_r]$ such that absolute values of these jumps lie in the interval $[\delta_l, \delta'_l)$. By the definition, $\beta_{\varepsilon T_r}$ is the difference between the numbers of jumps of the process $\xi_\varepsilon(t), t \geq 0$ in the interval $[0, T_r]$ such that absolute values of these jumps are greater than of equal to δ'_l and δ_l, respectively. By Lemma 1.6.10, **(f)** the random variables $\beta_{\varepsilon T_r} \Rightarrow \beta_{0 T_r}$ as $\varepsilon \to 0$, and, by Lemma 1.4.1, **(g)** $\beta_{0 T_r}$ is a finite random variable. Obviously, (2.4.42) follows from **(e)** – **(g)**.

The following estimate holds for every $k \geq 1$:

$$\begin{aligned}
&\mathsf{P}\{\alpha^{(\delta'_l)}_{\varepsilon k T_r} - c \leq \nu_\varepsilon < \alpha^{(\delta'_l)}_{\varepsilon k T_r}\} \\
&\quad \leq \sum_{j=1}^{n} \mathsf{P}\{\alpha^{(\delta_l)}_{\varepsilon j T_r} - c \leq \nu_\varepsilon < \alpha^{(\delta_l)}_{\varepsilon j T_r}, \mu_{\varepsilon k} = j\} + \mathsf{P}\{\mu_{\varepsilon k} > n\} \\
&\quad \leq \sum_{j=1}^{n} \mathsf{P}\{\alpha^{(\delta_l)}_{\varepsilon j T_r} - c \leq \nu_\varepsilon < \alpha^{(\delta_l)}_{\varepsilon j T_r}\} + \mathsf{P}\{\mu_{\varepsilon k} > n\}.
\end{aligned} \tag{2.4.43}$$

The proof of statement **(d)** for $T'_r = T_r$ follows from (2.4.42) and (2.4.43). The transition to the case $T'_r < T_r$ is obvious, since the probability $\mathsf{P}\{\alpha^{(\delta_l)}_{\varepsilon k} - c \leq \nu_\varepsilon < \alpha^{(\delta_l)}_{\varepsilon k}, \alpha^{(\delta_l)}_{\varepsilon k} < T_r\}$ is a nondecreasing function in T_r. □

2.4.3. Weakened continuity conditions for randomly stopped vector processes. The result formulated in Theorem 2.4.1 can be generalised to vector processes. Let, for every $\varepsilon \geq 0$, $\boldsymbol{\xi}_\varepsilon(t) = (\xi_{\varepsilon 1}(t), \ldots, \xi_{\varepsilon m}(t))$, $t \geq 0$, be a m-dimensional càdlàg process with real-valued components, and $\boldsymbol{\nu}_\varepsilon = (\nu_{\varepsilon 1}, \ldots, \nu_{\varepsilon m})$ be a random vector with non-negative components. Consider the random vectors $\boldsymbol{\zeta}_\varepsilon = (\xi_{\varepsilon 1}(\nu_{\varepsilon 1}), \ldots, \xi_{\varepsilon m}(\nu_{\varepsilon m}))$.

Let S_0 be the set of points of stochastic continuity of the vector process $\xi_0(t), t \geq 0$, and Y_0 the set of $t > 0$ that are points of continuity of the distribution functions of the random variables ν_{0i} for all $i = 1, \ldots, m$.

Let also $\alpha^{(\delta)}_{\varepsilon ikT} = \alpha^{(\delta)}_{\varepsilon kT}(\xi_{\varepsilon i}(\cdot))$ and $\alpha^{(\delta)}_{\varepsilon ik} = \alpha^{(\delta)}_{\varepsilon k}(\xi_{\varepsilon i}(\cdot))$ be, respectively, the truncated and non-truncated moments of large jumps of the processes $\xi_{\varepsilon i}(t), t \geq 0$, for $i = 1, \ldots, m$.

The following condition is a "vector" analogue of condition $\mathfrak{D}_2$:

$\mathfrak{D}_3$: There exist a sequence $\delta_l \in Z_0, \delta_l \to 0$ as $l \to \infty$ and a sequence $0 < T_r \in S_0 \cap Y_0, T_r \to \infty$ as $r \to \infty$ such that for every $l, k, r \geq 1$ and $i = 1, \ldots, m$, $\underline{\lim}_{\varepsilon \to 0} \mathsf{P}\{\alpha^{(\delta_l)}_{\varepsilon ikT_r} - \nu_{\varepsilon i} \leq 0\} \geq \mathsf{P}\{\alpha^{(\delta_l)}_{0kiT_r} - \nu_{0i} \leq 0\}$.

This condition can be formulated in equivalent forms that are "vector" analogues of the conditions $\mathfrak{D}_2' - \mathfrak{D}_2'''$:

$\mathfrak{D}_3'$: There exist a sequence $\delta_l \in Z_0, \delta_l \to 0$ as $l \to \infty$ and a sequence $0 < T_r \in S_0 \cap Y_0, T_r \to \infty$ as $r \to \infty$ such that, for every $n, k, r \geq 1$ and $i = 1, \ldots, m$, $\lim_{\varepsilon \to 0} \mathsf{P}\{\alpha^{(\delta_l)}_{\varepsilon ikT_r} - \nu_{\varepsilon i} \leq 0\} = \mathsf{P}\{\alpha^{(\delta_l)}_{0ikT_r} - \nu_0 \leq 0\}$.

$\mathfrak{D}_3''$: There exist a sequence $\delta_l \in Z_0, \delta_l \to 0$ as $l \to \infty$ and a sequence $0 < T_r \in S_0 \cap Y_0, T_r \to \infty$ as $r \to \infty$ such that, for every $l, k, r \geq 1$ and $i = 1, \ldots, m$, $\lim_{0<c\to 0} \overline{\lim}_{\varepsilon \to 0} \mathsf{P}\{\alpha^{(\delta_l)}_{\varepsilon ikT_r} - c \leq \nu_{\varepsilon i} < \alpha^{(\delta_l)}_{\varepsilon ikT_r}\} = 0$.

$\mathfrak{D}_3'''$: There exist a sequence $\delta_l \in Z_0, \delta_l \to 0$ as $l \to \infty$ and a sequence $0 < T_r \to \infty$ as $r \to \infty$ such that, for every $l, k, r \geq 1$ and $i = 1, \ldots, m$, $\lim_{0<c\to 0} \overline{\lim}_{\varepsilon \to 0} \mathsf{P}\{\alpha^{(\delta_l)}_{\varepsilon ik} - c \leq \nu_{\varepsilon i} < \alpha^{(\delta_l)}_{\varepsilon ik}, \alpha^{(\delta_l)}_{\varepsilon ik} < T_r\} = 0$.

The following condition is a "vector" analogue of the condition $\mathfrak{D}_2''''$:

$\mathfrak{D}_3''''$: There exist a sequence of $\delta_l \in Z_0, \delta_l \to 0$ as $l \to \infty$ such that, for every $l, k \geq 1$ and $i = 1, \ldots, m$, $\lim_{0<c\to 0} \overline{\lim}_{\varepsilon \to 0} \mathsf{P}\{\alpha^{(\delta_l)}_{\varepsilon ik} - c \leq \nu_{\varepsilon i} < \alpha^{(\delta_l)}_{\varepsilon ik}\} = 0$.

Let us now formulate a vector analogue of Theorem 2.4.1.

Theorem 2.4.2. *Let conditions* $\mathcal{A}_{20}$, $\mathfrak{J}_4$, *and* $\mathfrak{D}_3$ *hold. Then*

$$\zeta_\varepsilon = (\xi_{\varepsilon 1}(\nu_{\varepsilon 1}), \ldots, \xi_{\varepsilon m}(\nu_{\varepsilon m})) \Rightarrow \zeta_0 = (\xi_{01}(\nu_{\varepsilon 1}), \ldots, \xi_{0m}(\nu_{0m})) \ \text{as } \varepsilon \to 0. \tag{2.4.44}$$

Condition $\mathfrak{J}_4$ can be replaced with condition $\mathfrak{J}_8$ in Theorem 2.4.2.

Theorem 2.4.3. *Let conditions* $\mathcal{A}_{20}$, $\mathfrak{J}_8$, *and* $\mathfrak{D}_3$ *hold. Then*

$$\zeta_\varepsilon = (\xi_{\varepsilon 1}(\nu_{\varepsilon 1}), \ldots, \xi_{\varepsilon m}(\nu_{\varepsilon m})) \Rightarrow \zeta_0 = (\xi_{01}(\nu_{\varepsilon 1}), \ldots, \xi_{0m}(\nu_{0m})) \ \text{as } \varepsilon \to 0. \tag{2.4.45}$$

Proof of Theorems 2.4.2 and 2.4.3. Let us take arbitrary $n \geq 1$, vector $\bar{c} = (c_j, j = 1, \dots, 2m, c_{ik}, i = 1, \dots, m, k = 1, \dots, n) \in R_{2m+mn}$ and points $t_1, \dots, t_n \in U$, where U is the set which appears in condition $\mathcal{A}_{20}$. Let us consider the stochastic process

$$\xi_{\varepsilon 1,\bar{c}}(t) = c_1 \xi_{\varepsilon 1}(t) + \sum_{j=1}^{m} c_{m+j} \nu_{\varepsilon j} + \sum_{i=1}^{m} \sum_{k=1}^{n} c_{ik} \xi_{\varepsilon i}(t_k), t \geq 0.$$

Obviously, condition $\mathcal{A}_{20}$ implies the following relation

$$(\nu_{\varepsilon 1}, \xi_{\varepsilon 1,\bar{c}}(t)), t \in U \Rightarrow (\nu_{01}, \xi_{01,\bar{c}}(t)), t \in U \text{ as } \varepsilon \to 0. \tag{2.4.46}$$

Thus condition $\mathcal{A}_{17}$ holds for the random variables $\nu_{\varepsilon 1}$ and the processes $\xi_{\varepsilon 1,\bar{c}}(t), t \geq 0$ with the same set of weak convergence U as in condition $\mathcal{A}_{20}$.

Condition $\mathcal{J}_8$ implies that condition $\mathcal{J}_7$ holds for the processes $\xi_{\varepsilon 1,\bar{c}}(t), t \geq 0$. Indeed, the modulus of $\mathbf{J}$-compactness $\Delta_J(\xi_{\varepsilon 1,\bar{c}}(\cdot), c, T) \equiv |c_1| \Delta_J(\xi_{\varepsilon 1}(\cdot), c, T)$.

Finally, condition $\mathcal{D}_3'''$ implies that condition $\mathcal{D}_2'''$ holds for the random variables $\nu_{\varepsilon 1}$ and the processes $\xi_{\varepsilon 1,\bar{c}}(t), t \geq 0$. Indeed, the processes $\xi_{\varepsilon 1}(t), t \geq 0$ and $\xi_{\varepsilon 1,\bar{c}}(t)$ have the same moments and values of jumps. Thus, by applying Theorem 2.4.1 to the random variables $\nu_{\varepsilon 1}$ and the processes $\xi_{\varepsilon 1,\bar{c}}(t), t \geq 0$, we get

$$\begin{aligned} & c_1 \xi_{\varepsilon 1}(\nu_{\varepsilon 1}) + \sum_{j=1}^{m} c_{m+j} \nu_{\varepsilon j} + \sum_{i=1}^{m} \sum_{k=1}^{n} c_{ik} \xi_{\varepsilon i}(t_k) \\ & \quad \Rightarrow c_1 \xi_{01}(\nu_{01}) + \sum_{j=1}^{m} c_{m+j} \nu_{0j} + \sum_{i=1}^{m} \sum_{k=1}^{n} c_{ik} \xi_{0i}(t_k) \text{ as } \varepsilon \to 0. \end{aligned} \tag{2.4.47}$$

Let us consider the processes

$$\xi_{\varepsilon 2,\bar{c}}(t) = c_2 \xi_{\varepsilon 2}(t) + c_1 \xi_{\varepsilon 1}(\nu_{\varepsilon 1}) + \sum_{j=1}^{m} c_{m+j} \nu_{\varepsilon j} + \sum_{i=1}^{m} \sum_{k=1}^{n} c_{ik} \xi_{\varepsilon i}(t_k), t \geq 0.$$

Relation (2.4.47) implies, due to Lemma 1.2.1 and arbitrariness in the choice of $n \geq 1$, $\bar{c} \in \mathbb{R}_{2m+mn}$, and $t_1, \dots, t_n \in U$,

$$(\nu_{\varepsilon 2}, \xi_{\varepsilon 2,\bar{c}}(t)), t \in U \Rightarrow (\nu_{02}, \xi_{02,\bar{c}}(t)), t \in U \text{ as } \varepsilon \to 0. \tag{2.4.48}$$

Thus condition $\mathcal{A}_{17}$ holds for the random variables $\nu_{\varepsilon 2}$ and the processes $\xi_{\varepsilon 2,\bar{c}}(t), t \geq 0$ with the same set of weak convergence U as in condition $\mathcal{A}_{20}$.

Condition $\mathcal{J}_8$ implies that condition $\mathcal{J}_7$ holds for the processes $\xi_{\varepsilon 2,\bar{c}}(t), t \geq 0$. Indeed, the modulus of $\mathbf{J}$-compactness $\Delta_J(\xi_{\varepsilon 2,\bar{c}}(\cdot), c, T) \equiv |c_2| \Delta_J(\xi_{\varepsilon 2}(\cdot), c, T)$.

Finally, condition $\mathcal{D}_3'''$ implies that $\mathcal{D}_2'''$ holds for the random variables $\nu_{\varepsilon 2}$ and the processes $\xi_{\varepsilon 2,\bar{c}}(t), t \geq 0$. Indeed, the processes $\xi_{\varepsilon 2}(t), t \geq 0$ and $\xi_{\varepsilon 2,\bar{c}}(t)$ have the same moments and values of jumps. Thus, we get applying Theorem 2.4.1 to the random variables $\nu_{\varepsilon 2}$ and the processes $\xi_{\varepsilon 2,\bar{c}}(t), t \geq 0$,

$$\begin{aligned} & c_2\xi_{\varepsilon 2}(\nu_{\varepsilon 2}) + c_1\xi_{\varepsilon 1}(\nu_{\varepsilon 1}) + \sum_{j=1}^{m} c_{m+j}\nu_{\varepsilon j} + \sum_{i=1}^{m}\sum_{k=1}^{n} c_{ik}\xi_{\varepsilon i}(t_k) \\ & \quad \Rightarrow c_2\xi_{02}(\nu_{02}) + c_1\xi_{01}(\nu_{01}) + \sum_{j=1}^{m} c_{m+j}\nu_{0j} + \sum_{i=1}^{m}\sum_{k=1}^{n} c_{ik}\xi_{0i}(t_k) \text{ as } \varepsilon \to 0. \end{aligned} \tag{2.4.49}$$

By proceeding in the same way, we get, after m steps, the following relation

$$\begin{aligned} & \sum_{i=1}^{m} c_i\xi_{\varepsilon i}(\nu_{\varepsilon i}) + \sum_{j=1}^{m} c_{m+j}\nu_{\varepsilon j} + \sum_{i=1}^{m}\sum_{k=1}^{n} c_{ik}\xi_{\varepsilon i}(t_k) \\ & \quad \Rightarrow \sum_{i=1}^{m} c_i\xi_{0i}(\nu_{0i}) + \sum_{j=1}^{m} c_{m+j}\nu_{0j} + \sum_{i=1}^{m}\sum_{k=1}^{n} c_{ik}\xi_{0i}(t_k) \text{ as } \varepsilon \to 0. \end{aligned} \tag{2.4.50}$$

Relation (2.4.50) implies, due to Lemma 1.2.1 and arbitrariness in the choice of $\bar{c} \in \mathbb{R}_{2m+mn}$, the statement of the theorem. □

2.4.4. Random vectors $(\xi_{\varepsilon i}(\nu_{\varepsilon i} - 0), i = 1, \ldots, m)$. In this case, the conditions of weak convergence should be slightly modified, since the processes $(\xi_{\varepsilon i}(t - 0), i = 1, \ldots, m), t \geq 0$ (here $\xi_{\varepsilon i}(0 - 0) = \xi_{\varepsilon i}(0)$) are a.s. continuous from the left.

The inequalities $\alpha^{(\delta_l)}_{\varepsilon ikT_r} - \nu_{\varepsilon i} \leq 0$ in the conditions $\mathcal{D}_3$ and $\mathcal{D}'_3$ should be replaced with the inequalities $\alpha^{(\delta_l)}_{\varepsilon ikT_r} - \nu_{\varepsilon i} \geq 0$. Analogously, the inequalities $\alpha^{(\delta_l)}_{\varepsilon ik} - c \leq \nu_{\varepsilon i} < \alpha^{(\delta_l)}_{\varepsilon ik}$ in $\mathcal{D}''_3$ and $\mathcal{D}''''_3$ should be replaced with the inequalities $\alpha^{(\delta_l)}_{\varepsilon ik} + c \geq \nu_{\varepsilon i} > \alpha^{(\delta_l)}_{\varepsilon ik}$.

The conditions $\mathcal{A}_{20}$, $\mathcal{J}_4$ or $\mathcal{J}_8$, and the modified version of condition $\mathcal{D}_3$ (or one of the conditions $\mathcal{D}'_3 - \mathcal{D}''''_3$) imply that

$$(\xi_{\varepsilon i}(\nu_{\varepsilon i} - 0), i = 1, \ldots, m) \Rightarrow (\xi_{0i}(\nu_{0i} - 0), i = 1, \ldots, m) \text{ as } \varepsilon \to 0. \tag{2.4.51}$$

This can be proved by repeating the proofs above with obvious changes in the definitions of the corresponding functionals.

Also, the method of time reversion can be used. As in Subsection 2.3.8, the consideration can be reduced to the case where the stopping moments are bounded, i.e., **(a)** $\nu_{\varepsilon i} \in [0, T], i = 1, \ldots m$; **(b)** T is a point of stochastic continuity of the processes $\xi_{0i}(t), t \geq 0$, for $i = 1, \ldots, m$; and **(c)** T is a point of continuity of the distribution functions of the random variables $\nu_{0i}, i = 1, \ldots, m$.

Let us define the processes $\boldsymbol{\xi}^{(T)}_\varepsilon(t) = \boldsymbol{\xi}_\varepsilon(T - t - 0)$ for $t \in [0, T]$ and $\boldsymbol{\xi}^{(T)}_\varepsilon(t) = \boldsymbol{\xi}_\varepsilon(0)$ for $t > T$. Obviously, $\boldsymbol{\xi}^{(T)}_\varepsilon(t), t \geq 0$, is a càdlàg process. Let us also consider the random vectors $\boldsymbol{\nu}^{(T)}_\varepsilon = (\nu^{(T)}_{\varepsilon i}, i = 1, \ldots, m) = (T - \nu_{\varepsilon i}, i = 1, \ldots, m)$. By the definition of these processes and the random vectors,

$$(\xi^{(T)}_{\varepsilon i}(\nu^{(T)}_{\varepsilon i}), i = 1, \ldots, m) = (\xi_{\varepsilon i}(\nu_{\varepsilon i} - 0), i = 1, \ldots, m). \tag{2.4.52}$$

If the conditions $\mathcal{A}_{20}$ and $\mathfrak{J}_4$ (or $\mathfrak{J}_8$) hold for the processes $\xi_\varepsilon(t), t \geq 0$ and the random vectors $\mathbf{v}_\varepsilon$, then this condition also holds for the processes $\xi_\varepsilon^{(T)}(t), t \geq 0$ and the random vectors $\mathbf{v}_\varepsilon^{(T)}$.

Indeed, the functional $f_t^-(\mathbf{x}(\cdot)) = \mathbf{x}(t-0)$ is an a.s. **J**-continuous functional with respect to the measure generated by the process $\xi_0(t), t \geq 0$ for every point t where this process is stochastically continuous. So, the processes $(\mathbf{v}_\varepsilon^{(T)}, \xi_\varepsilon^{(T)}(t))$ weakly converge on the set that contains all points $t \in [0, T]$, such that $T - t$ are points of stochastic continuity of the process $\xi_0(t)$, and all points $t \geq T$.

For condition $\mathfrak{J}_4$ (or $\mathfrak{J}_8$), this implication follows from the equalities $\Delta_J(\xi_\varepsilon(\cdot), c, T) = \Delta_J(\xi_\varepsilon^{(T)}(\cdot), c, T)$ and $\Delta_J(\xi_{\varepsilon i}(\cdot), c, T) = \Delta_J(\xi_{\varepsilon i}^{(T)}(\cdot), c, T), i = 1, \ldots, m$, which hold with probability 1 under assumption **(b)**.

Finally, if the modified version of condition $\mathfrak{D}_3$ (or one of the conditions $\mathfrak{D}_3' - \mathfrak{D}_3''''$) described above holds for the processes $\xi_\varepsilon(t), t \geq 0$ and the random vectors $\mathbf{v}_\varepsilon$, then the corresponding condition $\mathfrak{D}_3$ (or one of the conditions $\mathfrak{D}_3' - \mathfrak{D}_3''''$) holds for the processes $\xi_\varepsilon^{(T)}(t), t \geq 0$ and the random vectors $\mathbf{v}_\varepsilon^{(T)}$.

Summarising the remarks made above one obtains relation (2.4.51).

Moreover, if conditions $\mathcal{A}_{20}$, $\mathfrak{J}_4$ (or $\mathfrak{J}_8$) hold, together with the modified versions of the condition $\mathfrak{D}_3$ (or one of the conditions $\mathfrak{D}_3' - \mathfrak{D}_3''''$), then one also obtains the following more general relation:

$$
\begin{aligned}
&(\xi_{\varepsilon i}(\nu_{\varepsilon i}), \xi_{\varepsilon i}(\nu_{\varepsilon i} - 0), i = 1, \ldots, m) \\
&\quad \Rightarrow (\xi_{0i}(\nu_{0i}), \xi_{0i}(\nu_{0i} - 0), i = 1, \ldots, m) \text{ as } \varepsilon \to 0.
\end{aligned} \tag{2.4.53}
$$

2.4.5. Examples. Let us return to the basic scalar case ($m = 1$). The following lemma shows that, under the assumption that conditions $\mathcal{A}_{17}$, $\mathfrak{J}_7$ hold, condition $\mathfrak{D}_2$ can be considered as a weakened form of the continuity condition $\mathfrak{C}_3$.

Lemma 2.4.4. *Let conditions $\mathcal{A}_{17}$ and $\mathfrak{J}_7$ hold. Then condition $\mathfrak{C}_3$ implies condition $\mathfrak{D}_2$.*

Proof of Lemma 2.4.4. Condition $\mathfrak{C}_3$ means that ν_0 can coincide with any discontinuity point of the process $\xi_0(t), t \geq 0$ only with probability 0. This actually means that $\mathsf{P}\{\alpha_{0kT}^{(\delta_l)} - \nu_0 = 0\} = 0$ for any $\delta_l \in Z_0$, $T_r \in Y_0$ and $k \geq 1$. So, the point 0 is a continuity point of the distribution function of the random variable $\alpha_{0kT_r}^{(\delta_l)} - \nu_0$ for every $\delta_l \in Z_0$, $T_r \in Y_0$ and $k \geq 1$. Thus, condition $\mathfrak{D}_2'$ holds. Recall that conditions $\mathfrak{D}_2$ and $\mathfrak{D}_2'$ are equivalent. □

However, condition $\mathfrak{D}_2$ can hold in cases where ν_0 is not a point of continuity of the process $\xi_0(t), t \geq 0$, but rather a point of jump of this process.

For example, condition $\mathfrak{D}_2$ is satisfied if $\nu_\varepsilon = \alpha_{\varepsilon nT}^{(\delta)}$ is itself a moment of large jump of the process $\xi_\varepsilon(t), t \geq 0$.

Indeed, in this case, the event $\{\alpha_{\varepsilon k}^{(\delta_l)} - c \leq \alpha_{\varepsilon nT}^{(\delta)} < \alpha_{\varepsilon k}^{(\delta_l)}, \alpha_{\varepsilon k}^{(\delta_l)} < T_r\} \subseteq \{\Delta_J(\xi_\varepsilon(\cdot), c, T_r) \geq \delta_l \wedge \delta\}$ if $n \leq k, T \leq T_r$. So, if condition $\mathfrak{D}_2$ does not hold, then the condition of **J**-compactness $\mathfrak{J}_7$, does not hold.

Let us reformulate this fact in a more general form. Assuming that the following condition holds:

$\bar{\mathfrak{O}}_5$: $\lim_{0<\sigma\to 0} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{|\Delta_{\nu_\varepsilon}(\xi_\varepsilon(\cdot))| \le \sigma\} = 0$.

This condition means that the stopping moments ν_ε are, asymptotically, moments of large jumps of the processes $\xi_\varepsilon(t), t \ge 0$.

Lemma 2.4.5. *Let conditions $\mathcal{A}_{17}$ and $\mathfrak{J}_7$ hold. Then condition $\bar{\mathfrak{O}}_5$ implies condition $\mathfrak{D}_2$.*

Proof of Lemma 2.4.5. Let us consider the events $A_{\varepsilon,c,klr} = \{\alpha_{\varepsilon k}^{(\delta_l)} - c \le \nu_\varepsilon < \alpha_{\varepsilon k}^{(\delta_l)}, \alpha_{\varepsilon k}^{(\delta_l)} < T_r\}$. Assume for the moment that condition $\mathfrak{D}_2'''$ does not hold. This means that there exist $0 < \delta_l \in Z_0$ and $T_r > 0$ such that **(a)** $\lim_{0<c\to 0} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}(A_{\varepsilon,c,klr}) = a_{klr} > 0$. Consider also the events $B_{\varepsilon,\sigma} = \{|\Delta_{\nu_\varepsilon}(\xi_\varepsilon(\cdot))| > \sigma\}$. Condition $\bar{\mathfrak{O}}_5$ implies that this $\sigma > 0$ can be chosen so that **(b)** $\overline{\lim}_{\varepsilon\to 0} \mathsf{P}(\overline{B}_{\varepsilon,\sigma}) < a_{klr}$. Obviously, **(c)** $A_{\varepsilon,c,klr} \cap B_{\varepsilon,\sigma} \subseteq \{\Delta_J(\xi_\varepsilon(\cdot), c, T_r) \ge \delta_l \wedge \sigma\}$. Using **(a)** – **(c)** one gets

$$
\begin{aligned}
&\lim_{0<c\to 0} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\Delta_J(\xi_\varepsilon(\cdot), c, T_r) \ge \delta_l \wedge \sigma\} \\
&\qquad \ge \lim_{0<c\to 0} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{A_{\varepsilon,c,klr} \cap B_{\varepsilon,\sigma}\} \ge \lim_{0<c\to 0} \overline{\lim_{\varepsilon\to 0}} (\mathsf{P}(A_{\varepsilon,c,klr}) + \mathsf{P}(B_{\varepsilon,\sigma}) - 1) \qquad (2.4.54) \\
&\qquad \ge \lim_{0<c\to 0} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}(A_{\varepsilon,c,klr}) - \overline{\lim_{\varepsilon\to 0}} \mathsf{P}(\overline{B}_{\varepsilon,\sigma}) > 0.
\end{aligned}
$$

This shows that the condition of **J**-compactness $\mathfrak{J}_7$ does not hold, which contradicts the conditions of the lemma. □

It can be shown in the same way that, under the conditions $\mathcal{A}_{17}$ and $\mathfrak{J}_7$, condition $\bar{\mathfrak{O}}_5$ also implies that the modified version of condition $\mathfrak{D}_2$, which was described in Subsection 2.4.4, holds.

Taking this remark into consideration one can write relation (2.4.53) under conditions $\mathcal{A}_{17}$, $\mathfrak{J}_7$, and $\bar{\mathfrak{O}}_5$. This relation obviously implies that

$$\Delta_{\nu_\varepsilon}(\xi_\varepsilon(\cdot)) \Rightarrow \Delta_{\nu_0}(\xi_0(\cdot)) \text{ as } \varepsilon \to 0. \qquad (2.4.55)$$

For non-random càdlàg functions, relation (2.4.55) was mentioned by Kolmogorov (1956). It was extended to the case of càdlàg processes by Anisimov (1975) with the use of Skorokhod's representation theorem for càdlàg processes.

Condition $\bar{\mathfrak{O}}_5$ and relation (2.4.55) imply, in an obvious way, that the random variable $\Delta_{\nu_0}(\xi_0(\cdot)) > 0$ with probability 1, i.e., ν_0 is a discontinuity point of the process $\xi_0(t), t \ge 0$ with probability 1.

Condition $\bar{\mathfrak{O}}_5$ is restrictive in the sense that it requires that the pre-limiting stopping moment ν_ε itself be a discontinuity point of the pre-limiting external process $\xi_\varepsilon(t), t \ge 0$ with a probability that tends to 1 as $\varepsilon \to 0$.

Let us weaken condition $\bar{\mathfrak{O}}_5$ and assume that the following condition holds:

$\bar{\mathfrak{O}}_6$: $\nu_\varepsilon = \nu'_\varepsilon + \nu''_\varepsilon$, where (a) $\bar{\mathfrak{O}}_5$ holds for the processes $\xi_\varepsilon(t), t \geq 0$ and the random variables ν'_ε; (b) ν''_ε are non-negative random variables; (c) $\nu''_\varepsilon \xrightarrow{\mathsf{P}} 0$ as $\varepsilon \to 0$.

Lemma 2.4.6. *Let conditions $\mathcal{A}_{17}$ and $\mathfrak{J}_7$ hold. Then condition $\bar{\mathfrak{O}}_6$ implies condition $\mathfrak{D}_2$.*

Proof of Lemma 2.4.6. by Lemma 2.4.5, condition $\mathfrak{D}_2'''$ holds for the processes $\xi_\varepsilon(t), t \geq 0$, and the random variables ν'_ε. This means that there exist a sequence $\delta_l \in Z_0, \delta_l \to 0$ as $l \to \infty$ and a sequence $0 < T_r \to \infty$ as $r \to \infty$ such that, for every $l, k, r \geq 1$,

$$\lim_{0<c\to 0} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\alpha_{\varepsilon k}^{(\delta_l)} - c \leq \nu'_\varepsilon < \alpha_{\varepsilon k}^{(\delta_l)}, \alpha_{\varepsilon k}^{(\delta_l)} < T_r\} = 0. \tag{2.4.56}$$

It is sufficient to show that relation (2.4.56) will hold for the same sequences $\delta_l \in Z_0, \delta_l \to 0$ as $l \to \infty$ and $0 < T_r \to \infty$ as $r \to \infty$ and for every $l, k, r \geq 1$ if the moments ν'_ε are replaced with the moments $\nu_\varepsilon = \nu'_\varepsilon + \nu''_\varepsilon$. Indeed, using $\bar{\mathfrak{O}}_6$ (b) we get

$$\begin{aligned} &\mathsf{P}\{\alpha_{\varepsilon k}^{(\delta_l)} - c \leq \nu_\varepsilon < \alpha_{\varepsilon k}^{(\delta_l)}, \alpha_{\varepsilon k}^{(\delta_l)} < T_r\} \\ &\quad \leq \mathsf{P}\{\nu''_\varepsilon > c\} + \mathsf{P}\{\nu''_\varepsilon \leq c, \alpha_{\varepsilon k}^{(\delta_l)} - c \leq \nu'_\varepsilon + \nu''_\varepsilon < \alpha_{\varepsilon k}^{(\delta_l)}, \alpha_{\varepsilon k}^{(\delta_l)} < T_r\} \\ &\quad \leq \mathsf{P}\{\nu''_\varepsilon > c\} + \mathsf{P}\{\alpha_{\varepsilon k}^{(\delta_l)} - 2c \leq \nu'_\varepsilon < \alpha_{\varepsilon k}^{(\delta_l)}, \alpha_{\varepsilon k}^{(\delta_l)} < T_r\}. \end{aligned} \tag{2.4.57}$$

Estimate (2.4.57) and the conditions $\bar{\mathfrak{O}}_6$ (a) and (c) imply, in an obvious way, that (2.4.56) holds for the stopping moments ν_ε. □

Note that conditions $\mathcal{A}_{17}$, $\mathfrak{J}_7$, and $\bar{\mathfrak{O}}_6$ also imply that the limiting stopping moment ν_0 is a discontinuity point of the limiting external process $\xi_0(t), t \geq 0$, with probability 1. However, condition $\bar{\mathfrak{O}}_6$ does not require that ν_ε be a discontinuity point of the pre-limiting process $\xi_\varepsilon(t), t \geq 0$ with a positive probability. Conversely, ν_ε can be a point of continuity of the process $\xi_\varepsilon(t), t \geq 0$ with probability 1 for all $\varepsilon > 0$.

It is possible that condition $\bar{\mathfrak{O}}_6$ holds for the processes $\xi_\varepsilon(t), t \geq 0$ and the stopping moments ν_ε, but condition $\bar{\mathfrak{O}}_5$ does not.

For example, let us assume that conditions $\mathcal{A}_{17}$, $\mathfrak{J}_7$, and $\bar{\mathfrak{O}}_6$ hold and, additionally, the following condition holds: **(d)** $\overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\nu''_\varepsilon \neq 0\} = a > 0$. In this case, condition $\bar{\mathfrak{O}}_5$ does not hold.

Indeed, let us assume, for the moment, that condition $\bar{\mathfrak{O}}_5$ holds. Conditions $\mathcal{A}_{17}$, $\bar{\mathfrak{O}}_6$ (a), and $\bar{\mathfrak{O}}_5$ imply that it is possible to choose $T > 0$ and $\sigma > 0$ such that **(e)** $\overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\Delta_{\nu'_\varepsilon}(\xi_\varepsilon(\cdot)) \leq 2\sigma\} + \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\Delta_{\nu_\varepsilon}(\xi_\varepsilon(\cdot)) \leq 2\sigma\} + \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\nu'_\varepsilon \geq T/2\} < a$, where a is the same as in condition **(d)**. Obviously, **(f)** $\{\nu''_\varepsilon \neq 0, \nu''_\varepsilon < c, \Delta_{\nu'_\varepsilon}(\xi_\varepsilon(\cdot)) \geq 2\sigma, \Delta_{\nu_\varepsilon}(\xi_\varepsilon(\cdot)) \geq 2\sigma, \nu'_\varepsilon \leq T/2\} \subseteq \{\Delta_J(\xi_\varepsilon(\cdot), c, T) \geq \sigma\}$ for $0 < c < T/2$. Using **(d)** – **(f)** and condition $\bar{\mathfrak{O}}_6$

(c) we get

$$\begin{aligned}
&\lim_{0<c\to 0}\overline{\lim_{\varepsilon\to 0}}\,\mathsf{P}\{\Delta_J(\xi_\varepsilon(\cdot),c,T)\geq\sigma\}\\
&\quad\geq\lim_{0<c\to 0}\overline{\lim_{\varepsilon\to 0}}(\mathsf{P}\{\nu''_\varepsilon\neq 0\}\\
&\quad+\mathsf{P}\{\nu''_\varepsilon<c,\Delta_{\nu'_\varepsilon}(\xi_\varepsilon(\cdot))\geq 2\sigma,\Delta_{\nu_\varepsilon}(\xi_\varepsilon(\cdot))\geq 2\sigma,\nu'_\varepsilon\leq T/2\}-1) \qquad (2.4.58)\\
&\quad\geq\overline{\lim_{\varepsilon\to 0}}\,\mathsf{P}\{\nu''_\varepsilon\neq 0\}-\lim_{0<c\to 0}\overline{\lim_{\varepsilon\to 0}}\,\mathsf{P}\{\nu''_\varepsilon\geq c\}-\overline{\lim_{\varepsilon\to 0}}\,\mathsf{P}\{\Delta_{\nu'_\varepsilon}(\xi_\varepsilon(\cdot))\leq 2\sigma\}\\
&\quad-\overline{\lim_{\varepsilon\to 0}}\,\mathsf{P}\{\Delta_{\nu_\varepsilon}(\xi_\varepsilon(\cdot))\leq 2\sigma\}-\overline{\lim_{\varepsilon\to 0}}\,\mathsf{P}\{\nu'_\varepsilon\geq T/2\}>0.
\end{aligned}$$

Thus, the condition of **J**-compactness $\mathfrak{J}_7$ does not hold. This contradicts the assumption that condition $\bar{\mathfrak{O}}_5$ holds.

Condition $\mathfrak{D}_2$ can also hold in situations where neither the continuity condition $\mathfrak{C}_3$ nor the condition of asymptotic discontinuity $\bar{\mathfrak{O}}_5$ or $\bar{\mathfrak{O}}_6$ holds.

Let us first consider the case where the stopping moment ν_ε can be represented in the following form for every $\varepsilon\geq 0$: $\nu_\varepsilon=\nu_{\varepsilon\mu_\varepsilon}$. Here (a) $\nu_{\varepsilon n}, n=0,1,\dots$ are non-negative random variables and (b) μ_ε is a random variable that takes the values $0,1,\dots$, (c) the random variables $\nu_{\varepsilon n}, n=0,1,\dots$ and μ_ε are defined on the same probability space (possibly different for different ε).

Lemma 2.4.7. *Let* **(α)** *condition* $\mathfrak{D}_2$ *hold for the processes* $\xi_\varepsilon(t), t\geq 0$ *and the stopping moments* $\nu_{\varepsilon n}$ *for every* $n=0,1,\dots$, *and* **(β)** $\lim_{N\to\infty}\overline{\lim}_{\varepsilon\to 0}\mathsf{P}\{\mu_\varepsilon>N\}=0$. *Then condition* $\mathfrak{D}_2$ *holds for the processes* $\xi_\varepsilon(t), t\geq 0$ *and the stopping moments* ν_ε.

Proof of Lemma 2.4.7. We use the following estimate that holds for any $\delta_l\in Z_0$ and $T_r>0$:

$$\begin{aligned}
&\mathsf{P}\{\alpha^{(\delta_l)}_{\varepsilon k}-c\leq\nu_\varepsilon<\alpha^{(\delta_l)}_{\varepsilon k},\alpha^{(\delta_l)}_{\varepsilon k}<T_r\}\\
&\quad\leq\mathsf{P}\{\mu_\varepsilon>n\}+\sum_{j=0}^{n}\mathsf{P}\{\alpha^{(\delta_l)}_{\varepsilon k}-c\leq\nu_{\varepsilon j}<\alpha^{(\delta_l)}_{\varepsilon k},\alpha^{(\delta_l)}_{\varepsilon k}<T_r,\mu_\varepsilon=j\} \qquad (2.4.59)\\
&\quad\leq\mathsf{P}\{\mu_\varepsilon>n\}+\sum_{j=0}^{n}\mathsf{P}\{\alpha^{(\delta_l)}_{\varepsilon k}-c\leq\nu_{\varepsilon j}<\alpha^{(\delta_l)}_{\varepsilon k},\alpha^{(\delta_l)}_{\varepsilon k}<T_r\}.
\end{aligned}$$

Due to Lemma 2.4.3, it is possible to choose sequences $\delta_l\in Z_0,\delta_l\to 0$ as $l\to\infty$ and $0<T_r\to\infty$ as $r\to\infty$ such that condition $\mathfrak{D}'''_2$ holds, with these sequences, for the processes $\xi_\varepsilon(t), t\geq 0$ and the stopping moments $\nu_{\varepsilon n}$ for every $n=0,1,\dots$. Then, using the conditions **(α)**, **(β)** and estimate (2.4.59) we get

$$\begin{aligned}
&\lim_{0<c\to 0}\overline{\lim_{\varepsilon\to 0}}\,\mathsf{P}\{\alpha^{(\delta_l)}_{\varepsilon k}-c\leq\nu_\varepsilon<\alpha^{(\delta_l)}_{\varepsilon k},\alpha^{(\delta_l)}_{\varepsilon k}<T_r\}\\
&\quad\leq\overline{\lim_{\varepsilon\to 0}}\,\mathsf{P}\{\mu_\varepsilon>N\}+\sum_{n=0}^{N}\lim_{0<c\to 0}\overline{\lim_{\varepsilon\to 0}}\,\mathsf{P}\{\alpha^{(\delta_l)}_{\varepsilon k}-c\leq\nu_{\varepsilon n}<\alpha^{(\delta_l)}_{\varepsilon k},\alpha^{(\delta_l)}_{\varepsilon k}<T_r\} \qquad (2.4.60)\\
&\quad\leq\overline{\lim_{\varepsilon\to 0}}\,\mathsf{P}\{\mu_\varepsilon>N\}\to 0 \text{ as } N\to\infty.
\end{aligned}$$

This completes the proof. □

Let us now give an example in which neither the continuity condition $\mathfrak{C}_3$ nor the discontinuity condition $\bar{\mathfrak{O}}_5$ or $\bar{\mathfrak{O}}_6$ holds but the condition $\mathfrak{D}_2$ does.

Let **(g)** the random variable μ_ε take only two values 0 and 1, i.e., the stopping moment ν_ε can be represented in the form

$$\nu_\varepsilon = \nu_{\varepsilon 0}\chi(\mu_\varepsilon = 0) + \nu_{\varepsilon 1}\chi(\mu_\varepsilon = 1).$$

Let us also assume that **(h)** $(\mu_\varepsilon, \nu_{\varepsilon i}, \xi_\varepsilon(t)), t \in U \Rightarrow (\mu_0, \nu_{0i}, \xi_0(t)), t \in U$ as $\varepsilon \to 0$ for $i = 0, 1$, where U is a subset of $[0, \infty)$, dense in this interval and containing the point 0; **(i)** $0 < p_0 < 1$, where $p_\varepsilon = \mathsf{P}\{\mu_\varepsilon = 0\}$; **(j)** condition $\mathfrak{C}_3$ holds for the processes $\xi_0(t), t \geq 0$ and the stopping moments ν_{00}; **(k)** condition $\bar{\mathfrak{O}}_5$ or $\bar{\mathfrak{O}}_6$ holds for the processes $\xi_\varepsilon(t), t \geq 0$ and the stopping moments $\nu_{\varepsilon 1}$, **(l)** condition $\mathfrak{J}_7$ holds for the processes $\xi_\varepsilon(t), t \geq 0$.

It is clear that condition $\mathcal{A}_{17}$ holds, in this case, for the processes $\xi_\varepsilon(t), t \geq 0$ and the stopping moments ν_ε, as well as the stopping moments $\nu_{\varepsilon 0}$ and $\nu_{\varepsilon 1}$.

Lemmas 2.4.4, 2.4.5, and 2.4.6 imply that, in this case, the conditions of Lemma 2.4.7 hold and, therefore, condition $\mathfrak{D}_2$ holds for the processes $\xi_\varepsilon(t), t \geq 0$, and the stopping moments ν_ε.

However, neither condition $\mathfrak{C}_3$ nor condition $\bar{\mathfrak{O}}_5$ or $\bar{\mathfrak{O}}_6$ holds for the processes $\xi_\varepsilon(t)$, $t \geq 0$ and the stopping moments ν_ε. Indeed, in this case, the limiting stopping moment ν_0 is either a point of continuity of the process $\xi_0(t), t \geq 0$ or a point of discontinuity of this process with probabilities p_0 and $1 - p_0$, respectively. By **(i)**, both probabilities are positive.

The example given above is, in some sense, artificially constructed. However, there is an important class of models, the so-called generalised exceeding processes, where the weakened continuity conditions of the type $\mathfrak{D}$ can be effectively used. Limit theorems for these processes are systematically studied in Chapter 4.

2.5 Iterated weak limits

In this section, we discuss some conditions of weak convergence for randomly stopped càdlàg processes, which are based on the so-called iterated weak limits.

2.5.1. Iterated weak limits. Let, for every $\varepsilon \geq 0$, $\boldsymbol{\xi}_\varepsilon(t) = (\xi_{\varepsilon 1}(t), \dots, \xi_{\varepsilon m}(t)), t \geq 0$ be a càdlàg process with real-valued components and $\boldsymbol{\nu}_\varepsilon = (\nu_{\varepsilon 1}, \dots, \nu_{\varepsilon m})$ be random vectors with non-negative components. Let us also introduce the random vectors $\boldsymbol{\zeta}_\varepsilon = (\xi_{\varepsilon 1}(\nu_{\varepsilon 1}), \dots, \xi_{\varepsilon m}(\nu_{\varepsilon m}))$.

The following question arises. Is it possible to avoid continuity conditions of the types $\mathfrak{C}$ or $\mathfrak{D}$ when proving weak convergence of the random vectors $\boldsymbol{\zeta}_\varepsilon$? In particular, is it possible to prove this if conditions $\mathcal{A}_{20}$ and $\mathfrak{J}_4$ hold but neither the continuity condition $\mathfrak{C}_4$ nor the condition $\mathfrak{D}_3$ does? The examples in the Section 2.1, give a negative answer to this question. However, some weaker statements concerning iterated weak limits of the random vectors $(\xi_{\varepsilon i}(\nu_{\varepsilon i} + c), i = 1, \dots, m)$ as $\varepsilon \to 0$ and then $0 < c \to 0$, can be proved without using any continuity type conditions.

Let $\boldsymbol{\zeta} = (\zeta_1, \dots, \zeta_m)$ and $\boldsymbol{\zeta}_{\varepsilon c} = (\zeta_{\varepsilon 1c}, \dots, \zeta_{\varepsilon ic})$, for $c > 0$ and $\varepsilon \geq 0$, be random variables that take values in $\mathbb{R}_m$.

Definition 2.5.1. The iterated weak convergence of the random variables $\boldsymbol{\zeta}_{\varepsilon c}$ to the random variable $\boldsymbol{\zeta}$ as $\varepsilon \to 0$ and then $0 < c \to 0$ ($\text{w-lim}_{0<c\to 0} \lim_{\varepsilon\to 0} \boldsymbol{\zeta}_{\varepsilon c} = \boldsymbol{\zeta}$), means that **(α)** $\lim_{0<c\to 0} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\zeta_{\varepsilon ic} \leq u_i, i = 1, \dots, m\} = \lim_{0<c\to 0} \underline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\zeta_{\varepsilon ic} \leq u_i, i = 1, \dots, m\}$ $= \mathsf{P}\{\zeta_i \leq u_i, i = 1, \dots, m\}$ for every continuity point $\mathbf{u} = (u_1, \dots, u_m)$ for the limiting distribution function.

Note that in the case where the random vectors $\boldsymbol{\zeta}_{\varepsilon c} = \boldsymbol{\zeta}_\varepsilon, c > 0$ do not depend on the parameter c, the convergence defined above is reduced to the usual weak convergence of the random vectors $\boldsymbol{\zeta}_\varepsilon$ to the random vectors $\boldsymbol{\zeta}$ as $\varepsilon \to 0$.

Let us prove two lemmas that will generalise Lemmas 1.2.5 and 1.2.6.

Suppose that, for every $c > 0$, the random vector $\boldsymbol{\zeta}_{\varepsilon c}$ can be represented in the form of a sum of two random vectors $\boldsymbol{\zeta}'_{\varepsilon c} = (\zeta'_{\varepsilon ic}, i = 1, \dots, m)$ and $\boldsymbol{\zeta}''_{\varepsilon c} = (\zeta''_{\varepsilon ic}, i = 1, \dots, m)$,

$$\boldsymbol{\zeta}_{\varepsilon c} = \boldsymbol{\zeta}'_{\varepsilon c} + \boldsymbol{\zeta}''_{\varepsilon c}. \tag{2.5.1}$$

Lemma 2.5.1. *Let* **(α)** $\text{w-lim}_{0<c\to 0} \lim_{\varepsilon\to 0} \boldsymbol{\zeta}'_{\varepsilon c} = \boldsymbol{\zeta}$, *and* **(β)** $\lim_{0<c\to 0} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{|\boldsymbol{\zeta}''_{\varepsilon c}| > \sigma\} = 0$, $\sigma > 0$. *Then* $\text{w-lim}_{0<c\to 0} \lim_{\varepsilon\to 0} \boldsymbol{\zeta}_{\varepsilon c} = \boldsymbol{\zeta}$.

Proof of Lemma 2.5.1. Let $\mathbf{u} = (u_1, \dots, u_m)$ be an arbitrary continuity point of the distribution function of the random vector $\boldsymbol{\zeta}$. One can always choose sequences of numbers $0 < \sigma_{in} \to 0$ as $n \to \infty$, $i = 1, \dots, m$, such that, for every $n \geq 1$, the point $(u_1 - \sigma_{1n}, \dots, u_m - \sigma_{mn})$ is a continuity point of the distribution function of the random vector $\boldsymbol{\zeta}$. Now, let us use the following estimate:

$$\begin{aligned}
&\mathsf{P}\{\zeta_{\varepsilon i} \leq u_i, i = 1, \dots, m\} \\
&\quad \geq \mathsf{P}\{\zeta'_{\varepsilon i} + |\zeta''_{\varepsilon i}| \leq u_i, i = 1, \dots, m\} \\
&\quad \geq \mathsf{P}\{\zeta'_{\varepsilon i} + \sigma_{in} \leq u_i, |\xi''_{\varepsilon i}| \leq \sigma_{in}, i = 1, \dots, m\} \\
&\quad \geq \mathsf{P}\{\zeta'_{\varepsilon i} \leq u_i - \sigma_{in}, i = 1, \dots, m\} - \sum\nolimits_{1\leq i\leq m} \mathsf{P}\{|\zeta''_{\varepsilon i}| > \sigma_{in}\}.
\end{aligned} \tag{2.5.2}$$

Passing to the limit in (2.5.2) as, first, $\varepsilon \to 0$ and then $0 < c \to 0$, and taking into account the conditions **(α)** and **(β)** of Lemma 2.5.1, we get

$$\begin{aligned}
&\lim_{0<c\to 0} \varliminf_{\varepsilon\to 0} \mathsf{P}\{\zeta_{\varepsilon i} \le u_i, i = 1,\dots,m\} \\
&\quad \ge \lim_{n\to\infty} \lim_{0<c\to 0} \varliminf_{\varepsilon\to 0} \mathsf{P}\{\zeta'_{\varepsilon i} \le u_i - \sigma_{in}, i = 1,\dots,m\} \\
&\quad - \sum\nolimits_{1\le i\le m} \lim_{n\to\infty} \lim_{0<c\to 0} \varlimsup_{\varepsilon\to 0} \mathsf{P}\{|\zeta''_{\varepsilon i}| > \sigma_{in}\} \\
&\quad \ge \lim_{n\to\infty} \mathsf{P}\{\zeta_{0i} \le u_i - \sigma_{in}, i = 1,\dots,m\} \\
&\quad = \mathsf{P}\{\zeta_{0i} \le u_i, i = 1,\dots,m\}.
\end{aligned} \tag{2.5.3}$$

Similarly to (2.5.3), it can be shown that, for an arbitrary continuity point $\mathbf{u} = (u_1,\dots,u_m)$ of the distribution function of the random vector $\boldsymbol{\zeta}$,

$$\lim_{0<c\to 0} \varlimsup_{\varepsilon\to 0} \mathsf{P}\{\zeta_{\varepsilon i} \le u_i, i = 1,\dots,m\} \le \mathsf{P}\{\zeta_{0i} \le u_i, i = 1,\dots,m\}. \tag{2.5.4}$$

Relations (2.5.3) and (2.5.4) prove the assertion of the lemma. □

The second lemma concerns the case where the random vectors $\boldsymbol{\zeta}_{\varepsilon c}$ possess upper and lower approximations $\boldsymbol{\zeta}^{\pm}_{\varepsilon c} = (\zeta^{\pm}_{\varepsilon ic}, i = 1,\dots,m)$ that are random vectors such that the following inequalities hold for every $c > 0$:

$$\zeta^{-}_{\varepsilon ic} \le \zeta_{\varepsilon ic} \le \zeta^{+}_{\varepsilon ic}, i = 1,\dots,m. \tag{2.5.5}$$

Lemma 2.5.2. *Let* **(α)** $\boldsymbol{\zeta}^{\pm}_{\varepsilon c} \Rightarrow \boldsymbol{\zeta}^{\pm}_{0c}$ *as* $\varepsilon \to 0$ *for every* $c > 0$, *and* **(β)** $\boldsymbol{\zeta}^{\pm}_{0c} \Rightarrow \boldsymbol{\zeta}$ *as* $0 < c \to 0$. *Then* $\text{w-lim}_{0<c\to 0} \lim_{\varepsilon\to 0} \boldsymbol{\zeta}_{\varepsilon c} = \boldsymbol{\zeta}$.

Proof of Lemma 2.5.2. Obviously, there is no loss of generality in assuming that the parameter c runs only over a countable number of values $0 < c_n \to 0$ as $n \to \infty$. Let $\mathbf{u} = (u_1,\dots u_m)$ be an arbitrary continuity point of distribution functions of the random vectors $\boldsymbol{\zeta}$ and $\boldsymbol{\zeta}^{\pm}_{0c_n}, n \ge 1$. The set U of such points is dense in $\mathbb{R}_m$.

Taking (2.5.5) into account we get

$$\begin{aligned}
&\lim_{n\to\infty} \varliminf_{\varepsilon\to 0} \mathsf{P}\{\zeta_{\varepsilon ic_n} \le u_i, i = 1,\dots,m\} \\
&\quad \ge \lim_{n\to\infty} \varliminf_{\varepsilon\to 0} \mathsf{P}\{\zeta^{+}_{\varepsilon ic_n} \le u_i, i = 1,\dots,m\} \\
&\quad = \lim_{n\to\infty} \mathsf{P}\{\zeta^{+}_{0ic_n} \le u_i, i = 1,\dots,m\} = \mathsf{P}\{\zeta_{0i} \le u_i, i = 1,\dots,m\}.
\end{aligned} \tag{2.5.6}$$

Similarly to (2.5.6), it can be shown that

$$\lim_{n\to\infty} \varlimsup_{\varepsilon\to 0} \mathsf{P}\{\zeta_{\varepsilon ic_n} \le u_i, i = 1,\dots,m\} \le \mathsf{P}\{\zeta_{0i} \le u_i, i = 1,\dots,m\}. \tag{2.5.7}$$

Relations (2.5.6) and (2.5.7) prove the assertion of the lemma. □

2.5.2. Iterated weak limits for randomly stopped càdlàg processes. Let us now assume that condition $\mathcal{A}_{19}$ holds. As was pointed out in Lemma 2.3.1, conditions $\mathcal{A}_{20}$ and $\mathcal{J}_4$ are sufficient for condition $\mathcal{A}_{19}$ to hold.

Let us introduce the random vectors $\boldsymbol{\zeta}_{\varepsilon c} = (\zeta_{\varepsilon ic}, i = 1, \dots, m)$, where $\zeta_{\varepsilon ic} = \xi_{\varepsilon i}(\nu_{\varepsilon i} + c), i = 1, \dots, m$, and $\boldsymbol{\zeta} = (\zeta_i, i = 1, \dots, m)$, $\zeta_i = \xi_{0i}(\nu_{0i}), i = 1, \dots, m$.

The following theorem is a vector variant of the corresponding statement from Mishura and Silvestrov (1978).

Theorem 2.5.1. *Let condition $\mathcal{A}_{19}$ hold. Then*

$$\mathrm{w-}\lim_{0<c\to 0}\lim_{\varepsilon\to 0} = \boldsymbol{\zeta}_{\varepsilon c} = \boldsymbol{\zeta}_0.$$

Proof of Theorem 2.5.1. Let S_i be the sets in the condition $\mathcal{A}_{19}$. For every $c > 0$, one can always construct a sequence of partitions $0 = z_{i,0,c} < z_{i,1,c} < \dots < z_{i,n,c} < \dots$ of the interval $[0, \infty)$ satisfying the following assumptions: **(a)** $z_{i,n,c} \in S_i$ for all $n \geq 1$ and $i = 1, \dots, m$; **(b)** $c/2 \leq z_{i,n+1,c} - z_{i,n,c} < c$ all $n \geq 1$ and $i = 1, \dots, m$.

Let us define random vectors $\boldsymbol{\zeta}^{\pm}_{\varepsilon c} = (\zeta^{\pm}_{\varepsilon ic}, i = 1, \dots, m)$, where

$$\zeta^{+}_{\varepsilon ic} = \sum_{n=0}^{\infty} \sup_{t\in[z_{i,n+1,c}, z_{i,n+3,c})} \xi_{\varepsilon i}(t)\chi(\nu_{\varepsilon i} \in [z_{i,n,c}, z_{i,n+1,c}))$$

and

$$\zeta^{-}_{\varepsilon ic} = \sum_{n=0}^{\infty} \inf_{t\in[z_{i,n+1,c}, z_{i,n+3,c})} \xi_{\varepsilon i}(t)\chi(\nu_{\varepsilon i} \in [z_{i,n,c}, z_{i,n+1,c})).$$

Obviously, if $\nu_{\varepsilon i} \in [z_{i,n,c}, z_{i,n+1,c})$, then, necessarily, $\nu_{\varepsilon i} + c \in [z_{i,n+1,c}, z_{i,n+3,c})$. Therefore, for any $c > 0$ and $\varepsilon \geq 0$,

$$\zeta^{-}_{\varepsilon ic} \leq \xi_{\varepsilon i}(\nu_{\varepsilon i} + c) \leq \zeta^{+}_{\varepsilon ic}, \ i = 1, \dots, m. \tag{2.5.8}$$

The random vector $\boldsymbol{\xi}^{\pm}_{\varepsilon c}$ has the distribution function given the following formula:

$$\begin{aligned} &\mathsf{P}\{\zeta^{+}_{\varepsilon ic} \leq u_i, i = 1, \dots, m\} \\ &\quad = \sum_{n_1,\dots,n_m=0}^{\infty} \mathsf{P}\{\sup_{t\in[z_{i,n_i+1,c}, z_{i,n_i+3,c})} \xi_{\varepsilon i}(t) \leq u_i, \nu_{\varepsilon i} \in [z_{i,n_i,c}, z_{i,n_i+1,c}), i = 1, \dots, m\}. \end{aligned} \tag{2.5.9}$$

The series in the right-hand side of (2.5.9) converges asymptotically and uniformly in $\varepsilon \to 0$, namely,

$$\begin{aligned} &\lim_{N\to\infty}\overline{\lim_{\varepsilon\to 0}} \sum_{\max(n_1,\dots,n_m)\geq N} \mathsf{P}\{\sup_{t\in[z_{i,n_i+1,c}, z_{i,n_i+3,c})} \xi_{\varepsilon i}(t) \leq u_i, \\ &\qquad \nu_{\varepsilon i} \in [z_{i,n_i,c}, z_{i,n_i+1,c}), i = 1, \dots, m\} \\ &\qquad \leq \lim_{N\to\infty}\overline{\lim_{\varepsilon\to 0}} \sum_{\max(n_1,\dots,n_m)\geq N} \mathsf{P}\{\nu_{\varepsilon i} \in [z_{i,n_i,c}, z_{i,n_i+1,c})\} \\ &\qquad = \lim_{N\to\infty}\overline{\lim_{\varepsilon\to 0}} \sum_{i=1}^{m} \mathsf{P}\{\nu_{\varepsilon i} \geq z_{i,N,c}\} = \lim_{N\to\infty} \sum_{i=1}^{m} \mathsf{P}\{\nu_{0i} \geq z_{i,N,c}\} = 0. \end{aligned} \tag{2.5.10}$$

At the same time, for every $N \geq 1$ by $\mathcal{A}_{19}$,

$$\begin{aligned} \lim_{\varepsilon \to 0} \sum_{n_1,\ldots,n_m=0}^{N} \mathsf{P}\{ \sup_{t \in [z_{i,n+1,c}, z_{i,n+3,c})} \xi_{\varepsilon i}(t) \leq u_i, \nu_{\varepsilon i} \in [z_{i,n,c}, z_{i,n+1,c}), i = 1, \ldots, m\} \\ = \sum_{n_1,\ldots,n_m=0}^{N} \mathsf{P}\{ \sup_{t \in [z_{i,n+1,c}, z_{i,n+3,c})} \xi_{0i}(t) \leq u_i, \nu_{0i} \in [z_{i,n,c}, z_{i,n+1,c}), i = 1, \ldots m\}. \end{aligned} \tag{2.5.11}$$

Relations (2.5.10) and (2.5.11) imply, in an obvious way, that for any $c > 0$,

$$\boldsymbol{\zeta}^{+}_{\varepsilon c} \Rightarrow \boldsymbol{\zeta}^{+}_{0c} \text{ as } \varepsilon \to 0. \tag{2.5.12}$$

Also, by the continuity of the càdlàg process $\boldsymbol{\xi}_0(t), t \geq 0$, from the right,

$$\boldsymbol{\zeta}^{+}_{0c} \Rightarrow \boldsymbol{\zeta} \text{ as } 0 < c \to 0. \tag{2.5.13}$$

So, the conditions **(α)** and **(β)** of Lemma 2.5.2 hold for the random vectors $\boldsymbol{\zeta}^{+}_{\varepsilon c}$ and $\boldsymbol{\zeta}$. Similarly, these conditions can be verified for the random vectors $\boldsymbol{\zeta}^{-}_{\varepsilon c}$ and $\boldsymbol{\zeta}$.

By applying Lemma 2.5.2 to the random vectors $\boldsymbol{\zeta}_{\varepsilon c}$, $\boldsymbol{\xi}^{\pm}_{\varepsilon c}$, and $\boldsymbol{\zeta}$, we get the assertion of the theorem. □

2.5.3. A condition of asymptotic stochastic continuity at a random stopping point. For simplicity, we restrict the consideration to the scalar case, $m = 1$. Let us introduce a condition that can be interpreted as a condition of asymptotic stochastic continuity of the processes $\xi_\varepsilon(t), t \geq 0$ at the random points ν_ε:

$\mathcal{O}_7$: $\lim_{0<c\to 0} \overline{\lim}_{\varepsilon \to 0} \mathsf{P}\{|\xi_\varepsilon(\nu_\varepsilon + c) - \xi_\varepsilon(\nu_\varepsilon)| > \sigma\} = 0, \sigma > 0.$

It was conjectured by Anisimov (1974b) that, assuming that $\mathcal{A}_{17}$ and $\mathcal{J}_7$ hold, condition $\mathcal{O}_7$ becomes necessary and sufficient for the random variables $\xi_\varepsilon(\nu_\varepsilon)$ to weakly converge to $\xi_0(\nu_0)$ as $\varepsilon \to 0$.

Condition $\mathcal{O}_7$ is sufficient, as was shown in Mishura and Silvestrov (1978), even if condition $\mathcal{A}_{18}$ holds (this condition is weaker than $\mathcal{A}_{17}$ and $\mathcal{J}_7$). So, the conditions $\mathcal{A}_{18}$ and $\mathcal{O}_7$ imply that $\xi_\varepsilon(\nu_\varepsilon) \Rightarrow \xi_0(\nu_0)$ as $\varepsilon \to 0$.

This can be simply proved by applying Lemma 2.5.1 to the random variables $\zeta_{\varepsilon c} = \zeta_\varepsilon = \xi_\varepsilon(\nu_\varepsilon)$, $\zeta'_{\varepsilon c} = \xi_\varepsilon(\nu_\varepsilon + c)$ and $\zeta''_{\varepsilon c} = \xi_\varepsilon(\nu_\varepsilon) - \xi_\varepsilon(\nu_\varepsilon + c)$. The condition **(α)** in Lemma 2.5.1 holds due to Theorem 2.5.1, while the condition **(β)** in this lemma coincides with $\mathcal{O}_7$.

Condition $\mathcal{O}_7$ is not, however, necessary, as the following simple example shows. Let, for every $\varepsilon \geq 0$, the process $\xi_\varepsilon(t), t \geq 0$, have two possible realisations $\chi(t \geq 1), t \geq 0$, and $1 - \chi(t \geq 1), t \geq 0$, that can occur with probability $1/2$. Let also $\nu_\varepsilon = 1 - \varepsilon$ for $\varepsilon \geq 0$. Obviously, conditions $\mathcal{A}_{17}$ and $\mathcal{J}_7$ hold. In this case, for every $\varepsilon \geq 0$, the random variable $\xi_\varepsilon(\nu_\varepsilon)$ takes the values 0 or 1 with probability $1/2$. Therefore, the random variables $\xi_\varepsilon(\nu_\varepsilon)$

weakly converge to $\xi_0(\nu_0)$ as $\varepsilon \to 0$. At the same time, $|\xi_\varepsilon(\nu_\varepsilon + c) - \xi_\varepsilon(\nu_\varepsilon)| = 1$ if $\varepsilon < c$ for every $c > 0$. Hence, $\mathcal{O}_7$ does not hold.

A corrected version of the corresponding necessary and sufficient conditions was given in Mishura and Silvestrov (1978). Denote $A_{\varepsilon,c,u} = \{\xi_\varepsilon(\nu_\varepsilon + c) \le u\}$. Evidently,

$$\mathsf{P}\{\xi_\varepsilon(\nu_\varepsilon + c) \le u\} - \mathsf{P}\{\xi_\varepsilon(\nu_\varepsilon) \le u\} = \mathsf{P}\{A_{\varepsilon,c,u} \setminus A_{\varepsilon,0,u}\} - \mathsf{P}\{A_{\varepsilon,0,u} \setminus A_{\varepsilon,c,u}\}. \quad (2.5.14)$$

Let us introduce the following condition that can also be interpreted as a condition of asymptotic weak stochastic continuity of the processes $\xi_\varepsilon(t), t \ge 0$ at the random points ν_ε:

$\mathcal{O}_8$: $\lim_{0<c\to 0} \overline{\lim}_{\varepsilon\to 0} |\mathsf{P}\{A_{\varepsilon,c,u} \setminus A_{\varepsilon,0,u}\} - \mathsf{P}\{A_{\varepsilon,0,u} \setminus A_{\varepsilon,c,u}\}| = 0$ for every u that is a continuity point of the distribution function of the random variable $\xi_0(\nu_0)$.

Let $\mathcal{A}_{18}$ hold. Then condition $\mathcal{O}_8$ is necessary and sufficient for the relation $\xi_\varepsilon(\nu_\varepsilon) \Rightarrow \xi_0(\nu_0)$ as $\varepsilon \to 0$ to hold. The proof follows in an obvious way from Theorem 2.5.1 and relation (2.5.14).

It should be noted, however, that the actual value of assertions that are based on conditions $\mathcal{O}_7$ and $\mathcal{O}_8$ must not be overestimated.

In fact, the reason for studying the weak convergence of compositions $\xi_\varepsilon(\nu_\varepsilon)$ is to simplify conditions that involve jointly the pre-limiting external processes $\xi_\varepsilon(t), t \ge 0$ and the stopping moments ν_ε. A reasonable variant is to use only the joint finite-dimensional distributions of the stopping moments ν_ε and the external processes $\xi_\varepsilon(t), t \ge 0$, i.e., condition $\mathcal{A}_{17}$.

Theorem 2.2.2 gives a well balanced version of conditions that provide weak convergence of random variables $\xi_\varepsilon(\nu_\varepsilon)$ to $\xi_0(\nu_0)$ as $\varepsilon \to 0$. One should supplement $\mathcal{A}_{17}$ with the continuity condition $\mathcal{C}_3$ and the condition of $\mathbf{J}$-compactness $\mathcal{J}_7$ that only involves the external processes $\xi_\varepsilon(t), t \ge 0$. The former continuity condition does involve jointly the limiting stopping moment ν_0 and the limiting external process $\xi_0(t), t \ge 0$. But, this pair, usually, has a much more simple structure than the corresponding pre-limiting stopping moments and the external processes. Condition $\mathcal{C}_3$ can be effectively verified in many cases and covers a significant part of applications.

Theorem 2.4.1 is another example. Here, the continuity condition $\mathcal{C}_3$ is weakened and replaced with condition $\mathcal{D}_2$. A drawback is that one should now use, in addition to condition $\mathcal{A}_{17}$, condition $\mathcal{D}_2$ that is based on joint distributions of the stopping moments ν_ε and moments of large jumps of the external processes $\xi_\varepsilon(t), t \ge 0$. These distributions are also not very complicated. Condition $\mathcal{D}_2$ can be effectively verified in some important cases not covered by condition $\mathcal{C}_3$.

Unfortunately, neither condition $\mathcal{O}_7$ nor $\mathcal{O}_8$ satisfies these requirements. The problem here is caused by a direct use of joint distributions of the random variables $\xi_\varepsilon(\nu_\varepsilon + c)$ and $\xi_\varepsilon(\nu_\varepsilon)$. These joint distributions are involved in a form that makes the conditions $\mathcal{O}_7$ and $\mathcal{O}_8$ too close to the tautology that the relation of weak convergence $\xi_\varepsilon(\nu_\varepsilon) \Rightarrow \xi_0(\nu_0)$

follows from itself. Moreover, conditions $\mathfrak{O}_7$ and $\mathfrak{O}_8$ involve repeated limits, which makes the asymptotic relations appearing in these conditions in a form even more complicated than the assertion of weak convergence, $\xi_\varepsilon(\nu_\varepsilon) \Rightarrow \xi_0(\nu_0)$. This indicates that these conditions can, at most, yield a very preliminary framework for potential proofs in weak limit theorems for compositions $\xi_\varepsilon(\nu_\varepsilon)$.

The same remarks can partially be applied to the conditions $\bar{\mathfrak{O}}_5$ and $\bar{\mathfrak{O}}_6$. These conditions also do not separate external processes and internal stopping moments and directly involve the joint distribution of the random variables $\xi_\varepsilon(\nu_\varepsilon)$ and $\xi_\varepsilon(\nu_\varepsilon - 0)$.

2.6 Scalar compositions of càdlàg processes

In this section, we formulate conditions of weak convergence for compositions of real-valued càdlàg processes. A special attention is paid to conditions that provide weak convergence of compositions on a set dense in the time interval $[0, \infty)$. Such weak convergence is one of necessary conditions for **J**-convergence of compositions of càdlàg stochastic processes.

2.6.1. Weak convergence of compositions on a preassigned set. Let, for every $\varepsilon \geq 0$, $\xi_\varepsilon(t)$, $t \geq 0$ be a real-valued càdlàg process and $\nu_\varepsilon(t)$, $t \geq 0$ be a non-negative and non-decreasing càdlàg process. We call $\xi_\varepsilon(t)$, $t \geq 0$ an *external process* and $\nu_\varepsilon(t)$, $t \geq 0$ a *internal stopping process*. We are interested in their composition $\zeta_\varepsilon(t) = \xi_\varepsilon(\nu_\varepsilon(t))$, $t \geq 0$. This process is also a càdlàg process.

Let U, V, $W \subseteq [0, \infty)$. The following condition of joint weak convergence makes a basis for further consideration:

$\mathcal{A}_{21}^{\mathbf{V}}$: $(\nu_\varepsilon(s), \xi_\varepsilon(t)), (s, t) \in V \times U \Rightarrow (\nu_0(s), \xi_0(t)), (s, t) \in V \times U$ as $\varepsilon \to 0$, where U is a subset of $[0, \infty)$ that is dense in this interval and contains the point 0.

It is useful to note that, under condition $\mathfrak{J}_7$, the set U in $\mathcal{A}_{21}^{\mathbf{V}}$ can be extended to the set $U \cup U_0$, where U_0 is the set of points of stochastic continuity of the processes $\xi_0(t)$, $t \geq 0$. Note that $U \cup U_0$ is $[0, \infty)$ except for at most a countable set, and $0 \in U \cup U_0$.

The following continuity condition also plays a principal role in what follows:

$\mathfrak{C}_5^{\mathbf{W}}$: $\mathsf{P}\{\nu_0(t) \in R[\xi_0(\cdot)]\} = 0$ for $t \in W$.

The following theorem can be found in Silvestrov (1971b, 1972a, 1972e).

Theorem 2.6.1. *Let conditions $\mathcal{A}_{21}^{\mathbf{V}}$, $\mathfrak{J}_7$, and $\mathfrak{C}_5^{\mathbf{W}}$ hold. Then, for the set $S = V \cap W$,*

$$\zeta_\varepsilon(t) = \xi_\varepsilon(\nu_\varepsilon(t)), t \in S \Rightarrow \zeta_0(t) = \xi_0(\nu_0(t)), t \in S \text{ as } \varepsilon \to 0.$$

Proof of Theorem 2.6.1. This theorem is a corollary of Theorem 2.3.3. Indeed, conditions $\mathcal{A}_{21}^{\mathbf{V}}$ and $\mathfrak{C}_5^{\mathbf{W}}$ imply that, for any $n \geq 1$ and points $t_i \in S$, $i = 1, \dots, n$, conditions $\mathcal{A}_{20}$ and $\mathfrak{C}_4$ hold for the n-dimensional vector processes with identical components $\xi_\varepsilon^{(n)}(t) = (\xi_\varepsilon(t), \dots, \xi_\varepsilon(t))$, $t \geq 0$ and the random vectors $(\nu_\varepsilon(t_1), \dots, \nu_\varepsilon(t_n))$.

It is obvious that $\Delta_J(\xi_\varepsilon^{(n)}(\cdot), c, T) = \sqrt{n}\Delta_J(\xi_\varepsilon(\cdot), c, T)$. Hence, condition $\mathfrak{J}_7$ implies that condition $\mathfrak{J}_4$ holds for the processes $\xi_\varepsilon^{(n)}(t)$, $t \geq 0$.

By applying Theorem 2.3.3, we get that for all $t_i \in S$, $i = 1, \dots, n$,

$$(\xi_{\varepsilon i}(\nu_{\varepsilon i}(t_i)), i = 1, \dots, n) \Rightarrow (\xi_{0i}(\nu_{0i}(t_i)), i = 1, \dots, n) \text{ as } \varepsilon \to 0. \tag{2.6.1}$$

Relation (2.6.1) is equivalent to the statement of Theorem 2.6.1. □

2.6.2. Weak convergence on a set dense in $[0, \infty)$. Theorem 2.6.1 implies weak convergence of compositions on a prescribed set $S = V \cap W$. Now, we would like to investigate conditions that would guarantee for the set S to be dense in the interval $[0, \infty)$.

Let V_0 be the set of points of stochastic continuity of the process $\nu_0(t)$, $t \geq 0$, and $V_0' = V_0 \setminus \{0\}$. Due to monotonicity of the processes $\nu_\varepsilon(t)$, if the set V from $\mathcal{A}_{21}^{\mathbf{V}}$ is dense in $[0, \infty)$, then V can be extended to the sets $V \cup V_0'$. The set $V \cup V_0'$ coincides with $[0, \infty)$ except for at most a countable set, namely, the set $\overline{V} \cap \overline{V}_0'$. Note that $0 \in V \cup V_0'$ does not necessarily holds, although $0 \in V \cup V_0'$ if $0 \in V$.

Let also introduce the following continuity condition:

$\mathcal{C}_5^{(\mathbf{w})}$: $\mathsf{P}\{\nu_0(w) \in R[\xi_0(\cdot)]\} = 0$.

Actually, $\mathcal{C}_5^{(\mathbf{w})}$ coincides with condition $\mathcal{C}_5^{\mathbf{W}}$ if the set $W = \{w\}$ contains only the point w.

Denote by W_0 the set of all points $t \geq 0$ satisfying condition $\mathcal{C}_5^{(\mathbf{w})}$. By the definition, $W \subseteq W_0$. As follows from Lemma 2.6.1, which we will formulate below, if condition $\mathcal{C}_5^{\mathbf{W}}$ holds for some set W dense in $[0, \infty)$, then the set W_0 is $[0, \infty)$ except for at most a countable set, namely the set $\overline{W}_0$. Note that $0 \in W_0$ does not necessarily holds, but $0 \in W_0$ if $0 \in W$.

So, if both sets V and W are dense in $[0, \infty)$, then the set $S_0 = (V \cup V_0') \cap W_0$ is $[0, \infty)$ except for at most a countable set, namely $(\overline{V} \cap \overline{V}_0') \cup \overline{W}_0$.

The following theorem is a variant of Theorem 2.6.1.

Theorem 2.6.2. *Let conditions $\mathcal{A}_{21}^{\mathbf{V}}$ and $\mathcal{C}_5^{\mathbf{W}}$ hold for some sets V, W dense in* $[0, \infty)$ *and let also condition $\mathfrak{J}_7$ hold. Then, for the set $S_0 = (V \cup V_0') \cap W_0$,*

$$\zeta_\varepsilon(t) = \xi_\varepsilon(\nu_\varepsilon(t)), t \in S_0 \Rightarrow \zeta_0(t) = \xi_0(\nu_0(t)), t \in S_0 \text{ as } \varepsilon \to 0.$$

Remark 2.6.1. The *set of weak convergence*, S_0, in Theorem 2.6.2 can differ from $[0, \infty)$ by at most a countable set. However, there is no certainty that this set contains some preassigned point $w \in [0, \infty)$ (in particular 0). In order for a point w to be in the set of convergence one should, for example, additionally assume that condition $\mathcal{C}_5^{(\mathbf{w})}$ holds and also require that $w \in V \cup V_0'$.

Now, we give a simple sufficient condition which implies that condition $\mathfrak{C}_5^{\mathbf{W}}$ holds with some set W dense in $[0, \infty)$.

First, let us consider the case where the following analogue of condition $\mathfrak{Q}_1$ holds:

$\mathfrak{Q}_2$: $\xi_0(t) = \xi_0'(t) + \xi_0''(t)$, $t \geq 0$, where (a) $\xi_0'(t), t \geq 0$ is a continuous process; (b) $\xi_0''(t), t \geq 0$ is a stochastically continuous càdlàg process; (c) the processes $\xi_0''(t), t \geq 0$ and $\nu_0(t), t \geq 0$ are independent.

The following lemma directly follows from Lemma 2.2.3.

Lemma 2.6.1. *Suppose that condition* $\mathfrak{Q}_2$ *holds. Then condition* $\mathfrak{C}_5^{\mathbf{W}}$ *holds with the set* $W = [0, \infty)$.

In condition $\mathfrak{Q}_2$, the assumption that the càdlàg process $\xi_0''(t)$, $t \geq 0$ is stochastically continuous can be weakened and replaced with the assumption that, for any point t_k of stochastic discontinuity of this process (the number of such points is always at most countable), $\mathsf{P}\{\nu_0(t) = t_k\} = 0$ for all $t \geq 0$. Of course, if this condition holds for all points t from some set W dense in $[0, \infty)$, then the condition $\mathfrak{C}_5^{\mathbf{W}}$ also holds for this set.

Let us now formulate a necessary and sufficient condition that implies that condition $\mathfrak{C}_5^{\mathbf{W}}$ holds for some set dense in $[0, \infty)$. This condition was introduced in Silvestrov and Teugels (1998) and Silvestrov (2000b):

$\mathcal{E}_1$: $\mathsf{P}\{\nu_0(t') = \nu_0(t'') \in R[\xi_0(\cdot)]\} = 0$ for $0 \leq t' < t'' < \infty$.

Conditions of type $\mathcal{E}$ will be referred to as the second-type continuity conditions, as to to distinguish them from conditions of type $\mathfrak{C}$ that are called first-type continuity conditions.

Let $\tau_{kn}, k \geq 1$, be successive moments of jumps of the process $\xi_0(t), t \geq 0$ with the absolute values of jumps belonging to the interval $[\frac{1}{n}, \frac{1}{n-1})$. Here $n = 1, 2, \ldots$.

We need the following lemma from Silvestrov and Teugels (1998) and Silvestrov (2000b).

Lemma 2.6.2. *The condition* $\mathcal{E}_1$ *is necessary for* $\mathfrak{C}_5^{\mathbf{W}}$ *to hold for some set W dense in* $[0, \infty)$. *It is sufficient for* $\mathfrak{C}_5^{\mathbf{W}}$ *to hold for some set W that coincides with* $[0, \infty)$ *except for at most a countable set.*

Proof of Lemma 2.6.2. Suppose $\mathcal{E}_1$ does not hold, i.e., the probability corresponding to this condition is positive for some $t' < t''$. Then the set W in condition $\mathfrak{C}_5^{\mathbf{W}}$ can not contain any point t from the interval $[t', t'']$. Indeed, $\{\nu_0(t') = \nu_0(t'') \in R[\xi_0(\cdot)]\} \subseteq \{\nu_0(t) \in R[\xi_0(\cdot)]\}$ for any $t' \leq t \leq t''$ and, therefore, $0 < \mathsf{P}\{\nu_0(t') = \nu_0(t'') \in R[\xi_0(\cdot)]\} \leq \mathsf{P}\{\nu_0(t) \in R[\xi_0(\cdot)]\}$. So, W can not be dense in $[0, \infty)$. This implies the necessity statement.

To prove sufficiency, let us suppose that $\mathcal{E}_1$ holds but the set $\overline{W}$ of points t for which $\mathsf{P}\{\nu_0(t) \in R[\xi_0(\cdot)]\} > 0$ is infinite and not countable. Then, at least for some k, n, and m, the set Z_{knm} of points t with $\mathsf{P}(B_{tkn}) > 1/m$ has to be infinite. Here $B_{tkn} = \{\nu_0(t) = \tau_{kn}\}$.

Since $\mathcal{E}_1$ holds, $\mathsf{P}\{B_{t'kn} \cap B_{t''kn}\} = 0$ for $t', t'' \in Z_{knm}, t' \neq t''$. Let us take $l > m$ and choose points $t_1 < \ldots < t_l$ from the set Z_{knm}. Then $\mathsf{P}(\cup_{1 \le r \le l} B_{t_r kn}) = \sum_{1 \le r \le l} \mathsf{P}(B_{t_r kn}) > \frac{l}{m} > 1$. This is impossible. Therefore, the set $\overline{W}$ must be empty, finite, or countable. □

Remark 2.6.2. The statement of Lemma 2.6.2 is valid if condition $\mathcal{E}_1$ is weakened by assuming that the relation $\mathsf{P}\{\nu_0(t') = \nu_0(t'') \in R[\xi_0(\cdot)]\} = 0$ holds only for $0 < t' < t'' < \infty$.

Indeed, take some sequence $0 < t_n \to 0$ as $n \to \infty$. Lemma 2.6.2 can be applied to the processes $\xi_0(t), t \ge 0$ and $\nu_0(t + t_n), t \ge 0$. In this way, it can be proved that for every $n \ge 1$ there exists a set W_n dense in interval $[t_n, \infty)$ such that condition $\mathcal{C}_5^{\mathbf{W_n}}$ holds. In this case, condition $\mathcal{C}_5^{\mathbf{W}}$ holds for the set $W = \cup_{n \ge 1} W_n$ that is dense in $[0, \infty)$.

We can reformulate now Theorem 2.6.2 in the following equivalent form, more suitable for applications.

Theorem 2.6.3. *Let condition $\mathcal{A}_{21}^{\mathbf{V}}$ hold for some set V dense in* $[0, \infty)$*, and also conditions $\mathcal{J}_7$ and $\mathcal{E}_1$ hold. Then, for the set $S_0 = (V \cup V_0') \cap W_0$,*

$$\zeta_\varepsilon(t) = \xi_\varepsilon(\nu_\varepsilon(t)), t \in S_0 \Rightarrow \zeta_0(t) = \xi_0(\nu_0(t)), t \in S_0 \text{ as } \varepsilon \to 0.$$

Remark 2.6.3. The set of weak convergence, S_0, in Theorem 2.6.2 differs from $[0, \infty)$ by at most a countable set. However, it is possible that this set does not contain a given point $w \in [0, \infty)$ (in particular, the point 0). In order to include a point w in the set S_0, it is sufficient to assume that condition $\mathcal{C}_5^{(\mathbf{w})}$ holds and $w \in V \cup V_0'$.

2.6.3. The continuity condition $\mathcal{E}_1$. As follows from the definition of the set $R[\xi_0(\cdot)]$, condition $\mathcal{E}_1$ can be reformulated in the following equivalent form:

$\mathcal{E}_1'$: $\mathsf{P}\{\nu_0(t') = \nu_0(t'') = \tau_{kn}\} = 0$ for $k, n = 1, 2, \ldots, 0 \le t' < t'' < \infty$.

Denote by $\mathbf{D}_+^{-1}$ the space of functions of the form $y^{-1}(t) = \inf(s \ge 0 \colon y(s) \ge t), t \ge 0$, where $y(t)$, $t \ge 0$ belongs to the space $\mathbf{D}_+$ of non-negative and non-decreasing càdlàg functions. It is easy to show that $\mathbf{D}_+^{-1}$ is the space of functions that take values in the interval $[0, \infty]$, non-decreasing, and continuous from the left. Let $R[y^{-1}(\cdot)]$ denote the set of points of discontinuity for the function $y^{-1}(t)$, $t \ge 0$. The set $R[y^{-1}(\cdot)]$ is an empty, finite, or countable subset of the interval $[0, \infty)$.

A very important property of the set $R[y^{-1}(\cdot)]$ is that **(a)** a point $t \in R[y^{-1}(\cdot)]$ if and only if the point $y^{-1}(t) = z_t'$ is the left endpoint of the interval $[z_t', z_t'']$ of positive length such that $y(s) < t$ for $s < z_t'$, $y(s) = t$ for $s \in [z_t', z_t'')$, and $y(s) > t$ for $s > z_t''$. The right endpoint $z_t'' = \inf(s \ge 0 \colon y(s) > t)$, so the case $z_t'' = \infty$ is also admitted.

Let us now introduce the inverse exceeding level process $\nu_0^{-1}(t) = \inf(s \ge 0 \colon \nu_0(s) \ge t)$, $t \ge 0$. By the definition, the process $\nu_0^{-1}(t)$, $t \ge 0$ has realisations that belong to the space $\mathbf{D}_+^{-1}$ with probability 1. The corresponding set $R[\nu_0^{-1}(\cdot)]$ is an empty, finite, or countable subset of random points of the interval $[0, \infty)$. These points can be enumerated in the same way as it was done for the points of the set $R[\xi_0(\cdot)]$ in condition $\mathcal{E}_1'$.

The following condition was introduced in Silvestrov (1974):

$\mathcal{E}_2$: $\mathsf{P}\{t \in R[\xi_0(\cdot)] \text{ for } t \in R[\nu_0^{-1}(\cdot)]\} = 0.$

In virtue of property **(a)**, condition $\mathcal{E}_2$ can also be reformulated in the following equivalent form given in Silvestrov (1974):

$\mathcal{E}_2'$: $\mathsf{P}\{\nu_0(t') = \nu_0(t'') \in R[\xi_0(\cdot)] \text{ for some } 0 \leq t' < t'' < \infty\} = 0.$

The following lemma is from Silvestrov (2000b).

Lemma 2.6.3. *The conditions $\mathcal{E}_1$ and $\mathcal{E}_2$ are equivalent.*

Proof of Lemma 2.6.3. Let $A[t', t'']$ denote the event $\{\nu_0(t') = \nu_0(t'') \in R[\xi_0(\cdot)]\}$, $t' < t''$. Let Z be some countable set dense in $[0, \infty)$ and containing 0. Also, denote by $A = \{\nu_0(t') = \nu_0(t'') \in R[\xi_0(\cdot)] \text{ for some } 0 \leq t' < t'' < \infty\}$. It is obvious that the event A occurs if and only if there exist points $t' < t''$, $t', t'' \in Z$ such that the event $A[t', t'']$ occurs, that is, $A = \cup_{t',t'' \in Z, t' < t''} A[t', t'']$.

Due to **(a)**, condition $\mathcal{E}_2$ is equivalent to the equality $\mathsf{P}(A) = 0$. Obviously, $A[t', t''] \subseteq A$ for all $t' < t''$. Therefore, $\mathsf{P}(A[t', t'']) = 0$ for all $t' < t''$ if $\mathsf{P}(A) = 0$. Hence, $\mathcal{E}_2$ implies $\mathcal{E}_1$.

Condition $\mathcal{E}_1$ means that $\mathsf{P}(A[t', t'']) = 0$ for all $t' < t''$. In this case, $\mathsf{P}(A) \leq \sum_{t',t'' \in Z, t' < t''} \mathsf{P}(A[t', t'']) = 0$. Therefore, $\mathcal{E}_1$ implies $\mathcal{E}_2$. □

It should be noted that, despite the equivalence of conditions $\mathcal{E}_1$ and $\mathcal{E}_2$, condition $\mathcal{E}_1$ is essentially simpler than condition $\mathcal{E}_2$ or its equivalent version $\mathcal{E}_2'$. As a matter of fact, $\mathcal{E}_2$ and $\mathcal{E}_2'$ deal with the whole internal stopping process $\nu_0(t), t \geq 0$, under the probability sign in these conditions. At the same time, $\mathcal{E}_1$ involves only values of the internal stopping process $\nu_0(t)$ at points $t' < t''$ under the probability sign in this condition. The latter form of probabilities is much simpler than any of the first ones.

Obviously, condition $\mathcal{Q}_2$ is sufficient for condition $\mathcal{E}_1$ to hold.

Instead of $\mathcal{Q}_2$, one can assume that, for each $k, n = 1, 2, \ldots$, the random variable τ_{kn} and the process $\nu_0(t)$, $t \geq 0$ are independent, and the distribution functions of τ_{kn} and $\nu_0(t)$ do not have common points of discontinuity for each $k, n \geq 1$ and $t \geq 0$. In this case, $\mathcal{E}_1$ also holds with the set $W = [0, \infty)$. Note that the process $\nu_0(t)$, $t \geq 0$ can depend on values of jumps of the processes $\xi_0(t)$, $t \geq 0$, as well as on the continuous component of this process.

But $\mathcal{E}_1$ can also hold in situations where no assumptions on independence are made. For example, the following condition obviously implies $\mathcal{E}_1$:

$\mathcal{I}_1$: $\nu_0(t)$, $t \geq 0$ is an a.s. strictly increasing process.

In some applications, e.g., random sums and extrema with random sample size, internal stopping processes can have the following structure: $\nu_\varepsilon(t) = t\nu_\varepsilon$, $t \geq 0$, where ν_ε are nonnegative random variables. The corresponding limiting random variable ν_0 is usually assumed to be positive with probability 1. In this case, the corresponding limiting process $\nu_0(t) = t\nu_0$, $t \geq 0$ satisfies condition $\mathfrak{I}_1$.

2.6.4. Conditions of weak convergence of compositions of càdlàg processes, based on M-topology. The proofs of Theorem 2.6.1, Theorems 2.6.2 and 2.6.3 were based on applying Theorem 2.3.3 to vector processes with n identical components, $\xi^{(n)}_\varepsilon(t) = (\xi_\varepsilon(t), \ldots, \xi_\varepsilon(t))$, $t \geq 0$ and the random vectors $(\nu_\varepsilon(t_1), \ldots, \nu_\varepsilon(t_n))$, where $t_i \in S$, $i = 1, \ldots, n$. By the reasons explained in Subsection 2.3.4, the condition of **J**-compactness $\mathcal{J}_4$ can be replaced in Theorem 2.3.3 with the condition of **M**-compactness $\mathcal{M}_5$. The corresponding statement is given in Theorem 2.3.5. Since the process $\xi^{(n)}_\varepsilon(t)$, $t \geq 0$ has the identical components, condition $\mathcal{M}_5$ reduces to the condition of **M**-compactness $\mathcal{M}_6$ for the scalar processes $\xi_\varepsilon(t)$, $t \geq 0$.

So, the condition of **J**-compactness $\mathcal{J}_7$ can be replaced in Theorems 2.6.1, 2.6.2, and 2.6.3 with the condition of **M**-compactness $\mathcal{M}_6$.

2.6.5. Weakened continuity conditions. Let us also formulate conditions of weak convergence, which are based on results of Section 2.4. The following Theorems 2.6.4, 2.6.5, and Lemma 2.6.4 are new.

We use below the notations introduced in Section 2.4, in particular, denote by $\alpha^{(\delta)}_{\varepsilon k}$ the successive moments of jumps of the process $\xi_\varepsilon(t)$, $t \geq 0$, with absolute values of jumps greater than or equal to δ.

Let us introduce the following condition:

$\mathcal{D}^{\mathbf{W}}_4$: There exist a sequence $\delta_l \in Z_0, \delta_l \to 0$ as $l \to \infty$ and a sequence $0 < T_r \to \infty$ as $r \to \infty$ such that, for every $l, k, r \geq 1$, $\lim_{0<c\to 0} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\alpha^{(\delta_l)}_{\varepsilon k} - c \leq \nu_\varepsilon(t) < \alpha^{(\delta_l)}_{\varepsilon k}, \alpha^{(\delta_l)}_{\varepsilon k} < T_r\} = 0$ for $t \in W$.

This condition replaces condition $\mathcal{C}^{\mathbf{W}}_5$.

Theorem 2.6.4. *Let conditions $\mathcal{A}^{\mathbf{V}}_{21}$, $\mathcal{J}_7$, and $\mathcal{D}^{\mathbf{W}}_4$ hold. Then, for the set $S = V \cap W$,*

$$\zeta_\varepsilon(t) = \xi_\varepsilon(\nu_\varepsilon(t)), t \in S \Rightarrow \zeta_0(t) = \xi_0(\nu_0(t)), t \in S \text{ as } \varepsilon \to 0.$$

Proof of Theorem 2.6.4. This theorem is a corollary of Theorem 2.4.2. Indeed, conditions $\mathcal{A}^{\mathbf{V}}_{21}$ and $\mathcal{D}^{\mathbf{W}}_4$ imply that, for any $n \geq 1$ and points $t_i \in S$, $i = 1, \ldots, n$, conditions $\mathcal{A}_{20}$ and $\mathcal{D}'''_3$ hold for the n-dimensional vector processes with the identical components $\xi^{(n)}_\varepsilon(t) = (\xi_\varepsilon(t), \ldots, \xi_\varepsilon(t))$, $t \geq 0$, and the random vectors $(\nu_\varepsilon(t_1), \ldots, \nu_\varepsilon(t_n))$.

It is obvious that $\Delta_J(\xi^{(n)}_\varepsilon(\cdot), c, T) = \sqrt{n}\Delta_J(\xi_\varepsilon(\cdot), c, T)$. So, condition $\mathcal{J}_7$ implies that condition $\mathcal{J}_4$ holds for the processes $\xi^{(n)}_\varepsilon(t)$, $t \geq 0$.

By applying Theorem 2.4.2, we get that for all $t_i \in S$, $i = 1, \ldots, n$,

$$(\xi_\varepsilon(\nu_\varepsilon(t_i)), i = 1, \ldots, n) \Rightarrow (\xi_0(\nu_0(t_i)), i = 1, \ldots, n) \text{ as } \varepsilon \to 0. \tag{2.6.2}$$

Relation (2.6.2) is equivalent to the statement of Theorem 2.6.4. □

Lemma 2.4.4 yields that, if condition $\mathcal{A}_{21}^{\mathbf{V}}$ and $\mathfrak{J}_7$ hold, then condition $\mathfrak{C}_5^{\mathbf{W}}$ implies that condition $\mathfrak{D}_4^{\mathbf{S}}$ holds. Here $S = V \cap W$.

Also note that condition $\mathfrak{C}_5^{\mathbf{W}}$ in Theorem 2.6.1 and condition $\mathfrak{D}_4^{\mathbf{W}}$ in Theorem 2.6.4 can be reduced to weaker conditions. Namely, the set W can be replaced with the set $S = V \cap W$ in these conditions. This does not change the statements of the theorems.

This shows that Theorem 2.6.4 can be regarded as an extension of Theorem 2.6.1.

Let us now introduce the following condition:

$\mathfrak{F}_2$: There exist sequences of $\delta_l \in Z_0$, $\delta_l \to 0$ as $l \to \infty$ and $0 < T_r \to \infty$ as $r \to \infty$ such that, for every $l, k, r \geq 1$, $\lim_{0<c\to 0} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\alpha_{\varepsilon k}^{(\delta_l)} - c \leq \nu_\varepsilon(t'), \nu_\varepsilon(t'') < \alpha_{\varepsilon k}^{(\delta_l)}, \alpha_{\varepsilon k}^{(\delta_l)} < T_r\} = 0$ for all $0 \leq t' < t'' < \infty$.

The following lemma is an analogue of Lemma 2.6.2.

Lemma 2.6.4. *The condition $\mathfrak{F}_2$ is necessary for $\mathfrak{D}_4^{\mathbf{W}}$ to hold for some set W dense in* $[0, \infty)$*, and sufficient for $\mathfrak{D}_4^{\mathbf{W}}$ to hold for some set W which is* $[0, \infty)$ *except for at most a countable set.*

Proof of Lemma2.6.4. Denote $B_{\varepsilon cklr,t} = \{\alpha_{\varepsilon k}^{(\delta_l)} - c \leq \nu_\varepsilon(t) < \alpha_{\varepsilon k}^{(\delta_l)}, \alpha_{\varepsilon k}^{(\delta_l)} < T_r\}$. Suppose $\mathfrak{F}_2$ does not hold, that is, the iterated limit of probabilities in this condition is positive for some $l, k, r \geq 1$ and $t' < t''$. Then the set W in condition $\mathfrak{D}_4^{\mathbf{W}}$ can not contain any point t of the interval $[t', t'']$. Indeed, $B_{\varepsilon cklr,t'} \cap B_{\varepsilon cklr,t''} \subseteq B_{\varepsilon cklr,t}$ for any $t' \leq t \leq t''$ and, therefore, $0 < \lim_{0<c\to 0} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}(B_{\varepsilon cklr,t'} \cap B_{\varepsilon cklr,t''}) \leq \lim_{0<c\to 0} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}(B_{\varepsilon cklr,t})$. Thus, W can not be dense in $[0, \infty)$, which implies the statement of necessity.

To prove sufficiency, let us suppose that $\mathfrak{F}_2$ holds but the set $\overline{W}$ of points t for which $\lim_{0<c\to 0} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}(B_{\varepsilon cklr,t}) > 0$ for some $k, l, r \geq 1$ is infinite and not countable. Then, at least for some k, l, r and $m \geq 1$, the set Z_{klrm} of points t for which $\lim_{0<c\to 0} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}(B_{\varepsilon cklr,t}) > \frac{1}{m}$ has to be infinite. Since condition $\mathfrak{F}_2$ holds, we have **(a)** $\lim_{0<c\to 0} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}(B_{\varepsilon cklr,t'} \cap B_{\varepsilon cklr,t''}) = 0$ for t', $t'' \in Z_{klrm}$, $t' \neq t''$. Let us take $n > m$ and choose points $t_1 < \ldots < t_n$ in the set Z_{klrm}. Relations **(a)** imply the following relation: **(b)** $\lim_{0<c\to 0} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}(\cup_{1\leq r\leq n} B_{\varepsilon cklr,t_r}) \geq \lim_{0<c\to 0} \overline{\lim}_{\varepsilon\to 0}(\sum_{1\leq r\leq n} \mathsf{P}(B_{\varepsilon cklr,t_r}) - \sum_{1\leq r'<r''\leq n} \mathsf{P}(B_{\varepsilon cklr,t_{r'}} \cap B_{\varepsilon cklr,t_{r''}})) = \lim_{0<c\to 0} \overline{\lim}_{\varepsilon\to 0} \sum_{1\leq r\leq n} \mathsf{P}(B_{\varepsilon cklr,t_r}) \geq \frac{n}{m} > 1$. This is impossible. Therefore, the set $\overline{W}$ must be empty, finite, or countable. □

Remark 2.6.4. The statement of Lemma 2.6.4 is valid if condition $\mathfrak{F}_2$ is weakened by assuming that the asymptotic relation in this condition holds only for $0 < t' < t'' < \infty$.

Indeed, take some sequence $0 < t_n \to 0$ as $n \to \infty$. Lemma 2.6.2 can be applied to the process $\xi_\varepsilon(t), t \geq 0$ and $\nu_\varepsilon(t + t_n), t \geq 0$, and this will prove that for every $n \geq 1$ there exists a set W_n dense in the interval $[t_n, \infty)$ such that condition $\mathfrak{D}_4^{\mathbf{W_n}}$ holds. In this case, condition $\mathfrak{D}_4^{\mathbf{W}}$ holds for set $W = \cup_{n\geq 1} W_n$ that is dense in $[0, \infty)$.

The following lemma shows a connection between conditions $\mathcal{F}_2$ and $\mathcal{E}_1$.

Lemma 2.6.5. *Let condition* $\mathcal{A}_{21}^{\mathbf{V}}$ *hold for some* V *that is dense in* $[0,\infty)$. *Let also condition* $\mathcal{J}_7$ *hold. Then condition* $\mathcal{E}_1$ *implies condition* $\mathcal{F}_2$.

Proof of Lemma 2.6.5. Condition $\mathcal{E}_1$ implies that condition $\mathcal{C}_5^{(\mathbf{w})}$ holds for some set W which is $[0,\infty)$ except for at most a countable set. Due to monotonicity of the processes $\nu_\varepsilon(t)$, the set V in condition $\mathcal{A}_{21}^{\mathbf{V}}$ can be extended to the set $V \cup V_0'$ that, again, differs from $[0,\infty)$ in except at most a countable set. Condition $\mathcal{C}_5^{(\mathbf{w})}$ holds for every point $w \in S$, where $S = (V \cup V_0') \cap W$. This implies, due to Lemma 2.4.4, that condition $\mathcal{D}_4^{(\mathbf{w})}$ holds for every point $w \in S$. It remains to note that the set S is $[0,\infty)$ except for at most a countable set and then to apply Lemma 2.6.4. □

Now, introduce a condition that is actually condition $\mathcal{D}_4^{\mathbf{W}}$ for the case where set W contains only one point w,

$\mathcal{D}_4^{(\mathbf{w})}$: There exist a sequence of $\delta_l \in Z_0, \delta_l \to 0$ as $l \to \infty$ and a sequence $0 < T_r \to \infty$ as $r \to \infty$ such that, for every $l,k,r \geq 1$, $\lim_{0<c\to 0} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\alpha_{\varepsilon k}^{(\delta_l)} - c \leq \nu_\varepsilon(w) < \alpha_{\varepsilon k}^{(\delta_l)}, \alpha_{\varepsilon k}^{(\delta_l)} < T_r\} = 0$.

Let W_0' denote the set of all points $w \geq 0$ that satisfy condition $\mathcal{D}_4^{(\mathbf{w})}$. According to Lemma 2.6.4, if condition $\mathcal{D}_4^{\mathbf{W}}$ holds for some set W dense in $[0,\infty)$, then the set W_0' is $[0,\infty)$ except for at most a countable set.

We can now formulate an analogue of Theorem 2.6.3 with condition $\mathcal{E}_1$ replaced with condition $\mathcal{F}_2$.

Theorem 2.6.5. *Let condition* $\mathcal{A}_{21}^{\mathbf{V}}$ *hold for some set* V *dense in* $[0,\infty)$, *and let also conditions* $\mathcal{J}_7$ *and* $\mathcal{F}_2$ *be fulfilled. Then, for the set* $S_0 = (V \cup V_0') \cap W_0'$,

$$\zeta_\varepsilon(t) = \xi_\varepsilon(\nu_\varepsilon(t)), t \in S_0 \Rightarrow \zeta_0(t) = \xi_0(\nu_0(t)), t \in S_0 \text{ as } \varepsilon \to 0.$$

Note that the set of weak convergence S_0 in Theorem 2.6.5 is $[0,\infty)$ except for at most a countable set.

Remark 2.6.5. Condition $\mathcal{F}_2$ does not necessarily imply that a given point w belongs to the set S_0. In order for a point w to be in the set S_0, it is sufficient to assume that condition $\mathcal{D}_4^{(\mathbf{w})}$ holds and $w \in V \cup V_0'$.

2.6.6. The time interval $[0,T]$. The results concerning weak convergence of compositions of càdlàg processes obtained so far deal with processes defined on the semi-infinite interval $[0,\infty)$. These results can also be obtained in the case where internal stopping càdlàg processes are defined on a finite interval $[0,T]$.

So, let $\xi_\varepsilon(t), t \geq 0$ be a real-valued càdlàg process and $\nu_\varepsilon(t), t \in [0,T]$ a non-negative and non-decreasing càdlàg process. We will consider their composition $\zeta_\varepsilon(t) = \xi_\varepsilon(\nu_\varepsilon(t))$, $t \in [0,T]$.

Note, first of all, that we can always continue the internal stopping process to the interval $[0, \infty)$ by the following formula:

$$\nu_\varepsilon(t) = \begin{cases} \nu_\varepsilon(t), & \text{if } 0 \le t \le T, \\ \nu_\varepsilon(T), & \text{if } t \ge T. \end{cases} \tag{2.6.3}$$

The case of weak convergence on a preassigned set is simple. Note that the sets V and W in conditions $\mathcal{A}_{21}^{\mathbf{V}}$ and $\mathcal{C}_5^{\mathbf{W}}$ are arbitrary subsets of the interval $[0, \infty)$. Here, these sets should be chosen so that $V, W \subseteq [0, T]$. In this case, it is obvious that $S = V \cap W \subseteq [0, T]$.

Taking into consideration the remarks made above we can conclude that conditions $\mathcal{A}_{21}^{\mathbf{V}}$, $\mathcal{C}_5^{\mathbf{W}}$, and $\mathcal{J}_7$, as well as Theorem 2.6.1, do not require any changes, except for an additional assumption that $V, W \subseteq [0, T]$. With these minor changes, Theorem 2.6.1 is valid for the composition of the càdlàg processes $\xi_\varepsilon(t)$, $t \ge 0$ and $\nu_\varepsilon(t)$, $t \in [0, T]$.

The situation with conditions of weak convergence on a set dense in $[0, T]$ is more complicated.

A direct application of the results of Subsections 2.6.1 – 2.6.5 to the processes $\xi_\varepsilon(t)$, $t \ge 0$ and $\nu_\varepsilon(t)$, $t \ge 0$ defined by formula (2.6.3) has a certain side effect. The assumption that the set of weak convergence V is dense in the interval $[0, \infty)$ would automatically imply that $T \in V$. However, it could be convenient to avoid an automatic use of this assumption. In such a case, we should repeat the analysis of the conditions and reformulate the results for a finite interval in the same way as it was done for the semi-infinite interval $[0, \infty)$.

Let V_0 be the set of points of stochastic continuity of the process $\nu_0(t)$, $t \in [0, T]$. Instead of the set $V_0' = V_0 \setminus \{0\}$, consider the set $V_0'' = V_0 \setminus \{0, T\}$. In the first case, the endpoint 0 of the interval $[0, \infty)$ is excluded, whereas in the second case, both endpoints of the interval $[0, 1]$, 0 and T, are excluded.

Since the processes $\nu_\varepsilon(t)$ is monotone, if the set V in $\mathcal{A}_{21}^{\mathbf{V}}$ is dense in $[0, T]$, V can be extended to the set $V \cup V_0''$. The set $V \cup V_0''$ is $[0, T]$ except for at most a countable set, namely the set $\overline{V} \cap \overline{V}_0''$. Note that it might happen that $0, T \in V \cup V_0''$. But $0, T \in V \cup V_0'$ if $0, T \in V$.

In this case, we also introduce W_0 as a set of all points $t \in [0, T]$ such that $\mathsf{P}\{\nu_0(t) \in R[\xi_0(\cdot)]\} = 0$. As follows from the remarks above, if condition $\mathcal{C}_5^{\mathbf{W}}$ holds for some set W dense in $[0, T]$, then the set W_0 is $[0, T]$ except for at most a countable set, namely the set $\overline{W}_0$. Note there is no guarantee that $0, T \in W_0$. But $0, T \in W_0$ if $0, T \in W$, i.e., conditions $\mathcal{C}_5^{(\mathbf{0})}$ and $\mathcal{C}_5^{(\mathbf{T})}$ hold.

So, if both sets V and W are dense in $[0, \infty)$, then the set $S_0 = (V \cup V_0'') \cap W_0$ is $[0, T]$ except for at most a countable set, namely $(\overline{V} \cap \overline{V}_0'') \cup \overline{W}_0$.

Taking into consideration the remarks above it is easy to see that Theorem 2.6.2 does not need any changes save for the assumption that the sets V, W are dense in $[0, T]$ (instead of $[0, \infty)$), with the corresponding changes in the definition of the set S_0 above.

The set of convergence S_0 in the new variant of Theorem 2.6.2 is $[0, T]$ except for at most a countable set. However, there is no guarantee that this set contains a preassigned point $w \in [0, T]$. For a point w to be in the set of convergence, one should make, in the conditions of Theorem 2.6.2, an additional assumption that $w \in S_0$, that is to require that $w \in V \cup V_0''$ and $w \in W$. In particular, $0, T \in S_0$ if $0, T \in V, W$.

Condition $\mathcal{E}_1$ requires an obvious change. It should be replaced with the following truncated version of this condition:

$\mathcal{E}_3$: $\mathsf{P}\{\nu_0(t') = \nu_0(t'') \in R[\xi_0(\cdot)]\} = 0$ for $0 \leq t' < t'' \leq T$.

Analogously to Lemma 2.6.2, it can be shown that condition $\mathcal{E}_3$ is necessary for $\mathcal{C}_5^{\mathbf{W}}$ to hold for some set W, which is everywhere dense in $[0, T]$, and sufficient for $\mathcal{C}_5^{\mathbf{W}}$ to hold for some set W, which is $[0, T]$ except for, perhaps, some finite or countable set.

Taking in consideration the remarks made above we see that Theorem 2.6.3 remains the same except for the assumption that the set V is dense in $[0, T]$, the change in the definition of the set S_0 described above, and the replacement of condition $\mathcal{E}_1$ with condition $\mathcal{E}_3$.

2.6.7. The time interval $(0, \infty)$. In the same way as above, the results given in Subsections 2.2.1 – 2.6.5 can be carried over to the case of the semi-infinite interval $(0, \infty)$. Here we will choose the sets V, $W \subseteq (0, \infty)$. We can directly use the set V_0 of points of stochastic continuity of the process $\nu_\varepsilon(t)$, $t \in (0, \infty)$ in the definition of the sets of convergence S and S_0. In condition $\mathcal{E}_1$, the assumption $0 \leq t' < t'' < \infty$ should be replaced with the assumption $0 < t' < t'' < \infty$. Finally, if in addition, $\nu_0(t) > 0$ with probability 1 for every $t > 0$, then the condition of $\mathbf{J}$-compactness $\mathcal{J}_7$ can also be weakened. The corresponding $\mathbf{J}$-compactness relation in this condition should be required to hold for every finite interval $[T', T'']$, where $0 < T' < T'' < \infty$.

With these changes, Theorems 2.6.1, 2.6.2, and 2.6.3 hold for the composition of càdlàg processes $\xi_\varepsilon(t)$, $t \geq 0$ and $\nu_\varepsilon(t)$, $t > 0$.

2.6.8. Random variables $\xi_\varepsilon(\nu_\varepsilon(t-0))$. Sometimes it is useful to include, in the statement of weak convergence of the compositions $\xi_\varepsilon(\nu_\varepsilon(t))$, $t \in S_0$, the random variables $\xi_\varepsilon(\nu_\varepsilon(t-0))$, $t \in \hat{S}_0$, where $\hat{S}_0$ is a subset of $[0, \infty)$. Here it is assumed that $\nu_\varepsilon(0-0) = \nu_\varepsilon(0)$.

For this to hold, one should include the random variable $\nu_\varepsilon(t-0)$, $t \in \hat{V}$ in the relation of weak convergence in condition $\mathcal{A}_{21}^{\mathbf{V}}$. Here $\hat{V}$ is a dense subset of $[0, \infty)$.

One should also add, in condition $\mathcal{C}_5^{\mathbf{W}}$, the assumption that $\mathcal{C}_5^{(\mathbf{w})}$ holds for the limiting process $\xi_0(t)$, $t \geq 0$ and the random variable $\nu_0(w-0)$ for $w \in \hat{W}$. Here, $\hat{W}$ is a dense subset of $[0, \infty)$. The set $\hat{W}_0$ should be introduced as a set of all $w \geq 0$ satisfying $\mathcal{C}_5^{(\mathbf{w})}$ for the process $\xi_0(t)$, $t \geq 0$ and the random variable $\nu_0(w-0)$. The set of weak convergence is $\hat{S}_0 = (\hat{V} \cup V_0') \cap \hat{W}_0$.

Another way would be to add, in condition $\mathcal{D}_4^{\mathbf{W}}$, the assumption that condition $\mathcal{D}_4^{(\mathbf{w})}$ holds for the processes $\xi_\varepsilon(t)$, $t \geq 0$ and the random variables $\nu_\varepsilon(w-0)$ for $w \in \hat{W}$. Here

$\hat{W}$ is a dense subset of $[0, \infty)$. The set $\hat{W}'_0$ should be introduced as a set of all $w \geq 0$ for which $\mathcal{D}_4^{(w)}$ holds for the processes $\xi_\varepsilon(t)$, $t \geq 0$ and the random variables $\nu_\varepsilon(w - 0)$. The set of weak convergence is $\hat{S}_0 = (\hat{V} \cup V'_0) \cap \hat{W}'_0$.

If condition $\mathcal{E}_1$ is used, then the same condition must be used and the random variables $\nu_0(t)$ must be replaced with $\nu_0(t - 0)$. If condition $\mathcal{F}_2$ is used, then the same condition should also be required, with the random variables $\nu_\varepsilon(t)$ being replaced with $\nu_\varepsilon(t - 0)$.

With the changes in the conditions, which were described above, the joint weak convergence of random variables $\xi_\varepsilon(\nu_\varepsilon(t)), t \in S_0$ and $\xi_\varepsilon(\nu_\varepsilon(t - 0)), t \in \hat{S}_0$ can be proved.

2.6.9. Non-monotone internal stopping processes. The requirement that $\nu_\varepsilon(t)$, $t \geq 0$ be a non-decreasing process is not essential in conditions $\mathcal{C}_5^{\mathbf{W}}$ and $\mathcal{D}_4^{\mathbf{W}}$ and, therefore, in Theorems 2.6.1, 2.6.2, and 2.6.4. These theorems also hold if the only assumptions made are that the random variables $\nu_\varepsilon(t)$ are non-negative for all $t \geq 0$. Of course, this does not guarantee, in this case, that the composition $\xi_\varepsilon(\nu_\varepsilon(t))$, $t \geq 0$ is a càdlàg process.

The requirement that $\nu_\varepsilon(t)$, $t \geq 0$ is a non-decreasing process is essential in conditions $\mathcal{E}_1$ and $\mathcal{F}_2$. Therefore, Theorems 2.6.3 and 2.6.5 do require the monotonicity assumption on the internal stopping processes.

2.7 Vector compositions of càdlàg processes

In this section, we formulate conditions for weak convergence of vector compositions of càdlàg processes. The results are similar to those obtained for one-dimensional compositions.

2.7.1. Weak convergence of vector compositions on a preassigned set. Let, for each $\varepsilon > 0$, $\boldsymbol{\xi}_\varepsilon(t) = (\xi_{\varepsilon i}(t), i = 1, \ldots, m)$, $t \geq 0$ be an m-dimensional càdlàg process with real-valued components and $\boldsymbol{\nu}_\varepsilon(t) = (\nu_{\varepsilon i}(t), i = 1, \ldots, m)$, $t \geq 0$ be an m-dimensional càdlàg processes with non-negative and non-decreasing components. We call $\boldsymbol{\xi}_\varepsilon(t)$, $t \geq 0$ an *external process* and $\boldsymbol{\nu}_\varepsilon(t)$, $t \geq 0$ a *internal stopping process*. We are interested in the *vector composition* $\boldsymbol{\zeta}_\varepsilon(t) = (\xi_{\varepsilon i}(\nu_{\varepsilon i}(t)), i = 1, \ldots, m)$, $t \geq 0$, which is also an m-dimensional càdlàg processes with real-valued components.

Let $V, U, W \subseteq [0, \infty)$. The following conditions are vector versions of the conditions $\mathcal{A}_{21}^{\mathbf{V}}$ and $\mathcal{C}_5^{\mathbf{W}}$:

$\mathcal{A}_{22}^{\mathbf{V}}$: $(\boldsymbol{\nu}_\varepsilon(s), \boldsymbol{\xi}_\varepsilon(t)), (s, t) \in V \times U \Rightarrow (\boldsymbol{\nu}_0(s), \boldsymbol{\xi}_0(t)), (s, t) \in V \times U$ as $\varepsilon \to 0$, where U is a subset of $[0, \infty)$ that is dense in this interval and contains the point 0.

and

$\mathcal{C}_6^{\mathbf{W}}$: $\mathsf{P}\{\nu_{0i}(t) \in R[\xi_{0i}(\cdot)]\} = 0$ for $t \in W$, $i = 1, \ldots, m$.

It is useful to note that, under $\mathcal{A}_{22}^{\mathbf{V}}$ and $\mathfrak{J}_4$ or $\mathfrak{J}_8$, the set U, in condition $\mathcal{A}_{19}^{\mathbf{V}}$, can be enlarged to the set $U \cup U_0$. Here U_0 is the set of points of stochastic continuity of the process $\boldsymbol{\xi}_0(t)$, $t \geq 0$. Note that $U \cup U_0$ is $[0, \infty)$ except for at most a countable set.

The following theorem from Silvestrov (19721b, 1972b, 1972e) is a vector analogue of Theorem 2.6.1.

Theorem 2.7.1. *Let conditions $\mathcal{A}_{22}^{\mathbf{V}}$, $\mathfrak{J}_4$, and $\mathfrak{C}_6^{\mathbf{W}}$ hold. Then, for the set $S = V \cap W$,*

$$\boldsymbol{\zeta}_\varepsilon(t), t \in S \Rightarrow \boldsymbol{\zeta}_0(t), t \in S \text{ as } \varepsilon \to 0.$$

Proof of Theorem 2.7.1. As in the one-dimensional case, Theorem 2.7.1 is a corollary of Theorem 2.3.3. Indeed, conditions $\mathcal{A}_{22}^{\mathbf{V}}$ and $\mathfrak{C}_6^{\mathbf{W}}$ imply that, for any $m, n \geq 1$, and points $t_{ij} \in S$, $i = 1, \ldots, m$, $j = 1, \ldots, n$, conditions $\mathcal{A}_{20}$ and $\mathfrak{J}_4$ hold for the mn-dimensional vector processes $\boldsymbol{\xi}_\varepsilon^{(mn)}(t) = (\xi_{\varepsilon ij}(t), i = 1, \ldots, m, j = 1, \ldots, n)$, $t \geq 0$ with the components $\xi_{\varepsilon ij}(t) = \xi_{\varepsilon i}(t)$, $t \geq 0$, and the random vectors $(\nu_{\varepsilon i}(t_{ij}), i = 1, \ldots, m, j = 1, \ldots, n)$.

It is obvious that $\Delta_J(\boldsymbol{\xi}_\varepsilon^{(mn)}(\cdot), c, T) = \sqrt{n}\Delta_J(\boldsymbol{\xi}_\varepsilon(\cdot), c, T)$. So, condition $\mathfrak{J}_4$, assumed for the processes $\boldsymbol{\xi}_\varepsilon(t)$, $t \geq 0$, implies that condition $\mathfrak{J}_4$ holds also for the processes $\boldsymbol{\xi}_\varepsilon^{(mn)}(t)$, $t \geq 0$.

By applying Theorem 2.3.3, we get that, for all $t_{ij} \in S$, $i = 1, \ldots, m$, $j = 1, \ldots, n$,

$$\begin{aligned} &(\xi_{\varepsilon i}(\nu_{\varepsilon i}(t_{ij})), i = 1, \ldots, m, j = 1, \ldots, n) \\ &\quad \Rightarrow (\xi_{0i}(\nu_{0i}(t_{ij})), i = 1 \ldots, m, j = 1, \ldots, n) \text{ as } \varepsilon \to 0. \end{aligned} \tag{2.7.1}$$

Relation (2.7.1) is equivalent to the statement of Theorem 2.7.1. □

Theorem 2.7.1 can be improved. The condition of **J**-compactness $\mathfrak{J}_4$ can be replaced with the weaker condition $\mathfrak{J}_8$.

Theorem 2.7.2. *Let conditions $\mathcal{A}_{22}^{\mathbf{V}}$, $\mathfrak{J}_8$, and $\mathfrak{C}_6^{\mathbf{W}}$ hold. Then, for the set $S = V \cap W$,*

$$\boldsymbol{\zeta}_\varepsilon(t), t \in S \Rightarrow \boldsymbol{\zeta}_0(t), t \in S \text{ as } \varepsilon \to 0.$$

Proof of Theorem 2.7.2. The proof repeats the proof of Theorem 2.7.1 with the only one change. The reference to Theorem 2.3.3 should be replaced with the reference to Theorem 2.3.4. □

2.7.2. Weak convergence of vector compositions on a set dense in $[0, \infty)$. The corresponding results are analogous to those given for the one-dimensional case in Section 2.6.

Let us denote by V_0 the set of points of stochastic continuity of the process $\boldsymbol{\nu}_0(t)$, $t \geq 0$, and $V_0' = V_0 \setminus \{0\}$. Recall that $\nu_{\varepsilon i}(t)$, $t \geq 0$ is a non-decreasing càdlàg process for every $i = 1, \ldots, m$ and $\varepsilon \geq 0$. Thus, if the set V in $\mathcal{A}_{22}^{\mathbf{V}}$ is dense in $[0, \infty)$, then V can be extended to the set $V \cup V_0'$. Note that $V \cup V_0'$ is $[0, \infty)$ except for at most a countable set, namely the set $\overline{V} \cap \overline{V}_0'$. Note, there is no guarantee that $0 \in V \cup V_0'$. But this is so if $0 \in V$.

Let us introduce the continuity condition:

$\mathfrak{C}_6^{(\mathbf{w})}$: $\mathsf{P}\{\nu_{0i}(w) \in R[\xi_{0i}(\cdot)]\} = 0$ for $i = 1, \dots, m$.

Actually, $\mathfrak{C}_6^{(\mathbf{w})}$ coincides with condition $\mathfrak{C}_6^{\mathbf{W}}$ for the set $W = \{w\}$ that contains only one point w.

For every $i = 1, \dots, m$, denote by W_{0i} the set of all points $t \geq 0$ such that $\mathsf{P}\{\nu_{0i}(t) \in R[\xi_{0i}(\cdot)]\} = 0$. Let also $W_0 = \cap_{i=1}^{m} W_{0i}$. As follows from Lemma 2.6.2, if condition $\mathfrak{C}_6^{\mathbf{W}}$ holds for some set W dense in $[0, \infty)$, then the set $\overline{W}_{0i}$ is empty, finite, or countable for every $i = 1, \dots, m$. So, W_0 is $[0, \infty)$ except for at most a countable set, namely the set $\overline{W}_0$. Note there is no guarantee that $0 \in W_0$. But this is so if $0 \in W$, i.e., condition $\mathfrak{C}_6^{(\mathbf{0})}$ holds.

If both sets V and W are dense in $[0, \infty)$, then the set $S_0 = (V \cup V_0') \cap W_0$ is $[0, \infty)$ except for at most a countable set, namely $(\overline{V} \cap \overline{V}_0') \cup \overline{W}_0$.

The following theorem is a vector analogue of Theorem 2.6.3.

Theorem 2.7.3. *Let conditions $\mathcal{A}_{22}^{\mathbf{V}}$, $\mathfrak{J}_4$, and $\mathfrak{C}_6^{\mathbf{W}}$ hold for some sets V, W dense in* $[0, \infty)$. *Then for the set $S_0 = (V \cup V_0') \cap W_0$,*

$$\zeta_\varepsilon(t), t \in S_0 \Rightarrow \zeta_0(t), t \in S_0 \text{ as } \varepsilon \to 0.$$

Remark 2.7.1. The set of weak convergence S_0 in Theorem 2.6.2 is $[0, \infty)$ except for at most a countable set. However, there is no guarantee that this set contains a preassigned point $w \in [0, \infty)$, in particular the point 0. In order to include a point w in the set of convergence, one should assume that condition $\mathfrak{C}_6^{(\mathbf{w})}$ holds and also to require that $w \in V \cup V_0'$.

Let us also formulate a variant of Theorem 2.7.3, in which the **J**-compactness condition $\mathfrak{J}_4$ is replaced with the weaker condition $\mathfrak{J}_8$.

Theorem 2.7.4. *Let conditions $\mathcal{A}_{22}^{\mathbf{V}}$, $\mathfrak{J}_8$, and $\mathfrak{C}_6^{\mathbf{W}}$ hold for some sets V, W dense in* $[0, \infty)$. *Then, for the set $S_0 = (V \cup V_0') \cap W_0$,*

$$\zeta_\varepsilon(t), t \in S_0 \Rightarrow \zeta_0(t), t \in S_0 \text{ as } \varepsilon \to 0.$$

The following condition is a vector analogue of the condition $\mathfrak{Q}_2$:

$\mathfrak{Q}_3$: $\xi_0(t) = \xi_0'(t) + \xi_0''(t)$, $t \geq 0$, where (a) $\xi_0'(t), t \geq 0$ is a continuous process; (b) $\xi_0''(t), t \geq 0$ is a stochastically continuous càdlàg process; (c) the processes $\xi_0''(t), t \geq 0$ and $\mathbf{v}_0(t)$, $t \geq 0$ are independent.

The following lemma is a vector analogue of Lemma 2.6.1. This follows from Lemma 2.2.3.

Lemma 2.7.1. *Suppose that condition $\mathfrak{Q}_3$ holds. Then condition $\mathfrak{C}_6^{\mathbf{W}}$ holds with the set* $W = [0, \infty)$.

It is useful to note that condition $\mathfrak{Q}_3$ can be replaced, in Lemma 2.7.1 by the following weaker condition:

$\mathfrak{Q}_4$: $\xi_0(t) = \xi'_0(t) + \xi''_0(t)$, $t \geq 0$, where (a) $\xi'_0(t), t \geq 0$ is a continuous process; (b) $\xi''_0(t), t \geq 0$ is a stochastically continuous càdlàg process; (c) for every $i = 1, \ldots, m$, the processes $\xi''_{0i}(t), t \geq 0$ and $\nu_{0i}(t)$, $t \geq 0$ are independent.

A vector analogue of the condition $\mathcal{E}_1$ is the following condition:

$\mathcal{E}_4$: $\mathsf{P}\{\nu_{0i}(t') = \nu_{0i}(t'') \in R[\xi_{0i}(\cdot)]\} = 0$ for $0 \leq t' < t'' < \infty$, $i = 1, \ldots, m$.

Lemma 2.7.2. *The condition $\mathcal{E}_4$ is necessary for $\mathfrak{C}_6^{\mathbf{W}}$ to hold for some set W, which is dense in* $[0, \infty)$*, and sufficient for $\mathfrak{C}_6^{\mathbf{W}}$ to hold for some set W, which is* $[0, \infty)$ *except for at most a countable set.*

Proof of Lemma 2.7.2. Lemma 2.7.2 directly follows from Lemma 2.6.2, since condition $\mathcal{E}_4$ implies that condition $\mathcal{E}_1$ holds for the processes $\xi_{0i}(t), t \geq 0$ and $\nu_{0i}(t), t \geq 0$, for every $i = 1, \ldots, m$. Thus the set W_{0i} is empty, finite, or countable for every $i = 1, \ldots, m$. That is why $W_0 = \cap_{i=1}^m W_{0i}$ is $[0, \infty)$ except for at most a countable set. □

The following theorem is a vector analogue of Theorem 2.6.3.

Theorem 2.7.5. *Let condition $\mathcal{A}_{22}^{\mathbf{V}}$ hold for some set V dense in* $[0, \infty)$*, and also conditions $\mathfrak{J}_4$ and $\mathcal{E}_4$ hold. Then, for the set $S_0 = (V \cup V'_0) \cap W_0$,*

$$\zeta_\varepsilon(t), t \in S_0 \Rightarrow \zeta_0(t), t \in S_0 \text{ as } \varepsilon \to 0.$$

Condition $\mathfrak{J}_4$ in the Theorems 2.7.5 can be replaced with the weaker condition $\mathfrak{J}_8$.

Theorem 2.7.6. *Let condition $\mathcal{A}_{22}^{\mathbf{V}}$ hold for some set V dense in* $[0, \infty)$*, and also conditions $\mathfrak{J}_8$ and $\mathcal{E}_4$ hold. Then, for the set $S_0 = (V \cup V'_0) \cap W_0$,*

$$\zeta_\varepsilon(t), t \in S_0 \Rightarrow \zeta_0(t), t \in S_0 \text{ as } \varepsilon \to 0.$$

Note that the set of weak convergence S_0 in Theorems 2.7.5 and 2.7.6 is $[0, \infty)$ except for at most a countable set.

The condition $\mathfrak{Q}_3$ as well as $\mathfrak{Q}_4$ is sufficient for $\mathcal{E}_4$ to hold with the set $W = [0, \infty)$.

But condition $\mathcal{E}_4$ can also hold if no assumptions about independence are made. For example, the following condition obviously implies $\mathcal{E}_4$:

$\mathfrak{I}_2$: $\nu_{0i}(t), t \geq 0$ is an a.s. strictly increasing process for every $i = 1, \ldots, m$.

2.7.3. Conditions for weak convergence of compositions of càdlàg processes, based on M-topology. The proofs of Theorems 2.7.1, 2.7.3, and 2.7.5 are based on applying Theorem 2.3.3. Similarly, the proofs of Theorems 2.7.2, 2.7.4, and 2.7.6 are based on application of Theorem 2.3.4.

For the reasons explained in Subsection 2.3.4, the condition of **J**-compactness $\mathfrak{J}_8$ can be replaced, in Theorems 2.3.3 and 2.3.4, with the condition of **M**-compactness $\mathcal{M}_5$. The corresponding statement is given in Theorem 2.3.5. As in the proof of Theorem 2.7.1, one should apply Theorem 2.3.5 to the process $\boldsymbol{\xi}_\varepsilon^{(mn)}(t)$, $t \geq 0$ with the components $\xi_{\varepsilon ij}(t) = \xi_{\varepsilon i}(t)$, $t \geq 0$ for $j = 1, \ldots, n$ and $i = 1, \ldots, m$. In this case, condition $\mathcal{M}_5$ for the processes $\boldsymbol{\xi}_\varepsilon^{(mn)}(t)$, $t \geq 0$ reduces to the same condition $\mathcal{M}_5$ for the processes $\boldsymbol{\xi}_\varepsilon(t)$, $t \geq 0$.

2.7.4. Weakened continuity conditions. Let us also formulate conditions of weak convergence, based on the results of Section 2.4. Theorems 2.7.7 and 2.7.9 given below are new results.

We use the notations introduced in that section, in particular, let $\alpha_{\varepsilon ik}^{(\delta)}$ be the successive moments of jumps of the process $\xi_{\varepsilon i}(t)$, $t \geq 0$, at which the absolute values of the jumps are greater than or equal to δ.

Let us introduce a condition that replaces condition $\mathcal{C}_6^{\mathbf{W}}$,

$\mathcal{D}_5^{\mathbf{W}}$: There exist a sequence $\delta_l \in Z_0, \delta_l \to 0$ as $l \to \infty$ and a sequence $0 < T_r \to \infty$ as $r \to \infty$ such that for every $l, k, r \geq 1$ and $i = 1, \ldots, m$, $\lim_{0<c\to 0} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\alpha_{\varepsilon ik}^{(\delta_l)} - c \leq \nu_\varepsilon(t) < \alpha_{\varepsilon ik}^{(\delta_l)}, \alpha_{\varepsilon ik}^{(\delta_l)} < T_r\} = 0$ for $t \in W$.

Theorem 2.7.7. *Let conditions $\mathcal{A}_{22}^{\mathbf{V}}$, $\mathfrak{J}_4$, and $\mathcal{D}_5^{\mathbf{W}}$ hold. Then for the set $S = V \cap W$,*

$$\zeta_\varepsilon(t), t \in S \Rightarrow \zeta_0(t), t \in S \text{ as } \varepsilon \to 0.$$

Proof of Theorem 2.7.7. Theorem 2.7.7 is a simple corollary of Theorem 2.4.2. Indeed, let us choose arbitrary $m, n \geq 1$, and points $t_{ij} \in S$, $i = 1, \ldots, m$, $j = 1, \ldots, n$. Conditions $\mathcal{A}_{22}^{\mathbf{V}}$ and $\mathcal{D}_5^{\mathbf{W}}$ imply that conditions $\mathcal{A}_{20}$ and $\mathcal{D}_3$ hold for the mn-dimensional vector processes $\boldsymbol{\xi}_\varepsilon^{(mn)}(t) = (\xi_{\varepsilon ij}(t), i = 1, \ldots, m, j = 1, \ldots, n)$, $t \geq 0$ with the components $\xi_{\varepsilon ij}(t) = \xi_{\varepsilon i}(t)$, $t \geq 0$, and the random vectors $(\nu_{\varepsilon i}(t_{ij}), i = 1, \ldots, m, j = 1, \ldots, n)$.

It is obvious that $\Delta_J(\boldsymbol{\xi}_\varepsilon^{(mn)}(\cdot), c, T) = \sqrt{n}\Delta_J(\boldsymbol{\xi}_\varepsilon(\cdot), c, T)$. Thus, condition $\mathfrak{J}_4$, assumed for the processes $\boldsymbol{\xi}_\varepsilon(t)$, $t \geq 0$, implies that condition $\mathfrak{J}_4$ holds also for the processes $\boldsymbol{\xi}_\varepsilon^{(mn)}(t)$, $t \geq 0$.

By applying Theorem 2.4.2, we get that, for all $t_{ij} \in S$, $i = 1, \ldots, m$, $j = 1, \ldots, n$,

$$\begin{aligned}&(\xi_{\varepsilon i}(\nu_{\varepsilon i}(t_{ij})), i = 1, \ldots, m, j = 1, \ldots, n) \\ &\quad \Rightarrow (\xi_{0i}(\nu_{0i}(t_{ij})), i = 1 \ldots, m, j = 1, \ldots, n) \text{ as } \varepsilon \to 0.\end{aligned} \tag{2.7.2}$$

Relation (2.7.2) is equivalent to the statement of Theorem 2.7.7. □

Let us also formulate a variant of Theorem 2.7.7, in which the condition of **J**-compactness $\mathfrak{J}_4$ is replaced with the weaker **J**-compactness condition $\mathfrak{J}_8$.

Theorem 2.7.8. *Let conditions $\mathcal{A}_{22}^{\mathbf{V}}$, $\mathfrak{J}_8$, and $\mathfrak{D}_5^{\mathbf{W}}$ hold. Then, for the set $S = V \cap W$,*

$$\zeta_\varepsilon(t), t \in S \Rightarrow \zeta_0(t), t \in S \text{ as } \varepsilon \to 0.$$

Proof of Theorem 2.7.8. The proof repeats the proof of Theorem 2.7.7 with only one change. The reference to Theorem 2.4.2 should be changed to the reference to Theorem 2.4.3. □

Introduce now a continuity condition that replaces condition $\mathcal{E}_4$,

$\mathcal{F}_3$: There exist a sequence $\delta_l \in Z_0, \delta_l \to 0$ as $l \to \infty$ and a sequence $0 < T_r \to \infty$ as $r \to \infty$ such that, for every $l, k, r \geq 1$ and $i = 1, \ldots, m$, $\lim_{0<c\to 0} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\alpha_{\varepsilon ik}^{(\delta_l)} - c \leq \nu_{\varepsilon i}(t'), \nu_{\varepsilon i}(t'') < \alpha_{\varepsilon ik}^{(\delta_l)}, \alpha_{\varepsilon ik}^{(\delta_l)} < T_r\} = 0$ for $0 \leq t' < t'' < \infty$.

The following lemma is a direct corollary of Lemma 2.6.4.

Lemma 2.7.3. *The condition $\mathcal{F}_3$ is necessary for $\mathfrak{D}_5^{\mathbf{W}}$ to hold for some set W, dense in $[0, \infty)$, and sufficient for $\mathfrak{D}_5^{\mathbf{W}}$ to hold for some set W which is $[0, \infty)$ except for at most a countable set.*

The following condition coincides with condition $\mathfrak{D}_5^{\mathbf{W}}$ for the case where set W contains only one point w:

$\mathfrak{D}_5^{(\mathbf{w})}$: There exist a sequence $\delta_l \in Z_0, \delta_l \to 0$ as $l \to \infty$ and a sequence $0 < T_r \to \infty$ as $r \to \infty$ such that, for every $l, k, r \geq 1$ and $i = 1, \ldots, m$, $\lim_{0<c\to 0} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\alpha_{\varepsilon ik}^{(\delta_l)} - c \leq \nu_{\varepsilon i}(w) < \alpha_{\varepsilon ik}^{(\delta_l)}, \alpha_{\varepsilon ik}^{(\delta_l)} < T_r\} = 0$.

Let W_0' denote the set of all points $w \geq 0$ that satisfy condition $\mathfrak{D}_5^{(\mathbf{w})}$. As follows from Lemma 2.7.3, if condition $\mathfrak{D}_5^{\mathbf{W}}$ holds for some set W, dense in $[0, \infty)$, then the set W_0' is $[0, \infty)$ except for at most a countable set, namely the set $\overline{W}_0$. Note that there is no guarantee that $0 \in W_0'$. But this is so if $0 \in W$, i.e., condition $\mathfrak{C}_6^{(\mathbf{0})}$ holds.

If both sets V and W are dense in $[0, \infty)$, then the set $S_0 = (V \cup V_0') \cap W_0$ is $[0, \infty)$ except for at most a countable set, namely $(\overline{V} \cap \overline{V}_0') \cup \overline{W}_0$.

We now give an analogue of Theorem 2.7.5, where condition $\mathcal{E}_4$ is replaced with condition $\mathcal{F}_3$.

Theorem 2.7.9. *Let condition $\mathcal{A}_{22}^{\mathbf{V}}$ hold for some sets V that are dense in $[0, \infty)$, and let also conditions $\mathfrak{J}_4$ and $\mathcal{F}_3$ hold. Then, for the set $S_0 = (V \cup V_0') \cap W_0'$,*

$$\zeta_\varepsilon(t), t \in S_0 \Rightarrow \zeta_0(t), t \in S_0 \text{ as } \varepsilon \to 0.$$

Condition $\mathfrak{F}_3$ does not guarantee that a point w belongs to the set S_0. In order for a particular point w to be in the set S_0, it is sufficient to assume that condition $\mathfrak{D}_5^{(\mathbf{w})}$ holds and $w \in V \cup V_0'$.

Let us also formulate an analogue of Theorem 2.7.9, where the condition of **J**-compactness $\mathfrak{J}_4$ is replaced with the weaker **J**-compactness condition $\mathfrak{J}_8$.

Theorem 2.7.10. *Let condition $\mathcal{A}_{22}^{\mathbf{V}}$ hold for some set V dense in* $[0,\infty)$*, and also conditions $\mathfrak{J}_8$ and $\mathfrak{F}_3$ hold. Then, for the set $S_0 = (V \cup V_0') \cap W_0'$,*

$$\boldsymbol{\zeta}_\varepsilon(t), t \in S_0 \Rightarrow \boldsymbol{\zeta}_0(t), t \in S_0 \text{ as } \varepsilon \to 0.$$

Note that the set of weak convergence S_0 in Theorems 2.7.9 and 2.7.10 is $[0,\infty)$ except for at most a countable set.

2.7.5. The time interval $[0, T]$. Subsection 2.6.6 contains remarks concerning weak convergence of compositions of scalar càdlàg processes defined on a finite interval. These remarks are still true in the case of vector compositions of càdlàg processes.

In this case, we are interested in the vector composition $\boldsymbol{\zeta}_\varepsilon(t) = (\xi_{\varepsilon i}(\nu_{\varepsilon i}(t)), i = 1,\dots,m)$, $t \in [0,T]$. Here $\boldsymbol{\xi}_\varepsilon(t) = (\xi_{\varepsilon i}(t), i = 1,\dots,m)$, $t \geq 0$ is a vector càdlàg process with real-valued components and $\mathbf{v}_\varepsilon(t) = (\nu_{\varepsilon i}(t), i = 1,\dots,m), t \in [0,T]$ is a vector càdlàg process with non-negative and non-decreasing components.

As in the case of scalar processes, one can always continue internal stopping process to the interval $[0,\infty)$ by the following formula:

$$\mathbf{v}_\varepsilon(t) = \begin{cases} \mathbf{v}_\varepsilon(t), & \text{if } 0 \leq t \leq T, \\ \mathbf{v}_\varepsilon(T), & \text{if } t \geq T. \end{cases} \tag{2.7.3}$$

The sets V and W are arbitrary subsets of the interval $[0,\infty)$ in conditions $\mathcal{A}_{22}^{\mathbf{V}}$ and $\mathfrak{C}_6^{\mathbf{W}}$. Here, these sets should be chosen such that $V, W \subseteq [0,T]$. In this case, it is obvious that $S = V \cap W \subseteq [0,T]$.

Taking into consideration the remarks made above we can conclude that the conditions $\mathcal{A}_{22}^{\mathbf{V}}$, $\mathfrak{C}_6^{\mathbf{W}}$, and $\mathfrak{J}_4$, $\mathfrak{J}_8$, as well as Theorems 2.7.1 and 2.7.2, do not require any changes, except for an additional assumption that V, $W \subseteq [0,T]$. With these minor changes, Theorems 2.7.1 and 2.7.2 are valid for the vector composition of the càdlàg processes $\boldsymbol{\xi}_\varepsilon(t)$, $t \geq 0$, and $\mathbf{v}_\varepsilon(t)$, $t \in [0,T]$.

A direct application of Theorems 2.7.3 and 2.7.4 to the processes $\boldsymbol{\xi}_\varepsilon(t)$, $t \geq 0$ and $\mathbf{v}_\varepsilon(t)$, $t \geq 0$, defined by formula (2.7.3), has some side effect. The assumption that the set of weak convergence V is dense in the interval $[0,\infty)$ would automatically imply that the point $T \in V$. However, it can be convenient to avoid an automatic use of this assumption.

Let V_0 be the set of points of stochastic continuity of the process $\mathbf{v}_0(t)$, $t \in [0,T]$. Instead of the set $V_0' = V_0 \setminus \{0\}$, we consider the set $V_0'' = V_0 \setminus \{0, T\}$. In the first case, the endpoint 0 of the interval $[0,\infty)$ is excluded, whereas in the second case, the two endpoints 0 and T of the interval $[0,T]$ are excluded.

Due to monotonicity of the processes $\nu_{\varepsilon i}(t)$, if the set V in $\mathcal{A}_{22}^{\mathbf{V}}$ is dense in $[0,T]$, then V can be extended to the sets $V \cup V_0''$. This set is $[0,T]$ except for at most a countable set, namely the set $\overline{V} \cap \overline{V}_0''$. Note there is no guarantee that $0, T \in V \cup V_0'$. But this is so if $0, T \in V$.

Let, for every $i = 1, \ldots, m$, the set W_{0i} be a set of all points $w \in [0,T]$ such that $\mathsf{P}\{\nu_{0i}(w) \in R[\xi_{0i}(\cdot)]\} = 0$. Let also $W_0 = \cap_{i=1}^m W_{0i}$. If condition $\mathfrak{C}_6^{\mathbf{W}}$ holds for some set W dense in $[0,T]$, then the set $\overline{W}_{0i}$ is empty, finite, or countable for every $i = 1, \ldots, m$. In the sequel, the set W_0 is $[0,T]$ except for at most a countable set, namely the set $\overline{W}_0$. Note there is no guarantee that $0 \in W_0$. But $0 \in W_0$ if $0 \in W$.

So, if both sets V and W are dense in $[0,\infty)$, then the set $S_0 = (V \cup V_0') \cap W_0$ is $[0,\infty)$ except for at most a countable set, namely $(\overline{V} \cap \overline{V}_0') \cup \overline{W}_0$.

Taking into consideration the remarks made above one can conclude that Theorems 2.7.3 and 2.7.4 do not require any changes, except for the assumption that the sets V, W are some sets dense in $[0,T]$ (instead of $[0,\infty)$) and for the change in the definition of the set S_0 described above.

In the new variant of Theorems 2.7.3 and 2.7.4, the set of convergence S_0 is $[0,T]$ except for at most a countable set. However, there is no guarantee that this set contains some preassigned point $w \in [0,T]$. In order for a point w to belong to the set of convergence, one also needs to add the assumption $w \in S_0$ to the conditions of Theorem 2.6.2, that is to require that $w \in V \cup V_0''$, W. In particular, both endpoints $0, T \in S_0$ if $0, T \in V$, W.

Condition $\mathcal{E}_4$ requires an obvious change. It should be replaced with the following truncated version of this condition:

$\mathcal{E}_5$: $\mathsf{P}\{\nu_{0i}(t') = \nu_{0i}(t'') \in R[\xi_{0i}(\cdot)]\} = 0$ for $0 \leq t' < t'' \leq T$, $i = 1, \ldots, m$.

Similar to Lemma 2.7.2, it can be shown that condition $\mathcal{E}_5$ is necessary for $\mathfrak{C}_6^{\mathbf{W}}$ to hold for some set W that is everywhere dense in $[0,T]$, and sufficient for $\mathfrak{C}_6^{\mathbf{W}}$ to hold for some set W that coincides with $[0,T]$ except for at most a countable set.

Using the remarks above we can conclude that Theorems 2.7.5 and 2.7.6 still remain true if to assume that the set V is dense in $[0,T]$, to make the changes in the definition of the set S_0 described above, and to replace condition $\mathcal{E}_4$ with condition $\mathcal{E}_5$.

Condition $\mathcal{F}_3$ also requires an obvious change. It should be replaced with the following truncated version of this condition:

$\mathcal{F}_4$: There exist a sequence $\delta_l \in Z_0, \delta_l \to 0$ as $l \to \infty$ and a sequence $0 < T_r \to \infty$ as $r \to \infty$ such that, for every $l, k, r \geq 1$ and $i = 1, \ldots, m$, $\lim_{0 < c \to 0} \overline{\lim}_{\varepsilon \to 0} \mathsf{P}\{\alpha_{\varepsilon ik}^{(\delta_l)} - c \leq \nu_{\varepsilon i}(t'), \nu_{\varepsilon i}(t'') < \alpha_{\varepsilon ik}^{(\delta_l)}, \alpha_{\varepsilon ik}^{(\delta_l)} < T_r\} = 0$ for $0 \leq t' < t'' \leq T$.

In the same way as in Lemma 2.7.3, it can be shown that condition $\mathcal{F}_4$ is necessary for $\mathfrak{D}_5^{\mathbf{W}}$ to hold for some set W that is everywhere dense in $[0,T]$, and sufficient for $\mathfrak{D}_5^{\mathbf{W}}$ to hold for some set W which is $[0,T]$ except for at most some finite or countable set.

As follows from the above, the only changes are to be made for Theorems 2.7.9 and 2.7.10 to hold is to require that the set V is a set dense in $[0, T]$, to make a change in the definition of the set S_0 described above, and to replace condition $\mathfrak{F}_3$ with condition $\mathfrak{F}_4$.

2.7.6. The time interval $(0, \infty)$. The results of the section can also be restated in the case of the semi-infinite interval $(0, \infty)$. In this case, it is necessary to choose V, $W \subseteq (0, \infty)$. One can use directly the set V_0 of points of stochastic continuity of the process $\mathbf{v}_\varepsilon(t)$, $t \in (0, \infty)$ in the definition of the sets S and S_0 of weak convergence. Also, the assumption that $0 \leq t' < t'' < \infty$ should be replaced with the assumption $0 < t' < t'' < \infty$ in conditions $\mathcal{E}_4$ and $\mathfrak{F}_3$. Finally, if $\nu_{0i}(t) > 0$ with probability 1 for every $t > 0$ and $i = 1, \dots, m$, then the conditions of $\mathbf{J}$-compactness $\mathfrak{J}_4$ and $\mathfrak{J}_8$ can be weakened. The corresponding $\mathbf{J}$-compactness relations in these conditions need to be assumed to hold for every finite interval $[T', T'']$, where $0 < T' < T'' < \infty$.

With these changes, Theorems 2.7.1 – 2.7.10 hold for the vector composition $\boldsymbol{\zeta}_\varepsilon(t)$, $t \in (0, \infty)$ of the càdlàg processes $\boldsymbol{\xi}_\varepsilon(t)$, $t \geq 0$ and $\mathbf{v}_\varepsilon(t)$, $t \in (0, \infty)$.

2.7.7. The random vectors $\boldsymbol{\zeta}_\varepsilon(t - 0)$. Sometimes it is useful to include, in the statement of weak convergence of the compositions $\boldsymbol{\zeta}_\varepsilon(t)$, $t \in S_0$, the random vectors $\boldsymbol{\zeta}_\varepsilon(t - 0)$ for $t \in \hat{S}_0$ where $\hat{S}_0 \subseteq [0, \infty)$. Here, we assume that $\mathbf{v}_\varepsilon(0 - 0) = \mathbf{v}_\varepsilon(0)$.

To provide this inclusion, one should include the random variables $\mathbf{v}_\varepsilon(t - 0)$, $t \in \hat{V}$ in the relation of weak convergence in condition $\mathcal{A}_{22}^{\mathbf{V}}$. Here $\hat{V}$ is a dense subset of $[0, \infty)$.

Also, one should add, in condition $\mathfrak{C}_6^{\mathbf{W}}$, the assumption that the limiting process $\xi_{0i}(t)$, $t \geq 0$ is continuous at the limiting random point $\nu_{0i}(t - 0)$ with probability 1 for $t \in \hat{W}$ and $i = 1, \dots, m$. Here $\hat{W}$ is a dense subset of $[0, \infty)$. Also, the set $\hat{W}_0$ should be introduced as a set of all $t \geq 0$ such that $\nu_{0i}(t - 0)$ is a point of continuity of the process $\xi_0(t)$, $t \geq 0$ with probability 1. In this case, the set of weak convergence is $\hat{S}_0 = (\hat{V} \cup V_0') \cap \hat{W}_0$.

If condition $\mathfrak{D}_5^{\mathbf{W}}$ is used, then the corresponding asymptotic relation in this condition should be additionally required for the processes $\xi_{\varepsilon i}(t)$, $t \geq 0$ and the random variables $\nu_{\varepsilon i}(t - 0)$ for every $t \in \hat{W}$ and $i = 1, \dots, m$. Here $\hat{W}$ is a dense subset of $[0, \infty)$. In this case, the set $\hat{W}_0'$ should be introduced as a set of all $w \geq 0$ for which condition $\mathfrak{D}_5^{\mathbf{W}}$ holds for the processes $\xi_\varepsilon(t)$, $t \geq 0$ and the random variables $\nu_\varepsilon(w - 0)$. In this case, the set of weak convergence is $\hat{S}_0 = (\hat{V} \cup V_0') \cap \hat{W}_0'$.

If the above conditions are extended, it is possible to prove the joint weak convergence of random variables $(\xi_{\varepsilon i}(\nu_{\varepsilon i}(t)), i = 1, \dots, m), t \in S_0$ and $(\xi_{\varepsilon i}(\nu_{\varepsilon i}(t - 0)), i = 1, \dots, m), t \in \hat{S}_0$.

2.7.8. A Polish phase space. The results in this section can be generalised to a model where the external stochastic processes $\xi_{\varepsilon i}(t)$, $t \geq 0$ take values in a Polish space X. The formulation of conditions $\mathcal{A}_{22}^{\mathbf{V}}$, $\mathfrak{C}_6^{\mathbf{W}}$, and $\mathfrak{D}_5^{\mathbf{W}}$ can be kept without a change. In the conditions $\mathfrak{J}_4$ and $\mathfrak{J}_8$, the Euclidian distance $|x - y|$ must be replaced, in the formula for the moduli of $\mathbf{J}$-compactness $\Delta_J(\xi_{\varepsilon i}(\cdot), c, T)$, $i = 1, \dots, m$, with the corresponding metric $d(x, y)$. Details of a procedure that allows to reduce the consideration to the case of real-valued processes can be found in Subsection 2.3.9.

2.8 Translation theorems

In this section we obtain the so-called "translation theorems" for randomly stopped càdlàg processes and compositions of càdlàg processes. These theorems play an essential role in applications.

2.8.1. Translation theorems for randomly stopped stochastic processes. We consider the same model of randomly stopped vector càdlàg processes as in Section 2.3, but assume that the following representation holds for the càdlàg process $\boldsymbol{\xi}_\varepsilon(t) = (\xi_{\varepsilon i}(t), i = 1, \dots, m)$, $t \geq 0$ and the random vector $\boldsymbol{\nu}_\varepsilon = (\nu_{\varepsilon i}, i = 1, \dots, m)$ for every $\varepsilon > 0$:

$$\nu_{\varepsilon i} = \frac{\mu_{\varepsilon i}}{n_{\varepsilon i}}, \quad \xi_{\varepsilon i}(t) = \frac{\eta_{\varepsilon i}(t n_{\varepsilon i})}{n_{\varepsilon i}^{\alpha_i} h_i(n_{\varepsilon i})}, \ t \geq 0, \ i = 1, \dots, m, \tag{2.8.1}$$

where: **(a)** $\alpha_i = \text{const} \geq 0$, $i = 1, \dots, m$; **(b)** $n_{\varepsilon i}$, $i = 1, \dots, m$, are non-random positive functions such that $n_{\varepsilon i} \to \infty$ as $\varepsilon \to 0$; **(c)** $h_i(x)$, $x \geq 0$, $i = 1, \dots, m$ are slowly varying functions.

In this section, we assume that the parameter $\varepsilon \to 0$ taking only positive values, as to avoid considering the expression in the right-hand side of (2.8.1) for $\varepsilon = 0$.

It should be noted that representation (2.8.1) admits the values $\alpha_i = 0$, $i = 1, \dots, m$, and functions $h_i(x) \equiv 1$, $i = 1, \dots, m$.

Also, some remark should be made about *slowly varying functions* $h_i(x)$. By the definition, such a function **(d)** should be Borel-measurable, and *slow variation* means that **(e)** $h_i(zy)/h_i(y) \to 1$ as $y \to \infty$ for every $x > 0$.

It follows from the definition that a slowly varying function is not equal to zero for all x large enough. Since the quantities $n_{\varepsilon i}^{\alpha_i} h_i(n_{\varepsilon i})$ are used as normalisation functions and $n_{\varepsilon i} \to \infty$ as $\varepsilon \to 0$, one can always assume that $h_i(x) \neq 0$ for all $x \geq 0$ and every $i = 1, \dots, m$.

As is known (see, for example, Bingham, Goldie, and Teugels (1989)), for any slowly varying function and, therefore, for the functions $h_i(x)$, $i = 1, \dots, m$, and every $0 < z' < z'' < \infty$,

$$\sup_{z' \leq z \leq z''} \left|\frac{h_i(zy)}{h_i(y)} - 1\right| \to 0 \text{ as } y \to \infty. \tag{2.8.2}$$

We are interested in the random vectors

$$\zeta'_{\varepsilon i} = \frac{\eta_{\varepsilon i}(\mu_{\varepsilon i})}{\mu_{\varepsilon i}^{\alpha_i} h_i(\mu_{\varepsilon i})} \cdot \chi(\mu_{\varepsilon i} \neq 0), \ i = 1, \dots, m.$$

Here, the product in the right-hand side should be interpreted as zero if $\mu_{\varepsilon i} = 0$ and, therefore, $\chi(\mu_{\varepsilon i} \neq 0) = 0$. The indicator $\chi(\mu_{\varepsilon i} \neq 0)$ is used to avoid considering improper random variables.

Let us also introduce the corresponding limiting random variables,

$$\zeta'_{0i} = \nu_{0i}^{-\alpha_i} \xi_{0i}(\nu_{0i}), \ i = 1, \dots, m.$$

We assume also that the following condition holds:

$\mathfrak{I}_3$: $\nu_{0i} > 0$ with probability 1 for $i = 1, \dots, m$.

Due to this condition, $\zeta'_{0i}, i = 1, \dots, m$ are proper random variables.
The following theorem can be found in Silvestrov (1972a, 1972e).

Theorem 2.8.1. *Let conditions $\mathcal{A}_{19}$, $\mathfrak{C}_4$, and $\mathfrak{I}_3$ hold. Then*

$$(\zeta'_{\varepsilon i}, i = 1, \dots, m) \Rightarrow (\zeta'_{0i}, i = 1, \dots, m) \text{ as } \varepsilon \to 0.$$

Proof of Theorem 2.8.1. By applying Theorem 2.3.2 we get the following relation:

$$(\nu_{\varepsilon i}, \xi_{\varepsilon i}(\nu_{\varepsilon i}), i = 1, \dots, m) \Rightarrow (\nu_{0i}, \xi_{0i}(\nu_{0i}), i = 1, \dots, m) \text{ as } \varepsilon \to 0. \tag{2.8.3}$$

Note also that conditions $\mathcal{A}_{19}$ and $\mathfrak{I}_3$ imply that the random variables

$$\chi(\mu_{\varepsilon i} \neq 0) = \chi(\nu_{\varepsilon i} \neq 0) \xrightarrow{\mathsf{P}} 1 \text{ as } \varepsilon \to 0. \tag{2.8.4}$$

Let us introduce the functions $f(x_i, y_i) = \sum_{i=1}^m u_i x_i^{-\alpha_i} y_i \cdot \chi(x_i \neq 0)$, $i = 1, \dots, m$, where $u_i \in \mathbb{R}_1$, $i = 1, \dots, m$. Due to condition $\mathfrak{I}_3$, every such a function is continuous almost everywhere with respect to the distribution of the random vector $(\nu_{0i}, \xi_{0i}(\nu_{0i}), i = 1, \dots, m)$. Thus, we get using Theorem 1.3.2 that

$$\begin{aligned} \sum_{i=1}^m u_i \nu_{\varepsilon i}^{-\alpha_i} \xi_{\varepsilon i}(\nu_{\varepsilon i}) \cdot \chi(\nu_{\varepsilon i} \neq 0) &= \sum_{i=1}^m u_i \frac{\eta_{\varepsilon i}(\mu_{\varepsilon i})}{\mu_{\varepsilon i}^{\alpha_i} h_i(n_{\varepsilon i})} \cdot \chi(\mu_{\varepsilon i} \neq 0) \\ &\Rightarrow \sum_{i=1}^m u_i \nu_{0i}^{-\alpha_i} \xi_{0i}(\nu_{0i}) \text{ as } \varepsilon \to 0. \end{aligned} \tag{2.8.5}$$

Since $u_i \in \mathbb{R}_1$, $i = 1, \dots, m$ can be chosen arbitrarily, (2.8.5) is equivalent to the following relation:

$$\Big(\frac{\eta_{\varepsilon i}(\mu_{\varepsilon i})}{\mu_{\varepsilon i}^{\alpha_i} h_i(n_{\varepsilon i})} \cdot \chi(\mu_{\varepsilon i} \neq 0), i = 1, \dots, m\Big) \Rightarrow (\nu_{0i}^{-\alpha_i} \xi_{0i}(\nu_{0i}), i = 1, \dots, m) \text{ as } \varepsilon \to 0. \tag{2.8.6}$$

Now let us show that

$$\frac{h_i(\mu_{\varepsilon i})}{h_i(n_{\varepsilon i})} \xrightarrow{\mathsf{P}} 1 \text{ as } \varepsilon \to 0, \ i = 1, \dots, m. \tag{2.8.7}$$

Choose an arbitrary $\delta > 0$. Since $\nu_{0i} > 0$, $i = 1, \dots, m$ with probability 1, one can choose points $0 < z' < z'' < \infty$, which are points of continuity of the distribution functions of the random variables $\nu_{0i}, i = 1, \dots, m$, such that

$$\mathsf{P}\{\nu_{0i} \notin [z', z'']\} \leq \delta/2, \ i = 1, \dots, m. \tag{2.8.8}$$

By (2.8.2), for an arbitrary $\sigma > 0$ there exists $\varepsilon_0 > 0$ such that, if $\varepsilon \leq \varepsilon_0$, then for every $x \in [z', z'']$,

$$\left|\frac{h_i(xn_{\varepsilon i})}{h_i(n_{\varepsilon i})} - 1\right| \leq \sigma. \tag{2.8.9}$$

We can always assume that ε_0 in (2.8.9) is chosen in such a way that, for $\varepsilon \leq \varepsilon_0$,

$$|\mathsf{P}\{\frac{\mu_{\varepsilon i}}{n_{\varepsilon i}} \notin [z', z'']\} - \mathsf{P}\{\nu_{0i} \notin [z', z'']\}| \leq \delta/2. \tag{2.8.10}$$

By using relations (2.8.8)–(2.8.10), we get, for $\varepsilon \leq \varepsilon_0$,

$$\begin{aligned}\mathsf{P}\{|\frac{h_i(\mu_{\varepsilon i})}{h_i(n_{\varepsilon i})} - 1| > \sigma\} &= \int_0^\infty \mathsf{P}\{|\frac{h_i(xn_{\varepsilon i})}{h_i(n_{\varepsilon i})} - 1| > \sigma\}\mathsf{P}\{\frac{\mu_{\varepsilon i}}{n_{\varepsilon i}} \in dx\} \\ &\leq \int_{z'}^{z''} \mathsf{P}\{|\frac{h_i(xn_{\varepsilon i})}{h_i(n_{\varepsilon i})} - 1| > \sigma\}\mathsf{P}\{\frac{\mu_{\varepsilon i}}{n_{\varepsilon i}} \in dx\} + \mathsf{P}\{\frac{\mu_{\varepsilon i}}{n_{\varepsilon i}} \notin [z', z'']\} \\ &\leq \mathsf{P}\{\nu_{0i} \notin [z', z'']\} + |\mathsf{P}\{\frac{\mu_{\varepsilon i}}{n_{\varepsilon i}} \in [z', z'']\} - \mathsf{P}\{\nu_{0i} \in [z', z'']\}| \leq \delta.\end{aligned} \tag{2.8.11}$$

Since δ and σ are arbitrary, relation (2.8.7) follows from (2.8.11). In virtue of Lemma 1.2.1 and relations (2.8.6)–(2.8.7),

$$\begin{aligned}&(\frac{\eta_{\varepsilon i}(\mu_{\varepsilon i})}{\mu_{\varepsilon i}^{\alpha_i} h_i(\mu_{\varepsilon i})}, i = 1, \dots, m) \\ &\quad = ((\frac{h_i(\mu_{\varepsilon i})}{h_i(n_{\varepsilon i})})^{-1} \cdot \frac{\eta_{\varepsilon i}(\mu_{\varepsilon i})}{\mu_{\varepsilon i}^{\alpha_i} h_i(n_{\varepsilon i})} \cdot \chi(\mu_{\varepsilon i} \neq 0), i = 1, \dots, m) \\ &\quad \Rightarrow (\nu_{0i}^{-\alpha_i}\xi_{0i}(\nu_{0i}), i = 1, \dots, m) \text{ as } \varepsilon \to 0.\end{aligned} \tag{2.8.12}$$

This completes the proof. □

Remark 2.8.1. Note that condition $\mathcal{A}_{\mathbf{19}}$ can be replaced with a combination of conditions $\mathcal{A}_{\mathbf{20}}$, $\mathfrak{J}_{\mathbf{8}}$, which implies condition $\mathcal{A}_{\mathbf{19}}$.

Remark 2.8.2. Note that, due to condition $\mathfrak{I}_{\mathbf{3}}$ and the remarks in Subsection 2.3.8, one can slightly weaken conditions $\mathcal{A}_{\mathbf{19}}$ or $\mathcal{A}_{\mathbf{20}}$ and $\mathfrak{J}_{\mathbf{8}}$. In condition $\mathcal{A}_{\mathbf{19}}$, the sets S_i can be assumed to be dense in $(0, \infty)$ and the assumption $0 \in S_i$ can be omitted. Analogously, in condition $\mathcal{A}_{\mathbf{20}}$, the set U can be assumed to be dense in $(0, \infty)$ and the assumption $0 \in U$ can be omitted. Also, in condition $\mathfrak{J}_{\mathbf{8}}$, the relation of **J**-compactness should be assumed only for for the intervals $[T', T'']$ where $0 < T' < T'' < \infty$.

2.8.2. Translation theorems for semi-vector compositions of càdlàg processes. The result of Theorem 2.8.1 can be generalised to the case of stochastic processes.

Let us first consider the case of semi-vector compositions. Consider the same model for randomly stopped vector càdlàg processes as in Section 2.7 but assume that the vector stopping processes have identical components, i.e., $\boldsymbol{\nu}_\varepsilon(t) = (\nu_\varepsilon(t), \dots, \nu_\varepsilon(t)), t \geq 0$.

Moreover, we assume that the external càdlàg process $\boldsymbol{\xi}_\varepsilon(t) = (\xi_{\varepsilon 1}(t), \ldots, \xi_{\varepsilon m}(t))$, $t \geq 0$ and the stopping process $\nu_\varepsilon(t), t \geq 0$ can be represented in the following form for every $\varepsilon > 0$:

$$\nu_\varepsilon(t) = t\nu_\varepsilon = t\frac{\mu_\varepsilon}{n_\varepsilon}, \quad \xi_{\varepsilon i}(t) = \frac{\eta_{\varepsilon i}(tn_\varepsilon)}{n_\varepsilon^\alpha h(n_\varepsilon)}, \ t \geq 0, \ i = 1, \ldots, m, \tag{2.8.13}$$

where: **(a)** $\alpha = \text{const} \geq 0$; **(b)** n_ε is a non-random positive function such that $n_\varepsilon \to \infty$ as $\varepsilon \to 0$; **(c)** $h(x)$, $x \geq 0$ is a slowly varying function.

It should be noted that representation (2.8.13) is still valid if $\alpha = 0$ and $h(x) \equiv 1$.

Condition $\mathcal{A}_{22}^{\mathbf{V}}$ takes, in this case, the following form:

$\mathcal{A}_{23}$: $(\nu_\varepsilon, \boldsymbol{\xi}_\varepsilon(t)), t \in U \Rightarrow (\nu_0, \boldsymbol{\xi}_0(t)), t \in U$ as $\varepsilon \to 0$, where U is a subset of $[0, \infty)$ that is dense in this interval and contains the point 0.

We also assume that the following condition holds:

$\mathfrak{I}_4$: $\nu_0 > 0$ with probability 1.

Consider the stochastic processes

$$\zeta'_{\varepsilon i}(t) = \frac{\eta_{\varepsilon i}(t\mu_\varepsilon)}{\mu_\varepsilon^\alpha h(\mu_\varepsilon)} \cdot \chi(\mu_\varepsilon \neq 0), \ t \geq 0, \ i = 1, \ldots, m.$$

Also introduce the corresponding limiting processes

$$\zeta'_{0i}(t) = \nu_0^{-\alpha}\xi_{0i}(t\nu_0), \ t \geq 0, \ i = 1, \ldots, m.$$

Denote by W_{0i} the set of $t \geq 0$ such that $\mathsf{P}\{\tau_{kni}/\nu_0 = t\} = 0$ for all $k, n = 1, 2, \ldots$, where τ_{kni}, $k = 1, 2 \ldots$ are successive moments of jumps of the process $\xi_{0i}(t)$, $t \geq 0$ such that absolute values of the jumps belong to the interval $[\frac{1}{n}, \frac{1}{n-1})$. Let also $W = \cap_{i=1}^m W_{0i}$. Obviously, the sets W_{0i}, $i = 1, \ldots, m$, and W are $[0, \infty)$ except for at most countable sets. Also, $0 \in W$.

Theorem 2.8.2. *Let conditions $\mathcal{A}_{23}$, $\mathfrak{J}_4$, and $\mathfrak{I}_4$ hold. Then*

$$\boldsymbol{\zeta}'_\varepsilon(t) = (\zeta'_{\varepsilon i}(t), i = 1, \ldots, m), t \in W \Rightarrow \boldsymbol{\zeta}'_0(t) = (\zeta'_{0i}(t), i = 1, \ldots, m), t \in W \text{ as } \varepsilon \to 0.$$

Proof of Theorem 2.8.2. Let us introduce the processes

$$\xi'_{\varepsilon i}(t) = \frac{\eta_{\varepsilon i}(tn_\varepsilon)}{\mu_\varepsilon^\alpha h(\mu_\varepsilon)} \cdot \chi(\mu_\varepsilon \neq 0), \ t \geq 0, \ i = 1, \ldots, m. \tag{2.8.14}$$

Obviously,

$$\zeta'_{\varepsilon i}(t) = \xi'_{\varepsilon i}(t\nu_\varepsilon), \ t \geq 0, \ i = 1, \ldots, m. \tag{2.8.15}$$

The processes $\xi'_{\varepsilon i}(t)$, $t \geq 0$ can be represented as

$$\xi'_{\varepsilon i}(t) = (h(\mu_\varepsilon)/h(n_\varepsilon))^{-1}\nu_\varepsilon^{-\alpha}\xi_{\varepsilon i}(t) \cdot \chi(\mu_\varepsilon \neq 0), \ t \geq 0, \ i = 1, \ldots, m. \tag{2.8.16}$$

Let us choose an arbitrary $n \geq 1$ and points $s_1, \ldots, s_n \geq 0$, and $t_1, \ldots, t_n \in U$. As was shown in the proof of Theorem 2.8.1, **(d)** $h(\mu_\varepsilon)/h(n_\varepsilon) \stackrel{\mathsf{P}}{\longrightarrow} 1$ as $\varepsilon \to 0$ and also **(e)** $\chi(\mu_\varepsilon \neq 0) = \chi(\nu_\varepsilon \neq 0) \stackrel{\mathsf{P}}{\longrightarrow} 1$ as $\varepsilon \to 0$. Using representation (2.8.16), the relations **(d)** and **(e)**, condition $\mathcal{A}_{23}$, and the Slutsky Theorem 1.2.3, we get

$$\begin{aligned}(s_j\nu_\varepsilon, \xi'_{\varepsilon i}(t_j), j = 1, \ldots, n, i = 1, \ldots, m)\\ \Rightarrow (s_j\nu_0, \xi'_{0i}(t_j), j = 1, \ldots, n, i = 1, \ldots, m) \text{ as } \varepsilon \to 0,\end{aligned} \tag{2.8.17}$$

where

$$\xi'_{0i}(t) = \nu_0^{-\alpha}\xi_{0i}(t), \ t \geq 0, \ i = 1, \ldots, m.$$

Since the points points $s_1, \ldots, s_n \geq 0$ and $t_1, \ldots, t_n \in U$ are arbitrary, relation (2.8.17) can be rewritten in the following form:

$$\begin{aligned}(s\nu_\varepsilon, \xi'_{\varepsilon i}(t), i = 1, \ldots, m), (s, t) \in [0, \infty) \times U\\ \Rightarrow (s\nu_0, \xi'_{0i}(t), i = 1, \ldots, m), (s, t) \in [0, \infty) \times U \text{ as } \varepsilon \to 0.\end{aligned} \tag{2.8.18}$$

Relation (2.8.18) means that condition $\mathcal{A}_{22}^{\mathbf{V}}$, with the set $V = [0, \infty)$, holds for the processes $\mathbf{v}_\varepsilon(t) = (t\nu_\varepsilon, \ldots, t\nu_\varepsilon)$, $t \geq 0$ and $\boldsymbol{\xi}'_\varepsilon(t) = (\xi'_{\varepsilon 1}(t), \ldots, \xi'_{\varepsilon m}(t))$, $t \geq 0$.

Let us show now that condition $\mathcal{J}_4$ holds for the processes $\boldsymbol{\xi}'_\varepsilon(t)$, $t \geq 0$.

We use the following simple equality, which is valid for an arbitrary càdlàg function $\mathbf{x}(t)$, $t \geq 0$ that takes values in $\mathbb{R}_m$ and a positive constant b,

$$\Delta_J(b\mathbf{x}(\cdot), c, T) = b\Delta_J(\mathbf{x}(\cdot), c, T), \ c, T > 0. \tag{2.8.19}$$

Let us denote

$$\beta_\varepsilon = \nu_\varepsilon^{-\alpha}(h(\mu_\varepsilon)/h(n_\varepsilon))^{-1} \cdot \chi(\mu_\varepsilon \neq 0).$$

Using this inequality (2.8.19) and formula (2.8.16) we get

$$\begin{aligned}&\mathsf{P}\{\Delta_J(\boldsymbol{\xi}'_\varepsilon(\cdot), c, T) > \delta\}\\ &\quad = \mathsf{P}\{\beta_\varepsilon \cdot \Delta_J(\boldsymbol{\xi}_\varepsilon(\cdot), c, T) > \delta\}\\ &\quad \leq \mathsf{P}\{\beta_\varepsilon \geq b\} + \mathsf{P}\{\Delta_J(\boldsymbol{\xi}_\varepsilon(\cdot), c, T) > \delta/b\}.\end{aligned} \tag{2.8.20}$$

Since $h(\mu_\varepsilon)/h(n_\varepsilon) \stackrel{\mathsf{P}}{\longrightarrow} 1$ as $\varepsilon \to 0$ and $\chi(\mu_\varepsilon \neq 0) \stackrel{\mathsf{P}}{\longrightarrow} 1$ as $\varepsilon \to 0$, condition $\mathcal{A}_{23}$ implies that

$$\beta_\varepsilon \Rightarrow \beta_0 = \nu_0^{-\alpha} \text{ as } \varepsilon \to 0. \tag{2.8.21}$$

For an arbitrary $\sigma > 0$, by (2.8.21) and condition $\mathcal{J}_4$, we can choose b such that $\overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\beta_\varepsilon > b\} \leq \sigma/2$. By fixing b and then using condition $\mathcal{J}_4$, we can find $c > 0$ such that $\overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\Delta_J(\boldsymbol{\xi}_\varepsilon(\cdot), c, T) > \delta/b\} \leq \sigma/2$. If we pass to limit in (2.8.20), making first $\varepsilon \to 0$ and then $c \to 0$, we find

$$\lim_{c\to 0} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\Delta_J(\boldsymbol{\xi}'_\varepsilon(\cdot), c, T) > \delta\} \leq \sigma. \tag{2.8.22}$$

Since σ is arbitrary, 2.8.22 proves that condition $\mathfrak{J}_4$ holds for the processes $\xi'_\varepsilon(t)$, $t \geq 0$.

Finally, one can note that condition $\mathfrak{C}_6^{\mathrm{W}}$ holds for the processes $\mathbf{v}_0(t)$, $t \geq 0$ and $\xi'_0(t)$, $t \geq 0$ with the set W described above. This is so, since the processes $\xi'_0(t)$, $t \geq 0$ and $\xi_0(t)$, $t \geq 0$ have the same set of discontinuity points, i.e., $R[\xi'_0(\cdot)] = R[\xi_0(\cdot)]$.

To complete the proof of the theorem, it remains to apply Theorem 2.7.1 to the processes $\xi'_\varepsilon(t)$, $t \geq 0$ and $\mathbf{v}_\varepsilon(t)$, $t \geq 0$. □

Remark 2.8.3. Note that the point 0 belongs to the set of weak convergence W. If the set of weak convergence in condition $\mathcal{A}_{23}$ does not contain 0 and the relation of **J**-compactness in condition $\mathfrak{J}_4$ holds only for intervals $[T', T'']$, where $0 < T' < T'' < \infty$, one should exclude the point 0 from the set W.

2.8.3. Translation theorems for vector compositions of càdlàg processes. Let us consider the general case of vector compositions where the stopping processes $\nu_{\varepsilon i}(t), t \geq 0$ can be different for $i = 1, \ldots, m$. We assume that the external càdlàg process $\xi_\varepsilon(t) = (\xi_{\varepsilon 1}(t), \ldots, \xi_{\varepsilon m}(t)), t \geq 0$ and the stopping process $\mathbf{v}_\varepsilon(t) = (\nu_{\varepsilon 1}(t), \ldots, \nu_{\varepsilon m}(t)), t \geq 0$ can be represented in the following form for every $\mathsf{E} > 0$:

$$\nu_{\varepsilon i}(t) = t\nu_{\varepsilon i} = t\frac{\mu_{\varepsilon i}}{n_{\varepsilon i}}, \quad \xi_{\varepsilon i}(t) = \frac{\eta_{\varepsilon i}(tn_{\varepsilon i})}{n_{\varepsilon i}^{\alpha_i} h_i(n_{\varepsilon i})}, \ i = 1, \ldots, m, \ t \geq 0, \tag{2.8.23}$$

where: **(a)** $\alpha_i = \text{const} \geq 0, i = 1, \ldots, m$; **(b)** $n_{\varepsilon i}, i = 1, \ldots, m$ are non-random positive functions such that $n_{\varepsilon i} \to \infty$ as $\varepsilon \to 0$; **(c)** $h_i(x)$, $x \geq 0, i = 1, \ldots, m$ are slowly varying functions.

Condition $\mathcal{A}_{23}$ should be replaced with the following vector analogue:

$\mathcal{A}_{24}$: $(\mathbf{v}_\varepsilon, \xi_\varepsilon(t)), t \in U \Rightarrow (\mathbf{v}_0, \xi_0(t)), t \in U$ as $\varepsilon \to 0$, where U is a subset of $[0, \infty)$ that is dense in this interval and contains the point 0.

We also assume that condition $\mathfrak{I}_3$ holds.

Now, consider the processes

$$\zeta''_{\varepsilon i}(t) = \frac{\eta_{\varepsilon i}(t\mu_{\varepsilon i})}{\mu_{\varepsilon i}^{\alpha_i} h_i(\mu_{\varepsilon i})} \cdot \chi(\mu_{\varepsilon i} \neq 0), \ t \geq 0, \ i = 1, \ldots, m.$$

Let us introduce also the corresponding limiting processes

$$\zeta''_{0i}(t) = \nu_{0i}^{-\alpha_i} \xi_{0i}(t\nu_{0i}), \ t \geq 0, \ i = 1, \ldots, m.$$

Denote by W'_{0i} the set of $t \geq 0$ such that $\mathsf{P}\{\tau_{kni}/\nu_{0i} = t\} = 0$ for all $k, n = 1, 2, \ldots$, where τ_{kni}, $k = 1, 2, \ldots$ are successive moments of jumps of the process $\xi_{0i}(t)$, $t \geq 0$ that have absolute values of the jumps in the interval $[\frac{1}{n}, \frac{1}{n-1})$. These moments were defined in Section 2.2.1. Let also $W' = \cap_{i=1}^m W'_{0i}$. Obviously, the sets W'_{0i}, $i = 1, \ldots m$, and W' coincide with $[0, \infty)$ except for, perhaps, some finite or countable sets.

Theorem 2.8.3. *Let conditions $\mathcal{A}_{24}$, $\mathfrak{J}_8$, and $\mathfrak{I}_3$ hold. Then*

$$\zeta''_\varepsilon(t) = (\zeta''_{\varepsilon i}(t), i = 1, \dots, m), t \in W' \Rightarrow \zeta''_0(t) = (\zeta''_{0i}(t), i = 1, \dots, m), t \in W' \text{ as } \varepsilon \to 0.$$

Proof of Theorem 2.8.3. The first step in the proof repeats, mainly, the proof of Theorem 2.8.2. Let us introduce the processes

$$\xi''_{\varepsilon i}(t) = \frac{\eta_{\varepsilon i}(tn_{\varepsilon i})}{\mu^{\alpha}_{\varepsilon i} h_i(\mu_{\varepsilon i})} \cdot \chi(\mu_{\varepsilon i} \neq 0),\ t \geq 0,\ i = 1, \dots, m. \tag{2.8.24}$$

Obviously,

$$\zeta''_{\varepsilon i}(t) = \xi''_{\varepsilon i}(t\nu_{\varepsilon i}),\ t \geq 0,\ i = 1, \dots, m. \tag{2.8.25}$$

Using a method, similar to the one used in the proof of Theorem 2.8.2 in relations (2.8.16), (2.8.17), and (2.8.18), we get

$$\begin{aligned} &(s\nu_{\varepsilon i}, \xi''_{\varepsilon i}(t), i = 1, \dots, m), (s, t) \in [0, \infty) \times U \\ &\quad \Rightarrow (s\nu_{0i}, \xi'_{0i}(t), i = 1, \dots, m), (s, t) \in [0, \infty) \times U \text{ as } \varepsilon \to 0, \end{aligned} \tag{2.8.26}$$

where

$$\xi''_{0i}(t) = \nu_{0i}^{-\alpha_i} \xi_{0i}(t),\ t \geq 0,\ i = 1, \dots, m.$$

So, condition $\mathcal{A}_{22}^{\mathbf{V}}$ holds for the processes $\mathbf{v}_\varepsilon(t) = (t\nu_{\varepsilon i}, i = 1, \dots, m)$, $t \geq 0$ and $\boldsymbol{\xi}''_\varepsilon(t) = (\xi'_{\varepsilon i}(t), i = 1, \dots, m)$, $t \geq 0$ with the set $V = [0, \infty)$.

The next step is slightly different. Since the stopping processes are different for different i, the equality (2.8.19) can not be used for proving **J**-compactness of the vector compositions $\boldsymbol{\zeta}''_\varepsilon(t) = (\xi''_{\varepsilon i}(t\nu_{\varepsilon i}), i = 1, \dots, m)$, $t \geq 0$. However, this equality can still be used to prove **J**-compactness of the scalar processes $\zeta''_{\varepsilon i}(t) = \xi''_{\varepsilon i}(t\nu_{\varepsilon i})$, $t \geq 0$, for every $i = 1, \dots, m$. So, using a reasoning similar to that in the proof of Theorem2.8.2 in relations(2.8.19) – (2.8.22), we can get

$$\lim_{c \to 0} \overline{\lim_{\varepsilon \to 0}} \mathsf{P}\{\Delta_J(\xi''_{\varepsilon i}(\cdot), c, T > \delta\} = 0,\ \delta, T > 0,\ i = 1, \dots, m. \tag{2.8.27}$$

Therefore, condition $\mathfrak{J}_8$ holds for the processes $\boldsymbol{\xi}''_\varepsilon(t)$, $t \geq 0$.

Finally, one can note that condition $\mathfrak{C}_6^{\mathbf{W}'}$ holds for the processes $\boldsymbol{\xi}''_0(t)$, $t \geq 0$ and $\mathbf{v}_0(t)$, $t \geq 0$ with the set W' described above. This is true, since the processes $\xi''_{0i}(t)$, $t \geq 0$ and $\xi_{0i}(t)$, $t \geq 0$ have the same set of discontinuity points, that is, $R[\xi''_{0i}(\cdot)] = R[\xi_{0i}(\cdot)]$ for every $i = 1, \dots, m$.

For completing the proof of theorem, it remains to apply Theorem 2.7.2 to the processes $\boldsymbol{\xi}''_\varepsilon(t)$, $t \geq 0$ and $\mathbf{v}_\varepsilon(t)$, $t \geq 0$. □

Remark 2.8.4. Note that point 0 belongs to the set of weak convergence W'. If the set of weak convergence in condition $\mathcal{A}_{24}$ does not contain 0 and the relation of **J**-compactness in condition $\mathfrak{J}_8$ holds only for intervals $[T', T'']$, where $0 < T' < T'' < \infty$, one should exclude the point 0 from the set W'.

2.8.4. Translation theorems for randomly stopped stochastic sequences. Let us consider an example that explains the appearance of power type normalising functions in the translation theorems given above. The results presented in this subsection, mainly, are due to Durrett and Resnik (1977). We present them with slight variations, in the context of the general translation theorems given above in Subsections 2.8.1 - 2.8.3.

Let $\xi_n, n = 0, 1, \ldots$ be a sequence of real-valued random variables and $\mu_n, n = 1, \ldots$ be a sequence of non-negative integer random variables. Let also $a_n > 0$ and b_n be two sequences of real numbers.

We assume that the random variables $\xi_n, n = 0, 1, \ldots$ and $\mu_n, n = 1, \ldots$ are defined on the same probability space. Let us now define, for $n \geq 1$,

$$\nu_n = \frac{\mu_n}{n}, \ \xi_n(t) = \frac{\xi_{[tn]} - b_n}{a_n}, t \geq 0.$$

Here we prefer to index the random variables and the processes by n. However, one can always define the random variables $\nu_\varepsilon = \nu_n$ and the processes $\xi_\varepsilon(t) = \xi_n(t), t \geq 0$, for $\varepsilon \in [\frac{1}{n}, \frac{1}{n-1})$, where $n \geq 1$, and use the index $\varepsilon > 0$ that runs over all positive real values.

Let us assume the following condition of weak convergence:

$\mathcal{A}_{25}$: $(\nu_n, \xi_n(t)), t \in U \Rightarrow (\nu_0, \xi_0(t)), t \in U$ as $n \to \infty$, where $\xi_0(t), t > 0$ is a càdlàg process and U is a subset of $(0, \infty)$ dense in this interval.

We also assume the following variant of the **J**-compactness condition:

$\mathfrak{J}_9$: $\lim_{c \to 0} \overline{\lim}_{n \to \infty} \mathsf{P}\{\Delta_J(\xi_n(\cdot), c, T', T'') > \delta\} = 0, \ \delta > 0, 0 < T' < T'' < \infty.$

Note that conditions $\mathcal{A}_{25}$ and $\mathfrak{J}_9$ are necessary and sufficient for the **J**-convergence

$$(\nu_n, \xi_n(t)), t > 0 \xrightarrow{\mathbf{J}} (\nu_0, \xi_0(t)), t > 0 \text{ as } n \to \infty. \tag{2.8.28}$$

We also assume the positivity condition $\mathfrak{I}_4$, i.e., that $\nu_0 > 0$ with probability 1.

Denote by W the set of $t > 0$ such that $\mathsf{P}\{\tau_{knr}/\nu_0 = t\} = 0$ for all $k, n, r = 1, 2, \ldots$, where τ_{knr}, $k = 1, 2\ldots$ are successive moments of jumps of the process $\xi_0(t)$, $t \geq r^{-1}$, such that absolute values of the jumps belong to the interval $[\frac{1}{n}, \frac{1}{n-1})$. It is clear that the set W is the interval $(0, \infty)$ except for at most a countable set.

By applying Theorem 2.6.1 to the composition of the processes $\xi_n(t), t \geq 0$ and $\nu_n(t) = t\nu_n, t \geq 0$, and taking into account Remark 2.8.3, we get the following relation:

$$\xi_n(t\nu_n) = \frac{\xi_{[t\mu_n]} - b_n}{a_n}, \ t \in W \Rightarrow \xi_0(t\nu_0), t \in W \text{ as } n \to \infty. \tag{2.8.29}$$

Let us now assume the following condition:

$\mathfrak{I}_5$: $\xi_0(t)$ has a non-degenerate distribution for each $t > 0$.

It follows from a convergence types theorem (See, Lamperti (1962b), Weissman (1975), and Durrett and Resnik (1977)) that, if the processes $\xi_n(t), t > 0 \xrightarrow{\mathbf{J}} \xi_0(t), t > 0$ as $n \to \infty$ and condition $\mathfrak{I}_5$ holds, then the normalisation constants a_n and the centralisation constants b_n must satisfy the following relations:

$$\frac{a_{[tn]}}{a_n} \to \alpha_\rho(s) \text{ as } n \to \infty, \ t > 0, \tag{2.8.30}$$

and

$$\frac{b_{[tn]} - b_n}{a_n} \to \beta_\rho(s) \text{ as } n \to \infty, \ t > 0. \tag{2.8.31}$$

Moreover, there exist only three possibilities **(a)** $\alpha_\rho(s) = s^\rho$, $\beta_\rho(s) = b \cdot (s^\rho - 1)$, $\rho > 0$; **(b)** $a_0(s) = 1$, $b_0(s) = b \cdot \ln s$, $\rho = 0$; and **(c)** $a_\rho(s) = s^\rho$, $b_\rho(s) = b \cdot (1 - s^\rho)$, $\rho < 0$, where $b = \text{const}$.

Also, as was shown in Durrett and Resnik (1977), in this case, $\xi_0(t), t > 0$, is a stochastically continuous càdlàg process.

Relation (2.8.30) implies that the function $a_{[s]}$ is a regularly varying function, that is, it can be represented in the form $a_{[s]} = s^\rho h(s)$, where $h(s)$ is a slowly varying function. Using this fact and (2.8.2) one can easily show that the convergence in (2.8.30) and (2.8.31) is uniform in every finite interval separated from zero, that is, for any $0 < s' < s'' < \infty$,

$$\sup_{s' \leq s \leq s''} |\frac{a_{[sn]}}{a_n} - \alpha_\rho(s)| \to 0 \text{ as } n \to \infty, \tag{2.8.32}$$

and

$$\sup_{s' \leq s \leq s''} |\frac{b_{[sn]} - b_n}{a_n} - \beta_\rho(s)| \to 0 \text{ as } n \to \infty. \tag{2.8.33}$$

As soon as the form of normalisation constants $a_n = n^\rho h(n)$ is obtained, the general translation Theorem 2.8.2 and Remark 2.8.3 can be applied. This yields the following asymptotic relation:

$$\frac{\xi_{[t\mu_n]} - b_n}{a_{\mu_n}} \cdot \chi(\mu_n \neq 0), \ t \in W \Rightarrow \nu_0^{-\rho} \xi_0(t\nu_0), t \in W \text{ as } n \to \infty. \tag{2.8.34}$$

It also follows from relation (2.8.33) that the non-random functions

$$b_n(t) = \frac{b_{[tn]} - b_n}{a_n}, t > 0 \xrightarrow{\mathbf{U}} \beta_\rho(t), t > 0 \text{ as } n \to \infty. \tag{2.8.35}$$

Due to Lemma 1.6.11, it follows from the relations (2.8.28) and (2.8.35) that

$$(\nu_n, \xi_n(t), b_n(t)), t > 0 \xrightarrow{\mathbf{J}} (\nu_0, \xi_0(t), \beta_\rho(t)), t > 0 \text{ as } n \to \infty. \tag{2.8.36}$$

Now, by applying the translation Theorem 2.8.2 and Remark 2.8.3 once more, one can get the following relation:

$$
\begin{aligned}
&\Big(\frac{\xi_{[t\mu_n]} - b_n}{a_{\mu_n}}, \frac{b_{[t\mu_n]} - b_n}{a_{\mu_n}}\Big) \cdot \chi(\mu_n \neq 0), t \in W \\
&\quad \Rightarrow (\nu_0^{-\rho}\xi_0(t\nu_0), \beta_\rho(t\nu_0)), t \in W \text{ as } n \to \infty.
\end{aligned}
\tag{2.8.37}
$$

As a corollary of (2.8.37), one gets, in an obvious way, the following relations:

$$
\begin{aligned}
&\frac{\xi_{[t\mu_n]} - b_{[t\mu_n]}}{a_{\mu_n}} \cdot \chi(\mu_n \neq 0), t \in W \\
&\quad \Rightarrow (\nu_0^{-\rho}\xi_0(t\nu_0) - \beta_\rho(t\nu_0)), t \in W \text{ as } n \to \infty,
\end{aligned}
\tag{2.8.38}
$$

as well as

$$
\begin{aligned}
&\frac{\xi_{[t\mu_n]} - b_{[\mu_n]}}{a_{\mu_n}} \cdot \chi(\mu_n \neq 0), t \in W \\
&\quad \Rightarrow (\nu_0^{-\rho}\xi_0(t\nu_0) - \beta_\rho(\nu_0)), t \in W \text{ as } n \to \infty.
\end{aligned}
\tag{2.8.39}
$$

Note also that not only weak convergence but also the convergence of the corresponding processes in the topology **J** can be proved by using theorems on **J**-convergence of compositions of càdlàg processes. We formulate such statements in Subsection 3.4.9.

2.9 Randomly stopped locally compact càdlàg processes

In this section, we obtain conditions for weak convergence of randomly stopped càdlàg processes and compositions of càdlàg processes for a model with asymptotically locally compact external processes. A standard combination of general conditions that provides weak convergence of randomly stopped càdlàg processes includes the condition of joint weak convergence of random stopping moments and external processes, $\mathcal{A}_{17}$, the condition of **J**-compactness of external processes, $\mathfrak{J}_7$, and the continuity condition $\mathfrak{C}_3$. These conditions can be effectively checked and are sufficient for a wide range of applications. Nevertheless, in the case under consideration, the conditions of **J**-compactness, $\mathfrak{J}_7$, and continuity, $\mathfrak{C}_3$, can be weakened. They can be replaced with a condition of local conditional compactness of external processes at every point of a set S. Here S is a set, where the distribution of the limiting stopping moment is concentrated.

2.9.1. A condition of local compactness for scalar randomly stopped processes. Let, for every $\varepsilon \geq 0$, $\xi_\varepsilon(t)$, $t \geq 0$ be a real-valued càdlàg process and ν_ε a non-negative random variable.

Let Y_0 be the set that contains all continuity points of the distribution function of the random variable ν_0 and the point 0. Then $\overline{Y}_0$ is the set of all points $t > 0$ such that $\mathsf{P}\{\nu_0 = t\} > 0$. This set contains at most a countable number of points.

We assume the following variant of condition $\mathcal{A}_{17}$:

$\mathcal{A}_{26}$: $(\nu_\varepsilon, \xi_\varepsilon(t)), t \in U \Rightarrow (\nu_0, \xi_0(t)), t \in U$ as $\varepsilon \to 0$, where U is a subset of $[0, \infty)$ such that the set $U \setminus \overline{Y}_0$ is dense in $[0, \infty)$, and $0 \in U$.

Note that, if the set of weak convergence U is $[0, \infty)$ except for at most a countable set, then the set $U \setminus \overline{Y}_0$ is also $[0, \infty)$ except for at most a countable set. So, condition $\mathcal{A}_{26}$ holds with the set U.

Let us define, for a function $x(t)$, $t \geq 0$ from the space $\mathbf{D}^{(1)}_{[0,\infty)}$, a functional that is the oscillation of the function $x(t)$ on the interval $[u, w)$, $0 \leq u < w < \infty$,

$$\Delta_{u,w}(x(\cdot)) = \sup_{t',t'' \in [u,w)} |x(t') - x(t'')|.$$

Let S_0 be the set of points $t \geq 0$ such that $\mathsf{P}\{\nu_0 \in [u, w)\} > 0$ for all $u \leq t < w$ (the set of points of growth of the distribution function of ν_0). It is not difficult to show that S_0 is a Borel-measurable subset of $[0, \infty)$ and $\mathsf{P}\{\nu_0 \in S_0\} = 1$.

We use the following condition that replaces the condition of $\mathbf{J}$-compactness $\mathcal{J}_7$:

$\mathcal{L}_1$: There exists a Borel-measurable subset $S \subseteq S_0$ such that (a) $\mathsf{P}\{\nu_0 \in S\} = 1$, (b) $\lim_{u \leq t < w, w-u \to 0} \overline{\lim}_{\varepsilon \to 0} \mathsf{P}\{\Delta_{u,w}(\xi_\varepsilon(\cdot)) > \delta / \nu_\varepsilon \in [u, w)\} = 0$, $\delta > 0$ for $t \in S$.

The main result of this section is the following theorem from Silvestrov (1979a).

Theorem 2.9.1. *Let conditions $\mathcal{A}_{26}$ and $\mathcal{L}_1$ hold. Then*

$$\xi_\varepsilon(\nu_\varepsilon) \Rightarrow \xi_0(\nu_0) \text{ as } \varepsilon \to 0.$$

Proof of Theorem 2.9.1. Let $0 = z_{0,n} < z_{1,n} < \ldots$ be, for every $n = 0, 1, \ldots$, a partition of the interval $[0, \infty)$ and let also these partitions satisfy the following conditions: **(a)** $z_{k,n} \in U \setminus \overline{Y}_0$ for all $k, n = 0, 1, \ldots$; **(b)** $z_{k,n} \to \infty$ as $k \to \infty$ for $n = 0, 1, \ldots$, **(c)** $\max_{k \geq 0}(z_{k+1,n} - z_{k,n}) \to 0$ as $n \to 0$. Such partitions exist, since the set $U \setminus \overline{Y}_0$ is dense in $[0, \infty)$ and contains 0.

Let us now define, for $n = 0, 1, \ldots$, the stochastic processes

$$\xi^{(n)}_\varepsilon(t) = \xi_\varepsilon(z_{k+1,n}) \text{ for } t \in [z_{k,n}, z_{k+1,n}), \quad k = 0, 1, \ldots. \tag{2.9.1}$$

We are going to use the approximation representation

$$\xi_\varepsilon(\nu_\varepsilon) = \xi^{(n)}_\varepsilon(\nu_\varepsilon) + (\xi_\varepsilon(\nu_\varepsilon) - \xi^{(n)}_\varepsilon(\nu_\varepsilon)), \tag{2.9.2}$$

and show that, for every $n = 0, 1, \ldots$,

$$\xi^{(n)}_\varepsilon(\nu_\varepsilon) \Rightarrow \xi^{(n)}_0(\nu_0) \text{ as } \varepsilon \to 0, \tag{2.9.3}$$

and

$$\lim_{n \to \infty} \overline{\lim_{\varepsilon \to 0}} \mathsf{P}\{|\xi_\varepsilon(\nu_\varepsilon) - \xi^{(n)}_\varepsilon(\nu_\varepsilon)| > \delta\} = 0, \ \delta > 0. \tag{2.9.4}$$

Theorem 2.9.1 follows from relations (2.9.1)–(2.9.4) and Lemma 1.2.5.

By the definition of the processes $\xi_\varepsilon^{(n)}(t), t \geq 0$,

$$\mathsf{P}\{\xi_\varepsilon^{(n)}(\nu_\varepsilon) < u\} = \sum_{k=0}^{\infty} \mathsf{P}\{\xi_\varepsilon(z_{k+1,n}) < u, \nu_\varepsilon \in [z_{k,n}, z_{k+1,n})\}. \tag{2.9.5}$$

For an arbitrary $\delta > 0$ and every $n = 0, 1, \dots$, one can always select a number m_n such that $\mathsf{P}\{\nu_0 \geq z_{m_n,n}\} \leq \delta$. Let X be the set of discontinuity points of the distribution functions of the random variables $\xi_0(z_{k,n})$, $k, n = 0, 1, \dots$. The set X is at most countable. Using (2.9.5), condition $\mathcal{A}_{\mathbf{26}}$, and the choice of points $z_{k,n}$, we have, for all $n = 0, 1, \dots$ and $u \in \overline{X}$,

$$\begin{aligned} &|\mathsf{P}\{\xi_\varepsilon^{(n)}(\nu_\varepsilon) < u\} - \mathsf{P}\{\xi_0^{(n)}(\nu_0) < u\}| \\ &\quad \leq \overline{\lim_{\varepsilon\to 0}}\, \mathsf{P}\{\nu_\varepsilon \geq z_{m_n,n}\} + \mathsf{P}\{\nu_0 \geq z_{m_n,n}\} \\ &\quad + \sum_{k=0}^{m_n-1} \overline{\lim_{\varepsilon\to 0}}\, |\mathsf{P}\{\xi_\varepsilon(z_{k+1,n}) < u, \nu_\varepsilon \in [z_{k,n}, z_{k+1,n})\} \\ &\quad - \mathsf{P}\{\xi_0(z_{k+1,n}) < u, \nu_0 \in [z_{k,n}, z_{k+1,n})\}| \leq 2\delta. \end{aligned} \tag{2.9.6}$$

Since δ is arbitrary, relation (2.9.6) implies that, for every $n = 0, 1, \dots$ and $u \in \overline{Z}$,

$$\lim_{\varepsilon\to 0} \mathsf{P}\{\xi_\varepsilon^{(n)}(\nu_\varepsilon) < u\} = \mathsf{P}\{\xi_0^{(n)}(\nu_0) < u\}. \tag{2.9.7}$$

Recall that weak convergence of distribution functions follows from their convergence on a countable everywhere dense set in $\mathbb{R}_1$. Hence, relation (2.9.3) follows from (2.9.7).

Let us show that conditions $\mathcal{A}_{\mathbf{26}}$ and $\mathcal{L}_{\mathbf{1}}$ imply relation (2.9.4). For the random variable $\xi_\varepsilon(\nu_\varepsilon) - \xi_\varepsilon^{(n)}(\nu_\varepsilon)$, we have the following estimate:

$$|\xi_\varepsilon(\nu_\varepsilon) - \xi_\varepsilon^{(n)}(\nu_\varepsilon)| \leq \sum_{k=0}^{\infty} \chi(\nu_\varepsilon \in [z_{k,n}, z_{k+1,n}))\Delta_{z_{k,n},z_{k+1,n}}(\xi_\varepsilon(\cdot)). \tag{2.9.8}$$

Denote by I_n the set of indices k such that $\mathsf{P}\{\nu_0 \in [z_{k,n}, z_{k+1,n})\} > 0$, and by $\overline{I}_n$ the set of all other natural k.

Let, as above, for arbitrary $\delta > 0$, a number m be chosen such that $\mathsf{P}\{\nu_0 \geq z_{m,n}\} \leq \delta$.

Using estimate (2.9.8), condition $\mathcal{A}_{26}$, and the choice of points $z_{k,n}$ we have

$$
\begin{aligned}
&\overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{|\xi_\varepsilon(\nu_\varepsilon) - \xi_\varepsilon^{(n)}(\nu_\varepsilon)| > \delta\} \\
&\quad \leq \overline{\lim_{\varepsilon\to 0}} \sum_{k=0}^{\infty} \mathsf{P}\{\nu_\varepsilon \in [z_{k,n}, z_{k+1,n}), \Delta_{z_{k,n}, z_{k+1,n}}(\xi_\varepsilon(\cdot)) > \delta\} \\
&\quad \leq \sum_{k<m, k\in I_n} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\Delta_{z_{k,n}, z_{k+1,n}}(\xi_\varepsilon(\cdot)) > \delta / \nu_\varepsilon \in [z_{k,n}, z_{k+1,n})\} \mathsf{P}\{\nu_\varepsilon \in [z_{k,n}, z_{k+1,n})\} \\
&\quad + \sum_{k<m, k\in \bar{I}_n} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\nu_\varepsilon \in [z_{k,n}, z_{k+1,n})\} + \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\nu_\varepsilon \geq z_{m,n}\} \\
&\quad \leq \sum_{k<m, k\in I_n} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\Delta_{z_{k,n}, z_{k+1,n}}(\xi_\varepsilon(\cdot)) > \delta / \nu_\varepsilon \in [z_{k,n}, z_{k+1,n})\} \mathsf{P}\{\nu_0 \in [z_{k,n}, z_{k+1,n})\} + \delta \\
&\quad \leq \int_0^\infty f_n(t) \mathsf{P}\{\nu_0 \in dt\} + \delta,
\end{aligned}
\tag{2.9.9}
$$

where

$$
\begin{aligned}
f_n(t) = &\sum_{k=0}^{\infty} \chi(t \in [z_{k,n}, z_{k+1,n}))\chi(k \in I_n) \\
&\times \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\Delta_{z_{k,n}, z_{k+1,n}}(\xi_\varepsilon(\cdot)) > \delta / \nu_\varepsilon \in [z_{k,n}, z_{k+1,n})\}.
\end{aligned}
\tag{2.9.10}
$$

By the definition, **(d)** $0 \leq f_n(t) \leq 1$ for $t \in [0, \infty)$, and, by condition $\mathcal{L}_1$, **(e)** $f_n(t) \to 0$ as $n \to \infty$ for $t \in S$.

Using the Lebesgue theorem we obtain, by **(d)** – **(e)**, and (2.9.9),

$$
\lim_{n\to\infty} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{|\xi_\varepsilon(\nu_\varepsilon) - \xi_\varepsilon^{(n)}(\nu_\varepsilon)| > \delta\} \leq \lim_{n\to\infty} \int_0^\infty f_n(t) \mathsf{P}\{\nu_0 \in dt\} + \delta = \delta. \tag{2.9.11}
$$

Since δ is arbitrary, the last relation proves (2.9.4). □

2.9.2. Asymptotically independent stopping moments and external processes. We will now study the case where the following condition holds for càdlàg processes $\xi_\varepsilon(t), t \geq 0$ and stopping moments ν_ε:

$\mathfrak{Q}_5$: $\nu_\varepsilon = \alpha_\varepsilon + \beta_\varepsilon$, where (a) for every $\varepsilon \geq 0$, the random variable α_ε and the process $\xi_\varepsilon(t), t \geq 0$ are independent; (b) the random variables $\beta_\varepsilon \xrightarrow{\mathsf{P}} 0$ as $\varepsilon \to 0$.

It follows from Lemma 1.2.4 that, under $\mathfrak{Q}_5$, condition $\mathcal{A}_{26}$ is equivalent to the following two conditions:

$\mathcal{A}_{27}$: $\nu_\varepsilon \Rightarrow \nu_0$ as $\varepsilon \to 0$.

and

$\mathcal{A}_{28}$: $\xi_\varepsilon(t), t \in U \Rightarrow \xi_0(t), t \in U$ as $\varepsilon \to 0$, where U is a subset of $[0, \infty)$ such that the set $U \setminus \overline{Y}_0$ is dense in $[0, \infty)$, and $0 \in U$.

In this case, the limiting process $\xi_0(t)$, $t \geq 0$ and the stopping moment ν_0 in $\mathcal{A}_{26}$ are independent.

Also, under condition $\mathfrak{Q}_5$, condition $\mathfrak{L}_1$ can be simplified and replaced with the following condition:

$\mathfrak{L}_2$: There exists a Borel-measurable set $S \subseteq S_0$ such that, (a) $\mathsf{P}\{\nu_0 \in S\} = 1$; (b) $\lim_{u \leq t < w, w-u \to 0} \overline{\lim}_{\varepsilon \to 0} \mathsf{P}\{\Delta_{u,w}(\xi_\varepsilon(\cdot)) > \delta\} = 0, \delta > 0$ for $t \in S$.

Theorem 2.9.2. *Let conditions $\mathfrak{Q}_5$, $\mathcal{A}_{27}$, $\mathcal{A}_{28}$, and $\mathfrak{L}_2$ hold. Then*

$$\xi_\varepsilon(\nu_\varepsilon) \Rightarrow \xi_0(\nu_0) \text{ as } \varepsilon \to 0,$$

where the random variable ν_0 and the process $\xi_0(t)$, $t \geq 0$ are independent.

Proof of Theorem 2.9.2. In order to prove the theorem, it would be sufficient to show that, under $\mathfrak{Q}_5$ and $\mathcal{A}_{27}$, condition $\mathfrak{L}_2$ implies condition $\mathfrak{L}_1$.

It is obvious that the outer limit (as $u \leq t < w, w - u \to 0$) is equal to 0 in $\mathfrak{L}_2$ if **(a)** this limit is equal to 0 under an additional assumption that the points $u, w \in U \setminus \overline{Y}_0$. So, it is enough to show that, with this additional assumption, for $t \in S$,

$$\begin{aligned} \lim_{u \leq t < w, w-u \to 0} \overline{\lim_{\varepsilon \to 0}} \mathsf{P}\{\Delta_{u,w}(\xi_\varepsilon(\cdot)) > \delta / \nu_\varepsilon \in [u, w)\} \\ \leq \lim_{u \leq t < w, w-u \to 0} \overline{\lim_{\varepsilon \to 0}} \mathsf{P}\{\Delta_{u,w}(\xi_\varepsilon(\cdot)) > \delta\}. \end{aligned} \tag{2.9.12}$$

Since t is a point of growth of the distribution function of ν_0, and $u, w \in U \setminus \overline{Y}_0$, by condition $\mathfrak{L}_2$ there exists $\varepsilon_0 > 0$ such that $\mathsf{P}\{\nu_\varepsilon \in [u, w)\} > 0$ for all $\varepsilon \leq \varepsilon_0$. For $\varepsilon \leq \varepsilon_0$ and $\gamma > 0$, the following estimate holds:

$$\begin{aligned} &\mathsf{P}\{\Delta_{u,v}(\xi_\varepsilon(\cdot)) > \delta / \nu_\varepsilon \in [u, w)\} \\ &\quad \leq \frac{\mathsf{P}\{\Delta_{u,w}(\xi_\varepsilon(\cdot)) > \delta, \alpha_\varepsilon \in [u - \gamma, w + \gamma)\} + \mathsf{P}\{|\beta_\varepsilon| \geq \gamma\}}{\mathsf{P}\{\nu_\varepsilon \in [u, w)\}} \\ &\quad \leq \frac{\mathsf{P}\{\Delta_{u,w}(\xi_\varepsilon(\cdot)) > \delta\}\mathsf{P}\{\alpha_\varepsilon \in [u - \gamma, w + \gamma)\} + \mathsf{P}\{|\beta_\varepsilon| \geq \gamma\}}{\mathsf{P}\{\nu_\varepsilon \in [u, w)\}}. \end{aligned} \tag{2.9.13}$$

For any $\sigma > 0$, one can always select $\gamma = \gamma(u, w) > 0$ such that the points $u - \gamma$ and $w + \gamma$ are points of continuity of the distribution function of ν_0 and

$$\mathsf{P}\{\nu_0 \in [u - \gamma, w + \gamma)\} \leq \mathsf{P}\{\nu_0 \in [u, w)\}(1 + \sigma). \tag{2.9.14}$$

Using (2.9.14) and passing to the limit in (2.9.13), as $\varepsilon \to \infty$ and then as $u \le t < w,\ u, w \in U \setminus \overline{Y}_0,\ w - u \to 0$, we have for $t \in S$,

$$\begin{aligned}
&\lim_{u \le t < w, w-u \to 0} \overline{\lim_{\varepsilon \to 0}} \mathsf{P}\{\Delta_{u,w}(\xi_\varepsilon(\cdot)) > \delta / \nu_\varepsilon \in [u, w)\} \\
&\quad \le \lim_{u \le t < w, w-u \to 0} \overline{\lim_{\varepsilon \to 0}} \mathsf{P}\{\Delta_{u,w}(\xi_\varepsilon(\cdot)) > \delta\} \frac{\mathsf{P}\{\nu_0 \in [u - \gamma, w + \gamma)\}}{\mathsf{P}\{\nu_0 \in [u, w)\}} \\
&\quad \le \lim_{u \le t < w, w-u \to 0} \overline{\lim_{\varepsilon \to 0}} \mathsf{P}\{\Delta_{u,w}(\xi_\varepsilon(\cdot)) > \delta\}(1 + \sigma).
\end{aligned} \tag{2.9.15}$$

Since $\sigma > 0$ is arbitrary, the last relation yields (2.9.12). The theorem is proved. □

2.9.3. A condition of local compactness for vector randomly stopped processes. Let, for every $\varepsilon \ge 0$, $\xi_\varepsilon(t) = (\xi_{\varepsilon i}(t), i = 1, \ldots, m), t \ge 0$ be a càdlàg process with real-valued components and $\boldsymbol{\nu}_\varepsilon = (\nu_{\varepsilon i}, i = 1, \ldots, m)$ a random vector with non-negative components.

Let Y_{0i} be the set that contains all continuity points of the distribution function of the random variable ν_{0i} and the point 0. Then $\overline{Y}_{0i}$ is the set of all points $t > 0$ such that $\mathsf{P}\{\nu_{0i} = t\} > 0$. This set contains at most a countable number of points for every $i = 1, \ldots m$.

We assume the following variant of condition $\mathcal{A}_{26}$:

$\mathcal{A}_{29}$: $(\boldsymbol{\nu}_\varepsilon, \xi_\varepsilon(t)), t \in U \Rightarrow (\boldsymbol{\nu}_0, \xi_0(t)), t \in U$ as $\varepsilon \to 0$, where U is a subset of $[0, \infty)$ such that the set $U \setminus \overline{Y}_{0i}$ is dense in $[0, \infty)$ for every $i = 1, \ldots m$, and $0 \in U$.

Let S_{0i} be a set of points $t \ge 0$ such that $\mathsf{P}\{\nu_{0i} \in [u, w)\} > 0$ for all $u \le t < w$ (the set of points of growth of the distribution function of ν_{0i}). It is not difficult to show that S_{0i} is a Borel-measurable subset of $[0, \infty)$ and $\mathsf{P}\{\nu_{0i} \in S_{0i}\} = 1$, $i = 1, \ldots m$.

We use the following local compactness condition:

$\mathcal{L}_3$: There exist Borel-measurable sets $S_i \subseteq S_{0i}$ such that (a) $\mathsf{P}\{\nu_{0i} \in S_i\} = 1$, $i = 1, \ldots, m$, (b) $\lim_{u \le t < w, w-u \to 0} \overline{\lim}_{\varepsilon \to 0} \mathsf{P}\{\Delta_{u,v}(\xi_{\varepsilon i}(\cdot)) > \delta / \nu_{\varepsilon i} \in [u, w)\} = 0$, $\delta > 0$ for $t \in S_i$, $i = 1, \ldots, m$.

Theorem 2.9.3. *Let conditions $\mathcal{A}_{29}$ and $\mathcal{L}_3$ hold. Then*

$$(\xi_{\varepsilon i}(\nu_{\varepsilon i}), i = 1, \ldots, m) \Rightarrow (\xi_{0i}(\nu_{0i}), i = 1, \ldots, m) \text{ as } \varepsilon \to 0.$$

Proof of Theorem 2.9.3. The proof of this theorem is similar to the proof of Theorem 2.9.1. Let $0 = z_{i,0,n} < z_{i,1,n} < \ldots$, for every $i = 1, \ldots, m$ and $n = 0, 1, \ldots$, be a partition of the interval $[0, \infty)$ and also let these partitions satisfy the following conditions: **(a)** $z_{i,k,n} \in U_i \setminus \overline{Y}_{0i}$ for all $i = 1, \ldots, m$ and k, $n = 0, 1, \ldots$, **(b)** $z_{i,k,n} \to \infty$ as $k \to \infty$ for $i = 1, \ldots, m$ and $n = 0, 1, \ldots$, **(c)** $\max_{k \ge 0}(z_{i,k+1,n} - z_{i,k,n}) \to 0$ as $n \to 0$ for $i = 1, \ldots, m$. Such partitions exist, since the set $U \setminus \overline{Y}_{0i}$ is dense in $[0, \infty)$ for every $i = 1, \ldots, m$.

Now we define, for $i = 1, \dots, m$ and $n = 0, 1, \dots$, the processes

$$\xi^{(n)}_{\varepsilon i}(t) = \xi_{\varepsilon i}(z_{i,k+1,n}) \text{ for } t \in [z_{i,k,n}, z_{i,k+1,n}),\ k = 0, 1, \dots. \quad (2.9.16)$$

As in the scalar case, it is enough to show that for every $n = 0, 1, \dots$,

$$(\xi^{(n)}_{\varepsilon i}(\nu_{\varepsilon i}), i = 1, \dots, m) \Rightarrow (\xi^{(n)}_{0i}(\nu_{0i}, i = 1, \dots, m) \text{ as } \varepsilon \to 0, \quad (2.9.17)$$

and

$$\lim_{n\to\infty} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{|\xi_{\varepsilon i}(\nu_{\varepsilon i}) - \xi^{(n)}_{\varepsilon i}(\nu_{\varepsilon i})| > \delta\} = 0,\ \delta > 0, i = 1, \dots, m. \quad (2.9.18)$$

The following formula is an analogue of (2.9.5):

$$\mathsf{P}\{\xi^{(n)}_{\varepsilon i}(\nu_{\varepsilon i}) < u_i, i = 1, \dots m\} = \sum_{i=1}^{m} \sum_{k=0}^{\infty} \mathsf{P}\{\xi_{\varepsilon i}(z_{i,k+1,n}) < u_i, \nu_{\varepsilon i} \in [z_{i,k,n}, z_{i,k+1,n})\}. \quad (2.9.19)$$

Using this formula one can prove relation (2.9.17) absolutely the same way as it was done in the proof of relation (2.9.3) in Theorem 2.9.1.

Relation (2.9.18) coincides with (2.9.4) for every $i = 1, \dots, m$. Therefore, it does not require a separate proof. □

2.9.4. Weak convergence of compositions of càdlàg processes based on a local compactness condition. Let, for every $\varepsilon \geq 0$, $\boldsymbol{\xi}_\varepsilon(t) = (\xi_{\varepsilon i}(t), i = 1, \dots, m), t \geq 0$ be a càdlàg process with real-valued components and $\boldsymbol{\nu}_\varepsilon(t) = (\nu_{\varepsilon i}(t), i = 1, \dots, m)$, $t \geq 0$ a càdlàg process with non-negative and non-decreasing components.

Let, for every $i = 1, \dots, m$ and $s \geq 0$, $Y_{i,s}$ be a set that contains all continuity points of the distribution function of the random variable $\nu_{0i}(s)$ and the point 0. Then $\overline{Y}_{i,s}$ is the set of all points $t > 0$ such that $\mathsf{P}\{\nu_{0i}(s) = t\} > 0$. This set contains at most a countable number of points for every $i = 1, \dots m$ and $s \geq 0$.

Below, U, $V \subseteq [0, \infty)$. Let us assume the following condition:

$\mathcal{A}^{\mathbf{V}}_{\mathbf{30}}$: $(\boldsymbol{\nu}_\varepsilon(s), \boldsymbol{\xi}_\varepsilon(t)), (s, t) \in V \times U \Rightarrow (\boldsymbol{\nu}_0(s), \boldsymbol{\xi}_0(t)), (s, t) \in V \times U$ as $\varepsilon \to 0$, where U is a subset of $[0, \infty)$ such that the sets $U \setminus \overline{Y}_{i,s}$ are dense in $[0, \infty)$ for every $i = 1, \dots, m$ and $s \in V$, and $0 \in U$.

Let $S_{0i,s}$ be a set of points $t \geq 0$ such that $\mathsf{P}\{\nu_{0i}(s) \in [u, w)\} > 0$ for all $u \leq t < w$ (the set of points of growth of the distribution function of $\nu_{0i}(s)$). It is not difficult to show that $S_{0i,s}$ is a Borel-measurable subset of $[0, \infty)$ and $\mathsf{P}\{\nu_{0i}(s) \in S_{0i}(s)\} = 1$, $i = 1, \dots m$, $s \geq 0$.

We use the following local compactness condition:

$\mathcal{L}_{\mathbf{4}}$: There exist Borel-measurable sets $S_{i,s} \subseteq S_{0i,s}$ such that (a) $\mathsf{P}\{\nu_{0i}(s) \in S_{i,s}\} = 1$ for $i = 1, \dots, m$, $s \in V$, (b) $\lim_{u\leq t<w, w-u\to 0} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\Delta_{u,w}(\xi_{\varepsilon i}(\cdot)) > \delta / \nu_{\varepsilon i}(s) \in [u, w)\} = 0, \delta > 0$ for $t \in S_{i,s}$, $i = 1, \dots, m$, $s \in V$.

Theorem 2.9.4. *Let conditions $\mathcal{A}_{30}^{\mathbf{V}}$ and $\mathfrak{L}_4$ hold. Then*

$$(\xi_{\varepsilon i}(\nu_{\varepsilon i}(t)), i = 1, \ldots, m), t \in V \Rightarrow (\xi_{0i}(\nu_{0i}(t)), i = 1, \ldots, m), t \in V \text{ as } \varepsilon \to 0.$$

Proof of Theorem 2.9.4. The proof follows from Theorem 2.9.3 that should be applied, for every sequence of points $t_1, \ldots, t_n \in V, n \geq 1$, to the processes $(\xi_{\varepsilon ij}(t), i = 1, \ldots, m, j = 1, \ldots, n)$, $t \geq 0$, where $\xi_{\varepsilon ij}(t) = \xi_{\varepsilon i}(t)$, $t \geq 0$, for $i = 1, \ldots, m$, $j = 1, \ldots, n$, and $(\nu_{\varepsilon i}(t_j), i = 1, \ldots, m, j = 1, \ldots, n)$. Obviously, conditions $\mathcal{A}_{30}^{\mathbf{V}}$ and $\mathfrak{L}_4$ imply that $\mathcal{A}_{29}$ and $\mathfrak{L}_3$ hold for these processes. □

2.9.5. Asymptotically independent internal stopping processes and external processes. The following condition is an analogue of condition $\mathfrak{Q}_5$:

$\mathfrak{Q}_6$: $\boldsymbol{\nu}_\varepsilon(t) = \boldsymbol{\alpha}_\varepsilon(t) + \boldsymbol{\beta}_\varepsilon(t)$, $t \geq 0$, where (a) the processes $\boldsymbol{\alpha}_\varepsilon(t)$, $t \geq 0$ and $\boldsymbol{\xi}_\varepsilon(t)$, $t \geq 0$ are independent for every $\varepsilon \geq 0$, (b) the random variables $\boldsymbol{\beta}_\varepsilon(t) \xrightarrow{\mathsf{P}} 0$ as $\varepsilon \to 0$ for every $t \geq 0$.

It follows from Lemma 1.2.4, under condition $\mathfrak{Q}_6$, that condition $\mathcal{A}_{30}^{\mathbf{V}}$ is equivalent to the following two conditions:

$\mathcal{A}_{31}$: $\boldsymbol{\nu}_\varepsilon(t), t \in V \Rightarrow \boldsymbol{\nu}_0(t), t \in V$ as $\varepsilon \to 0$,

and

$\mathcal{A}_{32}$: $\boldsymbol{\xi}_\varepsilon(t), t \in U \Rightarrow \boldsymbol{\xi}_0(t), t \in U$ as $\varepsilon \to 0$, where U is a subset of $[0, \infty)$ such that the set $U \setminus \overline{Y}_{0i}$ is dense in $[0, \infty)$ for every $i = 1, \ldots, m$, and $0 \in U$.

In this case, the limiting external process $\boldsymbol{\xi}_0(t)$, $t \geq 0$ and the limiting stopping process $\boldsymbol{\nu}_0(t), t \in V$ in $\mathcal{A}_{30}^{\mathbf{V}}$ are independent.

The following condition is an analogue of condition $\mathfrak{L}_4$:

$\mathfrak{L}_5$: There exist Borel-measurable subsets $S_{i,s} \subseteq S_{0i,s}$ such that (a) $\mathsf{P}\{\nu_{0i}(s) \in S_{i,s}\} = 1$ for $i = 1, \ldots, m$, $s \in V$, (b) $\lim_{u \leq t < w, w-u \to 0} \overline{\lim}_{\varepsilon \to 0} \mathsf{P}\{\Delta_{u,w}(\xi_{\varepsilon i}(\cdot)) > \delta\} = 0, \delta > 0$ for $t \in S_{i,s}$, $i = 1, \ldots, m$, $s \in V$.

Theorem 2.9.5. *Let conditions $\mathfrak{Q}_6$, $\mathcal{A}_{31}$, $\mathcal{A}_{32}$, and $\mathfrak{L}_5$ hold. Then*

$$(\xi_\varepsilon(\nu_\varepsilon(t)), i = 1, \ldots, m), t \in V \Rightarrow (\xi_0(\nu_0(t)), i = 1, \ldots, m), t \in V \text{ as } \varepsilon \to 0,$$

where the processes $\boldsymbol{\xi}_0(t)$, $t \geq 0$ and $\boldsymbol{\nu}_0(t)$, $t \in V$ are independent.

Proof of Theorem 2.9.5. It follows from Lemma 1.2.5 that, under conditions$\mathfrak{Q}_6$, $\mathcal{A}_{31}$, and $\mathcal{A}_{32}$, the condition $\mathcal{A}_{30}^{\mathbf{V}}$ holds, and the limiting processes $\boldsymbol{\xi}_0(t)$, $t \geq 0$ and $\boldsymbol{\nu}_0(t)$, $t \in V$ are independent. Also, conditions $\mathfrak{Q}_6$, $\mathcal{A}_{31}$, and $\mathfrak{L}_5$ imply condition $\mathfrak{L}_4$, which can be proved using the same reasoning as in the proof of Theorem 2.9.2. □

2.9.6. References. Conditions of weak convergence of randomly stopped càdlàg processes presented in Theorems 2.2.1, 2.2.2, 2.3.1, and 2.3.3 as well as Lemmas 2.2.3 and 2.3.1 can be found in Silvestrov (1971b, 1972a). These theorems cover the case where one of the first-type continuity conditions, $\mathfrak{C}$, introduced in Silvestrov (1971b), holds. This condition requires that the limiting stopping moment be a continuity point of the limiting external process with probability 1. A simpler case where the limiting external process is continuous was considered earlier by Billingsley (1968). Theorem 2.3.4 with a weakened **J**-compactness condition as well as Theorem 2.3.6 that extends the results to the case of external processes with a Polish phase space are new.

Theorems 2.4.1 and 2.4.2 are also new results announced in Silvestrov (2002b). These theorems give conditions for weak convergence of randomly stopped càdlàg processes in the case where continuity conditions of type $\mathfrak{C}$ are replaced with new weakened continuity conditions of type $\mathfrak{D}$.

Conditions of weak convergence for compositions of càdlàg processes presented in Theorems 2.6.1, 2.6.2, 2.7.1, 2.7.4 and Lemmas 2.6.1, 2.7.1 are from Silvestrov (1972a, 1972b, 1972e). The case where both the limiting external process and the limiting internal stopping process are continuous was considered earlier by Billigsley (1968).

Theorems 2.6.3, 2.7.3, and 2.7.5 are from Silvestrov (1974), where the continuity condition of the second-type $\mathfrak{E}_2$ was introduced. A new more convenient equivalent form of this condition, $\mathfrak{E}_1$, and Lemmas 2.6.2 and 2.6.3 are from Silvestrov and Teugels (1998a) and Silvestrov (2000b). The theorems mentioned above are given in a new and more convenient form, where condition $\mathfrak{E}_2$ is replaced with condition $\mathfrak{E}_1$. The versions of these theorems with the improved **J**-compactness condition, given in Theorems 2.7.2, 2.7.4, and 2.7.6, are new results. The more detailed analysis of the structure of the set of weak convergence, as compared to that in Silvestrov (1974), is partly due to Silvestrov and Teugels (1998a) and Silvestrov (2000b).

Theorems 2.6.4 – 2.6.5 and 2.7.7 – 2.7.10, which are based on a new weakened continuity conditions of types $\mathfrak{D}$ and $\mathfrak{F}$, and Lemma 2.6.4 are new results.

Translation theorems 2.8.1, 2.8.2, and 2.8.3 are from Silvestrov (1972a, 1972b, 1972e). Conditions for weak convergence of randomly stopped locally compact processes, given in Theorems 2.9.1 – 2.9.5, are from Silvestrov (1979a).

Chapter 3

J-convergence of compositions of stochastic processes

In this chapter, general conditions for **J**-convergence of compositions of càdlàg stochastic processes are presented.

The main results concerning **J**-convergence of compositions of càdlàg stochastic processes are Theorems 3.4.2, 3.6.1, and 3.6.2.

In Theorem 3.4.2, conditions for **J**-convergence of compositions of càdlàg processes are given in the case where the corresponding limiting internal stopping process is continuous. This theorem covers a significant part of applications. In Theorems 3.6.1 and 3.6.2, general conditions for **J**-compactness and **J**-convergence of compositions of càdlàg processes are given for the case where both the limiting external process and the limiting internal stopping process can be discontinuous.

The latter theorem gives the most general conditions that, together, provide **J**-convergence of compositions of càdlàg processes. These are **(a)** the condition of joint weak convergence of external stochastic processes and internal stopping processes; **(b)** the conditions of **J**-compactness of external and internal stopping processes; and the following two continuity conditions on the limiting processes: **(c)** the left and the right limiting values of the internal stopping process at points where the process has jumps are, with probability 1, points of continuity for the corresponding external process; and **(d)** there does not exist with probability 1 a time interval such that the internal stopping process takes a constant value in this interval and this value is a point of discontinuity for the corresponding external process.

These conditions have a good balance between conditions imposed on the pre-limiting processes and the corresponding limiting processes.

Pre-limiting joint distributions of external processes and internal stopping processes usually have a complicated structure. However, these joint distributions are involved only in the simplest and most natural way via the condition of their joint weak convergence. The conditions of **J**-compactness of pre-limiting external and internal processes involve only their distributions separately. These conditions were thoroughly studied for various classes of stochastic càdlàg processes. The continuity conditions described above involve joint distributions of the limiting external process and the limiting internal stopping process. These limiting distributions are usually simpler than the corresponding pre-limiting joint distributions. This permits to check these continuity conditions in various practically important cases. Because of a good balance, the conditions described above make an effective tool in establishing functional limit theorems for compositions of càdlàg stochastic processes.

In the theorems mentioned above, a model for compositions of scalar (one-dimensional) càdlàg processes was considered. In Theorems 3.8.1 and 3.8.2, analogous results are given for vector compositions of càdlàg processes. In this model, the composition of each component of the external vector process with its own internal stopping process is taken. There, some additional continuity conditions should be imposed on the corresponding limiting external and internal processes. That is, it should be assumed that **(e)** the components of the limiting vector external process do not have, with probability 1, simultaneous jumps at the corresponding limiting stopping points defined by components of the limiting internal stopping processes.

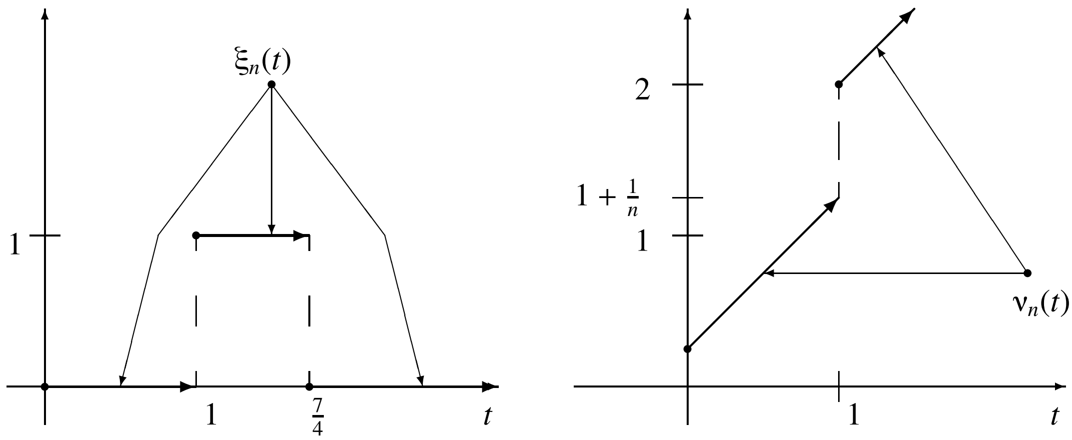
In Section 3.1, examples that clarify the formulation of the problem and conditions for **J**-convergence of compositions of càdlàg stochastic processes are given. In Section 3.2, conditions for **U**-compactness and **U**-convergence are given for compositions of asymptotically continuous càdlàg processes. In Sections 3.3 and 3.4, conditions for **J**-convergence are given for the cases where, respectively, the external limiting process or the internal limiting process is continuous. In Sections 3.5 and 3.6, conditions for **J**-compactness and **J**-convergence are given for general scalar compositions of non-random càdlàg functions and scalar compositions of càdlàg stochastic processes, respectively. This is the case where both the limiting external and internal functions or processes can be discontinuous. In Sections 3.7 and 3.8, similar results are given for vector compositions of non-random càdlàg functions and vector compositions of càdlàg processes, respectively. This section also contains reference remarks.

3.1 Introductory remarks

In this section we discuss some examples that clarify conditions for **J**-convergence of compositions of càdlàg stochastic processes presented in Chapter 3.

3.1.1. Conditions for joint weak convergence and J-compactness. Let us use a natural parameter n, instead of ε, to index the corresponding external processes and internal stopping processes. Actually, we can always assume that $\varepsilon = n^{-1}$ for $n \geq 1$ and $\varepsilon = 0$ for $n = 0$. Let $\xi_n(t)$, $t \geq 0$ and $\nu_n(t)$, $t \geq 0$ be for every $n = 0, 1, \ldots$, respectively, a real-valued càdlàg process and a non-negative and non-decreasing càdlàg process. We are interested in their composition $\xi_n(\nu_n(t)), t \geq 0$, which is also a real-valued càdlàg process.

We are interested in conditions that should be imposed on the processes $\xi_n(t)$, $t \geq 0$



(a) $\xi_n(t), t \geq 0$. (b) $\nu_n(t), t \geq 0$.

Figure 3.1: $\mathcal{G}$: third-type continuity condition.

and $\nu_n(t)$, $t \geq 0$, as to have the following **J**-convergence relation:

$$\xi_n(\nu_n(t)), t \geq 0 \xrightarrow{\mathbf{J}} \xi_0(\nu_0(t)), t \geq 0 \text{ as } n \to \infty. \tag{3.1.1}$$

Natural candidates that are expected to provide the relation (3.1.1) to hold are the following three conditions.

The first one is the condition for joint weak convergence of the external càdlàg processes and the internal stopping processes,

$\mathcal{A}_{16}$: $(\nu_n(t), \xi_n(t)), t \geq 0 \Rightarrow (\nu_0(t), \xi_0(t)), t \geq 0$ as $n \to \infty$.

The second one is the condition of **J**-compactness for the external processes,

$\mathcal{J}_6$: $\lim_{c\to 0} \overline{\lim}_{n\to\infty} \mathsf{P}\{\Delta_J(\xi_n(\cdot), c, T) > \delta\} = 0,\ \delta, T > 0$.

The third one is the condition of **J**-compactness for the internal stopping processes,

$\mathcal{J}_{10}$: $\lim_{c\to 0} \overline{\lim}_{n\to\infty} \mathsf{P}\{\Delta_J(\nu_n(\cdot), c, T) > \delta\} = 0,\ \delta, T > 0$.

Conditions $\mathcal{A}_{16}$ and $\mathcal{J}_6$ provide **J**-convergence for the processes $\xi_n(t)$, $t \geq 0$. Conditions $\mathcal{A}_{16}$ and $\mathcal{J}_{10}$ provide **J**-convergence for the processes $\nu_n(t)$, $t \geq 0$. But these three conditions together, $\mathcal{A}_{16}$, $\mathcal{J}_6$, and $\mathcal{J}_{10}$, do not imply that the vector processes $(\nu_n(t), \xi_n(t))$, $t \geq 0$ or the compositions $\xi_n(\nu_n(t))$, $t \geq 0$ **J**-converge.

Let us consider the following example illustrated in Figures 3.1 and 3.2. We define $\xi_n(t) = \chi_{[1,\frac{7}{4})}(t)$, $t \geq 0$, for $n \geq 1$, and $\nu_n(t) = t + n^{-1}$ if $t < 1$ and $t + 1$ if $t \geq 1$, for $n \geq 1$.

In this case condition $\mathcal{A}_{16}$ obviously holds. The corresponding limiting process $\xi_0(t) = \chi_{[1,\frac{7}{4})}(t)$, $t \geq 0$, and the limiting stopping process $\nu_0(t) = t$ if $t < 1$ and $t + 1$ if $t \geq 1$. Conditions $\mathcal{J}_6$ and $\mathcal{J}_{10}$ also hold.

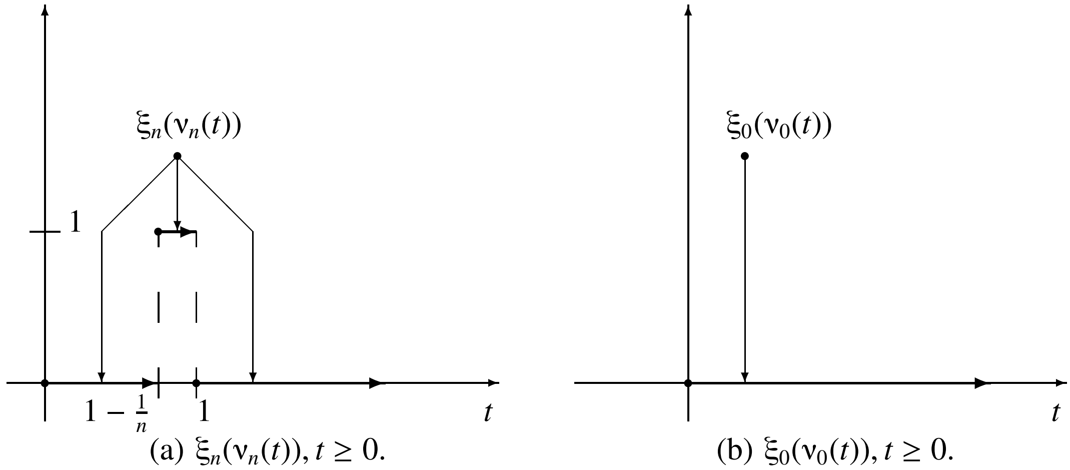


Figure 3.2: $\mathcal{G}$: third-type continuity condition.

In this case, the composition $\xi_n(\nu_n(t)) = \chi_{[1-n^{-1},1)}(t)$, $t \geq 0$, while $\xi_0(\nu_0(t)) = 0, t \geq 0$.

The process $\xi_n(\nu_n(t))$, $t \geq 0$ has two jumps with the values 1 and -1 in the close points $1 - n^{-1}$ and 1, respectively. So, $\Delta_J(\xi_n(\nu_n(\cdot)), c, T) = 1$ if $n^{-1} < c$ and, therefore, $\lim_{c\to 0} \overline{\lim}_{n\to\infty} \Delta_J(\xi_n(\nu_n(\cdot)), c, T) = 1$. This shows that the condition of **J**-compactness does not hold for the processes $\xi_n(\nu_n(t))$, $t \geq 0$.

Therefore, the processes $\xi_n(\nu_n(t))$, $t \geq 0$ do not **J**-converge, since the left limiting value of the limiting stopping process $\nu_0(t), t \geq 0$, at point 1, which is a point of discontinuity for the limiting stopping process, is $\nu_0(1 - 0) = 1$. This value is a point of discontinuity for the external limiting processes $\xi_0(t)$, $t \geq 0$. The example can be easily modified such that the right limiting value of the limiting stopping process at a discontinuity point would cause the same effect.

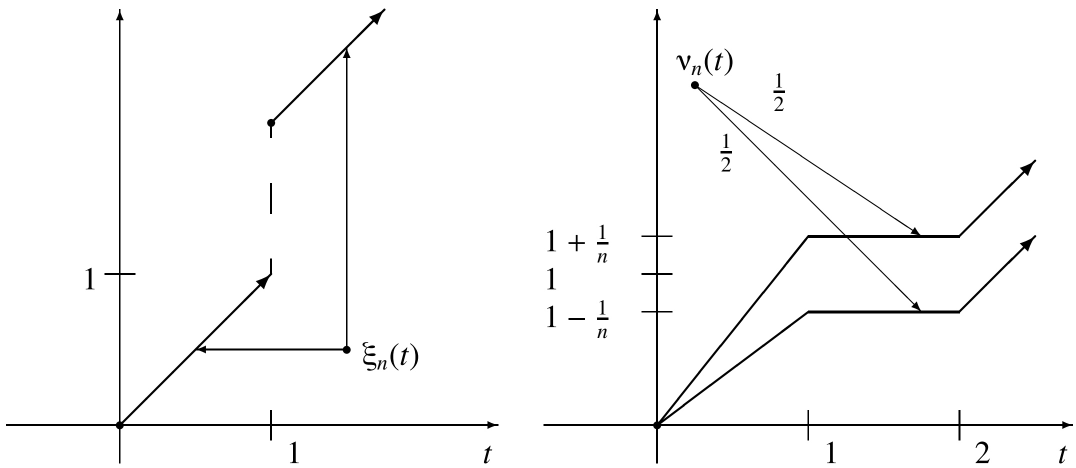
3.1.2. Third-type continuity conditions. The example considered above leads to the following hypothesis. In order to provide (3.1.1), it is enough to add, to the conditions $\mathcal{A}_{16}$, $\mathcal{J}_6$ and $\mathcal{J}_{10}$, the following condition:

$\mathcal{G}_1$: $\mathsf{P}\{\nu_0(t \pm 0) \notin R[\xi_0(\cdot)]$ for $t \in R[\nu_0(\cdot)]\} = 1$.

Here $R[\xi_0(\cdot)]$ and $R[\nu_0(\cdot)]$ are sets of discontinuity points, respectively, for the process $\xi_0(t)$, $t \geq 0$ and the process $\nu_0(t)$, $t \geq 0$.

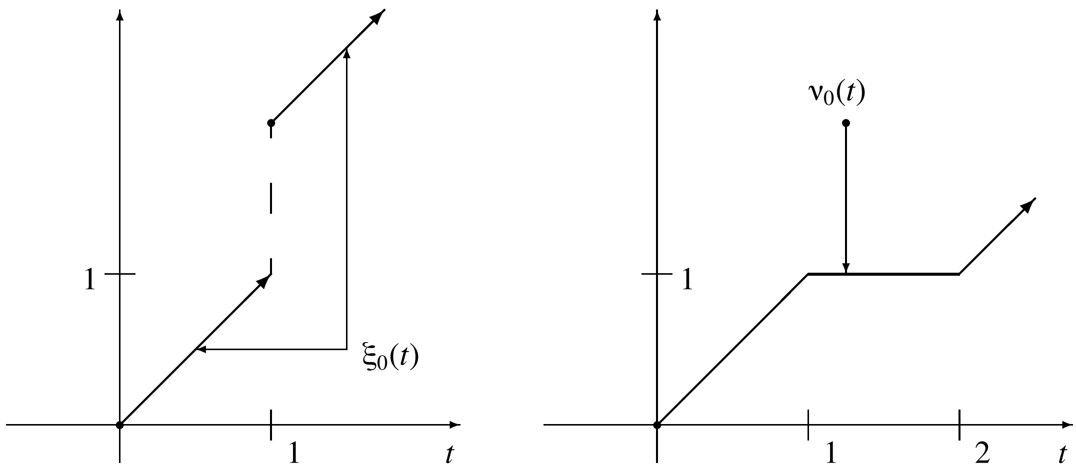
This hypothesis is not true. Conditions $\mathcal{A}_{16}$, $\mathcal{J}_6$, $\mathcal{J}_{10}$, and $\mathcal{G}_1$ do not provide **J**-convergence of the processes $\xi_n(\nu_n(t))$, $t \geq 0$. However, we prove in Theorem 3.6.1 that conditions $\mathcal{A}_{16}$, $\mathcal{J}_6$, $\mathcal{J}_{10}$, and $\mathcal{G}_1$ do provide **J**-compactness of the processes $\xi_n(\nu_n(t))$, $t \geq 0$, that is, the following relation holds:

$$\lim_{c\to 0} \overline{\lim_{n\to\infty}} \mathsf{P}\{\Delta_J(\xi_n(\nu_n(\cdot)), c, T) > \delta\} = 0, \ \delta, T > 0. \quad (3.1.2)$$



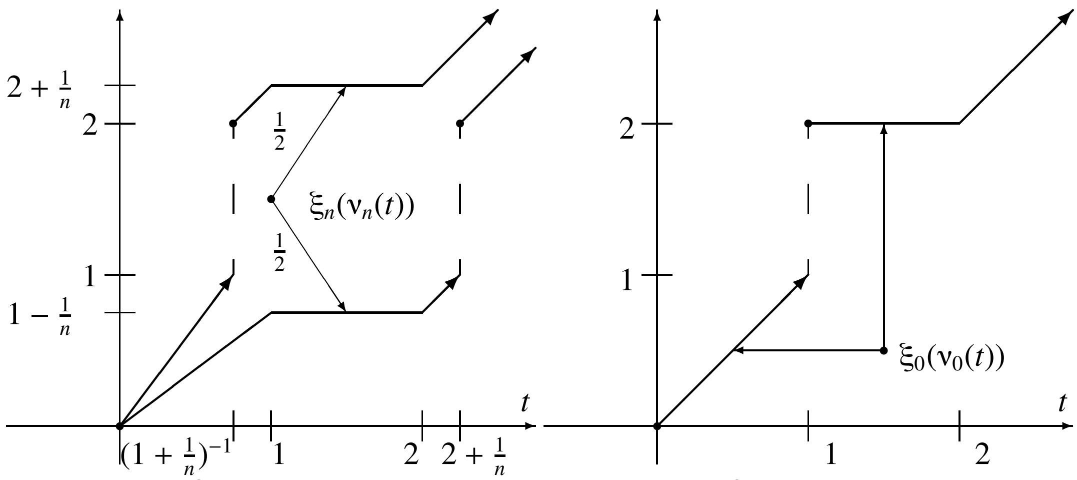
(a) $\xi_n(t), t \geq 0$. (b) $\nu_n(t), t \geq 0$.

Figure 3.3: $\mathcal{E}$: second-type continuity condition.



(a) $\xi_0(t), t \geq 0$. (b) $\nu_0(t), t \geq 0$.

Figure 3.4: $\mathcal{E}$: second-type continuity condition.



(a) $\xi_n(\nu_n(t)), t \geq 0$. (b) $\xi_0(\nu_0(t)), t \geq 0$.

Figure 3.5: $\mathcal{E}$: second-type continuity condition.

Let us consider the following example shown in Figures 3.3, 3.4, and 3.5. Let $\xi_n(t) = t + \chi_{[1,\infty)}(t)$, $t \geq 0$, for $n \geq 1$. Let also, for $n \geq 1$, the process $\nu_n(t), t \geq 0$ have two possible realisations that occur with probability $\frac{1}{2}$. These realisations are $(1 \pm n^{-1})t$ for $t < 1$; $1 \pm n^{-1}$ for $1 \leq t < 2$; $1 \pm n^{-1} + t - 2$ for $t \geq 2$.

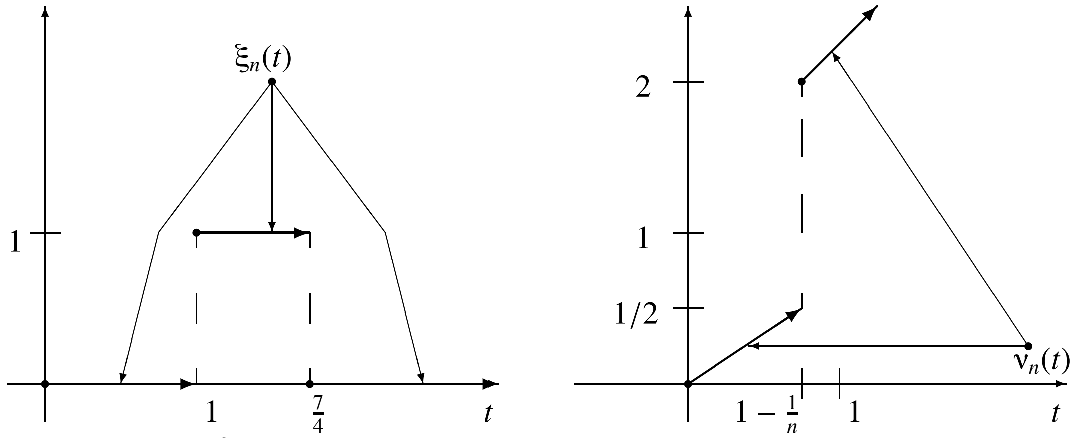
In this case, condition $\mathcal{A}_{\mathbf{16}}$ obviously holds. The limiting process $\xi_0(t) = \chi_{[1,\infty)}(t)$, $t \geq 0$. At the same time, the limiting stopping process $\nu_0(t), t \geq 0$ has only one realisation, which is t for $t < 1$; 1 for $1 \leq t < 2$; $t - 1$ for $t \geq 2$. The conditions of **J**-compactness $\mathcal{J}_{\mathbf{6}}$ and $\mathcal{J}_{\mathbf{10}}$ also hold. Condition $\mathcal{G}_{\mathbf{1}}$ holds as well, since the limiting stopping process $\nu_0(t)$, $t \geq 0$ is continuous.

In this case, the composition $\xi_n(\nu_n(t)), t \geq 0$ also has two possible realisations that occur with probability $\frac{1}{2}$. The first realisation is $(1 + n^{-1})t$ for $0 \leq t < (1 + n^{-1})^{-1}$; $1 + (1 + n^{-1})t$ for $(1 + n^{-1})^{-1} \leq t < 1$; $2 + n^{-1}$ for $1 \leq t < 2$; $n^{-1} + t$ for $t \geq 2$. The second one is $(1 - n^{-1})t$ for $0 \leq t < 1$; $1 - n^{-1}$ for $1 \leq t < 2$; $-n^{-1} - 1 + t$ for $2 \leq t < 2 + n^{-1}$; $-n^{-1} + t$ for $t \geq 2 + n^{-1}$. The composition $\xi_0(\nu_0(t)), t \geq 0$ has only one realisation, t for $0 \leq t < 1$; 2 for $1 \leq t < 2$; t for $t \geq 2$.

The relation of **J**-compactness (3.1.2) holds for these processes, which is consistent with the remarks made above.

At the same time, the processes $\xi_n(\nu_n(t))$, $t \geq 0$ do not **J**-converge to the corresponding limiting process $\xi_0(\nu_0(t))$, $t \geq 0$. Indeed, for every $t \in [1, 2)$, the random variable $\xi_n(\nu_n(t))$ takes two values, $2 + n^{-1}$ and $1 - n^{-1}$, with probability $\frac{1}{2}$. We also have that $\xi_0(\nu_0(t)) = 2$ with probability 1. So, for every $t \in [1, 2)$, the random variables $\xi_n(\nu_n(t))$ do not weakly converge to the random variable $\xi_0(\nu_0(t))$.

In this example, the condition of weak convergence does not hold for the processes



(a) $\xi_n(t), t \geq 0$. (b) $\nu_n(t), t \geq 0$.

Figure 3.6: Compositions, which **J**-converge.

$\xi_n(\nu_n(t))$, $t \geq 0$ in the interval $[1, 2)$, because the limiting process $\nu_0(t)$ takes the constant value 1 in the interval $[1, 2)$ and this value 1 is a point of discontinuity for the external limiting process $\xi_0(t)$, $t \geq 0$.

3.1.3. Second-type continuity conditions. The example considered above leads to the following hypothesis. In order to provide (3.1.1), it is enough to add, to $\mathcal{A}_{16}$, $\mathcal{J}_6$, and $\mathcal{J}_{10}$, condition $\mathcal{G}_1$ and the following condition already introduced in Section 2.1:

$\mathcal{E}_1$: $\mathsf{P}\{\nu_0(t') = \nu_0(t'') \in R[\xi_0(\cdot)]\} = 0$ for $0 \leq t' < t'' < \infty$.

Theorem 2.2.1 states that conditions $\mathcal{A}_{16}$, $\mathcal{J}_6$, and $\mathcal{E}_1$ imply that there exists some set S dense in $[0, \infty)$ such that

$$\xi_n(\nu_n(t)), t \in S \Rightarrow \xi_0(\nu_0(t)), t \in S \text{ as } n \to \infty. \tag{3.1.3}$$

In order for the set of weak convergence S to contain a point 0, one can additionally assume the following condition:

$\mathcal{C}_5^{(0)}$: $\mathsf{P}\{\nu_0(0) \in R[\xi_0(\cdot)]\} = 0$.

We prove in Theorem 3.6.2 that the conditions $\mathcal{A}_{16}$, $\mathcal{J}_6$, $\mathcal{J}_{10}$, together with the continuity conditions $\mathcal{G}_1$, $\mathcal{E}_1$, and $\mathcal{C}_5^{(0)}$, imply the desirable asymptotical relation (3.1.1), i.e., that the compositions $\xi_n(\nu_n(t))$, $t \geq 0$ **J**-converge to $\xi_0(\nu_0(t))$, $t \geq 0$ as $n \to \infty$.

In both examples given above, the vector processes $(\nu_n(t), \xi_n(t)), t \geq 0$ **J**-converge. However, as was mentioned in Subsections 1.5.11 and 1.6.15, the compositions $\xi_n(\nu_n(t))$, $t \geq 0$ can **J**-converge even if the vector processes $(\nu_n(t), \xi_n(t)), t \geq 0$ do not **J**-converge.

Let us modify the first example considered in Subsection 3.1.1. Figure 3.6 illustrates this modified example. We use the same external processes $\xi_n(t) = \chi_{[1,\frac{7}{4})}(t)$, $t \geq 0$, for

$n \geq 1$, but define new internal stopping processes $\nu_n(t) = \frac{1}{2}(1 - n^{-1})^{-1}t$ if $t < 1 - n^{-1}$ and $t + 1 + n^{-1}$ if $t \geq 1 - n^{-1}$, for $n \geq 1$.

In this case, the corresponding limiting process $\xi_0(t) = \chi_{[1,\frac{7}{4})}(t)$, $t \geq 0$, and the limiting stopping process $\nu_0(t) = \frac{1}{2}t$ if $t < 1$ and $t+1$ if $t \geq 1$. Hence, $\xi_n(\nu_n(t)) = 0, t \geq 0$, for $n \geq 1$ as well as for $n = 0$. Therefore, the compositions $\xi_n(\nu_n(t)), t \geq 0$ **J**-converge. This is consistent with Theorem 3.6.2, since all conditions of this theorem listed above, hold. However, the vector processes $(\nu_n(t), \xi_n(t)), t \geq 0$ do not **J**-converge, since the process $(\nu_n(t), \xi_n(t))$ has two large jumps with the absolute values $\frac{3}{2}$ and 1 in the close points $1 - n^{-1}$ and 1, respectively.

As the example above shows, Theorem 3.6.2 extends the setting of **J**-continuous mapping theorem with respect to the composition mapping. Additional comments are given in Subsections 3.5.3 and 3.6.3.

3.1.4. Weakened second-type continuity conditions. Let go back to the example considered in Subsection 2.1.8 and shown in Figures 2.3, 2.4, and 2.5. In this example, conditions $\mathcal{A}_{16}$, $\mathcal{J}_6$, $\mathcal{J}_{10}$, and the continuity conditions $\mathcal{G}_1$ hold.

In the case **(a)** $p_0 = 1$, conditions $\mathcal{E}_1$, and $\mathcal{C}_5^{(0)}$ hold. Therefore, according Theorem 3.6.2 mentioned above, the compositions $\xi_n(\nu_n(t))$, $t \geq 0$ **J**-converge to $\xi_0(\nu_0(t))$, $t \geq 0$ as $n \to \infty$.

In the case **(b)** $q_0 = 0, p_0 < 1$, conditions $\mathcal{E}_1$, and $\mathcal{C}_5^{(0)}$ do not hold. However, in this case, the following condition, which is weaker than $\mathcal{E}_1$, holds:

$\mathcal{F}_1$: $\lim_{0<c\to 0} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\alpha_{\varepsilon k}^{(\delta)} - c \leq \nu_\varepsilon(t'), \nu_\varepsilon(t'') < \alpha_{\varepsilon k}^{(\delta)}\} = 0$ for $0 \leq t' < t'' < \infty$, $\delta > 0$ and $k \geq 1$.

Here $\alpha_{nk}^{(\delta)}, k = 1, 2, \dots$ are the successive moments of jumps of the process $\xi_n(t), t \geq 0$, which have the absolute values of jumps greater than or equal to $\delta > 0$. By the definition, $\alpha_{nk}^{(\delta)} = \infty$ if there exist less than k such points.

Also, the following condition, which is weaker than $\mathcal{C}_5^{(0)}$, holds:

$\mathcal{D}_6^{(0)}$: $\lim_{c\to 0} \overline{\lim}_{n\to\infty} \mathsf{P}\{\nu_n(0) \in [\alpha_{nk}^{(\delta)} - c, \alpha_{nk}^{(\delta)})\} = 0$ for $\delta > 0$ and $k \geq 1$.

We prove in Theorem 3.4.3 that the conditions $\mathcal{A}_{16}$, $\mathcal{J}_6$, $\mathcal{J}_{10}$, together with the continuity conditions $\mathcal{G}_1$, $\mathcal{F}_1$, and $\mathcal{D}_6^{(0)}$, also imply the desirable asymptotical relation (3.1.1), i.e., that the compositions $\xi_n(\nu_n(t))$, $t \geq 0$ **J**-converge to $\xi_0(\nu_0(t))$, $t \geq 0$ as $n \to \infty$.

3.1.5. Vector compositions of càdlàg processes and fourth-type continuity conditions. In a model, one considers vector càdlàg process $\boldsymbol{\xi}_n(t) = (\xi_{ni}(t), i = 1, \dots, m)$, $t \geq 0$ with real-valued components, vector càdlàg process $\boldsymbol{\nu}_n(t) = (\nu_{ni}(t), i = 1, \dots, m)$, $t \geq 0$ with non-negative and non-decreasing components, and their vector composition $\boldsymbol{\zeta}_n(t) = (\xi_{ni}(\nu_{ni}(t)), i = 1, \dots, m)$, $t \geq 0$, which is also a vector càdlàg process with real-valued components.

Let us assume that the following condition of joint weak convergence holds:

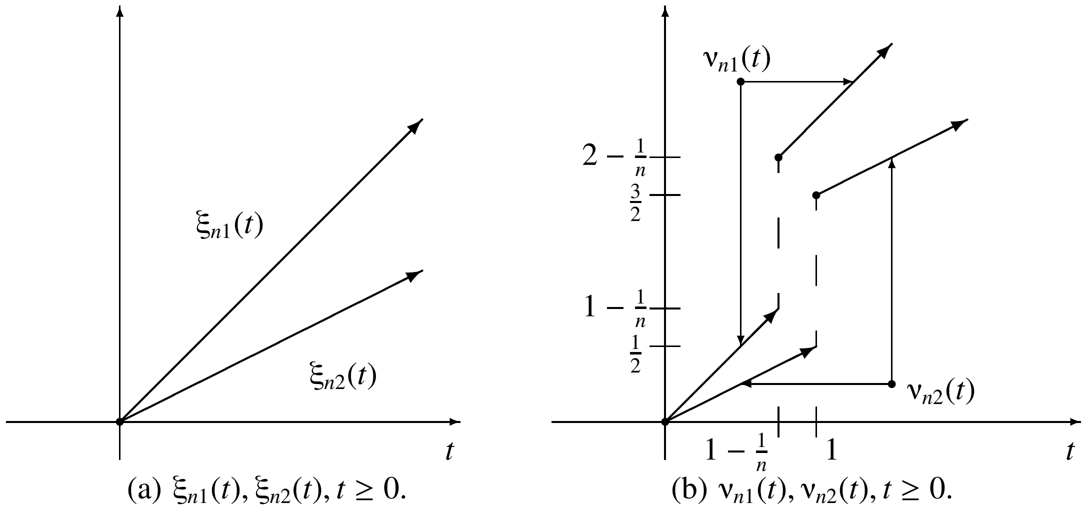


Figure 3.7: $\mathcal{H}$: fourth-type continuity condition.

$\mathcal{A}_{33}$: $(\mathbf{v}_n(t), \boldsymbol{\xi}_n(t)), t \geq 0 \Rightarrow (\mathbf{v}_0(t), \boldsymbol{\xi}_0(t)), t \geq 0$ as $n \to \infty$.

We are interested in additional conditions to be imposed on the processes $(\mathbf{v}_n(t), \boldsymbol{\xi}_\varepsilon(t))$, $t \geq 0$ as to provide the following relation of $\mathbf{J}$-convergence:

$$\begin{aligned} \boldsymbol{\zeta}_n(t) &= (\xi_{ni}(\nu_{ni}(t)), i = 1, \ldots, m), t \geq 0 \\ &\xrightarrow{\mathbf{J}} \boldsymbol{\zeta}_0(t) = (\xi_{0i}(\nu_{0i}(t)), i = 1, \ldots, m), t \geq 0 \text{ as } \varepsilon \to 0. \end{aligned} \tag{3.1.4}$$

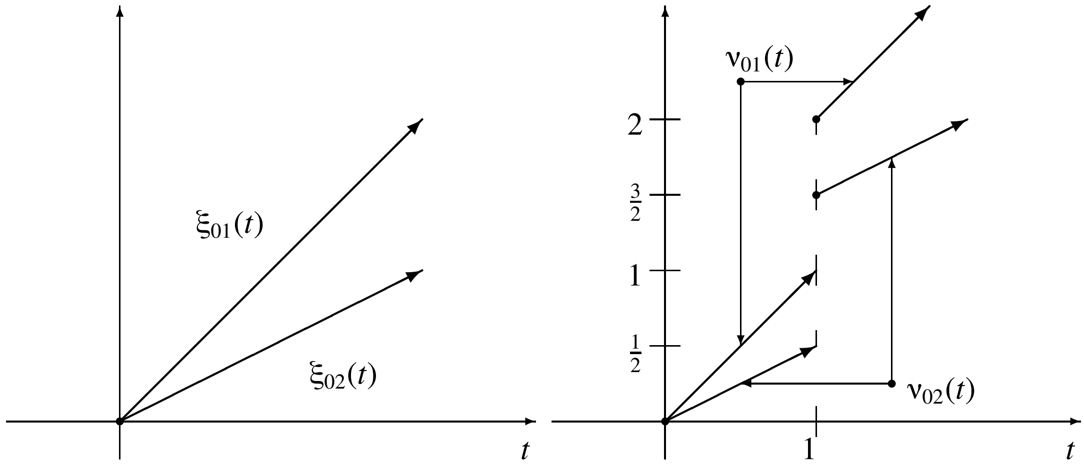
Let us also assume that conditions $\mathcal{J}_6$, $\mathcal{J}_{10}$, $\mathcal{G}_1$, $\mathcal{E}_1$, and $\mathcal{C}_5^{(0)}$ hold for the processes $\xi_{ni}(t), t \geq 0$ and $\nu_{ni}(t), t \geq 0$, for every $i = 1, \ldots m$.

These assumptions imply that, for every $i = 1, \ldots, m$,

$$\xi_{ni}(\nu_{ni}(t)), t \geq 0 \xrightarrow{\mathbf{J}} \xi_{0i}(\nu_{0i}(t)), t \geq 0 \text{ as } n \to \infty. \tag{3.1.5}$$

The following two examples show that condition $\mathcal{A}_{33}$, together with all conditions $\mathcal{J}_6$, $\mathcal{J}_{10}$, $\mathcal{G}_1$, $\mathcal{E}_1$, and $\mathcal{C}_5^{(0)}$, does not, however, provide $\mathbf{J}$-convergence of the vector compositions $\boldsymbol{\zeta}_n(t), t \geq 0$.

Figures 3.7, 3.8, and 3.9 illustrate the first example. Let $\xi_{n1}(t) = t, t \geq 0$ and $\xi_{n2}(t) = \frac{1}{2}t$, $t \geq 0$, for $n = 1, 2, \ldots$. Let also $\nu_{n1}(t) = t + \chi_{[1-n^{-1},\infty)}(t)$, $t \geq 0$, while $\nu_{n2}(t) = \frac{1}{2}t + \chi_{[1,\infty)}(t)$, $t \geq 0$, for $n = 1, 2, \ldots$. In this case, condition $\mathcal{A}_{33}$ obviously holds. The corresponding limiting processes are $\xi_{01}(t) = t, t \geq 0$ and $\xi_{02}(t) = \frac{1}{2}t$, $t \geq 0$, while $\nu_{01}(t) = t + \chi_{[1,\infty)}(t), t \geq 0$ and $\nu_{02}(t) = \frac{1}{2}t + \chi_{[1,\infty)}(t)$, $t \geq 0$. Also, conditions $\mathcal{J}_6$ and $\mathcal{J}_{10}$, as well as conditions $\mathcal{G}_1$, $\mathcal{E}_1$, and $\mathcal{C}_5^{(0)}$ hold for $i = 1, 2$. However, $\xi_{n1}(\nu_{n1}(t)) = t + \chi_{[1-n^{-1},\infty)}(t), t \geq 0$ and $\xi_{n2}(\nu_{n2}(t)) = \frac{1}{4}t + \frac{1}{2}\chi_{[1,\infty)}(t)$. Therefore, the vector process $\boldsymbol{\zeta}_n(t) = (\xi_{n1}(\nu_{n1}(t)), \xi_{n2}(\nu_{n2}(t)))$, $t \geq 0$ has two jumps with the absolute values 1 and $\frac{1}{2}$ in close points $1 - n^{-1}$ and 1, respectively.



(a) $\xi_{01}(t), \xi_{02}(t), t \geq 0$. (b) $\nu_{01}(t), \nu_{02}(t), t \geq 0$.

Figure 3.8: $\mathcal{H}$: fourth-type continuity condition.

This shows that the condition of **J**-compactness does not hold for the vector processes $\boldsymbol{\zeta}_n(t)$, $t \geq 0$ and, therefore, they do not **J**-converge.

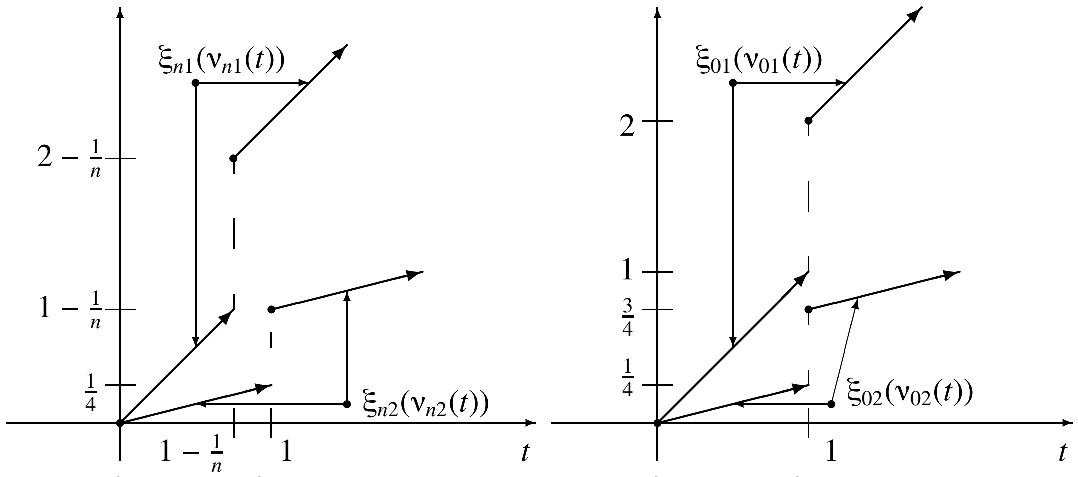
Figures 3.10, 3.11, and 3.12 illustrate the second example. Let $\xi_{n1}(t) = \frac{1}{4}t + \frac{1}{4}\chi_{[2,\infty)}(t)$, $t \geq 0$ and $\xi_{n2}(t) = t + \chi_{[1-n^{-1},\infty)}(t)$, $t \geq 0$, for $n = 1, 2 \ldots$. Let also $\nu_{n1}(t) = 2t$, $t \geq 0$ and $\nu_{n2}(t) = t$, $t \geq 0$, for $n = 1, 2, \ldots$. Again, condition $\mathcal{A}_{33}$ holds and the corresponding limiting processes are $\xi_{01}(t) = \frac{1}{4}t + \frac{1}{4}\chi_{[2,\infty)}(t)$, $t \geq 0$ and $\xi_{02}(t) = t + \chi_{[1,\infty)}(t)$, $t \geq 0$, while $\nu_{01}(t) = 2t$, $t \geq 0$ and $\nu_{01}(t) = t$, $t \geq 0$. Also, conditions $\mathcal{J}_6$ and $\mathcal{J}_{10}$, and also conditions $\mathcal{G}_1$, $\mathcal{E}_1$, and $\mathcal{C}_5^{(0)}$ hold for $i = 1, 2$. However, $\xi_{n1}(\nu_{n1}(t)) = \frac{1}{2}t + \frac{1}{4}\chi_{[2,\infty)}(2t) = \frac{1}{2}t + \frac{1}{4}\chi_{[1,\infty)}(t), t \geq 0$ and $\xi_{n2}(\nu_{n2}(t)) = t + \chi_{[1-n^{-1},\infty)}(t), t \geq 0$. The vector process $\boldsymbol{\zeta}_n(t) = (\xi_{n1}(\nu_{n1}(t)), \xi_{n2}(\nu_{n2}(t)))$, $t \geq 0$ has two jumps with the absolute values 1 and $\frac{1}{4}$ in the close points $1 - n^{-1}$ and 1, respectively.

So, the condition of **J**-compactness does not hold for the processes $\boldsymbol{\zeta}_n(t)$, $t \geq 0$ and, therefore, they do not **J**-converge.

In the first example, the vector process $\boldsymbol{\zeta}_n(t)$, $t \geq 0$ has two large jumps in the close points $1 - n^{-1}$ and 1. These jumps appear, because the first and the second components of the internal vector stopping process $\boldsymbol{\nu}_n(t) = (\nu_{n1}(t), \nu_{n2}(t))$, $t \geq 0$ has jumps in the close points $1 - n^{-1}$ and 1, respectively.

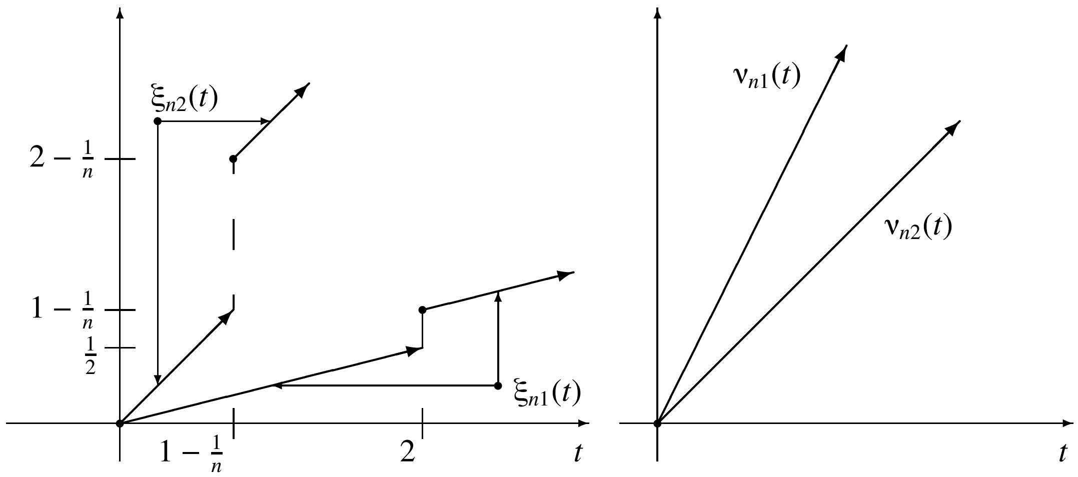
In the second example, the vector process $\boldsymbol{\zeta}_n(t)$, $t \geq 0$ also has two large jumps in the close points $1 - n^{-1}$ and 1. These jumps occur, since the first and the second components of the external vector process $\boldsymbol{\xi}_n(t) = (\xi_{n1}(t), \xi_{n2}(t))$, $t \geq 0$ has jumps in the points $\nu_{n1}(1) = 2$ and $\nu_{n2}(1 - n^{-1}) = 1 - n^{-1}$.

These examples lead to the following hypothesis. In order to provide **J**-convergence of the vector compositions $\boldsymbol{\zeta}_n(t)$, $t \geq 0$, it is sufficient to supplement the conditions listed above with the following fourth-type continuity condition:



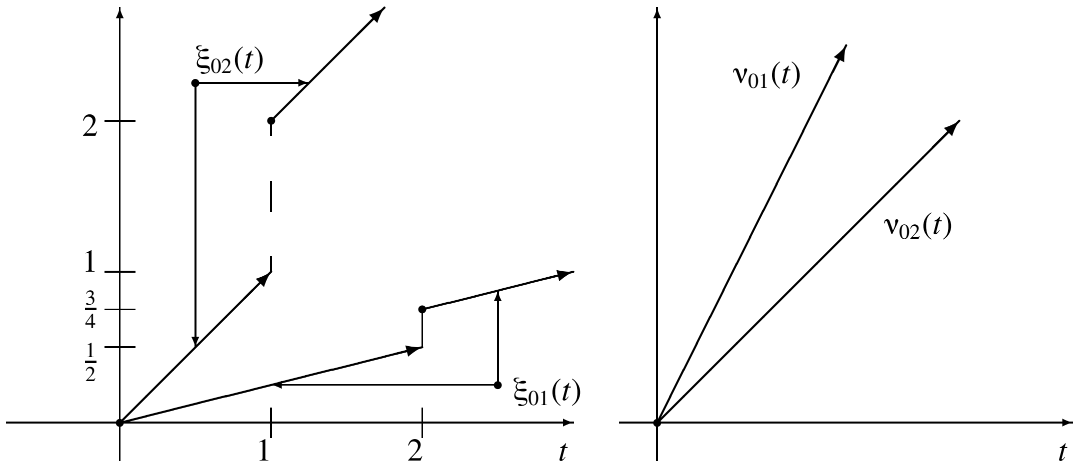
(a) $\xi_{n1}(\nu_{n1}(t)), \xi_{n2}(\nu_{n2}(t)), t \geq 0$. (b) $\xi_{01}(\nu_{01}(t)), \xi_{02}(\nu_{02}(t)), t \geq 0$.

Figure 3.9: $\mathcal{H}$: fourth-type continuity condition.



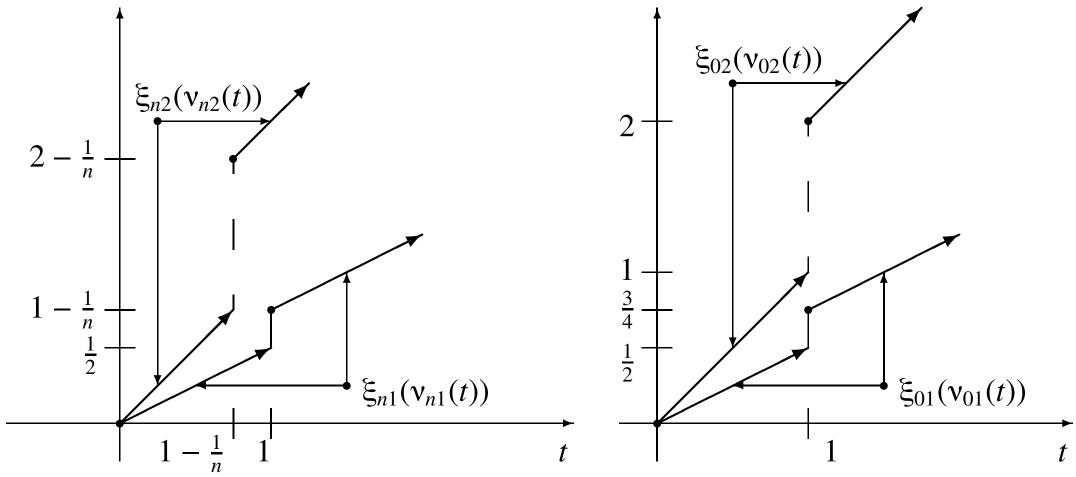
(a) $\xi_{n1}(t), \xi_{n2}(t), t \geq 0$. (b) $\nu_{n1}(t), \nu_{n2}(t), t \geq 0$.

Figure 3.10: $\mathcal{H}$: fourth-type continuity condition.



(a) $\xi_{01}(t), \xi_{02}(t), t \geq 0$. (b) $\nu_{01}(t), \nu_{02}(t), t \geq 0$.

Figure 3.11: $\mathcal{H}$: fourth-type continuity condition.



(a) $\xi_{n1}(\nu_{n1}(t)), \xi_{n2}(\nu_{n2}(t)), t \geq 0$. (b) $\xi_{01}(\nu_{01}(t)), \xi_{02}(\nu_{02}(t)), t \geq 0$.

Figure 3.12: $\mathcal{H}$: fourth-type continuity condition.

$\mathcal{H}_2$: $\mathsf{P}\{\sum_{i=1}^m \chi(t \in R[\xi_{0i}(\nu_{0i}(\cdot))]) \leq 1$ for $t \geq 0\} = 1$.

This hypothesis is true, as shown in Theorem 3.8.5. However, condition $\mathcal{H}_2$ is too restrictive. It usually prohibits the processes $\xi_{0i}(\cdot)$ to have synchronous jumps (for different i) at the points $\nu_{0i}(t)$ for every $t \geq 0$, and the processes $\nu_{0i}(\cdot)$ to have simultaneous jumps (for different i) at a point t for every $t \geq 0$. The latter requirement is restrictive. In many applications, the limiting internal vector stopping process has the form $\mathbf{v}_0(t) = (q_i\nu_0(t), i = 1, \dots, m)$, $t \geq 0$, where q_i, $i = 1, \dots, m$ are positive constants and $\nu_0(t)$, $t \geq 0$ is a scalar non-negative and non-decreasing càdlàg process. In this case, the processes $q_i\nu_0(t)$ have simultaneous jumps (for every i) at any jump point of the process $\nu_0(t)$.

In Theorem 3.8.2 we use a weaker modification of this condition. We show that under the natural additional assumption that

$$\mathbf{v}_n(t), t \geq 0 \xrightarrow{\mathbf{J}} \mathbf{v}_0(t), t \geq 0 \text{ as } n \to \infty, \tag{3.1.6}$$

condition $\mathcal{H}_2$ can be replaced with the following weaker continuity condition:

$\mathcal{H}_3$: $\mathsf{P}\{\sum_{i=1}^m \chi(\nu_{0i}(t) \in R[\xi_{0i}(\cdot)]) \leq 1$ for $t \geq 0\} = 1$.

Condition $\mathcal{H}_3$ only prohibits the processes $\xi_{0i}(\cdot)$ to have synchronous jumps (for different i) at the points $\nu_{0i}(t)$ for every $t \geq 0$.

Let us return to the first example considered in this subsection. Condition $\mathcal{H}_2$ does not hold in this case, but condition $\mathcal{H}_3$ does. The processes $\zeta_n(t)$, $t \geq 0$ do not **J**-converge, because condition (3.1.6) is not fulfilled. However, let us slightly modify the example and assume that $\nu_{n1}(t) = t + \chi_{[1-n^{-1},\infty)}(t)$, $t \geq 0$ and $\nu_{n2}(t) = \frac{1}{2}t + \chi_{[1-n^{-1},\infty)}(t)$, $t \geq 0$. In this case again, $\mathcal{H}_2$ does not hold, but condition $\mathcal{H}_3$ does. Also, condition (3.1.6) holds true. The processes $\zeta_n(t)$, $t \geq 0$ **J**-converge to the process $\zeta_0(t)$, $t \geq 0$ as $n \to \infty$.

In conclusion, we would like to note that continuity conditions $\mathcal{G}_1$, $\mathcal{E}_1$ (for every $i = 1, \dots, m$) and $\mathcal{H}_3$ are satisfied in many important cases. The corresponding examples are given in Subsection 3.8.3.

3.2 Compositions with asymptotically continuous components

In this section, we formulate conditions for **U**-convergence of compositions of asymptotically continuous processes. In the case of convergence to continuous processes, there is no essential distinction between scalar and vector compositions, hence we do not consider the scalar case separately.

3.2.1. U-convergence of vector compositions of asymptotically continuous càdlàg processes. Let, for every $\varepsilon \geq 0$, $\boldsymbol{\xi}_\varepsilon(t) = (\xi_{\varepsilon i}(t), i = 1, \dots, m)$, $t \geq 0$ be a vector càdlàg process with real-valued components, and $\mathbf{v}_\varepsilon(t) = (\nu_{\varepsilon i}(t), i = 1, \dots, m)$, $t \geq 0$ be a vector càdlàg process with non-negative and non-decreasing components. We will consider the *vector composition* $\boldsymbol{\zeta}_\varepsilon(t) = (\xi_{\varepsilon i}(\nu_{\varepsilon i}(t)), i = 1, \dots, m)$, $t \geq 0$ that is also a càdlàg process with real-valued components.

The following condition is a basis for subsequent considerations:

$\mathcal{A}_{34}$: $(\mathbf{v}_\varepsilon(s), \boldsymbol{\xi}_\varepsilon(t)), (s,t) \in V \times U \Rightarrow (\mathbf{v}_0(s), \boldsymbol{\xi}_0(t)), (s,t) \in V \times U$ as $\varepsilon \to 0$, where U and V are some subsets of $[0, \infty)$ that are dense in this interval and contain the point 0.

We also assume that the following conditions of **U**-compactness hold for the external processes and the internal stopping processes:

$\mathcal{U}_4$: $\lim_{c \to 0} \overline{\lim}_{\varepsilon \to 0} \mathsf{P}\{\Delta_U(\boldsymbol{\xi}_\varepsilon(\cdot), c, T) > \delta\} = 0, \ \delta, T > 0,$

and

$\mathcal{U}_5$: $\lim_{c \to 0} \overline{\lim}_{\varepsilon \to 0} \mathsf{P}\{\Delta_U(\mathbf{v}_\varepsilon(\cdot), c, T) > \delta\} = 0, \ \delta, T > 0.$

Conditions $\mathcal{A}_{34}$ and $\mathcal{U}_4$ imply **U**-convergence of the processes $\boldsymbol{\xi}_\varepsilon(t)$, $t \geq 0$.

Since $\mathcal{U}_4$ includes also the case $\varepsilon = 0$, this condition also implies that:

$\mathcal{B}_2$: $\boldsymbol{\xi}_0(t)$, $t \geq 0$ is an a.s. continuous process.

Conditions $\mathcal{A}_{34}$ and $\mathcal{U}_5$ imply **U**-convergence of the processes $\mathbf{v}_\varepsilon(t)$, $t \geq 0$.

Condition $\mathcal{U}_5$ implies also the following condition:

$\mathcal{B}_3$: $\mathbf{v}_0(t)$, $t \geq 0$ is an a.s. continuous process.

Since both limiting processes $\boldsymbol{\xi}_0(t)$, $t \geq 0$ and $\mathbf{v}_0(t)$, $t \geq 0$ are a.s. continuous, their composition $\boldsymbol{\zeta}_0(t)$, $t \geq 0$ is also an a.s. continuous process.

It follows from the remarks above and Theorem 1.6.11 that the sets U and V in $\mathcal{A}_{34}$ can be enlarged to the interval $[0, \infty)$ under conditions $\mathcal{U}_4$ and $\mathcal{U}_5$, respectively.

It is also useful to note that conditions $\mathcal{U}_4$ and $\mathcal{U}_5$ are equivalent, respectively, to the following conditions:

$\mathcal{U}_4'$: $\lim_{c \to 0} \overline{\lim}_{\varepsilon \to 0} \mathsf{P}\{\Delta_U(\xi_{\varepsilon i}(\cdot), c, T) > \delta\} = 0, \ \delta, T > 0, i = 1, \dots, m,$

and

$\mathcal{U}_5'$: $\lim_{c \to 0} \overline{\lim}_{\varepsilon \to 0} \mathsf{P}\{\Delta_U(\nu_{\varepsilon i}(\cdot), c, T) > \delta\} = 0, \ \delta, T > 0, i = 1, \dots, m.$

Let us introduce the following condition:

$\mathcal{A}_{35}$: $\mathbf{v}_\varepsilon(s), s \in V \Rightarrow \mathbf{v}_0(s), s \in V$ as $\varepsilon \to 0$, where V is a subset of $[0, \infty)$ that is dense in this interval and contains the point 0.

The following lemma allows to simplify conditions for **U**-convergence of compositions of càdlàg processes.

Lemma 3.2.1. *Let condition $\mathcal{A}_{35}$ hold. Then conditions $\mathcal{B}_3$ and $\mathcal{U}_5$ are equivalent.*

Proof of Lemma 3.2.1. It should only be proved that conditions $\mathcal{A}_{35}$ and $\mathcal{B}_3$ imply $\mathcal{U}_5$. Let us choose an arbitrary positive $T \in V$. It is always possible to construct a sequence of partitions $0 = t_{0,n} < t_{1,n} < \ldots < t_{n,n} = T$, $n \geq 1$, of the interval $[0, T]$ such that: **(a)** $t_{k,n} \in V$, $k = 0, \ldots, n$, $n \geq 1$; **(b)** $h_n = \max_{0 \leq k \leq n-1}(t_{k+1,n} - t_{k,n}) \to 0$ as $n \to \infty$.

Since the processes $\nu_{\varepsilon i}(t)$, $t \in [0, T]$, $i = 1, \ldots, m$ are monotone,

$$
\begin{aligned}
\Delta_U(\nu_{\varepsilon i}(\cdot), h_n, T) &\\
&\leq 2 \max_{1 \leq k \leq n} \sup_{t \in [t_{k-1,n}, t_{k,n}]} |\nu_{\varepsilon i}(t) - \nu_{\varepsilon i}(t_{k-1,n})| \\
&\leq 2 \max_{1 \leq k \leq n} (\nu_{\varepsilon i}(t_{k,n}) - \nu_{\varepsilon i}(t_{k-1,n})) \\
&= \alpha_{\varepsilon i}(n).
\end{aligned}
\tag{3.2.1}
$$

It readily follows from condition $\mathcal{A}_{35}$ that for all $i = 1, \ldots, m$ and $n \geq 1$,

$$
\alpha_{\varepsilon i}(n) \Rightarrow \alpha_{0i}(n) \text{ as } \varepsilon \to 0. \tag{3.2.2}
$$

The process $\nu_{0i}(t)$, $t \in [0, T]$ is continuous with probability 1 in the interval $[0, T]$ and, therefore, it is also uniformly continuous with probability 1 in this interval for every $i = 1, \ldots, m$. This implies that for every $i = 1, \ldots m$,

$$
\alpha_{0i}(n) \xrightarrow{\mathsf{P1}} 0 \text{ as } n \to \infty. \tag{3.2.3}
$$

For an arbitrary $\delta > 0$, one can always choose $\delta' \in (0, \delta)$ such that the point δ'/m is a continuity point for the distribution function of the random variable $\alpha_{0i}(n)$ for every $i = 1, \ldots, m$ and $n \geq 1$. By using (3.2.1), (3.2.2), and (3.2.3), we get

$$
\begin{aligned}
&\overline{\lim_{\varepsilon \to 0}} \mathsf{P}\{\Delta_U(\boldsymbol{\nu}_\varepsilon(\cdot), h_n, T) > \delta\} \\
&\quad \leq \sum_{i=1}^m \overline{\lim_{\varepsilon \to 0}} \mathsf{P}\{\Delta_U(\nu_{\varepsilon i}(\cdot), h_n, T) > \delta/m\} \\
&\quad \leq \sum_{i=1}^m \overline{\lim_{\varepsilon \to 0}} \mathsf{P}\{\alpha_{\varepsilon i}(n) > \delta'/m\} = \sum_{i=1}^m \mathsf{P}\{\alpha_{0i}(n) > \delta'/m\} \to 0 \text{ as } n \to \infty.
\end{aligned}
\tag{3.2.4}
$$

The proof is completed. □

The following lemma is a direct corollary of Lemma 3.2.1.

Lemma 3.2.2. *Let conditions* $\mathcal{A}_{35}$ *and* $\mathfrak{B}_3$ *hold. Then*

$$\mathbf{v}_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{U}} \mathbf{v}_0(t), t \geq 0 \text{ as } \varepsilon \to 0.$$

The following theorem is the main result of this section. In the case of scalar compositions of càdlàg processes it belongs to Billingsley (1968). An extension of this result to the case of vector compositions of càdlàg processes, presented below, was given in Silvestrov (1974).

Theorem 3.2.1. *Let conditions* $\mathcal{A}_{34}$, $\mathfrak{U}_4$, *and* $\mathfrak{B}_3$ *hold. Then*

$$\zeta_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{U}} \zeta_0(t), t \geq 0 \text{ as } \varepsilon \to 0.$$

Proof of Theorem 3.2.1. Condition $\mathcal{A}_{34}$ implies that condition $\mathcal{A}_{22}^{\mathbf{V}}$ holds with the set V as in $\mathcal{A}_{34}$. Condition $\mathfrak{U}_4$ obviously implies that condition $\mathfrak{J}_4$ holds. Since $\xi_0(t)$, $t \geq 0$ is an a.s. continuous process, condition $\mathfrak{C}_6^{\mathbf{W}}$ holds with the set $W = [0, \infty)$. Therefore, it follows from Theorem 2.7.1 that, for the set V,

$$\zeta_\varepsilon(t), t \in V \Rightarrow \zeta_0(t), t \in V \text{ as } \varepsilon \to 0. \tag{3.2.5}$$

By $\mathcal{A}_{34}$, the set V is everywhere dense in $[0, \infty)$ and contains 0.

To prove the theorem, we must also supplement the relation of weak convergence (3.2.5) with the following relation of **U**-compactness:

$$\lim_{c\to 0} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\Delta_U(\zeta_\varepsilon(\cdot), c, T) > \delta\} = 0, \ T, \delta > 0. \tag{3.2.6}$$

Now, we are going to use the following estimate which is valid for composition of any real-valued càdlàg function $x(t)$, $t \geq 0$ and any non-negative càdlàg function $y(t)$, $t \geq 0$:

$$\Delta_U(x(y(\cdot)), c, T)\chi(\Delta_U(y(\cdot), c, T) \leq c', \sup_{0\leq t\leq T} y(t) \leq T') \leq \Delta_U(x(\cdot), c', T'). \tag{3.2.7}$$

Using (3.2.7) and taking into account the monotonicity of the processes $\nu_{\varepsilon i}(t)$, $t \geq 0$, $i = 1, \dots, m$, we get

$$\begin{aligned}
&\mathsf{P}\{\Delta_U(\zeta_\varepsilon(\cdot), c, T) > \delta\} \\
&\quad \leq \sum_{i=1}^m \mathsf{P}\{\Delta_U(\xi_{\varepsilon i}(\nu_{\varepsilon i}(\cdot)), c, T) > \delta/m\} \\
&\quad \leq \sum_{i=1}^m (\mathsf{P}\{\Delta_U(\xi_{\varepsilon i}(\nu_{\varepsilon i}(\cdot)), c, T) > \delta/m, \Delta_U(\nu_{\varepsilon i}(\cdot), c, T) \leq c', \nu_{\varepsilon i}(T) \leq T'\} \\
&\quad + \mathsf{P}\{\Delta_U(\nu_{\varepsilon i}(\cdot), c, T) > c'\} + \mathsf{P}\{\nu_{\varepsilon i}(T) > T'\}) \\
&\quad \leq \sum_{i=1}^m (\mathsf{P}\{\Delta_U(\xi_{\varepsilon i}(\cdot), c', T') > \delta/m\} \\
&\quad + \mathsf{P}\{\Delta_U(\nu_{\varepsilon i}(\cdot), c, T) > c'\} + \mathsf{P}\{\nu_{\varepsilon i}(T) > T'\}).
\end{aligned} \tag{3.2.8}$$

For an arbitrary $\sigma > 0$, by condition $\mathcal{A}_{34}$, we can choose $T'' \in V$ and then T', which is a point of continuity for the distribution functions of the random variables $\nu_{0i}(T'')$, $i = 1, \ldots, m$ such that $\overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\nu_{\varepsilon i}(T) > T'\} \leq \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\nu_{\varepsilon i}(T'') > T'\} \leq \sigma/2m$. Then, fixing T' and using condition $\mathbf{U_4}$, we can find $c' > 0$ such that $\overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\Delta_U(\xi_{\varepsilon i}(\cdot), c', T') > \delta\} \leq \sigma/2m$. If we pass to limit in (3.2.8), first making $\varepsilon \to 0$ and then $c \to 0$, and use Lemma 3.2.1, we get

$$\begin{aligned}
&\lim_{c\to 0} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\Delta_U(\zeta_\varepsilon(\cdot), c, T) > \delta\} \\
&\quad \leq \sum_{i=1}^{m} (\overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\Delta_U(\xi_{\varepsilon i}(\cdot), c', T') > \delta/m\} \\
&\quad + \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\Delta_U(\nu_{\varepsilon i}(\cdot), c, T) > c'\} + \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\nu_{\varepsilon i}(T) > T'/2\}) \\
&\quad \leq \sigma + \sum_{i=1}^{m} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\Delta_U(\nu_{\varepsilon i}(\cdot), c, T) > c'\} = \sigma.
\end{aligned} \tag{3.2.9}$$

This proves (3.2.6), since σ is arbitrary. □

3.2.2. Conditions of U-compactness. It is useful to note that relation (3.2.6), i.e., **U**-compactness of the processes $\zeta_\varepsilon(t)$, $t \geq 0$ can be obtained without the use of the condition of weak convergence $\mathcal{A}_{34}$.

Let introduce the following condition:

$\mathcal{K}_3^{(0)}$: $\lim_{t\to\infty} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\nu_{\varepsilon i}(0) > t\} = 0$, $i = 1, \ldots, m$.

Lemma 3.2.3. *Let conditions $\mathfrak{U}_4$, $\mathfrak{U}_5$, and $\mathcal{K}_3^{(0)}$ hold. Then*

$$\lim_{c\to 0} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\Delta_U(\zeta_\varepsilon(\cdot), c, T) > \delta\} = 0, \ \delta > 0.$$

Proof of Lemma 3.2.3. Let $x(t)$, $t \geq 0$ be a real-valued càdlàg function. The following estimate is valid for every $0 < c < T < \infty$:

$$|x(T)| \leq |x(0)| + ([T/c] + 1)\Delta_U(x(\cdot), c, T). \tag{3.2.10}$$

Using (3.2.10) for $c = 1/T'$ and conditions $\mathfrak{U}_5$ and $\mathcal{K}_3^{(0)}$ we get, for every $i = 1, \ldots, m$ and $T > 0$,

$$\begin{aligned}
&\lim_{T'\to\infty} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\nu_{\varepsilon i}(T) \geq T'\} \\
&\quad \leq \lim_{T'\to\infty} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\nu_{\varepsilon i}(0) \geq T'/2\} \\
&\quad + \lim_{T'\to\infty} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\Delta_U(\nu_{\varepsilon i}(\cdot), 1/T', T) \geq T'/2([TT'] + 1)\} = 0,
\end{aligned} \tag{3.2.11}$$

since $T'/2([TT'] + 1) \to 1/2T > 0$ as $T' \to \infty$.

The proof of the lemma follows directly from relations (3.2.8) and (3.2.11). □

3.2.3. The set of weak convergence. It follows from Theorem 1.6.11 that, under conditions of Theorem 3.2.1,

$$\zeta_\varepsilon(t), t \geq 0 \Rightarrow \zeta_0(t), t \geq 0 \text{ as } \varepsilon \to 0. \tag{3.2.12}$$

3.2.4. Non-monotone internal stopping processes. Theorem 3.2.1 can be generalised to a model where the monotonicity of non-negative càdlàg processes $\nu_{\varepsilon i}(t)$, $t \geq 0$ is not assumed. In this case there is no guarantee that the composition $\zeta_\varepsilon(t)$, $t \geq 0$ is a càdlàg processes.

Conditions $\mathcal{A}_{34}$ and $\mathcal{U}_4$ still provide, due to Theorem 2.7.1, weak convergence of the compositions $\zeta_\varepsilon(t)$ on the set V.

Let $0 = t_{0,n} < t_{1,n} < \cdots < t_{n,n} = T$, $n \geq 1$ be a sequence of partitions of the interval $[0, T]$ such that: **(a)** $t_{k,n} \in V$, $k = 0, \ldots, n$, $n \geq 1$; **(b)** $h_n = \max_{0 \leq k \leq n-1} (t_{k+1,n} - t_{k,n}) \to 0$ as $n \to \infty$. The following estimate holds for all $0 < c < T < \infty$:

$$|\sup_{t \leq T} \nu_{\varepsilon i}(t) - \max_{0 \leq k \leq n} \nu_{\varepsilon i}(t_{k,n})| \leq \Delta_U(\nu_{\varepsilon i}(\cdot), h_n, T). \tag{3.2.13}$$

Relation (3.2.13) and conditions $\mathcal{A}_{34}$ and $\mathcal{U}_5$ imply, due to Lemma 1.2.5, the following relation for every $i = 1, \ldots, m$:

$$\sup_{t \leq T} \nu_{\varepsilon i}(t) \Rightarrow \sup_{t \leq T} \nu_{0i}(t) \text{ as } \varepsilon \to 0. \tag{3.2.14}$$

Relation (3.2.14) and conditions $\mathcal{U}_4$ and $\mathcal{U}_5$ permit to repeat the proof of the **U**-compactness relation (3.2.6) given above for relations (3.2.8) and (3.2.9). The only difference is that the random variables $\nu_{\varepsilon i}(T)$ should be replaced with the random variables $\sup_{t \leq T} \nu_{\varepsilon i}(t)$.

Note that the modulus Δ_U can be defined by the same formula not only for a càdlàg process for any real-valued function. Also, estimate 3.2.7 is valid for any real-valued functions.

So, under conditions $\mathcal{A}_{34}$, $\mathcal{U}_4$, and $\mathcal{U}_5$, the processes $\zeta_\varepsilon(t)$ weakly converge on the set V and satisfy the relation of **J**-compactness (3.3.2).

As was mentioned above, the pre-limiting composition $\zeta_\varepsilon(t)$, $t \geq 0$ may be not an a.s. càdlàg process for $\varepsilon > 0$. However, the limiting process $\zeta_0(t)$, $t \geq 0$ is an a.s. continuous process.

The question about the corresponding class of a.s. **U**-continuous functionals should belong to needs in this case, a special investigation. We refer here to the works by Borovkov (1976) and Borovkov, Mogul'skij, and Sakhanenko (1995).

3.2.5. The time interval $[0, T]$. In this case, we consider the vector composition $\zeta_\varepsilon(t) = (\xi_{\varepsilon i}(\nu_{\varepsilon i}(t)), i = 1, \ldots, m)$, $t \in [0, T]$ of a vector càdlàg process $\xi_\varepsilon(t) = (\xi_{\varepsilon i}(t), i = 1, \ldots, m)$, $t \geq 0$, with real-valued components, and a vector càdlàg process

$\mathbf{v}_\varepsilon(t) = (\nu_{\varepsilon i}(t), i = 1, \dots, m)$, $t \in [0, T]$, with non-negative and non-decreasing components.

We can always continue the internal stopping process to the interval $[0, \infty)$ by the following formula:

$$\mathbf{v}_\varepsilon(t) = \begin{cases} \mathbf{v}_\varepsilon(t) & \text{if } 0 \le t \le T, \\ \mathbf{v}_\varepsilon(T) & \text{if } t \ge T. \end{cases} \tag{3.2.15}$$

Now we can apply Theorem 3.2.1. Condition $\mathcal{A}_{34}$ should be replaced with a condition in which the set V is dense in $[0, T]$ and contains the points 0 and T. The condition of $\mathbf{U}$-compactness $\mathcal{U}_4$ does not require any changes. In the condition of $\mathbf{U}$-compactness $\mathcal{U}_5$ the corresponding asymptotic relation should be required to hold only for the interval $[0, T]$. Finally, by applying Theorem 3.2.1, we get

$$\zeta_\varepsilon(t), t \in [0, T] \xrightarrow{\mathbf{U}} \zeta_0(t), t \in [0, T] \text{ as } \varepsilon \to 0. \tag{3.2.16}$$

Also, in Lemma 3.2.3, conditions $\mathcal{U}_4$ and $\mathcal{K}_3^{(0)}$ remain the same, while $\mathcal{U}_5$ should be used in the modified form described above in order to prove $\mathbf{U}$-compactness of the processes $\zeta_\varepsilon(t)$, $t \in [0, T]$.

3.2.6. The time interval $(0, \infty)$. The results of the previous section can easily be translated to the case of the semi-infinite interval $(0, \infty)$ under the condition that the limiting internal stopping random variable $\nu_{0i}(t)$ is positive with probability 1 for every $t > 0$ and $i = 1, \dots, m$. In this case, the point 0 can be excluded from the sets U and V in condition $\mathcal{A}_{34}$. Also, in conditions $\mathcal{U}_4$ and $\mathcal{U}_5$, the relations of $\mathbf{U}$-compactness should be required to hold for any finite interval $[T', T'']$, where $0 < T' < T'' < \infty$.

By applying Theorem 2.7.1 and taking into account the remarks made in Subsection 2.7.6, one can prove weak convergence of the vector compositions

$$\zeta_\varepsilon(t), t \in V \Rightarrow \zeta_\varepsilon(t), t \in V \text{ as } \varepsilon \to 0. \tag{3.2.17}$$

As easily seen, the $\mathbf{U}$-compactness condition can be obtained by a slight modification of estimates (3.2.7) and (3.2.8).

The first one holds for any non-negative real-valued function $x(t)$, $t \ge 0$, and any non-decreasing function $y(t)$, $t \ge 0$ and $0 < T' < T'' < \infty$, $0 < T_1 < T_2 < \infty$,

$$\begin{aligned} &\Delta_U(x(y(\cdot)), c, T', T'')\chi(\Delta_U(y(\cdot), c, T', T'') \le c', y(T') \ge T_1, y(T'') \le T_2) \\ &\qquad \le \Delta_U(x(\cdot), c', T_1, T_2). \end{aligned} \tag{3.2.18}$$

The second one takes the form

$$
\begin{aligned}
&\mathsf{P}\{\Delta_U(\zeta_\varepsilon(\cdot), c, T', T'') > \delta\} \\
&\quad \le \sum_{i=1}^{m} \mathsf{P}\{\Delta_U(\xi_{\varepsilon i}(\nu_{\varepsilon i}(\cdot)), c, T', T'') > \delta/m\} \\
&\quad \le \sum_{i=1}^{m} (\mathsf{P}\{\Delta_U(\xi_{\varepsilon i}(\nu_{\varepsilon i}(\cdot)), c, T', T'') > \delta/m, \nu_{\varepsilon i}(T') \ge T_1, \\
&\qquad \nu_{\varepsilon i}(T'') \le T_2, \Delta_U(\nu_{\varepsilon i}(\cdot), c, T', T'') \le c'\} + \mathsf{P}\{\nu_{\varepsilon i}(T') < T_1\} \\
&\quad + \mathsf{P}\{\nu_{\varepsilon i}(T'') > T_2\} + \mathsf{P}\{\Delta_U(\nu_{\varepsilon i}(\cdot), c, T', T'') > c'\}) \\
&\quad \le \sum_{i=1}^{m} (\mathsf{P}\{\Delta_U(\xi_{\varepsilon i}(\cdot), c', T_1, T_2) > \delta/m\} \le \mathsf{P}\{\nu_{\varepsilon i}(T') < T_1\} \\
&\quad + \mathsf{P}\{\nu_{\varepsilon i}(T'') > T_2\} + \mathsf{P}\{\Delta_U(\nu_{\varepsilon i}(\cdot), c, T', T'') > c'\}).
\end{aligned}
\tag{3.2.19}
$$

By repeating the subsequent steps in the proof of Theorem 3.2.1, one can get a relation of **U**-compactness for $0 < T' < T'' < \infty$,

$$\lim_{c\to 0} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\Delta_U(\zeta_\varepsilon(\cdot), c, T', T'') > \delta\} = 0, \ \delta > 0. \tag{3.2.20}$$

The relations (3.2.17) and (3.2.20) imply that

$$\zeta_\varepsilon(t), t \in (0, \infty) \xrightarrow{\mathbf{U}} \zeta_0(t), t \in (0, \infty) \text{ as } \varepsilon \to 0. \tag{3.2.21}$$

3.2.7. A Polish phase space. The results presented in this section can be generalised to a model with external stochastic processes, $\xi_\varepsilon(t)$, $t \ge 0$, the components of which, $\xi_{\varepsilon i}(t)$, $t \ge 0$, take values in a Polish space X.

The formulation of condition $\mathcal{A}_{34}$ remains the same. In the condition $\mathcal{U}_4'$, the Euclidean distance $|x - y|$ must be replaced with the corresponding metric $d(x, y)$ in the formula for the modula of **U**-compactness, $\Delta_U(\xi_{\varepsilon i}(\cdot), c, T)$.

Conditions $\mathcal{A}_{34}$ and $\mathcal{U}_4'$, modified as described above, still imply weak convergence of the compositions $\zeta_\varepsilon(t)$ on the set V, as follows from Theorem 2.3.6.

All estimates for the modula of **U**-compactness $\Delta_U(\zeta_{\varepsilon i}(\cdot), c, T)$, given in (3.2.8)–(3.2.9), can be repeated and the **U**-compactness relation (3.2.6) can be written.

Finally, under conditions $\mathcal{A}_{34}$, $\mathcal{U}_4'$, and $\mathcal{U}_5$, we get

$$\zeta_\varepsilon(t), t \ge 0 \xrightarrow{\mathbf{U}} \zeta_0(t), t \ge 0 \text{ as } \varepsilon \to 0. \tag{3.2.22}$$

3.3 Asymptotically continuous external processes

In this section, we formulate conditions for **J**-convergence of compositions of càdlàg processes and asymptotically continuous external processes.

3.3.1. J-convergence of semi-vector compositions with an asymptotically continuous external component. Let, for every $\varepsilon \geq 0$, $\boldsymbol{\xi}_\varepsilon(t) = (\xi_{\varepsilon i}(t), i = 1, \ldots, m)$, $t \geq 0$ be a vector càdlàg process with real-valued components, and $\nu_\varepsilon(t)$, $t \geq 0$ a non-negative non-decreasing càdlàg process. Consider the *semi-vector composition* $\boldsymbol{\zeta}_\varepsilon(t) = (\xi_{\varepsilon i}(\nu_\varepsilon(t)), i = 1, \ldots, m)$, $t \geq 0$, which is also a vector càdlàg process with real-valued components.

We assume that the following analogue of the condition of joint weak convergence $\mathcal{A}_{34}$ holds:

$\mathcal{A}_{36}$: $(\nu_\varepsilon(s), \boldsymbol{\xi}_\varepsilon(t)), (s,t) \in V \times U \Rightarrow (\nu_0(s), \boldsymbol{\xi}_0(t)), (s,t) \in V \times U$ as $\varepsilon \to 0$, where U and V are some subsets of $[0, \infty)$ that are dense in this interval and contain the point 0.

We also assume that the condition of **U**-compactness $\mathcal{U}_4$ holds for the external processes $\boldsymbol{\xi}_\varepsilon(t)$, $t \geq 0$. It is also useful to note that condition $\mathcal{U}_4$ is equivalent to condition $\mathcal{U}'_4$.

Conditions $\mathcal{A}_{36}$ and $\mathcal{U}_4$ imply **U**-convergence of the processes $\boldsymbol{\xi}_\varepsilon(t)$, $t \geq 0$ and a.s. continuity of the limiting process $\boldsymbol{\xi}_0(t)$, $t \geq 0$.

For the internal component $\nu_\varepsilon(t)$, $t \geq 0$, we assume that the following condition of **J**-compactness holds:

$\mathcal{J}_{11}$: $\lim_{c \to 0} \overline{\lim}_{\varepsilon \to 0} \mathsf{P}\{\Delta_J(\nu_\varepsilon(\cdot), c, T) > \delta\} = 0, \ \delta, T > 0.$

Conditions $\mathcal{A}_{36}$ and $\mathcal{J}_{11}$ imply **J**-convergence of the processes $\nu_\varepsilon(t)$, $t \geq 0$. However, continuity of the corresponding càdlàg limiting processes $\nu_0(t)$, $t \geq 0$ is not required.

The following theorem presents a result given in Whitt (1973, 1980) and Silvestrov (1974).

Theorem 3.3.1. *Let conditions $\mathcal{A}_{36}$, $\mathcal{U}_4$, and $\mathcal{J}_{11}$ hold. Then*

$$\boldsymbol{\zeta}_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{J}} \boldsymbol{\zeta}_0(t), t \geq 0 \text{ as } \varepsilon \to 0.$$

Theorem 3.3.1 does not require a separate proof. This theorem is a particular case of Theorem 3.3.2 that gives a similar result for a more general model of vector compositions of càdlàg processes.

3.3.2. J-convergence of vector compositions with an asymptotically continuous external component. Let, for every $\varepsilon \geq 0$, $\boldsymbol{\xi}_\varepsilon(t) = (\xi_{\varepsilon i}(t), i = 1, \ldots, m)$, $t \geq 0$ be a vector càdlàg process with real-valued components, and $\boldsymbol{\nu}_\varepsilon(t) = (\nu_{\varepsilon i}(t), i = 1, \ldots, m)$, $t \geq 0$ a vector càdlàg process with non-negative and non-decreasing components. Consider the *vector composition* $\boldsymbol{\zeta}_\varepsilon(t) = (\xi_{\varepsilon i}(\nu_{\varepsilon i}(t)), i = 1, \ldots, m)$, $t \geq 0$, which is also a vector càdlàg process with real-valued components.

In this subsection, we consider a model where the external processes $\boldsymbol{\xi}_\varepsilon(t)$, $t \geq 0$ are asymptotically continuous.

We assume that the condition of joint weak convergence $\mathcal{A}_{34}$ holds, together with the condition of **U**-compactness $\mathcal{U}_4$ for the processes $\boldsymbol{\xi}_\varepsilon(t)$, $t \geq 0$.

Conditions $\mathcal{A}_{34}$ and $\mathfrak{U}_4$ imply **U**-convergence of the processes $\xi_\varepsilon(t)$, $t \geq 0$, and a.s. continuity of the limiting process $\xi_0(t)$, $t \geq 0$.

We assume that the internal component $\mathbf{v}_\varepsilon(t)$, $t \geq 0$ satisfies the following vector analogue of condition $\mathfrak{J}_{11}$:

$\mathfrak{J}_{12}$: $\lim_{c\to 0} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\Delta_J(\mathbf{v}_\varepsilon(\cdot), c, T) > \delta\} = 0, \ \delta, T > 0.$

Conditions $\mathcal{A}_{34}$ and $\mathfrak{J}_{12}$ imply **J**-convergence of the processes $\mathbf{v}_\varepsilon(t)$, $t \geq 0$. However, the continuity of the limiting càdlàg processes $\mathbf{v}_0(t)$, $t \geq 0$ is not required.

The following new theorem generalises the result of Theorem 3.3.1 to the model of vector compositions of càdlàg processes.

Theorem 3.3.2. *Let conditions $\mathcal{A}_{34}$, $\mathfrak{U}_4$, and $\mathfrak{J}_{12}$ hold. Then*

$$\zeta_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{J}} \zeta_0(t), t \geq 0 \text{ as } \varepsilon \to 0.$$

Proof of Theorem 3.3.2. Condition $\mathcal{A}_{34}$ implies that condition $\mathcal{A}_{22}^{\mathbf{V}}$ holds with the set V used in $\mathcal{A}_{34}$. Condition $\mathfrak{U}_4$ obviously implies that condition $\mathfrak{J}_4$ holds. Since $\xi_0(t)$, $t \geq 0$ is a continuous process, condition $\mathfrak{C}_6^{\mathbf{W}}$ holds with the set $W = [0, \infty)$. Therefore, it follows from Theorem 2.7.1 that, for the set V,

$$\zeta_\varepsilon(t), t \in V \Rightarrow \zeta_0(t), t \in V \text{ as } \varepsilon \to 0. \tag{3.3.1}$$

By $\mathcal{A}_{34}$, the set V is everywhere dense in $[0, \infty)$ and contains the point 0.

To prove the theorem, we must also supplement the relation of weak convergence (3.3.1) with the relation of **J**-compactness,

$$\lim_{c\to 0} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\Delta_J(\zeta_\varepsilon(\cdot), c, T) > \delta\} = 0, \ \delta, T > 0. \tag{3.3.2}$$

Let $\mathbf{x}(t) = (x_i(t), i = 1, \ldots, m)$, $t \geq 0$ be a vector càdlàg function with real-valued components and $\mathbf{y}(t) = (y_i(t), i = 1, \ldots, m)$, $t \geq 0$ be a vector càdlàg function with non-negative and non-decreasing components. Let also $\mathbf{z}(t) = (x_i(y_i(t)), i = 1, \ldots, m)$, $t \geq 0$ be their vector composition, which is also a vector càdlàg function with real-valued components.

It is clear that if $t' \leq t \leq t''$ and $\min(|\mathbf{y}(t)-\mathbf{y}(t')|, |\mathbf{y}(t)-\mathbf{y}(t'')|) \leq c'$, then $|y_i(t)-y_i(t')| \leq c'$, $i = 1, \ldots m$ or $|y_i(t) - y_i(t'')| \leq c'$, $i = 1, \ldots, m$. Hence,

$$\begin{aligned}
&\min(|\mathbf{z}(t) - \mathbf{z}(t')|, |\mathbf{z}(t) - \mathbf{z}(t'')|) \\
&\quad \times \chi(\min(|\mathbf{y}(t) - \mathbf{y}(t')|, |\mathbf{y}(t) - \mathbf{y}(t'')|) \leq c', \max_{1\leq i\leq m} y_i(t'') \leq T') \\
&\quad \leq \min(\sum_{i=1}^m |x_i(y_i(t)) - x_i(y_i(t'))|, \sum_{i=1}^m |x_i(y_i(t)) - x_i(y_i(t''))|) \\
&\quad \times \chi(\min(|\mathbf{y}(t) - \mathbf{y}(t')|, |\mathbf{y}(t) - \mathbf{y}(t'')|) \leq c', \max_{1\leq i\leq m} y_i(t'') \leq T') \\
&\quad \leq \sum_{i=1}^m \Delta_U(x_i(\cdot), c', T')
\end{aligned}$$

and, therefore,

$$\Delta_J(\mathbf{z}(\cdot), c, T)\chi(\Delta_J(\mathbf{y}(\cdot), c, T) \le c', \max_{1\le i\le m} y_i(T) \le T') \le \sum_{i=1}^{m} \Delta_U(x_i(\cdot), c', T'). \tag{3.3.3}$$

Using (3.3.3) and taking into account monotonicity of the processes $\nu_{\varepsilon i}(t), t \ge 0$, $i = 1, \dots, m$, we have

$$\begin{aligned}
&\mathsf{P}\{\Delta_J(\zeta_\varepsilon(\cdot), c, T) > \delta\} \\
&\quad \le \mathsf{P}\{\sum_{i=1}^{m} \Delta_U(\xi_{\varepsilon i}(\cdot), c, T) > \delta\} \\
&\quad + \mathsf{P}\{\Delta_J(\boldsymbol{\nu}_\varepsilon(\cdot), c, T) > c'\} + \mathsf{P}\{\max_{1\le i\le m} \nu_{\varepsilon i}(T) > T'\} \\
&\quad \le \sum_{i=1}^{m} \mathsf{P}\{\Delta_U(\xi_{\varepsilon i}(\cdot), c', T') > \delta/m\} \\
&\quad + \mathsf{P}\{\Delta_J(\boldsymbol{\nu}_\varepsilon(\cdot), c, T) > c'\} + \sum_{i=1}^{m} \mathsf{P}\{\nu_{\varepsilon i}(T) > T'\}.
\end{aligned} \tag{3.3.4}$$

For an arbitrary $\sigma > 0$, by condition $\mathcal{A}_{\mathbf{34}}$, we can choose $T'' \in V$ and then T', which is a point of continuity for the distribution functions of the random variables $\nu_{0i}(T''), i = 1, \dots, m$, such that $\overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\nu_{\varepsilon i}(T) > T'\} \le \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\nu_{\varepsilon i}(T'') > T'\} \le \sigma/2m$. Then, by using condition $\mathfrak{U}_{\mathbf{4}}$, we can find $c' > 0$ such that $\overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\Delta_U(\xi_{\varepsilon i}(\cdot), c', T') > \delta\} \le \sigma/2m$. If we pass to the limit in (3.3.4), first for $\varepsilon \to 0$ and then for $c \to 0$, and use condition $\mathfrak{J}_{\mathbf{12}}$, we get

$$\lim_{c\to 0} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\Delta_J(\zeta_\varepsilon(\cdot), c, T) > \delta\} \le \sigma + \lim_{c\to 0} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\Delta_J(\boldsymbol{\nu}_\varepsilon(\cdot), c, T) > c'\} = \sigma. \tag{3.3.5}$$

This proves (3.3.2), since σ is arbitrary. □

3.3.3. Conditions of J-compactness. It is useful to note that **J**-compactness of the processes $\zeta_\varepsilon(t)$, $t \ge 0$, can be proved without the use of the condition of weak convergence $\mathcal{A}_{\mathbf{34}}$.

Let us introduce the following condition:

$\mathcal{K}_{\mathbf{3}}^{(\mathbf{T})}$: $\lim_{t\to\infty} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\nu_{\varepsilon i}(T) > t\} = 0, i = 1, \dots, m$.

It follows directly from (3.3.4) that the relation of **J**-compactness (3.3.2) holds for a given $T > 0$ if **(a)** condition $\mathfrak{U}_{\mathbf{4}}$ holds, **(b)** the relation of **J**-compactness in condition $\mathfrak{J}_{\mathbf{12}}$ holds for this T, and **(c)** condition $\mathcal{K}_{\mathbf{3}}^{(\mathbf{T})}$ holds.

3.3.4. J-convergence of monotone càdlàg processes. In this subsection, we study conditions for **J**-compactness and **J**-convergence of monotone càdlàg processes.

Let, for every $\varepsilon \geq 0$, $\nu_\varepsilon(t)$, $t \in [0, T]$ be a non-negative and non-decreasing càdlàg process.

Let $x(t)$, $t \in [0, T]$ be a non-decreasing càdlàg function. For every $k \geq 1$, we define the functional

$$\kappa_{T,k}^{(\delta)}(x(\cdot)) = \kappa_k^{(\delta)} = \begin{cases} \inf(s \in (\kappa_{k-1}^{(\delta)}, T] : x(s) \geq x(\kappa_{k-1}^{(\delta)}) + \delta, & \text{if } x(T) \geq x(\kappa_{k-1}^{(\delta)}) + \delta, \\ \kappa_{k-1}^{(\delta)} + T, & \text{if } x(T) < x(\kappa_{k-1}^{(\delta)}) + \delta, \end{cases}$$

where $\kappa_{T,0}^{(\delta)}(x(\cdot)) = \kappa_0^{(\delta)} = 0$.

Let us also define, for every $r \geq 1$, the following functional, which is the minimal distance between successive moments of minimal δ-increments for the function $x(t)$, $t \in [0, T]$,

$$\pi_{T,r}^{(\delta)}(x(\cdot)) = \min_{1 \leq k \leq r}(\kappa_{T,k}^{(\delta)}(x(\cdot)) - \kappa_{T,k-1}^{(\delta)}(x(\cdot))).$$

Let us introduce the following condition:

$\mathcal{N}_1$: (a) $\lim_{T' \to \infty} \overline{\lim}_{\varepsilon \to 0} \mathsf{P}\{|\nu_\varepsilon(T) - \nu_\varepsilon(0)| > T'\} = 0$;

(b) $\lim_{c \to 0} \overline{\lim}_{\varepsilon \to 0} \mathsf{P}\{\pi_{T,r}^{(\delta_l)}(\nu_\varepsilon(\cdot)) \leq c\} = 0$, $r \geq 1$ for some sequence $0 < \delta_l \to 0$ as $l \to \infty$.

The following theorem is given in Silvestrov (1974), where one can also find some applications to monotone Markov type processes.

Theorem 3.3.3. *Let condition* $\mathcal{N}_1$ *hold. Then*

$$\lim_{c \to 0} \overline{\lim_{\varepsilon \to 0}} \mathsf{P}\{\Delta_J(\nu_\varepsilon(\cdot), c, T) > \sigma\} = 0, \ \sigma > 0.$$

Proof of Theorem 3.3.3. Let us also define the following functional

$$\pi_T^{(\delta)}(x(\cdot)) = \min_{k \geq 1}(\kappa_{T,k}^{(\delta)}(x(\cdot)) - \kappa_{T,k-1}^{(\delta)}(x(\cdot))).$$

Let us show that, for any non-decreasing càdlàg function $x(t)$, $t \in [0, T]$ and $c, \delta > 0$, if

$$\Delta_J(x(\cdot), c/2, T) \geq 2\delta, \tag{3.3.6}$$

then

$$\pi_T^{(\delta)}(x(\cdot)) \leq c. \tag{3.3.7}$$

Suppose that (3.3.6) holds. Then there exist three points $t', t, t'' \in [0, T]$, $t - c/2 \leq t' \leq t \leq t'' \leq t + c/2$ such that $x(t) - x(t') \geq \delta$ and $x(t'') - x(t) \geq \delta$.

There always exists $k = 0, 1, \ldots$ such that **(a)** $t \in [\kappa_k^{(\delta)}, \kappa_{k+1}^{(\delta)})$. Let us assume that **(b)** $\kappa_k^{(\delta)} \leq t'$. Then **(c)** $x(t) - x(\kappa_k^{(\delta)}) \geq x(t) - x(t') \geq \delta$. Obviously, **(c)** implies that **(d)**

$\kappa^{(\delta)}_{k+1} \leq t$. But, **(d)** contradicts **(a)**. So, assumption **(b)** does not hold, and, therefore, **(e)** $t' < \kappa^{(\delta)}_{k} \leq t'$. Then **(f)** $x(t'') - x(\kappa^{(\delta)}_{k}) \geq x(t'') - x(t) \geq \delta$. Thus, **(g)** $\kappa^{(\delta)}_{k+1} \leq t''$. Obviously, **(e)** and **(g)** imply that **(h)** $\kappa^{(\delta)}_{k+1} - \kappa^{(\delta)}_{k} \leq t'' - t' \leq c$.

Let us define

$$\mu^{(\delta)}_{T}(x(\cdot)) = \max(k\colon x(T) \geq x(\kappa^{(\delta)}_{T,k-1}(x(\cdot))) + h).$$

It follows from the definition of $\mu^{(\delta)}_{T}(x(\cdot))$ and $\kappa^{(\delta)}_{T,k}(x(\cdot))$ that

$$\mu^{(\delta)}_{T}(x(\cdot)) \leq \frac{x(T) - x(0)}{\delta}.$$

On the other hand, by the definition of the functionals $\pi^{(\delta)}_{T,r}(x(\cdot))$ and $\pi^{(\delta)}_{T}(x(\cdot))$, we have

$$\pi^{(\delta)}_{T,r}(x(\cdot)) - \pi^{(\delta)}_{T}(x(\cdot)) = 0 \text{ if } \mu^{(h)}_{T}(x(\cdot)) \leq r.$$

Hence, we get the following estimate:

$$\begin{aligned} &\mathsf{P}\{|\pi^{(\delta)}_{T,r}(\nu_\varepsilon(\cdot)) - \pi^{(\delta)}_{T}(\nu_\varepsilon(\cdot))| > 0\} \\ &\qquad \leq \mathsf{P}\{\mu^{(\delta)}_{T}(\nu_\varepsilon(\cdot)) > r\} \leq \mathsf{P}\{\nu_\varepsilon(T) - \nu_\varepsilon(0) > \delta r\}. \end{aligned} \tag{3.3.8}$$

By using (3.3.8) and condition $\mathbf{N_1}$, we get

$$\begin{aligned} &\lim_{r\to\infty} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{|\pi^{(\delta)}_{T,r}(\nu_\varepsilon(\cdot)) - \pi^{(\delta)}_{T}(\nu_\varepsilon(\cdot))| > 0\} \\ &\qquad \leq \lim_{r\to\infty} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\nu_\varepsilon(T) - \nu_\varepsilon(0) > \delta r\} = 0. \end{aligned} \tag{3.3.9}$$

By using (3.3.9) and condition $\mathbf{N_1}$, we get, for δ_l, $l \geq 1$ from this condition,

$$\begin{aligned} &\lim_{c\to 0} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\pi^{(\delta_l)}_{T}(\nu_\varepsilon(\cdot)) \leq c\} \\ &\quad \leq \lim_{c\to 0} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\pi^{(\delta_l)}_{T,r}(\nu_\varepsilon(\cdot)) - |\pi^{(\delta_l)}_{T,r}(\nu_\varepsilon(\cdot)) - \pi^{(\delta_l)}_{T}(\nu_\varepsilon(\cdot))| \leq c\} \\ &\quad \leq \lim_{c\to 0} \overline{\lim_{\varepsilon\to 0}} (\mathsf{P}\{\pi^{(\delta_l)}_{T,r}(\nu_\varepsilon(\cdot)) \leq c\} \\ &\quad + \mathsf{P}\{|\pi^{(\delta_l)}_{T,r}(\nu_\varepsilon(\cdot)) - \pi^{(\delta_l)}_{T}(\nu_\varepsilon(\cdot))| > 0\}) \\ &\quad = \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{|\pi^{(\delta_l)}_{T,r}(\nu_\varepsilon(\cdot)) - \pi^{(\delta_l)}_{T}(\nu_\varepsilon(\cdot))| > 0\} \to 0 \text{ as } r \to \infty, \end{aligned} \tag{3.3.10}$$

and, consequently,

$$\lim_{c\to 0} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\pi^{(\delta_l)}_{T}(\nu_\varepsilon(\cdot)) \leq c\} = 0, \ l \geq 1. \tag{3.3.11}$$

It clearly follows from (3.3.11) that, for arbitrary α, $\sigma > 0$, there exists $c > 0$ and $2\delta_l < \sigma$ such that

$$\overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\pi^{(\delta_l)}_{T}(\nu_\varepsilon(\cdot)) \leq c\} \leq \alpha. \tag{3.3.12}$$

By using (3.3.6) and (3.3.7), we get

$$\begin{aligned}
&\overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\Delta_J(\nu_\varepsilon(\cdot), c/2, T) > \sigma\} \\
&\quad \leq \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\Delta_J(\nu_\varepsilon(\cdot), c/2, T) \geq 2\delta_l\} \\
&\quad \leq \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\pi_T^{(\delta_l)}(\nu_\varepsilon(\cdot)) \leq c\} \leq \alpha,
\end{aligned} \tag{3.3.13}$$

and so, due to arbitrary choice of α

$$\lim_{c\to 0} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\Delta_J(\nu_\varepsilon(\cdot), c/2, T) > \sigma\} = 0. \tag{3.3.14}$$

The proof is completed. □

Let us assume the following condition:

$\mathcal{A}_{37}$: $\nu_\varepsilon(s), s \in V \Rightarrow \nu_0(s), s \in V$ as $\varepsilon \to 0$, where V is a subset of $[0, \infty)$ that is dense in this interval and contains the point 0.

Taking into account Theorem 3.3.3 we can formulate the following condition for **J**-convergence of monotone càdlàg processes.

Theorem 3.3.4. *Let conditions* $\mathcal{A}_{37}$ *and* $\mathfrak{N}_1$ *hold. Then*

$$\nu_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{J}} \nu_0(t), t \geq 0 \text{ as } \varepsilon \to 0.$$

3.3.5. The set of weak convergence. Let V_0 be the set of points of stochastic continuity of the limiting stopping process $\boldsymbol{\nu}_0(t)$, $t \geq 0$, and Z_0 be the set of points of stochastic continuity of the limiting stopping process $\zeta_0(t)$, $t \geq 0$. Since the limiting external process $\xi_0(t)$, $t \geq 0$ is a.s. continuous, $V_0 \subseteq Z_0$. Under conditions $\mathcal{A}_{34}$ and $\mathcal{J}_{12}$, the set of weak convergence V, in condition $\mathcal{A}_{34}$, can be enlarged, by Lemma 1.6.5, to the set $V \cup V_0$. In sequel, the set of weak convergence V will be replaced with the set $V \cup V_0$ from the proof of Theorem 3.3.1. This set coincides with $[0, \infty)$ except for at most a countable set. Also, $0 \in V \cup V_0$. The processes $\zeta_\varepsilon(t)$, $t \geq 0$ **J**-converge and, therefore, by Lemma 1.6.5, the set $V \cup V_0$ can be extended to the set $V \cup V_0 \cup Z_0 = V \cup Z_0$. Finally, under the conditions of Theorem 3.3.2, we get that

$$\zeta_\varepsilon(t), t \in V \cup Z_0 \Rightarrow \zeta_0(t), t \in V \cup Z_0 \text{ as } \varepsilon \to 0. \tag{3.3.15}$$

3.3.6. Non-monotone internal stopping processes. The result of Theorems 3.3.1 and 3.3.2 can be generalised to include the case where the monotonicity of the non-negative càdlàg processes $\nu_{\varepsilon i}(t)$, $t \geq 0$ is not assumed. In this case, it can be that the compositions $\zeta_\varepsilon(t)$, $t \geq 0$ are not càdlàg processes.

Conditions $\mathcal{A}_{34}$ and $\mathfrak{U}_4$ provide, due to Theorem 2.7.1, weak convergence of the compositions $\zeta_\varepsilon(t)$ on the set V.

Let $0 = t_{0,n} < t_{1,n} < \ldots < t_{n,n} = T$, $n \geq 1$ be a sequence of partitions of the interval $[0, T]$ such that: **(a)** $t_{k,n} \in V$, $k = 0, \ldots n$, $n \geq 1$; **(b)** $h_n = \max_{0 \leq k \leq n-1} (t_{k+1,n} - t_{k,n}) \to 0$ as $n \to \infty$. Then

$$|\sup_{t \leq T} \nu_{\varepsilon i}(t) - \max_{0 \leq k \leq n} \nu_{\varepsilon i}(t_{k,n})| \leq \Delta_J(\nu_{\varepsilon i}(\cdot), h_n, T). \tag{3.3.16}$$

Relation (3.3.16) and conditions $\mathcal{A}_{34}$ and $\mathfrak{J}_{12}$ imply, due to Lemma 1.2.5, the following relation for every $i = 1, \ldots, m$:

$$\sup_{t \leq T} \nu_{\varepsilon i}(t) \Rightarrow \sup_{t \leq T} \nu_{0i}(t) \text{ as } \varepsilon \to 0. \tag{3.3.17}$$

Relation (3.3.17) and conditions $\mathfrak{U}_4$ and $\mathfrak{J}_{12}$ permit to follow the proof of the relation of **J**-compactness (3.3.2) given in the proof of Theorem 3.3.2. One should only replace the random variables $\nu_{\varepsilon i}(T)$ with the random variables $\sup_{t \leq T} \nu_{\varepsilon i}(t)$ in relations (3.3.3) and (3.3.4). Note that estimate (3.3.3) does not require monotonicity of the functions $y_i(t)$ and, in sequel, estimate (3.3.4) will not require monotonicity of the processes $\nu_{\varepsilon i}(t)$.

Moreover, it is useful to note that the moduli Δ_U and Δ_J can be defined by the same formulas not only for càdlàg functions, but for any real-valued function. Also the estimate 3.2.7 is valid for any real-valued functions.

So, under conditions $\mathcal{A}_{34}$, $\mathfrak{U}_4$, and $\mathfrak{J}_{12}$, the processes $\zeta_\varepsilon(t)$ weakly converge on the set V and satisfy the relation of **J**-compactness (3.3.2).

As was mentioned above, it is not certain that the pre-limiting composition $\zeta_\varepsilon(t), t \geq 0$, is an a.s. càdlàg process for $\varepsilon > 0$. However, the limiting process $\zeta_0(t), t \geq 0$ is an a.s. càdlàg process, since the external limiting process $\xi_0(t), t \geq 0$ is a.s. continuous.

We refer to works by Borovkov (1976) and Borovkov, Mogul'skij, and Sakhanenko (1995) where one can find a discussion concerning **J**-convergence of stochastic processes in such a case.

3.3.7. The time interval $[0, T]$. In this case, we consider the vector composition $\zeta_\varepsilon(t) = (\xi_{\varepsilon i}(\nu_{\varepsilon i}(t)), i = 1, \ldots, m)$, $t \in [0, T]$ of a vector càdlàg process $\xi_\varepsilon(t) = (\xi_{\varepsilon i}(t), i = 1, \ldots, m)$, $t \geq 0$ with real-valued components, and a vector càdlàg process $\boldsymbol{\nu}_\varepsilon(t) = (\nu_{\varepsilon i}(t), i = 1, \ldots, m)$, $t \in [0, T]$ with non-negative and non-decreasing components.

The internal stopping process can be continued to the interval $[0, \infty)$ by the following formula:

$$\boldsymbol{\nu}_\varepsilon(t) = \begin{cases} \boldsymbol{\nu}_\varepsilon(t) & \text{if } 0 \leq t \leq T, \\ \boldsymbol{\nu}_\varepsilon(T) & \text{if } t \geq T. \end{cases} \tag{3.3.18}$$

We can apply Theorem 3.3.2 to the processes $\zeta_\varepsilon(t)$, $t \geq 0$ with the internal stopping processes $\boldsymbol{\nu}_\varepsilon(t)$, $t \geq 0$ defined in (3.3.18). Conditions of this theorem should be modified taking into account (3.3.18).

The condition of weak convergence $\mathcal{A}_{34}$ should be replaced with a condition in which V is a set dense in $[0, T]$ and contains the points 0 and T. The condition of **U**-compactness $\mathfrak{U}_4$ remains the same. In the condition of **J**-compactness $\mathfrak{J}_{12}$, the corresponding asymptotic relation should be required to hold only for the interval $[0, T]$. With these changes, conditions $\mathcal{A}_{34}$, $\mathfrak{U}_4$, and $\mathfrak{J}_{12}$ imply that $\zeta_\varepsilon(t), t \in [0, \infty) \xrightarrow{\mathbf{J}} \zeta_0(t), t \in [0, \infty)$ as $\varepsilon \to 0$.

Note that **J**-convergence of the processes $\zeta_\varepsilon(t)$ on the interval $[0, \infty)$ does automatically imply **J**-convergence of these processes on the interval $[0, T]$ if the point T is a point of stochastic continuity for the limiting process $\zeta_0(t)$, $t \geq 0$. In this case, no additional conditions are required.

However, it can happen that T is not a point of stochastic continuity for the limiting process $\zeta_0(t)$, $t \geq 0$. Since the process $\xi_0(t)$, $t \geq 0$ is a.s. continuous, this can occur if T is not a point of stochastic continuity for the limiting stopping process $\mathbf{v}_0(t)$, $t \geq 0$. Note that this process may be not an a.s. continuous process.

In this case, as follows from Theorem 1.6.3, the random variables $\zeta_\varepsilon(T - 0)$ must be added in the relation of weak convergence for the processes $\zeta_\varepsilon(t)$ on the set V. In order to provide this convergence, it is necessary add, in the relation of weak convergence in condition $\mathcal{A}_{34}$, the random variables $\mathbf{v}_\varepsilon(T - 0)$. These modified versions of conditions $\mathcal{A}_{34}$ and $\mathfrak{U}_4$ imply that $(\zeta_\varepsilon(t), \zeta_\varepsilon(T - 0)), t \in V \Rightarrow (\zeta_0(t), \zeta_0(T - 0)), t \in V$ as $\varepsilon \to 0$.

Finally, the modified versions of conditions $\mathcal{A}_{34}$, $\mathfrak{U}_4$, and $\mathfrak{J}_{12}$ imply that

$$\zeta_\varepsilon(t), t \in [0, T] \xrightarrow{\mathbf{J}} \zeta_0(t), t \in [0, T] \text{ as } \varepsilon \to 0. \tag{3.3.19}$$

3.3.8. The time interval $(0, \infty)$. The results of the section can also be restated for the case of the semi-infinite interval $(0, \infty)$ under the condition that the limiting stopping random variable $\nu_{0i}(t)$ is positive with probability 1 for every $t > 0$ and $i = 1, \dots m$. In this case, the point 0 can be excluded from the sets U and V in condition $\mathcal{A}_{34}$. Also, the relations of **U**- and **J**-compactness, respectively, in conditions of $\mathfrak{U}_4$ and $\mathfrak{J}_{12}$ should be required to hold for any finite interval $[T', T'']$, where $0 < T' < T'' < \infty$.

By applying Theorem 2.7.1 and taking into account the remarks in Subsection 2.7.6, one can prove weak convergence of the vector compositions

$$\zeta_\varepsilon(t), t \in V \Rightarrow \zeta_\varepsilon(t), t \in V \text{ as } \varepsilon \to 0. \tag{3.3.20}$$

The condition of **J**-compactness can be obtained by using the modified estimates (3.2.7) and (3.2.8).

The first one is valid for any m-dimensional càdlàg function $\mathbf{x}(t) = (x_i(t), i = 1, \dots, m)$, $t \geq 0$, any m-dimensional càdlàg function $\mathbf{y}(t) = (y_i(t), i = 1, \dots, m)$, $t \geq 0$, with non-negative components, and their vector composition $\mathbf{z}(t) = (x_i(y_i(t)), i = 1, \dots, m)$, $t \geq 0$

and $0 < T' < T'' < \infty$, $0 < T_1 < T_2 < \infty$,

$$\Delta_J(\mathbf{z}(\cdot), c, T', T'')\chi(\Delta_J(\mathbf{y}(\cdot), c, T', T'') \leq c',$$
$$\min_{1\leq i\leq m} y_i(T') \geq T_1, \max_{1\leq i\leq m} y_i(T'') \leq T_2) \leq \sum_{i=1}^{m} \Delta_U(x_i(\cdot), c', T_1, T_2). \tag{3.3.21}$$

Using (3.3.21) we have

$$\begin{aligned}
&\mathsf{P}\{\Delta_J(\boldsymbol{\zeta}_\varepsilon(\cdot), c, T', T'') > \delta\} \\
&\quad \leq \mathsf{P}\{\sum_{i=1}^{m} \Delta_U(\xi_{\varepsilon i}(\cdot), c, T_1, T_2) > \delta\} + \mathsf{P}\{\Delta_J(\mathbf{v}_\varepsilon(\cdot), c, T', T'') > c'\} \\
&\quad + \mathsf{P}\{\min_{1\leq i\leq m} \nu_{\varepsilon i}(T') < T_1\} + \mathsf{P}\{\max_{1\leq i\leq m} \nu_{\varepsilon i}(T'') > T_2\} \\
&\quad \leq \sum_{i=1}^{m} \mathsf{P}\{\Delta_U(\xi_{\varepsilon i}(\cdot), c', T_1, T_2) > \delta/m\} + \mathsf{P}\{\Delta_J(\mathbf{v}_\varepsilon(\cdot), c, T', T'') > c'\} \\
&\quad + \sum_{i=1}^{m} \mathsf{P}\{\nu_{\varepsilon i}(T') < T_1\} + \sum_{i=1}^{m} \mathsf{P}\{\nu_{\varepsilon i}(T'') > T_2\}.
\end{aligned} \tag{3.3.22}$$

By repeating the steps that follow in the proof of Theorem 3.3.2, one can get the relation of **J**-compactness for $0 < T' < T'' < \infty$,

$$\lim_{c\to 0} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\Delta_J(\boldsymbol{\zeta}_\varepsilon(\cdot), c, T', T'') > \delta\} = 0, \ \delta > 0. \tag{3.3.23}$$

The relations (3.3.20) and (3.3.23) imply that

$$\boldsymbol{\zeta}_\varepsilon(t), t \in (0, \infty) \xrightarrow{\mathbf{J}} \boldsymbol{\zeta}_0(t), t \in (0, \infty) \text{ as } \varepsilon \to 0. \tag{3.3.24}$$

3.3.8. A Polish phase space. Results presented in this section can be generalised to a model with external stochastic processes $\boldsymbol{\xi}_\varepsilon(t)$, $t \geq 0$, which components $\xi_{\varepsilon i}(t)$, $t \geq 0$, take values in a Polish space X.

The formulation of condition $\mathcal{A}_{\mathbf{34}}$ will be the same. In the conditions $\mathfrak{U}_{\mathbf{4}}'$, the Euclidian distance $|x-y|$ must be replaced with the corresponding metric $d(x, y)$ in the formulas that define the corresponding moduli of **U**-compactness. Condition $\mathfrak{J}_{\mathbf{12}}$ does not require any changes.

3.4 Asymptotically continuous internal stopping processes

In this section, we formulate conditions for **J**-convergence of compositions of càdlàg processes with asymptotically continuous internal stopping processes. This model covers a significant part of applications.

3.4.1. J-convergence of semi-vector compositions with asymptotically continuous internal stopping component. Let, for every $\varepsilon \geq 0$, $\xi_\varepsilon(t) = (\xi_{\varepsilon i}(t), i = 1, \dots, m)$, $t \geq 0$ be an m-dimensional càdlàg process with real-valued components and $\nu_\varepsilon(t)$, $t \geq 0$ be a non-negative non-decreasing càdlàg process. We will consider the *semi-vector composition* $\zeta_\varepsilon(t) = (\xi_{\varepsilon i}(\nu_\varepsilon(t)), i = 1, \dots, m)$, $t \geq 0$, which is also an m-dimensional càdlàg process with real-valued components.

We impose the following condition of **J**-compactness on the external processes $\xi_\varepsilon(t)$, $t \geq 0$. This condition was actually introduced in Subsection 1.6.11,

$\mathcal{J}_4$: $\lim_{c\to 0} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\Delta_J(\xi_\varepsilon(\cdot), c, T) > \delta\} = 0,\ \delta, T > 0.$

Conditions $\mathcal{A}_{36}$ and $\mathcal{J}_4$ imply **J**-convergence of the processes $\xi_\varepsilon(t)$, $t \geq 0$, but they do not require a.s. continuity of the corresponding limiting càdlàg processes $\xi_0(t)$, $t \geq 0$.

As follows from the remark above and Lemma 1.6.5, the set U in $\mathcal{A}_{36}$ can be enlarged, under condition $\mathcal{J}_4$, to the set $U \cup U_0$. Here U_0 is a set of points of stochastic continuity for the process $\xi_0(t)$, $t \geq 0$.

We assume that the internal stopping processes $\nu_\varepsilon(t)$, $t \geq 0$, satisfy a condition that is a scalar analogue of the condition $\mathcal{B}_3$,

$\mathcal{B}_4$: $\nu_0(t)$, $t \geq 0$ is an a.s. continuous process.

Lemma 3.2.1 implies that, under $\mathcal{A}_{36}$, condition $\mathcal{B}_4$ is equivalent to the following condition of **U**-compactness, a scalar analogue of condition $\mathcal{U}_5$:

$\mathcal{U}_6$: $\lim_{c\to 0} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\Delta_U(\nu_\varepsilon(\cdot), c, T) > \delta\} = 0, \delta, T > 0.$

We also use the following form of the first-type continuity condition $\mathcal{C}_6^{\mathbf{W}}$, in which the set W is not specified,

$\mathcal{C}_7$: There exists a set W such that (a) $\mathsf{P}\{\nu_0(t) \in R[\xi_0(\cdot)]\} = 0$ for $t \in W$, (b) W is a subset of $[0, \infty)$ that is dense in this interval and contains the point 0.

The following theorem can be found in Silvestrov (1972b, 1972e, 1973a).

Theorem 3.4.1. *Let conditions $\mathcal{A}_{36}$, $\mathcal{J}_4$, $\mathcal{B}_4$, and $\mathcal{C}_7$ hold. Then*

$$\zeta_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{J}} \zeta_0(t), t \geq 0 \text{ as } \varepsilon \to 0.$$

Proof of Theorem 3.4.1. We first apply Theorem 2.7.1 to the vector composition $\zeta_\varepsilon(t) = (\xi_{\varepsilon 1}(\nu_\varepsilon(t)), \dots, \xi_{\varepsilon m}(\nu_\varepsilon(t)))$, $t \geq 0$ of the external processes $\xi_\varepsilon(t) = (\xi_{\varepsilon 1}(t), \dots, \xi_{\varepsilon m}(t))$, $t \geq 0$ and the internal stopping processes $\boldsymbol{\nu}_\varepsilon(t) = (\nu_\varepsilon(t), \dots, \nu_\varepsilon(t))$, ≥ 0, with m identical components. It follows from Lemma 3.2.1 that conditions $\mathcal{A}_{36}$ and $\mathcal{B}_4$ imply condition $\mathcal{U}_5$. It also follows from Theorem 1.6.11 that, under condition $\mathcal{U}_5$, the set V in $\mathcal{A}_{36}$ can be taken to be the interval $[0, \infty)$. Therefore, condition $\mathcal{A}_{22}^{\mathbf{V}}$ holds with the set $V =$

$[0, \infty)$. Condition $\mathfrak{C}_7$ means that condition $\mathfrak{C}_6^{\mathbf{W}}$ holds with some set W dense in $[0, \infty)$ and containing 0. Therefore, it follows from Theorem 2.7.1 that, for this set W,

$$\zeta_\varepsilon(t), t \in W \Rightarrow \zeta_0(t), t \in W \text{ as } \varepsilon \to 0. \tag{3.4.1}$$

To prove the theorem we must also supplement the relation of weak convergence (3.4.1) with the relation of $\mathbf{J}$-compactness

$$\lim_{c\to 0} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\Delta_J(\zeta_\varepsilon(\cdot), c, T) > \delta\} = 0, \ \delta, T > 0. \tag{3.4.2}$$

Let $\mathbf{x}(t)$, $t \geq 0$ be a càdlàg function taking values in $\mathbb{R}_m$ and $y(t)$, $t \geq 0$ be a non-negative and non-decreasing càdlàg function.

It is clear that, if $t' \leq t \leq t''$, then

$$\begin{aligned}&\min(|\mathbf{x}(y(t)) - \mathbf{x}(y(t'))|, |\mathbf{x}(y(t)) - \mathbf{x}(y(t''))|)\\ &\quad \times \chi(\max(|y(t) - y(t')|, |y(t) - y(t'')|) \leq c', y(t'') \leq T')\\ &\quad \leq \Delta_J(\mathbf{x}(\cdot), c', T')\end{aligned}$$

and, therefore,

$$\Delta_J(\mathbf{x}(y(\cdot)), c, T)\chi(\Delta_U(y(\cdot), c, T) \leq c', y(T) \leq T') \leq \Delta_J(\mathbf{x}(\cdot), c', T'). \tag{3.4.3}$$

Using (3.4.3) we have

$$\begin{aligned}&\mathsf{P}\{\Delta_J(\xi_\varepsilon(\nu_\varepsilon(\cdot)), c, T) > \delta\}\\ &\quad \leq \mathsf{P}\{\Delta_J(\xi_\varepsilon(\nu_\varepsilon(\cdot)), c, T) > \delta, \Delta_U(\nu_\varepsilon(\cdot), c, T) \leq c', \nu_\varepsilon(T) \leq T'\}\\ &\quad + \mathsf{P}\{\Delta_U(\nu_\varepsilon(\cdot), c, T) > c'\} + \mathsf{P}\{\nu_\varepsilon(T) > T'\}\\ &\quad \leq \mathsf{P}\{\Delta_J(\xi_\varepsilon(\cdot), c', T') > \delta\}\\ &\quad + \mathsf{P}\{\Delta_U(\nu_\varepsilon(\cdot), c, T) > c'\} + \mathsf{P}\{\nu_\varepsilon(T) > T'\}.\end{aligned} \tag{3.4.4}$$

For an arbitrary $\sigma > 0$, by condition $\mathcal{A}_{36}$, we can choose $T'' \in V$ and then T', which is a point of continuity for the distribution function of the random variable $\nu_0(T'')$, such that $\overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\nu_\varepsilon(T) > T'\} \leq \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\nu_\varepsilon(T'') > T'\} \leq \sigma/2$. Then, fixing T' and using condition $\mathfrak{J}_4$, we can find $c' > 0$ such that $\overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\Delta_J(\xi_\varepsilon(\cdot), c', T') > \delta\} \leq \sigma/2$. If we pass to the limit in (3.4.4), first making $\varepsilon \to 0$ and then $c \to 0$, and use condition $\mathfrak{U}_5$, we find

$$\lim_{c\to 0} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\Delta_J(\xi_\varepsilon(\nu_\varepsilon(\cdot)), c, T > \delta\} \leq \sigma + \lim_{c\to 0} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\Delta_U(\nu_\varepsilon(\cdot), c, T) > c'\} = \sigma. \tag{3.4.5}$$

This proves (3.4.2), since σ is arbitrary. □

Condition $\mathfrak{Q}_3$ takes in this case the following form:

$\mathfrak{Q}_7$: $\xi_0(t) = \xi'_0(t) + \xi''_0(t)$, $t \geq 0$, where (a) $\xi'_0(t)$ is a continuous process, (b) $\xi''_0(t)$ is a stochastically continuous càdlàg process, (c) the processes $\xi''_0(t), t \geq 0$ and $\nu_0(t)$, $t \geq 0$ are independent.

It follows from Lemma 2.7.1 that condition $\mathfrak{Q}_7$ implies that condition $\mathfrak{C}_7$ holds with the set $W = [0, \infty)$.

Condition $\mathcal{E}_4$ takes the following form:

$\mathcal{E}_6$: $\mathsf{P}\{\nu_0(t') = \nu_0(t'') \in R[\xi_0(\cdot)]\} = 0$ for $0 \leq t' < t'' < \infty$.

Let us also introduce a condition that, actually, coincides with $\mathfrak{C}_6^{\mathbf{W}}$ in the case where the set $W = \{w\}$ contains only one point w and the process $\boldsymbol{\nu}_0(t) = (\nu_0(t), \dots, \nu_0(t)), t \geq 0$ has identical components,

$\mathfrak{C}_8^{(\mathbf{w})}$: $\mathsf{P}\{\nu_0(w) \in R[\xi_0(\cdot)]\} = 0$.

As follows from Lemma 2.7.2, conditions $\mathcal{E}_6$ and $\mathfrak{C}_8^{(0)}$ are necessary and sufficient for condition $\mathfrak{C}_7$ to hold.

The following theorem is the main result of this section. It is, actually, equivalent to Theorem 3.4.1 and does not require a separate proof.

Theorem 3.4.2. *Let conditions $\mathcal{A}_{36}$, $\mathcal{J}_4$, $\mathcal{B}_4$, $\mathcal{E}_6$, and $\mathfrak{C}_8^{(0)}$ hold. Then*

$$\zeta_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{J}} \zeta_0(t), t \geq 0 \text{ as } \varepsilon \to 0.$$

Despite that the Theorems 3.4.1 and 3.4.2 are equivalent, the latter one has an advantage, since conditions $\mathcal{E}_6$ and $\mathfrak{C}_8^{(0)}$ have a more explicit form than condition $\mathfrak{C}_7$ used in Theorem 3.4.1.

Condition $\mathcal{E}_6$ is satisfied if the following condition introduced in Subsection 2.6.3 holds:

$\mathfrak{I}_1$: $\nu_0(t)$, $t \geq 0$ is an a.s. strictly increasing process.

In applications to renewal type models, the limiting internal stopping process is often an exceeding time process. It has the following form: $\nu_0(t) = \sup(s : \kappa_0(s) \leq t), t \geq 0$, where $\kappa_0(s)$, $s \geq 0$ is a càdlàg process such that **(a)** $\kappa_0^+(s) = \sup_{u \leq s} \kappa_0(u)$, $s \geq 0$ is an a.s. strictly increasing process, **(b)** $\kappa_0^+(s) \xrightarrow{\mathsf{P}} \infty$ as $s \to \infty$.

Condition **(b)** implies that $\nu_0(t) < \infty$ with probability 1 for every $t \geq 0$, whereas the condition **(a)** implies that $\nu_0(t), t \geq 0$ is an a.s. continuous process.

In this case, condition $\mathfrak{I}_1$ usually prohibits the process $\kappa_0(s), s \geq 0$ to have positive jumps. This restricts applications of condition $\mathfrak{I}_1$.

Condition $\mathcal{E}_6$ can hold in situations where the process $\kappa_0(s), s \geq 0$ may possess positive jumps. For example, condition $\mathcal{E}_6$ holds if the process $\xi_0(s), s \geq 0$ can be decomposed into the sum $\xi_0(s) = \xi'_0(s) + \xi''_0(s), s \geq 0$, where the first component is an a.s. continuous process possibly dependent on the process $\kappa_0(s), s \geq 0$, whereas the second component is a stochastically continuous càdlàg process independent of the process $\kappa_0(s), s \geq 0$.

3.4.2. Conditions of J-compactness. It is useful to note that **J**-compactness of the processes $\zeta_\varepsilon(t), t \geq 0$, can be proved without the use of the weak convergence condition $\mathcal{A}_{36}$.

Let us introduce the following condition:

$\mathcal{K}_4^{(0)}$: $\lim_{t\to\infty} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\nu_\varepsilon(0) > t\} = 0$.

Lemma 3.4.1. *Let conditions $\mathcal{J}_4$, $\mathcal{U}_5$, and $\mathcal{K}_4^{(0)}$ hold. Then*

$$\lim_{c\to 0}\overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\Delta_J(\zeta_\varepsilon(\cdot), c, T) > \delta\} = 0, \ \delta, T > 0.$$

Proof of Lemma 3.4.1. Using estimate (3.2.10) for $c = 1/T$ and conditions $\mathcal{U}_5$ and $\mathcal{K}_4^{(0)}$ we get for every $T > 0$ that

$$\begin{aligned} \lim_{T'\to\infty}\overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\nu_\varepsilon(T) \geq T'\} &\leq \lim_{T'\to\infty}\overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\nu_\varepsilon(0) \geq T'/2\} \\ &+ \lim_{T'\to\infty}\overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\Delta_U(\nu_\varepsilon(\cdot), 1/T', T) \geq T'/2([TT'] + 1)\} = 0. \end{aligned} \tag{3.4.6}$$

Now, the proof of the lemma follows directly from estimate (3.4.4) and relation (3.4.6). □

3.4.3. Weakened second-type continuity conditions. Let us formulate an analogue of Theorem 3.4.2, in which the continuity conditions $\mathcal{E}_6$ and $\mathcal{C}_8^{(0)}$ are weakened. Introduce the following conditions:

$\mathcal{F}_4$: There exist sequences $\delta_l \in Z_0, \delta_l \to 0$ as $l \to \infty$ and $0 < T_r \to \infty$ as $r \to \infty$ such that, for every $l, k, r \geq 1$ and $i = 1, \dots, m$, $\lim_{0<c\to 0} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\alpha_{\varepsilon ik}^{(\delta_l)} - c \leq \nu_\varepsilon(t'), \nu_\varepsilon(t'') < \alpha_{\varepsilon ik}^{(\delta_l)}, \alpha_{\varepsilon ik}^{(\delta_l)} < T_r\} = 0$ for all $0 \leq t' < t'' < \infty$;

and

$\mathcal{D}_7^{(w)}$: There exist sequences $\delta_l \in Z_0, \delta_l \to 0$ as $l \to \infty$ and $0 < T_r \to \infty$ as $r \to \infty$ such that, for every $l, k, r \geq 1$ and $i = 1, \dots, m$, $\lim_{0<c\to 0} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\alpha_{\varepsilon ik}^{(\delta_l)} - c \leq \nu_\varepsilon(w) < \alpha_{\varepsilon ik}^{(\delta_l)}, \alpha_{\varepsilon ik}^{(\delta_l)} < T_r\} = 0$.

Theorem 3.4.3. *Let conditions $\mathcal{A}_{36}$, $\mathcal{J}_4$, $\mathcal{B}_4$, $\mathcal{F}_4$, and $\mathcal{D}_7^{(0)}$ hold. Then*

$$\zeta_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{J}} \zeta_0(t), t \geq 0 \text{ as } \varepsilon \to 0.$$

Proof of Theorem 3.4.3. Conditions $\mathcal{A}_{36}$, $\mathcal{J}_4$, $\mathcal{F}_4$, and $\mathcal{D}_7^{(0)}$ imply that conditions of Theorem 2.7.9 hold for the external processes $\boldsymbol{\xi}_\varepsilon(t)$, $t \geq 0$ and the internal stopping processes $\mathbf{v}_\varepsilon(t) = (\nu_\varepsilon(t), \dots, \nu_\varepsilon(t))$, $t \geq 0$, with m identical components. In particular, condition $\mathcal{A}_{36}$ implies that condition $\mathcal{A}_{22}^{\mathbf{V}}$ holds for the set V in $\mathcal{A}_{36}$. Condition $\mathcal{J}_4$ is required in both Theorems 3.4.3 and 2.7.9. Also, condition $\mathcal{F}_4$ implies that condition $\mathcal{F}_3$ holds. By applying Theorem 2.7.9, we prove that the processes $\zeta_\varepsilon(t)$ weakly converge to $\zeta_0(t)$ as $\varepsilon \to 0$ on the set S_0 defined in this theorem. This set is dense in $[0, \infty)$. Due to condition $\mathcal{D}_7^{(0)}$, the point 0 can also be included in the set S_0.

Conditions $\mathcal{A}_{36}$, $\mathcal{J}_4$, and $\mathcal{B}_4$ imply that the conditions of Lemma 3.4.1 hold. In particular, condition $\mathcal{A}_{36}$ implies condition $\mathcal{K}_4^{(0)}$. Also, by Lemma 3.2.1, conditions $\mathcal{A}_{36}$ and $\mathcal{B}_4$ imply condition $\mathcal{U}_5$. By applying Lemma 3.4.1, we prove **J**-compactness of the processes $\zeta_\varepsilon(t)$, $t \geq 0$, on any finite interval.

To complete the proof, it remains to use Theorem 1.6.6 that gives conditions for **J**-convergence of càdlàg processes defined on interval $[0, \infty)$. □

3.4.4. The set of weak convergence. Denote by W_0 the set of points w that satisfy condition $\mathcal{C}_8^{(\mathbf{w})}$. Obviously, $W \subseteq W_0$. Let also Z_0 be the set of points of stochastic continuity for the limiting stopping process $\zeta_0(t)$, $t \geq 0$. The set W can obviously be replaced with the set W_0 in the proof of Theorem 3.4.1. So, we can prove weak convergence of the processes $\zeta_\varepsilon(t)$, $t \geq 0$ on set W_0. This set is dense in $[0, \infty)$ and contains the point 0. The processes $\zeta_\varepsilon(t)$, $t \geq 0$ **J**-converge and therefore, by Lemma 1.6.5, the set W_0 can be extended to the set $W_0 \cup Z_0$. Finally, we get that, under conditions of Theorem 3.4.1, the following relation holds:

$$\zeta_\varepsilon(t), t \in W_0 \cup Z_0 \Rightarrow \zeta_0(t), t \in W_0 \cup Z_0 \text{ as } \varepsilon \to 0. \tag{3.4.7}$$

3.4.5. Non-monotone internal stopping processes. If the external processes are not asymptotically continuous, then monotonicity of the internal stopping processes plays an essential role. As a matter of fact, the key estimate (3.4.3) does require that the order $t' \leq t \leq t''$ be preserved for the values $y(t') \leq y(t) \leq y(t'')$. We discuss this problem in Subsection 3.6.8.

3.4.6. The time interval $[0, T]$. In this case, we consider the semi-vector composition $\zeta_\varepsilon(t) = (\xi_{\varepsilon i}(\nu_\varepsilon(t)), i = 1, \dots, m)$, $t \in [0, T]$ of a vector càdlàg process $\boldsymbol{\xi}_\varepsilon(t) = (\xi_{\varepsilon i}(t), i = 1, \dots, m)$, $t \geq 0$, with real-valued components, and a non-negative and non-decreasing càdlàg process $\nu_\varepsilon(t)$, $t \in [0, T]$.

We can always continue the internal stopping process to the interval $[0, \infty)$ by the following formula:

$$\nu_\varepsilon(t) = \begin{cases} \nu_\varepsilon(t) & \text{if } 0 \leq t \leq T, \\ \nu_\varepsilon(T) & \text{if } t \geq T. \end{cases} \tag{3.4.8}$$

No we can apply Theorems 3.4.1 or 3.4.2 to the processes $\zeta_\varepsilon(t)$, $t \geq 0$ with internal stopping processes $\nu_\varepsilon(t)$, $t \geq 0$, defined in (3.4.8). Conditions of this theorems should be modified taking into account (3.4.8).

The condition of weak convergence $\mathcal{A}_{36}$ should be replaced with a condition in which the set V is dense in $[0, T]$ and contains the points 0 and T. The condition of **J**-compactness $\mathfrak{J}_4$ does not require any changes. The condition of continuity $\mathfrak{B}_4$ should be restricted to the interval $[0, T]$.

Also, it should be assumed that the set W in condition $\mathfrak{C}_7$ is a dense set in $[0, T]$ and contains the points 0 and T. If condition $\mathcal{E}_6$ is used, instead of $\mathfrak{C}_7$, it should be assumed that the points t', t'' in this condition are taken such that $0 \leq t' < t'' \leq T$. Also, condition $\mathfrak{C}_8^{(0)}$ and, additionally, condition $\mathfrak{C}_8^{(T)}$ must be assumed to hold.

If condition $\mathcal{F}_4$ is used, instead of $\mathfrak{C}_7$, then it should be assumed that the points t', t'' in this condition are taken such that $0 \leq t' < t'' \leq T$. Also, it should be assumed that condition $\mathfrak{D}_7^{(0)}$ and, additionally, condition $\mathfrak{D}_7^{(T)}$ hold.

With these changes, conditions $\mathcal{A}_{36}$, $\mathfrak{J}_4$, $\mathfrak{B}_4$, and one of the following combinations of conditions **(a)** $\mathfrak{C}_7$, **(b)** $\mathcal{E}_6$, $\mathfrak{C}_8^{(0)}$, $\mathfrak{C}_8^{(T)}$, or **(c)** $\mathcal{F}_4$, $\mathfrak{D}_7^{(0)}$, $\mathfrak{D}_7^{(T)}$ imply that the processes $\zeta_\varepsilon(t), t \in [0, \infty)$ **J**-converge to the process $\zeta_0(t), t \in [0, \infty)$ as $\varepsilon \to 0$.

Again, **J**-convergence of the processes $\zeta_\varepsilon(t)$ on the interval $[0, \infty)$ automatically implies that these processes **J**-convergence on the interval $[0, T]$ if the point T is a point of stochastic continuity for the limiting process $\zeta_0(t)$, $t \in [0, T]$. In this case, no additional conditions are needed. This pertains to cases **(a)** and **(b)**. Indeed, conditions $\mathfrak{B}_4$ and $\mathfrak{C}_8^{(T)}$ imply that T is a point of stochastic continuity for the process $\zeta_0(t)$. In the case **(c)**, it is possible that T is a point of stochastic discontinuity for the process $\zeta_0(t)$. So, condition $\mathfrak{D}_7^{(T-)}$ i.e., condition $\mathfrak{D}_7^{(T)}$ where the random variables $\nu_\varepsilon(T)$ are replaced with $\nu_\varepsilon(T-0)$, should additionally be assumed. This allows to include the random variables $\zeta_\varepsilon(T-0)$ in the corresponding relation of weak convergence and, in the sequel, to get **J**-convergence of the processes $\zeta_\varepsilon(t), t \in [0, T]$.

3.4.7. The time interval $(0, \infty)$. The results of this section can also be recast in the case of the semi-infinite interval $(0, \infty)$. The condition that the limiting internal stopping random variable $\nu_0(t) > 0$ with probability 1 for every $t > 0$ should be imposed. In this case, the point 0 can be excluded from the sets U and V in condition $\mathcal{A}_{36}$. Also, the relations of **J** and **U**-compactness, respectively, in conditions of $\mathfrak{J}_4$ and $\mathfrak{U}_6$ should be requested to hold for any finite interval $[T', T'']$, where $0 < T' < T'' < \infty$. Finally, the set W in condition $\mathfrak{C}_7$ should be dense in the open interval $(0, \infty)$. Also, in conditions $\mathcal{E}_6$, the corresponding relation should be required to hold only for $0 < t' < t'' < \infty$ and condition $\mathfrak{C}_8^{(0)}$ should be omitted. Analogously, if condition $\mathcal{F}_4$ is employed, then the corresponding asymptotic relation in this condition should be requested to hold only for $0 < t' < t'' < \infty$ and condition $\mathfrak{D}_7^{(0)}$ should be omitted.

By applying Theorem 2.7.1 and taking into account remarks in Subsection 2.7.6, one can prove weak convergence of the vector compositions

$$\zeta_\varepsilon(t), t \in V \Rightarrow \zeta_\varepsilon(t), t \in V \text{ as } \varepsilon \to 0. \tag{3.4.9}$$

The **J**-compactness condition can be obtained with a slight modification of estimates (3.4.3) and (3.4.4).

The first one is valid for any m-dimensional càdlàg function $\mathbf{x}(t)$, $t \geq 0$, non-negative and non-decreasing càdlàg function $y(t)$, $t \geq 0$, and $0 < T' < T'' < \infty$, $0 < T_1 < T_2 < \infty$,

$$\begin{aligned}\Delta_J(\mathbf{x}(y(\cdot)), c, T', T'')\chi(\Delta_U(y(\cdot), c, T', T'') \leq c', y(T') \geq T_1, y(T'') \leq T_2) \\ \leq \Delta_J(\mathbf{x}(\cdot), c', T_1, T_2).\end{aligned} \tag{3.4.10}$$

Using (3.4.10) we have

$$\begin{aligned}&\mathsf{P}\{\Delta_J(\xi_\varepsilon(\nu_\varepsilon(\cdot)), c, T', T'') > \delta\} \\ &\quad \leq \mathsf{P}\{\Delta_J(\xi_\varepsilon(\nu_\varepsilon(\cdot)), c, T', T'') > \delta, \\ &\qquad \Delta_U(\nu_\varepsilon(\cdot), c, T', T'') \leq c', \nu_\varepsilon(T') \geq T_1, \nu_\varepsilon(T'') \leq T_2\} \\ &\quad + \mathsf{P}\{\Delta_U(\nu_\varepsilon(\cdot), c, T) > c'\} + \mathsf{P}\{\nu_\varepsilon(T') < T_1\} + \mathsf{P}\{\nu_\varepsilon(T'') > T_2\} \\ &\quad \leq \mathsf{P}\{\Delta_J(\xi_\varepsilon(\cdot), c', T_1, T_2) > \delta\} + \mathsf{P}\{\Delta_U(\nu_\varepsilon(\cdot), c, T', T'') > c'\} \\ &\quad + \mathsf{P}\{\nu_\varepsilon(T') < T_1\} + \mathsf{P}\{\nu_\varepsilon(T'') > T_2\}.\end{aligned} \tag{3.4.11}$$

By repeating the subsequent steps in the proof of Theorem 3.4.2, one can get relation of **J**-compactness for $0 < T' < T'' < \infty$,

$$\lim_{c \to 0} \overline{\lim_{\varepsilon \to 0}} \mathsf{P}\{\Delta_J(\zeta_\varepsilon(\cdot), c, T', T'') > \delta\} = 0, \ \delta > 0. \tag{3.4.12}$$

Relations (3.4.9) and (3.4.12) imply that

$$\zeta_\varepsilon(t), t \in (0, \infty) \xrightarrow{\mathbf{J}} \zeta_0(t), t \in (0, \infty) \text{ as } \varepsilon \to 0. \tag{3.4.13}$$

3.4.8. A Polish phase space. Results in this section can be generalised to a model with the external processes $\xi_\varepsilon(t)$, $t \geq 0$, that have values of the components $\xi_{\varepsilon i}(t)$, $t \geq 0$ in a Polish space X.

The formulation of condition $\mathcal{A}_{36}$ can be kept without changes. In the condition $\mathcal{J}_4$, the Euclidean distance $|x - y|$ must be replaced with the corresponding metric $d(x, y)$ in the formulas for the moduli $\Delta_J(\xi_{\varepsilon i}(\cdot), c, T)$, $i = 1, \dots, m$.

All other conditions $\mathcal{B}_4$, $\mathcal{E}_6$, and $\mathcal{C}_8^{(0)}$ of Theorems 3.4.1 and 3.4.2 remain the same. With these changes in the conditions, the proofs of these theorems can be repeated. In particular, the estimates for the modulus of **J**-compactness $\Delta_J(\zeta_{\varepsilon i}(\cdot), c, T)$, given in the proofs of Theorem 3.4.1 and 3.4.2, still hold. Finally, we get that, under conditions of Theorems 3.4.1 or Theorem 3.4.2,

$$\zeta_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{J}} \zeta_0(t), t \geq 0 \text{ as } \varepsilon \to 0. \tag{3.4.14}$$

3.4.9. J-convergence in translation theorems. As an example of application of Theorem 3.4.2, let us consider the model introduced in Section 2.8. We, therefore, assume that the following representation holds:

$$\nu_\varepsilon(t) = t\nu_\varepsilon = t\frac{\mu_\varepsilon}{n_\varepsilon}, \quad \xi_{\varepsilon i}(t) = \frac{\eta_{\varepsilon i}(tn_\varepsilon)}{n_\varepsilon^\alpha h(n_\varepsilon)}, \ t \geq 0, \ i = 1, \dots, m,$$

where **(a)** $\alpha = \text{const} \geq 0$; **(b)** n_ε is a non-random positive function such that $n_\varepsilon \to \infty$ as $\varepsilon \to 0$; **(c)** $h(x)$, $x \geq 0$ is a slowly varying function.

We will consider the processes $\zeta'_\varepsilon(t) = (\zeta'_{\varepsilon i}(t), i = 1, \dots, m)$, $t \geq 0$, where

$$\zeta'_{\varepsilon i}(t) = \frac{\eta_{\varepsilon i}(t\mu_\varepsilon)}{\mu_\varepsilon^\alpha h(\mu_\varepsilon)} \cdot \chi(\mu_\varepsilon \neq 0), \ t \geq 0, \ i = 1, \dots, m.$$

As was shown in Section 2.8, the processes $\zeta'_\varepsilon(t)$, $t \geq 0$ can be represented in the form of a semi-vector composition $\zeta'_\varepsilon(t) = \xi'_\varepsilon(\nu_\varepsilon(t))$, $t \geq 0$. Here, the external process $\xi'_\varepsilon(t) = \beta_\varepsilon \xi_\varepsilon(t)$, $t \geq 0$, where $\beta_\varepsilon = \nu_\varepsilon^{-\alpha}(h(\mu_\varepsilon)/h(n_\varepsilon))^{-1} \cdot \chi(\mu_\varepsilon \neq 0)$, and the internal stopping process is $\nu_\varepsilon(t) = t\nu_\varepsilon$, $t \geq 0$.

It was shown in the proof of Theorem 2.8.2 that, under conditions $\mathcal{A}_{23}$, $\mathfrak{J}_4$, and $\mathfrak{I}_4$ of this theorem,

$$(\nu_\varepsilon(s), \xi'_\varepsilon(t)), (s,t) \in [0,\infty) \times U \Rightarrow (\nu_0(t), \xi'_0(t)), t \in [0,\infty) \times U \text{ as } \varepsilon \to 0, \qquad (3.4.15)$$

where $\xi'_0(t) = \nu_0^{-\alpha}\xi_0(t), t \geq 0$.

This means that condition $\mathcal{A}_{36}$ holds for the processes $\nu_\varepsilon(t)$, $t \geq 0$, and $\xi'_\varepsilon(t)$, $t \geq 0$, with the set $V = [0,\infty)$. Also, it was shown in the proof of Theorem 2.8.2 that condition $\mathfrak{J}_4$ holds for the processes $\xi'_\varepsilon(t)$, $t \geq 0$. Conditions $\mathcal{B}_4$, $\mathcal{E}_6$, $\mathcal{C}_8^{(0)}$ obviously hold, since the limiting internal stopping process is $\nu_0(t) = t\nu_0$, $t \geq 0$. Therefore, Theorem 3.4.2 can be applied to the compositions $\zeta'_\varepsilon(t) = \xi'_\varepsilon(t\nu_\varepsilon)$, $t \geq 0$. This yields the following statement.

Theorem 3.4.4. *Let conditions $\mathcal{A}_{23}$, $\mathfrak{J}_4$, and $\mathfrak{I}_4$ of Theorem* 2.8.2 *hold. Then*

$$\zeta'_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{J}} \zeta'_0(t), t \geq 0 \text{ as } \varepsilon \to 0.$$

3.4.10. J-convergence for randomly stopped stochastic sequences. Let us also consider the model of randomly stopped stochastic sequences, introduced in Subsection 2.8.4. In this case, the conditions $\mathcal{A}_{25}$, $\mathfrak{J}_9$, and $\mathfrak{I}_5$, introduced in Subsection 2.8.4, and the condition $\mathfrak{I}_4$, introduced in Subsection 2.8.2, imply relation (2.8.36). This relation imply, in its turn, that conditions $\mathcal{A}_{36}$ and $\mathfrak{J}_4$ hold for the processes $\nu_n(t) = t\nu_n, t > 0$ and $\xi_n(t) = (\xi_n(t), b_n(t)), t > 0$. Also, conditions $\mathcal{B}_4$, $\mathcal{E}_6$, $\mathcal{C}_8^{(0)}$ obviously hold, since the limiting internal stopping process $\nu_0(t) = t\nu_0$, $t > 0$ is continuous. Therefore, Theorem

3.4.2 can be applied to the compositions $\xi_n(t\nu_n), t > 0$. The following relations, which improve relations (2.8.37), (2.8.38), and (2.8.39), can be written:

$$\begin{aligned}&\Big(\frac{\xi_{[t\mu_n]} - b_n}{a_{\mu_n}}, \frac{b_{[t\mu_n]} - b_n}{a_{\mu_n}}\Big) \cdot \chi(\mu_n \neq 0), t > 0 \\ &\qquad \xrightarrow{\mathbf{J}} (\nu_0^{-\rho}\xi_0(t\nu_0), \beta_\rho(t\nu_0)), t > 0 \text{ as } n \to \infty,\end{aligned} \tag{3.4.16}$$

and

$$\begin{aligned}&\frac{\xi_{[t\mu_n]} - b_{[t\mu_n]}}{a_{\mu_n}} \cdot \chi(\mu_n \neq 0), t > 0 \\ &\qquad \xrightarrow{\mathbf{J}} (\nu_0^{-\rho}\xi_0(t\nu_0) - \beta_\rho(t\nu_0)), t > 0 \text{ as } n \to \infty,\end{aligned} \tag{3.4.17}$$

as well as

$$\begin{aligned}&\frac{\xi_{[t\mu_n]} - b_{[\mu_n]}}{a_{\mu_n}} \cdot \chi(\mu_n \neq 0), t > 0 \\ &\qquad \xrightarrow{\mathbf{J}} (\nu_0^{-\rho}\xi_0(t\nu_0) - \beta_\rho(\nu_0)), t > 0 \text{ as } n \to \infty.\end{aligned} \tag{3.4.18}$$

3.5 Semi-vector compositions of càdlàg functions

In this section, we formulate general conditions for **J**-compactness and **J**-convergence of semi-vector compositions of non-random càdlàg functions for the general case where both the external and the internal limiting functions can be discontinuous. These conditions are used in an essential way in the next Section 3.6, where the corresponding results are obtained for semi-vector compositions of càdlàg stochastic processes.

3.5.1. Conditions for J-compactness of semi-vector compositions of non-random càdlàg functions. Let $\mathbf{x}(t)$, $t \geq 0$ be a function from the space $\mathbf{D}^{(m)}_{[0,\infty)}$. We denote by $R[\mathbf{x}(\cdot)]$ the set of points of discontinuity for the function $\mathbf{x}(t)$, $t \geq 0$. The set $R[\mathbf{x}(\cdot)]$ is empty, finite, or countable. The structure of this set can be described in the following way. Define recursively the functionals $\tau_{kn}(\mathbf{x}(\cdot)) = \inf(s > \tau_{k-1n}(\mathbf{x}(\cdot))\colon |\mathbf{x}(s) - \mathbf{x}(s-0)| \in [\frac{1}{n}, \frac{1}{n-1}))$, $k = 1, 2, \ldots$, where $\tau_{0n}(\mathbf{x}(\cdot)) = 0$. The functionals $\tau_{kn}(\mathbf{x}(\cdot)), k \geq 1$ take values in the interval $(0, \infty]$. If $\tau_{kn}(\mathbf{x}(\cdot)) < \infty$, then it is a point of the k-th jump of the function $\mathbf{x}(t)$, $t \geq 0$, with absolute value of the jump belonging to the interval $[\frac{1}{n}, \frac{1}{n-1})$. Denote by $\mu_n(\mathbf{x}(\cdot)) = \max(k\colon \tau_{kn}(\mathbf{x}(\cdot)) < \infty)$ the total number of such points of jumps. Obviously,

$$R[\mathbf{x}(\cdot)] = \{\tau_{kn}(\mathbf{x}(\cdot)), 1 \leq k < \mu_n(\mathbf{x}(\cdot)) + 1, n = 1, 2, \ldots\}.$$

The definitions above can also be applied to functions from the space $\mathbf{D}^{(1)}_{[0,\infty)+}$ of non-negative and non-decreasing càdlàg functions. Let $y(t)$, $t \geq 0$ be a function from this space. Let $R[y(\cdot)]$ denote the set of points of discontinuity for the function $y(t)$,

$t \geq 0$. The set $R[y(\cdot)]$ is empty, finite, or countable. Its structure can be described in the following way. Define recursively the functionals $\kappa_{kn}(y(\cdot)) = \inf(s > \kappa_{k-1n}(y(\cdot)) \colon y(s) - y(s-0) \in [\frac{1}{n}, \frac{1}{n-1}))$, $k = 1, 2, \ldots$, where $\kappa_{0n}(y(\cdot)) = 0$. The functionals $\kappa_{kn}(y(\cdot)), k \geq 1$ take values in the interval $(0, \infty]$. If $\kappa_{kn}(y(\cdot)) < \infty$, then it is a point of the k-th jump of the function $y(t)$, $t \geq 0$, with value of the jump belonging to the interval $[\frac{1}{n}, \frac{1}{n-1})$. Denote by $\mu_n(y(\cdot)) = \max(k \colon \kappa_{kn}(y(\cdot)) < \infty)$ the total number of such points of jumps. Obviously,

$$R[y(\cdot)] = \{\kappa_{kn}(y(\cdot)), 1 \leq k < \mu_n(y(\cdot)) + 1, n = 1, 2, \ldots\}.$$

Let $\mathbf{x}_n(t) = (x_{n1}(t), \ldots, x_{nm}(t)), t \geq 0$, $n = 0, 1, \ldots$, be a sequence of functions from $\mathbf{D}^{(m)}_{[0,\infty)}$, and let $y_n(t), t \geq 0$, $n = 0, 1, \ldots$ be a sequence of functions from $\mathbf{D}^{(1)}_{[0,\infty)+}$.

We assume that the following conditions of $\mathbf{J}$-convergence for the functions $\mathbf{x}_n(t)$ are verified:

$\mathcal{A}_{38}$: $\mathbf{x}_n(t) \to \mathbf{x}_0(t)$ as $n \to \infty$, $t \in X$, where X is a subset of $[0, \infty)$ that is dense in this interval and contains the point 0;

and

$\mathfrak{J}_{13}$: $\lim_{c\to 0} \overline{\lim}_{n\to\infty} \Delta_J(\mathbf{x}_n(\cdot), c, T) = 0, \ T > 0.$

We also impose the following conditions of $\mathbf{J}$-convergence on the functions $y_n(t)$:

$\mathcal{A}_{39}$: $y_n(t) \to y_0(t)$ as $n \to \infty$, $t \in Y$, where Y is a subset of $[0, \infty)$ that is dense in this interval and contains the point 0;

and

$\mathfrak{J}_{14}$: $\lim_{c\to 0} \overline{\lim}_{n\to\infty} \Delta_J(y_n(\cdot), c, T) = 0, \ T > 0.$

Note that the limiting functions $\mathbf{x}_0(t), t \geq 0$ and $y_0(t), t \geq 0$ are not assumed to be continuous.

Conditions $\mathcal{A}_{38}$ and $\mathfrak{J}_{13}$ imply $\mathbf{J}$-convergence of the functions $\mathbf{x}_n(t)$, $t \geq 0$, and conditions $\mathcal{A}_{39}$ and $\mathfrak{J}_{14}$ imply $\mathbf{J}$-convergence of the functions $y_n(t)$, $t \geq 0$. However, these conditions together do not imply either $\mathbf{J}$-convergence or $\mathbf{J}$-compactness of the vector functions $(y_n(t), \mathbf{x}_n(t))$, $t \geq 0$ and their compositions $\mathbf{x}(y_n(t))$, $t \geq 0$. The corresponding examples are given in Section 3.1.

The following condition plays a key role in the subsequent consideration:

$\mathcal{G}_2$: $y_0(t \pm 0) \notin R[\mathbf{x}_0(\cdot)]$ for $t \in R[y_0(\cdot)]$.

The following result is from Silvestrov (1974).

Lemma 3.5.1. *Let conditions* $\mathcal{A}_{38}$, $\mathfrak{J}_{13}$, $\mathcal{A}_{39}$, $\mathfrak{J}_{14}$, *and* $\mathcal{G}_2$ *hold. Then*

$$\lim_{c\to 0} \overline{\lim_{n\to\infty}} \Delta_J(\mathbf{x}_n(y_n(\cdot)), c, T) = 0, \ T > 0.$$

Proof of Lemma 3.5.1. Conditions $\mathcal{A}_{38}$ and $\mathcal{J}_{13}$ imply that $\mathbf{x}_n(\cdot), t \geq 0 \xrightarrow{\mathbf{J}} \mathbf{x}_0(t),\ t \geq 0$ as $\varepsilon \to 0$. This shows that the set X in condition $\mathcal{A}_{38}$ can be enlarged to the set $X \cup X_0$, where $X_0 = [0, \infty) \setminus R[\mathbf{x}_0(\cdot)]$ is the set of continuity points for the function $\mathbf{x}_0(t),\ t \geq 0$. Analogously, conditions $\mathcal{A}_{39}$ and $\mathcal{J}_{14}$ imply that $y_n(\cdot), t \geq 0 \xrightarrow{\mathbf{J}} y_0(t), t \geq 0$ as $\varepsilon \to 0$. Thus, the set Y in the condition $\mathcal{A}_{39}$ can be extended to the set $Y \cup Y_0$, where $Y_0 = [0, \infty) \setminus R[y_0(\cdot)]$ is the set of continuity points for the function $y_0(t),\ t \geq 0$. Both sets X_0 and Y_0 are dense in $[0, \infty)$, moreover, they coincide with $[0, \infty)$ except for at most countable sets.

It is sufficient to show that the compactness relation in Lemma 3.5.1 holds for any $T \in Y_0$. In this case, by $\mathcal{A}_{39}$ and $\mathcal{J}_{14}$, the functions $y_n(t), t \in [0, T] \xrightarrow{\mathbf{J}} y_0(t), t \in [0, T]$ as $n \to \infty$. This implies existence of a sequence of continuous one-to-one mappings $\lambda_n(t)$, $n \geq 1$ of the interval $[0, T]$ into itself such that

$$\lim_{n\to\infty} \sup_{t\in[0,T]} (|y_n(t) - y_0(\lambda_n(t))| + |\lambda_n(t) - t|) = 0. \tag{3.5.1}$$

Since $y_n(T) \to y_0(T)$ as $n \to \infty$, there exists T' such that $\max_n y_n(T) \leq T'$. Obviously, T' can be taken from the set X_0. In this case, by $\mathcal{G}_2$, the functions $\mathbf{x}_n(t), t \in [0, T'] \xrightarrow{\mathbf{J}} \mathbf{x}_0(t), t \in [0, T']$ as $n \to \infty$. This implies existence of a sequence of continuous one-to-one mappings $\lambda'_n(t)$, $n \geq 1$ of the interval $[0, T']$ into itself such that

$$\lim_{n\to\infty} \sup_{t\in[0,T']} (|\mathbf{x}_n(t) - \mathbf{x}_0(\lambda'_n(t))| + |\lambda'_n(t) - t|) = 0. \tag{3.5.2}$$

By using estimate (1.4.8), given in Lemma 1.4.9, we get

$$\begin{aligned} &\Delta_J(\mathbf{x}_n(y_n(\cdot)), c, T) \\ &\quad \leq \Delta_J(\mathbf{x}_0(\lambda'_n(y_n(\cdot))), c, T) + \sup_{t\in[0,T]} |\mathbf{x}_n(y_n(t)) - \mathbf{x}_0(\lambda'_n(y_n(t)))| \\ &\quad \leq \Delta_J(\mathbf{x}_0(\lambda'_n(y_n(\cdot))), c, T) + \sup_{t\in[0,T']} |\mathbf{x}_n(t) - \mathbf{x}_0(\lambda'_n(t))|. \end{aligned} \tag{3.5.3}$$

It follows from estimate (3.5.3) and (3.5.2) that, to prove the lemma, it will be sufficient to show that

$$\begin{aligned} &\lim_{c\to 0} \overline{\lim_{n\to\infty}} \Delta_J(\mathbf{x}_0(\lambda'_n(y_n(\cdot))), c, T) \\ &\qquad = \lim_{c\to 0} \overline{\lim_{n\to\infty}} \Delta_J(\mathbf{x}_0(\lambda'_n(y_n(\lambda_n{}^{-1}(\lambda_n(\cdot))))), c, T) = 0. \end{aligned} \tag{3.5.4}$$

Relation (3.5.4) will be proved if we apply Lemma 3.4.1 to the composition of càdlàg functions $\mathbf{x}_0(\lambda'_n(y_n(\lambda_n{}^{-1}(t))))$, $t \in [0, T]$ and $\lambda_n(t)$, $t \in [0, T]$. In order to show that Lemma 3.4.1 can be used, we must show that conditions $\mathcal{U}_5$, $\mathcal{K}_4^{(0)}$, and $\mathcal{J}_4$ are satisfied in this case.

Conditions $\mathfrak{U}_5$ and $\mathfrak{K}_4^{(0)}$ obviously hold for the functions $\lambda_n(t)$, $t \in [0, T]$. Condition $\mathfrak{J}_4$ takes in this case the following form:

$$\begin{aligned}\lim_{c\to 0}\overline{\lim_{n\to\infty}}\,\Delta_J(\mathbf{x}_0(\lambda_n'(y_n(\lambda_n{}^{-1}(\cdot)))), c, T)\\ = \lim_{c\to 0}\overline{\lim_{n\to\infty}}\,\Delta_J(\mathbf{x}_0(y_0(\cdot) + \beta_n(\cdot)), c, T) = 0,\end{aligned} \tag{3.5.5}$$

where

$$\beta_n(t) = \lambda_n'(y_n(\lambda_n{}^{-1}(t))) - y_0(t).$$

By (3.5.1) and (3.5.2), we have

$$\begin{aligned}\beta_n = \sup_{t\in[0,T]} |\beta_n(t)| &\le \sup_{t\in[0,T]} |\lambda_n'(y_n(\lambda_n{}^{-1}(t))) - y_n(\lambda_n{}^{-1}(t))|\\ &+ \sup_{t\in[0,T]} |y_n(\lambda_n{}^{-1}(t)) - y_0(t)| \le \sup_{t\in[0,T']} |\lambda_n'(t) - t|\\ &+ \sup_{t\in[0,T]} |y_n(t) - y_0(\lambda_n(t))| \to 0 \text{ as } n \to \infty.\end{aligned} \tag{3.5.6}$$

Take an arbitrary $\sigma > 0$ and choose h (see Lemma 1.4.1) such that

$$\Delta_J(\mathbf{x}_0(\cdot), h, T') \le \sigma. \tag{3.5.7}$$

Denote by $u_k = u_k^{(h)}$, $k = 1, \ldots r_h$, the points of the interval $(0, T)$ at which absolute values of the jumps of the function $y_0(t)$ are not less than $h/2$ (there is a finite number of such points).

Since, by condition $\mathfrak{G}_2$, the function $\mathbf{x}_0(t)$ is continuous at the points $y_0(u_k \pm 0)$, $k = 1, \ldots r_h$, there is $h' > 0$ such that

$$\max_{1\le k\le r_h} \sup_{|s'|,|s''|\le h'} |\mathbf{x}_0(y_0(u_k \pm 0) + s') - \mathbf{x}_0(y_0(u_k \pm 0) + s'')| \le \sigma. \tag{3.5.8}$$

By Lemma 1.4.2, there exists $c > 0$ such that, if the points t', t'' belong to one of the intervals $[u_0, u_1)$, $[u_1, u_2), \ldots, [u_{r_h-1}, u_{r_h})$, $[u_{r_h}, u_{r_h+1}]$ (here $u_0 = 0$, $u_{r_h+1} = T$) and $|t' - t''| \le c$, then

$$|y_0(t') - y_0(t'')| \le h/2. \tag{3.5.9}$$

Here we can assume that c is chosen such that

$$\max_{1\le k\le r_h} \sup_{0<t\le 2c} |y_0(u_k - 0) - y_0(u_k - t)| \le h'/2, \tag{3.5.10}$$

and

$$\max_{1\le k\le r_h} \sup_{0\le t\le 2c} |y_0(u_k) - y_0(u_k + t)| \le h'/2, \tag{3.5.11}$$

and, moreover,

$$\sup_{0<\le 2c} |y_0(T) - y_0(T - t)| \le h/4, \tag{3.5.12}$$

and

$$\sup_{0 \le t \le 2c} |y_0(0) - y_0(t)| \le h/4. \tag{3.5.13}$$

Let n_0 be such that, for $n \ge n_0$,

$$\beta_n = \sup_{t \in [0,T]} |\beta_n(t)| \le \min(h/4, h'/4). \tag{3.5.14}$$

To prove the lemma it is sufficient to show that, if $n \ge n_0$, for any three points t', t'', $t''' \in [0, T]$, $t - c \le t' < t < t'' \le t + c$, we have

$$R_n[t', t, t''] = \min(|\mathbf{x}_0(s_n(t)) - \mathbf{x}_0(s_n(t'))|, |\mathbf{x}_0(s_n(t)) - \mathbf{x}_0(s_n(t''))|) \le \sigma, \tag{3.5.15}$$

where

$$s_n(t) = y_0(t) + \beta_n(t) = \lambda'_n(y_n(\lambda_n{}^{-1}(t))).$$

The following three cases are possible.

(a) $t' = 0$ or $t'' = T$. Consider the case where $t' = 0$ (the case $t'' = T$ is treated similarly). By using (3.5.12) and (3.5.14), we get

$$|s_n(0) - s_n(t)| \le 2\beta_n + \sup_{0 \le s \le c} |y_0(0) - y_0(s)| \le h,$$

and

$$|s_n(t) - s_n(t'')| \le 2\beta_n + 2 \sup_{0 \le s \le 2c} |y_0(0) - y_0(s)| \le h,$$

whence, by (3.5.7), we obtain

$$R_n[t', t, t''] \le \Delta_J(\mathbf{x}_0(t), h, T') \le \sigma.$$

(b) $t' \le u_k \le t < t''$ or $t' < t < u_k \le t''$ for some $k = 1, \dots, r_n$. In the first case (the second one is similar), we get from (3.5.10), (3.5.11), (3.5.13), and (3.5.12) that

$$|s_n(t) - y_0(u_k)| \le \sup_{0 \le s \le c} |y_0(u_k + s) - y_0(u_k)| + \beta_n \le h',$$

and

$$|s_n(t'') - y_0(u_k)| \le \sup_{0 \le s \le 2c} |y_0(u_k + s) - y_0(u_k)| + \beta_n \le h',$$

whence, by (3.5.8),

$$R_n[t', t, t''] \le |\mathbf{x}_0(s_n(t)) - \mathbf{x}_0(s_n(t''))| \le \sigma.$$

(c) $t', t, t'' \in (u_k, u_{k+1})$ for some $k = 0, \dots, r_h$. Then, by (3.5.9) and (3.5.14), we get

$$|s_n(t') - s_n(t)| \le |y_0(t') - y_0(t)| + 2\beta_n \le h,$$

and

$$|s_n(t'') - s_n(t)| \leq |y_0(t'') - y_0(t)| + 2\beta_n \leq h,$$

whence, from (3.5.7) again,

$$R_n[t', t, t''] \leq \Delta_J(\mathbf{x}_0(\cdot), h, T') \leq \sigma.$$

This completes the proof. □

3.5.2. J-convergence of semi-vector compositions of non-random càdlàg functions. Note first of all that, as follows from the examples given in Section 3.1, conditions $\mathcal{A}_{38}$, $\mathfrak{J}_{13}$, $\mathcal{A}_{39}$, $\mathfrak{J}_{14}$, and $\mathfrak{G}_2$ do not guarantee $\mathbf{J}$-convergence of the compositions $\mathbf{x}_n(y_n(t))$, $t \geq 0$. These conditions do provide $\mathbf{J}$-compactness of these functions but they do not guarantee pointwise convergence of these functions on some set dense in $[0, \infty)$ and containing the point 0. Some additional conditions should be added.

Let us introduce the following conditions:

$\mathfrak{C}_9$: There exists a set W such that (a) $y_0(t) \notin R[\mathbf{x}_0(\cdot)]$ for $t \in W$, (b) W is a subset of $[0, \infty)$ that is dense in this interval and contains the point 0;

$\mathcal{E}_7$: There do not exist points $0 \leq t' < t'' < \infty$ such that $y_0(t') = y_0(t'') \in R[\mathbf{x}_0(\cdot)]$;

and

$\mathfrak{C}_{10}^{(w)}$: $y_0(w) \notin R[\mathbf{x}_0(\cdot)]$.

Conditions $\mathfrak{C}_9$, $\mathcal{E}_7$, and $\mathfrak{C}_{10}^{(w)}$ coincide, respectively, with conditions $\mathfrak{C}_7$, $\mathcal{E}_6$, and $\mathfrak{C}_8^{(w)}$ in the case of non-random functions $\mathbf{x}_0(t)$, $t \geq 0$ and $y_0(t)$, $t \geq 0$, which replace in this case, respectively, the stochastic processes $\xi_0(t)$, $t \geq 0$ and $\nu_0(t)$, $t \geq 0$.

As follows from Lemma 2.7.2, conditions $\mathcal{E}_7$ and $\mathfrak{C}_{10}^{(0)}$ are necessary and sufficient for existence of a set W such that condition $\mathfrak{C}_9$ holds with this set.

Let W_0 denote the set of all points for which condition $\mathfrak{C}_{10}^{(w)}$ holds. Obviously, $W \subseteq W_0$ for any set W that can appear in condition $\mathfrak{C}_9$. Hence, under condition $\mathfrak{C}_9$ or conditions $\mathcal{E}_7$ and $\mathfrak{C}_{10}^{(0)}$, the set W_0 is the interval $[0, \infty)$ except for at most a countable set, and $0 \in W_0$.

Denote by Y_0 the set of points of continuity for the function $y_0(t)$, $t \geq 0$. This set and therefore, the set $Y \cup Y_0$ is $[0, \infty)$ except for at most a countable set. Also $0 \in Y \cup Y_0$.

Denote $Z_0 = (Y \cup Y_0) \cap W_0$. If condition $\mathfrak{C}_9$ or conditions $\mathcal{E}_7$ and $\mathfrak{C}_{10}^{(0)}$ hold, then the set Z_0 is $[0, \infty)$ except for at most a countable set. Also, $0 \in Z_0$.

Lemma 3.5.2. *Let conditions $\mathcal{A}_{38}$, $\mathfrak{J}_{13}$, $\mathcal{A}_{39}$, $\mathcal{E}_7$, and $\mathfrak{C}_{10}^{(0)}$ hold. Then*

$$\mathbf{x}_n(y_n(t)) \to \mathbf{x}_0(y_0(t)) \text{ as } n \to \infty, \ t \in Z_0.$$

Proof of Lemma 3.5.2. The proof can be obtained by applying Theorem 2.7.5 to the vector compositions $\mathbf{z}_n(t) = (x_{n1}(y_n(t)), \ldots, x_{nm}(y_n(t))), t \geq 0$ of the vector càdlàg functions $\mathbf{x}_n(t) = (x_{n1}(t), \ldots, x_{nm}(t))$, $t \geq 0$, with real-valued components, and the vector càdlàg functions $\mathbf{y}_n(t) = (y_n(t), \ldots y_n(t)), t \geq 0$, with identical components that are non-negative and nondecreasing functions. Here n^{-1} can play the role of the parameter ε.

Conditions $\mathcal{A}_{38}$ and $\mathcal{A}_{39}$ obviously imply that condition $\mathcal{A}_{22}^{\mathbf{V}}$ holds with the set $V = Y$. Condition $\mathfrak{J}_{13}$ implies that the condition of **J**-compactness, $\mathfrak{J}_4$, holds. Finally, conditions $\mathcal{E}_7$ and $\mathfrak{C}_{10}^{(0)}$ imply that the continuity condition $\mathcal{E}_4$ holds. In this case, the set of weak convergence that enters Theorem 2.7.5, S_0, coincides with set Z_0. □

Now, general conditions for **J**-convergence of compositions of the càdlàg functions $\mathbf{x}_n(t)$, $t \geq 0$ and $y_n(t)$, $t \geq 0$ can be obtained by combining the conditions of Lemmas 3.5.1 and 3.5.2 and by applying Theorem 1.4.9 to these functions. These conditions were given in Silvestrov (1974).

Lemma 3.5.3. *Let conditions $\mathcal{A}_{38}$, $\mathfrak{J}_{13}$, $\mathcal{A}_{39}$, $\mathfrak{J}_{14}$, $\mathfrak{G}_2$, $\mathcal{E}_7$, and $\mathfrak{C}_{10}^{(0)}$ hold. Then*

$$\mathbf{x}_n(y_n(t)), t \geq 0 \xrightarrow{\mathbf{J}} \mathbf{x}_0(y_0(t)), t \geq 0 \text{ as } n \to \infty.$$

3.5.3. J-continuity properties of the composition mapping. Lemma 3.5.3 gives the most general conditions for **J**-convergence of compositions of càdlàg functions for the case where both limiting functions can be discontinuous. These conditions require **J**-convergence of components $y_n(t), t \geq 0$ and $\mathbf{x}_n(t), t \geq 0$, but they do not require **J**-convergence of vector càdlàg functions $(y_n(t), \mathbf{x}_n(t)), t \geq 0$. The corresponding example is given in Subsection 3.1.3. In this sense, Lemma 3.5.3 extends, with respect to the composition mapping, setting of the definition of a **J**-continuous mapping.

However, there are particular cases, where the composition mapping is **J**-continuous.

The first case is where **(a)** both limiting functions $y_0(t), t \geq 0$ and $\mathbf{x}_0(t), t \geq 0$ are continuous. This case was considered by Billingsley (1968). Here, conditions of Lemma 3.5.3 are reduced to the conditions of Theorem 3.2.1 applied to semi-vector compositions of non-random càdlàg functions. Conditions $\mathfrak{G}_2$, $\mathcal{E}_7$, and $\mathfrak{C}_{10}^{(0)}$ automatically hold. Condition $\mathfrak{J}_{13}$ is reduced to condition $\mathfrak{U}_4$. Condition $\mathfrak{J}_{14}$ also holds, due to Lemma 3.2.1.

The second case is where **(b)** the limiting external function $\mathbf{x}_0(t), t \geq 0$ is continuous. This case was considered by Whitt (1973, 1980) and Silvestrov (1974). Here, conditions of Lemma 3.5.3 are reduced to the conditions of Theorem 3.3.2 applied to semi-vector compositions of non-random càdlàg functions. Again, conditions $\mathfrak{G}_2$, $\mathcal{E}_7$, and $\mathfrak{C}_{10}^{(0)}$ automatically hold, and condition $\mathfrak{J}_{13}$ reduces to condition $\mathfrak{U}_4$.

The third case is where **(c)** the limiting internal stopping function $y_0(t), t \geq 0$ is continuous. This case was considered by Silvestrov (1972b, 1972e, 1973a, 1974). Here, conditions of Lemma 3.5.3 are reduced to the conditions of Theorem 3.4.2 applied to semi-vector compositions of non-random càdlàg functions. Condition $\mathfrak{G}_2$ automatically holds. Condition $\mathcal{J}_{14}$ also holds, due to Lemma 3.2.1. Conditions $\mathcal{E}_7$ and $\mathfrak{C}^{(0)}_{10}$ remain. As show the example given in Subsection 3.1.2, if condition $\mathcal{E}_7$ does not hold, then compositions may not converge pointwise on some interval. In the sequel, they do not **J**-converge. Condition $\mathcal{E}_7$ holds, for example, if the limiting internal stopping function $y_0(t), t \geq 0$ is not only continuous but also strictly monotone. This case was independently considered by Whitt (1973, 1980).

3.5.4. The finite interval $[0, T]$. The result of Lemmas 3.5.1 and 3.5.3 can be easily reduced to the case of a finite interval $[0, T]$.

Conditions $\mathcal{A}_{38}$ and $\mathcal{J}_{13}$ do not need any change, but condition $\mathcal{A}_{39}$ and $\mathcal{J}_{14}$ have to be reduced to the following form:

$\mathcal{A}_{40}$: $y_n(t) \to y_0(t)$ as $n \to \infty$, $t \in Y$, where Y is a subset of $[0, T]$ that is dense in this interval and contains the points 0 and T;

and

$\mathcal{J}_{15}$: $\lim_{c\to 0} \overline{\lim}_{n\to\infty} \Delta_J(y_n(\cdot), c, T) = 0$.

Let $R_T[y(\cdot)] = R[y(\cdot)] \cap [0, T]$ denote the set of points of discontinuity for the càdlàg function $y(t)$, $t \geq 0$ in the interval $[0, T]$. Condition $\mathcal{G}_2$ must be transformed to the following form:

$\mathcal{G}_3$: $y_0(t \pm 0) \notin R[\mathbf{x}_0(\cdot)]$ for $t \in R_T[y_0(\cdot)]$.

Let us first formulate a statement that is an analogue of Lemma 3.5.1.

Lemma 3.5.4. *Let conditions $\mathcal{A}_{38}$, $\mathcal{J}_{13}$, $\mathcal{A}_{40}$, $\mathcal{J}_{15}$, and $\mathcal{G}_3$ hold. Then*

$$\lim_{c\to 0} \overline{\lim_{n\to\infty}} \Delta_J(\mathbf{x}_n(y_n(\cdot)), c, T) = 0.$$

Proof of Lemma 3.5.4. The case of a finite interval $[0, T]$ can be reduced to the case of the semi-infinite interval $[0, \infty)$ by applying Lemma 3.5.1 to the functions $\mathbf{x}_n(t)$, $t \geq 0$ and $y_n(t) = y_n(t \wedge T)$, $t \geq 0$. Conditions $\mathcal{A}_{38}$ and $\mathcal{J}_{13}$ are not changed. It is obvious that conditions $\mathcal{A}_{40}$, $\mathcal{J}_{15}$, and $\mathcal{G}_3$ imply that conditions $\mathcal{A}_{39}$, $\mathcal{J}_{14}$, and $\mathcal{G}_2$ hold for the functions $\mathbf{x}_n(t)$, $t \geq 0$ and $y_n(t)$, $t \geq 0$. By applying Lemma 3.5.1, we get a relation of **J**-compactness for the functions $\mathbf{x}_n(y_n(t))$, $t \geq 0$, for all intervals $[0, T']$. For $T' \geq T$, this relation coincides with the relation given in Lemma 3.5.4. □

An analogous reduction to the case of a finite interval can be made in Lemma 3.5.2.

In this case, we should add to $\mathcal{A}_{40}$ the assumption of convergence of the values $y_n(T - 0)$, that is, to replace $\mathcal{A}_{40}$ by the following condition:

$\mathcal{A}_{41}$: (a) $y_n(t) \to y_0(t)$ as $n \to \infty$ for $t \in Y$, where Y is a subset of $[0, T]$ that is dense in this interval and contains the points 0 and T;

(b) $y_n(T-0) \to y_0(T-0)$ as $n \to \infty$.

Also conditions $\mathbf{\mathcal{C}_9}$ and $\mathbf{\mathcal{E}_7}$ should be modified in the following way:

$\mathbf{\mathcal{C}_{11}}$: There exists a set W such that (a) $y_0(t) \notin R[\mathbf{x}_0(\cdot)]$ for $t \in W$, (b) W is a subset of $[0, T]$ that is dense in this interval and contains the points 0 and T;

and

$\mathbf{\mathcal{E}_8}$: There do not exist points $0 \le t' < t'' \le T$ such that $y_0(t') = y_0(t'') \in R[\mathbf{x}_0(\cdot)]$.

Let us modify condition $\mathbf{\mathcal{C}_{10}^{(w)}}$ in the following way:

$\mathbf{\mathcal{C}_{10}^{(w\pm)}}$: $y_0(w \pm 0) \notin R[\mathbf{x}_0(\cdot)]$.

Note that $\mathbf{\mathcal{C}_{10}^{(w+)}}$ coincides with $\mathbf{\mathcal{C}_{10}^{(w)}}$.

Denote by $W_0(T) = W_0 \cap [0, T]$ the set of all points $w \in [0, T]$ that satisfy condition $\mathbf{\mathcal{C}_{10}^{(w)}}$.

It follows from Lemma 2.7.2 that conditions $\mathbf{\mathcal{E}_8}$, $\mathbf{\mathcal{C}_{10}^{(0)}}$, and $\mathbf{\mathcal{C}_{10}^{(T)}}$ are necessary and sufficient for condition $\mathbf{\mathcal{C}_{11}}$ to hold. Actually, it follows from Lemma 2.7.2 that, under condition $\mathbf{\mathcal{C}_{11}}$ or conditions $\mathbf{\mathcal{E}_8}$, $\mathbf{\mathcal{C}_{10}^{(0)}}$, and $\mathbf{\mathcal{C}_{10}^{(T)}}$, the set W, which appears in $\mathbf{\mathcal{C}_{11}}$, is the interval $[0, T]$ except for at most a countable set. Since $W \subseteq W_0(T)$, the set $W_0(T)$ is also the interval $[0, T]$ except for at most a countable set.

Let $Y_0(T) = Y_0 \cap [0, T]$. This set and the set $Y \cup Y_0(T)$ are $[0, T]$ except for at most countable sets. Finally, denote $Z_0 = (Y \cup Y_0(T)) \cap W_0(T)$. If condition $\mathbf{\mathcal{C}_{11}}$ or conditions $\mathbf{\mathcal{E}_8}$, $\mathbf{\mathcal{C}_{10}^{(0)}}$, and $\mathbf{\mathcal{C}_{10}^{(T)}}$ hold, then the set Z_0 coincides with $[0, T]$ except for at most a countable set. Also, $0, T \in Z_0$.

Lemma 3.5.5. *Let conditions* $\mathbf{\mathcal{A}_{38}}$, $\mathbf{\mathcal{J}_{13}}$, $\mathbf{\mathcal{A}_{41}}$, $\mathbf{\mathcal{E}_8}$, $\mathbf{\mathcal{C}_{10}^{(0)}}$, *and* $\mathbf{\mathcal{C}_{10}^{(T)}}$ *hold. Then*

$$\mathbf{x}_n(y_n(t)) \to \mathbf{x}_0(y_0(t)) \text{ as } n \to \infty, t \in Z_0.$$

If, additionally, condition $\mathbf{\mathcal{C}_{10}^{(T-)}}$ *holds, then also* $\mathbf{x}_n(y_n(T-0)) \to \mathbf{x}_0(y_0(T-0))$ *as* $n \to \infty$.

Proof of Lemma 3.5.5. To prove the first statement, i.e., to prove that $\mathbf{x}_n(y_n(t))$ converges to $\mathbf{x}_0(y_0(t))$ in points $t \in Z_0$, it is enough to apply Lemma 3.5.2 to the functions $\mathbf{x}_n(t)$, $t \ge 0$, and $y_n(t \wedge T)$, $t \ge 0$. Condition $\mathbf{\mathcal{A}_{41}}$ implies $\mathbf{\mathcal{A}_{40}}$. Conditions $\mathbf{\mathcal{E}_8}$ and $\mathbf{\mathcal{C}_{10}^{(T)}}$ imply condition $\mathbf{\mathcal{E}_7}$.

It should be noted that the proof of Lemma 3.5.2 given above is based on applying Theorem 2.7.5. This theorem, in its turn, is based on Theorem 2.3.3.

To prove the second statement of the lemma, i.e., that $\mathbf{x}_n(y_n(T-0))$ converges to $\mathbf{x}_0(y_0(T-0))$ as $n \to \infty$, we can apply Theorem 2.3.3 directly to the non-random functions $\mathbf{x}_n(t)$, $t \ge 0$ and the vector stopping points $\mathbf{x}_n(T-0) = (y_0(T-0), \dots y_0(T-0))$ with identical m components. Conditions $\mathbf{\mathcal{A}_{38}}$ and $\mathbf{\mathcal{A}_{41}}$ (b) imply condition $\mathbf{\mathcal{A}_{20}}$. Condition $\mathbf{\mathcal{J}_{13}}$ implies condition $\mathbf{\mathcal{J}_4}$. Condition $\mathbf{\mathcal{C}_{10}^{T-}}$ implies condition $\mathbf{\mathcal{C}_4}$. □

Lemma 3.5.6. *Let conditions* $\mathcal{A}_{38}$, $\mathcal{J}_{13}$, $\mathcal{A}_{41}$, $\mathcal{J}_{15}$, $\mathcal{G}_3$, $\mathcal{E}_8$, $\mathcal{C}_{10}^{(0)}$, *and* $\mathcal{C}_{10}^{(T)}$ *hold. Then*

$$\mathbf{x}_n(y_n(t)), t \in [0, T] \xrightarrow{\mathbf{J}} \mathbf{x}_0(y_0(t)), t \in [0, T] \text{ as } n \to \infty.$$

Proof of Lemma 3.5.6. The proof can be obtained by combining the conditions of Lemmas 3.5.4 and 3.5.5, and applying Theorem 1.4.4 to the functions $\mathbf{x}_n(y_n(t))$, $t \in [0, T]$. What remains to be explained is why condition $\mathcal{C}_{10}^{(T-)}$ is omitted in Lemma 3.5.6. As a matter of fact, conditions $\mathcal{G}_3$ and $\mathcal{C}_{10}^{(T)}$ imply this condition. Indeed, if $y_0(T - 0) = y_0(T)$, then condition $\mathcal{C}_{10}^{(T-)}$ coincides with $\mathcal{C}_{10}^{(T)}$. If $y_0(T - 0) \neq y_0(T)$, then condition $\mathcal{G}_3$ implies $\mathcal{C}_{10}^{(T-)}$. □

Remark 3.5.1. If the point T is a point of continuity for the limiting function $\mathbf{x}_0(y_0(t))$, condition $\mathcal{A}_{41}$ can be replaced in Lemma 3.5.6 by condition $\mathcal{A}_{40}$, i.e., condition $\mathcal{A}_{41}$ (b) $y_n(T - 0) \to y_0(T - 0)$ as $n \to \infty$ can be omitted.

3.6 Semi-vector compositions of càdlàg processes

In this section, we formulate conditions for **J**-convergence of general semi-vector compositions of càdlàg processes. We consider a model where both limiting external and internal stopping processes can be discontinuous.

3.6.1. J-compactness of semi-vector compositions of càdlàg processes. Let, for every $\varepsilon \geq 0$, $\xi_\varepsilon(t) = (\xi_{\varepsilon i}(t), i = 1, \ldots, m)$, $t \geq 0$ be an m-dimensional càdlàg process with real-valued components, and $\nu_\varepsilon(t)$, $t \geq 0$ be a non-negative and non-decreasing càdlàg process. Consider their *semi-vector composition* $\zeta_\varepsilon(t) = \xi_\varepsilon(\nu_\varepsilon(t))$, $t \geq 0$, which, in this case, is also an m-dimensional càdlàg process.

A basis for further considerations is the condition of joint weak convergence $\mathcal{A}_{36}$, and the conditions of **J**-compactness $\mathcal{J}_4$ and $\mathcal{J}_{11}$.

Conditions $\mathcal{A}_{36}$ and $\mathcal{J}_4$ imply **J**-convergence of the processes $\xi_\varepsilon(t)$, $t \geq 0$, while conditions $\mathcal{A}_{36}$ and $\mathcal{J}_{11}$ imply **J**-convergence of the processes $\nu_\varepsilon(t)$, $t \geq 0$. However, the examples given in Section 3.1 show that, together, conditions $\mathcal{A}_{36}$, $\mathcal{J}_4$, and $\mathcal{J}_{11}$ do not imply either **J**-convergence or **J**-compactness for the vector processes $(\nu_\varepsilon(t), \xi_\varepsilon(t))$, $t \geq 0$ and their compositions $\zeta_\varepsilon(t)$, $t \geq 0$.

We first give general conditions that would yield **J**-compactness of compositions. These conditions can be combined with various conditions that imply weak convergence of compositions, in order to get conditions for **J**-convergence. For this reason, we formulate **J**-compactness conditions separately.

Let us formulate a condition that is, actually, a stochastic analogue of condition $\mathcal{G}_2$,

$\mathcal{G}_4$: $\mathsf{P}\{\nu_0(t \pm 0) \notin R[\xi_0(\cdot)] \text{ for } t \in R[\nu_0(\cdot)]\} = 1$.

The first main result is the following theorem from Silvestrov (1974).

Theorem 3.6.1. *Let conditions $\mathcal{A}_{36}$, $\mathcal{J}_4$, $\mathcal{J}_{11}$, and $\mathcal{G}_4$ hold. Then*

$$\lim_{c\to 0}\overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\Delta_J(\zeta_\varepsilon(\cdot), c, T) > \delta\} = 0, \ \delta, T > 0.$$

Proof of Theorem 3.6.1. We are going to reduce the proof to the case of non-random càdlàg functions using Skorokhod's representation Theorem 1.6.16 and then Lemma 3.5.1.

Unfortunately, Theorem 1.6.16 can not be directly applied either to the vector processes $(\nu_\varepsilon(t), \xi_\varepsilon(t))$, $t \geq 0$, or to their compositions $\zeta_\varepsilon(t)$, $t \geq 0$. Indeed, as it was remarked above, conditions $\mathcal{A}_{36}$, $\mathcal{J}_4$, $\mathcal{J}_{11}$, and $\mathcal{G}_4$ do not necessarily imply **J**-convergence of these processes. So, this approach must be carried out in a more delicate modified way. The result can be achieved by applying first Theorem 1.6.14 to the vector processes $(\nu_\varepsilon(t), \xi_\varepsilon(t))$, $t \geq 0$, and then Theorem 1.6.16 to the processes $\xi_\varepsilon(t)$, $t \geq 0$ and $\nu_\varepsilon(t)$, $t \geq 0$.

Note, first of all, that conditions $\mathcal{A}_{36}$, $\mathcal{J}_4$, and $\mathcal{J}_{11}$ allow to enlarge the sets of weak convergence, U and V, in condition $\mathcal{A}_{36}$ to the sets $U' = U \cup U_0$ and $V' = V \cup V_0$. Here U_0 and V_0 are sets of stochastic continuity of the processes $\xi_0(t)$, $t \geq 0$ and $\nu_0(t)$, $t \geq 0$, respectively. Both sets U' and V' equal $[0, \infty)$ except for at most countable sets. Also, both sets, U' and V', contain the point 0. This implies that the set $S' = U' \cap V'$ is also $[0, \infty)$ except for at most a countable set, and $0 \in S'$. Also, the following relation holds:

$$(\nu_\varepsilon(t), \xi_\varepsilon(t)), t \in S' \Rightarrow (\nu_0(t), \xi_0(t)), t \in S' \text{ as } \varepsilon \to 0. \tag{3.6.1}$$

Choose a countable set $\tilde{S} \subseteq S'$ such that $\tilde{S}$ is dense in $[0, \infty)$ and contains the point 0. Relation (3.6.1) permits to apply Theorem 1.6.14 and construct some probability space $(\Omega, \mathfrak{F}, \mathsf{P})$ and an a.s. càdlàg processes $(\tilde{\nu}_\varepsilon(t), \tilde{\xi}_\varepsilon(t))$, $t \geq 0$, defined on this space for every $\varepsilon \geq 0$ and such that

$$(\tilde{\nu}_\varepsilon(t), \tilde{\xi}_\varepsilon(t)), t \geq 0 \stackrel{\text{d}}{=} (\nu_\varepsilon(t), \xi_\varepsilon(t)), t \geq 0, \tag{3.6.2}$$

and, for an arbitrary sequence $0 \leq \varepsilon_n \to 0$ as $n \to \infty$,

$$(\tilde{\nu}_{\varepsilon_n}(t), \tilde{\xi}_{\varepsilon_n}(t)) \xrightarrow{\text{a.s.}} (\tilde{\nu}_0(t), \tilde{\xi}_0(t)) \text{ as } n \to \infty, \ t \in \tilde{S}. \tag{3.6.3}$$

Let ε_n be an arbitrary sequence such $0 \leq \varepsilon_n \to 0$ as $n \to \infty$.

Relations (3.6.1), (3.6.2), (3.6.3) and condition $\mathcal{J}_4$ allow to apply Theorem 1.6.16 to the processes $\xi_\varepsilon(t)$, $t \geq 0$, and $\tilde{\xi}_\varepsilon(t)$, $t \geq 0$.

Therefore, there exists a subsequence $\varepsilon'_k = \varepsilon_{n_k} \to 0$ as $k \to \infty$ of the sequence ε_n such that $\mathsf{P}(A) = 1$, where $A \in \mathfrak{F}$ is the set of elementary events ω such that

$$\tilde{\xi}_{\varepsilon'_k}(t, \omega), t \geq 0 \xrightarrow{\mathbf{J}} \tilde{\xi}_0(t, \omega), t \geq 0 \text{ as } k \to \infty. \tag{3.6.4}$$

Due to relations (3.6.1), (3.6.2), (3.6.3) and condition $\mathcal{J}_4$, we can apply Theorem 1.6.16 to the processes $\nu_\varepsilon(t)$, $t \geq 0$ and $\tilde{\nu}_\varepsilon(t)$, $t \geq 0$.

Therefore, there exists a subsequence $\varepsilon''_r = \varepsilon'_{k_r} \to 0$ as $r \to \infty$ of the sequence ε'_k such that $\mathsf{P}(B) = 1$, where $B \in \mathfrak{F}$ is the set of elementary events ω such that

$$\tilde{\nu}_{\varepsilon''_r}(t, \omega), t \geq 0 \xrightarrow{\mathbf{J}} \tilde{\nu}_0(t, \omega), t \geq 0 \text{ as } r \to \infty. \tag{3.6.5}$$

Using relation (3.6.2) we see that condition $\mathcal{G}_4$ also implies that $\mathsf{P}(C) = 1$, where $C \in \mathfrak{F}$ is the set of elementary events ω for which

$$\tilde{\nu}_0(t \pm 0, \omega) \notin R[\tilde{\xi}_0(\cdot, \omega)] \text{ for } t \in R[\tilde{\nu}_0(\cdot, \omega)]. \tag{3.6.6}$$

Obviously, $\mathsf{P}(A \cap B \cap C) = 1$ and, for $\omega \in A \cap B \cap C$, conditions $\mathcal{A}_{38}$, $\mathcal{J}_{13}$, $\mathcal{A}_{39}$, $\mathcal{J}_{14}$, and $\mathcal{G}_2$ in Lemma 3.5.1 hold for sequences of the càdlàg functions $\tilde{\xi}_{\varepsilon''_r}(t, \omega)$, $t \geq 0$ and $\tilde{\nu}_{\varepsilon''_r}(t, \omega)$, $t \geq 0$. By applying Lemma 3.5.1 to their compositions, $\tilde{\zeta}_{\varepsilon''_r}(t) = \tilde{\xi}_{\varepsilon''_r}(\tilde{\nu}_{\varepsilon''_r}(t))$, $t \geq 0$, we get that, for $\omega \in A \cap B \cap C$,

$$\lim_{c \to 0} \overline{\lim_{r \to \infty}} \Delta_J(\tilde{\zeta}_{\varepsilon''_r}(\cdot, \omega), c, T) = 0, \ T > 0. \tag{3.6.7}$$

Relation (3.6.7) implies that

$$\lim_{c \to 0} \overline{\lim_{r \to \infty}} \mathsf{P}\{\Delta_J(\tilde{\zeta}_{\varepsilon''_r}(\cdot), c, T) \geq \delta\} = 0, \ \delta, T > 0. \tag{3.6.8}$$

In the case of the usual limits, this implication is obvious, since a.s. convergence of random variables implies their convergence in probability. In the case of iterated limits, the same implication takes place. The relation (3.6.7) means that the random variables $\Delta_c = \overline{\lim}_{r \to \infty} \Delta_J(\tilde{\zeta}_{\varepsilon''_r}(\cdot), c, T) \xrightarrow{\text{a.s.}} 0$ as $c \to 0$. Therefore, these random variables also converge in probability, and so **(a)** $\lim_{c \to 0} \mathsf{P}\{\Delta_c \geq \delta\} = 0$ for all $\delta > 0$. Set $\Delta_{c,r} = \max_{l \geq r} \Delta_J(\tilde{\zeta}_{\varepsilon''_l}(\cdot), c, T)$. The sequence of random variables $\Delta_{c,r}$, $r = 1, 2, \ldots$ is non-increasing in r and $\Delta_{c,r} \xrightarrow{\text{a.s.}} \Delta_c$ as $r \to \infty$. Hence, $\lim_{r \to \infty} \mathsf{P}\{\Delta_{c,r} \geq \delta\} = \mathsf{P}\{\Delta_c \geq \delta\}$. But $\mathsf{P}\{\Delta_{c,r} \geq \delta\} \geq \max_{l \geq r} \mathsf{P}\{\Delta_J(\tilde{\zeta}_{\varepsilon''_l}(\cdot), c, T) \geq \delta\}$ and, therefore, **(b)** $\mathsf{P}\{\Delta_c \geq \delta\} \geq \lim_{r \to \infty} \max_{l \geq r} \mathsf{P}\{\Delta_J(\tilde{\zeta}_{\varepsilon''_l}(\cdot), c, T) \geq \delta\} = \overline{\lim}_{r \to \infty} \mathsf{P}\{\Delta_J(\tilde{\zeta}_{\varepsilon''_r}(\cdot), c, T) \geq \delta\}$. Obviously, **(a)** and **(b)** imply (3.6.8).

Since the subsequence ε''_r was selected from an arbitrarily chosen sequence $0 \leq \varepsilon_n \to \infty$, relation (3.6.8) implies that

$$\lim_{c \to 0} \overline{\lim_{\varepsilon \to 0}} \mathsf{P}\{\Delta_J(\tilde{\zeta}_\varepsilon(\cdot), c, T) \geq \delta\} = 0, \ \delta, T > 0. \tag{3.6.9}$$

Again, this implication is obvious in the case of the usual limits. In the case of iterated limits, the same implication takes place. Assume that (3.6.9) does not hold. This means that there exist $\delta, T, \gamma > 0$ such that **(c)** $\lim_{c \to 0} \overline{\lim}_{\varepsilon \to 0} \mathsf{P}\{\Delta_J(\tilde{\zeta}_\varepsilon(\cdot), c, T) \geq \delta\} \geq \gamma$. Choose an arbitrary sequence $0 < c_n \to 0$ as $n \to \infty$. Note that $\mathsf{P}\{\Delta_J(\tilde{\zeta}_\varepsilon(\cdot), c, T) \geq \delta\}$ is a non-decreasing function in $c > 0$. Hence, **(c)** implies that **(d)** there exists a sequence

$0 < \varepsilon_n \to 0$ as $n \to \infty$ such that $\mathsf{P}\{\Delta_J(\tilde{\zeta}_{\varepsilon_n}(\cdot), c_n, T) \geq \delta\} \geq \gamma/2$. By (3.6.4) – (3.6.8), **(e)** there exists a subsequence $\varepsilon_{n_r} = \varepsilon''_r \to 0$ as $r \to \infty$ such that (3.6.8) holds. Obviously, **(e)** implies that **(f)** there exists $\tilde{c} = \tilde{c}(\gamma)$ such that $\overline{\lim}_{r\to\infty} \mathsf{P}\{\Delta_J(\tilde{\zeta}_{\varepsilon''_r}(\cdot), \tilde{c}, T) \geq \delta\} \leq \gamma/8$. As a consequence, **(f)** implies that **(g)** there exists $\tilde{r} = \tilde{r}(\gamma)$ such that $\max_{r\geq\tilde{r}} \mathsf{P}\{\Delta_J(\tilde{\zeta}_{\varepsilon''_r}(\cdot), \tilde{c}, T) \geq \delta\} \leq \gamma/4$. But $c_n \to 0$ as $n \to \infty$ and, therefore, **(h)** there exists $r \geq \tilde{r}$ such that $c_{n_r} \leq \tilde{c}$. Obviously, **(h)** implies that **(i)** $\gamma/4 \geq \mathsf{P}\{\Delta_J(\tilde{\zeta}_{\varepsilon''_r}(\cdot), \tilde{c}, T) \geq \delta\} \geq \mathsf{P}\{\Delta_J(\tilde{\zeta}_{\varepsilon_{n_r}}(\cdot), c_{n_r}, T) \geq \delta\}$. But **(i)** contradicts **(d)**. Therefore, relation (3.6.9) does hold.

The relation (3.6.9) implies the relation stated in the theorem, since, due to (3.6.2), $\Delta_J(\tilde{\zeta}_\varepsilon(\cdot), c, T) \stackrel{\mathrm{d}}{=} \Delta_J(\zeta_\varepsilon(\cdot), c, T)$. □

3.6.2. J-convergence of semi-vector compositions of càdlàg processes. In order to obtain general conditions for **J**-convergence, one can combine the conditions of **J**-compactness formulated above in Theorem 3.6.1 and the conditions of weak convergence for compositions of càdlàg processes formulated in Theorem 2.7.5.

The second main result in this section is the following theorem from Silvestrov (1974).

Theorem 3.6.2. *Let conditions* $\mathcal{A}_{36}$, $\mathfrak{J}_4$, $\mathfrak{J}_{11}$, $\mathfrak{G}_4$, $\mathcal{E}_6$, *and* $\mathfrak{C}_8^{(0)}$ *hold. Then*

$$\zeta_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{J}} \zeta_0(t), t \geq 0 \text{ as } \varepsilon \to 0.$$

Proof of Theorem 3.6.2. Conditions $\mathcal{A}_{36}$, $\mathfrak{J}_4$, $\mathfrak{J}_{11}$, and $\mathfrak{G}_4$ are conditions of Theorem 3.6.1. This theorem implies **J**-compactness of the processes $\zeta_\varepsilon(t)$, $t \geq 0$ on any finite interval.

Conditions $\mathcal{A}_{36}$, $\mathfrak{J}_4$, $\mathcal{E}_6$, and $\mathfrak{C}_8^{(0)}$ imply that conditions of Theorem 2.7.5 hold for the external processes $\xi_\varepsilon(t), t \geq 0$ and the internal stopping processes $\mathbf{v}_\varepsilon(t) = (\nu_\varepsilon(t), \ldots, \nu_\varepsilon(t))$, $t \geq 0$, with m identical components. In particular, condition $\mathcal{A}_{36}$ implies that condition $\mathcal{A}_{22}^{\mathbf{V}}$ holds for the set V in $\mathcal{A}_{36}$. Condition $\mathfrak{J}_4$ is required in both Theorems 3.6.2 and 2.7.5. Also, condition $\mathcal{E}_6$ implies, in this case, that condition $\mathcal{E}_4$ holds. By applying Theorem 2.7.5, we prove that the processes $\zeta_\varepsilon(t)$ weakly converge to $\zeta_0(t)$ as $\varepsilon \to 0$ on the set S_0 defined in this theorem. This set is dense in $[0, \infty)$. Due to condition $\mathfrak{C}_8^{(0)}$, the point 0 can also be included in the set S_0.

To complete the proof, it remains to refer to Theorem 1.6.6 which gives conditions for **J**-convergence of càdlàg processes defined on the interval $[0, \infty)$. □

3.6.3. Skorokhod's method of a single probability space. The proof of Theorem 3.6.2 can also be accomplished with the use of the modified method of a single probability space. One only needs to continue the proof of Theorem 3.6.1.

Due to relation (3.6.2), condition $\mathcal{E}_6$ implies that $\mathsf{P}(D) = 1$, where $D \in \mathfrak{F}$ is a set of elementary events ω for which there do not exist points $0 \leq t' < t'' \leq T$ such that

$\nu_0(t', \omega) = \nu_0(t'', \omega) \in R[\xi_0(\cdot, \omega)]$. Also, due to relation (3.6.2), condition $\mathcal{C}_{\mathbf{10}}^{(\mathbf{0})}$ implies that $\mathsf{P}(E) = 1$, where $E \in \mathfrak{F}$ is a set of elementary events ω for which $\nu_0(0, \omega) \notin R[\xi_0(\cdot, \omega)]$.

Obviously, $\mathsf{P}(A \cap B \cap C \cap D \cap E) = 1$ and, for $\omega \in A \cap B \cap C \cap D \cap E$, conditions $\mathcal{A}_{\mathbf{38}}$, $\mathcal{J}_{\mathbf{13}}$, $\mathcal{A}_{\mathbf{39}}$, $\mathcal{J}_{\mathbf{14}}$, $\mathcal{G}_{\mathbf{2}}$, $\mathcal{E}_{\mathbf{7}}$, and $\mathcal{C}_{\mathbf{10}}^{(\mathbf{0})}$ of Lemma 3.5.3 hold for the sequences of càdlàg functions $\tilde{\xi}_{\varepsilon''_r}(t, \omega)$, $t \geq 0$, and $\tilde{\nu}_{\varepsilon''_r}(t, \omega)$, $t \geq 0$. By applying Lemma 3.5.3 to their compositions $\tilde{\zeta}_{\varepsilon''_r}(t) = \tilde{\xi}_{\varepsilon''_r}(\tilde{\nu}_{\varepsilon''_r}(t))$, $t \geq 0$, we get, for $\omega \in A \cap B \cap C \cap D \cap E$, the following relation: $\tilde{\zeta}_{\varepsilon''_r}(t, \omega), t \geq 0 \xrightarrow{\mathbf{J}} \tilde{\zeta}_0(t, \omega), t \geq 0$ as $r \to \infty$. In terms of the metrics d_J, the last relation means that $d_J(\tilde{\zeta}_{\varepsilon''_r}(\cdot, \omega), \tilde{\zeta}_0(\cdot, \omega)) \to 0$. Since the initial sequence ε_n was arbitrary, this relation means that the random variables $d_J(\tilde{\zeta}_\varepsilon(\cdot), \tilde{\zeta}_0(\cdot)) \xrightarrow{\mathsf{P}} 0$ as $\varepsilon \to 0$. As it was pointed out in Lemma 1.3.1, convergence in probability implies weak convergence. So, we get that the processes $\tilde{\zeta}_\varepsilon = \{\tilde{\zeta}_\varepsilon(t), t \geq 0\}$, considered as random variables that take values in the space $\mathbf{D}_{[0,\infty)}^{(m)}$ with the metric d_J, weakly converge. Since $\tilde{\zeta}_\varepsilon \stackrel{\mathrm{d}}{=} \zeta_\varepsilon$, this completes the proof.

Let us compare the method described above and the "combined" method used in the proof of Theorem 3.6.2 given in Subsection 3.6.2. That method combines a modified version of the method of a single probability space, used to prove $\mathbf{J}$-compactness of the corresponding càdlàg processes, with the general conditions of weak convergence given Theorem 2.7.5 and based on the continuity condition $\mathcal{E}_{\mathbf{6}}$.

We think that the separation of the proof of $\mathbf{J}$-compactness and the proof of weak convergence of compositions of càdlàg processes on a set dense in $[0, \infty)$ is a significant advantage of the combined method. One can combine conditions that imply $\mathbf{J}$-compactness with various conditions that yield weak convergence, in particular, with conditions based on continuity conditions weaker than the second-type continuity condition $\mathcal{E}_{\mathbf{6}}$. For example, the conditions of $\mathbf{J}$-compactness, formulated above in Theorem 3.6.1, can be combined with the conditions for weak convergence formulated in Theorem 2.7.9 that are more general than conditions given in Theorem 2.7.5 (see Theorem 3.6.4 in Subsection 3.6.6). In this case the "pure" method of a single probability space, described above, can not be applied. Other examples are given in Chapter 4 of this book and in Silvestrov (1974).

The proof of Theorem 3.6.2, based on the modified method of a single probability space, was given in Silvestrov (1974). Some advantage of this approach is connected with a possibility to carry it over to some other topologies of convergence. For example, Anisimov (1977, 1988) gave a sketch of such an application to the topologies $\mathbf{M}, \mathbf{J}_2$ and some others. In the case of the $\mathbf{J}$-topology, the results replicated Theorem 3.6.2 but in a weaker form. In particular, the continuity condition $\mathcal{E}_{\mathbf{1}}$ has been used in an "awkward" form, $\mathcal{E}'_{\mathbf{2}}$ (see Subsection 2.6.3), together with an additional condition of stochastic boundedness of internal stopping processes.

Theorems 3.6.2 and 3.6.4 give the most general conditions for $\mathbf{J}$-convergence of compositions of càdlàg processes for the case **(d)** where both limiting processes can be discontinuous. These conditions require "separate" **J**-convergence of components $\nu_\varepsilon(t), t \geq 0$ and $\xi_\varepsilon(t), t \geq 0$, but they do not require **J**-convergence of vector càdlàg processes $(\nu_\varepsilon(t), \xi_\varepsilon(t)), t \geq 0$. The corresponding example is given in Subsection 3.1.3. In this sense, Theorems 3.6.2 and 3.6.4 extend, with respect to the composition mapping, setting of the continuous mapping theorem.

However, there are particular cases, where the continuous mapping theorem can be applied. These are the cases where at least one component of the limiting composition is a.s. continuous. Here, the conditions of joint weak convergence, $\mathcal{A}_{36}$, and **J**-compactness, $\mathfrak{J}_4$ and $\mathfrak{J}_{11}$, do imply **J**-convergence of the vector processes $(\nu_\varepsilon(t), \xi_\varepsilon(t)), t \geq 0$. This makes it possible to reduce the consideration to the case of non-random càdlàg functions using the continuous mapping theorem. It should be noted that the use of this theorem should be anticipated by the proof of **J**-continuity of the composition mapping in every particular case.

We prefer, however, to use, in these cases, the most simple "direct" method combining results on weak convergence of compositions with the direct check of the corresponding **J**-compactness conditions. The main advantage of this method is the same as for the combined method described above. It is connected with the separation of conditions of weak convergence and conditions of **J**-compactness. Another advantage of this method is that it provides an additional information about the structure of the corresponding sets of weak convergence.

Let us compare results that can be obtained with the use of Theorem 3.6.2 in the situations where at least one component of of the limiting composition is a.s. continuous.

There are two simplest cases where all methods give similar results.

The first case is where **(a)** both limiting processes $\nu_0(t), t \geq 0$ and $\xi_0(t), t \geq 0$ are a.s. continuous. This case was treated by Billingsley (1968) with the use of the continuous mapping theorem. Theorem 3.2.1, proved in Section 3.2 with the use of the direct method, yields a similar result for vector compositions of càdlàg processes. The conditions of Theorem 3.6.2 are reduced, in this case, to the conditions of Theorem 3.2.1 applied to semi-vector compositions of càdlàg processes. Conditions $\mathcal{G}_4$, $\mathcal{E}_6$, and $\mathcal{C}_8^{(0)}$ automatically hold. Condition $\mathfrak{J}_4$ are reduced to condition $\mathfrak{U}_4$. Condition $\mathfrak{J}_{11}$ also holds, due to Lemma 3.2.1.

The second case is where **(b)** the limiting external process $\xi_0(t), t \geq 0$ is a.s. continuous. This case was considered by Whitt (1973, 1980) with the use of the continuous mapping theorem and by Silvestrov (1974) with the use of the direct method. Theorem 3.3.2 proved, in Section 3.3, with the use of direct method gives a similar result for vector compositions of càdlàg processes. In this case, the conditions of Theorem 3.6.2 are reduced to the conditions of Theorem 3.3.2 applied to semi-vector compositions of càdlàg processes. Conditions $\mathcal{G}_4$, $\mathcal{E}_6$, and $\mathcal{C}_8^{(0)}$ automatically hold, and condition $\mathfrak{J}_4$ is reduced to condition $\mathfrak{U}_4$.

A situation is more interesting in the case where **(c)** the limiting internal stopping processes $\nu_0(t), t \geq 0$ is a.s. continuous. This case was considered by Silvestrov (1972b, 1972e, 1973a, 1974) with the use of the direct method. Here, conditions of Theorem 3.3.2 are reduced to the conditions of Theorem 3.4.2. Condition $\mathfrak{G}_4$ automatically holds. Condition $\mathfrak{J}_{11}$ also holds, due to Lemma 3.2.1. Conditions $\mathcal{E}_6$ and $\mathfrak{C}_8^{(0)}$ remain. As the example given in Subsection 3.1.2 shows, if condition $\mathcal{E}_6$ does not hold, then compositions may not weakly converge on some interval. In the sequel, they do not **J**-converge. Condition $\mathcal{E}_6$ holds, for example, if the limiting internal stopping process $\nu_0(t), t \geq 0$ is not only continuous but also strictly monotone. This case was independently considered by Whitt (1973, 1980) with the use of the continuous mapping theorem.

However, the direct method used in Section 3.4 and the combined method used in Subsection 3.6.2., also yield more general results that are not covered by the continuous mapping theorem. These are Theorems 3.4.3 and 3.6.4 based on the weakened second-type continuity condition $\mathcal{F}_4$.

3.6.4. The set of weak convergence. Let V_0 be the set of points of stochastic continuity of the limiting stopping process $\nu_0(t)$, $t \geq 0$, and $V_0' = V_0 \setminus \{0\}$. This set is the interval $[0, \infty)$, except for at most a countable set.

Let also W_0 be the set of all points for which condition $\mathfrak{C}_8^{(w)}$ holds. Conditions $\mathcal{E}_6$ and $\mathfrak{C}_8^{(0)}$ imply that set W_0 is the interval $[0, \infty)$, except for at most a countable set, and also that $0 \in W_0$.

According to Theorem 2.7.5, the set of weak convergence, used in the proof of Theorem 3.6.2, is $S_0 = (V \cup V_0') \cap W_0$. This set also is the interval $[0, \infty)$, except for at most a countable set. Also, $0 \in S_0$.

However, the set S_0 can be enlarged in the following way. Let Z_0 be the set of points of stochastic continuity for the limiting composition $\zeta_0(t)$, $t \geq 0$. The processes $\zeta_\varepsilon(t)$, $t \geq 0$ **J**-converge and, therefore, by Lemma 1.6.5, the set S_0 can be enlarged to the set $S_0 \cup Z_0$. Finally, we get that, under the conditions of Theorem 3.6.2,

$$\zeta_\varepsilon(t), t \in S_0 \cup Z_0 \Rightarrow \zeta_0(t), t \in S_0 \cup Z_0 \text{ as } \varepsilon \to 0. \tag{3.6.10}$$

3.6.5. The continuity conditions $\mathcal{E}_6$ and $\mathfrak{G}_4$. Let us consider moments of jumps of the process $\xi_0(t)$, $t \geq 0$, namely $\tau_{kn} = \inf(s > \tau_{k-1n} : |\xi_0(s) - \xi_0(s-0)| \in [\frac{1}{n}, \frac{1}{n-1}))$, $k, n = 1, 2, \ldots$, where $\tau_{0n} = 0$. By the definition, τ_{kn} are successive moments of jumps with absolute values in the interval $[\frac{1}{n}, \frac{1}{n-1})$ for $k < \mu_n + 1$ and $\tau_{kn} = \infty$ for $k \geq \mu_n + 1$, where $\mu_n = \max(k \geq 0 : \tau_{kn} < \infty)$ is the total number of such jumps in the interval $[0, \infty)$.

Similar notations can be introduced for moments of jumps of the process $\nu_0(t)$, $t \geq 0$, namely, $\kappa_{kn} = \inf(s > \kappa_{k-1n} : |\nu_0(s) - \nu_0(s-0)| \in [\frac{1}{n}, \frac{1}{n-1}))$, $k, n = 1, 2, \ldots$, where $\kappa_{0n} = 0$. By the definition, κ_{kn} are successive moments of such jumps with absolute values in the the interval $[\frac{1}{n}, \frac{1}{n-1})$ for $k < \lambda_n + 1$ and $\kappa_{kn} = \infty$ for $k \geq \lambda_n + 1$, where $\lambda_n = \max(k \geq 0 : \kappa_{kn} < \infty)$ is the total number of such jumps in the interval $[0, \infty)$.

Condition $\mathcal{E}_6$ can be reformulated in the following equivalent form (see, also Subsection 2.6.3):

$\mathcal{E}_6'$: $\mathsf{P}\{\nu_0(t') = \nu_0(t'') = \tau_{rl}\} = 0$ for $0 \leq t' < t'' < \infty$ and $r, l = 1, 2, \ldots$.

Condition $\mathcal{G}_4$ is equivalent to the following:

$\mathcal{G}_4'$: $\mathsf{P}\{\nu_0(\kappa_{kn} \pm 0) = \tau_{rl}\} = 0$ for $k, n, r, l = 1, 2, \ldots$.

Note that the random variables $\nu_0(\kappa_{kn} \pm 0)$ can take values in the interval $[0, \infty]$, since the random variables κ_{kn} can take the value $+\infty$. In this case, by the definition, $\nu_0(+\infty \pm 0) = \lim_{t\to\infty} \nu_0(t)$.

Let us recall that condition $\mathcal{Q}_7$. This condition means that the process $\xi_0(t)$, $t \geq 0$ can be decomposed in a sum of two processes $\xi_0(t) = \xi_0'(t) + \xi_0''(t)$, $t \geq 0$, where $\xi_0'(t)$, $t \geq 0$ is a continuous process, and $\xi_0''(t)$, $t \geq 0$ is a stochastically continuous càdlàg process independent of the process $\nu_0(t)$, $t \geq 0$.

As follows from Lemma 2.7.1, condition $\mathcal{Q}_7$ implies that condition $\mathcal{E}_6$ holds. Since Lemma 2.7.1 was given without a proof in Subsection 2.7.2, let us give it here.

Lemma 3.6.1. *Let condition $\mathcal{Q}_7$ hold. Then condition $\mathcal{E}_6$ also holds.*

Proof of Lemma 3.6.1. If condition $\mathcal{Q}_7$ holds, then every random variable τ_{rl} is a moment of jump of the process $\xi_0(t)$ if and only if it is a corresponding moment of jump of the second component $\xi_0''(t)$. This is so, since the first component $\xi_0'(t)$ is a continuous process. Therefore, the process $\nu_0(t)$, $t \geq 0$ and the random variable τ_{rl} are independent. Since the process $\xi_0''(t)$, $t \geq 0$ is stochastically continuous, the random variables τ_{rl} have continuous distribution functions. This implies that condition $\mathcal{E}_6$ holds. Note that, in this case, form of the distribution of random vector $(\nu_0(t'), \nu_0(t''))$ does not play any role. □

Let us also formulate a similar statement concerning condition $\mathcal{G}_4$.

Lemma 3.6.2. *Let condition $\mathcal{Q}_7$ hold. Then condition $\mathcal{G}_4$ also holds.*

Proof of Lemma 3.6.2. As was shown in the proof of Lemma 3.6.1, every random variable τ_{rl} is a moment of jump of the second component $\xi_0''(t)$. Therefore, the process $\nu_0(t)$, $t \geq 0$, and the random variable τ_{rl} are independent. Consequently, the random variables $\nu_0(\kappa_{kn} \pm 0)$ and τ_{rl} are independent. It was also shown in the proof of Lemma 3.6.1 that the random variables τ_{rl} have continuous distribution functions. Hence, condition $\mathcal{G}_4$ holds. In this case, form of the distribution functions of random variables $\nu_0(\kappa_{kn} \pm 0)$ do not play any role. □

If the process $\xi_0''(t)$, $t \geq 0$ is not stochastically continuous, then the distribution functions of the random variables τ_{rl} can possess discontinuity points.

In this case, in order to prove that condition $\mathcal{E}_6$ holds, it is enough to require that the distribution functions of the random variables $\nu_0(t)$ and τ_{rl} have not common points of discontinuity for any $t \geq 0$ and $r, l = 1, 2, \ldots$.

Analogously, to make the condition $\mathfrak{G}_4$ hold, it will suffice to require that the distribution functions of the random variables $\nu_0(\kappa_{kn} \pm 0)$ and τ_{rl} have not common points of discontinuity for any $k, n, r, l = 1, 2, \ldots$.

Also, condition $\mathfrak{Q}_7$ implies condition $\mathfrak{C}_8^{(w)}$ for any $w \geq 0$. This follows from Lemma 2.2.3.

Using the remarks made above and Lemmas 3.6.1 and 3.6.2, we can formulate the following theorem from Silvestrov (1974), which is a direct corollary of Theorem 3.6.2. This theorem is used in a significant number of applications.

Theorem 3.6.3. *Let conditions $\mathcal{A}_{36}$, $\mathfrak{J}_4$, $\mathfrak{J}_{11}$, $\mathfrak{Q}_7$ hold. Then*

$$\zeta_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{J}} \zeta_0(t), t \geq 0 \text{ as } \varepsilon \to 0.$$

3.6.6. Weakened second-type continuity conditions. Let us formulate an analogue of Theorem 3.6.2, in which the continuity conditions $\mathfrak{E}_6$ and $\mathfrak{C}_8^{(0)}$ are weakened and replaced with conditions $\mathfrak{F}_4$ and $\mathfrak{D}_7^{(0)}$.

Theorem 3.6.4. *Let conditions $\mathcal{A}_{36}$, $\mathfrak{J}_4$, $\mathfrak{J}_{11}$, $\mathfrak{G}_4$, $\mathfrak{F}_4$, and $\mathfrak{D}_7^{(0)}$ hold. Then*

$$\zeta_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{J}} \zeta_0(t), t \geq 0 \text{ as } \varepsilon \to 0.$$

Proof of Theorem 3.6.4. Conditions $\mathcal{A}_{36}$, $\mathfrak{J}_4$, $\mathfrak{J}_{11}$, and $\mathfrak{G}_4$ are conditions of Theorem 3.6.1. Using this theorem we prove **J**-compactness of the processes $\zeta_\varepsilon(t)$, $t \geq 0$, on any finite interval.

Conditions $\mathcal{A}_{36}$, $\mathfrak{J}_4$, $\mathfrak{F}_4$, and $\mathfrak{D}_7^{(0)}$ imply that the conditions of Theorem 2.7.9 hold for the external processes $\xi_\varepsilon(t)$, $t \geq 0$, and the internal stopping processes $\boldsymbol{\nu}_\varepsilon(t) = (\nu_\varepsilon(t), \ldots, \nu_\varepsilon(t))$, $t \geq 0$, with m identical components. In particular, condition $\mathcal{A}_{36}$ implies that condition $\mathcal{A}_{22}^{\mathbf{V}}$ holds for set V in $\mathcal{A}_{36}$. Condition $\mathfrak{J}_4$ is required for both Theorems 3.6.4 and 2.7.9. Also, condition $\mathfrak{F}_4$ implies condition $\mathfrak{F}_3$. Applying Theorem 2.7.9 we prove that the processes $\zeta_\varepsilon(t)$ weakly converge to $\zeta_0(t)$ as $\varepsilon \to 0$ on the set S_0 defined in this theorem. This set is dense in $[0, \infty)$. Due to condition $\mathfrak{D}_7^{(0)}$, the point 0 can also be included in set S_0.

To complete the proof we use Theorem 1.6.6 that gives conditions for **J**-convergence of càdlàg processes defined on interval $[0, \infty)$. □

Remark 3.6.1. Note that, in the case where the limiting stopping process $\nu_0(t), t \geq 0$ is a.s. continuous, conditions $\mathfrak{J}_{11}$ and $\mathfrak{G}_4$ automatically hold. In this case, Theorem 3.6.4 becomes Theorem 3.4.3.

3.6.7. The time interval $[0, T]$. In this case, we consider the semi-vector composition $\zeta_\varepsilon(t) = (\xi_{\varepsilon i}(\nu_\varepsilon(t)), i = 1, \ldots, m)$, $t \in [0, T]$ of a vector càdlàg process $\xi_\varepsilon(t) = (\xi_{\varepsilon i}(t), i = 1, \ldots, m)$, $t \geq 0$, with real-valued components, and a scalar non-negative and non-decreasing càdlàg process $\nu_\varepsilon(t)$, $t \in [0, T]$.

We can always continue the internal stopping process to the interval $[0, \infty)$ by the following formula:

$$\nu_\varepsilon(t) = \begin{cases} \nu_\varepsilon(t) & \text{if } 0 \le t \le T, \\ \nu_\varepsilon(T) & \text{if } t \ge T. \end{cases} \tag{3.6.11}$$

Formula (3.6.11) implies that

$$\xi_\varepsilon(\nu_\varepsilon(t)) = \begin{cases} \xi_\varepsilon(\nu_\varepsilon(t)) & \text{if } 0 \le t \le T, \\ \xi_\varepsilon(\nu_\varepsilon(T)) & \text{if } t \ge T. \end{cases} \tag{3.6.12}$$

The processes $\nu_\varepsilon(t)$ and $\xi_\varepsilon(\nu_\varepsilon(t))$ take, respectively, the values $\nu_\varepsilon(T)$ and $\xi_\varepsilon(\nu_\varepsilon(T))$ for $t \ge T$. This fact should be taken into account when modifying the conditions.

Condition $\mathcal{A}_{36}$ takes, in this case, the following form:

$\mathcal{A}_{42}$: $(\nu_\varepsilon(s), \xi_\varepsilon(t)), (s,t) \in V \times U \Rightarrow (\nu_0(s), \xi_0(t)), (s,t) \in V \times U$ as $\varepsilon \to 0$, where (a) U is a subset of $[0, \infty)$ that is dense in this interval and contains the point 0, (b) V is a subset of $[0, T]$ that is dense in this interval and contains the points 0 and T.

Condition $\mathcal{J}_4$ does not require any changes, whereas condition $\mathcal{J}_{11}$ takes the following form:

$\mathcal{J}_{16}$: $\lim_{c\to 0} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\Delta_J(\nu_\varepsilon(\cdot), c, T) > \delta\} = 0, \ \delta > 0.$

Denote by $R_T[\nu_0(\cdot)]$ the random set of points of discontinuity of the process $\nu_0(t), t \in [0, T]$.

Condition $\mathcal{G}_4$ takes the following form:

$\mathcal{G}_5$: $\mathsf{P}\{\nu_0(t \pm 0) \notin R[\xi_0(\cdot)] \text{ for } t \in R_T[\nu_0(\cdot)]\} = 1.$

The following theorem is an analogue of Theorem 3.6.1.

Theorem 3.6.5. *Let conditions $\mathcal{A}_{42}$, $\mathcal{J}_4$, $\mathcal{J}_{16}$, and $\mathcal{G}_5$ hold. Then*

$$\lim_{c\to 0} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\Delta_J(\zeta_\varepsilon(\cdot), c, T) > \delta\} = 0, \ \delta > 0.$$

Proof of Theorem 3.6.5. It is sufficient to apply Theorem 3.6.1 to the semi-vector composition of the processes $\xi_\varepsilon(t)$, $t \ge 0$ and $\nu_\varepsilon(t)$, $t \ge 0$, where the latter process is defined in (3.6.11). Condition $\mathcal{A}_{42}$ implies condition $\mathcal{A}_{36}$, condition $\mathcal{J}_{16}$ implies $\mathcal{J}_{11}$, and, finally, condition $\mathcal{G}_5$ implies $\mathcal{G}_4$. □

We also use the following modification of condition $\mathcal{A}_{42}$ in which the random variables $\nu_\varepsilon(T - 0)$ are additionally included in the relation of weak convergence:

$\mathcal{A}_{43}$: $(\nu_\varepsilon(s), \nu_\varepsilon(T-0), \xi_\varepsilon(t)), (s,t) \in V \times U \Rightarrow (\nu_0(s), \xi_0(t)), (s,t) \in V \times U$ as $\varepsilon \to 0$, where (a) U is a subset of $[0,\infty)$ that is dense in this interval and contains the point 0, (b) V is a subset of $[0,T]$ that is dense in this interval and contains the points 0 and T.

Condition $\mathcal{E}_6$ takes, in this case, the following form:

$\mathcal{E}_9$: $\mathsf{P}\{\nu_0(t') = \nu_0(t'') \in R[\xi_0(\cdot)]\} = 0$ for $0 \le t' < t'' \le T$.

Condition $\mathcal{C}_8^{(\mathbf{w})}$ does not require any changes. However, we also use the following modification of this condition:

$\mathcal{C}_8^{(\mathbf{w}\pm)}$: $\mathsf{P}\{\nu_0(w \pm 0) \in R[\xi_0(\cdot)]\} = 0$.

The following theorem is an analogue of Theorem 3.6.2.

Theorem 3.6.6. *Let conditions $\mathcal{A}_{43}$, $\mathcal{J}_4$, $\mathcal{J}_{16}$, $\mathcal{G}_5$, $\mathcal{E}_9$, $\mathcal{C}_8^{(\mathbf{0})}$, and $\mathcal{C}_8^{(\mathbf{T})}$ hold. Then*

$$\zeta_\varepsilon(t), t \in [0,T] \xrightarrow{\mathbf{J}} \zeta_0(t), t \in [0,T] \text{ as } \varepsilon \to 0.$$

Proof of Theorem 3.6.6. The proof can be obtained by applying Theorem 3.6.2 to the semi-vector composition of the processes $\xi_\varepsilon(t)$, $t \ge 0$ and $\nu_\varepsilon(t)$, $t \ge 0$, where the latter process is defined in (3.6.11). Condition $\mathcal{A}_{43}$ implies $\mathcal{A}_{36}$, condition $\mathcal{J}_{16}$ implies $\mathcal{J}_{11}$, and condition $\mathcal{G}_5$ implies $\mathcal{G}_4$. Also, conditions $\mathcal{E}_9$ and $\mathcal{C}_8^{(\mathbf{T})}$ imply $\mathcal{E}_6$. Conditions $\mathcal{J}_4$ and $\mathcal{C}_8^{(\mathbf{0})}$ are required in both Theorems 3.6.6 and 3.6.2. By applying Theorem 3.6.2 we prove that the processes $\zeta_\varepsilon(t), t \ge 0 \xrightarrow{\mathbf{J}} \zeta_0(t), t \ge 0$ as $\varepsilon \to 0$.

However, $\mathbf{J}$-convergence of the processes $\zeta_\varepsilon(t)$ on the interval $[0,\infty)$ does not automatically imply $\mathbf{J}$-convergence of these processes on the interval $[0,T]$. In order for the processes $\zeta_\varepsilon(t)$ to be $\mathbf{J}$-convergent on the interval $[0,T]$, the random variables $\zeta_\varepsilon(T)$ must be included in the relation of weak convergence for these processes on the set $S_0(T) = S_0 \cap [0,T]$. Note that this set is dense in $[0,T]$ and contains the point 0. Moreover, as follows from Theorem 1.6.3, if the point T is not a point of stochastic continuity of the limiting process $\zeta_0(t)$, then the random variables $\zeta_\varepsilon(T-0)$ must also be included in the corresponding relation of weak convergence.

The random variables $\zeta_\varepsilon(T)$ can be included due to condition $\mathcal{C}_8^{(\mathbf{T})}$. Also, conditions $\mathcal{G}_5$ and $\mathcal{C}_8^{(\mathbf{T})}$ imply that $\mathcal{C}_8^{(\mathbf{T}-)}$ holds. Indeed, let A_T denote the set of elementary events in condition $\mathcal{G}_5$ that has, according this condition, probability 1. Then we have

$$\begin{aligned} &\mathsf{P}\{\nu_0(T-0) \in R[\xi_0(\cdot)]\} \\ &\qquad = \mathsf{P}\{\nu_0(T-0) \in R[\xi_0(\cdot)], \nu_0(T-0) = \nu_0(T)\} \\ &\qquad + \mathsf{P}\{\nu_0(T-0) \in R[\xi_0(\cdot)], \nu_0(T-0) \neq \nu_0(T)\} \\ &\qquad = \mathsf{P}\{\nu_0(T) \in R[\xi_0(\cdot)], \nu_0(T-0) = \nu_0(T)\} \\ &\qquad + \mathsf{P}\{\nu_0(T-0) \in R[\xi_0(\cdot)], T \in R[\nu_0(\cdot)]\} \\ &\qquad \le \mathsf{P}\{\nu_0(T) \in R[\xi_0(\cdot)]\} + P(\overline{A}_T) = 0. \end{aligned} \tag{3.6.13}$$

Since condition $\mathfrak{C}_{\mathbf{8}}^{(\mathbf{T}-)}$ holds, the random variable $\zeta_\varepsilon(T-0)$ can also be included in the relation of weak convergence of the processes $\zeta_\varepsilon(t)$.

Reference to Theorem 1.6.3, which gives conditions for **J**-convergence of càdlàg processes defined on the interval $[0, T]$, completes the proof. □

Condition $\mathcal{A}_{\mathbf{43}}$ can be simplified if the point T is a point of continuity of the limiting function $\xi_0(\nu_0(t))$, i.e., the following condition holds:

$\mathfrak{O}_{\mathbf{9}}^{(\mathbf{T})}$: $\mathsf{P}\{\xi_0(\nu_0(T-0)) = \xi_0(\nu_0(T))\} = 1$.

In this case, $\mathcal{A}_{\mathbf{43}}$ can be replaced, in Theorem 3.6.6, with condition $\mathcal{A}_{\mathbf{42}}$.

Let us also give a description of the corresponding set of weak convergence. It follows from (3.6.10) that, in the case under consideration, the set of weak convergence is $S_0 \cup Z_0$, where $S_0 = (V \cup V_0'') \cap W_0$.

Here V is the set of weak convergence that appears in condition $\mathcal{A}_{\mathbf{43}}$, $V_0'' = V_0 \setminus \{0, T\}$, V_0 is a set of points of stochastic continuity for the process $\zeta_0(t), t \in [0, T]$. Also, W_0 is a set of all points in the interval $[0, T]$ that satisfy condition $\mathfrak{C}_{\mathbf{8}}^{(\mathbf{w})}$, and, finally, Z_0 is a set of all points of stochastic continuity for the limiting composition $\zeta_0(t), t \in [0, T]$. The set $S_0 \cup Z_0$ is the interval $[0, T]$ except for at most a countable set. Also, the points 0 an T belong to this set.

3.6.8. Non-monotone internal processes. In the case where the external processes are not asymptotically continuous, the assumption of monotonicity of the internal stopping processes plays an essential role.

Our conjecture is that the results formulated in Theorems 3.6.1 and 3.6.2 can be generalised to a model in which the internal stopping processes are piecewise monotone. This means that there exist random moments $0 = \varsigma_{\varepsilon 0} \le \varsigma_{\varepsilon 1} \le \ldots$, and a set of elementary events, A_ε, with $\mathsf{P}(A_\varepsilon) = 1$ such that for every $\omega \in A_\varepsilon$, **(a)** $\varsigma_{\varepsilon k}(\omega) \to \infty$ as $k \to \infty$, **(b)** the trajectory $\nu_\varepsilon(t, \omega), t \ge 0$ is a monotone function in every non-empty subinterval $[\varsigma_{\varepsilon k}(\omega), \varsigma_{\varepsilon k+1}(\omega))$.

Two conditions should be included in the conditions of Theorems 3.6.1 and 3.6.2. The first one is the condition **(c)** of joint weak convergence of the processes $(\nu_\varepsilon(t), \xi_\varepsilon(t))$, $t \ge 0$ and the random sequence $\varsigma_{\varepsilon k}, k = 0, 1, \ldots$. The second one is the condition **(d)** that the limiting external process $\xi_0(t), t \ge 0$ is continuous with probability 1 at the random point $\nu_0(\varsigma_{0k} \pm 0)$ for every $k = 0, 1, \ldots$.

Under these additional conditions, it will be possible to extend the proofs of Theorems 3.6.1 and 3.6.2, which are based on their reduction to the case of compositions of non-random càdlàg functions, to the case of piecewise monotone internal stopping processes. More precisely, it will be possible to prove the corresponding relation of **J**-compactness and weak convergence of the compositions on some set dense in $[0, \infty)$ and containing the point 0.

In this case, the pre-limiting processes $\zeta_\varepsilon(t) = \xi_\varepsilon(\nu_\varepsilon(t)), t \geq 0$, may not be càdlàg processes. However, the corresponding limiting process $\zeta_0(t) = \xi_0(\nu_0(t)), t \geq 0$ that verifies condition **(d)** is an a.s. process without discontinuities of the second kind.

We again refer to works by Borovkov (1976) and Borovkov, Mogul'skij, and Sakhanenko (1995), where one can find results concerning **J**-convergence of stochastic processes in such a case.

3.6.9. A Polish phase space. The results in this section can be generalised to a model with external stochastic processes $\xi_\varepsilon(t), t \geq 0$ that take values in a Polish space X.

The formulation of condition $\mathcal{A}_{36}$ or $\mathcal{A}_{43}$ does not change. In the condition $\mathcal{J}_4$, the Euclidean distance $|x - y|$ must be replaced with the corresponding metric $d(x, y)$ in the formula for the modulus $\Delta_J(\xi_\varepsilon(\cdot), c, T)$.

All other conditions of Theorems 3.6.1 – 3.6.6 remain without changes. With these modifications in the conditions, the proofs of these theorems can be repeated.

3.7 Vector compositions of càdlàg functions

In this section, we will study conditions for **J**-compactness and **J**-convergence of general vector compositions of non-random càdlàg functions. This conditions will be essentially used in the next Section 3.8, where the corresponding results are obtained for vector compositions of càdlàg stochastic processes.

3.7.1. J-compactness of vector compositions of non-random càdlàg functions. Let $\mathbf{x}_n(t) = (x_{ni}(t), i = 1, \ldots, m)$, $t \geq 0, n = 0, 1, \ldots$ be a sequence of vector càdlàg functions with real-valued components, $\mathbf{y}_n(t) = (y_{ni}(t), i = 1, \ldots, m), t \geq 0, n = 0, 1, \ldots$, a sequence of vector càdlàg functions with non-negative and non-decreasing components. Let also $\mathbf{z}_n(t) = (x_{ni}(y_{ni}(t)), i = 1, \ldots, m)$, $t \geq 0$ be *vector compositions* of the functions $\mathbf{x}_n(t)$ and $\mathbf{y}_n(t)$. The functions $\mathbf{z}_n(t)$, $t \geq 0, n = 0, 1, \ldots$ are also vector càdlàg functions with real-valued components.

We impose the following conditions on the functions $\mathbf{x}_n(t)$:

$\mathcal{A}_{44}$: $x_{ni}(t) \to x_{0i}(t)$ as $n \to \infty$, $t \in X_i$, $i = 1, \ldots, m$, where X_i are subsets of $[0, \infty)$ that are dense in this interval and contain the point 0;

and

$\mathcal{J}_{17}$: $\lim_{c\to 0} \overline{\lim}_{n\to\infty} \Delta_J(x_{ni}(\cdot), c, T) = 0, \ T > 0, \ i = 1, \ldots, m.$

We also assume that the functions $\mathbf{y}_n(t)$ satisfy the following conditions:

$\mathcal{A}_{45}$: $y_{ni}(t) \to y_{0i}(t)$ as $n \to \infty$, $t \in Y_i$, $i = 1, \ldots, m$, where Y_i are subsets of $[0, \infty)$ that are dense in this interval and contain the point 0;

and

$\mathcal{J}_{18}$: $\lim_{c\to 0}\overline{\lim}_{n\to\infty} \Delta_J(\mathbf{y}_n(\cdot), c, T) = 0,\ T > 0$.

Note that both limiting functions $\mathbf{x}_0(t), t \geq 0$ and $\mathbf{y}_0(t), t \geq 0$ are not assumed to be continuous.

Conditions $\mathcal{A}_{44}$ and $\mathcal{J}_{17}$ imply that the functions $x_{ni}(t),\ t \geq 0$ **J**-converge to $x_{0i}(t)$, $t \geq 0$ as $n \to \infty$ for every $i = 1, \dots, m$. Conditions $\mathcal{A}_{45}$ and $\mathcal{J}_{18}$ provide **J**-convergence of the functions $\mathbf{y}_n(t),\ t \geq 0$. However, these conditions together do not provide either **J**-convergence or **J**-compactness for the vector functions $\mathbf{x}_n(t),\ t \geq 0$ and $(\mathbf{y}_n(t), \mathbf{x}_n(t))$, $t \geq 0$, or the compositions $\mathbf{z}_n(t), t \geq 0$. The corresponding examples are given in Section 3.1.

The following continuity conditions play a key role in further consideration:

$\mathcal{G}_6$: $y_{0i}(t \pm 0) \notin R[x_{0i}(\cdot)],\ i = 1, \dots, m$ for $t \in \cup_{i=1}^m R[y_{0i}(\cdot)]$;

and

$$\mathcal{H}_4: \sum_{i=1}^{m} \chi(y_{0i}(t) \in R[x_{0i}(\cdot)]) \leq 1 \text{ for } t \geq 0.$$

Lemma 3.7.1. *Let conditions $\mathcal{A}_{44}$, $\mathcal{J}_{17}$, $\mathcal{A}_{45}$, $\mathcal{J}_{18}$, $\mathcal{G}_6$, and $\mathcal{H}_4$ hold. Then*

$$\lim_{c\to 0}\overline{\lim_{n\to\infty}} \Delta_J(\mathbf{z}_n(\cdot), c, T) = 0,\ T > 0.$$

Proof of Lemma 3.7.1. As in the case of one-dimensional functions, the proof consists of two parts. The first part reduces the proof to simpler functions and is similar to that given in the proof of Lemma 3.5.1. The second part gives a uniform estimate of the corresponding local modulus $R_n[t', t, t'']$. It is much more difficult than the corresponding part of the proof of Lemma3.5.1. This is due to a much more complicated relation between points of discontinuity of the internal functions $\mathbf{y}_n(t)$ and the external functions $\mathbf{x}_n(t)$ in the vector case.

Conditions $\mathcal{A}_{44}$ and $\mathcal{J}_{17}$ imply that the functions $x_{ni}(\cdot), t \geq 0 \xrightarrow{\mathbf{J}} x_{0i}(t), t \geq 0$ as $\varepsilon \to 0$ for every $i = 1, \dots, m$. Hence, the sets X_i in condition $\mathcal{A}_{44}$ can be enlarged to the set $X_i \cup X_{0i}$. Here $X_{0i} = [0, \infty) \setminus R[x_{0i}(\cdot)]$ is the set of continuity points for the function $x_{0i}(t), t \geq 0$. For every $i = 1, \dots, m$, the set X_{0i} is dense in $[0, \infty)$, moreover, it coincides with $[0, \infty)$ except for at most a countable set. Thus the set $X = \cap_{i=1}^m (X_i \cup X_{0i})$ is also $[0, \infty)$, except for at most a countable set.

Analogously, conditions $\mathcal{A}_{45}$ and $\mathcal{J}_{18}$ imply that the functions $y_{ni}(\cdot), t \geq 0 \xrightarrow{\mathbf{J}} y_{0i}(t), t \geq 0$ as $\varepsilon \to 0$ for every $i = 1, \dots, m$. So, for every $i = 1, \dots, m$, the set Y_i in the condition $\mathcal{A}_{45}$ can be enlarged to the set $Y_i \cup Y_{0i}$. Here $Y_{0i} = [0, \infty) \setminus R[y_{0i}(\cdot)]$ is a set of continuity points for the function $y_{0i}(t), t \geq 0$. For every $i = 1, \dots, m$, the set Y_{0i} is dense in $[0, \infty)$, moreover, it is the interval $[0, \infty)$, except for at most a countable set. Hence, the set $Y = \cap_{i=1}^m (Y_i \cup Y_{0i})$ is also $[0, \infty)$ except for at most a countable set.

It is sufficient to show that the compactness relation in Lemma 3.7.1 holds for any $T \in Y$.

Conditions $\mathcal{A}_{45}$ and $\mathfrak{J}_{18}$ also imply that the vector functions $\mathbf{y}_n(t) \to \mathbf{y}_0(t)$ as $n \to \infty$ for $t \in Y$. Since T is a point of continuity for the function $\mathbf{y}_0(t)$, this convergence, together with the conditions $\mathcal{A}_{45}$ and $\mathfrak{J}_{18}$, implies that the vector functions $\mathbf{y}_n(t), t \in [0, T] \xrightarrow{\mathbf{J}} \mathbf{y}_0(t), t \in [0, T]$ as $n \to \infty$. This means that there exists a sequence $\lambda_n(t)$, $n \geq 1$ of continuous one-to-one mappings of the interval $[0, T]$ into itself such that

$$\lim_{n\to\infty} \sup_{t\in[0,T]} (|\mathbf{y}_n(t) - \mathbf{y}_0(\lambda_n(t))| + |\lambda_n(t) - t|) = 0. \tag{3.7.1}$$

Let T_k, $k \geq 1$, be a sequence of points of the set X such that $T_k \to \infty$ as $k \to \infty$. For every $i = 1, \dots, m$, the function $x_{0i}(t)$ is continuous in the points T_k, $k \geq 1$. So, by conditions $\mathcal{A}_{44}$ and $\mathfrak{J}_{17}$, the functions $x_{ni}(t), t \in [0, T_k] \xrightarrow{\mathbf{J}} x_{0i}(t), t \in [0, T_k]$ as $n \to \infty$ for every $i = 1, \dots, m$ and $k \geq 1$.

Since the sequence $y_{ni}(T)$, $n \geq 1$ is bounded for every $i = 1, \dots, m$, there exists $T' = T_k$ such that $y_{ni}(T) \leq T'$ for every $n \geq 1$, $i = 1, \dots, m$.

Let $\lambda_{ni}(t)$, $n \geq 1$ be sequences of continuous one-to-one mappings of the interval $[0, T']$, for $i = 1, \dots, m$, such that

$$\lim_{n\to\infty} \sup_{t\in[0,T']} (|x_{ni}(t) - x_{0i}(\lambda_{ni}(t))| + |\lambda_{ni}(t) - t|) = 0, \ i = 1, \dots, m. \tag{3.7.2}$$

By using estimate (1.4.8) given in Lemma 1.4.9, we get

$$\begin{aligned}
&\Delta_J(\mathbf{z}_n(t), c, T) \leq \\
&\quad \leq \Delta_J(\mathbf{w}_n(t), c, T) + \sum_{i=1}^{m} \sup_{t\in[0,T]} |x_{ni}(y_{ni}(t)) - x_{0i}(\lambda_{ni}(y_{ni}(t)))| \\
&\quad \leq \Delta_J(\mathbf{w}_n(t), c, T) + \sum_{i=1}^{m} \sup_{t\in[0,T']} |x_{ni}(t) - x_{0i}(\lambda_{ni}(t))|,
\end{aligned} \tag{3.7.3}$$

where

$$\mathbf{w}_n(t) = (x_{0i}(\lambda_{ni}(y_{ni}(t))), i = 1, \dots, m), t \in [0, T].$$

In virtue of (3.7.2) and estimate (3.7.3), we see that it is sufficient to show that

$$\lim_{c\to 0} \overline{\lim_{n\to\infty}} \Delta_J(\mathbf{w}_n(t), c, T) = 0. \tag{3.7.4}$$

Let us introduce the functions

$$\mathbf{v}_n(t) = (x_{0i}(\lambda_{ni}(y_{ni}(\lambda_n^{-1}(t)))), i = 1, \dots, m), t \in [0, T],$$

and assume, for a moment, that we could show that

$$\lim_{c\to 0}\overline{\lim_{n\to\infty}}\,\Delta_J(\mathbf{v}_n(t), c, T) = 0. \tag{3.7.5}$$

Then, by applying Lemma 3.4.1 to the non-random functions $\mathbf{v}_n(t)$, $t \in [0, T]$, and $\lambda_n(t)$, $t \in [0, T]$, we would obtain (3.7.4).

Denote

$$R_n[t', t, t''] = \min\Big(\sum_{i=1}^{m} |x_{0i}(s_{ni}(t')) - x_{0i}(s_{ni}(t))|, \sum_{i=1}^{m} |x_{0i}(s_{ni}(t'')) - x_{0i}(s_{ni}(t))|\Big),$$

where

$$s_{ni}(t) = \lambda_{ni}(y_{ni}(\lambda_n^{-1}(t))), t \in [0, T],\ i = 1, \ldots, m.$$

To prove (3.7.5), it is sufficient to show that

$$\lim_{c\to 0}\overline{\lim_{n\to\infty}} \sup_{0\vee(t-c)\le t'\le t\le t''\le(t+c)\wedge T} R_n[t', t, t''] = 0. \tag{3.7.6}$$

Denote $\beta_{ni}(t) = s_{ni}(t) - y_{0i}(t)$, $t \in [0, T]$, $i = 1, \ldots, m$, and

$$\beta_n = \max_{1\le i\le m} \sup_{t\in[0,T]} |\beta_{ni}(t)|.$$

By using (3.7.1) and (3.7.2), we have

$$\begin{aligned}
\beta_n &\le \max_{1\le i\le m} \sup_{t\in[0,T]} |\lambda_{ni}(y_{ni}(\lambda_n^{-1}(t))) - y_{ni}(\lambda_n^{-1}(t))| \\
&\quad + \max_{1\le i\le m} \sup_{t\in[0,T]} |y_{ni}(\lambda_n^{-1}(t)) - y_{0i}(t)| \\
&\le \max_{1\le i\le m} \sup_{t\in[0,T']} |\lambda_{ni}(t) - t| \\
&\quad + \max_{1\le i\le m} \sup_{t\in[0,T]} |y_{ni}(t) - y_{0i}(\lambda_n(t))| \to 0 \text{ as } n \to \infty.
\end{aligned} \tag{3.7.7}$$

First of all we are going to estimate $R_n[t', t, t'']$ locally in a neighbourhood of a point $u \in [0, T]$.

We will say that the points t', t, $t'' \in [0, T]$ satisfy condition $\mathcal{A}_{u,c}$ if $t - c \le t' \le t \le t'' \le t + c$ and at least one of these points belongs to the interval $[u - c, u + c]$.

Take an arbitrary $\sigma > 0$. First we show that for every fixed point $u \in [0, T]$ there exist $c = c_u$ and a number n_u such that, if the points t', t, t'' satisfy condition $\mathcal{A}_{u,c_u}$ and $n \ge n_u$, then

$$R_n[t', t, t''] \le \sigma. \tag{3.7.8}$$

Three cases are possible.

(i). The point u is a point in which the function $\mathbf{y}_0(t)$ is discontinuous.

In this case, by condition $\mathcal{G}_6$, the function $x_{0i}(s)$ is continuous in the points $y_{0i}(u \pm 0)$, for every $i = 1, \dots, m$. Hence, there exists $\delta > 0$ such that, for all points s', $s'' \in [y_{0i}(u \pm 0) - \delta, y_{0i}(u \pm 0) + \delta]$, we have $|x_{0i}(s') - x_{0i}(s'')| \leq \sigma/m$, for every $i = 1, \dots, m$. There always exists $c = c_u$ such that

$$\max_{1 \leq i \leq m} \sup_{0 < s \leq 3c} |y_{0i}(u - 0) - y_{0i}(u - s)| \leq \delta/2 \tag{3.7.9}$$

and

$$\max_{1 \leq i \leq m} \sup_{0 \leq s \leq 3c} |y_{0i}(u) - y_{0i}(u + s)| \leq \delta/2. \tag{3.7.10}$$

If the points t', t, t'' satisfy condition $\mathcal{A}_{u,c_u}$, then $u - 2c \leq t' \leq t < u \leq t'' \leq u + c$ or $u - c \leq t' \leq u \leq t \leq t'' \leq u + 2c$.

Consider the first case (the second is absolutely similar). It follows from (3.7.9) and (3.7.10) that

$$\max_{1 \leq i \leq m} |y_{0i}(t) - y_{0i}(u - 0)| \leq \delta/2 \tag{3.7.11}$$

and

$$\max_{1 \leq i \leq m} |y_{0i}(t') - y_{0i}(u - 0)| \leq \delta/2. \tag{3.7.12}$$

Choose now n_u such that $\beta_n \leq \delta/2$ for $n \geq n_u$ (this can be done due to (3.7.7)). Then, by using (3.7.11) and (3.7.12), we get for $n \geq n_u$ that

$$|s_{ni}(t) - y_{0i}(u - 0)| \leq \beta_n + |y_{0i}(t) - y_{0i}(u - 0)| \leq \delta, \ i = 1, \dots, m, \tag{3.7.13}$$

and

$$|s_{ni}(t') - y_{0i}(u - 0)| \leq \beta_n + |y_{0i}(t) - y_{0i}(u - 0)| \leq \delta, \ i = 1, \dots, m. \tag{3.7.14}$$

By the choice of δ and relations (3.7.13) and (3.7.14),

$$R_n[t', t, t''] \leq \sum_{i=1}^{m} |x_{0i}(s_{ni}(t)) - x_{0i}(s_{ni}(t'))| \leq \sigma. \tag{3.7.15}$$

(ii). The function $\mathbf{y}_0(s)$ is continuous in the point u but there is i such that the function $x_{0i}(s)$ is discontinuous in the point $y_{0i}(u)$.

It is clear that there exists $\delta > 0$ such that

$$\sup_{0 < s', s'' \leq \delta} |x_{0i}(y_{0i}(u) - s') - x_{0i}(y_{0i}(u) - s'')| \leq \sigma/m, \tag{3.7.16}$$

and

$$\sup_{0 \leq s', s'' \leq \delta} |x_{0i}(y_{0i}(u) + s') - x_{0i}(y_{0i}(u) + s'')| \leq \sigma/m. \tag{3.7.17}$$

Let now c_u be chosen such that, if $t \in [u - 3c_u, u + 3c_u]$, then $|y_{0i}(t) - y_{0i}(u)| \leq \delta/2$. If n_u are chosen so that $\beta_n \leq \delta/2$ for $n \geq n_u$, then

$$|s_{ni}(t') - y_{0i}(u)| \vee |s_{ni}(t) - y_{0i}(u)| \vee |s_{ni}(t'') - y_{0i}(u)| \leq \delta \tag{3.7.18}$$

if the points t', t, t'' satisfy condition $\mathcal{A}_{u,c_u}$.

For the points $s_{ni}(t')$, $s_{ni}(t)$, and $s_{ni}(t')$, we have either $s_{ni}(t') \leq s_{ni}(t) < y_{0i}(u)$ or $y_{0i}(u) \leq s_{ni}(t) \leq s_{ni}(t'')$. By the definition, the functions $s_{ni}(t)$ are non-decreasing, and so in the first case, because of (3.7.16), (3.7.17), and (3.7.18), $|x_{0i}(s_{ni}(t'))-x_{0i}(s_{ni}(t))| \leq \sigma/m$. In the second case, by (3.7.16), (3.7.17), and (3.7.18), $|x_{0i}(s_{ni}(t)) - x_{0i}(s_{ni}(t''))| < \sigma/m$. In any case,

$$\min(|x_{0i}(s_{ni}(t)) - x_{0i}(s_{ni}(t'))|, |x_{0i}(s_{ni}(t)) - x_{0i}(s_{ni}(t''))|) \leq \sigma/m. \tag{3.7.19}$$

Because of condition $\mathcal{H}_4$, the function $x_{0j}(s)$ is continuous in the point $y_{0j}(u)$ for every $j \neq i$. Hence, there exists $\delta' > 0$ such that

$$\max_{j \neq i} \sup_{|s'|,|s''| \leq \delta'} |x_{0j}(y_{0j}(u) + s') - x_{0j}(y_{0j}(u) + s'')| \leq \sigma/m. \tag{3.7.20}$$

We can assume that c_u is chosen in such a way that

$$\max_{j \neq i} |y_{0j}(t) - y_{0j}(u)| \leq \delta'/2 \tag{3.7.21}$$

if $t \in [u - 3c_u, u + 3c_u]$, and also that the choice of n_u yields that $\beta_n \leq \delta'/2$ for $n \geq n_u$. Then,

$$\max_{j \neq i}(|s_{nj}(t) - y_{0j}(u)| \vee |s_{nj}(t') - y_{0j}(u)| \vee |s_{nj}(t'') - y_{0j}(u)|) \leq \delta' \tag{3.7.22}$$

for $n \geq n_u$ if the points t', t, t'' satisfy condition $\mathcal{A}_{u,c_u}$ (see also (3.7.13) and (3.7.14).

It follows from (3.7.20) and (3.7.22) that

$$\max_{j \neq i}(|x_{0j}(s_{nj}(t)) - x_{0j}(s_{nj}(t'))| \vee |x_{0j}(s_{nj}(t)) - x_{0j}(s_{nj}(t''))|) \leq \sigma/m. \tag{3.7.23}$$

Finally, we get using (3.7.19) and (3.7.23) that

$$\begin{aligned} R_n[t', t, t''] = & \\ &= \min\Big(\sum_{j=1}^{m} |x_{0j}(s_{nj}(t)) - x_{0j}(s_{nj}(t'))|, \sum_{j=1}^{m} |x_{0j}(s_{nj}(t)) - x_{0j}(s_{nj}(t''))|\Big) \\ &\leq \min(|x_{0i}(s_{ni}(t)) - x_{0i}(s_{ni}(t'))|, |x_{0i}(s_{ni}(t)) - x_{0i}(s_{ni}(t''))|) \\ &+ (m-1) \max_{j \neq i} \max(|x_{0j}(s_{nj}(t)) - x_{0j}(s_{nj}(t'))|, \\ &\quad |x_{0j}(s_{nj}(t)) - x_{0j}(s_{nj}(t''))|) \leq \sigma. \end{aligned} \tag{3.7.24}$$

(iii). The function $\mathbf{y}_0(s)$ is continuous in the point u and the function $x_{0j}(s)$ is continuous in the point $y_{0j}(u)$ for every $j = 1, \ldots, m$.

In this case, there exists $\delta > 0$ such that

$$\max_{1\leq j\leq m} \sup_{|s'|,|s''|\leq\delta} |x_{0j}(y_{0j}(u) + s') - x_{0j}(y_{0j}(u) + s'')| \leq \sigma/m, \tag{3.7.25}$$

and there is c_u such that

$$\max_{1\leq j\leq m} \sup_{|s|\leq 3c_u} |y_{0j}(u) - y_{0j}(u + s)| \leq \delta/2. \tag{3.7.26}$$

If we choose n_u in such a way that $\beta_n \leq \delta/2$ for $n \geq n_k$, then

$$\max_{1\leq j\leq m}(|s_{nj}(t') - y_{0j}(u)| \vee |s_{nj}(t) - y_{0j}(u)| \vee |s_{nj}(t'') - y_{0j}(u)|) \leq \delta \tag{3.7.27}$$

for points t', t, t'' satisfying condition $\mathcal{A}_{u,c_u}$ for $n \geq n_u$.

Finally, we get using and, by (3.7.25) and (3.7.27),

$$\begin{aligned} R_n[t', t, t''] \leq m \max_{1\leq j\leq m} \max(&|x_{0j}(s_{nj}(t')) - x_{0j}(s_{nj}(t))|, \\ &|x_{0j}(s_{nj}(t'')) - x_{0j}(s_{nj}(t))|) \leq \sigma. \end{aligned} \tag{3.7.28}$$

Now we split the interval $[0, T]$ into several domains depending on the location of points of discontinuity for the functions $\mathbf{x}(t)$ and $\mathbf{y}(t)$, and show that the corresponding estimates for $R_n[t', t, t'']$ are uniform in these domains.

Condition $\mathcal{A}_{u,c'}$ implies condition $\mathcal{A}_{u,c''}$ if $c' < c''$. So, for any $c' \leq c''$ and any finite collection of points $v_k \in [0, T]$, $k = 1, \ldots, r$, there exist c and a number n_0 (depending on the points v_k) such that, if points t', t, t'' satisfy one of conditions $\mathcal{A}_{v_k,c}$, $k = 1, \ldots, r$, and $n \geq n_0$, then $R_n[t', t, t''] \leq \sigma$.

Let $c_1 \in (0, T/2)$ and a number n_1 be chosen such that, if points t', t, t'' satisfy one of conditions $\mathcal{A}_{0,c_1}$ or $\mathcal{A}_{T,c_1}$ and $n \geq n_1$, then $R_n[t', t, t''] \leq \sigma$.

Let $z_{1i} < z_{2i} < \cdots < z_{k_i,i}$ be points, for every $i = 1, \ldots, m$, in which the function $x_{0i}(t)$ is discontinuous with absolute values of the jumps greater than or equal to σ/m. By Lemma 1.4.2, we can always choose h_0 such that, if $|t' - t''| \leq h_0$ and t' and t'' belong to one of the intervals $I_{0i} = [z_{0i}, z_{1i}), \ldots, I_{k_i-1,i} = [z_{k_i-1,i}, z_{k_i,i})$, $I_{k_i,i} = [z_{k_i,i}, z_{k_i+1,i}]$ (here $z_{0i} = 0$, $z_{k_i+1,i} = T$), then

$$\max_{1\leq i\leq m} |x_{0i}(t') - x_{0i}(t'')| \leq \sigma/m. \tag{3.7.29}$$

Let $J_i = \{r_{1,i}, \ldots, r_{l_i,i}\}$ be a set of indices r for which there exists $s \in [c_1, T - c_1]$ such that $y_{0i}(s) = z_{ri}$ (by condition $\mathcal{G}_6$, the functions $y_{0j}(\cdot)$, $j = 1, \ldots, m$ are continuous in each such point). By the definition, $l_i \leq k_i$, $i = 1, \ldots, m$. For $r \in J_i$, define

$$v^-_{ri} = \inf(s \in [c_1, T - c_1] \colon y_{0i}(s) = z_{ri}), \ v^+_{ri} = \sup(s \in [c_1, T - c_1] \colon y_{0i}(s) = z_{ri}).$$

By the foregoing remark, there exist $c_2 < c_1$ and a number $n_2 > n_1$ such that, if points t', t, t'' satisfy one of conditions $\mathcal{A}_{v^{\pm}_{ri},c_2}$, $r \in J_i$, $i = 1, \dots, m$, and $n \geq n_2$, then $R_n[t', t, t''] \leq \sigma$.

Denote

$$U = [c_1, T - c_1] \setminus \bigcup_{i=1}^{m} \bigcup_{r \in J_i} (v^-_{ri} - c_2, v^+_{ri} + c_2).$$

By the definition, the set $U = \cup_{1 \leq l \leq l_0} [a_l, b_l]$ is the union of a finite number of closed intervals.

By the construction of the set U and conditions $\mathcal{G}_6$ and $\mathcal{H}_4$, $y_{0i}(t \pm 0) \neq z_{ri}$ for all $r = 1, \dots, k_i$, $i = 1, \dots, m$, and every $t \in U$. Since the functions $y_{0i}(t)$ belong to the space $\mathbf{D}^{(1)}_{[0,\infty)+}$, this implies that there exists $\gamma > 0$ such that

$$\min_{i=1,\dots,m} \min_{1 \leq r \leq k_i} \inf_{t \in U} |y_{0i}(t \pm 0) - z_{ri}| > \gamma. \tag{3.7.30}$$

Now we show that every closed interval $[a, b] \subseteq U$ on which the function $y_{0i}(s)$ does not have jumps exceeding in magnitude $\gamma/2$ has the following property (recall that the functions $y_{0i}(s), i = 1, \dots, m$ are non-decreasing). If $y_{0i}(t_0) \in I_{ri}$ for some point $t_0 \in [a, b]$, then: **(a)** $y_{0i}(t) \in I_{ri}$ for all $t \in [a, b]$; **(b)** $y_{0i}(b) \leq z_{r+1i} - \frac{\gamma}{2}$ (if $r = 0, \dots, k_i - 1$); **(c)** $y_{0i}(a) \geq z_{ri} + \frac{\gamma}{2}$, $t \in [a, b]$ (if $r = 1, \dots, k_i$). This is true for every $i = 1, \dots, m$.

Indeed, if $y_{0i}(t_0) \in I_{ri}$, then $z_{r+1i} - z_{ri} \geq \gamma$. This follows from (3.7.30). Denote $\tau^+ = \inf(s \geq t_0 \colon y_{0i}(s) > z_{r+li} - \gamma/2)$ and $\tau^- = \sup(s \leq t_0 \colon y_{0i}(s - 0) < z_{ri} + \gamma/2)$. Clearly, it is sufficient to show that $\tau^+ \notin [a, b]$ if $r = 0, \dots, k_i - 1$, and $\tau^- \notin [a, b]$ if $r = 1, \dots, k_i$. Let us consider, for example, the first case (the second is similar). Suppose that $\tau^+ \in [a, b]$. Because $y_{0i}(t_0) < z_{r+1i} - \gamma$ by (3.7.30), $\tau^+ > t_0$. So $y_{0i}(\tau^+ - 0) \geq y_{0i}(t_0)$, since the function $y_{0i}(s)$ is monotone. On the other hand, $y_{0i}(\tau^+ - 0) \leq z_{r+1,i} - \gamma/2$. So $y_{0i}(\tau^+ - 0) \in I_{ri}$ and, hence, $y_{0i}(\tau^+ - 0) < z_{r+1i} - \gamma$ by (3.7.30). But then the function $y_{0i}(s)$ has, in the point τ^+, a jump of magnitude greater than or equal to $\gamma/2$, which contradicts the assumption.

Now choose $h < \min(\gamma/2, h_0/2)$, and let $u_k = u^{(h)}_k$, $k = 1, \dots, k_0$, be points in which at least one of the functions $y_{0i}(s)$ has a jump with magnitude greater than or equal to h (note that, in other points of the interval $[0, T]$, all the functions $y_{0i}(s)$, $i = 1, \dots, m$, do not have jumps with magnitude greater than or equal to h).

Choose now $c_3 \leq c_2$ and a number $n_3 \geq n_2$ such that, if points t', t, t'' satisfy one of conditions $\mathcal{A}_{u_k,c_3}$, $k = 1, \dots, k_0$, and $n \geq n_3$, then $R_n[t', t, t''] \leq \sigma$.

Denote

$$V = U \setminus \bigcup_{k=1}^{k_0} (u_k - c_3, u_k + c_3).$$

It is easy to see that three points t', t, $t'' \in [0, T]$ such that $t - c \leq t' \leq t \leq t'' \leq t + c$ can lie in the interval $[0, T]$ in one of the following ways: **(i)** at least one of the points t', t, t'' belongs to $[0, c_1]$ or $[T - c_1, T]$; **(ii)** at least one of the points t', t, t'' belongs to $[v^{\pm}_{ri} - c_2, v^{\pm}_{ri} + c_2]$ for some r, i; **(iii)** at least one of the points t', t, t'' belongs to $[u_k - c_3, u_k + c_3]$ for some k; **(iv)** all three points t', t, t'' belong to the set V; **(v)** all three points t', t, t'' belong to the interval $[v^{-}_{ri}, v^{+}_{ri}]$ for some r, i.

It was shown before that in cases **(i)** – **(iii)**, due to the choice of $c_1 \geq c_2 \geq c_3$ and $n_1 \leq n_2 \leq n_3$, $R_n[t', t, t''] \leq \sigma$ if $c \leq c_3$ and $n \geq n_3$. So, consider now cases **(iv)** and **(v)**.

By the definition, the set $V = \cup_{1 \leq l \leq l'_0}[a'_l, b'_l]$ is the union of a finite number of closed intervals. We can assume that $a'_1 \leq b'_1 < a'_2 \leq b'_2 < \dots a'_{l'_0} \leq b'_{l'_0}$.

On every interval $[a'_l, b'_l]$, the functions $y_{0i}(s)$, $i = 1, \dots, m$, do not have jumps with the values greater than or equal to $\gamma/2$ and, hence, for every $l = 1, \dots, l'_0$ and $i = 1, \dots, m$ there is an interval $I_{r_{l,i},i}$ such that: **(d)** $y_{0i}(t) \in I_{r_{l,i},i}$, $t \in [a'_l, b'_l]$; **(e)** $y_{0i}(b'_l) \leq z_{r_{l,i}+1,i} + \gamma/2$ (if $r_{l,i} = 0, \dots, k_i - 1$); **(f)** $y_{0i}(a'_l) \geq z_{r_{l,i},i} - \gamma/2$ (if $r_{l,i} = 1, \dots, k_i$).

Choose $n_4 \geq n_3$ such that $\beta_n \leq \min(\gamma/4, h_0/4)$ for $n \geq n_4$. Then it is clear that, for all $n \geq n_4$,

$$s_{ni}(t) \in I_{r_{l,i},i}, \ t \in [a'_l, b'_l], \ i = 1, \dots, m. \tag{3.7.31}$$

Since the functions $y_{0i}(s)$, $i = 1, \dots, m$, do not have jumps in the interval $[a'_l, b'_l]$ with the values greater than or equal to $h_0/2$, by the construction of the set V and Lemma 1.4.2, there exists $c_4 \leq c_3$ such that for $t', t'' \in [a'_l, b'_l]$ and $|t' - t''| \leq c_4$,

$$\max_{i=1,\dots,m} |y_{0i}(t') - y_{0i}(t'')| \leq h_0/2. \tag{3.7.32}$$

By the choice of n_4 and (3.7.32), it follows that if $t', t'' \in [a'_l, b'_l]$, $|t' - t''| \leq c_4$ then, for $n \geq n_4$,

$$\max_{i=1,\dots,m} |s_{ni}(t') - s_{ni}(t'')| \leq h_0. \tag{3.7.33}$$

It follows from (3.7.31), (3.7.33), and (3.7.29) that if $|t' - t''| \leq c_4$ then, for $n \geq n_4$,

$$\max_{1 \leq i \leq m} |x_{0i}(s_{ni}(t')) - x_{0i}(s_{ni}(t''))| \leq \sigma/m. \tag{3.7.34}$$

Relation (3.7.34) implies, in its turn, that if $t', t, t'' \in [a'_l, b'_l]$ and $t - c_4 \leq t' \leq t \leq t'' \leq t + c_4$ then, for $n \geq n_4$,

$$\begin{aligned} R_n[t', t, t''] \leq m \max_{1 \leq i \leq m} \max(&|x_{0i}(s_{ni}(t')) - x_{0i}(s_{ni}(t))|, \\ &|x_{0i}(s_{ni}(t'') - x_{0i}(s_{ni}(t))|) \leq \sigma. \end{aligned} \tag{3.7.35}$$

Clearly, c_4 and then the number n_4 can be chosen such that relations (3.7.31) – (3.7.35) hold for all $l = 1, \dots, l'_0$. We can also assume that c_4 is chosen to satisfy

$$d = \min_{0 \leq l \leq l'_0 - 1} (a'_{l+1} - b'_l) \geq 3c_4. \tag{3.7.36}$$

Relation (3.7.36) implies that, if $t', t, t'' \in V$ and $t - c_4 \leq t' \leq t \leq t'' \leq t + c_4$, then all three points t', t, t'' belong to one of the intervals $[a'_l, b'_l]$.

So, finally, we get that, if $t', t, t'' \in V$ and $t - c_4 \le t' \le t \le t'' \le t + c_4$, then $R_n[t', t, t''] \le \sigma$ for all $n \ge n_4$.

To finish the proof of the lemma, it only remains to consider the last case **(v)** and show that there exist $c_5 \le c_4$ and a number $n_5 \ge n_4$ such that, if the points t', t, t'' belong to the interval $[v^-_{r,i}, v^+_{r,i}]$ for some r and i, and $t - c_5 \le t' \le t \le t'' \le t + c_5$, then $R_n[t', t, t''] \le \sigma$ for $n \ge n_5$.

In this case, $y_{0i}(t) = z_{ri}$ and $t \in [v^-_{ri}, v^+_{ri}]$. Indeed, by the definition, $y_{0i}(t) = z_{ri}$ for $t \in (v^-_{ri}, v^+_{ri})$. But, condition $\mathcal{G}_6$ implies that the function $y_{0i}(t)$ is continuous in the points $v^{\pm}_{ri}$. Let $\delta > 0$ be such that

$$\sup_{0<s',s''\le\delta} |x_{0i}(z_{ri} - s') - x_{0i}(z_{ri} - s'')| \le \sigma/m \tag{3.7.37}$$

and

$$\sup_{0\le s',s''\le\delta} |x_{0i}(z_{ri} + s') - x_{0i}(z_{ri} + s'')| \le \sigma/m. \tag{3.7.38}$$

Choose now $n_5 \ge n_4$ so that $\beta_n \le \delta/2$ for $n \ge n_5$. Then, for all $t \in [v^-_{ri}, v^+_{ri}]$,

$$|s_{ni}(t) - z_{ri}| \le \delta. \tag{3.7.39}$$

Because $s_{ni}(t') \le s_{ni}(t) < z_{ri}$ or $z_{ri} \le s_{ni}(t) \le s_{ni}(t'')$, it follows from (3.7.37), (3.7.38), and (3.7.39) that

$$\min(|x_{0i}(s_{ni}(t')) - x_{0i}(s_{ni}(t))|, |x_{0i}(s_{ni}(t)) - x_{0i}(s_{ni}(t''))|) \le \sigma/m. \tag{3.7.40}$$

Condition $\mathcal{G}_6$ implies that the functions $y_{0j}(s)$, $j \ne i$ are continuous on the interval $[v^-_{ri}, v^+_{ri}]$. Indeed, assume that $s \in [v^-_{ri}, v^+_{ri}]$ and it is a point of discontinuity for the function $y_{0j}(s)$ for some $j \ne i$. Then, according $\mathcal{G}_6$, $y_{0i}(s) \notin R[x_{0i}(\cdot)]$ that contradicts the equality $y_{0i}(s) = z_{ri}$.

Also, condition $\mathcal{H}_4$ implies that the function $x_{0j}(s)$, is continuous on the interval $[y_{0j}(v^-_{ri}), y_{0j}(v^+_{ri})]$, for every $j \ne i$. Indeed, according $\mathcal{H}_4$, the function $x_{0j}(t)$ must be continuous in the point $y_{0j}(s)$ for every $s \in [v^-_{ri}, v^+_{ri}]$ and $j \ne i$, since $x_{0i}(s) = z_{ri} \in R[x_{0i}(\cdot)]$ for $s \in [v^-_{ri}, v^+_{ri}]$. But the functions $y_{0j}(s)$, $j \ne i$ are monotone and continuous on the interval $[v^-_{ri}, v^+_{ri}]$. Thus, $y_{0j}(s)$ takes all values in the interval $[y_{0j}(v^-_{ri}), y_{0j}(v^+_{ri})]$ when s runs through all values in the interval $[v^-_{ri}, v^+_{ri}]$.

Continuity of the functions $x_{0j}(s)$, $j \ne i$ on the interval $[y_{0j}(v^-_{ri}), y_{0j}(v^+_{ri})]$ implies that there is $h' > 0$ such that, for $|s' - s''| \le h'$, $s', s'' \in [y_{0j}(v^-_{ri}) - h', y_{0j}(v^+_{ri}) + h']$, and $j \ne i$,

$$|x_{0j}(s') - x_{0j}(s'')| \le \sigma/m. \tag{3.7.41}$$

Since the functions $y_{0j}(s)$, $j \ne i$ are continuous on the interval $[v^-_{ri}, v^+_{ri}]$, it follows that there is $c_5 \le c_4$ such that, for $|t' - t''| \le c_5$, $t', t'' \in [v^-_{ri}, v^+_{ri}]$, and $j \ne i$,

$$|y_{0j}(t') - y_{0j}(t'')| \le h'/2. \tag{3.7.42}$$

The number n_5 can be chosen in such a way that $\beta_n \leq h'/4$ for $n \geq n_5$. Then, obviously, $s_{nj}(t) \in [y_{0j}(v^-_{ri}) - h', y_{0j}(v^+_{ri}) + h']$ for $t \in [v^-_{ri}, v^+_{ri}]$, $j \neq i$, and for $|t' - t''| \leq c_5$, $t', t'' \in [v^-_{ri}, v^+_{ri}]$, and $j \neq i$,

$$|s_{nj}(t') - s_{nj}(t'')| \leq h'. \tag{3.7.43}$$

If follows from relation (3.7.42) and (3.7.43) that, for all points $|t' - t''| \leq c_5$, t', $t'' \in [v^-_{ri}, v^+_{ri}]$ and $n \geq n_5$,

$$\max_{j \neq i} |x_{0j}(s_{nj}(t')) - x_{0j}(s_{nj}(t''))| \leq \sigma/m. \tag{3.7.44}$$

Finally we see, by relations (3.7.40) and (3.7.44), that

$$\begin{aligned} R_n[t', t, t''] &\leq \\ &\leq \min(|x_{0i}(s_{ni}(t')) - x_{0i}(s_{ni}(t''))|, |x_{0i}(s_{ni}(t'')) - x_{0i}(s_{ni}(t))|) \\ &+ (m-1) \max_{j \neq i} \max(|x_{0j}(s_{nj}(t')) - x_{0j}(s_{nj}(t))|, \\ &\quad |x_{0j}(s_{nj}(t'')) - x_{0j}(s_{nj}(t))|) \leq \sigma \end{aligned} \tag{3.7.45}$$

for $n \geq n_5$ if the points $t', t, t'' \in [v^-_{ri}, v^+_{ri}]$, and $t - c_5 \leq t' \leq t \leq t'' \leq t + c_5$.

It only remains to note that c_5 and the number n_5 can be chosen for all r and i simultaneously.

The proof of the lemma is completed. □

3.7.2. J-convergence of vector compositions of non-random càdlàg functions. First of all note that, as follows from the examples considered in Section 3.1, conditions $\mathcal{A}_{44}$, $\mathcal{J}_{17}$, $\mathcal{A}_{45}$, $\mathcal{J}_{18}$, $\mathcal{G}_6$, and $\mathcal{H}_4$ of Lemma 3.7.1 do not necessarily imply **J**-convergence of the vector compositions $\mathbf{z}_n(t)$, $t \geq 0$. However, these conditions do imply **J**-compactness of these functions, although they do not guarantee pointwise convergence of the functions $\mathbf{z}_n(t)$ on a set dense in $[0, \infty)$ and containing the point 0. Some additional conditions should be imposed.

Let us introduce the following conditions:

$\mathcal{C}_{12}$: There exists a set W such that (a) $y_{0i}(t) \notin R[x_{0i}(\cdot)]$, $i = 1, \ldots, m$, for $t \in W$; (b) W is a subset of $[0, \infty)$ that is dense in this interval and contains the point 0;

$\mathcal{E}_{10}$: There do not exist points $0 \leq t' < t'' < \infty$ and $i = 1, \ldots, m$ such that $y_{0i}(t') = y_{0i}(t'') \in R[x_{0i}(\cdot)]$;

and

$\mathcal{C}^{(w)}_{13}$: $y_{0i}(w) \notin R[x_{0i}(\cdot)]$, $i = 1, \ldots, m$.

Conditions $\mathcal{E}_{10}$ and $\mathcal{C}_{13}^{(w)}$ coincide, respectively, with conditions $\mathcal{E}_4$ and $\mathcal{C}_6^{(w)}$ in the case of non-random functions $\mathbf{x}_0(t)$, $t \geq 0$, and any $\mathbf{y}_0(t)$, $t \geq 0$. These non-random functions replace in this case, respectively, the stochastic processes $\boldsymbol{\xi}_0(t), t \geq 0$ and $\mathbf{v}_0(t)$, $t \geq 0$.

As follows from Lemma 2.7.2, condition $\mathcal{E}_{10}$ and $\mathcal{C}_{13}^{(0)}$ are necessary and sufficient for existence of a set W dense in $[0, \infty)$, containing 0, and such that condition $\mathcal{C}_{12}$ holds with this set W.

Let W_0 denote the set of all point $w \geq 0$ for which condition $\mathcal{C}_{13}^{(w)}$ holds. Obviously, $W \subseteq W_0$ for any set W that can appear in condition $\mathcal{C}_{12}$. So, under condition $\mathcal{C}_{12}$ or conditions $\mathcal{E}_{10}$ and $\mathcal{C}_{13}^{(0)}$, the set W_0 is the interval $[0, \infty)$, except for at most a countable set, and $0 \in W_0$.

Denote $Y_0 = \cap_{i=1}^{m}(Y_i \cup Y_{0i})$, where Y_{0i} is the set of continuity points of the function $y_{0i}(t), t \geq 0$. Let also $Z_0 = Y_0 \cap W_0$. This set is also $[0, \infty)$, except for at most a countable set. Also, $0 \in Z_0$.

Lemma 3.7.2. *Let conditions $\mathcal{A}_{44}$, $\mathcal{J}_{17}$, $\mathcal{A}_{45}$, $\mathcal{E}_{10}$, and $\mathcal{C}_{13}^{(0)}$ hold. Then*

$$\mathbf{z}_n(t) \to \mathbf{z}_0(t) \text{ as } n \to \infty, \ t \in Z_0.$$

Proof of Lemma 3.7.2. The proof can be obtained by applying Theorem 2.7.6 to the vector compositions $\mathbf{z}_n(t) = (x_{n1}(y_{n1}(t)), \dots, x_{nm}(y_{nm}(t))), t \geq 0$ of the vector càdlàg functions $\mathbf{x}_n(t), t \geq 0$ and $\mathbf{y}_n(t), t \geq 0$. Here n^{-1} can be regarded as the parameter ε.

Note that, for every $i = 1, \dots, m$, the set Y_i in condition $\mathcal{A}_{45}$ can be enlarged to the set $Y_i \cup Y_{0i}$, since monotonicity of the functions $y_{n1}(t), i = 1, \dots, m$.

Conditions $\mathcal{A}_{44}$ and $\mathcal{A}_{45}$ imply that the weak convergence condition $\mathcal{A}_{22}^{\mathbf{V}}$ holds with the set $V = Y_0$. The condition $\mathcal{J}_{17}$ implies that the condition of **J**-compactness $\mathcal{J}_8$ holds. Finally, condition $\mathcal{E}_{10}$ and $\mathcal{C}_{13}^{(0)}$ imply that the continuity condition $\mathcal{E}_4$ holds. In this case, the set of weak convergence S_0 in Theorem 2.7.6 coincides with the set Z_0. □

General conditions of **J**-convergence for compositions of càdlàg functions can be obtained by combining the conditions of Lemmas 3.7.1 and 3.7.2.

Lemma 3.7.3. *Let conditions $\mathcal{A}_{44}$, $\mathcal{J}_{17}$, $\mathcal{A}_{45}$, $\mathcal{J}_{18}$, $\mathcal{G}_6$, $\mathcal{H}_4$, $\mathcal{E}_{10}$, and $\mathcal{C}_{13}^{(0)}$ hold. Then*

$$\mathbf{z}_n(t), t \geq 0 \xrightarrow{\mathbf{J}} \mathbf{z}_0(t), t \geq 0 \text{ as } n \to \infty.$$

3.7.3. J-convergence of vector non-random càdlàg functions. Let us consider the case where the internal functions $y_{ni}(t) = t$, $t \geq 0$, for $i = 1, \dots, m$. In this case, conditions $\mathcal{A}_{45}$, $\mathcal{J}_{18}$, $\mathcal{G}_6$, $\mathcal{E}_{10}$, and $\mathcal{C}_{13}^{(0)}$ obviously hold.

Condition $\mathcal{H}_4$ now takes the following form:

$\mathcal{H}_5$: $\sum_{i=1}^{m} \chi(t \in R[x_{0i}(\cdot)]) \leq 1$ for $t \geq 0$, or, equivalently,

$\mathcal{H}_5'$: $R[x_{0i}(\cdot)] \cap R[x_{0j}(\cdot)] = \varnothing$ for $i \neq j$.

In this case, the functions $\mathbf{x}_n(t)$, $t \geq 0$ and $\mathbf{z}_n(t)$, $t \geq 0$ coincide. By applying Lemma 3.7.3 one can obtain the following simple conditions for $\mathbf{J}$-convergence of the vector càdlàg functions $\mathbf{x}_n(t)$, $t \geq 0$. This result is due to Whitt (1973, 1980).

Lemma 3.7.4. *Let conditions $\mathcal{A}_{44}$, $\mathcal{J}_{17}$, and $\mathcal{H}_5$ hold. Then*

$$\mathbf{x}_n(t), t \geq 0 \xrightarrow{\mathbf{J}} \mathbf{x}_0(t), t \geq 0 \text{ as } n \to \infty.$$

Note that condition $\mathcal{J}_{17}$ requires $\mathbf{J}$-compactness of the components $x_{ni}(t)$, $t \geq 0$ separately for every $i = 1, \ldots, m$. It may happen that, under conditions $\mathcal{A}_{44}$ and $\mathcal{J}_{17}$, the vector functions $\mathbf{x}_n(t)$, $t \geq 0$ are not $\mathbf{J}$-compact and do not $\mathbf{J}$-converge. Condition $\mathcal{H}_5$ is a condition additional to conditions $\mathcal{A}_{44}$ and $\mathcal{J}_{17}$, in order to provide $\mathbf{J}$-compactness and $\mathbf{J}$-convergence of the vector functions $\mathbf{x}_n(t)$, $t \geq 0$.

Let us now go back to the case of general compositions $\mathbf{z}_n(t)$, $t \geq 0$. Condition $\mathcal{H}_5$ takes the following form:

$\mathcal{H}_6$: $\sum_{i=1}^m \chi(t \in R[x_{0i}(y_{0i}(\cdot))]) \leq 1$ for all $t \geq 0$;

or, equivalently,

$\mathcal{H}_6'$: $R[x_{0i}(y_{0i}(\cdot))] \cap R[x_{0j}(y_{0j}(\cdot))] = \varnothing$ for $i \neq j$.

Lemma 3.7.4 allows to formulate the following conditions of $\mathbf{J}$-convergence for vector compositions of non-random càdlàg functions. These conditions make an alternative to those given above in Lemma 3.7.1.

Let us replace condition $\mathcal{J}_{17}$ with the weaker condition:

$\mathcal{J}_{19}$: $\lim_{c\to 0} \overline{\lim_{n\to\infty}} \Delta_J(y_{ni}(\cdot), c, T) = 0,\ T > 0,\ i = 1, \ldots, m.$

Lemma 3.7.5. *Let conditions $\mathcal{A}_{44}$, $\mathcal{J}_{17}$, $\mathcal{A}_{45}$, $\mathcal{J}_{19}$, $\mathcal{G}_6$, $\mathcal{H}_6$, $\mathcal{E}_{10}$, and $\mathcal{C}_{13}^{(0)}$ hold. Then*

$$\mathbf{z}_n(t), t \geq 0 \xrightarrow{\mathbf{J}} \mathbf{z}_0(t), t \geq 0 \text{ as } n \to \infty.$$

Proof of Lemma 3.7.5. Conditions $\mathcal{A}_{44}$, $\mathcal{J}_{17}$, $\mathcal{A}_{45}$, $\mathcal{J}_{19}$, $\mathcal{G}_6$, $\mathcal{E}_{10}$, and $\mathcal{C}_{13}^{(0)}$ imply that the functions $x_{ni}(t)$, $t \geq 0$ and $y_{ni}(t)$, $t \geq 0$ satisfy conditions $\mathcal{A}_{38}$, $\mathcal{J}_{13}$, $\mathcal{A}_{39}$, $\mathcal{J}_{14}$, $\mathcal{G}_2$, $\mathcal{E}_7$, and $\mathcal{C}_{10}^{(0)}$ of Lemma 3.5.3 for every $i = 1, \ldots, m$. By applying Lemma 3.5.3 to these functions, we prove that $x_{ni}(y_{ni}(t)), t \geq 0 \xrightarrow{\mathbf{J}} x_{0i}(y_{0i}(t)), t \geq 0$ as $n \to \infty$ for every $i = 1, \ldots, m$. Also, condition $\mathcal{H}_6$ coincides with condition $\mathcal{H}_5$ for the functions $z_{ni}(t) = x_{ni}(y_{ni}(t))$, $t \geq 0$, $i = 1, \ldots, m$. Now, by applying Lemma 3.7.4 to the functions $z_{ni}(t) = x_{ni}(y_{ni}(t))$, $t \geq 0$, $i = 1, \ldots, m$, we prove Lemma 3.7.5. □

Let us explain the difference between conditions of **J**-convergence given in Lemmas 3.7.3 and 3.7.5.

Conditions $\mathcal{H}_6$ and $\mathcal{H}_4$ used in these lemmas are not equivalent.

Condition $\mathcal{H}_4$ only excludes the situation where **(a)** two or more functions $x_{0i}(\cdot)$ have synchronous jumps in points $y_{0i}(t)$ for some $t \geq 0$.

Condition $\mathcal{H}_6$ prohibits the case **(a)** and, usually, also the case where **(b)** two or more functions $y_{0i}(\cdot)$ have simultaneous jumps in a point t for some $t \geq 0$.

At the same time, condition $\mathcal{J}_{18}$ is stronger than condition $\mathcal{J}_{19}$.

However, conditions $\mathcal{J}_{18}$ and $\mathcal{J}_{19}$ are equivalent if **(c)** two or more functions $y_{0i}(\cdot)$ have not simultaneous jumps, i.e., condition $\mathcal{H}_5$ holds for these functions.

3.7.4. The finite interval $[0, T]$. The statements of Lemmas 3.7.1 – 3.7.3 can easily be reduced to the case of a finite interval $[0, T]$ in the same way as it was done for semi-vector compositions of non-random càdlàg functions in Section 3.5.

Conditions $\mathcal{A}_{44}$ and $\mathcal{J}_{17}$ do not require any changes. But conditions $\mathcal{A}_{45}$ and $\mathcal{J}_{18}$ have to be taken in the following form:

$\mathcal{A}_{46}$: $y_{ni}(t) \to y_{0i}(t)$ as $n \to \infty$, $t \in Y_i$, $i = 1, \ldots, m$, where Y_i are subsets of $[0, T]$ that are dense in this interval and contain the points 0 and T;

and

$\mathcal{J}_{20}$: $\lim_{c\to 0} \overline{\lim}_{n\to\infty} \Delta_J(\mathbf{y}_n(\cdot), c, T) = 0,\ T > 0.$

Denote by $R_T[y(\cdot)] = R[y(\cdot)] \cap [0, T]$ the set of points of discontinuity for a càdlàg function $y(t)$, $t \geq 0$, in the interval $[0, T]$. Conditions $\mathcal{G}_6$ and $\mathcal{H}_4$ must be taken in the following form:

$\mathcal{G}_7$: $y_{0i}(t \pm 0) \notin R[x_{0i}(\cdot)]$, $i = 1, \ldots, m$ for $t \in \cup_{i=1}^m R_T[y_{0i}(\cdot)]$;

and

$\mathcal{H}_7$: $\sum_{i=1}^m \chi(y_{0i}(t) \in R[x_{0i}(\cdot)]) \leq 1$ for $t \in [0, T]$.

Let us first formulate an analogue of Lemma 3.7.1.

Lemma 3.7.6. *Let conditions $\mathcal{A}_{44}$, $\mathcal{J}_{17}$, $\mathcal{A}_{46}$, $\mathcal{J}_{20}$, $\mathcal{G}_7$, and $\mathcal{H}_7$ hold. Then*

$$\lim_{c\to 0} \overline{\lim_{n\to\infty}} \Delta_J(\mathbf{z}_n(\cdot), c, T) = 0.$$

Proof of Lemma 3.7.6. The consideration can be reduced to the case of the semi-infinite interval $[0, \infty)$ by applying Lemma 3.7.1 to the functions $\mathbf{x}_n(t)$, $t \geq 0$ and $\mathbf{y}_n(t) = \mathbf{y}_n(t \wedge T)$, $t \geq 0$. It is obvious that conditions $\mathcal{A}_{44}$, $\mathcal{J}_{17}$, $\mathcal{A}_{46}$, $\mathcal{J}_{20}$, $\mathcal{G}_7$, and $\mathcal{H}_7$ imply that these functions satisfy conditions $\mathcal{A}_{44}$, $\mathcal{J}_{17}$, $\mathcal{A}_{45}$, $\mathcal{J}_{18}$, $\mathcal{G}_6$, and $\mathcal{H}_4$. By applying Lemma 3.7.1, we get the relation of **J**-compactness for the functions $\mathbf{z}_n(t)$, $t \geq 0$ on the intervals $[0, T']$ for $T' > 0$. For $T' \geq T$, this relation coincides with the relation of **J**-compactness given in Lemma 3.7.6. □

An analogous reduction to the case of a finite interval can be carried out for Lemmas 3.7.2 and 3.7.3.

In this case, we should add to $\mathcal{A}_{46}$ the assumption that the left limits $\mathbf{y}_n(T-0)$ converge,

$\mathcal{A}_{47}$: (a) $y_{ni}(t) \to y_{0i}(t)$ as $n \to \infty$ for $t \in Y_i$, $i = 1, \dots, m$, where Y_i are subsets of $[0, T]$ that are dense in this interval and contain the points 0 and T;

(b) $y_{ni}(T-0) \to y_{0i}(T-0)$ as $n \to \infty$, $i = 1, \dots, m$.

Conditions $\mathcal{C}_{12}$ and $\mathcal{E}_{10}$ should also be modified in the following way:

$\mathcal{C}_{14}$: There exists a set W such that (a) $y_{0i}(t) \notin R[x_{0i}(\cdot)]$, $i = 1, \dots, m$, for $t \in W$, (b) W is a subset of $[0, T]$ that is dense in this interval and contains the points 0 and T;

and

$\mathcal{E}_{11}$: There do not exist points $0 \leq t' < t'' \leq T$ and $i = 1, \dots, m$ such that $y_{0i}(t') = y_{0i}(t'') \in R_T[x_{0i}(\cdot)]$.

As follows from Lemma 2.7.2, conditions $\mathcal{E}_{11}$, $\mathcal{C}_{13}^{(0)}$, and $\mathcal{C}_{13}^{(T)}$ are necessary and sufficient for existence of a set W dense in $[0, T]$, containing $0, T$ and such that condition $\mathcal{C}_{14}$ holds with this set W.

Let $W_0(T)$ denote the set of all points $w \in [0, T]$ that satisfy condition $\mathcal{C}_{13}^{(w)}$. Obviously, $W \subseteq W_0(T)$ for any set W that can appear in condition $\mathcal{C}_{14}$. So, if condition $\mathcal{C}_{14}$ or conditions $\mathcal{E}_{11}$, $\mathcal{C}_{13}^{(0)}$, and $\mathcal{C}_{13}^{(T)}$ hold, then the set $W_0(T)$ coincides with the interval $[0, T]$, except for at most a countable set. Also, $0, T \in W_0$.

Denote $Y_0(T) = \cap_{i=1}^m (Y_i \cup Y_{0i}(T))$, where $Y_{0i}(T)$ is the set of continuity points of function $y_{0i}(t)$, $t \in [0, T]$. Let also $Z_0(T) = Y_0(T) \cap W_0(T)$. This set is $[0, T]$, except for at most a countable set. Also, $0, T \in Z_0(T)$.

Lemma 3.7.7. *Let conditions $\mathcal{A}_{44}$, $\mathcal{J}_{17}$, $\mathcal{A}_{47}$, $\mathcal{E}_{11}$, $\mathcal{C}_{13}^{(0)}$, and $\mathcal{C}_{13}^{(T)}$ hold. Then*

$$\mathbf{z}_n(t) \to \mathbf{z}_0(t) \text{ as } n \to \infty, \ t \in Z_0(T).$$

If, additionally, condition $\mathcal{C}_{13}^{(T-)}$ holds, then also $\mathbf{z}_n(T-0) \to \mathbf{z}_0(T-0)$ as $n \to \infty$.

Proof of Lemma 3.7.7. To obtain pointwise convergence of the compositions $\mathbf{z}_n(t)$ to the functions $\mathbf{z}_0(t)$ at points from the set $Z_0(T)$, it will suffice to apply Lemma 3.7.2 to the functions $\mathbf{x}_n(t)$, $t \geq 0$ and $\mathbf{y}_n(t) = \mathbf{y}_n(t \wedge T)$, $t \geq 0$. Conditions $\mathcal{A}_{44}$, $\mathcal{J}_{17}$, $\mathcal{A}_{47}$, $\mathcal{E}_{11}$, $\mathcal{C}_{13}^{(0)}$, and $\mathcal{C}_{13}^{(T)}$ imply that these functions satisfy conditions $\mathcal{A}_{44}$, $\mathcal{J}_{17}$, $\mathcal{A}_{45}$, $\mathcal{E}_{10}$, $\mathcal{C}_{13}^{(0)}$, and $\mathcal{C}_{13}^{(T)}$. The proof of Lemma 3.7.2 is based Theorem 2.7.6. This theorem, in its turn, is based on Theorem 2.3.4. To prove that $\mathbf{z}_n(T-0)$ converges to $\mathbf{z}_0(T-0)$ as $n \to \infty$, one can apply Theorem 2.3.4 to the non-random functions $\mathbf{x}_n(t)$, $t \geq 0$ and the vector stopping moments $\mathbf{y}_n(T-0)$. In this case, the conditions of this theorem are reduced to conditions $\mathcal{A}_{44}$, $\mathcal{A}_{47}$ (b), $\mathcal{J}_{17}$, and $\mathcal{C}_{13}^{(T-)}$. □

Lemma 3.7.8. *Let conditions* $\mathcal{A}_{44}$, $\mathcal{J}_{17}$, $\mathcal{A}_{47}$, $\mathcal{J}_{20}$, $\mathcal{G}_7$, $\mathcal{H}_7$, $\mathcal{E}_{11}$, $\mathcal{C}_{13}^{(0)}$, *and* $\mathcal{C}_{13}^{(T)}$ *hold. Then*

$$\mathbf{z}_n(t), t \in [0, T] \xrightarrow{\mathbf{J}} \mathbf{z}_0(t), t \in [0, T] \text{ as } n \to \infty.$$

Proof of Lemma 3.7.8. The proof can be obtained by combining the conditions of Lemmas 3.7.6 and 3.7.7 and applying Theorem 1.4.4 to the functions $\mathbf{z}_n(t)$, $t \in [0, T]$.

It only remains to show why condition $\mathcal{C}_{13}^{(T-)}$ is omitted in Lemma 3.7.8. As a matter of fact, conditions $\mathcal{G}_7$ and $\mathcal{C}_{13}^{(T)}$ imply this condition. Indeed, if $\mathbf{y}_0(T - 0) = \mathbf{y}_0(T)$, then condition $\mathcal{C}_{13}^{(T-)}$ coincides with $\mathcal{C}_{13}^{(T)}$. If $\mathbf{y}_0(T - 0) \neq \mathbf{y}_0(T)$, then condition $\mathcal{G}_7$ implies $\mathcal{C}_{13}^{(T-)}$. □

Remark 3.7.1. If the point T is a point of continuity of the limiting function $\mathbf{z}_0(t)$, then condition $\mathcal{A}_{47}$ in Lemma 3.7.8 can be replaced with condition $\mathcal{A}_{46}$.

3.8 Vector compositions of càdlàg processes

In this section, we study conditions for **J**-convergence of general vector compositions of càdlàg processes. This model is more complicated than the model for semi-vector compositions of càdlàg processes considered in Section 3.6.

3.8.1. J-compactness of vector compositions of càdlàg processes. Let, for every $\varepsilon \geq 0$, $\xi_\varepsilon(t) = (\xi_{\varepsilon i}(t), i = 1, \ldots, m)$, $t \geq 0$ be an m-dimensional càdlàg process with real-valued components and $\mathbf{v}_\varepsilon(t) = (\nu_{\varepsilon i}(t), i = 1, \ldots, m)$, $t \geq 0$ be an m-dimensional càdlàg process with non-negative and non-decreasing components. Consider the *vector compositions* $\zeta_\varepsilon(t) = (\xi_{\varepsilon i}(\nu_{\varepsilon i}(t), i = 1, \ldots, m)$, $t \geq 0$. This process is also an m-dimensional càdlàg process with real-valued components.

The subsequent considerations will be based on the condition of joint weak convergence $\mathcal{A}_{34}$, and the conditions of **J**-compactness $\mathcal{J}_8$ and $\mathcal{J}_{12}$. Let us recall here condition $\mathcal{J}_8$ that was introduced in Subsection 2.3.2,

$\mathcal{J}_8$: $\lim_{c\to 0} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\Delta_J(\xi_{\varepsilon i}(\cdot), c, T) > \delta\} = 0, \ \delta, T > 0, \ i = 1, \ldots, m.$

Conditions $\mathcal{A}_{34}$ and $\mathcal{J}_8$ imply **J**-convergence of the processes $\xi_{\varepsilon i}(t)$, $t \geq 0$, for every $i = 1, \ldots, m$. At the same time, conditions $\mathcal{A}_{34}$ and $\mathcal{J}_{12}$ imply **J**-convergence of the processes $\mathbf{v}_\varepsilon(t)$, $t \geq 0$. However, the examples given in Section 3.1 show that all conditions together, $\mathcal{A}_{34}$, $\mathcal{J}_8$, and $\mathcal{J}_{12}$, do not imply that either **J**-convergence of the vector processes $(\mathbf{v}_\varepsilon(t), \xi_\varepsilon(t))$, $t \geq 0$, or their vector compositions $\zeta_\varepsilon(t)$, $t \geq 0$.

We first give general conditions that would provide **J**-compactness of the compositions $\zeta_\varepsilon(t)$, $t \geq 0$. These conditions can be combined with various other conditions, which imply weak convergence of these processes, in order to get conditions for their **J**-convergence. This suggests that it makes sense to formulate the **J**-compactness conditions separately.

Let us formulate a continuity condition that is a vector variant of condition $\mathcal{G}_4$ and a stochastic analogue of condition $\mathcal{G}_6$,

$\mathcal{G}_8$: $\mathsf{P}\{\nu_{0i}(t \pm 0) \notin R[\xi_{0i}(\cdot)], i = 1, \dots, m \text{ for } t \in \cup_{i=1}^m R[\nu_{0i}(\cdot)]\} = 1$.

Let us also formulate a continuity condition that is a stochastic analogue of condition $\mathcal{H}_4$,

$\mathcal{H}_8$: $\mathsf{P}\{\sum_{i=1}^m \chi(\nu_{0i}(t) \in R[\xi_{0i}(\cdot)]) \leq 1 \text{ for } t \geq 0\} = 1$.

The first main result is the following theorem from Silvestrov (1974).

Theorem 3.8.1. *Let conditions $\mathcal{A}_{34}$, $\mathcal{J}_8$, $\mathcal{J}_{12}$, $\mathcal{G}_8$, and $\mathcal{H}_8$ hold. Then*

$$\lim_{c\to 0} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\Delta_J(\zeta_\varepsilon(\cdot), c, T) > \delta\} = 0, \ \delta, T > 0.$$

Proof of Theorem 3.8.1. The proof is similar to that of Theorem 3.6.1. We are going to reduce the proof to the case of non-random functions using Skorokhod's method of a single probability space, which is based on his representation Theorem 1.6.16, and then use Lemma 3.7.1. Unfortunately, Theorem 1.6.16 can not be directly applied either to the vector processes $(\mathbf{v}_\varepsilon(t), \xi_\varepsilon(t))$, $t \geq 0$, or to their compositions $\zeta_\varepsilon(t)$, $t \geq 0$. As was mentioned above, conditions $\mathcal{A}_{34}$, $\mathcal{J}_8$, $\mathcal{J}_{12}$, $\mathcal{G}_8$, and $\mathcal{H}_8$ do not guarantee $\mathbf{J}$-convergence of these processes. So, this method must be realised in a more sophisticated way. This can be done by first applying Theorem 1.6.14 to the vector processes $(\mathbf{v}_\varepsilon(t), \xi_\varepsilon(t))$, $t \geq 0$, and then Theorem 1.6.16, separately, to the processes $\xi_{\varepsilon i}(t)$, $t \geq 0$, for every $i = 1, \dots, m$, and to the processes $\mathbf{v}_\varepsilon(t)$, $t \geq 0$.

Note, first of all, that conditions $\mathcal{A}_{34}$ and $\mathcal{J}_8$, and $\mathcal{J}_{12}$ and Theorem 1.6.8 permit to extend the sets of weak convergence, U and V, in condition $\mathcal{A}_{34}$ to the sets $U' = U \cup U_0$ and $V' = V \cup V_0$. Here $U_0 = \cap_{i=1}^m U_{0i}$, where U_{0i} is the set of points of stochastic continuity for the process $\xi_{\varepsilon i}(t)$, $t \geq 0$, for $i = 1, \dots, m$, and V_0 is the set of stochastic continuity for the process $\mathbf{v}_0(t)$, $t \geq 0$.

Both sets, U' and V', coincide with $[0, \infty)$, except for at most countable sets. Also, both sets U' and V' contain the point 0. Hence, the set $S' = U' \cap V'$ is also the interval $[0, \infty)$, except for at most a countable set, and $0 \in S'$.

Condition $\mathcal{A}_{34}$ implies the following relation:

$$(\mathbf{v}_\varepsilon(t), \xi_\varepsilon(t)), t \in S' \Rightarrow (\mathbf{v}_0(t), \xi_0(t)), t \in S' \text{ as } \varepsilon \to 0. \tag{3.8.1}$$

Let us choose a countable subset $\tilde{S} \subseteq S'$ that is dense in $[0, \infty)$ and contains the point 0. Using relation (3.8.1) we can apply Theorem 1.6.14 and construct some probability space $(\Omega, \mathfrak{F}, \mathsf{P})$ and a.s. càdlàg processes $(\tilde{\mathbf{v}}_\varepsilon(t), \tilde{\xi}_\varepsilon(t))$, $t \geq 0$, defined on this space for every $\varepsilon \geq 0$ and such that

$$(\tilde{\mathbf{v}}_\varepsilon(t), \tilde{\xi}_\varepsilon(t)), t \geq 0 \stackrel{d}{=} (\mathbf{v}_\varepsilon(t), \xi_\varepsilon(t)), t \geq 0, \tag{3.8.2}$$

and, for an arbitrary sequence $0 \leq \varepsilon_n \to 0$ as $n \to \infty$,

$$(\tilde{\mathbf{v}}_{\varepsilon_n}(t), \tilde{\xi}_{\varepsilon_n}(t)) \xrightarrow{\text{a.s.}} (\tilde{\mathbf{v}}_0(t), \tilde{\xi}_0(t)) \text{ as } n \to \infty,\ t \in \tilde{S}. \tag{3.8.3}$$

Let $\varepsilon_{0,n}$ be an arbitrary sequence such that $0 \leq \varepsilon_{0,n} \to 0$ as $n \to \infty$.

Relations (3.8.1), (3.8.2), (3.8.3), and condition $\mathfrak{J}_8$ permit to apply Theorem 1.6.16 to the processes $\xi_{\varepsilon i}(t), t \geq 0$ and $\tilde{\xi}_{\varepsilon i}(t), t \geq 0$, for every $i = 1, \ldots, m$.

Therefore, there is a subsequence $\varepsilon_{1,n} \to 0$ as $n \to \infty$ of the sequence $\varepsilon_{0,n}$ such that $\mathsf{P}(A_1) = 1$, where $A_1 \in \mathfrak{F}$ is the set of elementary events ω such that

$$\tilde{\xi}_{\varepsilon_{1,n}1}(t, \omega), t \geq 0 \xrightarrow{\mathbf{J}} \tilde{\xi}_{01}(t, \omega), t \geq 0 \text{ as } n \to \infty. \tag{3.8.4}$$

Then there is a subsequence $\varepsilon_{2,n} \to 0$ as $n \to \infty$ of the subsequence $\varepsilon_{1,n}$ such that $\mathsf{P}(A_2) = 1$, where $A_2 \in \mathfrak{F}$ is the set of elementary events ω such that

$$\tilde{\xi}_{\varepsilon_{2,n}2}(t, \omega), t \geq 0 \xrightarrow{\mathbf{J}} \tilde{\xi}_{02}(t, \omega), t \geq 0 \text{ as } n \to \infty. \tag{3.8.5}$$

By continuing this procedure, one can select, for every $i = 1, \ldots, m$, a subsequence $\varepsilon_{i,n} \to 0$ as $n \to \infty$ from the subsequence $\varepsilon_{i-1,n}$ such that $\mathsf{P}(A_i) = 1$, where $A_i \in \mathfrak{F}$ is the set of elementary events ω such that

$$\tilde{\xi}_{\varepsilon_{i,n}i}(t, \omega), t \geq 0 \xrightarrow{\mathbf{J}} \tilde{\xi}_{0i}(t, \omega), t \geq 0 \text{ as } n \to \infty. \tag{3.8.6}$$

Let $A = \cap_{i=1}^m A_i$. Obviously, $\mathsf{P}(A) = 1$. Since $\varepsilon_{m,n}$ is a subsequence of all preceding subsequences, we have for every elementary event $\omega \in A$ that

$$\tilde{\xi}_{\varepsilon_{m,n}i}(t, \omega), t \geq 0 \xrightarrow{\mathbf{J}} \tilde{\xi}_{0i}(t, \omega), t \geq 0 \text{ as } n \to \infty,\ i = 1, \ldots, m. \tag{3.8.7}$$

Relations (3.8.1), (3.8.2), (3.8.3), and condition $\mathfrak{J}_{12}$ also permit to apply Theorem 1.6.16 to the processes $\mathbf{v}_\varepsilon(t), t \geq 0$ and $\tilde{\mathbf{v}}_\varepsilon(t), t \geq 0$.

Therefore, one can choose a subsequence $\varepsilon'_n \to 0$ as $n \to \infty$ from the subsequence $\varepsilon_{m,n}$ such that $\mathsf{P}(B) = 1$, where $B \in \mathfrak{F}$ is the set of elementary events ω such that

$$\tilde{\mathbf{v}}_{\varepsilon'_n}(t, \omega), t \geq 0 \xrightarrow{\mathbf{J}} \tilde{\mathbf{v}}_0(t, \omega), t \geq 0 \text{ as } n \to \infty. \tag{3.8.8}$$

Due to relation (3.8.2), condition $\mathcal{G}_8$ implies that $\mathsf{P}(C) = 1$, where $C \in \mathfrak{F}$ is the set of elementary events ω satisfying

$$\tilde{\nu}_{0i}(t \pm 0, \omega) \notin R[\tilde{\xi}_{0i}(\cdot, \omega)], i = 1, \ldots, m \text{ for } t \in \cup_{i=1}^m R[\tilde{\nu}_{0i}(\cdot, \omega)]. \tag{3.8.9}$$

Also, due to relation (3.8.2), condition $\mathcal{H}_8$ implies that $\mathsf{P}(D) = 1$, where $D \in \mathfrak{F}$ is the set of elementary events ω for which

$$\sum_{i=1}^m \chi(\tilde{\nu}_{0i}(t, \omega) \in R[\tilde{\xi}_{0i}(\cdot, \omega)]) \leq 1 \ \text{ for } \ t \geq 0. \tag{3.8.10}$$

Obviously, $\mathsf{P}(A \cap B \cap C \cap D) = 1$ and, for $\omega \in A \cap B \cap C \cap D$, conditions $\mathcal{A}_{44}$, $\mathfrak{J}_{17}$, $\mathcal{A}_{45}$, $\mathfrak{J}_{18}$, $\mathcal{G}_6$, and $\mathcal{H}_4$ hold for the sequences of functions $\tilde{\xi}_{\varepsilon'_n}(t, \omega)$, $t \geq 0$ and $\tilde{\mathbf{v}}_{\varepsilon'_n}(t, \omega)$, $t \geq 0$. By applying Lemma 3.7.1 to their vector compositions $\tilde{\zeta}_{\varepsilon'_n}(t) = (\tilde{\xi}_{\varepsilon'_n i}(\tilde{\nu}_{\varepsilon'_n i}(t)), i = 1, \dots, m)$, $t \geq 0$, we get for $\omega \in A \cap B \cap C \cap D$ that

$$\lim_{c \to 0} \overline{\lim_{n \to \infty}} \Delta_J(\tilde{\zeta}_{\varepsilon'_n}(\cdot, \omega), c, T) = 0, \ T > 0. \tag{3.8.11}$$

Relation (3.8.11) implies the following relation:

$$\lim_{c \to 0} \overline{\lim_{n \to \infty}} \mathsf{P}\{\Delta_J(\tilde{\zeta}_{\varepsilon_n}(\cdot), c, T) \geq \delta\} = 0, \ \delta, T > 0, \tag{3.8.12}$$

which, due to arbitrariness in the choice of the sequence $\varepsilon_n \to 0$, implies in its turn that

$$\lim_{c \to 0} \overline{\lim_{\varepsilon \to 0}} \mathsf{P}\{\Delta_J(\tilde{\zeta}_{\varepsilon}(\cdot), c, T) \geq \delta\} = 0, \ \delta, T > 0. \tag{3.8.13}$$

The proof that relation (3.8.11) implies relations (3.8.12) and (3.8.13) is absolutely analogous to the proof that relation (3.6.7) implies relations (3.6.8) and (3.6.9), which was given in Theorem 3.6.1.

Relation (3.8.13) implies the relation stated in the theorem since, due to (3.8.2), $\Delta_J(\tilde{\zeta}_{\varepsilon}(\cdot), c, T) \stackrel{d}{=} \Delta_J(\zeta_{\varepsilon}(\cdot), c, T)$. □

3.8.2. J-convergence of vector compositions of càdlàg processes. To obtain general conditions for **J**-convergence of vector compositions of càdlàg processes, it is sufficient to combine the conditions of **J**-compactness formulated in Theorem 3.8.1 with the conditions of weak convergence for compositions obtained in Chapter 2, in particular, those formulated in Theorem 2.7.6.

Let also recall the conditions that were introduced in Subsection 2.7.2:

$\mathcal{E}_4$: $\mathsf{P}\{\nu_{0i}(t') = \nu_{0i}(t'') \in R[\xi_{0i}(\cdot)]\} = 0$ for $0 \leq t' < t'' < \infty$, $i = 1, \dots, m$;

and

$\mathcal{C}_6^{(\mathbf{w})}$: $\mathsf{P}\{\nu_{0i}(w) \in R[\xi_{0i}(\cdot)]\} = 0$ for $i = 1, \dots, m$.

The second main result of the section is the following theorem from Silvestrov (1974).

Theorem 3.8.2. *Let conditions $\mathcal{A}_{34}$, $\mathfrak{J}_8$, $\mathfrak{J}_{12}$, $\mathcal{G}_8$, $\mathcal{H}_8$, $\mathcal{E}_4$, and $\mathcal{C}_6^{(\mathbf{0})}$ hold. Then*

$$\zeta_{\varepsilon}(t), t \geq 0 \xrightarrow{\mathbf{J}} \zeta_0(t), t \geq 0 \text{ as } \varepsilon \to 0.$$

Proof of Theorem 3.8.2. Conditions $\mathcal{A}_{34}$, $\mathcal{J}_8$, $\mathcal{J}_{12}$, $\mathcal{G}_8$, and $\mathcal{H}_8$ are conditions of Theorem 3.8.1. By applying this theorem, we prove **J**-compactness of the processes $\zeta_\varepsilon(t)$, $t \geq 0$, on any finite interval.

Conditions $\mathcal{A}_{34}$, $\mathcal{J}_8$, $\mathcal{E}_4$, and $\mathcal{C}_6^{(0)}$ imply that the conditions of Theorem 2.7.6 hold for the external processes $\xi_\varepsilon(t)$, $t \geq 0$ and the internal stopping processes $\mathbf{v}_\varepsilon(t)$, $t \geq 0$. In particular, condition $\mathcal{A}_{34}$ implies that condition $\mathcal{A}_{22}^{\mathbf{V}}$ holds with the set V in condition $\mathcal{A}_{34}$. This set is dense in $[0, \infty)$ and contains the point 0. Conditions $\mathcal{J}_8$ and $\mathcal{E}_4$ are required in both Theorems 3.8.2 and 2.7.6. The corresponding set of weak convergence, S_0, is dense in $[0, \infty)$. Condition $\mathcal{C}_6^{(0)}$ permits to include the point 0 in S_0. By applying Theorem 2.7.6, we prove weak convergence of the processes $\zeta_\varepsilon(t)$ to $\zeta_0(t)$ as $\varepsilon \to 0$ on the set S_0.

To complete the proof one should apply Theorem 1.6.6, which gives conditions for **J**-convergence of càdlàg processes.

The proof of Theorem 3.8.2 can also be accomplished with the use of the method of a single probability space. One only needs to continue the proof of Theorem 3.8.1.

Due to relation (3.8.2), condition $\mathcal{E}_4$ implies that $\mathsf{P}(E) = 1$, where $E \in \mathfrak{F}$ is a set of elementary events ω for which there do not exist points $0 \leq t' < t'' \leq T$ and $i = 1, \dots, m$ such that $\nu_{0i}(t', \omega) = \nu_{0i}(t'', \omega) \in R[\xi_{0i}(\cdot, \omega)]$. Also, due to relation (3.8.2), condition $\mathcal{C}_6^{(0)}$ implies that $\mathsf{P}(F) = 1$, where $F \in \mathfrak{F}$ is a set of elementary events ω for which $\nu_{0i}(0, \omega) \notin R[\xi_{0i}(\cdot, \omega)]$ for $i = 1, \dots, m$.

Obviously $\mathsf{P}(A \cap B \cap C \cap D \cap E \cap F) = 1$ and, for $\omega \in A \cap B \cap C \cap D \cap E \cap F$, conditions $\mathcal{A}_{44}$, $\mathcal{J}_{17}$, $\mathcal{A}_{45}$, $\mathcal{J}_{18}$, $\mathcal{G}_6$, $\mathcal{H}_4$, $\mathcal{E}_{10}$, and $\mathcal{C}_{13}^{(0)}$ of Lemma 3.7.3 hold for the sequences of càdlàg functions $\tilde{\xi}_{\varepsilon'_n}(t, \omega)$, $t \geq 0$, and $\tilde{\mathbf{v}}_{\varepsilon'_n}(t, \omega)$, $t \geq 0$. By applying Lemma 3.7.3 to their compositions $\tilde{\zeta}_{\varepsilon'_n}(t) = (\tilde{\xi}_{\varepsilon'_n i}(\tilde{\nu}_{\varepsilon'_n i}(t)), i = 1, \dots, m)$, $t \geq 0$, we get, for $\omega \in A \cap B \cap C \cap D \cap E \cap F$, the following relation: $\tilde{\zeta}_{\varepsilon'_n}(t, \omega), t \geq 0 \xrightarrow{\mathbf{J}} \tilde{\zeta}_0(t, \omega), t \geq 0$ as $r \to \infty$. In terms of the metrics d_J, the last relation means that $d_J(\tilde{\zeta}_{\varepsilon'_n}(\cdot, \omega), \tilde{\zeta}_0(\cdot, \omega)) \to 0$. Since the initial sequence ε_n was arbitrary, this relation means that the random variables $d_J(\tilde{\zeta}_\varepsilon(\cdot), \tilde{\zeta}_0(\cdot)) \xrightarrow{\mathsf{P}} 0$ as $\varepsilon \to 0$. As it was pointed out in Lemma 1.3.1, convergence in probability implies weak convergence. So, we get that the processes $\tilde{\zeta}_\varepsilon = \{\tilde{\zeta}_\varepsilon(t), t \geq 0\}$, considered as random variables that take values in the space $\mathbf{D}_{[0,\infty)}^{(m)}$ with the metric d_J, weakly converge. Since $\tilde{\zeta}_\varepsilon \overset{\mathrm{d}}{=} \zeta_\varepsilon$, this completes the proof. □

In conclusion, let us compare the conditions of Theorems 3.8.2 and 3.6.2 in the semi-vector case, where the internal stopping process $\mathbf{v}_\varepsilon(t) = (\nu_\varepsilon(t), \dots, \nu_\varepsilon(t))$, $t \geq 0$ has identical components.

In this case, condition $\mathcal{A}_{34}$ is reduced to condition $\mathcal{A}_{36}$, condition $\mathcal{J}_{12}$ to $\mathcal{J}_{11}$, condition $\mathcal{E}_4$ to $\mathcal{E}_6$, condition $\mathcal{G}_8$ to $\mathcal{G}_4$, and condition $\mathcal{C}_6^{(0)}$ to $\mathcal{C}_8^{(0)}$.

The condition of **J**-compactness of external processes, $\mathcal{J}_8$, used in Theorem 3.8.2 is weaker than condition $\mathcal{J}_4$ used in Theorem 3.6.2. However, this is compensated by the use of the additional condition $\mathcal{H}_8$ in Theorem 3.8.2.

In the scalar case, where $m = 1$, the conditions of Theorems 3.8.2 and 3.6.2 coincide. In this case, conditions $\mathfrak{J}_8$ and $\mathfrak{J}_4$ coincide while condition $\mathcal{H}_8$ holds automatically.

3.8.3. The continuity conditions $\mathcal{E}_4$, $\mathcal{G}_8$, and $\mathcal{H}_8$. Let us formulate these conditions in a more convenient form and give some simple sufficient conditions.

Determine the moments of jumps of the process $\xi_{0i}(t)$, $t \geq 0$, namely $\tau_{kni} = \inf(s > \tau_{k-1ni}\colon |\xi_{0i}(s) - \xi_{0i}(s-0)| \in [\frac{1}{n}, \frac{1}{n-1}))$, $k = 1, 2, \ldots$, where $\tau_{0ni} = 0$. By the definition, τ_{kni} are successive moments of jumps with absolute values in the interval $[\frac{1}{n}, \frac{1}{n-1})$ for $k < \mu_{ni} + 1$ and $\tau_{kni} = \infty$ for $k \geq \mu_{ni} + 1$. Here $\mu_{ni} = \max(k \geq 0\colon \tau_{kni} < \infty)$ is the total number of such jumps in the interval $[0, \infty)$.

Similar notations can be introduced for moments of jumps of the process $\nu_{0i}(t)$, $t \geq 0$, namely, $\kappa_{kni} = \inf(s > \kappa_{k-1ni}\colon |\nu_{0i}(s) - \nu_{0i}(s-0)| \in [\frac{1}{n}, \frac{1}{n-1}))$, $k = 1, 2, \ldots$, where $\kappa_{0ni} = 0$. By the definition, κ_{kni} are successive moments of jumps with absolute values in the interval $[\frac{1}{n}, \frac{1}{n-1})$ for $k < \lambda_n + 1$ and $\kappa_{kni} = \infty$ for $k \geq \lambda_{ni} + 1$. Here $\lambda_{ni} = \max(k \geq 0\colon \kappa_{kni} < \infty)$ is the total number of such jumps in the interval $[0, \infty)$.

The condition $\mathcal{E}_4$ can be rewritten in an equivalent form,

$\mathcal{E}_4'$: $\mathsf{P}\{\nu_{0i}(t') = \nu_{0i}(t'') = \tau_{rli}\} = 0$ for $0 \leq t' < t'' < \infty$, $r, l = 1, 2, \ldots$ and $i = 1, \ldots, m$.

Note, first of all, that the following condition, introduced in Subsection 2.7.2, is obviously sufficient for condition $\mathcal{E}_4$ to hold:

$\mathfrak{I}_2$: $\nu_{0i}(t)$, $t \geq 0$ is an a.s. strictly increasing process for every $i = 1, \ldots, m$.

The following condition, introduced in the same Subsection 2.7.2, is also sufficient for condition $\mathcal{E}_4$ to hold:

$\mathfrak{Q}_4$: $\xi_0(t) = \xi_0'(t) + \xi_0''(t)$, $t \geq 0$, where (a) $\xi_0'(t), t \geq 0$ is a continuous process, (b) $\xi_0''(t), t \geq 0$ is a stochastically continuous càdlàg process, (c) for every $i = 1, \ldots, m$, the processes $\xi_{0i}''(t), t \geq 0$ and $\nu_{0i}(t)$, $t \geq 0$ are independent.

Lemma 3.8.1. *Let condition $\mathfrak{Q}_4$ hold. Then condition $\mathcal{E}_4$ holds.*

Proof of Lemma 3.8.1. If condition $\mathfrak{Q}_4$ holds, then the random variable τ_{rli} is a point of jump of the process $\xi_{0i}(t)$ if and only if it is the corresponding point of jump of the second component $\xi_{0i}''(t)$ in the decomposition. This is so, because the first component $\xi_{0i}'(t)$ is continuous. Therefore **(a)** the process $\nu_{0i}(t)$, $t \geq 0$ and the random variable τ_{rli} are independent. The process $\xi_{0i}(t)$, $t \geq 0$ is stochastically continuous. So, **(b)** the random variables τ_{rli} have continuous distribution functions. Obviously, **(a)** and **(b)** imply that condition $\mathcal{E}_4$ holds. □

Suppose that condition $\mathfrak{Q}_4$ holds without the assumption that the processes $\xi_{0i}''(t)$, $i = 1, \ldots, m$ are stochastically continuous. Then the distribution functions of the random variables τ_{rli} can possess discontinuity points. To make condition $\mathcal{E}_4$ hold, it is enough to require in this case that **(c)** the random variables $\nu_{0i}(t)$ and τ_{rli} be independent for every $t \geq 0$, $r, l = 1, 2, \ldots$, and $i = 1, \ldots, m$, and that **(d)** their distribution functions have not common points of discontinuity. Note that, in this case, the processes $\xi_0(t), t \geq 0$ and $\nu_0(t)$, $t \geq 0$ can be dependent.

Note also that condition $\mathfrak{Q}_4$ implies that condition $\mathfrak{C}_6^{(w)}$ holds for any $w \geq 0$.

Condition $\mathcal{G}_8$ can be rewritten in an equivalent form,

$\mathcal{G}_8'$: $\mathsf{P}\{\nu_{0i}(\kappa_{knj} \pm 0) = \tau_{rli}\} = 0$ for $k, n, r, l = 1, 2, \ldots$ and $i, j = 1, \ldots, m$.

Note that the random variables $\nu_{0i}(\kappa_{knj} \pm 0)$ can take values in the interval $[0, \infty]$, since the random variables κ_{knj} can take the value $+\infty$. In this case by definition, $\nu_{0i}(+\infty \pm 0) = \lim_{t\to\infty} \nu_{0i}(t)$.

The following condition, which is slightly stronger than $\mathfrak{Q}_4$, is sufficient for $\mathcal{G}_8$ to hold:

$\mathfrak{Q}_8$: $\xi_0(t) = \xi_0'(t) + \xi_0''(t)$, $t \geq 0$, where (a) $\xi_0'(t), t \geq 0$ is a continuous process, (b) $\xi_0''(t), t \geq 0$ is a stochastically continuous càdlàg process, (c) for every $i = 1, \ldots, m$, the processes $\xi_{0i}''(t), t \geq 0$ and the vector process $\mathbf{v}_0(t)$, $t \geq 0$ are independent.

Lemma 3.8.2. *Let condition $\mathfrak{Q}_8$ hold. Then condition $\mathcal{G}_8$ holds.*

Proof of Lemma 3.8.2. If condition $\mathfrak{Q}_8$ holds, then the random variable τ_{rli} is a moment of jump of the process $\xi_{0i}(t)$ if and only if it is the corresponding point of jump of the second component in the decomposition, $\xi_{0i}''(t)$. Therefore, the process $\mathbf{v}_0(t)$, $t \geq 0$ and the random variable τ_{rli} are independent. In sequel, **(e)** the random variables $\nu_{0i}(\kappa_{knj} \pm 0)$ and τ_{rli} are independent. The process $\xi_{0i}(t)$, $t \geq 0$ is stochastically continuous. Thus, **(f)** the random variables τ_{rli} have continuous distribution functions. Obviously, **(e)** and **(f)** imply that condition $\mathcal{G}_8$ holds. □

Suppose that $\mathfrak{Q}_8$ holds without the assumption that the processes $\xi_{0i}''(t)$, $i = 1, \ldots, m$ are stochastically continuous. Then the distributions of the random variables τ_{rli} can possess discontinuity points. To make condition $\mathcal{G}_8$ hold, it is enough to require in this case that the distribution functions of the random variables $\nu_{0i}(\kappa_{knj} \pm 0)$ and τ_{rli} have not common points of discontinuity for every $k, n, r, l = 1, 2, \ldots$ and $i, j = 1, \ldots, m$.

The analysis of condition $\mathcal{H}_8$ is more complicated. Define, for every $a \geq 0$, the random functionals $\gamma_i(a) = \inf(t \geq 0\colon \nu_{0i}(t) = a)$. By the definition, $\gamma_i(a)$ is the left endpoint of the interval where the process $\nu_{0i}(t)$ takes the value a. Note that it can happen that this interval consists only of the point $\gamma_i(a)$ itself. Also, $\gamma_i(a) = \infty$ if such an interval does not exist. It is clear that there exist a point $t \geq 0$ and $i \neq j$ such that $\nu_{0i}(t) \in R[\xi_{0i}(\cdot)]$ and $\nu_{0j}(t) \in R[\xi_{0j}(\cdot)]$ if and only if there exist some k, n, r, $l \geq 1$ and $i \neq j$ such that $\nu_{0j}(\gamma_i(\tau_{rli})) = \tau_{knj}$.

So, condition $\mathcal{H}_8$ can be rewritten in the following equivalent form:

$\mathcal{H}'_8$: $\mathsf{P}\{\nu_{0j}(\gamma_i(\tau_{rli})) = \tau_{knj}\} = 0$ for $k, n, r, l = 1, 2, \ldots$ and $i \neq j$.

The following condition, which is stronger than $\mathfrak{Q}_4$ and $\mathfrak{Q}_8$, is sufficient for $\mathcal{H}'_8$ to hold:

$\mathfrak{Q}_9$: $\xi_0(t) = \xi'_0(t) + \xi''_0(t)$, $t \geq 0$, where (a) $\xi'_0(t), t \geq 0$ is a continuous process, (b) $\xi''_0(t), t \geq 0$ is a stochastically continuous càdlàg process, (c) the processes $\xi''_{0i}(t)$, $t \geq 0$, for $i = 1, \ldots, m$ and $\mathbf{v}_0(t)$, $t \geq 0$ are mutually independent.

Lemma 3.8.3. *Let condition $\mathfrak{Q}_9$ hold. Then condition $\mathcal{H}_8$ holds.*

Proof of Lemma 3.8.3. If condition $\mathfrak{Q}_9$ holds, then the random variable τ_{rli} is a point of jump of the process $\xi_{0i}(t)$ if and only if it is the corresponding point of jump of the second component in the decomposition, $\xi''_{0i}(t)$. Therefore, by condition $\mathfrak{Q}_9$ (c), the process $\mathbf{v}_0(t)$, $t \geq 0$ and the random variables τ_{rli}, $i = 1, \ldots, m$ are mutually independent. In sequel, **(g)** the random variables $\nu_{0j}(\gamma_i(\tau_{rli}))$ and τ_{knj} are independent for every $i \neq j$. The process $\xi_{0i}(t)$, $t \geq 0$ is stochastically continuous. So, **(h)** the random variables τ_{knj} have continuous distribution functions. Obviously, **(g)** and **(h)** imply that condition $\mathcal{H}'_8$ holds. □

Suppose that condition $\mathfrak{Q}_9$ holds without imposing the assumption on the processes $\xi''_{0j}(t)$, $t \geq 0$ to be stochastically continuous. Then the distributions of the random variables τ_{knj} can possess discontinuity points. To provide condition $\mathcal{H}'_8$, it is enough to require in this case that the distribution functions of the random variables $\nu_{0j}(\gamma_i(\tau_{rli}))$ and τ_{knj} have not common points of discontinuity for every $k, n, r, l = 1, 2, \ldots$ and $i \neq j$.

Note that conditions $\mathfrak{Q}_4$, $\mathfrak{Q}_8$, and $\mathfrak{Q}_9$ admit dependence of the processes $\xi'_0(t)$, $t \geq 0$ and $\mathbf{v}_0(t)$, $t \geq 0$. Due to this dependence, the processes $\xi_0(t)$, $t \geq 0$ and $\mathbf{v}_0(t)$, $t \geq 0$ can be dependent.

The following theorem from Silvestrov (1974) is applicable in many cases. It is a direct corollary of Theorem 3.8.2 and Lemmas 3.8.1 – 3.8.3.

Theorem 3.8.3. *Let conditions $\mathcal{A}_{34}$, $\mathcal{J}_8$, $\mathcal{J}_{12}$, and $\mathfrak{Q}_9$ hold. Then*

$$\zeta_\varepsilon(t), t \geq 0 \stackrel{\mathbf{J}}{\longrightarrow} \zeta_0(t), t \geq 0 \text{ as } \varepsilon \to 0.$$

3.8.4. J-convergence of vector càdlàg processes. Let us assume that the processes $\nu_{\varepsilon i}(t) = t, t \geq 0$, for all $i = 1, \ldots, m$. In this case, the processes $\zeta_\varepsilon(t) = \xi_\varepsilon(t)$, $t \geq 0$.

Condition $\mathcal{A}_{34}$ takes, in this case, the following form:

$\mathcal{A}_{48}$: $\xi_\varepsilon(t), t \in U \Rightarrow \xi_0(t), t \in U$ as $\varepsilon \to 0$, where U is a subset of $[0, \infty)$ that is dense in this interval and contains the point 0.

The condition of **J**-compactness $\mathcal{J}_8$, which requires **J**-compactness of the processes $\xi_{\varepsilon i}(t)$, $t \geq 0$, separately for every $i = 1, \dots, m$, does not change.

Condition $\mathcal{H}_8$ takes the following form:

$\mathcal{H}_9$: $\sum_{i=1}^{m} \chi(t \in R[\xi_{0i}(\cdot)]) \leq 1$ for $t \geq 0\} = 1$.

It can also be formulated in the following equivalent forms:

$\mathcal{H}_9'$: $\mathsf{P}\{R[\xi_{0i}(\cdot)] \cap R[\xi_{0j}(\cdot)] = \varnothing\} = 1$ for $i \neq j$;

or

$\mathcal{H}_9''$: $\mathsf{P}\{\tau_{kni} = \tau_{rlj}\} = 0$ for $k, n, r, l = 1, 2, \dots$ and $i \neq j$.

The condition of **J**-compactness $\mathcal{J}_{12}$ obviously holds, as well as conditions $\mathcal{G}_8$, $\mathcal{E}_4$, and $\mathcal{C}_6^{(0)}$.

Theorems 3.8.2 yields, in this case, the following simple sufficient conditions for **J**-convergence of vector càdlàg processes. This result belongs to Whitt (1973, 1980).

Theorem 3.8.4. *Let conditions $\mathcal{A}_{48}$, $\mathcal{J}_8$, and $\mathcal{H}_9$ hold. Then*

$$\xi_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{J}} \xi_0(t), t \geq 0 \text{ as } \varepsilon \to 0.$$

Note that condition $\mathcal{J}_8$ requires **J**-compactness for the components $\xi_{\varepsilon i}(t)$, $t \geq 0$ separately for every $i = 1, \dots, m$. It can happen that, under conditions $\mathcal{A}_{48}$ and $\mathcal{J}_8$, the vector processes $\xi_\varepsilon(t)$, $t \geq 0$ are not **J**-compact and do not **J**-converge. Condition $\mathcal{H}_9$ is an additional condition that should be added to conditions $\mathcal{A}_{48}$ and $\mathcal{J}_8$ in order to provide **J**-compactness and also **J**-convergence of the vector processes $\xi_\varepsilon(t)$, $t \geq 0$.

Let us now go back to the case of general vector compositions $\zeta_\varepsilon(t)$, $t \geq 0$. Condition $\mathcal{H}_9$ takes the following form:

$\mathcal{H}_{10}$: $\mathsf{P}\{\sum_{i=1}^{m} \chi(t \in R[\xi_{0i}(\nu_{0i}(\cdot))]) \leq 1$ for $t \geq 0\} = 1$;

or the equivalent form:

$\mathcal{H}_{10}'$: $\mathsf{P}\{R[\xi_{0i}(\nu_{0i}(\cdot))] \cap R[\xi_{0j}(\nu_{0j}(\cdot))] = \varnothing\} = 1$ for $i \neq j$.

Let us introduce the following condition for **J**-compactness of the internal stopping processes, which is weaker than condition $\mathcal{J}_{12}$:

$\mathcal{J}_{21}$: $\lim_{c \to 0} \overline{\lim}_{n \to \infty} \mathsf{P}\{\Delta_J(\nu_{\varepsilon i}(\cdot), c, T) > \delta\} = 0,\ \delta, T > 0,\ i = 1, \dots, m.$

Theorem 3.8.4 permits to formulate conditions for **J**-convergence of vector compositions of càdlàg processes, which would be alternative to the conditions given in Theorem 3.8.2.

Theorem 3.8.5. *Let conditions $\mathcal{A}_{34}$, $\mathcal{J}_8$, $\mathcal{J}_{21}$, $\mathcal{G}_8$, $\mathcal{H}_{10}$, $\mathcal{E}_4$, and $\mathcal{C}_6^{(0)}$ hold. Then*

$$\zeta_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{J}} \zeta_0(t), t \geq 0 \text{ as } \varepsilon \to 0.$$

Proof of Theorem 3.8.5. Conditions $\mathcal{A}_{34}$, $\mathcal{J}_{18}$, $\mathcal{J}_{21}$, $\mathcal{G}_8$, $\mathcal{E}_4$, and $\mathcal{C}_6^{(0)}$ imply that conditions of Theorem 3.6.2 hold for the scalar processes $\xi_{\varepsilon i}(t)$, $t \geq 0$ and $\nu_{\varepsilon i}(t)$, $t \geq 0$, for every $i = 1, \ldots, m$. By applying Theorem 3.6.2 to these processes, we prove that $\xi_{\varepsilon i}(\nu_{\varepsilon i}(t)), t \geq 0 \xrightarrow{\mathbf{J}} \xi_{0i}(\nu_{0i}(t)), t \geq 0$ as $\varepsilon \to 0$, for every $i = 1, \ldots, m$. Finally, condition $\mathcal{H}_{10}$ permits to apply Theorem 3.8.4 to the vector processes $\zeta_\varepsilon(t) = (\xi_{\varepsilon i}(\nu_{\varepsilon i}(t)), i = 1, \ldots, m)$, $t \geq 0$, and to prove Theorem 3.8.5. □

Let us explain the difference between conditions of **J**-convergence in Theorems 3.8.2 and 3.8.5. Conditions $\mathcal{H}_{10}$ and $\mathcal{H}_8$ used in these theorems are not equivalent.

Condition $\mathcal{H}_8$ prohibits only the case where **(a)** two or more processes $\xi_{0i}(\cdot)$ have synchronous jumps at random points $\nu_{0i}(t)$ for some $t \geq 0$. This means that the probability of the event described in **(a)** equals 0.

Condition $\mathcal{H}_{10}$ does not allow for the case **(a)** and usually also for the case where **(b)** two or more processes $\nu_{0i}(\cdot)$ have simultaneous jumps at a point t for some $t \geq 0$. This means that both probabilities of the events described in **(a)** and **(b)** equal 0.

At the same time, condition $\mathcal{J}_{12}$ is stronger than condition $\mathcal{J}_{21}$.

However, conditions $\mathcal{J}_{12}$ and $\mathcal{J}_{21}$ are equivalent if **(c)** two or more processes $\nu_{0i}(\cdot)$ have not simultaneous jumps with probability 1, i.e., condition $\mathcal{H}_9$ holds for processes the $\boldsymbol{\nu}_0(t)$, $t \geq 0$.

Let consider two examples that illustrate the difference between Theorems 3.8.5 and 3.8.2.

In the model of semi-vector composition of càdlàg processes, the internal stopping process $\boldsymbol{\nu}_\varepsilon(t) = (\nu_\varepsilon(t), \ldots, \nu_\varepsilon(t)), t \geq 0$ has identical components. If the corresponding limiting process $\nu_0(t)$, $t \geq 0$ is discontinuous, then condition $\mathcal{H}_9$ does not hold for the processes $\boldsymbol{\nu}_0(t)$, $t \geq 0$. In this case, condition $\mathcal{H}_{10}$ may not hold for the processes $\zeta_0(t), t \geq 0$. Theorem 3.8.5 does not work. At the same time, conditions $\mathcal{J}_{12}$ and $\mathcal{H}_8$ may hold and Theorem 3.8.2 can be used.

In many applications, the components of the internal stopping processes $\boldsymbol{\nu}_\varepsilon(t), t \geq 0$ are asymptotically proportional. This means that the limiting process $\boldsymbol{\nu}_0(t)$, $t \geq 0$ has the following structure: $\boldsymbol{\nu}_0(t) = (q_i\nu_0(t), i = 1, \ldots, m)$, $t \geq 0$, where q_i, $i = 1, \ldots, m$ are positive constants. If the corresponding limiting process $\nu_0(t)$, $t \geq 0$ is discontinuous, then condition $\mathcal{H}_9$ does not hold for the processes $\boldsymbol{\nu}_0(t)$, $t \geq 0$. In this case, condition $\mathcal{H}_{10}$ may not hold for the processes $\zeta_0(t), t \geq 0$. In such situations, Theorem 3.8.5 does not work. At the same time, conditions $\mathcal{J}_{12}$ and $\mathcal{H}_8$ may hold. Theorem 3.8.2 can be applicable.

3.8.5. Weakened second-type continuity conditions. Let us formulate an analogue of Theorem 3.8.2 in which the continuity conditions $\mathcal{E}_4$ and $\mathcal{C}_6^{(0)}$ are weakened.

Theorem 3.8.6. *Let conditions $\mathcal{A}_{34}$, $\mathcal{J}_8$, $\mathcal{J}_{12}$, $\mathcal{G}_8$, $\mathcal{H}_8$, $\mathcal{F}_3$, and $\mathcal{D}_5^{(0)}$ hold. Then*

$$\zeta_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{J}} \zeta_0(t), t \geq 0 \text{ as } \varepsilon \to 0.$$

Proof of Theorem 3.8.6. Conditions $\mathcal{A}_{34}$, $\mathcal{J}_8$, $\mathcal{J}_{12}$, $\mathcal{G}_8$, and $\mathcal{H}_8$ are conditions of Theorem 3.8.1. By applying this theorem, we prove **J**-compactness of the processes $\zeta_\varepsilon(t)$, $t \geq 0$, for any finite interval.

Conditions $\mathcal{A}_{34}$, $\mathcal{J}_8$, $\mathcal{F}_3$, and $\mathcal{D}_5^{(0)}$ imply that the conditions of Theorem 2.7.10 hold for the external processes $\xi_\varepsilon(t)$, $t \geq 0$, and the internal stopping processes $\mathbf{v}_\varepsilon(t)$, $t \geq 0$. In particular, condition $\mathcal{A}_{34}$ implies that condition $\mathcal{A}_{22}^{\mathbf{V}}$ holds with the set V that enters condition $\mathcal{A}_{34}$. This set is dense in $[0, \infty)$ and contains the point 0. Condition $\mathcal{F}_3$ is required in both Theorems 3.8.6 and 2.7.10. The corresponding set of weak convergence, S_0, is dense in $[0, \infty)$. Condition $\mathcal{D}_5^{(0)}$ permits to include the point 0 in S_0. By applying Theorem 2.7.10, we prove that the processes $\zeta_\varepsilon(t)$ weakly converge to $\zeta_0(t)$ as $\varepsilon \to 0$ on the set S_0.

To complete the proof, it remains to apply Theorem 1.6.6 that gives conditions for **J**-convergence of càdlàg processes. □

3.8.6. The time interval $[0, T]$. In this case, we consider the vector composition $\zeta_\varepsilon(t) = (\xi_{\varepsilon i}(\nu_{\varepsilon i}(t)), i = 1, \dots, m)$, $t \in [0, T]$ of a vector càdlàg process $\xi_\varepsilon(t) = (\xi_{\varepsilon i}(t), i = 1, \dots, m)$, $t \geq 0$, with real-valued components and a vector càdlàg process $\mathbf{v}_\varepsilon(t) = (\nu_{\varepsilon i}(t), i = 1, \dots, m)$, $t \in [0, T]$, with non-negative and non-decreasing components.

We can always continue internal stopping process to the interval $[0, \infty)$ by the following formula:

$$\mathbf{v}_\varepsilon(t) = \begin{cases} \mathbf{v}_\varepsilon(t), & \text{if } 0 \leq t \leq T, \\ \mathbf{v}_\varepsilon(T), & \text{if } t \geq T. \end{cases} \tag{3.8.14}$$

Formula (3.8.14) implies that, for every $i = 1, \dots, m$,

$$\xi_{\varepsilon i}(\nu_{\varepsilon i}(t)) = \begin{cases} \xi_{\varepsilon i}(\nu_{\varepsilon i}(t)), & \text{if } 0 \leq t \leq T, \\ \xi_{\varepsilon i}(\nu_{\varepsilon i}(T)), & \text{if } t \geq T. \end{cases} \tag{3.8.15}$$

It follows from formulas (3.8.14) and (3.8.15) that the processes $\mathbf{v}_\varepsilon(t)$ and $\zeta_\varepsilon(t)$ take the values, respectively, $\mathbf{v}_\varepsilon(T)$ and $\zeta_\varepsilon(T)$ for $t \geq T$.

Formulas (3.8.14) and (3.8.15) allow to derive conditions for **J**-compactness and **J**-convergence of compositions of càdlàg processes defined on finite intervals from the corresponding results for the case of the semi-infinite interval $[0, \infty)$.

Condition $\mathcal{A}_{34}$ takes, in this case, the following form:

$\mathcal{A}_{49}$: $(\mathbf{v}_\varepsilon(s), \xi_\varepsilon(t)), (s, t) \in V \times U \Rightarrow (\mathbf{v}_0(s), \xi_0(t)), (s, t) \in V \times U$ as $\varepsilon \to 0$, where (a) U is a subset of $[0, \infty)$ that is dense in this interval and contains the point 0, (b) V is a subset of $[0, T]$ that is dense in this interval and contains the points 0 and T.

The condition for **J**-compactness of external processes $\mathcal{J}_8$ does not require any changes. Condition for **J**-compactness of internal stopping processes $\mathcal{J}_{12}$, however, should be modified to the following form:

$\mathcal{J}_{22}$: $\lim_{c\to 0}\overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\Delta_J(\mathbf{v}_\varepsilon(\cdot), c, T) > \delta\} = 0,\ \delta > 0.$

Denote by $R_T[\nu_0(\cdot)]$ the random set of discontinuity points for the process $\nu_0(t)$, $t \in [0, T]$.

Continuity conditions $\mathcal{E}_4$, $\mathcal{G}_8$, and $\mathcal{H}_8$ take the following form:

$\mathcal{E}_{12}$: $\mathsf{P}\{\nu_{0i}(t') = \nu_{0i}(t'') \in R[\xi_{0i}(\cdot)]\} = 0$ for $0 \le t' < t'' \le T$, $i = 1, \dots, m$;

$\mathcal{G}_9$: $\mathsf{P}\{\nu_{0i}(t \pm 0) \notin R[\xi_{0i}(\cdot)], i = 1, \dots, m$ for $t \in \cup_{i=1}^m R_T[\nu_{0i}(\cdot)]\} = 1;$

and

$\mathcal{H}_{11}$: $\mathsf{P}\{\sum_{i=1}^m \chi(\nu_{0i}(t) \in R[\xi_{0i}(\cdot)]) \le 1$ for $t \in [0, T]\} = 1.$

The following theorem is an analogue of Theorem 3.8.1.

Theorem 3.8.7. *Let conditions $\mathcal{A}_{49}$, $\mathcal{J}_8$, $\mathcal{J}_{22}$, $\mathcal{G}_9$, and $\mathcal{H}_{11}$ hold. Then*

$$\lim_{c\to 0}\overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\Delta_J(\zeta_\varepsilon(\cdot), c, T) > \delta\} = 0,\ \delta > 0.$$

Proof of Theorem 3.8.7. It is enough to apply Theorem 3.8.1 to the vector composition of the processes $\xi_\varepsilon(t)$, $t \ge 0$ and $\mathbf{v}_\varepsilon(t)$, $t \ge 0$, where the latter process is defined in (3.8.14). Condition $\mathcal{A}_{49}$ implies $\mathcal{A}_{34}$, condition $\mathcal{J}_{22}$ implies $\mathcal{J}_{12}$, condition $\mathcal{G}_9$ implies $\mathcal{G}_8$, and condition $\mathcal{H}_{11}$ implies $\mathcal{H}_8$. Condition $\mathcal{J}_8$ is the same in Theorems 3.8.1 and 3.8.7. The relation of **J**-compactness given in Theorem 3.8.1 yields, for $T' > T$, the relation of **J**-compactness given in Theorem 3.8.7. □

We also use the following modification of condition $\mathcal{A}_{49}$ where the random variables $\mathbf{v}_\varepsilon(T-0)$ are additionally included in the relation of weak convergence:

$\mathcal{A}_{50}$: $(\mathbf{v}_\varepsilon(s), \mathbf{v}_\varepsilon(T-0), \xi_\varepsilon(t)), (s, t) \in V \times U \Rightarrow (\mathbf{v}_0(s), \mathbf{v}_0(T-0), \xi_0(t)), (s, t) \in V \times U$ as $\varepsilon \to 0$, where (a) U is a subset of $[0, \infty)$ that is dense in this interval and contains the point 0, (b) V is a subset of $[0, T]$ that is dense in this interval and contains the points 0 and T.

The following theorem is an analogue of Theorem 3.8.2.

Theorem 3.8.8. *Let conditions $\mathcal{A}_{50}$, $\mathcal{J}_8$, $\mathcal{J}_{22}$, $\mathcal{G}_9$, $\mathcal{H}_{11}$, $\mathcal{E}_{12}$, $\mathcal{C}_6^{(0)}$, and $\mathcal{C}_6^{(T)}$ hold. Then*

$$\zeta_\varepsilon(t), t \in [0, T] \xrightarrow{\mathbf{J}} \zeta_0(t), t \in [0, T] \ \textit{as}\ \varepsilon \to 0.$$

Proof of Theorem 3.8.8. Let us apply Theorems 3.8.2 to the vector composition $\zeta_\varepsilon(t)$, $t \geq 0$ of the processes $\xi_\varepsilon(t)$, $t \geq 0$ and $\mathbf{v}_\varepsilon(t)$, $t \geq 0$, where the latter process is defined in (3.8.14). Condition $\mathcal{A}_{50}$ implies $\mathcal{A}_{34}$, condition $\mathcal{J}_{22}$ implies $\mathcal{J}_{12}$, condition $\mathcal{G}_9$ implies $\mathcal{G}_8$, and condition $\mathcal{H}_{11}$ implies $\mathcal{H}_8$. Conditions $\mathcal{J}_8$, $\mathcal{C}_6^{(0)}$, and $\mathcal{C}_6^{(T)}$ are the same in Theorems 3.8.2 and 3.8.8. Also, conditions $\mathcal{E}_{12}$ and $\mathcal{C}_6^{(T)}$ imply $\mathcal{E}_4$. Now, by applying Theorem 3.8.2, we prove **J**-convergence of the processes $\zeta_\varepsilon(t)$ on the interval $[0, \infty)$.

However, this does not automatically yields **J**-convergence of these processes on the interval $[0, T]$. In order to have **J**-convergence on the interval $[0, T]$, the random variables $\zeta_0(T)$ must be included in the relation of weak convergence of these processes. Moreover, if the point T is not a point of stochastic continuity for the limiting process $\zeta_0(t)$, then the random variables $\zeta_\varepsilon(T - 0)$ should also be included in the relation of weak convergence on the set $S_0(T) = S_0 \cap [0, T]$. The random variables $\zeta_\varepsilon(T)$ can be included due to condition $\mathcal{C}_6^{(T)}$. Also, conditions $\mathcal{G}_9$ and $\mathcal{C}_6^{(T)}$ imply that $\mathcal{C}_6^{(T-)}$ holds. To prove this, one can apply (3.6.13) in the case of scalar processes and show that **(a)** $\mathsf{P}\{\nu_{0i}(T - 0) \in R[\xi_{0i}(\cdot)]\} = 0$ for every $i = 1, \ldots, m$. These relations are equivalent to condition $\mathcal{C}_6^{(T-)}$. This condition allows to include the random variables $\zeta_\varepsilon(T - 0)$ in the relation of weak convergence of the processes $\zeta_\varepsilon(t)$.

The proof is completed by referring to Theorem 1.6.3 that gives conditions for **J**-convergence of càdlàg processes defined on a finite interval. □

Let us introduce the following condition:

$\mathcal{O}_{10}^{(T)}$: $\mathsf{P}\{\zeta_0(T - 0) = \zeta_0(T)\} = 1$.

Remark 3.8.1. Condition $\mathcal{A}_{50}$ can be replaced in Theorem 3.8.8 with condition $\mathcal{A}_{49}$ if the point T is a point of stochastic continuity for the limiting process $\zeta_0(t)$, which is equivalent to condition $\mathcal{O}_{10}^{(T)}$.

3.8.7. A Polish phase space. Results in this section can be generalised to a model with external stochastic processes $\xi_{\varepsilon i}(t)$, $t \geq 0$ that take values in a Polish space X.

The formulation of condition $\mathcal{A}_{34}$ remains without changes. In the conditions $\mathcal{J}_8$, the Euclidean distance $|x - y|$ must be replaced with the corresponding metric $d(x, y)$ in the formula for the moduli $\Delta_J(\xi_{\varepsilon i}(\cdot), c, T)$, $i = 1, \ldots, m$. With these changes in the conditions, the formulations and the proofs of Theorems 3.8.1 – 3.8.8 can be repeated.

3.8.8. References. Conditions for **U**-convergence of scalar compositions of càdlàg processes were obtained by Billingsley (1968). Theorem 3.2.1 and Lemmas 3.2.1 – 3.2.3 present vector versions of these results in the form given in Silvestrov (1974).

Conditions for **J**-convergence of semi-vector compositions with a continuous limiting external process, formulated in Theorem 3.3.1, are from Silvestrov (1974). Similar results were obtained by Whitt (1973, 1980). Theorem 3.3.2 and Lemma 3.2.3 present a new improved version of Theorem 3.3.1 for vector compositions of càdlàg processes.

Conditions for **J**-compactness and **J**-convergence of monotone processes given in Theorems 3.3.3 and 3.3.4 are from Silvestrov (1974).

Conditions for **J**-convergence of compositions of càdlàg processes with a continuous limiting internal stopping process, formulated in Theorems 3.4.1 and 3.4.4, are from Silvestrov (1972b, 1972e). These theorems cover an essential part of applications.

Theorem 3.4.2, which is an equivalent version of Theorem 3.4.1, and Lemma 3.4.1 are from Silvestrov (1974), where condition $\mathcal{E}_2$ was introduced and used, instead of $\mathcal{E}_1$. The latter condition, equivalent to $\mathcal{E}_2$ but more convenient for application, was given in Silvestrov and Teugels (1998a) and Silvestrov (2000b). Theorem 3.4.2 is given in a new form where condition $\mathcal{E}_2$ is replaced with condition $\mathcal{E}_1$. A weaker form of Theorems 3.4.1 and 3.4.2 was also given by Whitt (1973, 1980) under an additional condition that the limiting internal process is not only continuous but also strictly monotone. Theorem 3.4.3, with the weakened continuity condition $\mathcal{F}_4$ used instead of $\mathcal{E}_1$, is a new.

Conditions for **J**-compactness and **J**-convergence of compositions of càdlàg processes for a general model, where both limiting external and internal stopping processes can be discontinuous, were obtained in Silvestrov (1974). These results are formulated in Theorems 3.6.1 and 3.6.5, which give conditions for **J**-compactness of semi-vector compositions of càdlàg processes, and Theorem 3.6.2, 3.6.3, and 3.6.6, which give conditions for **J**-convergence of semi-vector compositions of càdlàg processes. A key role is played, in these theorems, by continuity conditions of type $\mathcal{G}_4$ also introduced in Silvestrov (1974). Theorem 3.6.4, with the weakened continuity condition $\mathcal{F}_3$, is a new result announced in Silvestrov (2002b).

A vector form of these results is also given in Silvestrov (1974). Theorems 3.8.1 and 3.8.7 give conditions for **J**-compactness of vector compositions of càdlàg processes, whereas Theorems 3.8.2 and 3.8.8 give conditions for **J**-convergence of vector compositions of càdlàg processes. In these theorems, an important role is played by condition $\mathcal{H}_8$ introduced in Silvestrov (1974). It should be noted that, in the case of vector compositions, the corresponding theorems are given in a new improved form with a weaker version of **J**-compactness condition for external processes, $\mathcal{J}_8$, used instead of condition $\mathcal{J}_4$ employed in Silvestrov (1974). It should also be mentioned that analogues of the theorems mentioned above for compositions of non-random càdlàg functions, given in Sections 3.5 and 3.7, are also from Silvestrov (1974). Theorem 3.8.4, which gives a simple sufficient condition for **J**-convergence of vector càdlàg processes, belongs to Whitt (1973, 1980). Theorem 3.8.6, where the weakened continuity condition $\mathcal{F}_3$ is used instead of $\mathcal{E}_4$, is new. This theorem is from Silvestrov (2002a).

Chapter 4

Summary of applications

This chapter gives a summary of applications of general limit theorems on randomly stopped stochastic processes and compositions of stochastic processes. The goal is to show how the general limit theorems given in Chapters 2 and 3 can be applied to some classical models of càdlàg processes with random stopping. These models include sum-processes (random sums), randomly stopped max-processes (extremes with random sample size), generalised exceeding processes, and various renewal models, namely, sum-processes and max-processes with renewal stopping and the so-called shock models. We also consider some related models, for example, accumulation processes.

First, we present results in the most general form with no special independence assumptions imposed on the random variables that are used to construct the corresponding processes. Then we proceed to the most important case where the corresponding processes are constructed from sequences of *independent identically distributed* (i.i.d.) random variables. We will not extend here the examples to processes defined on Markov chains, semi-Markov processes, etc. This would overload the book. Bibliographical remarks reflect our interest in these applications.

The most well known results for classical models of random sums and extremes with random sample size relate to two classes of models based on i.i.d. random variables. The first one concerns the model where random stopping indices and the corresponding external processes are independent. The second one deals with a model in which the random indices depend on the external processes but, being properly normalised, converge in probability. This provides asymptotic independence of the corresponding external processes and normalised random stopping indices.

We consider a general model where the corresponding external sum- or max-processes and stopping indices can be dependent in an arbitrary way. We show that weak convergence as well as **J**-convergence of the corresponding randomly stopped sum- or max-processes can be obtained under only two conditions. The first one is the condition of joint weak convergence of the normalised random stopping indices and external sum- or max-processes with non-random stopping indices, and the second one is the condition of **J**-compactness of the external processes. No extra assumptions on their independence or even on their asymptotic independence is required. In some sense, these results are surprising. They give a unified approach to various concrete models including those mentioned above. Theorems 4.2.1, 4.2.2 and 4.7.1, 4.7.2 contain results for general random sums and extremes with random sample size, whereas Theorems 4.2.3, 4.2.4 and 4.7.3, 4.7.4 cover the case of random sums and extremes with random sample size in the models constructed from sequences of i.i.d. random variables.

For renewal models, we first concentrate on the model of generalised exceeding processes. Such a process is constructed by random stopping of a càdlàg process at the moments when another non-decreasing càdlàg process exceeds the levels $t \geq 0$. This class of processes includes many various renewal models. In particular, sum-processes and max-processes with renewal stopping, as well as shock processes, are examples of the generalised exceeding processes. We show that weak convergence as well **J**-convergence of the generalised exceeding processes can be obtained under the only condition of **J**-convergence of two-dimensional càdlàg processes used to construct the generalised exceeding processes. The main results concerning weak convergence are given in Theorems 4.3.1 – 4.3.3 and 4.3.6. The main results concerning the **J**-convergence are given in Theorems 4.3.4, 4.3.5 and 4.3.7. The case of step generalised exceeding processes requires a special consideration. This is done in Theorems 4.4.1 and 4.4.2.

Application of these results to renewal models constructed from sequences of i.i.d. random variables yields very natural and general conditions of weak and **J**-convergence of renewal type processes in general triangular array mode. These results cover many results in the area, in particular those related to the classical case when external processes converge to a Wiener process and internal stopping processes converge to non-random functions. Our main results concerning the classical model of sum-processes with renewal stopping is Theorem 4.5.5 that covers the case where the limiting external process is a Wiener process, and Theorems 4.5.6 and 4.5.7 that treat the general case where the limiting external process can be an arbitrary càdlàg homogeneous process with independent increments. In both cases, the corresponding limiting stopping process is an exceeding process constructed from a non-negative càdlàg homogeneous process with independent increments.

As was mentioned above, we will also consider some models related to renewal type processes, namely accumulation processes. Here, the main results for general accumulation processes are given in Theorem 4.6.1 and 4.6.2. The case of accumulation processes with embedded regeneration cycles is covered in Theorems 4.6.3 and 4.6.4.

We also consider two types of models constructed from two dimensional càdlàg processes which has a sum-process as its first component and a max-process as the second one.

The first class is represented by max-processes with renewal stopping. Here our main results concerning weak and **J**-convergence for these processes are given in Theorems 4.9.1 and 4.9.2. Theorems 4.9.3 and 4.9.4 cover the case of max-processes with renewal stopping based on sequences of i.i.d. random variables.

The second class is represented by so-called shock processes. Here the main result is Theorem 4.10.1, which gives conditions of weak and **J**-convergence for general shock processes. Theorems 4.10.2 – 4.10.4 cover the case of shock processes based on sequences of i.i.d. random variables.

In this context, we would like also to mention Theorems 4.8.1 and 4.8.2, which gives general conditions of weak convergence and **J**-convergence for mixed sum-max processes based on sequences of i.i.d. random variables.

It seems us, results presented in Chapter 4 illustrate in a spectacular way a power of general limit theorems for randomly stopped processes and compositions of càdlàg processes presented in Chapters 2 and 3.

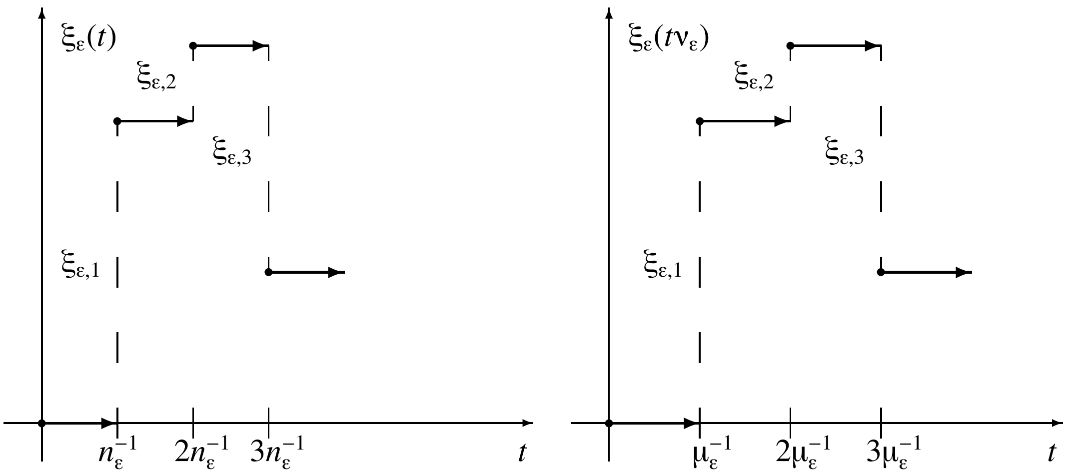
In Section 4.1, we introduce the models of càdlàg processes with random stopping mentioned above. Section 4.2 contains results concerned randomly stopped sum-processes (random sums). Limit theorems for generalised exceeding processes are given in Sections 4.3 and 4.4. Section 4.5 contains results concerned sum-processes with renewal stopping. Limit theorems for accumulation processes are given in Section 4.6. Limit theorems for extremes with random sample size are given in Section 4.7. In Section 4.8, limit theorems for mixed sum-max processes are given. In Section 4.9 limit theorems for max-processes with renewal stopping are given. The last Section 4.10 contains results on limit theorems for shock processes. The reference remarks are also given in the end of this section.

4.1 Introductory remarks

In this section, we describe some basic classes of stochastic processes used in applications of general limit theorems for compositions of càdlàg stochastic processes.

Let us make the following remark. In Chapter 4, we systematically study the so-called triangular array model in which the stochastic processes, say $\zeta_\varepsilon(t), t \geq 0$, depend on some small series parameter $\varepsilon \geq 0$. In the introductory section, however, we restrict consideration to a simpler model where the dependence of the processes on the parameter ε is introduced in terms of non-random scale normalisation coefficients $u_\varepsilon, t_\varepsilon > 0$, and the processes have the following structure: $\zeta_\varepsilon(t) = \zeta(tt_\varepsilon)/u_\varepsilon, t \geq 0$. This will permit to concentrate on the structure of the model of the corresponding processes.

We would also like to mention a general convention concerning notations used in Chapter 4. Henceforth, except for Sections 4.3 and 4.4, the parameter ε takes only positive values and, therefore, the symbol $\varepsilon \to 0$ means that $0 < \varepsilon \to 0$. As a matter of fact, in Sections 4.3 and 4.4, we consider general càdlàg processes that have the same structure in the pre-limiting case ($\varepsilon > 0$) as well as in the limiting case ($\varepsilon = 0$). In other sections, we consider various models constructed from sequences of random variables, for example, sum-processes, renewal processes, etc. At the same time, the corresponding limiting processes, usually, are constructed from some homogeneous processes with independent increments. Their structure differs from the structure of the corresponding



(a) A sum-process. (b) A random stopping.

Figure 4.1: A randomly stopped sum-process.

pre-limiting processes.

4.1.1. Randomly stopped sum-processes (random sums). Let $\xi_k, k = 1, 2, \ldots$ be a sequence of real-valued random variables. The classical object of studies in probability theory is sums of random variables $\xi(n) = \xi_1 + \ldots \xi_n, n = 0, 1, \ldots$, where $\xi(0) = 0$. The case, where $\xi_k, k = 1, 2, \ldots$ are i.i.d. random variables, is the most important and well investigated.

In order to study the whole trajectory of the sum-sequence $\xi(n), n = 0, 1, 2, \ldots$, it is convenient to connect with these sums the stochastic *sum-process* $\xi(t) = \sum_{k=1}^{[t]} \xi_k,\ t \geq 0$.

Studies of the asymptotic behaviour are usually concerned with a model in which the number of summands tends to infinity and the sums are normalised in a proper way. These elements can be introduced in the following way. Let $n_\varepsilon, u_\varepsilon$ be positive functions of a "small" parameter $\varepsilon > 0$ such that $n_\varepsilon, u_\varepsilon \to \infty$ as $\varepsilon \to 0$ (here $t_\varepsilon = n_\varepsilon$). We will consider the asymptotic behaviour of the stochastic *sum-processes*

$$\xi_\varepsilon(t) = \sum_{k=1}^{[tn_\varepsilon]} \xi_{\varepsilon,k},\ t \geq 0, \tag{4.1.1}$$

where the random variables $\xi_{\varepsilon,k} = \xi_k / u_\varepsilon, k \geq 1$.

Note that the normalised random variables $\xi_{\varepsilon,k} = \xi_k / u_\varepsilon, k \geq 1$ represent a particular case of the so-called triangular array model in which the random variables $\xi_{\varepsilon,k}$ depend on a small parameter ε.

A natural generalisation is a model in which the non-random indices n_ε, which determine the number of summands, are replaced with non-negative random variables μ_ε. In this model, the random variables $\mu_\varepsilon, \varepsilon > 0$ should be defined on the same probability space as the random variables $\xi_k, k \geq 1$.

Sum-processes with *random stopping index* (*random sums*) appear, for example, in various sample models with *random sample size*. In order to have a consistent model, it is natural to normalise the random variables μ_ε by n_ε and consider the normalised random stopping indices $\nu_\varepsilon = \mu_\varepsilon/n_\varepsilon$, The object of interest is the *randomly stopped sum-processes*

$$\zeta_\varepsilon(t) = \sum_{k=1}^{[t\mu_\varepsilon]} \xi_{\varepsilon,k} = \xi_\varepsilon(t\nu_\varepsilon), \ t \geq 0. \tag{4.1.2}$$

Relation (4.1.2) shows that the randomly stopped sum-process $\zeta_\varepsilon(t), t \geq 0$ can be represented in the form of a composition of the external sum-process $\xi_\varepsilon(t), t \geq 0$ and the internal stopping process $\nu_\varepsilon(t) = t\nu_\varepsilon, t \geq 0$.

Figure 4.1 shows the behaviour of the trajectories of a sum-process with non-random and random stopping.

4.1.2. Randomly stopped max-processes (extremes with random sample size). Let $\rho_k, k = 1, 2, \ldots$ be a sequence of real-valued random variables. Another classical object of studies in probability theory is maxima of random variables $\rho(n) = \max(\rho_1, \ldots, \rho_n), n = 0, 1, \ldots$, where $\rho(0) = 0$. Again, the case, where $\rho_k, k = 1, 2, \ldots$ are i.i.d. random variables, is the most important and well investigated.

In order to study the whole trajectory of the max-sequence $\rho(n), n = 0, 1, 2, \ldots$, it is convenient to consider the *max-processes* $\rho(t) = \max_{1\leq k\leq[t]} \rho_k, t \geq 0$.

Studies of the asymptotics are usually concerned with a model in which the sample size tends to infinity and the maxima are normalised in a proper way. These elements can be introduced in the following way. Let $n_\varepsilon, u_\varepsilon$ be positive functions of a "small" parameter $\varepsilon > 0$ such that $n_\varepsilon, u_\varepsilon \to \infty$ as $\varepsilon \to 0$. Consider the asymptotics of the stochastic *max-processes*

$$\rho_\varepsilon(t) = \max_{1\leq k\leq[tn_\varepsilon]} \rho_{\varepsilon,k}, \ t \geq 0, \tag{4.1.3}$$

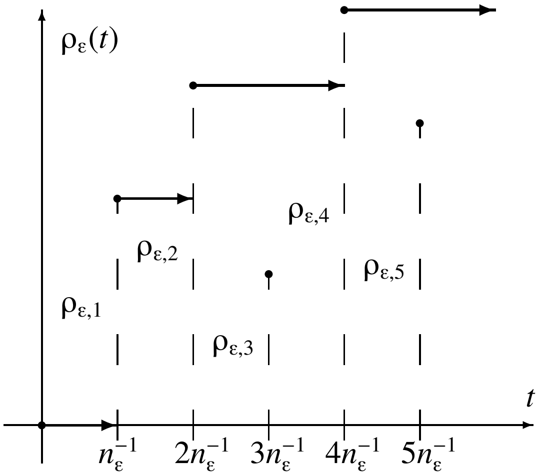
where the random variables $\rho_{\varepsilon,k} = \rho_k/u_\varepsilon, k \geq 1$.

A natural generalisation is a model in which the non-random indices n_ε are replaced with non-negative random variables μ_ε. The random variables $\mu_\varepsilon, \varepsilon > 0$ should be defined on the same probability space as the random variables $\rho_k, k \geq 1$.

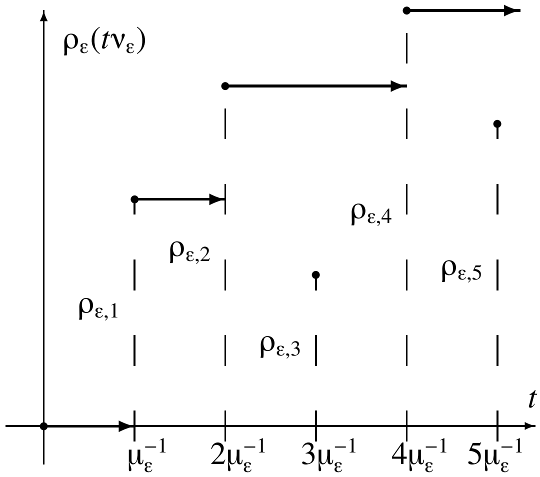
Max-processes with random stopping indices (extremes with random sample size) appear in sample models with random sample size. In order to have a consistent model, it is natural to normalise the random variables μ_ε by n_ε and consider the normalised random stopping indices $\nu_\varepsilon = \mu_\varepsilon/n_\varepsilon$. In this case, the object of interest is the *randomly stopped max-processes*

$$\zeta_\varepsilon(t) = \max_{1\leq k\leq[t\mu_\varepsilon]} \rho_{\varepsilon,k} = \rho_\varepsilon(t\nu_\varepsilon), \ t \geq 0. \tag{4.1.4}$$

Relation (4.1.4) shows that the randomly stopped max-process $\zeta_\varepsilon(t), t \geq 0$ can be represented in the form of a composition of the external max-process $\rho_\varepsilon(t), t \geq 0$, and the internal stopping process $\nu_\varepsilon(t) = t\nu_\varepsilon, t \geq 0$.



(a) A max-process.



(b) A random stopping.

Figure 4.2: A randomly stopped max-process.

Figure 4.2 shows the behaviour of trajectories of max-processes with non-random and random stopping.

4.1.3. Renewal type processes. This is one of the models of stochastic processes widely used in applications, e.g., queuing theory, reliability theory, etc. Let $\kappa_k, k = 1, 2, \dots$ be a sequence of non-negative random variables. Let also $\kappa(n) = \kappa_1 + \dots \kappa_n, n = 0, 1, \dots$, where $\kappa(0) = 0$. The random variables $\kappa(n)$ are usually interpreted as "renewal" moments. A standard additional assumption is that the random variables $\kappa(n) \xrightarrow{\mathsf{P}} \infty$ as $n \to \infty$. The case, where $\kappa_k, k = 1, 2, \dots$ are i.i.d. random variables, is the most important and well investigated. As above, the corresponding sum-processes can be constructed as $\kappa(t) = \sum_{k=1}^{[t]} \kappa_k,\ t \geq 0$.

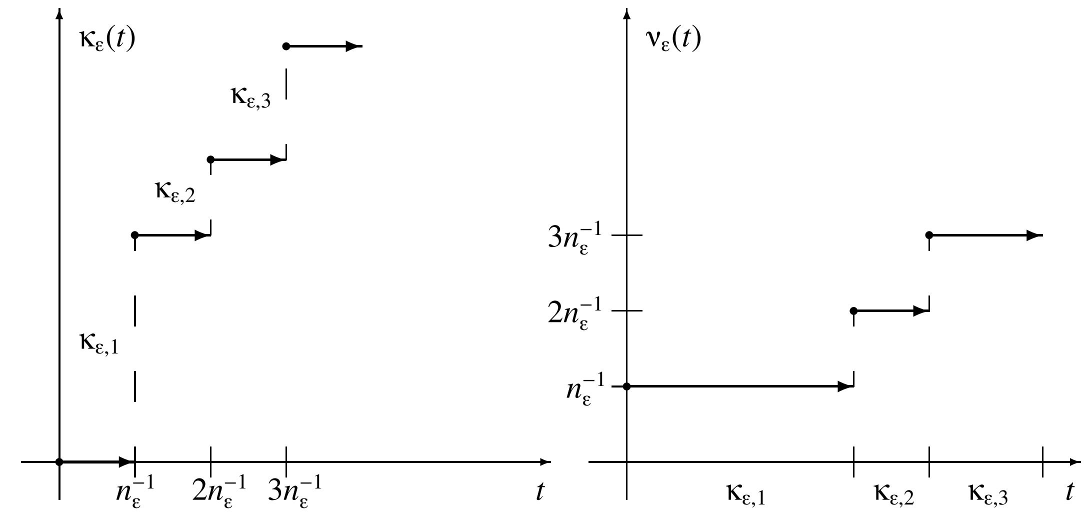
Let $n_\varepsilon, t_\varepsilon$ be positive functions of a "small" parameter $\varepsilon > 0$ such that $n_\varepsilon, t_\varepsilon \to \infty$ as $\varepsilon \to 0$. The corresponding normalised version of the process $\kappa(t), t \geq 0$ can be defined by

$$\kappa_\varepsilon(t) = \sum_{k=1}^{[tn_\varepsilon]} \kappa_{\varepsilon,k},\ t \geq 0, \tag{4.1.5}$$

where the random variables $\kappa_{\varepsilon,k} = \kappa_k / t_\varepsilon, k \geq 1$.

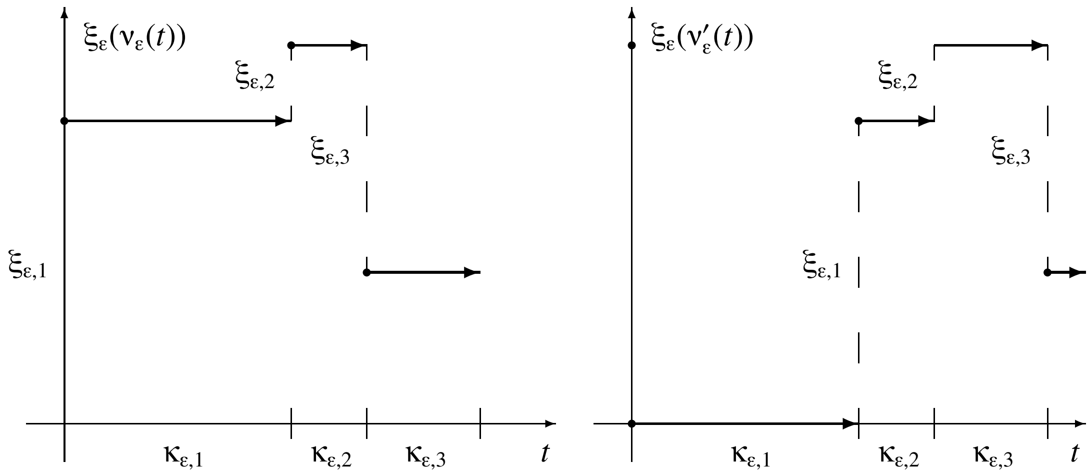
The *renewal process* can be defined by $\mu(t) = \min(n : \kappa(n) > t) = \inf(s : \kappa(s) > t) = \sup(s : \kappa(s) \leq t), t \geq 0$. The corresponding normalised version of the *renewal process* with rescaled time can be defined by

$$\nu_\varepsilon(t) = \mu(tt_\varepsilon)/n_\varepsilon = \sup(s : \kappa_\varepsilon(s) \leq t),\ t \geq 0. \tag{4.1.6}$$



(a) A non-negative sum-process. (b) A renewal stopping.

Figure 4.3: A renewal process.



(a) A standard renewal stopping. (b) A modified renewal stopping.

Figure 4.4: A sum-process with renewal stopping.

Figure 4.3 shows the behaviour of trajectories of a non-negative sum-process and the corresponding renewal process.

There is a slightly *modified version of the renewal process*, which is defined by $\mu'(t) = \mu(t) - 1 = \max(n : \kappa(n) \leq t)$. The process $\mu'(t) = \mu(t) - 1$ counts the number of renewal moments in the interval $[0, t]$. The corresponding normalised version of the renewal process with rescaled time can be defined by $\nu'_\varepsilon(t) = \mu'(tt_\varepsilon)/n_\varepsilon = \nu_\varepsilon(t) - 1/n_\varepsilon,\ t \geq 0$.

4.1.4. Sum-processes with renewal stopping. We are now in a position to introduce a model for randomly stopped processes, which generalises the model of renewal processes. This model is also widely used in queuing theory, insurance mathematics, etc. Let $(\kappa_k, \xi_k), k = 1, 2, \ldots$ be a sequence of random vectors with, respectively, the first component non-negative and the second one real-valued. As above, $\kappa(n) = \kappa_1 + \ldots \kappa_n, \xi(n) = \xi_1 + \ldots \xi_n,\ n = 0, 1, \ldots$, where $\kappa(0) = \xi(0) = 0$. The case, where $(\kappa_k, \xi_k), k = 1, 2, \ldots$ are i.i.d. random vectors, is the most important and well investigated. Let us introduce *sum-process with renewal stopping* by $\zeta(t) = \sum_{k=1}^{[\mu(t)]} \xi_k = \xi(\mu(t)), t \geq 0$, where $\xi(t) = \sum_{k=1}^{[t]} \xi_k$ and $\mu(t) = \sup(s : \kappa(s) \leq t), t \geq 0$.

The corresponding normalised version of *sum-process with renewal stopping* and rescaled time can be defined by

$$\zeta_\varepsilon(t) = \xi(\mu(tt_\varepsilon))/u_\varepsilon = \xi_\varepsilon(\nu_\varepsilon(t)),\ t \geq 0, \tag{4.1.7}$$

where $\xi_\varepsilon(t) = \xi_\varepsilon(tn_\varepsilon)/u_\varepsilon$ and $\nu_\varepsilon(t) = \mu(tt_\varepsilon)/n_\varepsilon$.

A slightly modified version of *sum-process with renewal stopping* and rescaled time can be defined by

$$\zeta'_\varepsilon(t) = \xi_\varepsilon(\nu'_\varepsilon(t)) = \xi_\varepsilon(\nu_\varepsilon(t) - 1/n_\varepsilon),\ t \geq 0. \tag{4.1.8}$$

Figure 4.4 shows the difference in the behaviour of trajectories of the processes $\zeta_\varepsilon(t), t \geq 0$ and $\zeta'_\varepsilon(t), t \geq 0$.

4.1.5. Accumulation processes. This model deals with a càdlàg stochastic process $\zeta(t), t \geq 0$ and a sequence of random renewal moments $0 = \tau_0 < \tau_1 < \tau_2 < \ldots$ such that $\tau_n \xrightarrow{\mathsf{P}} \infty$ as $n \to \infty$. Both the process and the sequence are defined on the same probability space. Let us also define random variables $\kappa_k = \tau_k - \tau_{k-1}, \xi_k = \zeta(\tau_k) - \zeta(\tau_{k-1})$ and $\varsigma_k = \sup_{t \in [\tau_{k-1}, \tau_k)} |\zeta(t) - \zeta(\tau_{k-1})|$, for $k \geq 1$. The basic case is where $(\kappa_k, \xi_k, \varsigma_k), k \geq 1$ is a sequence of i.i.d. random vectors.

The random variables κ_k are usually interpreted as times between successive "renewals", τ_k as successive renewal moments, ζ_k as accumulations between the successive renewals, and ς_k as oscillations of the *accumulation process* $\zeta(t), t \geq 0$ between successive renewal moments.

The key role in studying limit theorems for accumulation processes is played by the following representation of the accumulation process $\zeta(t), t \geq 0$ in the form of a renewal type random sum: **(a)** $\zeta(t) = \zeta(0) + \sum_{k=1}^{\mu(t)-1} \xi_k + \varsigma(t), t \geq 0$, where **(b)** $\varsigma(t) = \zeta(t) - \zeta(0) - \sum_{k=1}^{\mu(t)-1} \xi_k, t \geq 0$, and **(c)** $\mu(t) = \min(n : \tau_n > t), t \geq 0$. This representation

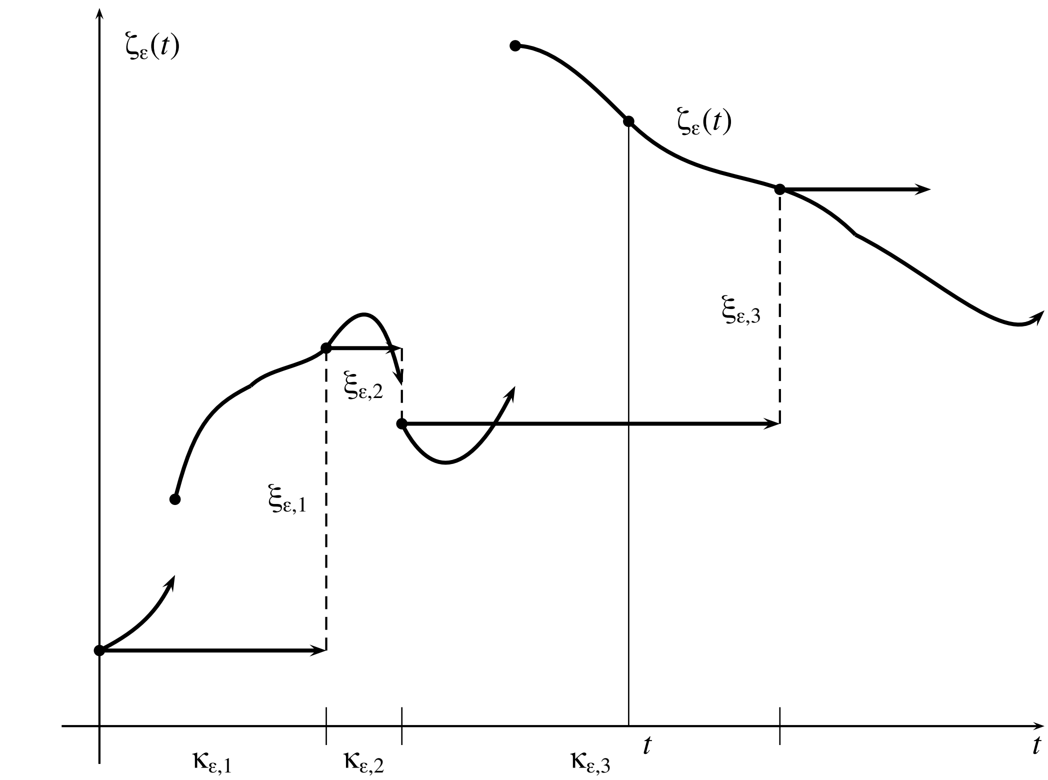


Figure 4.5: An accumulation process.

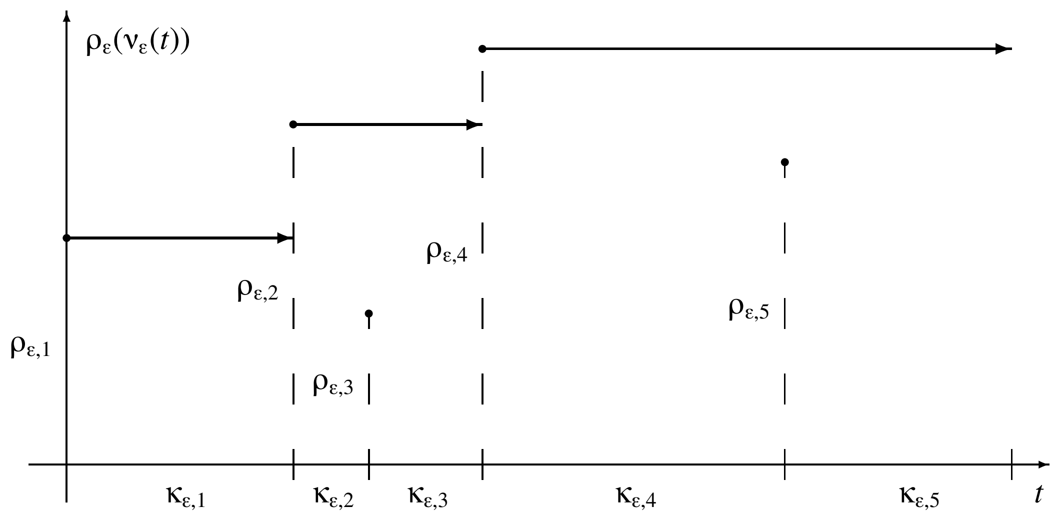


Figure 4.6: A max-process with renewal stopping.

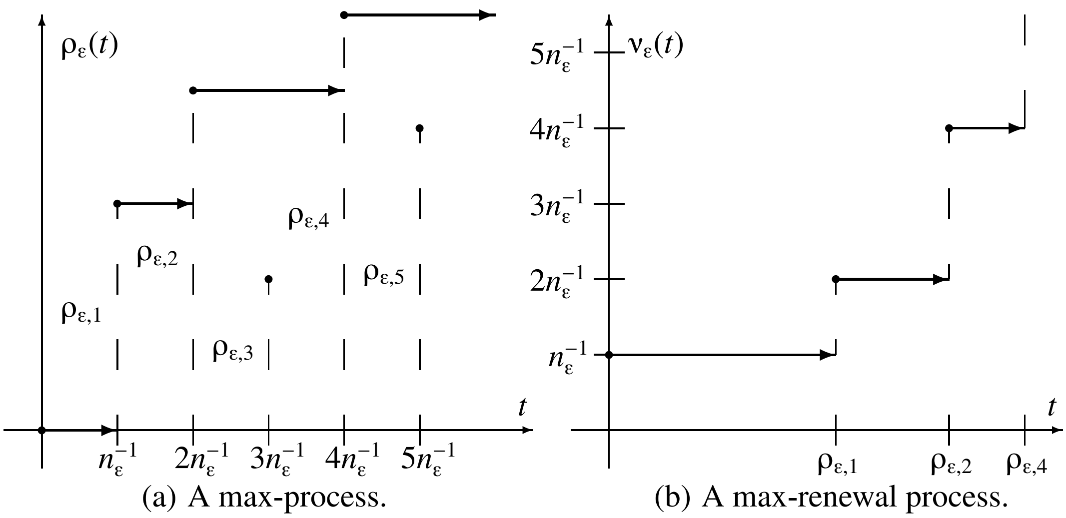


Figure 4.7: A max-renewal process.

also implies the following estimate for the *residual accumulation process* $\varsigma(t), t \geq 0$: **(d)** $|\varsigma(t)| \leq \max_{1\leq k\leq \mu(t)} \varsigma_k,\ t \geq 0$.

The corresponding normalised version of representation **(a)** takes the form

$$\zeta_\varepsilon(t) = \zeta(tt_\varepsilon)/u_\varepsilon = \xi_\varepsilon(\nu_\varepsilon(t) - 1/n_\varepsilon) + \varsigma(tt_\varepsilon)/u_\varepsilon, t \geq 0, \tag{4.1.9}$$

where $\xi_\varepsilon(t) = \xi_\varepsilon(tn_\varepsilon)/u_\varepsilon = \sum_{k=0}^{[tn_\varepsilon]} \xi_k/u_\varepsilon, \xi_0 = \xi(0)$ and $\nu_\varepsilon(t) = \mu(tt_\varepsilon)/n_\varepsilon$.

Figure 4.5 illustrates the behaviour of trajectories of an accumulation process and the corresponding *embedded sum-process* with renewal stopping.

4.1.6. Max-processes with renewal stopping. Let $(\kappa_k, \rho_k), k = 1, 2, \ldots$ be a sequence of random vectors with, respectively, non-negative and real-valued first and second components. As above, $\kappa(n) = \kappa_1 + \ldots \kappa_n, \rho(n) = \max(\rho_1, \ldots, \rho_n), n = 0, 1, \ldots$, where $\kappa(0) = \rho(0) = 0$. Again, the case, where $(\kappa_k, \rho_k), k = 1, 2, \ldots$ are i.i.d. random vectors, is the most important and well investigated. The *max-processes with renewal stopping* can be defined by $\zeta(t) = \max_{1\leq k\leq[\mu(t)]} \rho_k = \rho(\mu(t)), t \geq 0$, where $\rho(t) = \max_{1\leq k\leq[t]} \rho_k, t \geq 0$ and $\mu(t) = \sup(s : \kappa(s) \leq t), t \geq 0$.

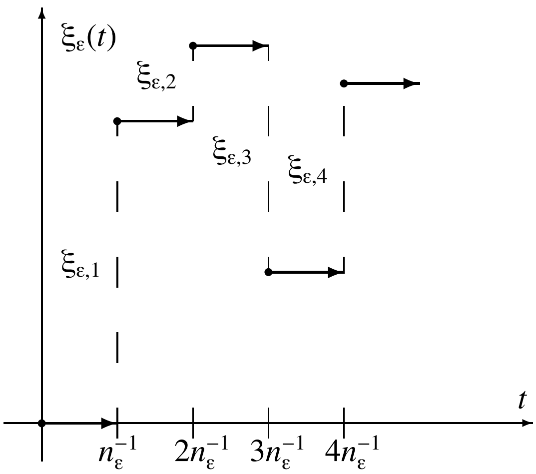
The corresponding normalised version of this process with rescaled time is defined by

$$\zeta_\varepsilon(t) = \rho(\mu(tt_\varepsilon))/u_\varepsilon = \rho_\varepsilon(\nu_\varepsilon(t)),\ t \geq 0, \tag{4.1.10}$$

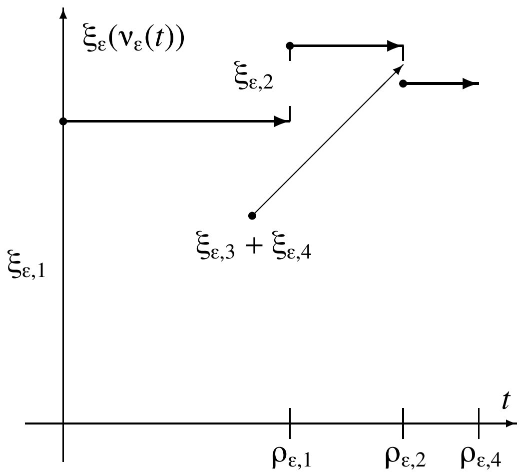
where $\rho_\varepsilon(t) = \rho_\varepsilon(tn_\varepsilon)/u_\varepsilon$ and $\nu_\varepsilon(t) = \mu(tt_\varepsilon)/n_\varepsilon$.

Figure 4.6 shows a typical trajectory for a max-process with renewal stopping.

4.1.7. Shock processes. Shock processes are constructed in a way opposite, in some sense, to the one employed for max-processes with renewal stopping. In this model, a



(a) A sum-process.



(b) A shock process.

Figure 4.8: A shock process.

sum-process is randomly stopped at moments when the max-process exceeds the levels $t > 0$. In this model, one should first define the so-called max-renewal process. Let $(\xi_k, \rho_k), k = 1, 2, \ldots$ be a sequence of random vectors with the first component being non-negative and the second one real-valued. As above, $\xi(n) = \xi_1 + \ldots \xi_n, \rho(n) = \max(\rho_1, \ldots, \rho_n), n = 0, 1, \ldots$, where $\xi(0) = \rho(0) = 0$. The most important and well investigated is again the case where $(\xi_k, \rho_k), k = 1, 2, \ldots$ are i.i.d. random vectors. To avoid the situation where the random variables are improper, one should assume that the random variables $\rho(n) \xrightarrow{\mathsf{P}} \infty$ as $n \to \infty$. A *max-renewal process* can be defined by $\mu(t) = \min(n : \rho(n) > t) = \sup(s : \rho(s) \le t), t \ge 0$. Then, the corresponding *shock process* can be defined as $\zeta(t) = \xi(\mu(t)), t \ge 0$, where $\xi(t) = \sum_{k=1}^{[t]} \xi_k$.

The corresponding normalised version of a *max-renewal process* with rescaled time can be defined by

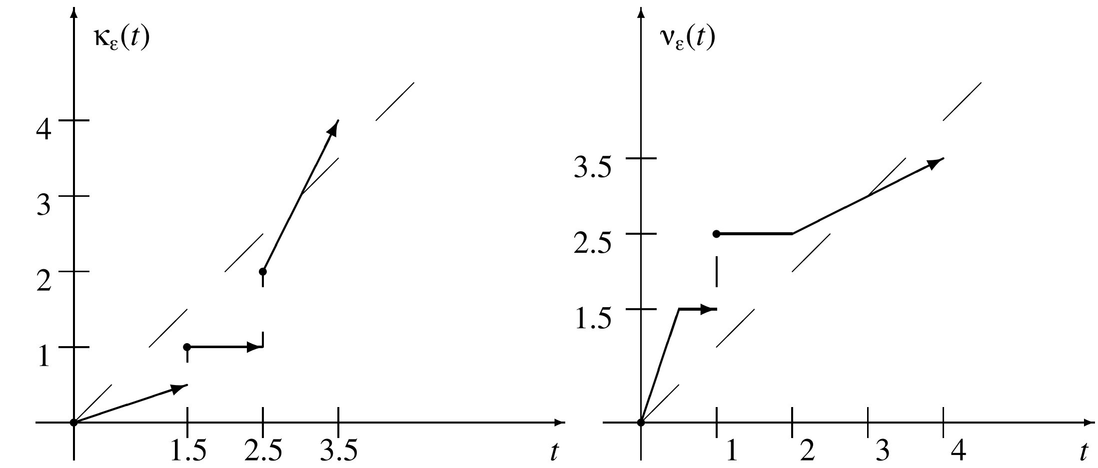
$$\nu_\varepsilon(t) = \mu(tt_\varepsilon)/n_\varepsilon = \sup(s : \rho_\varepsilon(s) \le t), \ t \ge 0, \tag{4.1.11}$$

where $\rho_\varepsilon(t) = \rho(tn_\varepsilon)/t_\varepsilon$ (note that the normalisation function t_ε is used instead of u_ε, since the latter function is used as a normalisation function for the external process $\xi_\varepsilon(t) = \sum_{k=1}^{[tn_\varepsilon]} \xi_k/u_\varepsilon$).

Then the corresponding normalised version of the *shock process* with rescaled time can be defined by

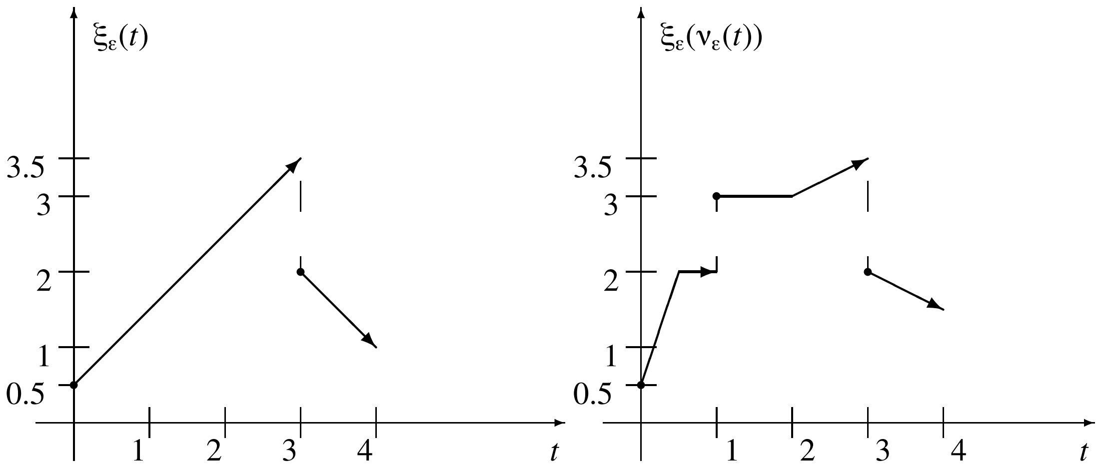
$$\zeta_\varepsilon(t) = \xi(\mu(tt_\varepsilon))/u_\varepsilon = \xi_\varepsilon(\nu_\varepsilon(t)), \ t \ge 0. \tag{4.1.12}$$

Figures 4.7 and 4.8 illustrate the behaviour of the trajectories of a max-renewal process and a shock process.



(a) A non-decreasing basic process. (b) An exceeding time process.

Figure 4.9: An exceeding time process.



(a) An external process. (b) A generalised exceeding process.

Figure 4.10: A generalised exceeding process.

4.1.8. Generalised exceeding processes. Such processes are constructed with the use of random stopping of a càdlàg process $\xi(t), t \geq 0$ at the moments when another non-decreasing càdlàg process $\kappa(s), s \geq 0$ exceeds the level $t \geq 0$. So, we first introduce the *exceeding time process* $\mu(t) = \sup(s : \kappa(s) \leq t), t \geq 0$, and then the *generalised exceeding process* as the composition $\zeta(t) = \xi(\mu(t)), t \geq 0$.

The corresponding normalised versions of the processes $\xi(t), t \geq 0$ and $\kappa(t), t \geq 0$ can be defined by

$$\xi_\varepsilon(t) = \xi(tn_\varepsilon)/u_\varepsilon, \ \kappa_\varepsilon(t) = \kappa(tn_\varepsilon)/t_\varepsilon, \ t \geq 0. \tag{4.1.13}$$

Then the normalised versions of the *exceeding time process* and the *generalised exceeding time process* can be defined by

$$\nu_\varepsilon(t) = \mu(tt_\varepsilon)/n_\varepsilon = \sup(s : \kappa_\varepsilon(s) \leq t), \ t \geq 0, \tag{4.1.14}$$

and

$$\zeta_\varepsilon(t) = \zeta(tt_\varepsilon)/u_\varepsilon = \xi_\varepsilon(\nu_\varepsilon(t)), \ t \geq 0. \tag{4.1.15}$$

This class of processes includes many renewal type models. In particular, sum-processes and max-processes with renewal stopping, as well as shock processes, are examples of the generalised exceeding processes.

Figures 4.9 and 4.10 show the behaviour of trajectories of an exceeding time process and a generalised exceeding process, respectively. To show the relation between the corresponding trajectories, we took an example of concrete realisations shown in these figures.

4.2 Randomly stopped sum-processes

In this section, we will study limit theorems for the classical model of randomly stopped sum-processes.

4.2.1. Sum-processes with random stopping indices. Let, for every $\varepsilon > 0$, $\xi_{\varepsilon,n}$, $n = 1, 2, \ldots$ be a sequence of real-valued random variables and μ_ε a non-negative random variable. Further, we need a non-random function $n_\varepsilon > 0$ of parameter ε such that $n_\varepsilon \to \infty$ as $\varepsilon \to 0$.

Consider a *sum-process* with non-random stopping index,

$$\xi_\varepsilon(t) = \sum_{k \leq tn_\varepsilon} \xi_{\varepsilon,k}, \ t \geq 0.$$

We will be interested in an analogue of this process the stopping index of which is also random. So, define the càdlàg process

$$\zeta_\varepsilon(t) = \sum_{k \leq t\mu_\varepsilon} \xi_{\varepsilon,k}, \ t \geq 0.$$

Denote by $\nu_\varepsilon = \mu_\varepsilon/n_\varepsilon$ the normalised *random stopping index*. Then the process $\zeta_\varepsilon(t) = \xi_\varepsilon(t\nu_\varepsilon)$, $t \geq 0$ can be represented in the form of a composition of two processes $\xi_\varepsilon(t), t \geq 0$ and $\nu_\varepsilon(t) = t\nu_\varepsilon, t \geq 0$.

Consider the following weak convergence condition:

$\mathcal{A}_{51}$: $(\nu_\varepsilon, \xi_\varepsilon(t)), t \in U \Rightarrow (\nu_0, \xi_0(t)), t \in U$ as $\varepsilon \to 0$, where (a) ν_0 ia a non-negative random variable; (b) $\xi_0(t), t \geq 0$ is a càdlàg process; (c) U is a subset of $[0, \infty)$ that is dense in this interval and contains the point 0.

Let us also assume that the following condition of **J**-compactness holds for the sum-processes $\xi_\varepsilon(t), t \geq 0$:

$\mathcal{J}_{23}$: $\lim_{c\to 0} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\Delta_J(\xi_\varepsilon(\cdot), c, T) > \delta\} = 0, \ \delta, T > 0.$

Denote by W_0 the set of $t \geq 0$ such that $\mathsf{P}\{\tau_{kn}/\nu_0 = t\} = 0$ for all $k, n = 1, 2, \ldots$, where τ_{kn}, $k = 1, 2, \ldots$ are successive moments of jumps of the process $\xi_0(t)$, $t \geq 0$, with absolute values of the jumps lying in the interval $[\frac{1}{n}, \frac{1}{n-1})$ (see Subsection 2.2.6 for details). Recall that the random variables τ_{kn} take values in the interval $(0, \infty]$ and the random variable ν_0 takes values in the interval $[0, \infty)$. So, the random variable τ_{kn}/ν_0 takes values in the interval $(0, \infty]$, that is, it is positive and, possibly, improper.

The set W_0 coincides with $[0, \infty)$ except for at most a countable set. Also, $0 \in W_0$. Indeed, the set $\overline{W}_0 = [0, \infty) \setminus W_0$ coincides with the set of all atoms of the distribution functions of the random variables τ_{kn}/ν_0, $k, n = 1, 2, \ldots$. This set is at most countable and $0 \notin \overline{W}_0$. Therefore, the set W_0 equals $[0, \infty)$ except for the countable set $\overline{W}_0$. Also, $0 \in W_0$.

Note that W_0 is a set of points of stochastic continuity for the process $\xi_0(t\nu_0), t \geq 0$.

The following theorem is a direct corollary of the results in Silvestrov (1971b, 1972a, 1972b).

Theorem 4.2.1. *Let conditions $\mathcal{A}_{51}$ and $\mathcal{J}_{23}$ hold. Then*

$$\zeta_\varepsilon(t) = \xi_\varepsilon(t\nu_\varepsilon), t \in W_0 \Rightarrow \zeta_0(t) = \xi_0(t\nu_0), t \in W_0 \ \textit{as}\ \varepsilon \to 0.$$

Theorem 4.2.2. *Let conditions $\mathcal{A}_{51}$ and $\mathcal{J}_{23}$ hold. Then*

$$\zeta_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{J}} \zeta_0(t), t \geq 0 \ \textit{as}\ \varepsilon \to 0.$$

Proof of Theorems 4.2.1 and 4.2.2. These theorems are direct corollaries of Theorems 2.6.1 and 3.4.1 applied to the processes $\xi_\varepsilon(t), t \geq 0$ and $\nu_\varepsilon(t) = t\nu_\varepsilon, t \geq 0$. Condition $\mathcal{A}_{51}$ obviously implies in this case that condition $\mathcal{A}_{21}^{\mathbf{V}}$ holds with the set $V = [0, \infty)$. Condition $\mathcal{J}_{23}$ coincides with condition $\mathcal{J}_{7}$. By the definition, the set W_0 is a set of all points $w \geq 0$ that satisfy condition $\mathcal{C}_5^{(\mathbf{w})}$. Therefore, by applying Theorem 2.6.1, we obtain weak convergence of the compositions $\zeta_\varepsilon(t) = \xi_\varepsilon(t\nu_\varepsilon)$ on the set W_0.

Condition $\mathcal{A}_{51}$ obviously implies in this case that condition $\mathcal{A}_{36}$ holds. Also, condition $\mathfrak{J}_{23}$ coincides with condition $\mathfrak{J}_4$. Condition $\mathfrak{C}_7$ holds, since W_0 equals $[0, \infty)$ except for at most a countable set, and also $0 \in W_0$. Finally, condition $\mathfrak{B}_4$ holds, since the limiting stopping process $\nu_0(t) = t\nu_0, t \geq 0$ is continuous. Therefore, applying Theorem 3.4.1 proves that the compositions $\zeta_\varepsilon(t), t \geq 0$ **J**-converge to the process $\zeta_0(t), t \geq 0$ as $\varepsilon \to 0$. □

Remark 4.2.1. Theorems 4.2.1 and 4.2.2 are also direct corollaries of the translation Theorems 2.8.2 and 3.4.4. This can be seen by applying these theorems to the compositions $\zeta_\varepsilon(t) = \xi_\varepsilon(t\nu_\varepsilon)$, $t \geq 0$ in the case where the constant $\alpha = 0$ and $h(x) \equiv 1$ is taken as the slowly varying function. However, one should involve in this case the additional condition $\mathfrak{I}_4$ that requires a.s. positivity of the random variable ν_0.

Note that Theorems 4.2.1 and 4.2.2 do not require any independence conditions to be imposed on the random variables $\xi_{\varepsilon,n}, n = 1, 2, \ldots$ and the stopping indices ν_ε.

It should also be noted that there is an advantage to formulate Theorems 4.2.1 and 4.2.2 separately. As a matter of fact, Theorem 4.2.1 gives additional information about the set of weak convergence of the processes $\zeta_\varepsilon(t), t \geq 0$.

4.2.2. Sum-processes based on i.i.d. random variables. Let us now consider the classical case where the following condition holds:

$\mathcal{T}_1$: $\xi_{\varepsilon,n}, n = 1, 2, \ldots$ is (for every $\varepsilon > 0$) a sequence of real-valued i.i.d. random variables .

In this case, the process $\xi_\varepsilon(t), t > 0$ is a sum-process of i.i.d. random variables.

Let us recall conditions for weak convergence of such processes known as the *central criterion for convergence* in the form given, for example, in Loève (1955). These conditions involve the tail probabilities, the truncated means, and the truncated variances of the random variables $\xi_{\varepsilon,1}$:

$\mathcal{S}_1$: (a) $n_\varepsilon \mathsf{P}\{\xi_{\varepsilon,1} > v\} \to \pi_2(v)$ as $\varepsilon \to 0$ for all $v > 0$ that are points of continuity of the limiting function $\pi_2(v)$;

(b) $n_\varepsilon \mathsf{P}\{\xi_{\varepsilon,1} \leq v\} \to \pi_2(v)$ as $\varepsilon \to 0$ for all $v < 0$ that are points of continuity of the limiting function $\pi_2(v)$.

$\mathcal{S}_2$: $n_\varepsilon \mathsf{E}\xi_{\varepsilon,1}\chi(|\xi_{\varepsilon,1}| \leq v) \to a(v)$ as $\varepsilon \to 0$ for some $v > 0$ for which the points $\pm v$ are points of continuity of the limiting function $\pi_2(v)$.

$\mathcal{S}_3$: $n_\varepsilon \operatorname{Var} \xi_{\varepsilon,1}\chi(|\xi_{\varepsilon,1}| \leq v) \to b^2$ as $\varepsilon \to 0$ and then $v \to 0$. This expression refers to two iterated limits of the form $\lim_{0<v\to 0} \overline{\lim}_{\varepsilon\to 0}$ and $\lim_{0<v\to 0} \underline{\lim}_{\varepsilon\to 0}$.

The limits above satisfy a number of conditions: **(a)** the function $\pi_2(v)$ is non-negative, non-increasing, and right-continuous for $v > 0$ and $\pi_2(\infty) = 0$; **(b)** the function $\pi_2(v)$ is non-negative, non-decreasing, and right-continuous for $v < 0$ and $\pi_2(-\infty) = 0$; **(c)** for these functions, define a measure $\Pi_2(A)$ on $\tilde{\mathfrak{B}}_1$, the Borel σ-algebra of subsets of $(-\infty, 0) \cup (0, \infty)$, by the relations $\Pi_2((v_1, v_2]) = \pi_2(v_1) - \pi_2(v_2)$ for $0 < v_1 \leq v_2 < \infty$ and $\Pi_2((v_1, v_2]) = \pi_2(v_2) - \pi_2(v_1)$ for $-\infty < v_1 \leq v_2 < 0$; **(d)** this measure possesses the following property: $⨍_{\mathbb{R}_1} \frac{s^2}{1+s^2}\Pi_2(ds) < \infty$, where $⨍$ is the integral over the corresponding interval with the point 0 excluded from the interval of integration; **(e)** under $\mathcal{S}_1$, condition $\mathcal{S}_2$ can hold only simultaneously for all points $v > 0$ such that $\pm v$ are points of continuity of $\pi_2(v)$ and, for any such points satisfying $0 < v_1 < v_2 < \infty$, the following equality holds: $a(v_1) = a(v_2) - \int_{v_1 \leq |s| < v_2} \frac{s^3}{1+s^2}\Pi_2(ds) - \int_{v_1 < |s| \leq v_2} \frac{s}{1+s^2}\Pi_2(ds)$; **(f)** the function $a(v)$ is real-valued and b^2 is a non-negative constant.

The central criterion for convergence states (in a form that extends the corresponding one-dimensional result) that conditions $\mathcal{S}_1$ - $\mathcal{S}_3$ are necessary and sufficient for the following condition of weak convergence to hold:

$\mathcal{A}_{52}$: $\xi_\varepsilon(t), t \geq 0 \Rightarrow \xi_0(t), t \geq 0$ as $\varepsilon \to 0$, where $\xi_0(t), t \geq 0$ is a càdlàg homogeneous process with independent increments.

The limiting process $\xi_0(t), t \geq 0$ in $\mathcal{A}_{52}$ has the characteristic function given, for every $t \geq 0$, by the following *Lévy–Khintchine representation* formula:

$$\begin{aligned} \mathsf{E}\exp\{\mathrm{i}z\xi_0(t)\} &= \phi_2(t, z) \\ &= \exp\{t(\mathrm{i}az - \frac{1}{2}b^2z^2 + ⨍_{\mathbb{R}_1}(e^{\mathrm{i}zs} - 1 - \frac{\mathrm{i}zs}{1+s^2})\Pi_2(ds))\} \end{aligned} \tag{4.2.1}$$

with the constant

$$a = a(v) - ⨍_{|s|<v} \frac{s^3}{1+s^2}\Pi_2(ds) + \int_{|s|>v} \frac{s}{1+s^2}\Pi_2(ds) \tag{4.2.2}$$

that does not depend on the choice of the point v in $\mathcal{S}_2$.

Moreover, as was shown by Skorokhod (1957, 1964), conditions $\mathcal{S}_1$ - $\mathcal{S}_3$ without any additional assumptions imply that

$$\xi_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{J}} \xi_0(t), t \geq 0 \text{ as } \varepsilon \to 0. \tag{4.2.3}$$

In conclusion, let us also recall that the càdlàg homogeneous *process with independent increments* $\xi_0(t), t \geq 0$ can be decomposed into a sum of two independent processes,

$$\xi_0(t) = \xi_0'(t) + \xi_0''(t), \ t \geq 0, \tag{4.2.4}$$

where $\xi_0'(t)$, $t \geq 0$, is a *Wiener process*, that is, a continuous homogeneous process with independent increments and the characteristic function given, for every $t \geq 0$, by the formula

$$\mathsf{E}\exp\{iz\xi_0'(t)\} = \phi_2'(t,z) = \exp\{itaz - \frac{1}{2}tb^2z^2\}, \qquad (4.2.5)$$

and $\xi_0''(t)$, $t \geq 0$ is a càdlàg homogeneous *process with independent increments of Poisson type* and the characteristic function given, for every $t \geq 0$, by the formula

$$\mathsf{E}\exp\{iz\xi_0''(t)\} = \phi_2''(t,z) = \exp\{t\int_{\mathbb{R}_1}(e^{izs} - 1 - \frac{izs}{1+s^2})\Pi_2(ds)\}. \qquad (4.2.6)$$

In conclusion, note a is usually referred as a *drift*, b as a *diffusion* (coefficient) and $\Pi_2(A)$ as a *jump measure* for the process $\xi_0(t)$, $t \geq 0$.

4.2.3. Randomly stopped sum-process based on i.i.d. random variables. We now generalise the limit theorems given in condition $\mathcal{A}_{52}$ and relation (4.2.3) to sum-processes with random stopping indices. Of course, we have to put some condition on the asymptotic behaviour of the random stopping indices. Such a minimal condition would be

$\mathcal{A}_{53}$: $\nu_\varepsilon = \mu_\varepsilon/n_\varepsilon \Rightarrow \nu_0$ as $\varepsilon \to 0$, where ν_0 is a non-negative random variable.

Conditions $\mathcal{A}_{52}$ and $\mathcal{A}_{53}$ are sufficient to imply weak convergence of sum-processes with random stopping indices in the case where the sum-process $\xi_\varepsilon(t)$, $t \geq 0$, and the random stopping index ν_ε are independent. However, it is clear that, if they are dependent, conditions $\mathcal{A}_{52}$ and $\mathcal{A}_{53}$ should be replaced with a stronger condition expressed in terms of the joint distribution of ν_ε and $\xi_\varepsilon(t)$, $t \geq 0$. The following condition plays a key role in subsequent consideration:

$\mathcal{A}_{54}$: $(\nu_\varepsilon, \xi_\varepsilon(t)), t \geq 0 \Rightarrow (\nu_0, \xi_0(t)), t \geq 0$ as $\varepsilon \to 0$, where (a) ν_0 is a non-negative random variable, and (b) $\xi_0(t)$, $t \geq 0$ is a càdlàg homogeneous process with independent increments.

Due to relation (4.2.3), condition $\mathcal{A}_{54}$ implies that condition $\mathcal{J}_{23}$ holds. So, applying Theorems 4.2.1 and 4.2.2 we can formulate the following two theorems.

Theorem 4.2.3. *Let conditions $\mathcal{T}_1$ and $\mathcal{A}_{54}$ hold. Then*

$$\zeta_\varepsilon(t) = \xi_\varepsilon(t\nu_\varepsilon), t \in W_0 \Rightarrow \zeta_0(t) = \xi_0(t\nu_0), t \in W_0 \text{ as } \varepsilon \to 0.$$

Theorem 4.2.4. *Let conditions $\mathcal{T}_1$ and $\mathcal{A}_{54}$ hold. Then*

$$\zeta_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{J}} \zeta_0(t), t \geq 0 \text{ as } \varepsilon \to 0.$$

We remark once more that external sum-processes and random stopping indices in Theorems 4.2.3 and 4.2.4 can have an arbitrary form of dependence. The only condition required is that of joint weak convergence of these processes. No independence or asymptotic independence conditions for random stopping indices and external sum-processes are needed.

4.2.4. The structure of the set W_0. Let us recursively define $\tau_{kn} = \inf(s > \tau_{k-1n}: |\xi_0''(s) - \xi_0''(s-0)| \in [\frac{1}{n}, \frac{1}{n-1}))$, $k = 1, 2, \dots$, where $\tau_{0n} = 0$. The random variables τ_{kn} are successive moments of jumps of the process $\xi_0''(t), t \geq 0$ that have the absolute values in the interval $[\frac{1}{n}, \frac{1}{n-1})$. By the definition, $\tau_{kn} = \infty$ if such a jump does not exist. Since the process $\xi_0'(t)$, $t \geq 0$ is continuous, the processes $\xi_0(t)$, $t \geq 0$ and $\xi_0''(t)$, $t \geq 0$ have the same points of jumps. So, the set of points of discontinuity $R[\xi_0(\cdot)] = R[\xi_0''(\cdot)] = \{\tau_{kn}: k, n = 1, 2, \dots\}$.

By the definition, the set W_0 is a set of all $t \geq 0$ such that $\mathsf{P}\{t\nu_0 = \tau_{kn}\} = 0$ for all k, $n = 1, 2, \dots$. So, we can define the set W_0 as a set of all $t \geq 0$ satisfying $\mathsf{P}\{t\nu_0 \in R[\xi_0''(\cdot)]\} = 0$, that is, as a set of all $w \geq 0$ satisfying condition $\mathbf{C}_5^{(w)}$.

It is useful to generalise the indexing scheme for points of jumps of the process $\xi_0''(t)$, $t \geq 0$. Let A_n, $n = 1, 2, \dots$ be a sequence of sets such that **(a)** $A_n \subseteq \{ x: |x| \geq a_n \}$, where $0 < a_n \to 0$ as $n \to \infty$; **(b)** $A_{n'} \cap A_{n''} = \varnothing$ if $n' \neq n''$; **(c)** $\cup_{n=1}^{\infty} A_n = \mathbb{R}_1 \setminus \{0\}$. Define now, recursively, $\tilde{\tau}_{kn} = \inf(s > \tilde{\tau}_{k-1n}: \xi_0''(s) - \xi_0''(s-0) \in A_n)$, $k = 1, 2, \dots$, where $\tilde{\tau}_{0n} = 0$. By the definition, $\tilde{\tau}_{kn} = \infty$ if such a jump does not exist.

Obviously, the random points $\tilde{\tau}_{kn}$ index the same random set $R[\xi_0''(\cdot)]$ of discontinuity points of the process $\xi_0''(t)$, $t \geq 0$, i.e., the random set $R[\xi_0(\cdot)] = R[\xi_0''(\cdot)] = \{ \tilde{\tau}_{kn}: k, n = 1, 2, \dots \}$ for any sequence A_n, $n = 1, 2, \dots$ that satisfies conditions **(a)**–**(c)**. By the definition, $\tilde{\tau}_{kn}$, $k = 1, 2, \dots$ are successive moments of jumps of the process $\xi_0''(t)$, $t \geq 0$, with values of the jumps in the set A_n. If $a_n = n^{-1}$, $n = 1, 2, \dots$, and the sets $A_n = \{ x: |x| \in [\frac{1}{n}, \frac{1}{n-1}) \}$, $n = 1, 2, \dots$, then the random variables $\tilde{\tau}_{kn} = \tau_{kn}$, $k, n \geq 1$.

The set W_0 coincides with the set of all $t \geq 0$ such that $\mathsf{P}\{t\nu_0 = \tilde{\tau}_{kn}\} = 0$ for all k, $n = 1, 2, \dots$. Therefore, the set $\overline{W}_0 = [0, \infty) \setminus W_0$ coincides with the set of all atoms of the distribution functions of the random variables $\tilde{\tau}_{kn}/\nu_0$, $k, n = 1, 2, \dots$. This set is at most countable. The set W_0 equals $[0, \infty)$ except for the set $\overline{W}_0$. Also, $0 \in W_0$.

Denote $\lambda_n = \Pi_2(A_n)$. Since $\xi_0''(t)$, $t \geq 0$ is a càdlàg homogeneous process with independent increments, there are only two alternatives for every $n \geq 1$: **(d)** if $\lambda_n = 0$, then $\tilde{\tau}_{kn} = \infty$ with probability 1 for all $k \geq 1$; **(e)** if $\lambda_n > 0$, then $\tilde{\tau}_{kn} < \infty$ with probability 1 for all $k \geq 1$. In the latter case, the random variables $\tilde{\tau}_{kn}$, $k \geq 1$ are successive moments of jumps of the Poisson process $\mu_n(t) = \max(k \geq 0: \tilde{\tau}_{kn} \leq t)$, $t \geq 0$ that counts jumps of the process $\xi_0''(t)$ in the interval $[0, t]$ with values in the set A_n. The Poisson process $\mu_n(t)$, $t \geq 0$ has an intensity parameter λ_n. Thus, the random variable $\tilde{\tau}_{kn}$ has Erlang distribution with the parameters k and λ_n.

If the process $\xi_0''(t)$, $t \geq 0$ and the random variable ν_0 are independent, then the set $\overline{W}_0$ is empty and, therefore, the set $W_0 = [0, \infty)$. This follows from the fact that the random variables $\tilde{\tau}_{kn}$ have continuous distributions and are independent of ν_0. Note that the process $\xi'_0(t)$, $t \geq 0$ and the random variable ν_0 can have an arbitrary form of dependence.

However, the assumption of independence of the process $\xi''_0(t)$, $t \geq 0$ and the random variable ν_0 can be replaced with a weaker assumption that the random variables $\tilde{\tau}_{kn}$ and ν_0 are independent for every $k, n = 1, 2, \ldots$. In this case, the process $\xi''_0(t)$, $t \geq 0$, and the random variable ν_0 can be dependent. For example, the random variable ν_0 can depend on values of jumps of the process $\xi''_0(t)$, $t \geq 0$ at the points $\tilde{\tau}_{kn}$, $k, n = 1, 2, \ldots$. Also, as was mentioned above, the process $\xi'_0(t)$, $t \geq 0$ and the random variable ν_0 can also be dependent. Still, the set $W_0 = [0, \infty)$.

It is also clear that the distribution functions of the random variables $\tilde{\tau}_{kn}/\nu_0$ can be continuous even in the case where the random variables $\tilde{\tau}_{kn}$ and ν_0 are dependent.

First of all note that the sets of random variables $\{\tilde{\tau}_{kn}, k = 1, 2, \}$ are mutually independent for $n = 1, 2, \ldots$. This is so because the corresponding Poisson processes $\mu_n(t)$, $t \geq 0$ are mutually independent for $n = 1, 2, \ldots$ due to the fact that the sets $A_{n'} \cap A_{n''} = \emptyset$ for $n' \neq n''$.

Let us consider an example where $\nu_0 = \tilde{\tau}_{k_0 n_0}$. It follows from the remark above that the random variable $\tilde{\tau}_{kn}/\tilde{\tau}_{k_0 n_0}$ has a continuous distribution function if $n \neq n_0$ for every $k = 1, 2, \ldots$. Consider the case where $n = n_0$ but $k \neq k_0$. If $k > k_0$, then the random variable $\tilde{\kappa}_{k_0 kn} = \tilde{\tau}_{kn} - \tilde{\tau}_{k_0 n_0}$ is independent of the random variable $\tilde{\tau}_{k_0 n_0}$. It has Erlang distribution with the parameters $k - k_0$ and λ_n. So, the random variable $\tilde{\tau}_{kn}/\tilde{\tau}_{k_0 n_0} = 1 + \tilde{\kappa}_{kk_0 n}/\tilde{\tau}_{k_0 n_0}$ has a continuous distribution function. If $k < k_0$, then the random variable $\tilde{\kappa}_{kk_0 n} = \tilde{\tau}_{k_0 n_0} - \tilde{\tau}_{kn}$ is independent of the random variable $\tilde{\tau}_{k_0 n_0}$. It has Erlang distribution with the parameters $k_0 - k$ and λ_n. Again the random variable $\tilde{\tau}_{kn}/\tilde{\tau}_{k_0 n_0} = 1 - \tilde{\kappa}_{k_0 kn}/\tilde{\tau}_{k_0 n_0}$ has a continuous distribution function. It remains to consider the case where $(n, k) = (n_0, k_0)$. In this case, the random variable $\tilde{\tau}_{kn}/\tilde{\tau}_{k_0 n_0} \equiv 1$. This random variable has the only one unit atom at the point $t = 1$. Taking into consideration the remarks above one can conclude that, in the case where $\nu_0 = \tilde{\tau}_{k_0 n_0}$, the set $W_0 = [0, \infty) \setminus \{1\} = [0, 1) \cup (1, \infty)$.

It should be noted that in this particular example the condition $\mathfrak{D}_4^{(\mathbf{w})}$ holds for the corresponding external processes $\xi_\varepsilon(t)$, $t \geq 0$ and the stopping processes $\nu_\varepsilon(t) = t\nu_\varepsilon, t \geq 0$, for every $w \geq 0$. Thus, applying Theorem 2.6.4 would yield a better result, namely, it would prove weak convergence of the compositions $\xi_\varepsilon(t\nu_\varepsilon)$ on the whole interval $[0, \infty)$.

Let us modify the above example and consider the case $\nu_0 = \tilde{\tau}_{k_0 n_0} + t_0$, where t_0 is a positive constant. In this case, using the same reasoning one can show that the random variable $\tilde{\tau}_{kn}/(\tilde{\tau}_{k_0 n_0} + t_0)$ has a continuous distribution function if $(n, k) \neq (n_0, k_0)$. However, even in the case where $(n, k) = (n_0, k_0)$, the random variable $\tilde{\tau}_{kn}/(\tilde{\tau}_{k_0 n_0} + t_0)$ has a continuous distribution function. Taking this into consideration one can conclude that, in the case where $\nu_0 = \tilde{\tau}_{k_0 n_0} + t_0$, the set $W_0 = [0, \infty)$.

One can easily generalise this example to the case where the random variable $\nu_0 = f(\tilde{\tau}_{k_0 n_0})$. Here $f(x)$ is a non-random measurable function defined on $[0, \infty)$ and positive almost everywhere with respect to the Lebesgue measure on $[0, \infty)$.

If $(n,k) \neq (n_0,k_0)$, independence of the random variables $\tilde{\tau}_{kn} - \tilde{\tau}_{k_0 n_0}$ and $f(\tilde{\tau}_{k_0 n_0})$ implies that the random variable $\tilde{\tau}_{kn}/f(\tilde{\tau}_{k_0 n_0})$ has a continuous distribution function. Let us now consider the case where $(n,k) = (n_0,k_0)$. Denote $B_s = \{ t \geq 0 \colon t/f(t) = s \}$, $s \geq 0$. Obviously, $\mathsf{P}\{\tilde{\tau}_{k_0 n_0}/f(\tilde{\tau}_{k_0 n_0}) = s\} = \mathsf{P}\{\tilde{\tau}_{k_0 n_0} \in B_s\} = \int_0^\infty \chi_{B_s}(t) p_{k_0,\lambda_{n_0}}(t)\, dt$, where $p_{k_0 n_0}(t)$ is the probability density of Erlang distribution with the parameters k_0 and λ_{n_0}. Since $p_{k_0,\lambda_{n_0}}(t) > 0$ for $t \geq 0$, this probability equals 0 if and only if the Lebesgue measure $m(B_s) = 0$. Hence, the random variable $\tilde{\tau}_{k_0 n_0}/f(\tilde{\tau}_{k_0 n_0})$ has a continuous distribution function if and only if $m(B_s) = 0$ for all $s \geq 0$. So, in the case $\nu_0 = f(\tilde{\tau}_{k_0 n_0})$, the set $\overline{W}_0 = [0,\infty)$ if and only if $m(B_s) = 0$ for all $s \geq 0$. Otherwise, the set $\overline{W}_0$, which is at most countable, is a set of $s \geq 0$ such that $m(B_s) > 0$.

A similar analysis can be carried out in the case where ν_0 is a function of several random variables $\tilde{\tau}_{kn}$.

4.2.5. Random stopping indices converging in probability. Let us now consider a model where the random stopping indices ν_ε converge in probability.

It is natural to assume in this case that the random variables $\xi_{\varepsilon,n}$, $n = 1,2,\ldots$ and μ_ε are defined on the same probability space for all $\varepsilon > 0$. We also assume that the independence condition imposed on the random variables $\xi_{\varepsilon,n}$ is satisfied in the following stronger form:

$\mathcal{T}_2$: The sets of random variables $\{\xi_{\varepsilon,n}, \varepsilon > 0\}$ are mutually independent for $n \geq 1$.

It is obvious that conditions $\mathcal{T}_1$ and $\mathcal{T}_2$ hold for the scale-location model. In this case, the random variables $\xi_{\varepsilon,n}$ have the form $\xi_{\varepsilon,n} = (\xi_n - a_\varepsilon)/b_\varepsilon$, where ξ_n, $n = 1,2,\ldots$ are i.i.d. random variables and a_ε and b_ε are some non-random centralisation and normalisation constants. It also holds for a more general model with random the variables $\xi_{\varepsilon,n} = h_\varepsilon(\xi_n)$, $n = 1,2,\ldots$, where $h_\varepsilon(\cdot)$ are non-random measurable real-valued functions.

The condition of weak convergence $\mathcal{A}_{54}$ is replaced with two conditions. The first one is the condition $\mathcal{A}_{52}$ of weak convergence of sum-processes with non-random stopping indices. The second one is the following condition of convergence of normalised stopping indices in probability, which is stronger than $\mathcal{A}_{53}$:

$\mathcal{P}_1$: $\nu_\varepsilon = \mu_\varepsilon/n_\varepsilon \xrightarrow{\mathsf{P}} \nu_0$ as $\varepsilon \to 0$, where ν_0 is a non-negative random variable.

The simplest classical case, first studied by Anscombe (1952), is where ν_0 is a constant with probability 1.

The following lemma shows that the model with normalised stopping indices converging in probability is a particular case of the model where the condition of joint weak convergence of external sum-processes and stopping indices $\mathcal{A}_{54}$ is involved. The following lemma is a generalisation of the well known result of Rényi (1958, 1960) to the triangular array mode.

Lemma 4.2.1. *Let conditions* $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{A}_{52}$ *(or* $\mathcal{S}_1$ - $\mathcal{S}_3$*), and* $\mathcal{P}_1$ *hold. Then condition* $\mathcal{A}_{54}$ *holds, moreover,* **(α)** *the limiting process* $\xi_0(t)$*,* $t \geq 0$ *and the limiting random variable*

ν_0 *are independent*; **(β)** $\xi_0(t)$, $t \geq 0$ *is a càdlàg homogeneous process with independent increments which has the same finite-dimensional distribution as the corresponding process in condition* $\mathcal{A}_{52}$; **(γ)** ν_0 *is a random variable which has the same distribution as the the corresponding random variable in condition* $\mathcal{P}_1$.

Proof of Lemma 4.2.1. Take an arbitrary subsequence $0 < \varepsilon_n \to 0$ as $n \to \infty$ and choose some $0 < t_1 < \ldots < t_r < \infty$ and $s_1 \leq \ldots \leq s_r < \infty$ that are points of continuity for the distribution functions of the random variables $\xi_0(t_1), \ldots, \xi_0(t_r)$, respectively. Define

$$A_n = \{\xi_{\varepsilon_n}(t_j) = \sum_{k \leq t_j n_{\varepsilon_n}} \xi_{\varepsilon_n,k} \leq s_j,\, j = 1, \ldots, r\},$$

and

$$A = \{\xi_0(t_j) \leq s_j,\, j = 1, \ldots, r\}.$$

First, we are going to prove that the sequence of events $\{A_n\}$ is a *mixing sequence* in the sense of Rényi (1958), that is, for any $l \geq 1$,

$$\lim_{n\to\infty} \mathsf{P}(A_n \cap A_l) = \mathsf{P}(A)\mathsf{P}(A_l). \tag{4.2.7}$$

Let us introduce the random variables

$$\xi^-_{nrl} = \sum_{k \leq t_r n_{\varepsilon_l}} \xi_{\varepsilon_n,k},\ \ \xi^+_{njrl} = \sum_{t_r n_{\varepsilon_l} < k \leq t_j n_{\varepsilon_n}} \xi_{\varepsilon_n,k},\ \ j = 1, \ldots, r.$$

Also denote

$$A_{nrl} = \mathsf{P}\{\xi^+_{njrl} < s_j,\, j = 1, \ldots, r\}.$$

By the definition, the events A_{nrl} and A_l are independent for n large enough, more precisely, if $t_r n_{\varepsilon_l} < t_1 n_{\varepsilon_n}$.

From conditions $\mathcal{A}_{52}$ and $\mathcal{T}_1$, it follows that $\xi_{\varepsilon_n,k} \xrightarrow{\mathsf{P}} 0$ as $n \to \infty$ and, consequently,

$$\xi^-_{nrl} \xrightarrow{\mathsf{P}} 0 \text{ as } n \to \infty. \tag{4.2.8}$$

From conditions $\mathcal{A}_{52}$, $\mathcal{T}_1$, $\mathcal{T}_2$, relation (4.2.8), and the remark about independence of the events A_{nrl} and A_l, it follows that

$$\begin{aligned}\lim_{n\to\infty} \mathsf{P}(A_n \cap A_l) &= \lim_{n\to\infty} \mathsf{P}(A_{nrl} \cap A_l) \\ &= \lim_{n\to\infty} \mathsf{P}(A_{nrl})\mathsf{P}(A_l) = \lim_{n\to\infty} \mathsf{P}(A_n)\mathsf{P}(A_l) = \mathsf{P}(A)\mathsf{P}(A_l).\end{aligned} \tag{4.2.9}$$

Relation (4.2.9) means that the sequence A_n is mixing in the sense of Rényi (1958) and so, for an arbitrary random event B,

$$\mathsf{P}(A_n \cap B) \to \mathsf{P}(A)\mathsf{P}(B) \text{ as } n \to \infty. \tag{4.2.10}$$

Consider the event as $B_z = \{\nu_0 \leq z\}$. Let also $B_{z,n} = \{\nu_{\varepsilon_n} \leq z\}$. Condition $\mathcal{P}_1$ implies that probability of the symmetric difference of these events tends to zero for any z if it is a point of continuity for the distribution function of the random variable ν_0, i.e.,

$$\mathsf{P}(B_z \,\Delta\, B_{z,n}) \to 0 \text{ as } n \to \infty. \tag{4.2.11}$$

Using asymptotic relations (4.2.10) and (4.2.11) we finally get

$$\lim_{n\to\infty} \mathsf{P}(A_n \cap B_{z,n}) = \lim_{n\to\infty} \mathsf{P}(A_n \cap B_z) = \mathsf{P}(A)\mathsf{P}(B_z). \tag{4.2.12}$$

Since the points $0 < t_1 < \ldots < t_m < \infty$ and $s_1 \leq \ldots \leq s_m < \infty$, and the subsequence $\varepsilon_n \to 0$ were chosen arbitrarily, relation (4.2.12) yields the statement of Lemma 4.2.1. □

The following theorem gives a triangular array version of the results known in different variants for the case of the scale-location model. Because of Lemma 4.2.1, these theorems directly follow from Theorems 4.2.3 and 4.2.4. Note that we use the fact that the set $W_0 = [0, \infty)$ if the limiting external process and limiting stopping index are independent.

Theorem 4.2.5. *Let conditions* $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{A}_{52}$ (*or* $\mathcal{S}_1$ - $\mathcal{S}_3$), and $\mathcal{P}_1$ hold. Then condition $\mathcal{A}_{54}$ holds with the process $\xi_0(t), t \geq 0$ and the random variable ν_0 which are independent, and

$$\zeta_\varepsilon(t) = \xi_\varepsilon(t\nu_\varepsilon), t \geq 0 \Rightarrow \zeta_0(t) = \xi_0(t\nu_0), t \geq 0 \text{ as } \varepsilon \to 0.$$

Theorem 4.2.6. *Let conditions* $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{A}_{52}$ (*or* $\mathcal{S}_1$ - $\mathcal{S}_3$), and $\mathcal{P}_1$ hold. Then

$$\zeta_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{J}} \zeta_0(t), t \geq 0 \text{ as } \varepsilon \to 0.$$

Remark 4.2.2. In the particular case, where the limiting stopping index ν_0 is a constant, Theorem 4.2.1 can be associated with well known Anscombe theorem. This theorem (see Anscombe (1952)) gives conditions for weak convergence of randomly indexed stochastic sequences in the model with stopping indices asymptotically degenerating in a constant. Applied to randomly stopped sum-processes based on i.i.d. random variables (in the scale-location model), Anscombe theorem gives the result equivalent to Theorem 4.2.1, in the case where ν_0 = const with probability 1.

4.2.6. Translation theorems for sum-processes with random stopping indices. Let us consider the case where the random variables $\xi_{\varepsilon,n} = (\xi'_{\varepsilon,n} - a_\varepsilon)/n_\varepsilon^\alpha h(n_\varepsilon)$, $n = 1, 2, \ldots$, where **(a)** $\alpha = \text{const} > 0$, and **(b)** $h(x)$ is a slowly varying function.

Consider the processes

$$\zeta'_\varepsilon(t) = \sum_{k \leq t\mu_\varepsilon} \frac{\xi'_{\varepsilon,k} - a_\varepsilon}{\mu_\varepsilon^\alpha h(\mu_\varepsilon)}, \ t \geq 0.$$

Applying Theorems 2.8.2 and 3.4.4 to these processes yields the following translation theorems.

Theorem 4.2.7. *Let conditions $\mathcal{T}_1$ and $\mathcal{A}_{54}$ hold. Then*

$$\zeta'_\varepsilon(t), t \in W_0 \Rightarrow \zeta'_0(t) = \nu_0^{-\alpha}\xi_0(t\nu_0), t \in W_0 \text{ as } \varepsilon \to 0.$$

Theorem 4.2.8. *Let conditions $\mathcal{T}_1$ and $\mathcal{A}_{54}$ hold. Then*

$$\zeta'_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{J}} \zeta'_0(t), t \geq 0 \text{ as } \varepsilon \to 0.$$

Consider the scale-location model where $\varepsilon = n^{-1}, n_\varepsilon = n$, and $\xi_{n,k} = (\xi'_k - a_n)/n^\alpha h(n)$. Here ξ_k, $k = 1, 2, \dots$ are i.i.d. random variables, a_ε are some non-random centralisation constants, $h(x)$ is a slowly varying function.

It is well known (see, for example, Feller (1966)) that in this case $\alpha \in [1/2, \infty)$ and the limiting process with independent increments $\xi_0(t)$, $t \geq 0$ is a *stable* with parameter α. With a minor exclusion (non-symetrical case with $\alpha = 1$), it has the characteristic function of the form $\mathsf{E}\exp\{is\xi_0(t)\} = \exp\{-|s|^{\alpha^{-1}}c(s)t\}$, $t \geq 0$, where $c(s) = c_+\chi(s > 0) + c_-\chi(s < 0)$. In this case, if the random variable ν_0 and the process $\xi_0(t)$, $t \geq 0$ are independent, then the process $\nu_0^{-\alpha}\xi_0(t\nu_0), t \geq 0 \stackrel{d}{=} \xi_0(t), t \geq 0$.

4.2.7. Asymptotics of the sample mean for a sample with random sample size. One of typical examples related to this model concerns the asymptotic distribution of the sample mean for a sample with random sample size. Let $\xi_1, \xi_2, \dots$ be a sequence of i.i.d. random variables with $\mathsf{E}\xi_1 = m, \mathsf{Var}\,\xi_1 = \sigma^2 < \infty$. Let also $\mu_n, n = 1, 2, \dots$ be non-negative integer random variables defined on the same probability space.

In this case, it is convenient to index the corresponding processes with the index n, that is, to use $\varepsilon = \frac{1}{n}$ as the small parameter. Without loss of generality, one can also use $n_\varepsilon = n$ as the normalisation constant.

Let us introduce normalised stopping indices and sum-processes with non-random indices in the following form:

$$\nu_n = \frac{\mu_n}{n},\ \xi_n(t) = \sum_{k \leq tn} \frac{\xi_k - m}{\sigma\sqrt{n}}, t \geq 0.$$

Condition $\mathcal{A}_{54}$ takes the following form:

$\mathcal{A}_{55}$: $(\nu_n, \xi_n(t)), t \geq 0 \Rightarrow (\nu_0, \xi_0(t)), t \geq 0,$ as $n \to \infty$, where (a) ν_0 is a positive random variable, and (b) $\xi_0(t) = w(t), t \geq 0$ is a standard Wiener process.

Let us consider the processes

$$\zeta'_n(t) = \sum_{k \leq t\mu_n} \frac{\xi_k - m}{\sigma\sqrt{\mu_n}}, t \geq 0.$$

We can apply Theorems 4.2.7 and 4.2.8. Since $\xi_0(t), t \geq 0$ is a continuous process, the set of weak convergence $W_0 = [0, \infty)$. Also, since the corresponding limiting process $\xi_0(t\nu_0), t \geq 0$ is continuous, **J**-convergence and **U**-convergence are equivalent.

Theorem 4.2.9. *Let* $\mathsf{Var}\,\xi_1 = \sigma^2 < \infty$ *and condition* $\mathcal{A}_{55}$ *hold. Then*

$$\zeta_n'(t), t \geq 0 \Rightarrow \zeta_0'(t) = \nu_0^{-\frac{1}{2}} \xi_0(t\nu_0), t \geq 0 \textit{ as } n \to \infty.$$

Theorem 4.2.10. *Let* $\mathsf{Var}\,\xi_1 = \sigma^2 < \infty$ *and condition* $\mathcal{A}_{55}$ *hold. Then*

$$\zeta_n'(t), t \geq 0 \xrightarrow{\mathbf{U}} \zeta_0'(t), t \geq 0 \textit{ as } n \to \infty.$$

If the random variable ν_0 and the process $\xi_0(t), t \geq 0$ are independent, then the process $\nu_0^{-\frac{1}{2}} \xi_0(\nu_0 t), t \geq 0 \stackrel{\mathrm{d}}{=} \xi_0(t), t \geq 0$.

Theorems 4.2.9 and 4.2.10 can be used to asymptotic confidence intervals and tests for the unknown mean m in the model with a random sample size in the case where the variance σ^2 is known. However, these theorems can easily be modified to cover the case of unknown variance.

Let us define a *sample mean* and a *sample variance* as follows:

$$\overline{x}_n = \frac{\xi_1 + \ldots + \xi_n}{n}, \; S_n^2 = \frac{1}{n-1} \sum_{k=1}^{n} (\xi_k - \overline{x}_n)^2.$$

Consider the stochastic process

$$\zeta_n''(t) = \sum_{k \leq t\mu_n} \frac{\xi_k - m}{S_{\mu_n} \sqrt{\mu_n}}, t \geq 0.$$

Theorem 4.2.11. *Let* $\mathsf{Var}\,\xi_1 = \sigma^2 < \infty$ *and condition* $\mathcal{A}_{55}$ *hold. Then*

$$\zeta_n''(t), t \geq 0 \Rightarrow \zeta_0''(t) = \nu_0^{-\frac{1}{2}} \xi_0(t\nu_0), t \geq 0 \textit{ as } n \to \infty.$$

Theorem 4.2.12. *Let* $\mathsf{Var}\,\xi_1 = \sigma^2 < \infty$ *and condition* $\mathcal{A}_{55}$ *hold. Then*

$$\zeta_n''(t), t \geq 0 \xrightarrow{\mathbf{U}} \zeta_0''(t), t \geq 0 \textit{ as } n \to \infty.$$

Proof of Theorems 4.2.11 and 4.2.12. It is well known that $\overline{x}_n \xrightarrow{\mathsf{P1}} m$ as $n \to \infty$ and $S_n^2 \xrightarrow{\mathsf{P1}} \sigma^2$ as $n \to \infty$. Due to Lemma 1.3.5, these two relations imply that

$$\overline{x}_{\mu_n} \xrightarrow{\mathsf{P}} m \text{ as } n \to \infty, \tag{4.2.13}$$

and

$$S_{\mu_n}^2 \xrightarrow{\mathsf{P}} \sigma^2 \text{ as } n \to \infty. \tag{4.2.14}$$

Obviously,

$$\zeta_n''(t) = (S_{\mu_n}/\sigma)^{-1} \zeta_n'(t), \; t \geq 0. \tag{4.2.15}$$

This representation and relation (4.2.14) show that Theorem 4.2.11 is a direct corollary of Theorem 4.2.9 and Slutsky Theorem 1.2.3.

Use the following simple inequality: $\Delta_U(bx(\cdot), c, T) \leq b\Delta_U(x(\cdot), c, T)$, which holds for an arbitrary càdlàg function $x(t), t \geq 0$ and $b > 0$. This inequality and representation (4.2.15) yield, for $\delta, b > 0$, that

$$\mathsf{P}\{\Delta_U(\zeta''_n(\cdot), c, T) > \delta\} \leq \mathsf{P}\{\sigma/S_{\mu_n} \geq b\} + P\{\Delta_U(\zeta'_n(\cdot), c, T) > \delta/b\}. \tag{4.2.16}$$

Take an arbitrary $\sigma > 0$. By (4.2.14), we can choose b such that $\overline{\lim}_{n\to\infty} \mathsf{P}\{\sigma/S_{\mu_n} > b\} \leq \sigma$. If we pass to the limit in (4.2.16), first as $n \to \infty$ and then as $c \to 0$, we find using Theorem 4.2.10 that $\lim_{c\to 0} \overline{\lim}_{n\to\infty} P\{\Delta_U(\zeta''_n(\cdot), c, T > \delta\} \leq \sigma$. Since σ is arbitrary, this proves **J**-compactness of the processes $\zeta''_n(t), t \geq 0$. □

4.3 Generalised exceeding processes

In this section, we consider limit theorems for the so-called generalised exceeding processes. Such a process is constructed from a two-dimensional càdlàg process by stopping the first component of this process at the moment when the second component exceeds the level $t \geq 0$ for the first time. In this model, t is being interpreted as time. The class of generalised exceeding processes includes many models of renewal type processes, in particular, sum-processes with renewal stopping, max-processes with renewal stopping, and shock processes.

4.3.1. Weak convergence of generalised exceeding processes. Let, for every $\varepsilon \geq 0$, $\boldsymbol{\alpha}_\varepsilon(t) = (\kappa_\varepsilon(t), \xi_\varepsilon(t)), t \geq 0$ be a two-dimensional càdlàg process with real-valued components. We also assume that the first component $\kappa_\varepsilon(t), t \geq 0$ is a non-decreasing process. Note that it is not required for this process to be non-negative.

Let us now introduce an *exceeding time process* by

$$\nu_\varepsilon(t) = \sup(s \geq 0 : \kappa_\varepsilon(s) \leq t) = \inf(s \geq 0 : \kappa_\varepsilon(s) > t), t \geq 0$$

This formula requires some comments. First of all, let us remark that we will prefer to use the second expression in the defining formula in the right-hand side. As a matter of fact, the second formula automatically yields the correct value $\nu_\varepsilon(t) = 0$ if $\kappa_\varepsilon(0) > t$. At the same time, the use of the first expression does require the additional convention that $\sup(s : \kappa_\varepsilon(s) \leq t) = 0$ if $\kappa_\varepsilon(0) > t$.

Also note that one can assume that the parameter s in both formulas runs over the domain $s > 0$, instead of $s \geq 0$. This modification does not change values of process $\nu_\varepsilon(t), t \geq 0$. At the same time, such a modification permits to introduce exceeding time processes using the same formula also in the case where the initial process $\kappa_\varepsilon(s), s > 0$ is defined on the open interval $(0, \infty)$, instead of the semi-open interval $[0, \infty)$.

Now, introduce a process, which we call a *generalised exceeding process*, by the following formula:

$$\zeta_\varepsilon(t) = \xi_\varepsilon(\nu_\varepsilon(t)), t \geq 0.$$

Let us assume that the following weak convergence condition holds:

$\mathcal{A}_{56}$: $(\kappa_\varepsilon(s), \xi_\varepsilon(t)), (s,t) \in V \times U \Rightarrow (\kappa_0(s), \xi_0(t)), (s,t) \in V \times U$ as $\varepsilon \to 0$, where (a) V is a subset of $(0, \infty)$, dense in this interval, and (b) U is a subset of $[0, \infty)$, dense in this interval and containing the point 0.

To avoid the need of considering the case where the random variables $\nu_\varepsilon(t)$ can be improper, let us also impose the following condition:

$\bar{\mathcal{K}}_5$: $\kappa_\varepsilon(t) \xrightarrow{\mathsf{P}} \infty$ as $t \to \infty$ for every $\varepsilon \geq 0$.

Since we are interested in limit theorems for the generalised exceeding processes $\zeta_\varepsilon(t) = \xi_\varepsilon(\nu_\varepsilon(t)), t \geq 0$ as $\varepsilon \to 0$, it will be sufficient to require that $\kappa_\varepsilon(t) \xrightarrow{\mathsf{P}} \infty$ as $t \to \infty$ in condition $\bar{\mathcal{K}}_5$ for all ε small enough. We require this relation to hold for all $\varepsilon \geq 0$ just in order to simplify the formulations.

Condition $\bar{\mathcal{K}}_5$ implies that the random variable $\nu_\varepsilon(t)$ is finite with probability 1 for every $t \geq 0$.

Denote by V_0 the set of points $t > 0$ that are points of stochastic continuity for the process $\nu_0(t), t \geq 0$. Note that V_0 coincides with $(0, \infty)$, except for at most a countable set of points.

Lemma 4.3.1. *Let conditions $\mathcal{A}_{56}$ and $\bar{\mathcal{K}}_5$ hold. Then*

$$(\nu_\varepsilon(s), \xi_\varepsilon(t)), (s,t) \in V_0 \times U \Rightarrow (\nu_0(s), \xi_0(t)), (s,t) \in V_0 \times U \text{ as } \varepsilon \to 0.$$

Proof of Lemma 4.3.1. For every $\varepsilon \geq 0$, by the definition of the process $\nu_\varepsilon(t), t \geq 0$, the following relation holds for any $s, t \geq 0$:

$$\{\kappa_\varepsilon(s) < t\} \subseteq \{\nu_\varepsilon(t) > s\} \subseteq \{\kappa_\varepsilon(s) \leq t\}. \tag{4.3.1}$$

Using (4.3.1) we get that for any $t_1, \ldots, t_n \geq 0$, $u_1, \ldots, u_n \in \mathbb{R}_1$, $0 \leq s_1 \leq \ldots \leq s_n$, $0 \leq w_1 \leq \ldots \leq w_n$, $n \geq 1$,

$$\begin{aligned}
&\mathsf{P}\{\kappa_\varepsilon(w_i) < s_i, \xi_\varepsilon(t_i) \leq u_i, i = 1, \ldots n\} \\
&\qquad \leq \mathsf{P}\{\nu_\varepsilon(s_i) > w_i, \xi_\varepsilon(t_i) \leq u_i, i = 1, \ldots n\} \\
&\qquad \leq \mathsf{P}\{\kappa_\varepsilon(w_i) \leq s_i, \xi_\varepsilon(t_i) \leq u_i, i = 1, \ldots n\}.
\end{aligned} \tag{4.3.2}$$

Denote by C_t the set of $u \in \mathbb{R}_1$ such that $\mathsf{P}\{\xi_0(t) = u\} = 0$. Choose a countable set of points $X = \{x_1, x_2, \ldots\}$ such that **(a)** it is a subset of V and dense in the interval $(0, \infty)$. Since any distribution function has not more than a countable set of discontinuity points, we can choose a countable set of points $Y = \{y_1, y_2, \ldots\} \subset (0, \infty)$ in such a way that **(b)** it is dense in $(0, \infty)$, and **(c)** $\mathsf{P}\{\kappa_0(x_i) = y_j\} = 0$ for all $i, j \geq 1$. Using condition $\mathcal{A}_{56}$ we have, for arbitrary $n \geq 1$ and points $t_1, \ldots, t_n \in U$, $u_i \in C_{t_i}$, $w_i \in X$, $s_i \in Y$, $i = 1, \ldots n$, that

$$\begin{aligned}\lim_{\varepsilon\to 0} \mathsf{P}\{\kappa_\varepsilon(w_i) < s_i, \xi_\varepsilon(t_i) \leq u_i, i = 1, \ldots n\}& \\ = \mathsf{P}\{\kappa_0(w_i) < s_i, \xi_0(t_i) \leq u_i, i = 1, \ldots n\}& \\ = \lim_{\varepsilon\to 0} \mathsf{P}\{\kappa_\varepsilon(w_i) \leq s_i, \xi_\varepsilon(t_i) \leq u_i, i = 1, \ldots n\}& \\ = \mathsf{P}\{\kappa_0(w_i) \leq s_i, \xi_0(t_i) \leq u_i, i = 1, \ldots n\},&\end{aligned} \tag{4.3.3}$$

and then using (4.3.2) we find

$$\begin{aligned}\mathsf{P}\{\nu_\varepsilon(s_i) > w_i, \xi_\varepsilon(t_i) \leq u_i, i = 1, \ldots n\}& \\ \to \mathsf{P}\{\nu_0(s_i) > w_i, \xi_0(t_i) \leq u_i, i = 1, \ldots n\}& \\ = \mathsf{P}\{\kappa_0(w_i) \leq s_i, \xi_0(t_i) \leq u_i, i = 1, \ldots n\} \text{ as } \varepsilon \to 0.&\end{aligned} \tag{4.3.4}$$

Since weak convergence of distribution functions of random vectors follows from their convergence on some countable set dense in the corresponding phase space and the random variables $\nu_0(s_i)$ are non-negative, we get from (4.3.4), for $t_i \in U$, $s_i \in Y$, $i = 1, \ldots n$, that

$$(\nu_\varepsilon(s_i), \xi_\varepsilon(t_i), i = 1, \ldots, n) \Rightarrow (\nu_0(s_i), \xi_0(t_i), i = 1, \ldots, n) \text{ as } \varepsilon \to 0. \tag{4.3.5}$$

Because $n \geq 1$ and $t_i \in U$, $s_i \in Y$, $i = 1, \ldots, n$, were chosen arbitrarily, relation (4.3.5) implies that

$$(\nu_\varepsilon(s), \xi_\varepsilon(t)), (s, t) \in Y \times U \Rightarrow (\nu_0(s), \xi_0(t)), (s, t) \in Y \times U \text{ as } \varepsilon \to 0. \tag{4.3.6}$$

Since $\nu_\varepsilon(t), t \geq 0$ are non-decreasing processes and the set Y is dense in $(0, \infty)$, relation (4.3.6) can be extended, by an obvious argument, to the relation given in Lemma 4.3.1. □

Remark 4.3.1. It is useful to mention that formula (4.3.4) also express the finite-dimensional distributions of the process $(\nu_0(s), \xi_0(t)), t \geq 0$ in terms of the corresponding finite-dimensional distributions of the process $(\kappa_0(s), \xi_0(t)), t \geq 0$.

Note that the point 0 can not be automatically included in the set Y constructed in the proof. In what follows, weak convergence can only be proved for the set V_0 of all positive points of stochastic continuity for the limiting exceeding process $\nu_0(t), t \geq 0$.

Taking into account this remark, we first give conditions for weak convergence of generalised exceeding processes on a set dense in the interval $(0, \infty)$.

The question about whether it is possible to include the point 0 in the set of weak convergence does require a special investigation that will be postponed until later subsections.

We also assume that the following condition of **J**-compactness introduced in Subsection 2.2.2 holds:

$\mathfrak{J}_7$: $\lim_{c\to 0} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\Delta_J(\xi_\varepsilon(\cdot), c, T) > \delta\} = 0,\ \delta, T > 0.$

Let us first consider the case where the following decomposition condition holds:

$\mathfrak{Q}_{10}$: $\xi_0(t) = \xi_0'(t)+\xi_0''(t), t \geq 0$, where (a) the process $\xi_0'(t), t \geq 0$ is a continuous process, (b) $\xi_0''(t), t \geq 0$ is a stochastically continuous càdlàg process, (c) the processes $\xi_0''(t), t \geq 0$ and $\kappa_0(t), t \geq 0$ are independent.

Theorem 4.3.1. *Let conditions $\mathcal{A}_{56}$, $\bar{\mathcal{K}}_5$, $\mathfrak{J}_7$, and $\mathfrak{Q}_{10}$ hold. Then*

$$\zeta_\varepsilon(t), t \in V_0 \Rightarrow \zeta_0(t), t \in V_0 \ as\ \varepsilon \to 0.$$

Proof of Theorem 4.3.1. Lemma 4.3.1 implies that condition $\mathcal{A}_{21}^{\mathbf{V}}$ holds for the processes $\xi_\varepsilon(t), t \geq 0$ and $\nu_\varepsilon(t), t \geq 0$ with the set $V = V_0$.

It is obvious that, in this case, the processes $\xi_0''(t), t \geq 0$ and $\nu_0(t), t \geq 0$ are independent. This means that condition $\mathfrak{Q}_2$ holds for the processes $\xi_0(t), t \geq 0$ and $\nu_0(t), t \geq 0$. Due to Lemma 2.6.1, this implies that the continuity condition $\mathfrak{C}_5^{\mathbf{W}}$ holds for these processes with the set $W = [0, \infty)$. Now, applying Theorem 2.6.1 or Theorem 2.6.2 finishes the proof of Theorem 4.3.1. □

Let us consider the case where the following continuity condition is satisfied:

$\mathcal{E}_{13}$: $\mathsf{P}\{\nu_0(t') = \nu_0(t'') \in R[\xi_0(\cdot)]\} = 0$ for $0 < t' < t'' < \infty$.

Obviously, condition $\mathcal{E}_{13}$ coincides with condition $\mathcal{E}_1$ applied to the process $\xi_0(t), t \geq 0$ and the exceeding time process $\nu_0(t), t \geq 0$. Here, we used Remark 2.6.2.

Let W_0 be the set of $t > 0$ that satisfy condition $\mathfrak{C}_5^{(\mathrm{w})}$, that is, $\mathsf{P}\{\nu_0(t) \in R[\xi_0(\cdot)]\} = 0$. Due to Lemma 2.6.2, condition $\mathcal{E}_{13}$ implies that the set W_0 is the interval $(0, \infty)$ except for at most a countable set.

Condition $\mathcal{E}_{13}$ holds, if the process $\nu_0(t), t \geq 0$ is an a.s. strictly monotone process.

For example, this is so if $\kappa_0(t), t \geq 0$ is an a.s. non-decreasing continuous càdlàg process and, in the sequel, $\nu_0(t), t \geq 0$, will be an a.s. strictly monotone càdlàg process. A particular case is where $\kappa_0(t), t \geq 0$ is a non-decreasing continuous non-random function and, therefore, $\nu_0(t), t \geq 0$ is a strictly monotone càdlàg non-random function.

Also, condition $\mathfrak{Q}_{10}$ implies that condition $\mathcal{E}_{13}$ holds. As was mentioned above, $W_0 = (0, \infty)$ in this case.

Obviously, the set $V_0 \cap W_0$ is $(0, \infty)$ except for at most a countable set of points.

Theorem 4.3.2. *Let conditions $\bar{\mathcal{K}}_5$, $\mathcal{A}_{56}$, $\mathcal{J}_7$, and $\mathcal{E}_{13}$ hold. Then*

$$\zeta_\varepsilon(t), t \in V_0 \cap W_0 \Rightarrow \zeta_0(t), t \in V_0 \cap W_0 \text{ as } \varepsilon \to 0.$$

Proof of Theorem 4.3.2. A direct application of Theorem 2.6.3 to the processes $\xi_\varepsilon(t)$, $t \geq 0$ and $\nu_\varepsilon(t), t \geq 0$ proves the theorem. □

Finally, let us consider the case where the following continuity condition, which is weaker than $\mathcal{E}_{13}$, holds:

$\mathcal{F}_5$: There exist a sequences of $\delta_l \in Z_0, \delta_l \to 0$ as $l \to \infty$ and $0 < T_r \to \infty$ as $r \to \infty$ such that, for every $l, k, r \geq 1$, we have $\lim_{0<c\to 0} \overline{\lim}_{0\leq\varepsilon\to 0} \mathsf{P}\{\alpha^{(\delta_l)}_{\varepsilon k} - c \leq \nu_\varepsilon(t'), \nu_\varepsilon(t'') < \alpha^{(\delta_l)}_{\varepsilon k}, \alpha^{(\delta_l)}_{\varepsilon k} < T_r\} = 0$ for all $0 < t' < t'' < \infty$.

Here, $\alpha^{(\delta_l)}_{\varepsilon k}$ are successive moments of jumps of the sum-process $\xi_\varepsilon(t), t \geq 0$ with absolute values of the jumps greater than or equal to δ (see Subsection 2.4.1 for details).

It is clear that condition $\mathcal{F}_5$ coincides with condition $\mathcal{F}_2$ applied to the processes $\xi_\varepsilon(t), t \geq 0$ and the exceeding time processes $\nu_\varepsilon(t), t \geq 0$. Here, we make use of Remark 2.6.4.

Let W'_0 be the set of $t > 0$ such that condition $\mathcal{D}^{(\mathbf{w})}_4$ holds for the processes $\xi_\varepsilon(t), t \geq 0$ and $\nu_\varepsilon(t), t \geq 0$. As follows from Lemma 2.6.4, condition $\mathcal{F}_5$ implies that set W'_0 equals $(0, \infty)$ except for at most a countable set.

The set $V_0 \cap W'_0$ also coincides with the interval $(0, \infty)$ except for at most a countable set.

Theorem 4.3.3. *Let conditions $\bar{\mathcal{K}}_5$, $\mathcal{A}_{56}$, $\mathcal{J}_7$, and $\mathcal{F}_5$ hold. Then*

$$\zeta_\varepsilon(t), t \in V_0 \cap W'_0 \Rightarrow \zeta_0(t), t \in V_0 \cap W'_0 \text{ as } \varepsilon \to 0.$$

Proof of Theorem 4.3.3. A direct application of Theorem 2.6.5 to the processes $\xi_\varepsilon(t)$, $t \geq 0$ and $\nu_\varepsilon(t), t \geq 0$ proves the theorem. □

4.3.2. Convergence of generalised exceeding processes at the point 0. Let us now discuss the possibility of including the point 0 in the set of weak convergence.

First of all, it should be possible to add this point to the set V_0 in the statement of weak convergence given in Lemma 4.3.1.

Recall that V_0 is the set of points $t > 0$ that are points of stochastic continuity for the process $\nu_0(t), t \geq 0$. Also, V is the set that enters condition $\mathcal{A}_{56}$.

The most important for applications is the case where the following conditions holds:

$\mathcal{I}_6$: $\kappa_0(0) \geq 0$ with probability 1.

and

$\mathbf{J}_7$: $\kappa_0(t), t \geq 0$ is a.s. strictly monotonic at the point 0.

Conditions $\mathbf{J}_6$ and $\mathbf{J}_7$ imply that **(a)** $\nu_0(0) = 0$ with probability 1.

Let us assume that conditions $\mathcal{A}_{56}$ and $\bar{\mathcal{K}}_5$ hold. Choose a sequence of points $s_n \in V_0$ such that $0 < s_n \to 0$ as $n \to \infty$. Obviously, **(b)** $0 \leq \nu_\varepsilon(0) \leq \nu_\varepsilon(s_n)$. Under the conditions of Lemma 4.3.1, **(c)** $\nu_\varepsilon(s_n) \Rightarrow \nu_0(s_n)$ as $\varepsilon \to 0$ for $n \geq 1$. Also, conditions $\mathbf{J}_6$ and $\mathbf{J}_7$ imply that **(d)** $\nu_0(s_n) \xrightarrow{\mathsf{P}} \nu_0(0) = 0$ as $n \to \infty$. By Lemma 1.2.6, **(b)** - **(d)** imply that **(e)** $\nu_\varepsilon(0) \Rightarrow \nu_0(0) = 0$ as $\varepsilon \to 0$. Due to Slutsky Theorem 1.2.3, it follows from **(d)** that the set of weak convergence in Lemma 4.3.1 can be extended to the set $V_0 \cup \{0\}$.

The càdlàg process $\xi_0(t), t \geq 0$ is continuous at the point $\nu_0(0) = 0$ with probability 1, i.e., condition $\mathbf{C}_5^{(0)}$ holds. So, by Remark 2.6.3, the point 0 can also be included in the set of weak convergence V_0 in Theorem 4.3.1 and in the set of weak convergence $V_0 \cap W_0$ in Theorem 4.3.2.

As it was pointed out in Subsection 2.4.5, condition $\mathbf{C}_5^{(0)}$ implies condition $\mathbf{D}_4^{(0)}$. So, by Remark 2.6.5, the point 0 can be added to the set of weak convergence $V_0 \cap W_0'$ in Theorem 4.3.3.

More complicated is the case where the random variable $\nu_0(0)$ can take positive values. Let us introduce the condition

$\mathbf{J}_8$: $\mathsf{P}\{\kappa_0(x_i) = 0\} = 0, \ x_i \in V$.

Now, assume that conditions $\mathcal{A}_{56}$ and $\mathbf{J}_8$ hold. Then the point 0 can be included in the set Y that enters relations (4.3.3) - (4.3.6) in the proof of Lemma 4.3.1.

So, the set of weak convergence V_0 in the statement of weak convergence in Lemma 4.3.1 can be extended to the set $V_0 \cup \{0\}$.

Note also that, as it was shown in Silvestrov (1974), if the processes $\kappa_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{J}} \kappa_0(t), t \geq 0$ as $\varepsilon \to 0$, then condition $\mathbf{J}_8$ can be replaced by a similar but weaker condition. This condition requires that $\mathsf{P}\{\kappa_0(x_i') = \kappa_0(x_i'') = 0\} = 0, \ x_i', x_i'' \in V, x_i' \neq x_i''$.

In some cases, the following condition of asymptotic local continuity of the processes $\nu_\varepsilon(t), t \geq 0$ can be used:

$\mathbf{O}_{11}$: $\lim_{0<s\to 0} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\nu_\varepsilon(s) - \nu_\varepsilon(0) > \delta\} = 0, \ \delta > 0$.

Let us show that conditions $\mathcal{A}_{56}$, $\bar{\mathcal{K}}_5$, and $\mathbf{O}_{11}$ imply the following relation of weak convergence:

$$\begin{aligned} &(\nu_\varepsilon(s), \xi_\varepsilon(t)), (s,t) \in (V_0 \cup \{0\}) \times U \\ &\quad \Rightarrow (\nu_0(s), \xi_0(t)), (s,t) \in (V_0 \cup \{0\}) \times U \text{ as } \varepsilon \to 0. \end{aligned} \tag{4.3.7}$$

Chose a sequence of points $s_n \in V_0$ such that $0 < s_n \to 0$ as $n \to \infty$, and also take an arbitrary $k \geq 1$ and $0 < t_1 < \ldots < t_k, t_1, \ldots, t_k \in V_0$. Consider the random vectors $\tilde{\nu}_{\varepsilon,n,k} = (\nu_\varepsilon(s_n), \nu_\varepsilon(t_1), \ldots, \nu_\varepsilon(t_k))$ and $\tilde{\nu}_{\varepsilon,0,k} = (\nu_\varepsilon(0), \nu_\varepsilon(t_1), \ldots, \nu_\varepsilon(t_k))$. By Lemma 4.3.1,

(f) $\tilde{\nu}_{\varepsilon,n,k} \Rightarrow \tilde{\nu}_{0,n,k}$ as $\varepsilon \to 0$. Also, by condition $\mathbf{O_{11}}$, **(g)** $\lim_{n\to\infty} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\tilde{\nu}_{\varepsilon,n,k} - \tilde{\nu}_{\varepsilon,0,k} > \delta\} = 0, \ \delta > 0$. Relation **(g)**, taken for $\varepsilon = 0$, implies that **(h)** $\tilde{\nu}_{0,n,k} \Rightarrow \tilde{\nu}_{0,0,k}$ as $n \to \infty$. Relations **(f)** - **(h)** and Lemma 1.2.5 imply that **(i)** $\tilde{\nu}_{\varepsilon,0,k} \Rightarrow \tilde{\nu}_{0,0,k}$ as $\varepsilon \to 0$. It follows from **(i)** that the set of weak convergence in Lemma 4.3.1 can be extended to the set $V_0 \cup \{0\}$.

It should be noted that, if $\mathcal{A}_{56}$ and $\bar{\mathcal{K}}_5$ hold, relation (4.3.7) implies that condition $\mathbf{O_{11}}$ holds.

Indeed, relation (4.3.7) implies that, for any sequence $s_n \in V_0$ such that $0 < s_n \to 0$ as $n \to \infty$, **(j)** $\nu_\varepsilon(s_n) - \nu_\varepsilon(0) \Rightarrow \nu_0(s_n) - \nu_0(0)$ as $\varepsilon \to 0$. Since $\nu_0(t), t \geq 0$ is an a.s. càdlàg process, **(k)** $\nu_0(s_n) \xrightarrow{\mathsf{P}} \nu_0(0)$ as $n \to \infty$. Relations **(j)** and **(k)** imply condition $\mathbf{O_{11}}$ in an obvious way.

These remarks show that, under $\mathcal{A}_{56}$ and $\bar{\mathcal{K}}_5$, conditions $\mathbf{J_6}$ and $\mathbf{J_7}$, as well as condition $\mathbf{J_8}$, are sufficient for condition $\mathbf{O_{11}}$ to hold.

At the same time, condition $\mathbf{O_{11}}$ should not be overestimated. It is just a convenient way of imposing on the processes $\nu_\varepsilon(t), t \geq 0$ a condition that would allow to include the point 0 in the relation of weak convergence given in Lemma 4.3.1.

Nevertheless, condition $\mathbf{O_{11}}$ can be applied in some cases where the exceeding time processes have a simple structure. For example, this is so in the case of step exceeding time processes considered in Section 4.4.

It should be pointed out that even if the point 0 can be included in the set of weak convergence V_0 in Lemma 4.3.1, it is not certain that the process $\xi_0(t), t \geq 0$ is continuous at the point $\nu_0(0)$ with probability 1.

The process $\xi_0(t), t \geq 0$ is continuous at the random point $\nu_0(0)$ with probability 1 if condition $\mathbf{Q_{10}}$ holds. In this case, by Remark 2.6.3, the point 0 can be added to the set of weak convergence, V_0, in Theorem 4.3.1.

In the general case, one should require condition $\mathbf{C_5^{(0)}}$ to hold (for the stopping moment $\nu_0(0)$ and the process $\xi_0(t), t \geq 0$) in order to include the point 0 in the set W_0. In this case, by Remark 2.6.3, the point 0 can be added to the set of weak convergence $V_0 \cap W_0$ in Theorem 4.3.2.

Analogously, one should require that condition $\mathbf{D_4^{(0)}}$ holds (for the stopping moments $\nu_\varepsilon(0)$ and the processes $\xi_\varepsilon(t), t \geq 0$) in order for the point 0 to be in the set W_0'. In this case, by Remark 2.6.5, the point 0 can be added to the set of weak convergence $V_0 \cap W_0'$ in Theorem 4.3.3.

In some cases, the following condition of asymptotic local continuity of the processes $\zeta_\varepsilon(t), t \geq 0$ can be used:

$\mathbf{O_{12}}$: $\lim_{0<s\to 0} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{|\zeta_\varepsilon(s) - \zeta_\varepsilon(0)| > \delta\} = 0, \ \delta > 0$.

In the same way as it was done above for exceeding time processes, it can be shown that, by adding condition $\mathbf{O_{12}}$ to conditions of Theorem 4.3.2 or 4.3.3, one can include the point 0 in the corresponding set of weak convergence, $V_0 \cap W_0$ or $V_0 \cap W_0'$, respectively.

Moreover, if the weak convergence of the generalised exceeding processes takes place on the extended set $(V_0 \cap W_0) \cup \{0\}$ or $(V_0 \cap W_0') \cup \{0\}$, then condition $\mathbf{O_{12}}$ holds.

Thus, if $\bar{\mathfrak{K}}_5$, $\mathcal{A}_{56}$, and $\mathfrak{J}_7$ hold, then condition $\mathfrak{C}_5^{(0)}$, as well as $\mathfrak{D}_4^{(0)}$, is a sufficient condition for $\mathfrak{O}_{12}$ to hold.

Condition $\mathfrak{O}_{12}$, as well as condition $\mathfrak{O}_{11}$, should not be overestimated. It is just a convenient way to impose on the processes $\zeta_\varepsilon(t), t \geq 0$ a condition that would permit to include the point 0 in the relation of weak convergence given in Theorems 4.3.2 or 4.3.3.

Nevertheless, condition $\mathfrak{O}_{12}$ can be applied in some cases where the exceeding time processes have a simple structure. For example, this is the case for step generalised exceeding processes considered in Section 4.4.

4.3.3. J-convergence of generalised exceeding processes. There are two basic cases for which we give conditions for **J**-convergence of generalised exceeding processes. The first one is where the limiting process $\kappa_0(t), t \geq 0$ is an a.s. strictly monotone càdlàg process. The second one is where the limiting process $\kappa_0(t), t \geq 0$ is a step càdlàg process. The latter case is considered in the next section.

For first case, the following condition holds:

$\mathfrak{I}_9$: $\kappa_0(t), t \geq 0$ is an a.s. strictly monotone process.

Condition $\mathfrak{I}_9$ implies that **(a)** $\nu_0(t), t \geq 0$ is an a.s. continuous process. It is also obvious that condition $\mathfrak{I}_9$ implies condition $\mathfrak{I}_7$.

Theorem 4.3.4. *Let conditions $\bar{\mathfrak{K}}_5$, $\mathcal{A}_{56}$, $\mathfrak{J}_7$, $\mathfrak{I}_6$, $\mathfrak{I}_9$, and $\mathcal{E}_{13}$ hold. Then*

$$\zeta_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{J}} \zeta_0(t), t \geq 0 \text{ as } \varepsilon \to 0.$$

Proof of Theorem 4.3.4. A direct application of Theorem 3.4.2 to the processes $\xi_\varepsilon(t)$, $t \geq 0$ and $\nu_\varepsilon(t), t \geq 0$ proves the theorem. □

Let Y_0 denote the set of points of stochastic continuity for the limiting process $\zeta_0(t) = \xi_0(\nu_0(t)), t \geq 0$.

Remark 4.3.2. Obviously, $(V_0 \cap W_0) \cup \{0\} \subseteq Y_0$. Due to Theorem 4.3.4, the set of weak convergence, $(V_0 \cap W_0) \cup \{0\}$, which is guaranteed by Theorem 4.3.2, can be extended to the set Y_0.

The key condition $\mathcal{E}_{13}$ has some limitation. It does not cover the case when the limiting process $\kappa_0(t), t \geq 0$ has positive jumps simultaneous with jumps of the process $\xi_0(t), t \geq 0$. Indeed, if τ is such a point and $\mathsf{P}\{\kappa_0(\tau - 0) \leq t' < t'' \leq \kappa_0(\tau), \Delta_\tau(\xi_0(\cdot)) \neq 0\} > 0$, then $\mathsf{P}\{\nu_0(t') = \nu_0(t'') = \tau\} > 0$. Therefore, condition $\mathcal{E}_{13}$ does not hold.

The following theorem covers the case in which the limiting process $\kappa_0(t), t \geq 0$ may possess positive jumps simultaneous with the corresponding jumps of the process $\xi_0(t), t \geq 0$.

Theorem 4.3.5. *Let conditions $\bar{\mathfrak{K}}_5$, $\mathcal{A}_{56}$, $\mathfrak{J}_7$, $\mathfrak{I}_6$, $\mathfrak{I}_9$, and $\mathfrak{F}_5$ hold. Then*

$$\zeta_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{J}} \zeta_0(t), t \geq 0 \text{ as } \varepsilon \to 0.$$

Proof of Theorem 4.3.5. The proof follows by directly applying Theorem 3.4.3 to the processes $\xi_\varepsilon(t), t \geq 0$ and $\nu_\varepsilon(t), t \geq 0$. □

Remark 4.3.3. Condition $\mathfrak{I}_6$ can be replaced by condition $\mathfrak{I}_8$ or by condition $\mathfrak{O}_{11}$ in Theorems 4.3.4 and 4.3.5. However, in both cases, one should also require that at least one of the conditions $\mathfrak{Q}_{10}$, $\mathfrak{C}_5^{(0)}$ (for the stopping moment $\nu_0(0)$ and the process $\xi_0(t), t \geq 0$) or $\mathfrak{D}_4^{(0)}$ (for the stopping moments $\nu_\varepsilon(0)$ and the processes $\xi_\varepsilon(t), t \geq 0$) holds. Alternatively, one can replace condition $\mathfrak{I}_6$ by condition $\mathfrak{O}_{12}$.

Remark 4.3.4. Due to Theorem 4.3.5, the set of weak convergence $(V_0 \cap W_0') \cup \{0\}$, which is guaranteed by Theorem 4.3.3, can be extended to the set $(V_0 \cap W_0') \cup Y_0$.

4.3.4. Generalised exceeding processes based on J-convergent processes. In this subsection we improve Theorem 4.3.5. This theorem is used for the processes $\kappa_\varepsilon(t), t \geq 0$ and $\xi_\varepsilon(t), t \geq 0$ that have simultaneous jumps. It is not convenient in this theorem that condition $\mathfrak{F}_5$ involves the exceeding time processes $\nu_\varepsilon(t), t \geq 0$ instead of the processes $\kappa_\varepsilon(t), t \geq 0$.

Let us now strengthen the condition of **J**-compactness $\mathfrak{J}_7$ and replace it by the following condition of **J**-compactness for the bivariate processes $\boldsymbol{\alpha}_\varepsilon(t), t \geq 0$:

$\mathfrak{J}_{24}$: $\lim_{c\to 0} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\Delta_J(\boldsymbol{\alpha}_\varepsilon(\cdot), c, T', T'') > \delta\} = 0,\ \delta > 0, 0 < T' < T'' < \infty.$

We also supplement it by the following condition of local continuity for the external processes $\xi_\varepsilon(t), t \geq 0$ at the point 0:

$\mathfrak{O}_{13}$: $\lim_{0<c\to 0} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\sup_{0\leq t\leq c} |\xi_\varepsilon(t) - \xi_\varepsilon(0)| > \delta\} = 0,\ \delta > 0.$

Note that conditions $\mathfrak{J}_{24}$ and $\mathfrak{O}_{13}$ imply that the condition of **J**-compactness $\mathfrak{J}_7$ holds for the processes $\xi_\varepsilon(t), t \geq 0$.

Let us now prove the following useful lemma.

Lemma 4.3.2. *Conditions $\bar{\mathfrak{K}}_5$, $\mathfrak{J}_{24}$, and $\mathfrak{O}_{13}$ imply condition $\mathfrak{F}_5$ to hold.*

Proof of Lemma 4.3.2. Assume that condition $\mathfrak{F}_5$ does not hold. This means that there exist $0 < t' < t'' < \infty$, $\delta_l \in Z_0$, $T_r > 0$, and $k \geq 1$ such that

$$\lim_{0<c\to 0} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\alpha_{\varepsilon k}^{(\delta_l)} - c \leq \nu_\varepsilon(t'), \nu_\varepsilon(t'') < \alpha_{\varepsilon k}^{(\delta_l)}, \alpha_{\varepsilon k}^{(\delta_l)} < T_r\} > 0. \tag{4.3.8}$$

Condition $\mathfrak{O}_{13}$ implies that, for every $\delta_l \in Z_0$,

$$\lim_{0<c\to 0} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\alpha_{\varepsilon k}^{(\delta_l)} \leq c\} \leq \lim_{0<c\to 0} \overline{\lim_{\varepsilon\to 0}} 2\mathsf{P}\{\sup_{0\leq t\leq c} |\xi_\varepsilon(t) - \xi_\varepsilon(0))| \geq \delta_l/2\} = 0. \tag{4.3.9}$$

Relations (4.3.8) and (4.3.9) imply that there exist small enough $0 < T < T_r$ such that

$$\begin{aligned}&\lim_{0<c\to 0}\overline{\lim_{\varepsilon\to 0}}\,\mathsf{P}\{T < \alpha_{\varepsilon k}^{(\delta_l)} - c \le \nu_\varepsilon(t'), \nu_\varepsilon(t'') < \alpha_{\varepsilon k}^{(\delta_l)}, \alpha_{\varepsilon k}^{(\delta_l)} < T_r\}\\ &\quad\ge \lim_{0<c\to 0}\overline{\lim_{\varepsilon\to 0}}(\mathsf{P}\{\alpha_{\varepsilon k}^{(\delta_l)} - c \le \nu_\varepsilon(t'), \nu_\varepsilon(t'') < \alpha_{\varepsilon k}^{(\delta_l)}, \alpha_{\varepsilon k}^{(\delta_l)} < T_r\}\\ &\quad - \lim_{0<c\to 0}\overline{\lim_{\varepsilon\to 0}}\,\mathsf{P}\{\alpha_{\varepsilon 1}^{(\delta_l)} \le T + c\} > 0.\end{aligned} \tag{4.3.10}$$

Let us choose some $0 < \sigma \le \frac{t''-t'}{2} \wedge \delta_l$ and show that

$$\begin{aligned}A_{\varepsilon,klr,T} &= \{T < \alpha_{\varepsilon k}^{(\delta_l)} - c \le \nu_\varepsilon(t'), \nu_\varepsilon(t'') < \alpha_{\varepsilon k}^{(\delta_l)}, \alpha_{\varepsilon k}^{(\delta_l)} < T_r\}\\ &\subseteq \{\Delta_J(\boldsymbol{\alpha}_\varepsilon(\cdot), 2c, T_r) > \sigma\}.\end{aligned} \tag{4.3.11}$$

Indeed, since $\kappa_\varepsilon(t), t \ge 0$ is a càdlàg process, we have $\kappa_\varepsilon(\nu_\varepsilon(t)) \ge t$ and $\kappa_\varepsilon(\nu_\varepsilon(t) - 0) \le t$ if $\nu_\varepsilon(t) > 0$. It follows from the definition that $\nu_\varepsilon(t) = \sup(s : \kappa_\varepsilon(s) \le t) = \inf(s : \kappa_\varepsilon(s) > t)$. Let the random event $A_{\varepsilon,klr,T}$ occur. If **(a)** $\kappa_\varepsilon(\nu_\varepsilon(t')) \le \frac{t'+t''}{2}$, then $\kappa_\varepsilon(\alpha_{\varepsilon k}^{(\delta_l)} - 0) - \kappa_\varepsilon(\nu_\varepsilon(t') - 0) \ge \kappa_\varepsilon(\nu_\varepsilon(t'')) - \kappa_\varepsilon(\nu_\varepsilon(t')) \ge \frac{t''-t'}{2} \ge \sigma$. If **(b)** $\kappa_\varepsilon(\nu_\varepsilon(t')) \ge \frac{t'+t''}{2}$, then $\kappa_\varepsilon(\alpha_{\varepsilon k}^{(\delta_l)} - 0) - \kappa_\varepsilon(\nu_\varepsilon(t') - 0) \ge \kappa_\varepsilon(\nu_\varepsilon(t')) - \kappa_\varepsilon(\nu_\varepsilon(t') - 0) \ge \frac{t''-t'}{2} \ge \sigma$. Also, $|\xi_\varepsilon(\alpha_{\varepsilon k}^{(\delta_l)}) - \xi_\varepsilon(\alpha_{\varepsilon k}^{(\delta_l)} - 0)| \ge \delta_l \ge \sigma$. In any case, $|\boldsymbol{\alpha}_\varepsilon(t' - 0) - \boldsymbol{\alpha}_\varepsilon(\alpha_{\varepsilon k}^{(\delta_l)} - 0)| \wedge |\boldsymbol{\alpha}_\varepsilon(\alpha_{\varepsilon k}^{(\delta_l)} - 0) - \boldsymbol{\alpha}_\varepsilon(\alpha_{\varepsilon k}^{(\delta_l)})| \ge \sigma$. But $0 < T < \alpha_{\varepsilon k}^{(\delta_l)} - c < \nu_\varepsilon(t') < \alpha_{\varepsilon k}^{(\delta_l)} < T_r$. Hence, $\Delta_J(\boldsymbol{\alpha}_\varepsilon(\cdot), c, T, T_r) \ge \sigma$. This means that relation (4.3.11) holds.

Using (4.3.11) we get

$$\begin{aligned}&\lim_{0<c\to 0}\overline{\lim_{\varepsilon\to 0}}\,\mathsf{P}\{\Delta_J(\boldsymbol{\alpha}_\varepsilon(\cdot), 2c, T_r) > \sigma\}\\ &\quad\ge \lim_{0<c\to 0}\overline{\lim_{\varepsilon\to 0}}\,\mathsf{P}\{\alpha_{\varepsilon k}^{(\delta_l)} - c \le \nu_\varepsilon(t'), \nu_\varepsilon(t'') < \alpha_{\varepsilon k}^{(\delta_l)}, \alpha_{\varepsilon k}^{(\delta_l)} \le T_r\} > 0.\end{aligned} \tag{4.3.12}$$

Relation (4.3.12) contradicts condition $\mathcal{J}_{24}$. Therefore, condition $\mathcal{F}_5$ holds. □

Remark 4.3.5. It follows from monotonicity of the sequence of random variables $\alpha_{\varepsilon k}^{(\delta_l)}, k = 1, 2, \ldots$, and relation (4.3.9), given in the proof of Lemma 4.3.2, that condition $\mathcal{O}_{13}$ can be replaced by the following weaker condition:

$\mathcal{O}_{14}$: $\lim_{0<c\to 0}\overline{\lim}_{\varepsilon\to 0}\,\mathsf{P}\{\alpha_{\varepsilon 1}^{(\delta_l)} \le c\} = 0,\ \delta_l \in Z_0$.

Conditions $\mathcal{A}_{56}$ and $\mathcal{J}_{24}$ are necessary and sufficient for the following **J**-convergence relation to hold:

$$\boldsymbol{\alpha}_\varepsilon(t), t > 0 \xrightarrow{\mathbf{J}} \boldsymbol{\alpha}_0(t), t > 0 \text{ as } \varepsilon \to 0. \tag{4.3.13}$$

Together with condition $\mathcal{O}_{13}$, these conditions also imply that

$$\xi_\varepsilon(t), t \ge 0 \xrightarrow{\mathbf{J}} \xi_0(t), t \ge 0 \text{ as } \varepsilon \to 0. \tag{4.3.14}$$

Also note that the relation of **J**-convergence (4.3.14) implies that condition $\mathfrak{O}_{13}$ holds.

Lemma 4.3.2 allows to improve Theorems 4.3.3 and 4.3.5 and state them in the following form.

Theorem 4.3.6. *Let conditions $\bar{\mathfrak{K}}_5$, $\mathcal{A}_{56}$, $\mathfrak{J}_{24}$, and $\mathfrak{O}_{13}$ hold. Then*

$$\zeta_\varepsilon(t), t \in V_0 \cap W_0' \Rightarrow \zeta_0(t), t \in V_0 \cap W_0' \text{ as } \varepsilon \to 0.$$

Theorem 4.3.7. *Let conditions $\bar{\mathfrak{K}}_5$, $\mathcal{A}_{56}$, $\mathfrak{J}_{24}$, $\mathfrak{O}_{13}$, $\mathfrak{I}_6$, and $\mathfrak{I}_9$ hold. Then*

$$\zeta_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{J}} \zeta_0(t), t \geq 0 \text{ as } \varepsilon \to 0.$$

Remark 4.3.6. According to Remark 4.3.3, condition $\mathfrak{I}_6$ can be replaced by condition $\mathfrak{I}_8$ or by condition $\mathfrak{O}_{11}$ in Theorem 4.3.7. However, in both cases one should also require that at least one of the conditions $\mathfrak{Q}_{10}$, $\mathfrak{C}_5^{(0)}$ (for the stopping moment $\nu_0(0)$ and the process $\xi_0(t), t \geq 0$) or $\mathfrak{D}_4^{(0)}$ (for the stopping moments $\nu_\varepsilon(0)$ and the processes $\xi_\varepsilon(t), t \geq 0$) holds. Alternatively, one can replace condition $\mathfrak{I}_6$ by condition $\mathfrak{O}_{12}$.

Remark 4.3.7. According to Remark 4.3.4, the set of weak convergence $(V_0 \cap W_0') \cup \{0\}$, which is guaranteed by Theorem 4.3.6, can be extended to the set $(V_0 \cap W_0') \cup Y_0$.

4.3.5. Generalised exceeding processes based on non-negative exceeding time processes. In this subsection, we show that in the case of **J**-convergent processes $\boldsymbol{\alpha}_\varepsilon(t)$, $t \geq 0$, the consideration can be reduced to the case of non-decreasing and non-negative exceeding time processes $\kappa_\varepsilon(t), t \geq 0$.

Let us define the process $\kappa_\varepsilon^{++}(t) = \max(0, \kappa_\varepsilon(t)), t \geq 0$ and the corresponding *exceeding time process* $\nu_\varepsilon^{++}(t) = \sup(s : \kappa_\varepsilon^{++}(s) \leq t), t \geq 0$.

By the definition, $\kappa_\varepsilon^{++}(t), t \geq 0$ is a non-negative and non-decreasing càdlàg process. But $\nu_\varepsilon(t) = \sup(s : \kappa_\varepsilon(s) \leq t) = \nu_\varepsilon^{++}(t), t \geq 0$. So, we can consider the *generalised exceeding process* $\zeta_\varepsilon(t) = \xi_\varepsilon(\nu_\varepsilon(t)) = \xi_\varepsilon(\nu_\varepsilon^{++}(t)), t \geq 0$ as a process based on the bivariate process $\boldsymbol{\alpha}_\varepsilon^{++}(t) = (\kappa_\varepsilon^{++}(t), \xi_\varepsilon(t))$, $t \geq 0$ with the second component $\kappa_\varepsilon^{++}(t), t \geq 0$ being a non-negative and non-decreasing càdlàg process.

Let us replace condition $\mathcal{A}_{56}$ by the following condition:

$\mathcal{A}_{57}$: $(\kappa_\varepsilon^{++}(s), \xi_\varepsilon(t)), (s, t) \in V \times U \Rightarrow (\kappa_0^{++}(s), \xi_0(t)), (s, t) \in V \times U$ as $\varepsilon \to 0$, where (a) V is a subset of $(0, \infty)$, dense in this interval; (b) U is a subset of $[0, \infty)$, dense in this interval and containing the point 0.

If condition $\mathcal{A}_{56}$ holds for the processes $\boldsymbol{\alpha}_\varepsilon(t), t \geq 0$, then condition $\mathcal{A}_{57}$ holds for the processes $\boldsymbol{\alpha}_\varepsilon^{++}(t), t \geq 0$, that is, condition $\mathcal{A}_{57}$ is weaker than condition $\mathcal{A}_{56}$. This fact follows from continuity of the function $f(x) = \max(0, x)$ and Theorem 1.3.2.

Let us now introduce the following **J**-compactness condition:

$\mathfrak{J}_{25}$: $\lim_{c\to 0} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\Delta_J(\boldsymbol{\alpha}_\varepsilon^{++}(\cdot), c, T', T'') > \delta\} = 0,\ \delta > 0, 0 < T' < T'' < \infty.$

It is obvious that $0 \leq \kappa_\varepsilon^{++}(t'') - \kappa_\varepsilon^{++}(t') \leq \kappa_\varepsilon(t'') - \kappa_\varepsilon(t')$ for any $0 < t' \leq t'' < \infty$. Thus we see that the following estimate is valid for any $c, \delta > 0$ and $0 < T' < T'' < \infty$:

$$\mathsf{P}\{\Delta_J(\boldsymbol{\alpha}_\varepsilon^{++}(\cdot), c, T', T'') > \delta\} \leq \mathsf{P}\{\Delta_J(\boldsymbol{\alpha}_\varepsilon(\cdot), c, T', T'') > \delta\}. \tag{4.3.15}$$

It follows from (4.3.15) that condition $\mathfrak{J}_{24}$ always implies condition $\mathfrak{J}_{25}$, that is, condition $\mathfrak{J}_{25}$ is weaker than condition $\mathfrak{J}_{24}$.

Note also that condition $\bar{\mathfrak{K}}_5$ holds for the processes $\kappa_\varepsilon^{++}(t), t \geq 0$, if and only if this condition holds for the processes $\kappa_\varepsilon(t), t \geq 0$.

Condition $\mathfrak{I}_6$ automatically holds for the processes $\kappa_0^{++}(t), t \geq 0$.

A direct analogue of $\mathfrak{I}_9$ is the following condition:

$\mathfrak{I}_{10}$: $\kappa_0^{++}(t), t \geq 0$ is an a.s. strictly increasing càdlàg process.

Condition $\mathfrak{I}_{10}$ implies that random variable $\nu_0(0) = 0$ with probability 1.

Taking in account the remarks above we can improve Theorems 4.3.6 and 4.3.7 to the following form.

Theorem 4.3.8. *Let conditions $\bar{\mathfrak{K}}_5$, $\mathcal{A}_{57}$, $\mathfrak{J}_{25}$, and $\mathfrak{O}_{13}$ hold. Then*

$$\zeta_\varepsilon(t), t \in V_0 \cap W_0' \Rightarrow \zeta_0(t), t \in V_0 \cap W_0' \text{ as } \varepsilon \to 0.$$

Theorem 4.3.9. *Let conditions $\bar{\mathfrak{K}}_5$, $\mathcal{A}_{57}$, $\mathfrak{J}_{25}$, $\mathfrak{O}_{13}$, and $\mathfrak{I}_{10}$ hold. Then*

$$\zeta_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{J}} \zeta_0(t), t \geq 0 \text{ as } \varepsilon \to 0.$$

As was pointed out above, condition $\mathfrak{I}_{10}$ implies that $\nu_0(0) = 0$ with probability 1. This excludes from the consideration the case where $\nu_0(0)$ can take positive values. This may happen if the random variable $\kappa_0(0)$ can take negative values.

Let us assume that the process $\kappa_0(t), t \geq 0$ is a.s. strictly monotone but we allow the random variable $\kappa_0(0)$ to take negative values.

In this case, $\kappa_0^{++}(t) = 0$ for $t < \nu_0(0) = \sup(s : \kappa_\varepsilon(s) \leq 0)$, while this process strictly increases for $t \geq \nu_0(0)$ with probability 1.

According this remark, we modify condition $\mathfrak{I}_{10}$ and replace it by the following weaker condition:

$\mathfrak{I}_{11}$: $\kappa_0^{++}(t) = 0$ for $t < \nu_0(0)$, where $\nu_0(0) = \inf(s : \kappa_0^{++}(s) > 0)$ and the càdlàg process $\kappa_0^{++}(t)$ strictly increases for $t \geq \nu_0(0)$ with probability 1.

By condition $\mathfrak{I}_{11}$, the process $\nu_0^{++}(t) = \nu_0(t), t \geq 0$ is again an a.s. continuous process. Therefore, Theorem 3.4.2 can be applied.

In this case, one should additionally provide for the possibility to include the point 0 in the set of weak convergence for the generalised exceeding processes $\zeta_\varepsilon(t), t \geq 0$.

One can require that condition $\mathfrak{I}_8$ holds for the process $\kappa_0^{++}(t), t \geq 0$, or that condition $\mathfrak{O}_{11}$ holds. Additionally, at least one of the conditions $\mathfrak{Q}_{10}$, $\mathfrak{C}_5^{(0)}$ (for the stopping moment $\nu_0(0)$ and the process $\xi_0(t), t \geq 0$) or $\mathfrak{D}_4^{(0)}$ (for the stopping moments $\nu_\varepsilon(0)$ and the processes $\xi_\varepsilon(t), t \geq 0$) holds. Then the point 0 can be included in the set of weak convergence. Alternatively, one can assume that condition $\mathfrak{O}_{12}$ holds.

4.3.6. Weak and J-convergence on the interval $(0, \infty)$. Theorems 4.3.1, 4.3.2, 4.3.3, and 4.3.6 give conditions for weak convergence of generalised exceeding processes on sets that are dense in $(0, \infty)$. Also, conditions of Theorems 4.3.4, 4.3.5, and 4.3.7 imply **J**-compactness of the processes $\zeta_\varepsilon(t)$ on any finite interval $[T', T'']$ for $0 < T' < T'' < \infty$. This follows directly from Lemma 3.4.1, if applied to the processes $\xi_\varepsilon(t), t \geq 0$ and $\nu_\varepsilon(T' + t), t \geq 0$. So, omitting condition $\mathfrak{I}_6$ in Theorems 4.3.4, 4.3.5 or 4.3.7 one obtains **J**-convergence of the processes $\zeta_\varepsilon(t)$ on the open interval $(0, \infty)$.

Using Theorem 4.3.9 one can prove **J**-convergence of the processes $\zeta_\varepsilon(t)$ on the open interval $(0, \infty)$, instead of $[0, \infty)$, if condition $\mathfrak{I}_{10}$ in this theorem is replaced by condition $\mathfrak{I}_{11}$. Conditions $\mathfrak{Q}_{10}$, $\mathfrak{C}_5^{(0)}$, or $\mathfrak{D}_4^{(0)}$ can be omitted in this case.

4.3.7. Exceeding time processes defined on the interval $(0, \infty)$. The results of Lemmas 4.3.1 – 4.3.2 and Theorems 4.3.1 – 4.3.9 can be generalised to a model in which the process $\kappa_\varepsilon(t), t \geq 0$ is defined on the interval $[0, \infty)$ but the value of this process, $\kappa_\varepsilon(0)$, at the point 0 may be a proper or an improper random variable (the value $-\infty$ has positive probability).

The same can be true if the process $\kappa_\varepsilon(t), t \geq 0$ is initially defined on the open interval $(0, \infty)$. In this case, one can always use monotonicity of the processes $\kappa_\varepsilon(t), t > 0$, and define $\kappa_\varepsilon(0) = \kappa_\varepsilon(0 + 0) = \lim_{0<t\to 0} \kappa_\varepsilon(0)$. This limit exists with probability 1. Such a definition needs to allow the random variable $\kappa_\varepsilon(0)$ to be improper.

Note that neither the definition of an exceeding time process given above nor conditions $\mathcal{A}_{56}$ and $\mathcal{J}_{24}$ involve the random variables $\kappa_\varepsilon(0)$. Also, the formulations of conditions $\mathfrak{Q}_{10}$, $\mathcal{E}_{13}$, and $\mathcal{F}_5$, as well as conditions $\bar{\mathcal{K}}_5$ and $\mathfrak{I}_6$–$\mathfrak{I}_{11}$, remain the same.

Thus, Lemmas 4.3.1 – 4.3.2 and Theorems 4.3.1 – 4.3.9 also remain unchanged.

It should also be noted that one can always reduce the model to the case of non-negative exceeding time processes. This can be achieved by the use of the truncation transformation described in Subsection 4.3.6.

4.3.8. External processes that do not converge at the point 0. Results of Lemmas 4.3.1 – 4.3.2 and Theorems 4.3.1 – 4.3.9 can also be generalised to the case when the processes $\xi_\varepsilon(t), t \geq 0$ are defined on the interval $[0, \infty)$ for $\varepsilon > 0$ but do not weakly converge at the point 0. In this case, the corresponding limiting process $\xi_0(t), t \geq 0$ can be a càdlàg process on the open interval $(0, \infty)$ but the point 0 can be a point of discontinuity for this process. Moreover, it can occur that this process a.s. has a right limit at the point 0 that is an improper random variable, or it can even happen that this limit does not exist at all.

In such a case, one can assign the standard value, $\xi_0(0) \equiv 0$. This will not affect the limiting composition $\zeta_0(t) = \xi_0(\nu_0(t))$ if the corresponding limiting internal stopping process $\nu_0(t), t \geq 0$ is an a.s. strictly positive process for $t > 0$ or $t \geq 0$. In this case, one can get weak convergence or **J**-convergence of the corresponding generalised exceeding processes $\zeta_\varepsilon(t) = \xi_\varepsilon(\nu_\varepsilon(t)$ on the interval $(0, \infty)$ or $[0, \infty)$, respectively.

Also, we can admit the case when the limit $\kappa_\varepsilon(0) = \kappa_\varepsilon(0+0) = \lim_{0<t\to 0} \kappa_\varepsilon(0)$, which exists with probability 1, is a proper or an improper random variable.

Condition $\mathcal{A}_{\mathbf{56}}$ should be modified to the following form:

$\mathcal{A}_{\mathbf{58}}$: $(\kappa_\varepsilon(s), \xi_\varepsilon(t)), (s,t) \in V \times U \Rightarrow (\kappa_0(s), \xi_0(t)), (s,t) \in V \times U$ as $\varepsilon \to 0$, where (a) V and U are subsets of $(0, \infty)$ dense in this interval.

Conditions $\mathfrak{Q}_{\mathbf{10}}$ and $\mathcal{E}_{\mathbf{13}}$ can be preserved.

Condition $\mathcal{F}_{\mathbf{5}}$ should be modified in a more complicated way, because it is necessary to change the definition of successive moments of jumps for the process $\xi_0(t)$ in the situation where this process needs to be considered on the open interval $(0, \infty)$.

Let us take some $s_n > 0$ and introduce the process

$$\xi_\varepsilon^{(s_n)}(t) = \xi_\varepsilon(t \vee s_n) = \begin{cases} \xi_\varepsilon(s_n) & \text{if } t < s_n, \\ \xi_\varepsilon(t) & \text{if } t \geq s_n, \end{cases}$$

and then the corresponding generalised exceeding process

$$\zeta_\varepsilon^{(s_n)}(t) = \xi_\varepsilon^{(s_n)}(\nu_\varepsilon(t)), t \geq 0.$$

Let $\alpha_{\varepsilon kn}^{(\delta)}, k = 1, 2, \ldots$ be successive moments of jumps of the càdlàg process $\xi_\varepsilon^{(s_n)}(t)$, $t \geq 0$, such that absolute values of the jumps are greater than or equal to δ.

Let also U_0 be the set of points $t > 0$ that are points of stochastic continuity for the process $\xi_0(t), t > 0$. This set coincides with $(0, \infty)$ except for at most a countable set.

Let also Z_0 be the set of $\delta > 0$ for which the process $\xi_0(t), t > 0$ has, with probability 1, no jumps with absolute values equal to δ. Since the càdlàg process $\xi_0(t), t > 0$ has at most a countable set of jump points, the set Z_0 is $(0, \infty)$ except for at most a countable set.

Condition $\mathcal{F}_{\mathbf{5}}$ should be modified in the following way:

$\mathcal{F}_{\mathbf{6}}$: There exist sequences $s_n \in U_0, 0 < s_n \to 0$ as $n \to \infty$ such that condition $\mathcal{F}_{\mathbf{5}}$ holds for the processes $\xi_\varepsilon^{(s_n)}(t), t \geq 0$, and $\nu_\varepsilon(t), t \geq 0$ for every $n \geq 1$.

Let W''_{0n} be the set of $t > 0$ for which condition $\mathfrak{D}_{\mathbf{4}}^{(\mathbf{w})}$ holds for the processes $\xi_\varepsilon^{(s_n)}(t), t \geq 0$ and $\nu_\varepsilon(t), t \geq 0$. Due to Lemma 2.6.4, condition $\mathcal{F}_{\mathbf{6}}$ implies that, for every $n \geq 1$, the set W''_{0n} is $(0, \infty)$ except for at most a countable set. Hence, the set $W''_0 = \cap_{n \geq 1} W''_{0n}$ also is $(0, \infty)$ except for at most a countable set.

It is possible to show that if condition $\mathfrak{F}_6$ holds for some sequence $s_n \in U_0, 0 < s_n \to 0$ as $n \to \infty$, it also holds for any other sequence $s'_n \in U_0, 0 < s'_n \to 0$ as $n \to \infty$ and the set W''_0 defined as above is the same for any such a sequence.

Recall that V_0 is the set of points $t > 0$ that are points of stochastic continuity for the process $\nu_0(t), t \geq 0$. This set equals $(0, \infty)$ except for at most a countable set.

Finally, $V_0 \cap W''_0$ is also the interval $(0, \infty)$ except for at most a countable set.

Conditions $\mathfrak{J}_{24}$ and $\mathfrak{J}_{25}$ do not need any changes, since they involve the processes $\alpha_\varepsilon(t)$ and $\alpha^{++}_\varepsilon(t)$ only for $t \in (0, \infty)$.

Condition $\bar{\mathcal{K}}_5$ also does not need to be modified.

Condition $\mathfrak{O}_{13}$ should be omitted in the corresponding theorems.

There are two cases that need to be considered. The first one is where **(a)** $\nu_0(t) > 0$ with probability 1 for $t > 0$. In the second case, **(b)** $\nu_0(0) > 0$ with probability 1.

If **(a)** is satisfied and **(b)** can not be guaranteed to hold, the point 0 needs to be excluded from the set of weak convergence, and **J**-convergence is guaranteed only on the interval $(0, \infty)$. If **(b)** is satisfied, one can include the point 0 in the set of weak convergence, and prove **J**-convergence of the generalised exceeding processes on the interval $[0, \infty)$.

Let us impose the following condition:

$\mathfrak{I}_{13}$: $\kappa_0(0) \leq 0$ with probability 1.

Condition $\mathfrak{I}_{13}$ holds if and only if the random variables $\nu_0(t) > 0$ with probability 1 for every $t > 0$.

Let us formulate analogues of Theorems 4.3.6 and 4.3.7.

Theorem 4.3.10. *Let conditions $\bar{\mathcal{K}}_5$, $\mathcal{A}_{58}$, $\mathfrak{J}_{24}$, and $\mathfrak{I}_{13}$ hold. Then*

$$\zeta_\varepsilon(t), t \in V_0 \cap W''_0 \Rightarrow \zeta_0(t), t \in V_0 \cap W''_0 \text{ as } \varepsilon \to 0.$$

Theorem 4.3.11. *Let conditions $\bar{\mathcal{K}}_5$, $\mathcal{A}_{58}$, $\mathfrak{J}_{24}$, $\mathfrak{I}_9$, and $\mathfrak{I}_{13}$ hold. Then*

$$\zeta_\varepsilon(t), t > 0 \xrightarrow{\mathbf{J}} \zeta_0(t), t > 0 \text{ as } \varepsilon \to 0.$$

Proof of Theorems 4.3.10 and 4.3.11. Conditions $\bar{\mathcal{K}}_5$, $\mathcal{A}_{58}$, and $\mathfrak{J}_{24}$ imply that $\mathfrak{F}_6$ holds for every sequence $s_n \in U_0, 0 < s_n \to 0$. Indeed, conditions $\mathcal{A}_{58}$ and $\mathfrak{J}_{24}$ imply that the processes $\boldsymbol{\alpha}_\varepsilon(t)$ **J**-converge on the open interval $(0, \infty)$. This permits to extend the set U in condition $\mathcal{A}_{58}$ by including all points of stochastic continuity for the process $\xi_0(t), t > 0$. This shows that condition $\mathcal{A}_{56}$ holds for the processes $\xi^{(s_n)}_\varepsilon(t), t \geq 0$ and $\kappa_\varepsilon(t), t \geq 0$, for every $n \geq 1$. It is also obvious that condition $\mathfrak{O}_{13}$ holds for the processes $\xi^{(s_n)}_\varepsilon(t), t \geq 0$, for every $n \geq 1$. Hence, condition $\mathfrak{F}_5$ holds for the processes $\xi^{(s_n)}_\varepsilon(t), t \geq 0$ and $\nu_\varepsilon(t), t \geq 0$, for every $n \geq 1$.

By applying Theorem 4.3.6 to the processes $\xi_\varepsilon^{(s_n)}(t), t \geq 0$ and $\nu_\varepsilon(t), t \geq 0$, and taking into consideration that $W_0'' \subseteq W_{0n}''$, we get the following relation for every $n \geq 1$:

$$\zeta_\varepsilon^{(s_n)}(t), t \in V_0 \cap W_0'' \Rightarrow \zeta_0^{(s_n)}(t), t \in V_0 \cap W_0'' \text{ as } \varepsilon \to 0. \tag{4.3.16}$$

Let us introduce the processes

$$\boldsymbol{\alpha}_\varepsilon^{(s_n)}(t) = (\kappa_\varepsilon(t), \xi_\varepsilon^{(s_n)}(t)), t \geq 0.$$

We are going to prove that condition $\mathcal{J}_{24}$ holds for these processes, i.e. for every $n \geq 1$ and $\delta > 0$, $0 < T' < T'' < \infty$,

$$\lim_{0<c\to 0} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\Delta_J(\boldsymbol{\alpha}_\varepsilon^{(s_n)}(\cdot), c, T', T'') > \delta\} = 0. \tag{4.3.17}$$

Relation (4.3.17) is obvious if **(c)** $s_n \geq T''$. Indeed, in this case,

$$\Delta_J(\boldsymbol{\alpha}_\varepsilon^{(s_n)}(\cdot), c, T', T'') = \Delta_J(\kappa_\varepsilon(\cdot), c, T', T''). \tag{4.3.18}$$

Relation (4.3.17) is also obvious if **(d)** $s_n \leq T'$. Indeed, in this case,

$$\Delta_J(\boldsymbol{\alpha}_\varepsilon^{(s_n)}(\cdot), c, T', T'') = \Delta_J(\boldsymbol{\alpha}_\varepsilon(\cdot), c, T', T''). \tag{4.3.19}$$

The only non-trivial case is **(e)** $T' < s_n < T''$. The following estimate holds for any two-dimensional càdlàg function $z(t) = (x(t), y(t)), t \geq 0$, and the function $z^{(s_n)}(t) = (x(t \vee s_n), y(t)), t \geq 0$, for $c < T'/2$:

$$\begin{aligned} &\Delta_J(z^{(s_n)}(\cdot), c, T', T'') \\ &\quad \leq \Delta_J(y(\cdot), c, T', s_n) + \Delta_J(z(\cdot), c, s_n, T'') \\ &\quad + \sup_{s_n - 2c \leq t', t'' \leq s_n + 2c} |z(t') - z(t'')|. \end{aligned} \tag{4.3.20}$$

Recall again that conditions $\mathcal{A}_{58}$ and $\mathcal{J}_{24}$ provide **J**-convergence of the processes $\boldsymbol{\alpha}_\varepsilon(t)$ on the open interval $(0, \infty)$. Recall also that s_n is a point of stochastic continuity for the process $\boldsymbol{\alpha}_0(t), t \geq 0$. Taking this into account and using condition $\mathcal{J}_{24}$ and estimate (4.3.20) we get, for every $\delta > 0$ and $0 < T' < T'' < \infty$, that

$$\begin{aligned} &\lim_{0<c\to 0} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\Delta_J(\boldsymbol{\alpha}_\varepsilon^{(s_n)}(\cdot), c, T', T'') > \delta\} \\ &\quad \leq \lim_{0<c\to 0} \overline{\lim_{\varepsilon\to 0}} (\mathsf{P}\{\Delta_J(\kappa_\varepsilon(\cdot), c, T', s_n) > \delta/3\} + \mathsf{P}\{\Delta_J(\boldsymbol{\alpha}_\varepsilon(\cdot), c, s_n, T'') > \delta/3\} \\ &\quad + \mathsf{P}\{\sup_{s_n - 2c \leq t', t'' \leq s_n + 2c} |\boldsymbol{\alpha}_\varepsilon(t') - \boldsymbol{\alpha}_\varepsilon(t'')| > \delta/3\}) = 0. \end{aligned} \tag{4.3.21}$$

Now, by applying Theorem 4.3.7 to the truncated generalised exceeding processes $\zeta_\varepsilon^{(s_n)}(t)$, $t \geq 0$, and taking into consideration the remarks in Subsection 4.3.6, we get **J**-convergence of these processes on the interval $(0, \infty)$ for every $n \geq 1$,

$$\zeta_\varepsilon^{(s_n)}(t), t > 0 \xrightarrow{\mathbf{J}} \zeta_0^{(s_n)}(t), t > 0 \text{ as } \varepsilon \to 0. \tag{4.3.22}$$

Take now any $T > 0$. The following estimate follows from the definition of the truncated processes $\xi_\varepsilon^{(s_n)}(t), t \geq 0$, and condition $\mathfrak{I}_{13}$:

$$\begin{aligned}
&\lim_{n\to\infty}\overline{\lim_{\varepsilon\to0}}\mathsf{P}\{\sup_{t\geq T}|\zeta_\varepsilon(t)-\zeta_\varepsilon^{(s_n)}(t)|>0\} \\
&\quad\leq \lim_{n\to\infty}\overline{\lim_{\varepsilon\to0}}\mathsf{P}\{\nu_\varepsilon(T)\leq s_n\}\leq \lim_{n\to\infty}\overline{\lim_{\varepsilon\to0}}\mathsf{P}\{\kappa_\varepsilon(s_n)\geq T/2\}=0.
\end{aligned} \tag{4.3.23}$$

Relation (4.3.23) also holds in the case where $\varepsilon = 0$ and implies, in this case, that

$$\zeta_0^{(s_n)}(t), t>0 \Rightarrow \zeta_0(t), t>0 \text{ as } n\to\infty. \tag{4.3.24}$$

Relations (4.3.16), (4.3.23), (4.3.24), and Lemma 1.2.5 imply relation of weak convergence given in Theorem 4.3.10.

Using the estimate in Lemma 1.4.9 and (4.3.23) we get, for every $0 < T' < T'' < \infty$, that

$$\begin{aligned}
&\lim_{0<c\to0}\overline{\lim_{\varepsilon\to0}}\mathsf{P}\{\Delta_J(\zeta_\varepsilon(\cdot),c,T',T'')>\delta\} \\
&\quad\leq \lim_{0<c\to0}\overline{\lim_{\varepsilon\to0}}(\mathsf{P}\{\Delta_J(\zeta_\varepsilon^{(s_n)}(\cdot),c,T',T'')>\delta/2\} \\
&\quad+\mathsf{P}\{\Delta_U(\zeta_\varepsilon(\cdot)-\zeta_\varepsilon^{(s_n)}(\cdot),c,T',T'')>\delta/2\}) \\
&\quad\leq \overline{\lim_{\varepsilon\to0}}\mathsf{P}\{\sup_{t\geq T'}|\zeta_\varepsilon(t)-\zeta_\varepsilon^{(s_n)}(t)|>\delta/4\}\to0 \text{ as } n\to\infty.
\end{aligned} \tag{4.3.25}$$

Relation of weak convergence, given in Theorem 4.3.10, and estimate (4.3.25) prove Theorem 4.3.11. □

Let us now consider the case when the following condition holds:

$\mathfrak{I}_{14}$: $\nu_0(0) > 0$ with probability 1.

Note that condition $\mathfrak{I}_{14}$ implies that the process $\zeta_0(t) = \xi_0(\nu_0(t)), t \geq 0$ is an a.s. càdlàg process despite a possible discontinuity of the process $\xi_0(t), t \geq 0$ at the point 0.

In this case, in order to include the point 0 in the set of weak convergence, $V_0 \cap W''_{0n}$, of the generalised exceeding processes $\zeta_\varepsilon^{(s_n)}(t), t \geq 0$, for every $n \geq 1$, one should require that condition $\mathfrak{I}_8$ or $\mathfrak{O}_{11}$ holds. Additionally, one should require that at least one of conditions $\mathfrak{Q}_{10}$ or $\mathfrak{C}_5^{(0)}$ (for the stopping moment $\nu_0(0)$ and the process $\xi_0(t), t > 0$) or $\mathfrak{D}_4^{(0)}$ (for the stopping moments $\nu_\varepsilon(0)$ and the processes $\xi_\varepsilon^{(s_n)}(t), t \geq 0$, for every $n \geq 1$) holds.

Note that condition $\mathfrak{Q}_{10}$ or $\mathfrak{C}_5^{(0)}$ holds for the stopping moment $\nu_0(0)$ and the process $\xi_0(t), t > 0$, if and only if it holds for the stopping moment $\nu_0(0)$ and the process $\xi_0^{(s_n)}(t), t > 0$, for every $n \geq 1$.

Alternatively, one can assume that conditions $\mathfrak{O}_{11}$ and $\mathfrak{O}_{12}$ hold.

For example, let us formulate versions of Theorems 4.3.10 and 4.3.11 in which conditions $\mathfrak{O}_{11}$ and $\mathfrak{O}_{12}$ are assumed to hold.

Theorem 4.3.12. *Let conditions $\bar{\mathfrak{K}}_5$, $\mathcal{A}_{58}$, $\mathfrak{J}_{24}$, $\mathfrak{I}_{14}$, $\mathfrak{O}_{11}$, and $\mathfrak{O}_{12}$ hold. Then*

$$\zeta_\varepsilon(t), t \in (V_0 \cap W_0'') \cap \{0\} \Rightarrow \zeta_0(t), t \in (V_0 \cap W_0'') \cap \{0\} \text{ as } \varepsilon \to 0.$$

Theorem 4.3.13. *Let conditions $\bar{\mathfrak{K}}_5$, $\mathcal{A}_{58}$, $\mathfrak{J}_{24}$, $\mathfrak{I}_9$, $\mathfrak{I}_{14}$, $\mathfrak{O}_{11}$, and $\mathfrak{O}_{12}$ hold. Then*

$$\zeta_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{J}} \zeta_0(t), t \geq 0 \text{ as } \varepsilon \to 0.$$

Proof of Theorem 4.3.12 and 4.3.13. Take again an arbitrary sequence $s_n \in U_0, 0 < s_n \to 0$. Using condition $\mathfrak{O}_{12}$ and the remarks made in Subsection 4.3.2 and repeating the reasoning used in the proof of Theorem 4.3.10 for getting relation (4.3.16) we can prove that, for every $n \geq 1$,

$$\zeta_\varepsilon^{(s_n)}(t), t \in (V_0 \cap W_0'') \cup \{0\} \Rightarrow \zeta_0^{(s_n)}(t), t \in (V_0 \cap W_0'') \cup \{0\} \text{ as } \varepsilon \to 0. \qquad (4.3.26)$$

It was shown in the proof of Theorem 4.3.11 that for every $\delta > 0$, $0 < T' < T'' < \infty$ and every $n \geq 1$,

$$\lim_{0<c\to 0} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\Delta_J(\boldsymbol{\alpha}_\varepsilon^{(s_n)}(\cdot), c, T', T'') > \delta\} = 0. \qquad (4.3.27)$$

Also, it is obvious that condition $\mathfrak{O}_{13}$ holds for the processes $\xi_\varepsilon^{(s_n)}(t), t \geq 0$, for every $n \geq 1$. Relation (4.3.27) and condition $\mathfrak{O}_{13}$ imply that, for every $\delta > 0$, $0 < T < \infty$, and every $n \geq 1$,

$$\lim_{0<c\to 0} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\Delta_J(\boldsymbol{\alpha}_\varepsilon^{(s_n)}(\cdot), c, T) > \delta\} = 0. \qquad (4.3.28)$$

Relations (4.3.26) and (4.3.28) imply that for every $n \geq 1$,

$$\zeta_\varepsilon^{(s_n)}(t), t \geq 0 \xrightarrow{\mathbf{J}} \zeta_0^{(s_n)}(t), t \geq 0 \text{ as } \varepsilon \to 0. \qquad (4.3.29)$$

Taking into consideration condition $\mathfrak{O}_{11}$ and the remarks made in Subsection 4.3.2 we get, by using condition $\mathfrak{I}_{14}$, that

$$\lim_{n\to\infty} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\nu_\varepsilon(0) < s_n\} \leq \lim_{n\to\infty} \mathsf{P}\{\nu_0(0) < s_n/2\} = 0. \qquad (4.3.30)$$

The following estimate follows from the definition of the truncated processes $\xi_\varepsilon^{(s_n)}(t)$, $t \geq 0$, and relation (4.3.30):

$$\lim_{n\to\infty} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\sup_{t\geq 0} |\zeta_\varepsilon(t) - \zeta_\varepsilon^{(s_n)}(t)| > 0\} \leq \lim_{n\to\infty} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\nu_\varepsilon(0) < s_n\} = 0. \qquad (4.3.31)$$

Relation (4.3.31) can also be applied in the case where $\varepsilon = 0$ to give

$$\zeta_0^{(s_n)}(t), t \geq 0 \Rightarrow \zeta_0(t), t \geq 0 \text{ as } n \to \infty. \qquad (4.3.32)$$

Relations (4.3.26), (4.3.31), (4.3.32), and Lemma 1.2.5 imply the relation of weak convergence given in Theorem 4.3.12.

Using estimate obtained in Lemma 1.4.9 and (4.3.31) we get, for every $\delta > 0$ and $0 < T < \infty$, that

$$\begin{aligned}
&\lim_{0<c\to 0}\overline{\lim_{\varepsilon\to 0}}\,\mathsf{P}\{\Delta_J(\zeta_\varepsilon(\cdot), c, T) > \delta\} \\
&\quad \le \lim_{0<c\to 0}\overline{\lim_{\varepsilon\to 0}}(\mathsf{P}\{\Delta_J(\zeta_\varepsilon^{(s_n)}(\cdot), c, T) > \delta/2\} \\
&\quad + \mathsf{P}\{\Delta_U(\zeta_\varepsilon(\cdot) - \zeta_\varepsilon^{(s_n)}(\cdot), c, T) > \delta/2\}) \\
&\quad \le \overline{\lim_{\varepsilon\to 0}}\,\mathsf{P}\{\sup_{t\ge 0}|\zeta_\varepsilon(t) - \zeta_\varepsilon^{(s_n)}(t)| > \delta/4\} \to 0 \text{ as } n \to \infty.
\end{aligned} \tag{4.3.33}$$

The relation of weak convergence in Theorem 4.3.12 and estimate (4.3.33) prove Theorem 4.3.13. □

4.3.9. Generalised exceeding processes based on non-monotone exceeding time processes. Let, for every $\varepsilon \ge 0$, $\boldsymbol{\alpha}_\varepsilon(t) = (\kappa_\varepsilon(t), \xi_\varepsilon(t))$, $t \ge 0$ be a two-dimensional càdlàg process with real-valued components. Here, neither monotonicity nor non-negativity of the component $\kappa_\varepsilon(t), t \ge 0$ is required.

Let us introduce an *exceeding time process* $\nu_\varepsilon(t) = \sup(s : \kappa_\varepsilon(s) \le t), t \ge 0$, and a *generalised exceeding process* $\zeta_\varepsilon(t) = \xi_\varepsilon(\nu_\varepsilon(t)), t \ge 0$.

This model can be reduced to a model with the non-decreasing component $\kappa_\varepsilon(t), t \ge 0$ in the following way. Define the process $\kappa_\varepsilon^+(t) = \sup_{s\le t}\kappa_\varepsilon(s), t \ge 0$ and the corresponding exceeding time process $\nu_\varepsilon^+(t) = \sup(s : \kappa_\varepsilon^+(s) \le t), t \ge 0$. By the definition, $\kappa_\varepsilon^+(t), t \ge 0$ is a non-decreasing càdlàg process. But $\nu_\varepsilon(t) = \nu_\varepsilon^+(t), t \ge 0$. Thus, one can consider the generalised exceeding process $\zeta_\varepsilon(t) = \xi_\varepsilon(\nu_\varepsilon(t)) = \xi_\varepsilon(\nu_\varepsilon^+(t)), t \ge 0$ as a process based on the bivariate process $\boldsymbol{\alpha}_\varepsilon^+(t) = (\kappa_\varepsilon^+(t), \xi_\varepsilon(t))$, $t \ge 0$, whose second component $\kappa_\varepsilon^+(t), t \ge 0$ is a nondecreasing càdlàg process.

An analogue of condition $\bar{\mathcal{K}}_5$ takes the following form:

$\bar{\mathcal{K}}_6$: $\kappa_\varepsilon^+(t) \xrightarrow{\mathsf{P}} \infty$ as $t \to \infty$ for every $\varepsilon \ge 0$.

In this case, it is reasonable to try to replace conditions $\mathcal{A}_{56}$ and $\mathcal{J}_{24}$ by similar conditions formulated in terms of the initial processes $\boldsymbol{\alpha}_\varepsilon(t) = (\kappa_\varepsilon(t), \xi_\varepsilon(t)), t \ge 0$.

Let us denote by V_0 the set of points $t > 0$ that are points of stochastic continuity for the process $\kappa_0(t), t \ge 0$, and by U_0 the set of points $t \ge 0$ that are points of stochastic continuity for the process $\xi_0(t), t \ge 0$. The set V_0 equals $(0, \infty)$ except for at most a countable set and also the set U_0 coincides with $[0, \infty)$ except for at most a countable set. Also, $0 \in U_0$.

Lemma 4.3.3. *If conditions* $\mathcal{A}_{56}$ *and* $\mathcal{J}_{24}$ *hold for the processes* $\boldsymbol{\alpha}_\varepsilon(t) = (\kappa_\varepsilon(t), \xi_\varepsilon(t))$, $t \ge 0$, *then* **(α)** *condition* $\mathcal{A}_{56}$ *holds for the processes* $\boldsymbol{\alpha}_\varepsilon^+(t) = (\kappa_\varepsilon^+(t), \xi_\varepsilon(t)), t \ge 0$, *with the sets* $V = V_0$ *and* $U = U_0$; **(β)** *condition* $\mathcal{J}_{24}$ *holds for the processes* $\boldsymbol{\alpha}_\varepsilon^+(t), t \ge 0$.

Proof of Lemma 4.3.3. Take an arbitrary point $0 < T < \infty$ that is a point of stochastic continuity for the process $\boldsymbol{\alpha}_0(t), t \geq 0$. By applying Lemma 1.6.14 to the processes $\boldsymbol{\alpha}_\varepsilon(t+T), t \geq 0$ we prove that **(a)** the processes $\boldsymbol{\alpha}^+_\varepsilon(t), t \in [T, \infty) \xrightarrow{\mathbf{J}} \boldsymbol{\alpha}^+_0(t), t \in [T, \infty)$ as $\varepsilon \to 0$. Since the choice of $0 < T < \infty$ was arbitrary, **(a)** implies that **(b)** $\boldsymbol{\alpha}^+_\varepsilon(t), t > 0 \xrightarrow{\mathbf{J}} \boldsymbol{\alpha}^+_0(t), t > 0$ as $\varepsilon \to 0$.

Note also that **(c)** every point of stochastic continuity of the process $\kappa_0(t), t \geq 0$ is also a point of stochastic continuity for the process $\kappa^+_0(t), t \geq 0$. Obviously, **(b)** and **(c)** imply the statement **(α)** of Lemma 4.3.3. Also, **(b)** implies the statement **(β)** of Lemma 4.3.3. □

Since $\kappa_0(0) \equiv \kappa^+_0(0)$, condition $\mathfrak{I}_6$, in which $\kappa_0(0) \geq 0$ with probability 1, does not need any changes.

Condition $\mathfrak{I}_9$ takes in this case the following form:

$\mathfrak{I}_{15}$: $\kappa^+_0(t), t \geq 0$ is an a.s. strictly increasing process.

The following theorem is a direct corollary of Lemma 4.3.3 and Theorem 4.3.7.

Theorem 4.3.14. *Let conditions $\bar{\mathcal{K}}_6$, $\mathcal{A}_{56}$, $\mathfrak{J}_{24}$, $\mathfrak{O}_{13}$, $\mathfrak{I}_6$, and $\mathfrak{I}_{15}$ hold. Then*

$$\zeta_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{J}} \zeta_0(t), t \geq 0 \text{ as } \varepsilon \to 0.$$

Remark 4.3.8. Condition $\mathfrak{I}_6$ can be replaced by condition $\mathfrak{I}_8$, which should be required to hold for the process $\kappa^+_0(t), t \geq 0$, or by condition $\mathfrak{O}_{11}$. In both cases, one should also require that at least one of conditions $\mathfrak{Q}_{10}$, $\mathfrak{C}^{(0)}_5$ (for the stopping moment $\nu_0(0)$ and the process $\xi_0(t), t \geq 0$) or $\mathfrak{D}^{(0)}_4$ (for the stopping moments $\nu_\varepsilon(0)$ and the processes $\xi_\varepsilon(t), t \geq 0$) holds. In this case, the point 0 can be included in the set of weak convergence $V_0 \cap W'_0$. Alternatively, condition $\mathfrak{O}_{12}$ can be assumed to hold.

Let us illustrate the theorem by the following example. Let, for every $\varepsilon \geq 0$, $\boldsymbol{\alpha}_\varepsilon(t) = (\kappa_\varepsilon(t), \xi_\varepsilon(t)), t \geq 0$ be a càdlàg homogeneous process with independent increments. For simplicity, let us assume that $\boldsymbol{\alpha}_\varepsilon(0) = (0, 0)$ with probability 1.

Condition $\mathcal{A}_{56}$ can be formulated in an equivalent form in terms of characteristics in Lévy–Khintchine representation for the characteristic function of the process $\boldsymbol{\alpha}_\varepsilon(t), t \geq 0$. The corresponding formulations can be found, for example, in Skorokhod (1964) or Gikhman and Skorokhod (1971). As is known (see, for example, Skorokhod (1964)), condition $\mathcal{A}_{56}$ (it is actually enough to require in this condition that the random variables $\boldsymbol{\alpha}_\varepsilon(1)$ weakly converge) implies in this case, without any additional assumptions, that **(d)** the processes $\boldsymbol{\alpha}_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{J}} \boldsymbol{\alpha}_0(t), t \geq 0$ as $\varepsilon \to 0$. Relation **(d)** implies that conditions $\mathcal{A}_{56}$, $\mathfrak{J}_{24}$, and $\mathfrak{O}_{12}$ hold for the processes $\boldsymbol{\alpha}_\varepsilon(t), t \geq 0$.

Therefore, conditions $\bar{\mathcal{K}}_6$, $\mathcal{A}_{56}$, and $\mathfrak{I}_{15}$ imply that the corresponding generalised exceeding processes $\zeta_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{J}} \zeta_0(t), t \geq 0$ as $\varepsilon \to 0$.

Note that, in this case, any assumptions about independence of the processes $\kappa_0(t), t \geq 0$ and $\xi_0(t), t \geq 0$ are not needed.

4.3.10. Generalised exceeding processes based on improper exceeding time processes. This is the case where condition $\bar{\mathcal{K}}_5$ does not hold. Denote $\kappa_\varepsilon(\infty) = \lim_{t\to\infty} \kappa_\varepsilon(t)$. This limit exists with probability 1, since $\kappa_\varepsilon(t), t \geq 0$ is a non-decreasing process. Condition $\bar{\mathcal{K}}_5$ does not hold if and only if $\mathsf{P}\{\kappa_\varepsilon(\infty) < \infty\} > 0$. In this case, the exceeding time process $\nu_\varepsilon(t) = \sup(s : \kappa_\varepsilon(s) \leq t), t \geq 0$ can be improper. Moreover, $\mathsf{P}\{\nu_\varepsilon(t) = +\infty\} = \mathsf{P}\{\kappa_\varepsilon(\infty) \leq t\}$.

To avoid this situation, one can choose some $T \in (0, \infty)$ and consider the truncated exceeding time processes $\nu_{\varepsilon,T}(t) = \min(\nu_\varepsilon(t), T), t \geq 0$ and the truncated generalised exceeding processes $\zeta_{\varepsilon,T}(t) = \xi_\varepsilon(\nu_{\varepsilon,T}(t)), t \geq 0$. The results formulated in Subsections 4.3.1 - 4.3.9 can be carried over to this case with some modifications.

We refer to Silvestrov (1974, 2000a), where one can find results concerning truncated generalised exceeding processes.

4.3.11. An alternative approach to limit theorems for generalised exceeding processes. A more general model for generalised exceeding processes was studied in Silvestrov (1974, 2000a). This model is based on a study of homogeneous families of functionals $\mu_t(\mathbf{x}(\cdot)), t \geq 0$, defined on the space $\bar{\mathbf{D}}^{(m)}_{[0,\infty)}$ of m-dimensional càdlàg functions $\mathbf{x}(t), t \geq 0$ that are invariant with respect to monotone transformations of time. The homogeneity property mentioned above means that $\mu_{\lambda(t)}(\mathbf{x}(\cdot)) = \mu_t(\mathbf{x}(\lambda(\cdot))), t \geq 0$, for any continuous one-to-one mapping $\lambda(t)$ of the interval $[0, \infty)$ into itself, and any càdlàg function $\mathbf{x}(t), t \geq 0$. An exceeding time process based on a m-dimensional càdlàg processes $\boldsymbol{\xi}_\varepsilon(t), t \geq 0$ is defined as $\nu_\varepsilon(t) = \inf(s : \mu_s(\boldsymbol{\xi}_\varepsilon(\cdot)) > t), t \geq 0$, and a generalised exceeding process is defined as $\boldsymbol{\zeta}_\varepsilon(t) = g(\boldsymbol{\xi}_\varepsilon(\nu_\varepsilon(t))), t \geq 0$, where $g(\mathbf{x})$ is a continuous function acting from $\mathbb{R}_m$ to $\mathbb{R}_l$. It is easily seen that the generalised exceeding processes considered in Section 4.3 is a particular case of the model described above. The method used in Silvestrov (1974, 2000a) is based on thorough studies of $\mathbf{J}$-continuity properties of the random functionals $\inf(s : \mu_s(\boldsymbol{\xi}_\varepsilon(\cdot)) > t)$ and $g(\boldsymbol{\xi}_\varepsilon(\nu_\varepsilon(t)))$.

For the model of generalised exceeding processes considered in Section 4.3, the results obtained in Silvestrov (1974, 2000a) are similar to those given in Theorems 4.3.6 – 4.3.7. Note that these theorems require stronger $\mathbf{J}$-compactness conditions than, for example, the preceding Theorems 4.3.1 – 4.3.5.

4.4 Step generalised exceeding processes

In this section, we continue studies of limit theorems for generalised exceeding processes. In Section 4.3, a model for generalised exceeding processes with asymptotically continuous exceeding time stopping processes was considered. In the present section, a model with step stopping exceeding time processes is considered.

4.4.1. Conditions for weak and J-convergence based on imbedded sequences. Consider an important class of generalised exceeding processes where the process $\kappa_\varepsilon(t)$, $t \geq 0$ is a step càdlàg process for every $\varepsilon \geq 0$. This means that there exists an a.s. strictly increasing sequence of random variables $0 = \tau_{\varepsilon 0} < \tau_{\varepsilon 1} < \tau_{\varepsilon 2} < \dots$ such that $\kappa_\varepsilon(t) = \kappa_\varepsilon(\tau_{\varepsilon n})$ for $t \in [\tau_{\varepsilon n}, \tau_{\varepsilon n+1}), n = 0, 1, \dots$. The random variables $\tau_{\varepsilon n}, n = 1, 2, \dots$ are successive moments moments of jumps, whereas the random variables $\kappa_\varepsilon(\tau_{\varepsilon n}), n = 1, 2, \dots$ are values of the process $\kappa_\varepsilon(t), t \geq 0$ at the moments of jumps. We exclude the case of fictitious jumps, that is, assume that $\kappa_\varepsilon(\tau_{\varepsilon 0}) < \kappa_\varepsilon(\tau_{\varepsilon 1}) < \kappa_\varepsilon(\tau_{\varepsilon 2}) < \dots$ with probability 1.

We summarise the above assumptions as the following condition:

$\mathfrak{I}_{16}$: $\tau_{\varepsilon n}, n = 0, 1, \dots$ and $\kappa_\varepsilon(\tau_{\varepsilon n}), n = 0, 1, \dots$ are a.s. strictly increasing sequences of random variables for every $\varepsilon \geq 0$.

We also restrict the consideration to the most important case where the process $\kappa_\varepsilon(t), t \geq 0$ a.s. has a finite number of jumps in any finite interval, that is,

$\bar{\mathcal{K}}_7$: $\tau_{\varepsilon n} \xrightarrow{\mathsf{P}} \infty$ as $n \to \infty$ for every $\varepsilon \geq 0$.

It is clear that in this case the exceeding time process $\nu_\varepsilon(t) = \sup(s : \kappa_\varepsilon(s) \leq t), t \geq 0$ can be represented in the following form:

$$\nu_\varepsilon(t) = \begin{cases} 0 & \text{if } t < \kappa_\varepsilon(\tau_{\varepsilon 0}), \\ \tau_{\varepsilon k+1} & \text{if } \kappa_\varepsilon(\tau_{\varepsilon k}) \leq t < \kappa_\varepsilon(\tau_{\varepsilon k+1}), k = 0, 1, \dots. \end{cases} \tag{4.4.1}$$

The process $\nu_\varepsilon(t), t \geq 0$ is also a càdlàg step process.

The process $\nu_\varepsilon(t), t \geq 0$ a.s. has a finite number of jumps in any finite interval if the following condition holds:

$\bar{\mathcal{K}}_8$: $\kappa_\varepsilon(\tau_{\varepsilon n}) \xrightarrow{\mathsf{P}} \infty$ as $n \to \infty$ for every $\varepsilon \geq 0$.

It is useful to note that, under $\bar{\mathcal{K}}_7$, conditions $\bar{\mathcal{K}}_8$ and $\bar{\mathcal{K}}_5$ are equivalent. Indeed, since the process $\kappa_\varepsilon(t)$ is monotone, we have **(a)** $\kappa_\varepsilon(t) \xrightarrow{\text{a.s.}} \kappa_\varepsilon(\infty)$, where $\kappa_\varepsilon(\infty)$ is a random variable (possibly improper) that a.s. takes values in the interval $(-\infty, \infty]$. Since the sequence of random variables $\tau_{\varepsilon n}$ is monotone, condition $\bar{\mathcal{K}}_7$ is equivalent to the relation **(b)** $\tau_{\varepsilon n} \xrightarrow{\text{a.s.}} \infty$ as $n \to \infty$. Obviously, **(a)** and **(b)** imply that **(c)** $\kappa_\varepsilon(\tau_{\varepsilon n}) \xrightarrow{\text{a.s.}} \kappa_\varepsilon(\infty)$ as $n \to \infty$. Condition $\bar{\mathcal{K}}_5$, as well as condition $\bar{\mathcal{K}}_8$, holds if and only if $\kappa_\varepsilon(\infty) = \infty$ with probability 1. So, these conditions are equivalent.

Condition $\bar{\mathcal{K}}_8$ implies that the random variable $\nu_\varepsilon(t) < \infty$ with probability 1 for every $t \geq 0$.

Let us now consider the *generalised exceeding process* $\zeta_\varepsilon(t) = \xi_\varepsilon(\nu_\varepsilon(t)), t \geq 0$. Since $\nu_\varepsilon(t), t \geq 0$ is a step càdlàg process, $\zeta_\varepsilon(t) = \xi_\varepsilon(\nu_\varepsilon(t)), t \geq 0$ is also a step càdlàg process. Obviously,

$$\zeta_\varepsilon(t) = \begin{cases} \xi_\varepsilon(0) & \text{if } t < \kappa_\varepsilon(\tau_{\varepsilon 0}), \\ \xi_\varepsilon(\tau_{\varepsilon k+1}) & \text{if } \kappa_\varepsilon(\tau_{\varepsilon k}) \leq t < \tau_\varepsilon(\tau_{\varepsilon k+1}), k = 0, 1, \dots. \end{cases} \tag{4.4.2}$$

Let us consider the case where the limiting exceeding time process $\nu_0(t)$, $t \geq 0$ is a step process. We give conditions for weak convergence and **J**-convergence of generalised exceeding processes formulated in terms of the"embedded" sequence of random vectors $(\kappa_\varepsilon(\tau_{\varepsilon n}), \xi(\tau_{\varepsilon n})), n = 0, 1, \dots$. Let us assume the following condition:

$\mathcal{A}_{59}$: $(\kappa_\varepsilon(\tau_{\varepsilon n}), \xi(\tau_{\varepsilon n})), n = 0, 1, \dots \Rightarrow (\kappa_0(\tau_{0n}), \xi(\tau_{0n})), n = 0, 1, \dots$ as $\varepsilon \to 0$.

Denote by Z_0 the set of all $t \geq 0$ that are points of continuity of distribution functions of the random variables $\kappa_0(\tau_{0n}), n = 0, 1, \dots$. This set is $[0, \infty)$ except for at most a countable set.

Lemma 4.4.1. *Let conditions $\mathfrak{I}_{16}$, $\bar{\mathcal{K}}_7$, $\bar{\mathcal{K}}_8$, and $\mathcal{A}_{59}$ holds. Then*

$$\zeta_\varepsilon(t), t \in Z_0 \Rightarrow \zeta_0(t), t \in Z_0 \text{ as } \varepsilon \to 0.$$

Proof of Lemma 4.4.1. Let us choose arbitrary $n \geq 1$, $t_k \in Z_0$, and x_k such that $\mathsf{P}\{\xi_0(t_k) = x_k\} = 0$ for $k = 1, \dots n$. By conditions $\mathfrak{I}_{16}$ and $\bar{\mathcal{K}}_7$,

$$\begin{aligned} &\mathsf{P}\{\zeta_\varepsilon(t_k) \leq x_k, k = 1, \dots n\} \\ &\quad = \sum_{k=1}^{n} \sum_{r_k=0}^{\infty} \mathsf{P}\{\xi_\varepsilon(\tau_{\varepsilon r_k}) \leq x_k, \kappa_\varepsilon(\tau_{\varepsilon r_k - 1}) \leq t_k < \kappa_\varepsilon(\tau_{\varepsilon r_k}), k = 1, \dots, n\}, \end{aligned} \tag{4.4.3}$$

where $\kappa_\varepsilon(\tau_{\varepsilon -1}) = -\infty$.

By conditions $\mathcal{A}_{59}$ and $\bar{\mathcal{K}}_8$, for any $t \geq 0$,

$$\lim_{N\to\infty} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\kappa_\varepsilon(\tau_{\varepsilon N}) \leq t\} \leq \lim_{N\to\infty} \mathsf{P}\{\kappa_0(\tau_{0N}) \leq 2t\} = 0. \tag{4.4.4}$$

Condition $\mathcal{A}_{59}$ and relation (4.4.4) imply that

$$\begin{aligned} &\lim_{\varepsilon\to 0} \mathsf{P}\{\zeta_\varepsilon(t_k) \leq x_k, k = 1, \dots n\} \\ &\quad = \lim_{N\to\infty} \lim_{\varepsilon\to 0} \sum_{k=1}^{n} \sum_{r_k=0}^{N} \mathsf{P}\{\xi_\varepsilon(\tau_{\varepsilon r_k}) \leq x_k, \kappa_\varepsilon(\tau_{\varepsilon r_k - 1}) \leq t_k < \kappa_\varepsilon(\tau_{\varepsilon r_k}), k = 1, \dots, n\} \\ &\quad = \lim_{N\to\infty} \sum_{k=1}^{n} \sum_{r_k=0}^{N} \mathsf{P}\{\xi(\tau_{0 r_k}) \leq x_k, \kappa_0(\tau_{0 r_k - 1}) \leq t_k < \kappa_0(\tau_{0 r_k}), k = 1, \dots, n\} \\ &\quad = \sum_{k=1}^{n} \sum_{r_k=0}^{\infty} \mathsf{P}\{\xi_0(\tau_{0 r_k}) \leq x_k, \kappa_0(\tau_{0 r_k - 1}) \leq t_k < \kappa_0(\tau_{0 r_k}), k = 1, \dots, n\} \\ &\quad = \mathsf{P}\{\zeta_0(t_k) \leq x_k, k = 1, \dots n\}. \end{aligned} \tag{4.4.5}$$

Since the choice of $t_k \in Z_0$ and x_k such that $\mathsf{P}\{\zeta_0(t_k) = x_k\} = 0$ for $k = 1, \ldots n, n \geq 1$ was arbitrary, relation (4.4.5) is equivalent to the weak convergence relation given in Lemma 4.4.1. □

Conditions of **J**-compactness can also be formulated in terms of the "embedded" sequence of random variables $\kappa_\varepsilon(\tau_{\varepsilon n}), n = 0, 1, \ldots$. It is sufficient to assume validity of the following condition, which is weaker than $\mathcal{A}_{59}$:

$\mathcal{A}_{60}$: $\kappa_\varepsilon(\tau_{\varepsilon n}), n = 0, 1, \ldots \Rightarrow \kappa_0(\tau_{0n}), n = 0, 1, \ldots$ as $\varepsilon \to 0$.

Lemma 4.4.2. *Let conditions $\mathfrak{I}_{16}$, $\bar{\mathcal{K}}_7$, $\bar{\mathcal{K}}_8$, and $\mathcal{A}_{60}$ hold. Then*

$$\lim_{c\to 0}\overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\Delta_J(\zeta_\varepsilon(\cdot), c, T) > \delta\} = 0, \ \delta, T > 0.$$

Proof of Lemma 4.4.2. Consider the random functionals $\eta_\varepsilon(T) = \max(n : \kappa_\varepsilon(\tau_{\varepsilon n}) \leq T)$ and $\theta_\varepsilon(n) = \min_{1\leq k\leq n}(\kappa_\varepsilon(\tau_{\varepsilon k}) - \kappa_\varepsilon(\tau_{\varepsilon k-1}))$.

It is readily seen that $\theta_\varepsilon(\eta_\varepsilon(T))$ is greater than or equal to the minimal length of the intervals between the moments of jumps of the process $\zeta_\varepsilon(t)$ in the interval $[0, T]$. So, we have the following implication for the random events:

$$\{\Delta_J(\zeta_\varepsilon(\cdot), c, T) > 0\} \subseteq \{\theta_\varepsilon(\eta_\varepsilon(T)) \leq c\}. \tag{4.4.6}$$

Relation (4.4.6) implies that

$$\begin{aligned}\mathsf{P}\{\Delta_J(\zeta_\varepsilon(\cdot), c, T) \geq \delta\} &\leq \mathsf{P}\{\theta_\varepsilon(\eta_\varepsilon(T)) \leq c\} \\ &\leq \mathsf{P}\{\theta_\varepsilon(n) \leq c\} + \mathsf{P}\{\eta_\varepsilon(T) > n\} \\ &= \mathsf{P}\{\theta_\varepsilon(n) \leq c\} + \mathsf{P}\{\kappa_\varepsilon(\tau_{\varepsilon n}) \leq T\}.\end{aligned} \tag{4.4.7}$$

Since $f(x_1, \ldots, x_n) = \min_{1\leq k\leq n}(x_k - x_{k-1})$ is a continuous function for every $n \geq 1$, condition $\mathcal{A}_{60}$ implies that

$$\theta_\varepsilon(n) \Rightarrow \theta_0(n) \text{ as } \varepsilon \to 0, \ n \geq 1. \tag{4.4.8}$$

Note also that, by condition $\mathfrak{I}_{16}$, the random variable $\theta_0(n) > 0$ with probability 1, for every $n \geq 1$. Using this fact and conditions $\mathfrak{I}_{16}$, $\bar{\mathcal{K}}_7$, $\bar{\mathcal{K}}_8$, and $\mathcal{A}_{60}$ we get

$$\begin{aligned}&\lim_{0<c\to 0}\overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\Delta_J(\zeta_\varepsilon(\cdot), c, T) \geq \delta\} \\ &\quad\leq \lim_{0<c\to 0}\overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\theta_\varepsilon(\eta_\varepsilon(T)) \leq c\} \\ &\quad\leq \lim_{0<c\to 0}\overline{\lim_{\varepsilon\to 0}}(\mathsf{P}\{\theta_\varepsilon(n) \leq c\} + \mathsf{P}\{\kappa_\varepsilon(\tau_{\varepsilon n}) \leq T\}) \\ &\quad\leq \lim_{0<c\to 0} \mathsf{P}\{\theta_0(n) \leq 2c\} + \mathsf{P}\{\kappa_0(\tau_{0n}) \leq 2T\} \\ &\quad= \mathsf{P}\{\kappa_0(\tau_{0n}) \leq 2T\} \to 0 \text{ as } n \to \infty.\end{aligned} \tag{4.4.9}$$

This completes the proof. □

If condition $\mathfrak{I}_{16}$ holds, then condition $\mathcal{A}_{60}$ can be replaced by the following weaker condition:

$\mathcal{N}_2$: (a) $\lim_{n\to\infty} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\kappa_\varepsilon(\tau_{\varepsilon n}) \leq T\} = 0$ for $T < \infty$;

(b) $\lim_{0<c\to 0} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\kappa_\varepsilon(\tau_{\varepsilon n}) - \kappa_\varepsilon(\tau_{\varepsilon n-1}) \leq c\} = 0$ for $n \geq 1$.

Lemma 4.4.3. *Let conditions* $\mathfrak{I}_{16}$, $\bar{\mathcal{K}}_7$, $\bar{\mathcal{K}}_8$, *and* $\mathcal{N}_2$ *hold. Then*

$$\lim_{c\to 0} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\Delta_J(\zeta_\varepsilon(\cdot), c, T) > \delta\} = 0, \ \delta, T > 0.$$

Proof of Lemma 4.4.3. The first part of the proof repeats the proof of Lemma 4.4.2 up to the estimate (4.4.7). This estimate can be continued in the following way:

$$\begin{aligned} \mathsf{P}\{\Delta_J(\zeta_\varepsilon(\cdot), c, T) \geq \delta\} &\leq \mathsf{P}\{\theta_\varepsilon(\eta_\varepsilon(T)) \leq c\} \\ &\leq \mathsf{P}\{\theta_\varepsilon(n) \leq c\} + \mathsf{P}\{\kappa_\varepsilon(\tau_{\varepsilon n}) \leq T\} \\ &\leq \sum_{k=1}^{n} \mathsf{P}\{\kappa_\varepsilon(\tau_{\varepsilon k}) - \kappa_\varepsilon(\tau_{\varepsilon k-1}) \leq c\} + \mathsf{P}\{\kappa_\varepsilon(\tau_{\varepsilon n}) \leq T\}. \end{aligned} \tag{4.4.10}$$

Take an arbitrary $\sigma > 0$. Using condition $\mathcal{N}_2$ (a) we can choose n so large that $\overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\kappa_\varepsilon(\tau_{\varepsilon n}) \leq T\} \leq \sigma$. Then we get, using condition $\mathcal{N}_2$ (b), that

$$\begin{aligned} &\lim_{0<c\to 0} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\Delta_J(\zeta_\varepsilon(\cdot), c, T) \geq \delta\} \\ &\quad\leq \lim_{0<c\to 0} \overline{\lim_{\varepsilon\to 0}} \Big(\sum_{k=1}^{n} \mathsf{P}\{\kappa_\varepsilon(\tau_{\varepsilon k}) - \kappa_\varepsilon(\tau_{\varepsilon k-1}) \leq c\} + \mathsf{P}\{\kappa_\varepsilon(\tau_{\varepsilon n}) \leq T\}\Big) \\ &\quad\leq \sum_{k=1}^{n} \lim_{0<c\to 0} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\kappa_\varepsilon(\tau_{\varepsilon k}) - \kappa_\varepsilon(\tau_{\varepsilon k-1}) \leq c\} + \sigma = \sigma. \end{aligned} \tag{4.4.11}$$

Since the choice of $\sigma > 0$ is arbitrary, relation (4.4.11) implies the relation of **J**-compactness stated in the lemma. □

The question about possibility to include the point 0 in the set of weak convergence Z_0 requires a special consideration. Let us formulate the following condition:

$\mathcal{R}_1$: $\lim_{0<t\to 0} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{0 < \kappa_\varepsilon(\tau_{\varepsilon n}) \leq t\} = 0$ for $n = 0, 1, \ldots$.

Lemma 4.4.4. *Let conditions* $\mathfrak{I}_{16}$, $\bar{\mathcal{K}}_7$, $\bar{\mathcal{K}}_8$, $\mathcal{N}_2$ (a)*, and* $\mathcal{R}_1$ *hold. Then*

$$\lim_{0<t\to 0} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\nu_\varepsilon(t) - \nu_\varepsilon(0) > \delta\} = 0, \ \delta > 0.$$

Proof of Lemma 4.4.4. By using conditions $\mathcal{N}_2$ (a) and $\mathcal{R}_1$, we get

$$\begin{aligned}
&\lim_{0<t\to 0} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\nu_\varepsilon(t) - \nu_\varepsilon(0) > \delta\} \leq \lim_{0<t\to 0} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\nu_\varepsilon(t) \neq \nu_\varepsilon(0)\} \\
&\quad \leq \sum_{k=0}^{n} \lim_{0<t\to 0} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{0 < \kappa_\varepsilon(\tau_{\varepsilon k}) \leq t\} + \lim_{0<t\to 0} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\kappa_\varepsilon(\tau_{\varepsilon n+1}) \leq t\} \qquad (4.4.12) \\
&\quad \leq \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\kappa_\varepsilon(\tau_{\varepsilon n+1}) \leq 0\} \to 0 \text{ as } n \to \infty.
\end{aligned}$$

The relation given in Lemma 4.4.4 follows from estimate (4.4.12). □

Lemma 4.4.5. *Let conditions* $\mathcal{I}_{16}$, $\bar{\mathcal{K}}_7$, $\bar{\mathcal{K}}_8$, $\mathcal{N}_2$ (a)*, and* $\mathcal{R}_1$ *hold. Then*

$$\lim_{c\to 0} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{|\zeta_\varepsilon(t) - \zeta_\varepsilon(0)| > \delta\} = 0, \ \delta > 0.$$

Proof of Lemma 4.4.5. Using relation (4.4.12) we get

$$\lim_{0<t\to 0} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{|\zeta_\varepsilon(t) - \zeta_\varepsilon(0)| > \delta\} \leq \lim_{0<t\to 0} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\nu_\varepsilon(t) \neq \nu_\varepsilon(0)\} = 0. \qquad (4.4.13)$$

□

As was mentioned above, if condition $\mathcal{I}_{16}$ holds, then condition $\mathcal{A}_{60}$ implies condition $\mathcal{N}_2$.

The following theorem is a corollary of Lemmas 4.4.1 – 4.4.5.

Theorem 4.4.1. *Let conditions* $\mathcal{I}_{16}$, $\bar{\mathcal{K}}_7$, $\bar{\mathcal{K}}_8$, $\mathcal{A}_{59}$*, and* $\mathcal{R}_1$ *hold. Then*

$$\zeta_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{J}} \zeta_0(t), t \geq 0 \text{ as } \varepsilon \to 0.$$

The most important for applications is the case where the following condition holds:

$\mathcal{I}_{17}$: $\kappa_\varepsilon(0) \geq 0$ with probability 1 for every $\varepsilon \geq 0$.

Obviously, if condition $\mathcal{I}_{17}$ holds, then the following condition is sufficient for condition $\mathcal{R}_1$ to hold:

$\mathcal{R}_2$: $\lim_{0<t\to 0} \overline{\lim}_{\varepsilon\to 0}(\mathsf{P}\{0 < \kappa_\varepsilon(0) \leq t\} + \mathsf{P}\{\kappa_\varepsilon(\tau_{\varepsilon 1}) \leq t\}) = 0$.

Remark 4.4.1. Under conditions $\mathcal{I}_{16}$ and $\mathcal{I}_{17}$, condition $\mathcal{A}_{60}$ implies condition $\mathcal{R}_2$ to hold if **(a)** $\mathsf{P}\{\kappa_\varepsilon(0) = 0\} \to \mathsf{P}\{\kappa_0(0) = 0\}$ as $\varepsilon \to 0$. Note that **(a)** and condition $\mathcal{I}_{17}$ hold if **(b)** $\mathsf{P}\{\kappa_\varepsilon(0) = 0\} = 1$ for every $\varepsilon \geq 0$.

Remark 4.4.2. If **(b)** holds, then condition $\mathcal{N}_2$ (b) implies that condition $\mathcal{R}_1$ holds.

4.4.2. General conditions for weak and J-convergence of step generalised exceeding processes. It should be noted that the results on weak convergence of generalised exceeding processes, given in Theorems 4.3.1 – 4.3.3, 4.3.6, 4.3.8, 4.3.10, and 4.3.12, as well as in Lemmas 4.3.1 – 4.3.3, can also be applied to step generalised exceeding processes.

As a matter of fact, the only assumption that the processes $\kappa_\varepsilon(t), t \geq 0$ are non-decreasing is involved. No other assumptions about the character of trajectories of these processes were used in these theorems.

Considering the **J**-convergence, one should be more careful. Theorems 4.3.4, 4.3.5, 4.3.7, 4.3.9, 4.3.11, and 4.3.13 can be applied in the case where the corresponding limiting exceeding time process $\nu_0(t), t \geq 0$ is an a.s. continuous process. Instead of using these theorems, one can combine the conditions of Theorems 4.3.1 – 4.3.3, 4.3.6, 4.3.8, 4.3.10, and 4.3.12 with the conditions of Lemmas 4.4.3 – 4.4.5 in order to get conditions for **J**-convergence of the step generalised exceeding processes.

For example, the following theorem combines the conditions of Theorem 4.3.7 and the ones pointed out in Remark 4.3.6 with the conditions of Lemmas 4.4.3 – 4.4.5. Note that condition $\mathfrak{R}_1$ replaces conditions $\mathfrak{O}_{11}$ and $\mathfrak{O}_{12}$. This is possible due to Lemmas 4.4.4 and 4.4.5.

Theorem 4.4.2. *Let conditions $\mathfrak{I}_{16}$, $\bar{\mathfrak{K}}_7$, $\bar{\mathfrak{K}}_8$, $\mathcal{A}_{56}$, $\mathfrak{J}_{24}$, $\mathfrak{O}_{13}$, $\mathfrak{N}_2$, and $\mathfrak{R}_1$ hold. Then*

$$\zeta_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{J}} \zeta_0(t), t \geq 0 \text{ as } \varepsilon \to 0.$$

Remark 4.4.3. Omitting condition $\mathfrak{R}_1$ in Theorem 4.4.2 we get conditions for **J**-convergence of the processes $\zeta_\varepsilon(t)$ on the interval $(0, \infty)$.

The case considered in Subsection 4.3.8 requires a special consideration. In this case, the conditions of Theorems 4.3.11 or 4.3.13 can be combined with the conditions of Lemmas 4.4.3 – 4.4.5.

Theorem 4.4.3. *Let conditions $\mathfrak{I}_{16}$, $\bar{\mathfrak{K}}_7$, $\bar{\mathfrak{K}}_8$, $\mathcal{A}_{58}$, $\mathfrak{J}_{24}$, $\mathfrak{I}_{13}$, and $\mathfrak{N}_2$ hold. Then*

$$\zeta_\varepsilon(t), t > 0 \xrightarrow{\mathbf{J}} \zeta_0(t), t > 0 \text{ as } \varepsilon \to 0.$$

Theorem 4.4.4. *Let conditions $\mathfrak{I}_{16}$, $\bar{\mathfrak{K}}_7$, $\bar{\mathfrak{K}}_8$, $\mathcal{A}_{58}$, $\mathfrak{J}_{24}$, $\mathfrak{I}_{14}$, $\mathfrak{N}_2$, and $\mathfrak{R}_1$ hold. Then*

$$\zeta_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{J}} \zeta_0(t), t \geq 0 \text{ as } \varepsilon \to 0.$$

There is an essential difference between Theorem 4.4.1 and Theorems 4.4.2 – 4.4.4. Theorem 4.4.1 is based on conditions for weak convergence of the embedded random sequence $(\kappa_\varepsilon(\tau_{\varepsilon n}), \xi(\tau_{\varepsilon n})), n = 0, 1, \ldots$. At the same time, Theorems 4.4.2 – 4.4.4 are based on the condition for **J**-convergence of the bivariate processes $(\kappa_\varepsilon(t), \xi_\varepsilon(t)), t \geq 0$. Both variants have their own advantages in applications.

4.4.3. The case of step exceeding time processes defined on $(0, \infty)$. Let us also consider a model where, for every $\varepsilon \geq 0$, $\kappa_\varepsilon(t), t > 0$ is a step càdlàg process defined on the interval $(0, \infty)$. This means that for every $s > 0$ there exists an a.s. strictly increasing sequence of random variables $s = \tau^{(s)}_{\varepsilon 0} < \tau^{(s)}_{\varepsilon 1} < \tau^{(s)}_{\varepsilon 2} < \dots$ such that $\kappa_\varepsilon(t) = \kappa_\varepsilon(\tau^{(s)}_{\varepsilon n})$ for $t \in [\tau^{(s)}_{\varepsilon n}, \tau^{(s)}_{\varepsilon n+1}), n = 0, 1, \dots$. The random variables $\tau^{(s)}_{\varepsilon n}, n = 1, 2 \dots$ are moments of jumps and the random variables $\kappa_\varepsilon(\tau^{(s)}_{\varepsilon n}), n = 1, 2, \dots$ are values of the process $\kappa_\varepsilon(t), t \geq s$ at the moments of jumps. We exclude the case of fictitious jumps, that is, we assume that $\kappa_\varepsilon(\tau^{(s)}_{\varepsilon 0}) < \kappa_\varepsilon(\tau^{(s)}_{\varepsilon 1}) < \kappa_\varepsilon(\tau^{(s)}_{\varepsilon 2}) < \dots$ with probability 1. Let us summarise the assumptions made above in the following condition:

$\mathfrak{I}_{\mathbf{18}}$: $\tau^{(s)}_{\varepsilon n}, n = 0, 1, \dots$ and $\kappa_\varepsilon(\tau^{(s)}_{\varepsilon 0}), n = 0, 1, \dots$ are a.s. strictly increasing sequences of random variables for every $s > 0$ and every $\varepsilon \geq 0$.

Note that we allow here for the random variable $\kappa_\varepsilon(0 + 0) = \lim_{0<s\to 0} \kappa_\varepsilon(s)$ to be improper, i.e., to take the value $-\infty$ with a positive probability. Note that this limit exists with probability 1, since $\xi_\varepsilon(t)$ is a non-decreasing process. As far as the first component $\xi_\varepsilon(t), t \geq 0$, is concerned, we restrict the consideration to a basic case where this process is a càdlàg process defined on the interval $[0, \infty)$.

Conditions $\bar{\mathfrak{K}}_{\mathbf{7}}$ and $\bar{\mathfrak{K}}_{\mathbf{8}}$ take in this case the following forms:

$\bar{\mathfrak{K}}_{\mathbf{9}}$: $\tau^{(s)}_{\varepsilon n} \xrightarrow{\mathsf{P}} \infty$ as $n \to \infty$, $s > 0$ for every $\varepsilon \geq 0$,

and

$\bar{\mathfrak{K}}_{\mathbf{10}}$: $\kappa_\varepsilon(\tau^{(s)}_{\varepsilon n}) \xrightarrow{\mathsf{P}} \infty$ as $n \to \infty$, $s > 0$ for every $\varepsilon \geq 0$.

Note that, under $\bar{\mathfrak{K}}_{\mathbf{9}}$, condition $\bar{\mathfrak{K}}_{\mathbf{10}}$ is equivalent to condition $\bar{\mathfrak{K}}_{\mathbf{5}}$. The proof is absolutely analogous to the one in Subsection 4.4.1 for conditions $\bar{\mathfrak{K}}_{\mathbf{8}}$ and $\bar{\mathfrak{K}}_{\mathbf{5}}$.

Let us now take some $s_r > 0$ and define the process

$$\kappa^{(s_r)}_\varepsilon(t) = \kappa_\varepsilon(t \vee s_r) = \begin{cases} \kappa_\varepsilon(s_r) & \text{if } t < s_r, \\ \kappa_\varepsilon(t) & \text{if } t \geq s_r, \end{cases} \tag{4.4.14}$$

and then the corresponding exceeding time process

$$\hat{\nu}^{(s_r)}_\varepsilon(t) = \sup(s : \kappa^{(s_r)}_\varepsilon(s) \leq t), t \geq 0,$$

as well as the generalised exceeding process

$$\hat{\zeta}^{(s_r)}_\varepsilon(t) = \xi_\varepsilon(\hat{\nu}^{(s_r)}_\varepsilon(t)), t \geq 0.$$

By the definition,

$$\nu_\varepsilon(t) \geq \hat{\nu}^{(s_r)}_\varepsilon(t) = \begin{cases} 0 & \text{if } t < \kappa(s_r), \\ \nu_\varepsilon(t) & \text{if } t \geq \kappa(s_r). \end{cases} \tag{4.4.15}$$

Note that relation (4.4.15) does not imply that $\hat{\nu}_\varepsilon^{(s_r)}(0) = 0$, since it can occur that the event $\{t < \kappa(s_r)\} = \varnothing$.

Also by the definition of the exceeding time processes, **(a)** $\nu_\varepsilon(t) \le s_r$ for $t < \kappa(s_r)$. Relations (4.4.15) and **(a)** imply that the following estimate holds:

$$0 = \sup_{t<\kappa(s_r)} \nu_\varepsilon^{(s_r)}(t) \le \sup_{t<\kappa(s_r)} \nu_\varepsilon(t) \le s_r. \tag{4.4.16}$$

Here the supremum over the empty set should be interpreted as 0.

Take a sequence $0 < s_r \to 0$ as $r \to \infty$. The following lemma gives a useful estimate for generalised exceeding processes at zero.

Lemma 4.4.6. *Let conditions* $\mathfrak{I}_{18}$, $\bar{\mathcal{K}}_9$, $\bar{\mathcal{K}}_{10}$, *and* $\mathfrak{O}_{13}$ *hold. Then*

$$\lim_{r\to\infty} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\sup_{t\ge 0} |\zeta_\varepsilon(t) - \hat{\zeta}_\varepsilon^{(s_r)}(t)| > \delta\} = 0, \ \ \delta > 0.$$

Proof of Lemma 4.4.6. Using relations (4.4.15) – (4.4.16) and condition $\mathfrak{O}_{13}$ we get, for $\delta > 0$, that

$$\begin{aligned}
&\lim_{n\to\infty} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\sup_{t\ge 0} |\zeta_\varepsilon(t) - \hat{\zeta}_\varepsilon^{(s_r)}(t)| > \delta\} \\
&\quad = \lim_{n\to\infty} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\sup_{t<\kappa(s_r)} |\xi_\varepsilon(\nu_\varepsilon(t)) - \xi_\varepsilon(\nu_\varepsilon^{(s_r)}(t))| > \delta\} \\
&\quad \le \lim_{n\to\infty} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\sup_{t',t''\le s_r} |\xi_\varepsilon(t') - \xi_\varepsilon(t'')| > \delta\} \\
&\quad \le \lim_{n\to\infty} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\sup_{t\le s_r} |\xi_\varepsilon(t) - \xi_\varepsilon(0)| > \delta/2\} = 0.
\end{aligned} \tag{4.4.17}$$

This estimate completes the proof. □

Let us assume the following condition:

$\mathcal{A}_{61}$: $(\kappa_\varepsilon(\tau_{\varepsilon n}^{(s_r)}), \xi(\tau_{\varepsilon n}^{(s_r)})), n = 0, 1, \dots \Rightarrow (\kappa_0(\tau_{0n}^{(s_r)}), \xi(\tau_{0n}^{(s_r)})), n = 0, 1, \dots$ as $\varepsilon \to 0$, for $r \ge 1$.

For every $r \ge 1$, let Z_{0r} be the set of all $t > 0$ which are points of discontinuity for distribution functions of the random variables $\kappa_0(\tau_{0n}^{(s_r)}), n = 0, 1, \dots$. The set Z_{0r} equals $[0, \infty)$ except for at most a countable set. Let us also denote $Z_0 = \cap_{r\ge 1} Z_{0r}$. This set is also $[0, \infty)$ except for at most a countable set.

Lemma 4.4.7. *Let conditions* $\mathfrak{I}_{18}$, $\bar{\mathcal{K}}_9$, $\bar{\mathcal{K}}_{10}$, $\mathcal{A}_{61}$, *and* $\mathfrak{O}_{13}$ *hold. Then*

$$\zeta_\varepsilon(t), t \in Z_0 \Rightarrow \zeta_0(t), t \in Z_0 \text{ as } \varepsilon \to 0.$$

Proof of Lemma 4.4.7. Conditions $\mathbf{J_{18}}$, $\bar{\mathcal{K}}_{\mathbf{9}}$, $\bar{\mathcal{K}}_{\mathbf{10}}$, and $\mathcal{A}_{\mathbf{61}}$ imply that conditions of Lemma 4.4.1 hold for the processes $\xi_\varepsilon(t), t \geq 0$ and $\kappa_\varepsilon^{(s_r)}(t), t \geq 0$ for every $r \geq 1$. By applying Lemma 4.4.1 to these processes and taking into consideration that the set $Z_0 \subseteq Z_{0r}$ for $r \geq 1$, we get the following relation for every $r \geq 1$:

$$\hat{\zeta}_\varepsilon^{(s_r)}(t), t \in Z_0 \Rightarrow \hat{\zeta}_0^{(s_r)}(t), t \in Z_0 \text{ as } \varepsilon \to 0. \tag{4.4.18}$$

Lemma 4.4.6 also implies that

$$\hat{\zeta}_0^{(s_r)}(t), t \geq 0 \Rightarrow \zeta_0(t), t \geq 0 \text{ as } r \to \infty. \tag{4.4.19}$$

Lemmas 1.2.5 and 4.4.6, together with relations (4.4.18) and (4.4.19), imply the relation given in Lemma 4.4.7. □

We now formulate a lemma which is an analogue of Lemma 4.4.3. The following condition replaces, in this case, condition $\mathbf{N_2}$:

$\mathbf{N_3}$: There exists a sequence $0 < s_r \to 0$ as $r \to \infty$ such that

(a) $\lim_{n\to\infty} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\kappa_\varepsilon(\tau_{\varepsilon n}^{(s_r)}) \leq T\} = 0$ for $T < \infty, r \geq 1$;

(b) $\lim_{0<c\to 0} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\kappa_\varepsilon(\tau_{\varepsilon n}^{(s_r)}) - \kappa_\varepsilon(\tau_{\varepsilon n-1}^{(s_r)}) \leq c\} = 0$ for $n, r \geq 1$.

Lemma 4.4.8. *Let conditions $\bar{\mathcal{K}}_{\mathbf{9}}$, $\bar{\mathcal{K}}_{\mathbf{10}}$, $\mathbf{N_3}$, and $\mathbf{O_{13}}$ hold. Then*

$$\lim_{c\to 0} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\Delta_J(\zeta_\varepsilon(\cdot), c, T) > \delta\} = 0, \ \delta, T > 0.$$

Proof of Lemma 4.4.8. Condition $\mathbf{N_3}$ implies that condition $\mathbf{N_2}$ holds for the generalised exceeding processes $\hat{\zeta}_\varepsilon^{(s_r)}(t), t \geq 0$ for every $r \geq 1$. Thus Lemma 4.4.3 can be applied to these processes. This yields the following relation for every $r \geq 1$:

$$\lim_{c\to 0} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\Delta_J(\hat{\zeta}_\varepsilon^{(s_r)}(\cdot), c, T) > \delta\} = 0, \ \delta, T > 0. \tag{4.4.20}$$

Using the estimate obtained in Lemma 4.4.6 and relation (4.4.20) we get, for $0 < T < \infty$, that

$$\begin{aligned}
&\lim_{0<c\to 0} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\Delta_J(\zeta_\varepsilon(\cdot), c, T) > \delta\} \\
&\quad \leq \lim_{0<c\to 0} \overline{\lim_{\varepsilon\to 0}} (\mathsf{P}\{\Delta_J(\hat{\zeta}_\varepsilon^{(s_r)}(\cdot), c, T) > \delta/2\} \\
&\quad + \mathsf{P}\{\Delta_U(\zeta_\varepsilon(\cdot) - \hat{\zeta}_\varepsilon^{(s_r)}(\cdot), c, T) > \delta/2\}) \\
&\quad \leq \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\sup_{t\geq 0} |\zeta_\varepsilon(t) - \hat{\zeta}_\varepsilon^{(s_r)}(t)| > \delta/4\} \to 0 \text{ as } r \to \infty.
\end{aligned} \tag{4.4.21}$$

This estimate completes the proof. □

Let us also formulate a lemma which is an analogue of Lemma 4.4.4. The following condition replaces, in this case, condition $\mathfrak{R}_1$:

$\mathfrak{R}_3$: There exists a sequence $0 < s_r \to 0$ as $r \to \infty$ such that $\lim_{0<t\to 0} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{0 < \kappa_\varepsilon(\tau_{\varepsilon n}^{(s_r)}) \leq t\} = 0$ for $n = 0, 1, \ldots, r \geq 1$.

Theorem 4.4.5. *Let conditions $\mathfrak{I}_{18}$, $\bar{\mathfrak{K}}_9$, $\bar{\mathfrak{K}}_{10}$, $\mathcal{A}_{61}$, $\mathfrak{N}_3$, $\mathfrak{R}_3$, and $\mathfrak{O}_{13}$ hold. Then*

$$\zeta_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{J}} \zeta_0(t), t \geq 0 \text{ as } \varepsilon \to 0.$$

Proof of Theorem 4.4.5. Condition $\mathfrak{R}_3$ implies that condition $\mathfrak{R}_1$ holds for the generalised exceeding processes $\hat{\zeta}_\varepsilon^{(s_r)}(t), t \geq 0$, for every $r \geq 1$. So, we apply Lemma 4.4.5 to these processes. This yields, together with Lemma 4.4.7, the following relation for every $r \geq 1$:

$$\hat{\zeta}_\varepsilon^{(s_r)}(t), t \in Z_0 \cup \{0\} \Rightarrow \hat{\zeta}_0^{(s_r)}(t), \in Z_0 \cup \{0\} \text{ as } \varepsilon \to 0. \tag{4.4.22}$$

Relation (4.4.22) and Lemma 4.4.8 imply the statement of Theorem 4.4.5. □

As was mentioned above, the most important for applications is the case where condition $\mathfrak{I}_{16}$ holds.

Obviously, if condition $\mathfrak{I}_{16}$ holds, then the following condition is sufficient for condition $\mathfrak{R}_3$ to hold:

$\mathfrak{R}_4$: There exists a sequence $0 < s_r \to 0$ as $r \to \infty$ such that (a) $\lim_{0<t\to 0} \overline{\lim}_{\varepsilon\to 0}(\mathsf{P}\{0 < \kappa_\varepsilon(s_r) \leq t\} + \mathsf{P}\{\kappa_\varepsilon(\tau_{\varepsilon 1}^{(s_r)}) \leq t\}) = 0$ for $r \geq 1$.

The results given in Section 4.3 can also be applied to step generalised exceeding processes defined on the interval $(0, \infty)$. The remarks made in Subsection 4.3.7 are also valid in this case.

Let us only formulate an analogue of Theorem 4.4.2.

Theorem 4.4.6. *Let conditions $\mathfrak{I}_{18}$, $\bar{\mathfrak{K}}_9$, $\bar{\mathfrak{K}}_{10}$, $\mathcal{A}_{56}$, $\mathfrak{J}_{24}$, $\mathfrak{O}_{13}$, $\mathfrak{N}_3$, and $\mathfrak{R}_3$ hold. Then*

$$\zeta_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{J}} \zeta_0(t), t \geq 0 \text{ as } \varepsilon \to 0.$$

Remark 4.4.4. Omitting condition $\mathfrak{R}_3$ in Theorem 4.4.6 one gets conditions for **J**-convergence of the processes $\zeta_\varepsilon(t)$ on the interval $(0, \infty)$.

4.5 Sum-processes with renewal stopping

In this section, we get limit theorems for renewal processes and sum-processes with renewal stopping. This model gives the most important examples of exceeding time processes and generalised exceeding processes.

4.5.1. General sum-processes with renewal stopping. Let, for every $\varepsilon > 0$, $(\kappa_{\varepsilon,n}, \xi_{\varepsilon,n})$, $n = 1, 2, \ldots$ be a sequence of random vectors taking values in $[0, \infty) \times \mathbb{R}_1$. Further, let $n_\varepsilon > 0$ be a non-random function such that $n_\varepsilon \to \infty$ as $\varepsilon \to 0$.

We first introduce a sum-process with non-random stopping index,

$$\boldsymbol{\alpha}_\varepsilon(t) = (\kappa_\varepsilon(t), \xi_\varepsilon(t)) = (\sum_{k \leq tn_\varepsilon} \kappa_{\varepsilon,k}, \sum_{k \leq tn_\varepsilon} \xi_{\varepsilon,k}), \ t \geq 0.$$

In this case, the following process is usually referred as a *renewal* (*stopping*) *process*:

$$\nu_\varepsilon(t) = \sup(s : \kappa_\varepsilon(s) \leq t), \ t \geq 0,$$

The following process is called a *sum-process with renewal stopping*:

$$\zeta_\varepsilon(t) = \xi_\varepsilon(\nu_\varepsilon(t)), t \geq 0.$$

All theorems formulated in Section 4.3 can be directly carried over to sum-processes with renewal stopping.

Let us just repeat here the formulations of the theorems that will be directly applied to sum-processes with renewal stopping constructed from i.i.d. random variables.

Condition $\mathcal{A}_{56}$ takes, in this case, the following form:

$\mathcal{A}_{62}$: $(\kappa_\varepsilon(s), \xi_\varepsilon(t)), (s, t) \in V \times U \Rightarrow (\kappa_0(s), \xi_0(t)), (s, t) \in V \times U$ as $\varepsilon \to 0$, where (a) V is a subset of $(0, \infty)$, dense in this interval, (b) U is a subset of $[0, \infty)$ that is dense in this interval and contains the point 0, (c) $(\kappa_0(t), \xi_0(t)), t \geq 0$ is a càdlàg process with non-negative and non-decreasing first component and real-valued second component.

It should be noted that **(a)** $\kappa_\varepsilon(t), t \geq 0$ is a non-negative and non-decreasing process for $\varepsilon > 0$. So, it is only necessary to require in condition $\mathcal{A}_{62}$ (c) that the limiting process $(\kappa_0(s), \xi_0(t)), t \geq 0$ be a càdlàg process omitting the requirement for this process to be non-negative and no-decreasing. Indeed, the relation of weak convergence given in condition $\mathcal{A}_{62}$ and **(a)** imply that the first component of the limiting process $\kappa_0(t), t \geq 0$ should be an a.s. non-negative and non-decreasing càdlàg process at least for $t \in V$. The set V is dense in $(0, \infty)$ and $\kappa_0(t), t \geq 0$ is a càdlàg process. So, $\kappa_0(t), t \geq 0$ is an a.s. non-negative and non-decreasing càdlàg process. This process can always be replaced by a stochastically equivalent càdlàg modification in condition $\mathcal{A}_{62}$.

Since the process $\kappa_0(t), t \geq 0$ is non-negative, condition $\mathfrak{I}_6$ automatically holds, i.e., $\kappa_0(0) \geq 0$ with probability 1.

However, it is useful to note that condition $\mathcal{A}_{62}$ does not require weak convergence of the processes $\kappa_\varepsilon(t), t \geq 0$ at the point 0. It can occur that the random variable $\kappa_0(0)$ takes positive values, although the random variable $\kappa_\varepsilon(0) = 0$ with probability 1 for every $\varepsilon > 0$. It should also be noted that the assumption $0 < \varepsilon \to 0$ does not affect conditions $\mathfrak{J}_{24}$ and $\mathfrak{O}_{13}$. The asymptotic relations that enter these conditions also hold for $\varepsilon = 0$. Indeed, by condition $\mathcal{A}_{62}$, $(\kappa_0(s), \xi_0(t)), t \geq 0$ is a càdlàg process.

Let us reformulate Theorem 4.3.7.

Theorem 4.5.1. *Let conditions $\mathcal{A}_{62}$, $\bar{\mathcal{K}}_5$, $\mathfrak{J}_{24}$, $\mathfrak{O}_{13}$, and $\mathfrak{I}_9$ hold for the sum-processes $\boldsymbol{\alpha}_\varepsilon(t)$, $t \geq 0$. Then*

$$\zeta_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{J}} \zeta_0(t), t \geq 0 \text{ as } \varepsilon \to 0.$$

Note that, in this case, $\nu_0(0) = 0$ with probability 1. Also, by the definition, $\xi_0(0) = 0$ with probability 1 for $\varepsilon > 0$. Since $0 \in U$, condition $\mathcal{A}_{62}$ also implies that $\xi_0(0) = 0$ with probability 1 and, consequently, $\zeta_0(0) = \xi_0(\nu_0(0)) = 0$ with probability 1.

Let us also consider the case when the limiting renewal process is a step càdlàg process.

In this case, the random variables $\tau_{\varepsilon n}, n = 0, 1, \dots$ should be defined in the same way as in Subsection 4.4.1, that is, as successive moments of positive jumps of the process $\kappa_\varepsilon(t), t \geq 0$, for $\varepsilon > 0$ as well as for $\varepsilon = 0$.

Conditions $\mathfrak{I}_{16}$, $\bar{\mathcal{K}}_7$, $\bar{\mathcal{K}}_8$ do not require any changes in the formulations. They should be required to hold.

Condition $\mathcal{A}_{59}$ takes, in this case, the following form:

$\mathcal{A}_{63}$: $(\kappa_\varepsilon(\tau_{\varepsilon n}), \xi(\tau_{\varepsilon n})), n = 0, 1, \dots \Rightarrow (\kappa_0(\tau_{0n}), \xi(\tau_{0n})), n = 0, 1, \dots$ as $\varepsilon \to 0$, where $(\kappa_0(\tau_{0n}), \xi(\tau_{0n})), n = 0, 1, \dots$ is a sequence of random vectors with non-negative first and real-valued second component.

Since, **(a)** $\kappa_\varepsilon(t), t \geq 0$ is a non-negative and non-decreasing process for $\varepsilon > 0$, the requirement of non-negativity of the corresponding random variables $\kappa_0(\tau_{0n})$ in conditions $\mathcal{A}_{63}$ can be omitted. This automatically follows from the relation of weak convergence given in condition $\mathcal{A}_{63}$. Also note that **(a)** implies that condition $\mathfrak{I}_{17}$ holds. Hence, condition $\mathfrak{R}_1$ can be replaced by condition $\mathfrak{R}_2$.

Let us reformulate Theorems 4.4.1 and 4.4.4.

Theorem 4.5.2. *Let conditions $\mathfrak{I}_{16}$, $\bar{\mathcal{K}}_7$, $\bar{\mathcal{K}}_8$, $\mathcal{A}_{63}$, and $\mathfrak{R}_2$ hold for the sum-processes $\boldsymbol{\alpha}_\varepsilon(t)$, $t \geq 0$. Then*

$$\zeta_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{J}} \zeta_0(t), t \geq 0 \text{ as } \varepsilon \to 0.$$

Theorem 4.5.3. *Let conditions $\mathfrak{I}_{16}$, $\bar{\mathcal{K}}_7$, $\bar{\mathcal{K}}_8$, $\mathcal{A}_{62}$, $\mathcal{J}_{24}$, $\mathcal{O}_{13}$, $\mathcal{N}_2$, and $\mathcal{R}_2$ hold for the sum-processes $\boldsymbol{\alpha}_\varepsilon(t), t \geq 0$. Then*

$$\zeta_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{J}} \zeta_0(t), t \geq 0 \text{ as } \varepsilon \to 0.$$

Let us also consider a general model in which $\kappa_{\varepsilon,n}$, $n = 1, 2, \ldots$ are real-valued random variables. This case does not require any changes in the definitions of the renewal processes and the sum-processes with renewal stopping. In this case, $\kappa_\varepsilon(t), t \geq 0$, is not a non-decreasing process. However, the process $\kappa_\varepsilon^+(t) = \sup_{s \leq t} \kappa_\varepsilon(s), t \geq 0$, is non-decreasing. At the same time, the corresponding renewal process $\nu_\varepsilon(t) = \sup(s : \kappa_\varepsilon(s) \leq t) = \sup(s : \kappa_\varepsilon^+(s) \leq t)$, $t \geq 0$ can be considered as an exceeding time process constructed from the process $\kappa_\varepsilon^+(t), t \geq 0$.

Let us just reformulate Theorem 4.3.14. Note that $\kappa_\varepsilon^+(0) = 0$ which allows to one omit condition $\mathfrak{I}_6$.

Theorem 4.5.4. *Let conditions $\mathcal{A}_{57}$, $\bar{\mathcal{K}}_6$, $\mathcal{J}_{25}$, $\mathcal{O}_{13}$, and $\mathfrak{I}_{15}$ hold for the sum-processes $\boldsymbol{\alpha}_\varepsilon(t)$, $t \geq 0$. Then*

$$\zeta_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{J}} \zeta_0(t), t \geq 0 \text{ as } \varepsilon \to 0.$$

Here we do not formulate separately conditions for weak convergence. Only note that, under condition $\mathcal{Q}_{10}$, the corresponding set of weak convergence in Theorem 4.5.1 – 4.5.4 is the interval $[0, \infty)$.

The renewal stopping processes are defined above as $\nu_\varepsilon(t) = \sup(s : \kappa_\varepsilon(s) \leq t) = \inf(s : \kappa_\varepsilon(s) > t) = \frac{1}{n_\varepsilon} \min(n : \sum_{k \leq n} \kappa_{\varepsilon,k} > t)$, $t \geq 0$. In some applications, slightly *modified sum-processes with renewal stopping* are used. In that case, the renewal stopping moments are defined as $\nu'_\varepsilon(t) = \frac{1}{n_\varepsilon} \max(n : \sum_{k \leq n} \kappa_{\varepsilon,k} \leq t) = \nu_\varepsilon(t) - 1/n_\varepsilon$, $t \geq 0$. Respectively, a slightly modified version of the *generalised exceeding process*, $\zeta'_\varepsilon(t) = \xi_\varepsilon(\nu'_\varepsilon(t)) = \xi_\varepsilon(\nu_\varepsilon(t) - 1/n_\varepsilon), t \geq 0$, is considered.

The modification of sum-processes with renewal stopping affects neither the conditions nor the formulations of the theorems given in Section 4.3. In particular, one can replace the processes $\nu_\varepsilon(t), t \geq 0$ by the processes $\nu'_\varepsilon(t), t \geq 0$ and, as a consequence, the processes $\zeta_\varepsilon(t), t \geq 0$ by the processes $\zeta'_\varepsilon(t), t \geq 0$ in all lemmas and theorems given in this section.

The only slight changes are required in the case of step sum-processes with renewal stopping, i.e., where condition $\mathcal{A}_{58}$ is employed. As a matter of fact, in this case $\kappa_\varepsilon(\tau_{\varepsilon 0}) = 0$ and, therefore, the modified sum-process with renewal stopping $\zeta'_\varepsilon(t) = \xi_\varepsilon(\tau_{\varepsilon k+1} - 1/n_\varepsilon)$ if $\kappa_\varepsilon(\tau_{\varepsilon k}) \leq t < \kappa_\varepsilon(\tau_{\varepsilon k+1})$, $k = 0, 1, \ldots$. So, the random variables $\xi(\tau_{\varepsilon n})$ should be replaced, in this condition, by the random variables $\xi(\tau_{\varepsilon n} - 1/n_\varepsilon)$ and the limiting random variables $\xi(\tau_{0n})$ by the random variables $\xi(\tau_{0n} - 0)$ for $n = 0, 1, \ldots$.

4.5.2. Non-negative sum-processes based on i.i.d. random variables. Let us consider the sum-processes $\kappa_\varepsilon(t) = \sum_{k \leq tn_\varepsilon} \kappa_{\varepsilon,k}, t \geq 0$. We assume that the following condition holds:

$\mathcal{T}_3$: $\kappa_{\varepsilon,k}, k = 1, 2, \ldots$ is (for every $\varepsilon > 0$) a sequence of non-negative i.i.d. random variables.

Conditions $\mathcal{S}_1 - \mathcal{S}_3$, which provide marginal weak convergence of the sum-processes $\xi_\varepsilon(t), t \geq 0$, were formulated in Subsection 4.2.2. Let us now formulate similar conditions for the sum-processes $\kappa_\varepsilon(t), t \geq 0$.

The process $\kappa_\varepsilon(t), t \geq 0$ is a sum-process of i.i.d. random variables. As easily seen, this is a particular case of the sum-process $\xi_\varepsilon(t), t \geq 0$. However, due to its non-negativity, the process $\kappa_\varepsilon(t), t > 0$ is simpler to deal with. Conditions for weak convergence of such processes involve the tail probabilities and the truncated means for the random variables $\kappa_{\varepsilon,1}$ but not their truncated variances,

$\mathcal{S}_4$: $n_\varepsilon \mathsf{P}\{\kappa_{\varepsilon,1} > u\} \to \pi_1(u)$ as $\varepsilon \to 0$ for all $u > 0$, which are points of continuity of the limiting function $\pi_1(u)$.

$\mathcal{S}_5$: $n_\varepsilon \mathsf{E}\kappa_{\varepsilon,1}\chi(\kappa_{\varepsilon,1} \leq u) \to c(u)$ as $\varepsilon \to 0$ for some $u > 0$, which is a point of continuity of $\pi_1(u)$.

Also here the limits satisfy a number of conditions: **(a)** $\pi_1(u)$ is a non-negative, non-increasing, and right-continuous function for $u > 0$ and $\pi_1(\infty) = 0$; **(b)** the measure $\Pi_1(A)$ on σ-algebra $\mathfrak{B}_1^+$, the Borel σ-algebra of subsets of $(0, \infty)$, defined by the relation $\Pi_1((u_1, u_2]) = \pi_1(u_1) - \pi_1(u_2), 0 < u_1 \leq u_2 < \infty$, satisfies the condition $\int_0^\infty \frac{s}{1+s}\Pi_1(ds) < \infty$; **(c)** under $\mathcal{S}_4$, condition $\mathcal{S}_5$ can only hold simultaneously for all continuity points of $\pi_1(u)$ and $c(u_1) = c(u_2) - \int_{u_1}^{u_2} s\Pi_1(ds)$ for any such points $0 < u_1 < u_2 < \infty$; **(d)** $c(u)$ is a non-negative function.

Note that, due to non-negativity of the random variables $\kappa_{\varepsilon,1}$, conditions $\mathcal{S}_4$ and $\mathcal{S}_5$ imply condition $\mathcal{S}_6$ to hold with the constant $b^2 = 0$.

Indeed, it follows from $\mathcal{S}_5$ in an obvious way that $n_\varepsilon(\mathsf{E}\kappa_{\varepsilon,1}\chi(\kappa_{\varepsilon,1} \leq u))^2 \to 0$ as $\varepsilon \to 0$. Also, $n_\varepsilon \mathsf{E}\kappa_{\varepsilon,1}^2\chi(\kappa_{\varepsilon,1} \leq u) \leq u n_\varepsilon \mathsf{E}\kappa_{\varepsilon,1}\chi(\kappa_{\varepsilon,1} \leq u) \to 0$ as $\varepsilon \to 0$ and then $0 < u \to 0$.

According to the central criterium of convergence, conditions $\mathcal{S}_4$ and $\mathcal{S}_5$ are necessary and sufficient for the following condition of weak convergence to hold:

$\mathcal{A}_{64}$: $\kappa_\varepsilon(t), t \geq 0 \Rightarrow \kappa_0(t), t \geq 0$ as $\varepsilon \to 0$, where $\kappa_0(t), t \geq 0$ is a non-negative and non-decreasing càdlàg homogeneous process with independent increments.

The limiting process $\kappa_0(t), t \geq 0$ has the following characteristic function for $t \geq 0$,

$$
\begin{aligned}
\mathsf{E}\exp\{\mathrm{i}z\kappa_0(t)\} &= \phi_1(t, y) \\
&= \exp\{t(\mathrm{i}cy + \int_0^\infty (e^{\mathrm{i}ys} - 1)\Pi_1(ds))\} \\
&= \exp\{t(\mathrm{i}dy + \int_0^\infty (e^{\mathrm{i}ys} - 1 - \frac{\mathrm{i}ys}{1+s^2})\Pi_1(ds))\},
\end{aligned}
\tag{4.5.1}
$$

where the constants

$$c = c(u) - \int_0^u s\Pi_1(ds) \geq 0, \ d = c + \int_0^\infty \frac{s}{1+s^2}\Pi_1(ds) \qquad (4.5.2)$$

do not depend of the choice of the point u in condition $\mathbf{\mathcal{S}_5}$.

As was shown by Skorokhod (1957, 1964), conditions $\mathbf{\mathcal{S}_4}$ – $\mathbf{\mathcal{S}_5}$ imply, without any additional assumptions, that

$$\kappa_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{J}} \kappa_0(t), t \geq 0 \text{ as } \varepsilon \to 0. \qquad (4.5.3)$$

4.5.3. Renewal processes based on i.i.d. random variables. To exclude the trivial case where the process $\kappa_0(t) = 0, t > 0$, we also assume the following condition:

$\mathbf{\mathfrak{I}_{19}}$: (a) $c > 0$, or (b) $\pi_1(0+) = \lim_{0<w\to 0} \pi_1(w) \in (0, \infty]$.

It is easy to show that condition $\mathbf{\mathfrak{I}_{19}}$ holds if and only if **(a)** $\kappa_0(t) \xrightarrow{\mathsf{P}} \infty$ as $t \to \infty$.

Conditions $\mathbf{\mathcal{S}_4}$ – $\mathbf{\mathcal{S}_5}$ and $\mathbf{\mathfrak{I}_{19}}$ also imply that **(b)** $\mathsf{P}\{\kappa_{\varepsilon,1} > 0\} > 0$ for all ε small enough. Without loss of generality, one can assume that **(b)** holds and, therefore, **(c)** $\kappa_\varepsilon(t) \xrightarrow{\mathsf{P}} \infty$ as $t \to \infty$ for every $\varepsilon > 0$.

So, we can assume that condition $\bar{\mathbf{\mathcal{K}}}_\mathbf{5}$ holds.

Let us consider the corresponding pre-limiting renewal stopping processes $\nu_\varepsilon(t) = \sup(s : \kappa_\varepsilon(s) \leq t), t > 0$. Note that we can interpret the random variables $\kappa_{\varepsilon,k}$ as inter-renewal times. Hence, $n_\varepsilon \nu_\varepsilon(t) - 1$ can be interpreted as the number of renewals in the interval $[0, tn_\varepsilon]$.

The corresponding limiting process $\nu_0(t) = \sup(s : \kappa_0(s) \leq t), t > 0$ is an exceeding time process for the process $\kappa_0(t), t \geq 0$.

Denote by V_0 the set of points $t > 0$ that are points of stochastic continuity of the process $\nu_0(t), t \geq 0$.

The following statement is a direct corollary of Lemma 4.3.1.

Lemma 4.5.1. *Let conditions* $\mathbf{\mathcal{T}_3}$, $\mathbf{\mathcal{A}_{64}}$ *(or* $\mathbf{\mathcal{S}_4}$, $\mathbf{\mathcal{S}_5}$*) and* $\mathbf{\mathfrak{I}_{19}}$ *hold. Then*

$$\nu_\varepsilon(t), t \in V_0 \Rightarrow \nu_0(t), t \in V_0 \text{ as } \varepsilon \to 0.$$

The process $\nu_0(t), t \geq 0$ is stochastically continuous and $V_0 = (0, \infty)$ if the following condition, which is stronger than $\mathbf{\mathfrak{I}_{19}}$, holds:

$\mathbf{\mathfrak{I}_{20}}$: (a) $c > 0$, or (b) $\pi_1(0+) = \infty$, or (c) $\pi_1(0+) \in (0, \infty)$ and $\pi_1(u), u > 0$ is a continuous function.

If condition $\mathbf{\mathfrak{I}_{20}}$ does not hold, then the set V_0 is $(0, \infty)$ except for at most a countable set. Namely, $\overline{V}_0$ is the set of points of discontinuity for the distribution function of

$\kappa_0(1)$. It can be described in the following way: $\overline{V}_0 = \{v = v_1 l_1 + \dots v_n l_n : l_1, \dots l_n = 0, 1, \dots, n, l_1 + \dots + l_n \geq 1, n \geq 1\}$, where $v_1, v_2, \dots$ are points of discontinuity of the function $\pi_1(u), u > 0$.

Condition $\mathfrak{I}_{19}$ contains two alternatives. The first one corresponds to the case when the following condition holds:

$\mathfrak{I}_{21}$: (a) $c > 0$ or (b) $\pi_1(0+) = \infty$.

In this case, $\kappa_0(t), t > 0$ is an a.s. strictly increasing càdlàg process and, therefore, the corresponding renewal process $\nu_0(t) = \sup(s : \kappa_0(t) \leq t), t > 0$ is a.s. continuous.

The second one corresponds to the case when the following condition holds:

$\mathfrak{I}_{22}$: (a) $c = 0$ and (b) $\pi_1(0+) \in (0, \infty)$.

In this case, $\kappa_0(t), t > 0$ is a *compound Poisson process*. It is a step càdlàg process with positive jumps and, therefore, the corresponding renewal process $\nu_0(t) = \sup(s \geq 0 : \kappa_0(t) \leq t), t > 0$ is also a step càdlàg process with positive jumps.

Lemma 4.5.2. *Let conditions $\mathcal{T}_3$, $\mathcal{A}_{64}$ (or $\mathcal{S}_4$, $\mathcal{S}_5$) and $\mathfrak{I}_{21}$ hold. Then*

$$\nu_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{U}} \nu_0(t), t \geq 0 \text{ as } \varepsilon \to 0.$$

Proof of Lemma 4.5.2. In this case, $\nu_\varepsilon(0) = 0$ for $\varepsilon > 0$, and also for $\varepsilon = 0$. So, the point 0 can be included in the set V_0 that appears in the relation of weak convergence given in Lemma 4.5.1. Since the limiting process $\nu_0(t), t \geq 0$ is a.s. continuous, Lemma 3.2.1 implies the statement of Lemma 4.5.2. Note that condition $\mathfrak{I}_{21}$ implies $\mathfrak{I}_{20}$. Hence, the set of weak convergence V_0, described in Lemma 4.5.1, can be extended to $[0, \infty)$. □

Let us introduce the following condition:

$\mathcal{S}_6$: $n_\varepsilon \mathsf{P}\{\kappa_{\varepsilon,1} > 0\} \to \pi_1(0+) < \infty$ as $\varepsilon \to 0$.

Lemma 4.5.3. *Let conditions $\mathcal{T}_3$, $\mathcal{A}_{64}$ (or $\mathcal{S}_4$, $\mathcal{S}_5$), $\mathcal{S}_6$ and $\mathfrak{I}_{22}$ hold. Then*

$$\nu_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{J}} \nu_0(t), t \geq 0 \text{ as } \varepsilon \to 0.$$

Proof of Lemma 4.5.3. The process $\kappa_\varepsilon(t), t \geq 0$ is a step càdlàg process for $\varepsilon > 0$, as well as for $\varepsilon = 0$. Obviously, $\kappa_\varepsilon(0) = 0$ and this process has only positive jumps. Let $\alpha_{\varepsilon k}, k = 1, 2, \dots$ be the successive inter-jump times for this process and $\beta_{\varepsilon k}, k = 1, 2, \dots$, the corresponding successive jumps. Obviously, **(d)** the random variables $\alpha_{\varepsilon k}, k = 1, 2, \dots$ and $\beta_{\varepsilon k}, k = 1, 2, \dots$ are mutually independent, **(e)** the random variables $\alpha_{\varepsilon k}, k = 1, 2, \dots$ have the same distribution, **(f)** the random variables $\beta_{\varepsilon k}, k = 1, 2, \dots$ have the same distribution.

Let us describe the process $\kappa_\varepsilon(t), t \geq 0$, for $\varepsilon > 0$. Denote $\mathsf{P}_\varepsilon = \mathsf{P}\{\kappa_{\varepsilon,1} > 0\}$. In this case, **(g)** the random variable $\alpha_{\varepsilon 1}$ takes the value l/n_ε with probability $(1 - \mathsf{P}_\varepsilon)^{l-1}\mathsf{P}_\varepsilon$ for $l = 1, 2, \ldots$; **(h)** the random variable $\beta_{\varepsilon 1}$ has the distribution function $G_\varepsilon(u) = \mathsf{P}\{\kappa_{\varepsilon 1} \leq u/\kappa_{\varepsilon 1} > 0\}, u > 0$.

The limiting process $\kappa_0(t), t \geq 0$, is a compound Poisson process. In this case, **(i)** the random variable α_{01} has the exponential distribution with parameter $\pi_1(0+)$, **(j)** the random variable β_{01} has the distribution function $G_0(u) = 1 - \pi_1(u)/\pi_1(0+), u > 0$.

Obviously, conditions $\mathcal{S}_4 - \mathcal{S}_6$ and $\mathfrak{I}_{20}$ imply that **(k)** the random variables $\alpha_{\varepsilon 1} \Rightarrow \alpha_{01}$ as $\varepsilon \to 0$ and **(l)** the random variables $\beta_{\varepsilon 1} \Rightarrow \beta_{01}$ as $\varepsilon \to 0$.

The process $\nu_\varepsilon(t), t \geq 0$ is also a càdlàg process with step trajectories and positive jumps for $\varepsilon > 0$, as well as for $\varepsilon = 0$. The inter-jump times are, in this case, $\beta_{\varepsilon k}, k = 1, 2, \ldots$, and $\nu_\varepsilon(0) = \alpha_{\varepsilon 1}$. Values of the successive jumps are the random variables $\alpha_{\varepsilon k}, k = 2, 3, \ldots$.

Lemmas 4.4.1 and 4.4.2 can be applied to the processes $\nu_\varepsilon(t), t \geq 0$. Here, we should assume that the external processes $\xi_\varepsilon(t) = t, t \geq 0$ and, therefore, the process $\zeta_\varepsilon(t) = \nu_\varepsilon(t), t \geq 0$.

In this case, the random variables $\tau_{\varepsilon n} = \sum_{k=1}^{n} \alpha_{\varepsilon k}$ and $\kappa_\varepsilon(\tau_{\varepsilon n}) = \xi(\tau_{\varepsilon n}) = \sum_{k=1}^{n} \beta_{\varepsilon k}$ for $n = 0, 1, \ldots$. Relations **(d)** – **(l)** imply, in an obvious way, that conditions $\bar{\mathcal{K}}_7$, $\mathfrak{I}_{16}$, $\bar{\mathcal{K}}_8$, and $\mathcal{A}_{59}$ hold. Also relations **(d)** – **(l)** imply that conditions $\mathcal{N}_2$ and $\mathcal{R}_2$ hold.

So, Lemma 4.5.3 follows from Theorems 4.4.1 or 4.5.2. Alternatively, Theorems 4.4.4 and 4.5.3 can be employed. Condition $\mathcal{A}_{64}$ implies conditions $\mathcal{A}_{56}$ and $\mathcal{A}_{62}$. Also, relation (4.5.3) implies that conditions $\mathcal{J}_{24}$ and $\mathcal{O}_{13}$ hold. □

Note that condition $\mathcal{S}_6$ plays an essential role in Lemma 4.5.3. The following two examples show that without this condition **J**-convergence of the processes $\nu_\varepsilon(t), t \geq 0$ can not be guaranteed.

Let the random variables $\kappa_{\varepsilon,k}$ take values 0 and 1, respectively, with probabilities $1 - p_\varepsilon = 1 - 1/n_\varepsilon$ and $p_\varepsilon = 1/n_\varepsilon$. In this case, conditions $\mathcal{S}_4$ - $\mathcal{S}_5$ and $\mathfrak{I}_{20}$ hold. Simple calculations yield that the functions $\pi_1(u) = 1 - \chi_{[1,\infty)}(u)$ and $c(u) = \chi_{[1,\infty)}(u)$. Therefore, $c = 0$, $\pi_1(0+) = 1$ and $G_0(u) = \chi_{[1,\infty)}(u)$.

In this case, for every $\varepsilon > 0$, the random variable $\alpha_{\varepsilon 1}$ has the geometrical distribution with parameter p_ε, i.e., it takes the value l/n_ε with probability $(1 - p_\varepsilon)^{l-1}p_\varepsilon$ for $l = 1, 2, \ldots$, and the random variable $\beta_{\varepsilon 1} = 1$ with probability 1. The corresponding limiting random variable α_{01} has the exponential distribution with parameter 1, and $\beta_{01} = 1$ with probability 1.

In this case, $\kappa_0(t), t \geq 0$ is a standard Poisson process with parameter 1. The process $\nu_0(t), t \geq 0$ is also a càdlàg process with step trajectories. The random variable $\nu_0(0) = \alpha_{01}$ and the inter-jump times for this process all equal to 1. At the same time, the values of the successive jumps at moments $1, 2, \ldots$ are, respectively, $\alpha_{02}, \alpha_{03}, \ldots$.

In this case, the set $\overline{V}_0 = \{1, 2, \ldots\}$ and $V_0 = (0, \infty) \setminus \overline{V}_0$. Lemma 4.5.1 guarantees that $\nu_\varepsilon(t), t \in V_0 \Rightarrow \nu_0(t), t \in V_0$ as $\varepsilon \to 0$. In this case, the set of weak convergence V_0

can be extended to the interval $[0, \infty)$. Indeed, the pre-limiting processes $\nu_\varepsilon(t)$ have the same fixed moments of jumps, $1, 2, \ldots$, for every $\varepsilon > 0$ as well as for $\varepsilon = 0$. The value at 0, which is $\alpha_{\varepsilon 1}$, and the values of the successive jumps, which are $\alpha_{\varepsilon 2}, \alpha_{\varepsilon 3}, \ldots$, weakly converge to the corresponding limiting random variables.

Condition $\mathbf{S_6}$ also holds, since $n_\varepsilon \mathsf{P}\{\kappa_{\varepsilon,1} > 0\} = n_\varepsilon p_\varepsilon = 1$. Therefore, Lemma 4.5.3 implies that the processes $\nu_\varepsilon(t), t \geq 0 \stackrel{\mathbf{J}}{\longrightarrow} \nu_0(t), t \geq 0$ as $\varepsilon \to 0$.

Now, let us slightly modify the model. Let the random variables $\kappa_{\varepsilon,k}$ take values h_ε and 1, respectively, with probabilities $1 - p_\varepsilon = 1 - 1/n_\varepsilon$ and $p_\varepsilon = 1/n_\varepsilon$. Here, $h_\varepsilon > 0$ and $h_\varepsilon = o(p_\varepsilon)$ as $\varepsilon \to 0$. In this case again, conditions $\mathbf{S_4}$ - $\mathbf{S_5}$ and $\mathbf{J_{22}}$ hold. Simple calculations yield that the functions $\pi_1(u) = 1 - \chi_{[1,\infty)}(u)$ and $c(u) = \chi_{[1,\infty)}(u)$ are the same as above. Consequently, $c = 0$, $\pi_1(0+) = 1$ and $G_0(u) = \chi_{[1,\infty)}(u)$.

Therefore, the limiting processes $\kappa_0(t), t \geq 0$ and $\nu_0(t), t \geq 0$ are the same as in the first example. Again, due to Lemma 4.5.1, $\nu_\varepsilon(t), t \in V_0 \Rightarrow \nu_0(t), t \in V_0$ as $\varepsilon \to 0$, where the sets $V_0 = (0, \infty) \setminus \overline{V}_0$ and $\overline{V}_0 = \{1, 2, \ldots\}$.

However, in this case, condition $\mathbf{S_6}$ does not hold, since $\mathsf{P}\{\kappa_{\varepsilon,1} > 0\} = 1$ and, therefore, $n_\varepsilon \mathsf{P}\{\kappa_{\varepsilon,1} > 0\} = n_\varepsilon \to \infty$ as $\varepsilon \to 0$.

In this case, for every $\varepsilon > 0$, the pre-limiting random variables $\alpha_{\varepsilon 1} = 1/n_\varepsilon$ and the random variable $\beta_{\varepsilon 1}$ take the values h_ε and 1, respectively, with probabilities $1 - p_\varepsilon$ and p_ε. The corresponding limiting random variables are the same as in the first example, i.e., α_{01} has the exponential distribution with parameter 1 and $\beta_{01} = 1$ with probability 1. The random variables $\alpha_{\varepsilon 1}$ do not weakly converge to α_{01} as $\varepsilon \to 0$ and, hence, the random variables $\nu_\varepsilon(t)$ do not weakly converge to $\nu_0(t)$ as $\varepsilon \to 0$ for every point $t = 0, 1, \ldots$.

Moreover, let $\tilde{\alpha}_{\varepsilon k}$ be the successive moments of jumps, with value 1, of the process $\kappa_\varepsilon(t), t \geq 0$. These random variables are independent and have the same distribution. Obviously, $\tilde{\alpha}_{\varepsilon 1}$ takes the value l/n_ε with probability $(1 - p_\varepsilon)^{l-1} p_\varepsilon$ for $l = 1, 2, \ldots$. Let also $\tau_{\varepsilon n} = \sum_{1 \leq k \leq n} (n_\varepsilon \tilde{\alpha}_{\varepsilon k} - 1) h_\varepsilon, n = 0, 1, \ldots$. The process $\nu_\varepsilon(t)$ can be described in the following way. Its trajectories have the same step structure in each interval $[\tau_{\varepsilon n}, \tau_{\varepsilon n+1} + 1)$ for $n = 0, 1, \ldots$. In each such interval, a trajectory first has positive jumps of the value $1/n_\varepsilon$ at the moments $\tau_{\varepsilon n}, \tau_{\varepsilon n} + h_\varepsilon, \tau_{\varepsilon n} + 2h_\varepsilon, \ldots \tau_{\varepsilon n+1} - h_\varepsilon$ and then takes the value $\tilde{\alpha}_{\varepsilon n+1}$ in the sub-interval $[\tau_{\varepsilon n+1}, \tau_{\varepsilon n+1} + 1)$. The length of each sub-interval with small jumps of the value $1/n_\varepsilon$ is $(n_\varepsilon \tilde{\alpha}_{\varepsilon n} - 1) h_\varepsilon$. Obviously, $(n_\varepsilon \tilde{\alpha}_{\varepsilon n} - 1) h_\varepsilon \stackrel{\mathsf{P}}{\longrightarrow} 0$ as $\varepsilon \to 0$ since $h_\varepsilon = o(p_\varepsilon)$. However, the value of the increment of the process $\nu_\varepsilon(t)$ in this sub-interval is $(n_\varepsilon \tilde{\alpha}_{\varepsilon n} - 1)/n_\varepsilon$. Obviously, $(n_\varepsilon \tilde{\alpha}_{\varepsilon n} - 1)/n_\varepsilon \Rightarrow \tilde{\alpha}_{0n}$ as $\varepsilon \to 0$, where the limiting random variable has exponential distribution with parameter 1. It can be easily derived from this that the processes $\nu_\varepsilon(t)$ are not **J**-compact on any finite interval. So, these processes do not **J**-converge.

4.5.4. Two-dimensional sum-processes based on i.i.d. random variables. Let us consider the case when the following condition holds:

$\mathcal{T}_4$: $(\kappa_{\varepsilon,k}, \xi_{\varepsilon,k}), k = 1, 2, \ldots$ is (for every $\varepsilon > 0$) a sequence of i.i.d. random vectors that take values in $[0, \infty) \times \mathbb{R}_1$.

First consider an important particular case when the limiting process $\xi_0(t), t \geq 0$ is a Wiener process.

So, assume that conditions $\mathcal{S}_1$ - $\mathcal{S}_3$ hold in the following specific form:

$\mathcal{S}_7$: $n_\varepsilon \mathsf{P}\{|\xi_{\varepsilon,1}| > u\} \to 0$ as $\varepsilon \to 0$ for every $u > 0$.

$\mathcal{S}_8$: $n_\varepsilon \mathsf{E}\xi_{\varepsilon,1}\chi(|\xi_{\varepsilon,1}| \leq u) \to a$ as $\varepsilon \to 0$ for some $u > 0$.

$\mathcal{S}_9$: $n_\varepsilon \operatorname{Var} \xi_{\varepsilon,1}\chi(|\xi_{\varepsilon,1}| \leq u) \to b^2$ as $\varepsilon \to 0$ for some $u > 0$.

Note that, under condition $\mathcal{S}_7$, the asymptotic relation in conditions $\mathcal{S}_8$ and $\mathcal{S}_9$ hold simultaneously for all $u > 0$ and the constants a and b^2 do not depend on the choice of $u > 0$.

According to the central criterium of convergence (in the form extending the corresponding one-dimensional result), conditions $\mathcal{S}_7$ – $\mathcal{S}_9$ are necessary and sufficient for the following condition for weak convergence to hold:

$\mathcal{A}_{65}$: $\xi_\varepsilon(t), t \geq 0 \Rightarrow \xi_0(t), t \geq 0$ as $\varepsilon \to 0$, where $\xi_0(t) = at + bw(t), t > 0$ and $w(t), t \geq 0$ is a standard Wiener process.

We are interested to improve the classical bivariate criterion of convergence which gives necessary and sufficient conditions for the following condition to hold:

$\mathcal{A}_{66}$: $(\kappa_\varepsilon(t), \xi_\varepsilon(t)), t \geq 0 \Rightarrow (\kappa_0(t), \xi_0(t)), t \geq 0$ as $\varepsilon \to 0$, where $(\kappa_0(t), \xi_0(t))$, $t \geq 0$ is a càdlàg homogeneous process with independent increments, non-negative and non-decreasing first component, and real-valued second component.

Let us first formulate the following useful lemma.

Lemma 4.5.4. *Let conditions $\mathcal{T}_4$, $\mathcal{A}_{64}$ (or $\mathcal{S}_4$, $\mathcal{S}_5$) and $\mathcal{A}_{65}$ (or $\mathcal{S}_7$ – $\mathcal{S}_9$) hold. Then condition $\mathcal{A}_{66}$ holds, moreover,* **(α)** *the limiting processes $\kappa_0(t), t \geq 0$ and $\xi_0(t), t \geq 0$ are independent*; **(β)** *$\kappa_0(t), t \geq 0$ is a non-negative càdlàg homogeneous process with independent increments which has the same finite-dimensional distributions as the corresponding process in condition $\mathcal{A}_{64}$*; **(γ)** *$\xi_0(t), t \geq 0$ is a Wiener process which has the same finite-dimensional distributions as the corresponding process in condition $\mathcal{A}_{65}$.*

Proof of Lemma 4.5.4. Let us take some $t > 0$. Conditions $\mathcal{S}_4$ - $\mathcal{S}_5$ imply that **(a)** the random variables $\kappa_\varepsilon(t) \Rightarrow \kappa_0(t)$ as $\varepsilon \to 0$. Also conditions $\mathcal{S}_7$ - $\mathcal{S}_8$ imply that **(b)** the random variables $\xi_\varepsilon(t) \Rightarrow \xi_0(t)$ as $\varepsilon \to 0$. Relations **(a)** and **(b)** imply that the family of distributions of the random vectors $(\kappa_\varepsilon(t), \xi_\varepsilon(t))$ as $\varepsilon \to 0$ is tight. The corresponding compacts can be chosen as $K_n = [0, k'_n] \times [-k''_n, k''_n]$, where $0 < k'_n, k''_n \to \infty$ as $n \to$

∞. Due to Theorem 1.3.4, the tightness of this family implies its relative compactness. Hence, any sequence $0 < \varepsilon_n \to 0$ as $n \to \infty$ contains a subsequence $\varepsilon'_k = \varepsilon_{n_k} \to 0$ as $k \to \infty$ such that **(c)** the random vectors $(\kappa_{\varepsilon'_k}(t), \xi_{\varepsilon'_k}(t)) \Rightarrow (\tilde{\kappa}_0(t), \tilde{\xi}_0(t))$ as $k \to \infty$.

By the definition, $(\kappa_{\varepsilon'_k}(t), \xi_{\varepsilon'_k}(t))$ is a sum of i.i.d. random vectors. Due to the central criterium of convergence, **(c)** implies that **(d)** the distribution of the limiting random vector $(\tilde{\kappa}_0(t), \tilde{\xi}_0(t))$ is infinitely divisible. Recall that the random variables $\kappa_\varepsilon(t)$ are non-negative. Conditions $\mathcal{S}_4$ - $\mathcal{S}_5$ and $\mathcal{S}_7$ - $\mathcal{S}_9$ imply that **(e)** the components of this vector, $\tilde{\kappa}_0(t))$ and $\tilde{\xi}_0(t)$, have distributions of Poisson and Gaussian types, respectively. Therefore, **(f)** the random variables $\tilde{\kappa}_0(t)$ and $\tilde{\xi}_0(t)$ are independent. This is so, because Poisson type and Gaussian components of a vector with an infinitely divisible distribution should be independent. Conditions $\mathcal{S}_4$ - $\mathcal{S}_5$ **(g)** show that the random variables $\tilde{\kappa}_0(t)$ and $\kappa_0(t)$ have the same distribution. Also, using $\mathcal{S}_7$ - $\mathcal{S}_9$ **(h)** we see that the random variables $\tilde{\xi}_0(t)$ and $\xi_0(t)$ have the same distribution.

It follows from **(e)** - **(h)** that the distribution of the limiting random vector $(\tilde{\kappa}_0(t), \tilde{\xi}_0(t))$ does not depend on the choice of the sequence ε_n and the subsequence $\varepsilon'_k = \varepsilon_{n_k}$. As was shown above, the components of this vector are independent and have distributions of Poisson and Gaussian types. So, the random vectors $(\kappa_\varepsilon(t), \xi_\varepsilon(t)) \Rightarrow (\tilde{\kappa}_0(t), \tilde{\xi}_0(t))$ as $\varepsilon \to 0$. This completes the proof for one-dimensional distributions. The proof for multi-dimensional distributions is absolutely analogous. □

4.5.5. Two-dimensional sum-processes based on i.i.d. random variables. The general case. In the general case, the following condition for the off-boundary sets should be added to conditions $\mathcal{S}_1 - \mathcal{S}_3$ and $\mathcal{S}_4 - \mathcal{S}_5$:

$\mathcal{S}_{10}$: (a) $n_\varepsilon \mathsf{P}\{\kappa_{\varepsilon,1} > u, \xi_{\varepsilon,1} > v\} \to \pi_{1,2}(u, v)$ as $\varepsilon \to 0$ for all $u > 0, v > 0$ that are points of continuity of the limiting function $\pi_{1,2}(u, v)$.

(b) $n_\varepsilon \mathsf{P}\{\kappa_{\varepsilon,1} > u, \xi_{\varepsilon,1} \leq v\} \to \pi_{1,2}(u, v)$ as $\varepsilon \to 0$ for all $u > 0, v < 0$ that are points of continuity of the limiting function $\pi_{1,2}(u, v)$.

The limits above satisfy a number of conditions: **(a)** $\pi_{1,2}(u, v)$ is a function that is non-negative, non-increasing, right-continuous in every argument for $u > 0, v > 0$, and $\pi_{1,2}(\infty, v) = \pi_{1,2}(u, \infty) = 0$; **(b)** $\pi_{1,2}(u, v)$ is a function that is non-negative, non-decreasing in $v < 0$, and non-increasing in $u > 0$, right-continuous in every argument for $u > 0, v < 0$, and $\pi_{1,2}(\infty, v) = \pi_{1,2}(u, -\infty) = 0$; **(c)** $\Pi_{1,2}((u_1, u_2] \times (v_1, v_2]) = \pi_{1,2}(u_1, v_1) - \pi_{1,2}(u_2, v_1) - \pi_{1,2}(u_1, v_2) + \pi_{1,2}(u_2, v_2)$ is a non-negative function for $0 < u_1 \leq u_2 < \infty, 0 < v_1 \leq v_2 < \infty$, which defines the measure $\Pi_{1,2}(A)$ on the σ-algebra $\mathfrak{B}_1^+ \times \mathfrak{B}_1^+$ (the Borel σ-algebra of subsets of $(0, \infty) \times (0, \infty)$); **(d)** similarly, $\Pi_{1,2}((u_1, u_2] \times (v_1, v_2]) = \pi_{1,2}(u_2, w_2) - \pi_{1,2}(u_1, v_2) - \pi_{1,2}(u_2, v_1) + \pi_{1,2}(u_1, v_1)$ is a non-negative function for $0 < u_1 \leq u_2 < \infty, -\infty < v_1 \leq v_2 < 0$, which defines the measure $\Pi_{1,2}(A)$ on the σ-algebra $\mathfrak{B}_1^+ \times \mathfrak{B}_1^+$; **(e)** the measure $\Pi_{1,2}(A) \leq \Pi_1(B) \wedge \Pi_2(C)$ for any Borel set $A = B \times C \subseteq ((-\infty, 0) \cup (0, \infty)) \times (0, \infty)$; **(f)** the measure $\Pi_{1,2}(A)$ is extended to $\tilde{\mathfrak{B}}_2^+$, the σ-algebra of subsets of $\tilde{\mathbb{R}}_2^+ = ((-\infty, \infty) \times [0, \infty)) \setminus \{(0,0)\}$, first by additionally defining its values on boundary sets $\Pi_{1,2}(B \times [0, \infty)) = \Pi_1(B)$ for Borel subsets $B \subseteq (0, \infty)$, and $\Pi_{1,2}((-\infty, \infty) \times C) = \Pi_2(C)$ for Borel subsets $C \subseteq (-\infty, 0) \cup (0, \infty)$, and second by using the standard extension procedure of the measure theory; **(g)** the projection measure $\Pi_1(B)$ possesses properties **(a)** – **(d)** listed in connection with conditions $\mathcal{S}_4 - \mathcal{S}_5$ (in Subsection 4.5.2); **(h)** the projection measure $\Pi_2(C)$ possesses properties **(a)** – **(f)** listed in connection with conditions $\mathcal{S}_1 - \mathcal{S}_3$ (in Subsection 4.2.2).

According to the central criterium of convergence, conditions $\mathcal{S}_1 - \mathcal{S}_5$ and $\mathcal{S}_{10}$ are necessary and sufficient for condition $\mathcal{A}_{66}$ to hold.

If the random variables $\kappa_{\varepsilon,1}$ could take positive and negative values, then one should add to the above conditions also conditions on convergence of the truncated variances $n_\varepsilon \operatorname{Var} \kappa_{\varepsilon,1}\chi(|\kappa_{\varepsilon,1}| \leq u)$ and a similar condition on convergence of the truncated covariances $n_\varepsilon \mathsf{E}(\kappa_{\varepsilon,k}\chi(|\kappa_{\varepsilon,1}| \leq u) - \mathsf{E}\kappa_{\varepsilon,k}\chi(|\kappa_{\varepsilon,1}| \leq u))(\xi_{\varepsilon,1}\chi(|\xi_{\varepsilon,1}| \leq u) - \mathsf{E}\xi_{\varepsilon,1}\chi(|\xi_{\varepsilon,1}| \leq u))$. However, due to non-negativity of the random variables $\kappa_{\varepsilon,k}$, both repeated limits equal to 0. It was shown in Subsection 4.2.3 that **(g)** $n_\varepsilon \operatorname{Var} \kappa_{\varepsilon,1}\chi(\kappa_{\varepsilon,1} \leq u) \to 0$ as $\varepsilon \to 0$ and then $0 < u \to 0$. It follows from relation **(g)** and condition $\mathcal{S}_9$ that $(n_\varepsilon \mathsf{E}(\kappa_{\varepsilon,k}\chi(\kappa_{\varepsilon,1} \leq u) - \mathsf{E}\kappa_{\varepsilon,k}\chi(\kappa_{\varepsilon,1} \leq u))(\xi_{\varepsilon,1}\chi(|\xi_{\varepsilon,1}| \leq u) - \mathsf{E}\xi_{\varepsilon,1}\chi(|\xi_{\varepsilon,1}| \leq u)))^2 \leq n_\varepsilon \operatorname{Var} \kappa_{\varepsilon,1}\chi(\kappa_{\varepsilon,1} \leq u) \times n_\varepsilon \operatorname{Var} \xi_{\varepsilon,1}\chi(|\xi_{\varepsilon,1}| \leq u) \to 0$ as $\varepsilon \to 0$.

The limiting process $(\kappa_0(t), \xi_0(t)), t \geq 0$ has the characteristic function given, for every $t > 0$, by the following formula:

$$\begin{aligned} \mathsf{E}\exp\{i(y\kappa_0(t) + z\xi_0(t))\} &= \phi_{1,2}(t, y, z) \\ &= \exp\{t(idy + iaz - \frac{1}{2}b^2z^2 + \\ &+ \int_{\tilde{\mathbb{R}}_2^+} (e^{i(yu+zw)} - 1 - \frac{i(yu + zw)}{1 + u^2 + w^2})\Pi_{1,2}(du \times dw))\}. \end{aligned} \tag{4.5.4}$$

As was shown by Skorokhod (1957, 1964), conditions $\mathcal{S}_1$ - $\mathcal{S}_5$ and $\mathcal{S}_{10}$ imply, without any additional assumptions, that

$$(\kappa_\varepsilon(t), \xi_\varepsilon(t)), t \geq 0 \xrightarrow{\mathbf{J}} (\kappa_0(t), \xi_0(t)), t \geq 0 \text{ as } \varepsilon \to 0. \tag{4.5.5}$$

4.5.6. Sum-processes with renewal stopping based on i.i.d. random variables. In this subsection, we present the main applications to renewal models, which can be obtained as corollaries of the general limit theorems given in Sections 4.3 and 4.4 and Subsection 4.5.1.

Let us first consider the case when conditions $\mathcal{T}_4$, $\mathcal{A}_{64}$, $\mathcal{A}_{65}$, and $\mathcal{I}_{21}$ hold. In this case, **(a)** both the external limiting processes $\xi_0(t), t \geq 0$ and the limiting internal stopping process $\nu_0(t), t \geq 0$ are a.s. continuous, as well as their composition $\zeta_0(t) = \xi_0(\nu_0(t)), t \geq 0$. Also, Lemmas 4.5.4 and 4.3.1 (see formula (4.3.4)) imply that **(b)** the limiting external process $\xi_0(t), t \geq 0$ and the limiting internal stopping process $\nu_0(t), t \geq 0$ are independent.

Theorem 4.5.5. *Let conditions $\mathcal{T}_4$, $\mathcal{A}_{64}$ (or $\mathcal{S}_4 - \mathcal{S}_5$), $\mathcal{A}_{65}$ (or $\mathcal{S}_7 - \mathcal{S}_9$) and $\mathcal{I}_{21}$ hold. Then*

$$\zeta_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{U}} \zeta_0(t), t \geq 0 \text{ as } \varepsilon \to 0.$$

Proof of Theorem 4.5.5. Theorem follows from the relation of **J**-convergence (4.5.5), Lemma 4.5.4, Theorem 4.5.1, and Lemma 1.6.15. □

Note that it follows from Theorem 1.6.11 that the corresponding set of weak convergence in Theorem 4.5.5 is the interval $[0, \infty)$.

Let us now consider the general case, where condition $\mathcal{A}_{66}$ holds. We formulate below general conditions for weak and **J**-convergence of sum-processes with renewal stopping constructed from i.i.d. random variables. The following theorem covers the most essential part of applications and many preceding results in the area.

Theorem 4.5.6. *Let conditions $\mathcal{T}_4$, $\mathcal{A}_{66}$ (or $\mathcal{S}_1 - \mathcal{S}_5$ and $\mathcal{S}_{10}$) and $\mathcal{I}_{21}$ hold. Then*

$$\zeta_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{J}} \zeta_0(t), t \geq 0 \text{ as } \varepsilon \to 0.$$

Proof of Theorem 4.5.6. Let us apply Theorem 4.5.1. Condition $\mathcal{A}_{66}$ and the relation of **J**-convergence (4.5.5) imply that conditions $\mathcal{A}_{62}$, $\mathcal{J}_{24}$, and $\mathcal{O}_{13}$ hold. Conditions $\mathcal{S}_4$ - $\mathcal{S}_5$ and condition $\mathcal{I}_{21}$ imply that condition $\bar{\mathcal{K}}_5$ holds (at least $\kappa_\varepsilon(t) \xrightarrow{\mathsf{P}} \infty$ as $t \to \infty$ for all ε small enough). Also, condition $\mathcal{I}_{21}$ implies that condition $\mathcal{I}_9$ holds. Therefore, Theorem 4.5.1 can be applied to yield the statement of Theorem 4.5.6. □

Let Y_0 be the set of points of stochastic continuity of the limiting process $\zeta_0(t) = \xi_0(\nu_0(t)), t \geq 0$. This set is $[0, \infty)$ except for at most a countable set. Also $0 \in Y_0$, because $\nu_0(0) = 0$ with probability 1. It follows from Lemma 1.6.5 that the processes $\zeta_\varepsilon(t), t \geq 0$ weakly converge on the set Y_0.

The question about the structure of the set Y_0 does require a special consideration. In particular, it would be interesting to know whether condition $\mathcal{I}_{19}$ implies that $Y_0 = [0, \infty)$.

Let us also consider the case when condition $\mathcal{I}_{22}$ holds, i.e., the limiting renewal stopping process $\nu_0(t), t \geq 0$ is a step càdlàg process.

Theorem 4.5.7. *Let conditions $\mathcal{T}_4$, $\mathcal{A}_{66}$ (or $\mathcal{S}_1$ - $\mathcal{S}_6$ and $\mathcal{S}_{10}$) and $\mathcal{I}_{22}$ hold. Then*

$$\zeta_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{J}} \zeta_0(t), t \geq 0 \text{ as } \varepsilon \to 0.$$

Proof of Theorems 4.5.7. Let us apply Theorem 4.5.3. Condition $\mathcal{A}_{66}$ and the relation of **J**-convergence (4.5.5) imply that conditions $\mathcal{A}_{62}$, $\mathcal{J}_{24}$, and $\mathcal{O}_{13}$ hold. The structure of the step processes $\kappa_\varepsilon(t), t \geq 0$ is described in the proof of Lemma 4.5.3. It is readily seen that relations **(d)** - **(l)** given in this proof imply that conditions $\bar{\mathcal{K}}_7$, $\bar{\mathcal{K}}_8$ hold (at least for all ε small enough) and that conditions $\mathcal{N}_2$ and $\mathcal{R}_1$ hold. Therefore, Theorem 4.5.3 can be applied, and this yields the statement of the Theorem 4.5.7. □

Let Y_0 be the set of all points of stochastic continuity for the process $\zeta_0(t), t \geq 0$. Recall that $V_0 \cup \{0\}$, which is the set points of stochastic continuity for the process $\nu_0(t), t \geq 0$. This set is $[0, \infty)$ except for at most a countable set which was described in Subsection 4.5.3. Obviously, the process $\zeta_0(t) = \xi_0(\nu_0(t)), t \geq 0$ is stochastically continuous at points of the set $V_0 \cup \{0\}$, i.e., $V_0 \cup \{0\} \subseteq Y_0$. It follows from Lemma 1.6.5 that the processes $\zeta_\varepsilon(t), t \geq 0$ weakly converge on the set Y_0.

The question about the structure of the set Y_0 does also require a special consideration. In particular, it would be interesting to know whether, under condition $\mathfrak{I}_{16}$, $Y_0 = V_0 \cup \{0\}$ if the external limiting process $\xi_0(t), t \geq 0$ does not degenerate, i.e. $\mathsf{P}\{\xi_0(1) \neq 0\} > 0$.

4.5.7. Markov property for renewal stopping processes based on i.i.d. random variables. An alternative approach to limit theorems for this class of processes was introduced in Silvestrov (1974) and also used in Silvestrov and Teugels (2001). As a matter of fact, the stopping moment $\nu_\varepsilon(t)$ is a Markov moment for the two-dimensional process with independent increments $(\kappa_\varepsilon(t), \xi_\varepsilon(t)), t \geq 0$. This makes it possible to apply, to sum-processes with renewal stopping based on i.i.d. random variables, the general limit theorems for càdlàg processes with random Markov type stopping given in Silvestrov (1974). The results obtained using this method are similar to those given above in Theorems 4.5.5 – 4.5.7.

4.6 Accumulation processes

In this section, we give general limit theorems for the so-called accumulation processes. The results of this section are based on the results obtained in Silvestrov (1971c, 1972c, 1972d, 1972e).

4.6.1. General accumulation processes. Let, for every $\varepsilon > 0$, $\zeta_\varepsilon(t)$, $t \geq 0$, be a m-dimensional càdlàg process and let $\kappa_{\varepsilon,k}$, $k = 1, 2, \ldots$, be a sequence of non-negative random variables.

We will also consider the random variables

$$\tau_{\varepsilon,k} = \kappa_{\varepsilon,1} + \ldots + \kappa_{\varepsilon,k}, \ \xi_{\varepsilon,k} = \zeta_\varepsilon(\tau_{\varepsilon,k}) - \zeta_\varepsilon(\tau_{\varepsilon,k-1}), k = 1, 2, \ldots,$$

and

$$\varsigma_{\varepsilon,k} = \sup_{t \in [\tau_{\varepsilon,k-1}, \tau_{\varepsilon,k})} |\zeta_\varepsilon(t) - \zeta_\varepsilon(\tau_{\varepsilon,k-1})|, k = 1, 2, \ldots,$$

where $\tau_{\varepsilon,0} = \kappa_{\varepsilon,0} = 0$, $\xi_{\varepsilon,0} = \zeta_\varepsilon(0)$.

The random variables $\tau_{\varepsilon,k}$ can be interpreted as "*renewal moments*" for the process $\zeta_\varepsilon(t)$, $t \geq 0$. Then $\kappa_{\varepsilon,k}$ is the *inter-renewal time* between the renewal moments $\tau_{\varepsilon,k-1}$ and $\tau_{\varepsilon,k}$, and $\varsigma_{\varepsilon,k}$ is the maximal absolute value of the *oscillation* of the process $\zeta_\varepsilon(t)$ in the renewal interval $[\tau_{\varepsilon,k-1}, \tau_{\varepsilon,k})$ for $k = 1, 2, \ldots$.

It often occurs in applications that the times $\tau_{\varepsilon,k}$ are indeed renewal moments for the process $\zeta_\varepsilon(t), t \geq 0$, and the sequence $(\kappa_{\varepsilon,k}, \xi_{\varepsilon,k}, \varsigma_{\varepsilon,k})$, $k = 1, 2, \dots$ is a sequence of i.i.d. random vectors.

Let also t_ε, u_ε, n_ε be non-random positive functions. We assume that $n_\varepsilon \to \infty$ as $\varepsilon \to 0$ but do not require this to hold for the functions t_ε and u_ε.

We shall study the *accumulation processes* $\zeta_\varepsilon(tt_\varepsilon)/u_\varepsilon, t \geq 0$. The normalisation functions t_ε and u_ε are explicitly included in the model in order to simplify application of results to the scale-location model. In this model, the accumulation process $\zeta(t), t \geq 0$ and the random variables ξ_k, κ_k and ς_k do not depend on the series parameter ε. In principle, one can always reduce the consideration to the case where $t_\varepsilon, u_\varepsilon \equiv 1$ by considering the accumulation process $\zeta'_\varepsilon(t) = \zeta_\varepsilon(tt_\varepsilon)/u_\varepsilon, t \geq 0$ instead of the accumulation process $\zeta_\varepsilon(t), t \geq 0$. In the sequel, this would lead to the embedded random variables $\xi'_{\varepsilon,k} = \xi_{\varepsilon,k}/u_\varepsilon$ and $\kappa'_{\varepsilon,k} = \kappa_{\varepsilon,k}/t_\varepsilon$.

Let us define the "*embedded*" *sum-process*

$$(\kappa_\varepsilon(t), \xi_\varepsilon(t)) = (\sum_{k=0}^{[tn_\varepsilon]} \kappa_{\varepsilon,k}/t_\varepsilon, \sum_{k=0}^{[tn_\varepsilon]} \xi_{\varepsilon,k}/u_\varepsilon), \ t \geq 0.$$

Introduce the following weak convergence condition:

$\mathcal{A}_{67}$: $(\kappa_\varepsilon(t), \xi_\varepsilon(t)), t \geq 0 \Rightarrow (\kappa_0(t), \xi_0(t)), t \geq 0$ as $\varepsilon \to 0$, where $(\kappa_0(t), \xi_0(t))$, $t \geq 0$ is a càdlàg process such that: (a) $\kappa_0(t), t \geq 0$ is an a.s. strictly monotone process; (b) $\kappa_0(t) \stackrel{\mathsf{P}}{\longrightarrow} \infty$ as $t \to \infty$; (c) $\xi_0(t), t \geq 0$ is an a.s. continuous process.

We also assume the following condition of **U**-compactness:

$\mathcal{U}_6$: $\lim_{c\to 0} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\Delta_U(\xi_\varepsilon(\cdot), c, T) > \delta\} = 0, \ \delta, T > 0.$

Let us introduce a condition that makes the normalised outliers of accumulation processes stochastically negligible,

$\mathcal{K}_{11}$: $\sum_{k=1}^{[Tn_\varepsilon]} \mathsf{P}\{\varsigma_{\varepsilon,k} > \delta u_\varepsilon\} \to 0$ as $\varepsilon \to 0$, $\delta > 0$ for $T > 0$.

Define the renewal process $\nu_0(t) = \sup(s : \kappa_0(s) \leq t), t \geq 0$. Due to condition $\mathcal{A}_{67}$ (b), $\nu_0(t) < \infty$ with probability 1 for all $t \geq 0$. Due to condition $\mathcal{A}_{67}$ (a), the process $\nu_0(t), t \geq 0$ is an a.s. continuous process.

Let us also introduce the process $\zeta_0(t) = \xi_0(\nu_0(t)), t \geq 0$. This composition is a.s. continuous due to conditions $\mathcal{A}_{67}$ (a) and (c).

Theorem 4.6.1. *Let conditions $\mathcal{A}_{67}$, $\mathcal{U}_6$, and $\mathcal{K}_{11}$ hold. Then*

$$\zeta_\varepsilon(tt_\varepsilon)/u_\varepsilon, t \geq 0 \stackrel{\mathbf{U}}{\longrightarrow} \zeta_0(t), t \geq 0 \text{ as } \varepsilon \to 0.$$

Proof of Theorem 4.6.1. Introduce the stochastic process

$$\mu_\varepsilon(t) = \min(n : \sum_{k=0}^{n} \kappa_{\varepsilon,k} > t),\ t \geq 0.$$

In principle, this process can be improper, since the random variable $\mu_\varepsilon(t)$ can take the value $+\infty$ with a positive probability. This can happened if the random series $\kappa_\varepsilon = \sum_{k=1}^{\infty} \kappa_{\varepsilon,k}$ converges with a positive probability. To avoid dealing with improper random variables, we truncate the random variables $\mu_\varepsilon(t)$. So, let $0 < T_\varepsilon \to \infty$ as $\varepsilon \to 0$. Choose T_ε in such a way that $n_\varepsilon T_\varepsilon$ take positive integer values. We consider the process

$$\hat{\mu}_\varepsilon(t) = \mu_\varepsilon(t) \wedge n_\varepsilon T_\varepsilon,\ t \geq 0,$$

and also define the process

$$\nu_\varepsilon(t) = \frac{\hat{\mu}_\varepsilon(tt_\varepsilon)}{n_\varepsilon} = \frac{\mu_\varepsilon(tt_\varepsilon)}{n_\varepsilon} \wedge T_\varepsilon,\ t \geq 0.$$

.

By the definition of the process $\nu_\varepsilon(t), t \geq 0$,

$$\begin{aligned} &\mathsf{P}\{\nu_\varepsilon(s_k) > x_k, \xi_\varepsilon(y_k) \leq \mathbf{z}_k, k = 1, \dots, r\} \\ &= \prod_{k=1}^{r} \chi(T_\varepsilon > x_k) \cdot \mathsf{P}\{\kappa_\varepsilon(x_k) \leq s_k,\ \xi_\varepsilon(y_k) \leq \mathbf{z}_k, k = 1, \dots, r\}. \end{aligned} \tag{4.6.1}$$

Choose $X = \{x_k, k = 1, 2, \dots\}$ to be some countable set of positive numbers, dense in $(0, \infty)$. Since any distribution function has at most a countable set of discontinuity points, there exists a set $S = \{s_1, s_2, \dots\}$, dense in $[0, \infty)$, such that $\mathsf{P}\{\kappa_0(x_k) = s_r\} = 0$ for all $s_r \in S, r \geq 1$ and $x_k \in X,\ k \geq 1$. Here we can assume that $0 \in S$ because, by condition $\mathcal{A}_{67}$, the random variables $\kappa_0(x_k),\ k \geq 1$, are positive with probability 1.

Recall that l-dimensional distribution functions weakly converge if these functions converge on a countable set dense in $\mathbb{R}_l$. Thus, it follows from condition $\mathcal{A}_{67}$ and relation (4.6.1) that, for all $s_k \in S, y_k \geq 0, k = 1, \dots r, r \geq 1$,

$$(\nu_\varepsilon(s_k), \xi_\varepsilon(y_k), k = 1, \dots, r) \Rightarrow (\nu_0(s_k), \xi_0(y_k), k = 1, \dots, r) \text{ as } \varepsilon \to 0. \tag{4.6.2}$$

Since $s_k \in S, y_k \geq 0, k = 1, \dots r, r \geq 1$, are arbitrary, this relation means that

$$(\nu_\varepsilon(s), \xi_\varepsilon(t)), (s,t) \in S \times [0, \infty) \Rightarrow (\nu_0(s), \xi_0(t)), (s,t) \in S \times [0, \infty) \text{ as } \varepsilon \to 0. \tag{4.6.3}$$

The limiting process $\nu_0(t), t \geq 0$ is a.s. continuous. So, relation (4.6.3) and Lemma 3.2.2 imply that $\nu_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{U}} \nu_0(t), t \geq 0$ as $\varepsilon \to 0$. Hence, the set S can be replaced by the interval $[0, \infty)$ in relation (4.6.3). So, (4.6.3) can be rewritten in the following extended form:

$$(\nu_\varepsilon(t), \xi_\varepsilon(t)), t \geq 0 \Rightarrow (\nu_0(t), \xi_0(t)), t \geq 0 \text{ as } \varepsilon \to 0. \tag{4.6.4}$$

Obviously, the pre-limiting stochastic process $\nu_\varepsilon(t), t \geq 0$ can be replaced, in relations (4.6.4), by the process $\nu_\varepsilon(t) - 1/n_\varepsilon, t \geq 0$.

Relation (4.6.4) and condition $\mathfrak{U}_6$ permit to apply Theorem 3.2.1 to the composition of the processes $\xi_\varepsilon(t), t \geq 0$ and $\nu_\varepsilon(t) - 1/n_\varepsilon, t \geq 0$. This yields the following relation:

$$\xi_\varepsilon(\nu_\varepsilon(t) - 1/n_\varepsilon) = \sum_{k=0}^{\hat{\mu}_\varepsilon(tt_\varepsilon)-1} \xi_{\varepsilon,k}/u_\varepsilon, t \geq 0 \xrightarrow{\mathbf{U}} \xi_0(\nu_0(t)), t \geq 0 \text{ as } \varepsilon \to 0. \tag{4.6.5}$$

Consider the *residual accumulation process*

$$\varsigma_\varepsilon(t) = \frac{\zeta(tt_\varepsilon)}{u_\varepsilon} - \sum_{k=0}^{\hat{\mu}_\varepsilon(tt_\varepsilon)-1} \frac{\xi_{\varepsilon,k}}{u_\varepsilon}, \ t \geq 0.$$

By the definition of the processes $\mu_\varepsilon(t)$ and $\hat{\mu}_\varepsilon(t)$, if $\hat{\mu}_\varepsilon(t) = \mu_\varepsilon(t) = k$, then $t \in [\tau_{\varepsilon,k-1}, \tau_{\varepsilon,k})$ and, hence, $|\varsigma_\varepsilon(t)| \leq \varsigma_{\varepsilon,k}/u_\varepsilon$. Thus, for any $T > 0$, if $\mu_\varepsilon(Tt_\varepsilon) \leq n \wedge n_\varepsilon T_\varepsilon$, then $\sup_{0 \leq t \leq T} |\varsigma_\varepsilon(t)| \leq \max_{1 \leq k \leq n} \varsigma_{\varepsilon,k}/u_\varepsilon$.

Take an arbitrary $u > 0$. Obviously, $[un_\varepsilon] \leq n_\varepsilon T_\varepsilon$ for ε small enough. Taking into account the estimates given above we have for such ε the following estimate:

$$\begin{aligned} \mathsf{P}\{\sup_{0 \leq t \leq T} |\varsigma_\varepsilon(t)| > \delta\} \\ &\leq \mathsf{P}\{\mu_\varepsilon(TT_\varepsilon) > [un_\varepsilon] \wedge n_\varepsilon T_\varepsilon\} + \mathsf{P}\{\max_{1 \leq k \leq [un_\varepsilon]} \varsigma_{\varepsilon,k} > \delta u_\varepsilon\} \\ &\leq \mathsf{P}\{\nu_\varepsilon(T) > [un_\varepsilon]/n_\varepsilon\} + \sum_{k=1}^{[un_\varepsilon]} \mathsf{P}\{\varsigma_{\varepsilon,k} > \delta u_\varepsilon\}. \end{aligned} \tag{4.6.6}$$

By choosing, for any $\sigma > 0$, a sufficiently large u (we can always choose u to be a point of continuity of the random variable $\nu_0(T)$), we can make $\mathsf{P}\{\nu_0(T) > u\} \leq \sigma$. Then, by passing in (4.6.6) to limit as $\varepsilon \to 0$ and using condition $\mathcal{K}_{11}$ and relation (4.6.4), we obtain

$$\overline{\lim_{\varepsilon \to 0}} \mathsf{P}\{\sup_{0 \leq t \leq T} |\varsigma_\varepsilon(t)| > \delta\} \leq \mathsf{P}\{\nu_0(T) > u\} + \overline{\lim_{\varepsilon \to 0}} \sum_{k=1}^{[un_\varepsilon]} \mathsf{P}\{\varsigma_{\varepsilon,k} > \delta u_\varepsilon\} \leq \sigma. \tag{4.6.7}$$

Since $\delta, \sigma > 0$ are arbitrary, relation (4.6.7) means that

$$\sup_{0 \leq t \leq T} |\varsigma_\varepsilon(t)| \xrightarrow{\mathsf{P}} 0 \text{ as } \varepsilon \to 0, \ T > 0. \tag{4.6.8}$$

Relation (4.6.8) implies, in an obvious way, that

$$\varsigma_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{U}} \varsigma_0(t), t \geq 0 \text{ as } \varepsilon \to 0, \tag{4.6.9}$$

where $\varsigma_0(t) = (0, \dots, 0), t \geq 0$.

Relations (4.6.5) and (4.6.8) imply (see, for example, Lemma 1.6.16) that

$$\zeta_\varepsilon(tT_\varepsilon)/u_\varepsilon, t \geq 0 \xrightarrow{\mathbf{U}} \xi_0(\nu_0(t)), t \geq 0 \text{ as } \varepsilon \to 0. \tag{4.6.10}$$

The proof is completed. □

4.6.2. Centralised accumulation processes. Let us introduce the *centralised accumulation processes* $\zeta'_\varepsilon(t) = \zeta_\varepsilon(t) - \mathbf{c}_\varepsilon t, t \geq 0$, where $\mathbf{c}_\varepsilon = \text{const} \in \mathbb{R}_m$. In order to formulate conditions for **U**-convergence of these processes, let us consider the following embedded sum-processes:

$$(\kappa_\varepsilon(t), \xi'_\varepsilon(t)) = (\sum_{k=0}^{[tn_\varepsilon]} \kappa_{\varepsilon,k}/t_\varepsilon, \sum_{k=0}^{[tn_\varepsilon]} (\xi_{\varepsilon,k} - \mathbf{c}_\varepsilon \kappa_{\varepsilon,k})/u_\varepsilon), \ t \geq 0.$$

Let us introduce the following weak convergence condition:

$\mathcal{A}_{68}$: $(\kappa_\varepsilon(t), \xi'_\varepsilon(t)), t \geq 0 \Rightarrow (\kappa_0(t), \xi'_0(t)), t \geq 0$ as $\varepsilon \to 0$, where $(\kappa_0(t), \xi'_0(t)), t \geq 0$ is a càdlàg process such that: (a) $\kappa_0(t), t \geq 0$ is an a.s. strictly monotone process; (b) $\kappa_0(t) \xrightarrow{\mathsf{P}} \infty$ as $t \to \infty$; (c) $\xi'_0(t), t \geq 0$ is an a.s. continuous process.

We also assume the following condition of **U**-compactness:

$\mathcal{U}_7$: $\lim_{c\to 0} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\Delta_U(\xi'_\varepsilon(\cdot), c, T) > \delta\} = 0, \ \delta, T > 0.$

Now, introduce a condition that, together with $\mathcal{K}_{11}$, implies that the normalised outliers for accumulation processes are stochastically negligible,

$\mathcal{K}_{12}$: $\sum_{k=1}^{[Tn_\varepsilon]} \mathsf{P}\{|\mathbf{c}_\varepsilon|\kappa_{\varepsilon,k} > \delta u_\varepsilon\} \to 0$ as $\varepsilon \to 0$, $\delta > 0$ for $T > 0$.

Introduce the process $\zeta'_0(t) = \xi'_0(\nu_0(t)), t \geq 0$. This process is a.s. continuous, due to conditions $\mathcal{A}_{68}$ (a) and (c).

The following version of Theorem 4.6.1 is also useful in applications.

Theorem 4.6.2. *Let conditions $\mathcal{A}_{68}$, $\mathcal{U}_7$, $\mathcal{K}_{11}$, and $\mathcal{K}_{12}$ hold. Then*

$$\zeta'_\varepsilon(tt_\varepsilon)/u_\varepsilon, t \geq 0 \xrightarrow{\mathbf{U}} \zeta'_0(t), t \geq 0 \textit{ as } \varepsilon \to 0.$$

Proof of Theorem 4.6.2. The proof is based on the application of Theorem 4.6.1 to the processes $\zeta'_\varepsilon(t)$, $t \geq 0$, and the random variables $\kappa_{\varepsilon,k}$, $k = 0, 1, \ldots$.

In this case, the random variables $\xi_{\varepsilon,k}$ should be replaced by the random variables $\xi'_{\varepsilon,k} = \xi_{\varepsilon,k} - \mathbf{c}_\varepsilon \kappa_{\varepsilon,k}$ and the random variables $\varsigma_{\varepsilon,k}$ by the random variables

$$\begin{aligned}\varsigma'_{\varepsilon,k} &= \sup_{t \in [\tau_{\varepsilon,k-1}, \tau_{\varepsilon,k})} |\zeta'_\varepsilon(t) - \zeta'_\varepsilon(\tau_{\varepsilon,k-1})| \\ &\leq \sup_{t \in [\tau_{\varepsilon,k-1}, \tau_{\varepsilon,k})} |\xi_\varepsilon(t) - \xi_\varepsilon(\tau_{\varepsilon,k-1})| + |\mathbf{c}_\varepsilon| \kappa_{\varepsilon,k} = \varsigma_{\varepsilon,k} + |\mathbf{c}_\varepsilon| \kappa_{\varepsilon,k}.\end{aligned} \tag{4.6.11}$$

Respectively, the sum-process $(\kappa_\varepsilon(t), \xi_\varepsilon(t)), t \geq 0$ should be replaced by the sum-process $(\kappa_\varepsilon(t), \xi'_\varepsilon(t)), t \geq 0$.

Condition $\mathcal{A}_{68}$ implies condition $\mathcal{A}_{67}$, condition $\mathfrak{U}_7$ implies $\mathfrak{U}_6$. Also, conditions $\mathcal{K}_{11}$ and $\mathcal{K}_{12}$ imply that condition $\mathcal{K}_{11}$ holds for the random variables $\varsigma'_{\varepsilon,k}, k \geq 1$. □

4.6.3. Accumulation processes with embedded regeneration cycles. Let us consider the basic case where the following condition holds:

$\mathcal{T}_5$: $(\kappa_{\varepsilon,k}, \xi_{\varepsilon,k}, \varsigma_{\varepsilon,k}), k = 1, 2, \ldots$ is (for every $\varepsilon > 0$) a sequence of i.i.d. random vectors taking values in $[0, \infty) \times \mathbb{R}_m \times [0, \infty)$.

Typical examples are supplied by various models in which the process $\zeta_\varepsilon(t) = \varphi_t(\eta_\varepsilon(\cdot))$, $t \geq 0$, where $\eta_\varepsilon(t), t \geq 0$ is a regenerative process with regenerative moments $\tau_{\varepsilon,k}, k = 1, 2, \ldots$, and $\varphi_t(\cdot), t \geq 0$ is a family of additive type functionals defined on trajectories of this process.

For example, let $\eta_\varepsilon(t), t \geq 0$ be a càdlàg regenerative process with a Polish phase space X and regenerative moments $0 = \tau_{\varepsilon,0} \leq \tau_{\varepsilon,1} \leq \ldots$, and $\psi_\varepsilon(x)$ be a measurable function acting from X to $\mathbb{R}_m$. Let also

$$\zeta_\varepsilon(t) = \int_0^t \psi_\varepsilon(\eta_\varepsilon(s)) ds, t \geq 0,$$

where we use the Lebesgue integration for every component of vector process $\psi_\varepsilon(\eta_\varepsilon(s))$.

In this case, the random variables are

$$\kappa_{\varepsilon,k} = \tau_{\varepsilon,k} - \tau_{\varepsilon,k-1}, \ \xi_{\varepsilon,k} = \int_{\tau_{\varepsilon,k-1}}^{\tau_{\varepsilon,k}} \psi_\varepsilon(\eta_\varepsilon(s)) ds, k = 1, 2, \ldots$$

and

$$\varsigma_{\varepsilon,k} = \sup_{t \in [\tau_{\varepsilon,k-1}, \tau_{\varepsilon,k})} \left| \int_{\tau_{\varepsilon,k-1}}^t \psi_\varepsilon(\eta_\varepsilon(s)) ds \right|, k = 1, 2, \ldots.$$

If $\mathcal{T}_5$ holds, then the formulations of Theorems 4.6.1 and 4.6.2 take an especially simple form.

Recall that the embedded sum-process $\xi_\varepsilon(t) = \sum_{k=0}^{[tn_\varepsilon]} \xi_{\varepsilon,k}/u_\varepsilon, t \geq 0$ includes the term $\xi_{\varepsilon,0} = \zeta_\varepsilon(0)/u_\varepsilon$. Let us change the definition of these processes and define the embedded sum-process

$$\xi_\varepsilon(t) = \sum_{k=1}^{[tn_\varepsilon]} \xi_{\varepsilon,k}/u_\varepsilon, t \geq 0.$$

Usually, the random variables $\xi_{\varepsilon,0} = \zeta_\varepsilon(0)/u_\varepsilon$ are asymptotically negligible. So, it is natural to use the following condition:

$\mathcal{A}_{69}$: (a) $\zeta_\varepsilon(0)/u_\varepsilon \xrightarrow{\mathsf{P}} 0$ as $\varepsilon \to 0$;

(b) $\xi_\varepsilon(t), t \geq 0 \Rightarrow \xi_0(t), t \geq 0$ as $\varepsilon \to 0$, where $\xi_0(t) = \mathbf{a}t + \mathbf{w}(t), t \geq 0$ is a m-dimensional Wiener process with drift $\mathbf{a}$ and covariance matrix Σ.

Condition $\mathcal{A}_{69}$ implies (see, for example, Prokhorov (1956), Skorokhod (1957, 1964)), without any additional assumptions, that

$$\xi_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{U}} \xi_0(t), t \geq 0 \text{ as } \varepsilon \to 0. \tag{4.6.12}$$

This result is known as an *invariance principle* after the work of Donsker (1951), who obtained it for continuous piecewise linear sum-processes under conditions of the standard central limit theorem.

The embedded sum-process $\kappa_\varepsilon(t) = \sum_{k=0}^{[tn_\varepsilon]} \kappa_{\varepsilon,k}/u_\varepsilon, t \geq 0$ includes the summand $\kappa_{\varepsilon,0} = 0$. The corresponding weak convergence condition can be formulated in the following form:

$\mathcal{A}_{70}$: $\kappa_\varepsilon(t), t \geq 0 \Rightarrow \kappa_0(t), t \geq 0$ as $\varepsilon \to 0$, where $\kappa_0(t), t \geq 0$ is a non-negative and a.s. strictly monotone càdlàg homogeneous process with independent increments.

Finally, condition $\mathcal{K}_{11}$ takes in this case the following form:

$\mathcal{K}_{13}$: $n_\varepsilon \mathsf{P}\{\varsigma_{\varepsilon 1} > \delta u_\varepsilon\} \to 0$ as $\varepsilon \to 0, \delta > 0$ for $T > 0$.

The corresponding limiting process $\zeta_0(t) = \xi_0(\nu_0(t)), t \geq 0$, where $\nu_0(t) = \sup(s : \kappa_0(s) \leq t), t \geq 0$. In this case, **(a)** the processes $\xi_0(t), t \geq 0$ and $\nu_0(t), t \geq 0$ are independent. Obviously, **(b)** $\nu_0(t), t \geq 0$, as well as $\zeta_0(t) = \xi_0(\nu_0(t)), t \geq 0$, are a.s. continuous processes.

Theorem 4.6.3. *Let conditions $\mathcal{T}_5$, $\mathcal{A}_{69}$, $\mathcal{A}_{70}$, and $\mathcal{K}_{13}$ hold. Then*

$$\zeta_\varepsilon(tt_\varepsilon)/u_\varepsilon, t \geq 0 \xrightarrow{\mathbf{U}} \zeta_0(t), t \geq 0 \text{ as } \varepsilon \to 0.$$

Proof of Theorem 4.6.3. The proof is based on the application of Theorem 4.6.1. The first step in the proof is to show that conditions $\mathcal{A}_{69}$ and $\mathcal{A}_{70}$ imply condition $\mathcal{A}_{67}$ to hold with the independent limiting processes $\kappa_0(t), t \geq 0$ and $\xi_0(t), t \geq 0$. This can be done by reducing the proof to the case of scalar processes considered in Lemma 4.5.4.

Let us take an arbitrary $s_0 \in \mathbb{R}_1$ and a vector $\mathbf{s} = (s_1, \ldots s_m) \in \mathbb{R}_m$. Introduce, for every $\varepsilon > 0$, the scalar càdlàg processes $\xi_\varepsilon^{(\mathbf{s})}(t) = (\mathbf{s}, \xi_\varepsilon(t)) = \sum_{1 \leq k \leq tn_\varepsilon} (\mathbf{s}, \xi_{\varepsilon,k}), t \geq 0$ and $s_0\kappa_\varepsilon(t) = \sum_{1 \leq k \leq tn_\varepsilon} s_0\kappa_{\varepsilon,k}, t \geq 0$. Condition $\mathcal{A}_{69}$ implies that **(c)** the processes $\xi_\varepsilon^{(\mathbf{s})}(t), t \geq 0 \Rightarrow \xi_0^{(\mathbf{s})}(t), t \geq 0$ as $\varepsilon \to 0$. Obviously, $\xi_0^{(\mathbf{s})}(t), t \geq 0$ is a scalar a.s. continuous homogeneous process with independent increments. Also, condition $\mathcal{A}_{70}$ implies that **(d)** $s_0\kappa_\varepsilon(t), t \geq 0 \Rightarrow s_0\kappa_0(t), t \geq 0$ as $\varepsilon \to 0$. So, we can apply Lemma 4.5.4 to the processes $s_0\kappa_\varepsilon(t), t \geq 0$ and $\xi_\varepsilon^{(\mathbf{s})}(t), t \geq 0$. This yields the relation **(e)** $(s_0\kappa_\varepsilon(t), \xi_\varepsilon^{(\mathbf{s})}(t)), t \geq 0 \Rightarrow (s_0\kappa_0(t), \xi_0^{(\mathbf{s})}(t)), t \geq 0$ as $\varepsilon \to 0$, where the processes $s_0\kappa_\varepsilon(t), t \geq 0$ and $\xi_\varepsilon^{(\mathbf{s})}(t)), t \geq 0$ are independent. Obviously **(e)** implies that **(f)** $s_0\kappa_\varepsilon(t) + \xi_\varepsilon^{(\mathbf{s})}(t), t \geq 0 \Rightarrow s_0\kappa_0(t) + \xi_0^{(\mathbf{s})}(t), t \geq 0$ as $\varepsilon \to 0$. Using **(f)** and taking into account arbitrariness of the choice of $s_0 \in \mathbb{R}_1$ and $\mathbf{s} \in \mathbb{R}_m$, we get, by applying Lemma 1.2.1, that **(g)** for every $t \geq 0$, the random variables $(\kappa_\varepsilon(t), \xi_\varepsilon(t)) \Rightarrow (\kappa_0(t), \xi_0(t))$ as $\varepsilon \to 0$, where the random variables $\kappa_0(t)$ and $\xi_0(t)$ are independent. Since $(\kappa_\varepsilon(t), \xi_\varepsilon(t)), t \geq 0$ is a process with independent increments, **(g)** implies that **(h)** the processes $(\kappa_\varepsilon(t), \xi_\varepsilon(t)), t \geq 0 \Rightarrow (\kappa_0(t), \xi_0(t)), t \geq 0$ as $\varepsilon \to 0$, where the processes $\kappa_0(t), t \geq 0$ and $\xi_0(t), t \geq 0$ are independent.

So, condition $\mathcal{A}_{67}$ holds. Condition $\mathcal{A}_{69}$ implies condition $\mathcal{U}_6$, as it was pointed out in (4.6.12). Also, condition $\mathcal{K}_{13}$ coincides with condition $\mathcal{K}_{11}$. So, Theorem 4.6.1 can be applied to the accumulation processes $\zeta_\varepsilon(tt_\varepsilon)/u_\varepsilon, t \geq 0$, which completes the proof. □

4.6.4. Centralised accumulation processes with embedded regeneration cycles. Let us consider the centralised accumulation processes $\zeta'_\varepsilon(t) = \zeta_\varepsilon(t) - \mathbf{c}_\varepsilon t, t \geq 0$, where $\mathbf{c}_\varepsilon = \text{const} \in \mathbb{R}_m$. In order to formulate conditions for **U**-convergence of these processes, let us introduce the following step embedded processes:

$$(\kappa_\varepsilon(t), \xi'_\varepsilon(t)) = (\sum_{k=1}^{[tn_\varepsilon]} \kappa_{\varepsilon,k}/t_\varepsilon, \sum_{k=1}^{[tn_\varepsilon]} (\xi_{\varepsilon,k} - \mathbf{c}_\varepsilon \kappa_{\varepsilon,k})/u_\varepsilon), \ t \geq 0.$$

We introduce the following weak convergence condition:

$\mathcal{A}_{71}$: (a) $\zeta_\varepsilon(0)/u_\varepsilon \xrightarrow{\mathsf{P}} 0$ as $\varepsilon \to 0$;

(b) $\xi'_\varepsilon(t), t \geq 0 \Rightarrow \xi'_0(t), t \geq 0$ as $\varepsilon \to 0$, where $\xi'_0(t) = \mathbf{a}t + \mathbf{w}(t), t \geq 0$ is a m-dimensional Wiener process with drift $\mathbf{a}$ and covariance matrix Σ.

Let us also introduce the following analogue of condition $\mathcal{K}_{12}$:

$\mathcal{K}_{14}$: $n_\varepsilon \mathsf{P}\{|\mathbf{c}_\varepsilon|\kappa_{\varepsilon 1} > \delta u_\varepsilon\} \to 0$ as $\varepsilon \to 0$, $\delta > 0$ for $T > 0$.

The corresponding limiting process $\zeta'_0(t) = \xi'_0(\nu_0(t)), t \geq 0$, where $\nu_0(t) = \sup(s : \kappa_0(s) \leq t), t \geq 0$. In this case, **(a)** the processes $\xi'_0(t), t \geq 0$ and $\nu_0(t), t \geq 0$ are independent. Obviously, **(b)** $\nu_0(t), t \geq 0$, as well as $\zeta'_0(t), t \geq 0$, are a.s. continuous processes.

Theorem 4.6.4. *Let conditions $\mathcal{T}_5$, $\mathcal{A}_{70}$, $\mathcal{A}_{71}$, $\mathcal{K}_{13}$, and $\mathcal{K}_{14}$ hold. Then*

$$\zeta'_\varepsilon(tt_\varepsilon)/u_\varepsilon, t \geq 0 \xrightarrow{\mathbf{U}} \zeta'_0(t), t \geq 0 \text{ as } \varepsilon \to 0.$$

Proof of Theorem 4.6.4. It is enough to refer to Theorems 4.6.2 and 4.6.3. □

4.6.5. Renewal and risk type accumulation processes. Consider the case when the accumulation process $\zeta_\varepsilon(t)$, $t \geq 0$ is a step càdlàg process and $\kappa_{\varepsilon,k}$, $k = 1, 2, \ldots$ are successive inter-jump times for this process. For simplicity, we assume that $\zeta_\varepsilon(0) = 0$.

In this case, $\zeta_\varepsilon(t) = \zeta_\varepsilon(\tau_{\varepsilon,k-1})$ for $t \in [\tau_{\varepsilon,k-1}, \tau_{\varepsilon,k})$, $k = 0, 1, \ldots$, where $\tau_{\varepsilon,0} = 0$ and $\tau_{\varepsilon,k} = \kappa_{\varepsilon,1} + \ldots + \kappa_{\varepsilon,k}, k = 1, 2, \ldots$ are the successive moments of jumps of the process $\zeta_\varepsilon(t), t \geq 0$. Also, $\xi_{\varepsilon,k} = \zeta_\varepsilon(\tau_{\varepsilon,k}) - \zeta_\varepsilon(\tau_{\varepsilon,k-1})$ and $\varsigma_{\varepsilon,k} = \sup_{t \in [\tau_{\varepsilon,k-1}, \tau_{\varepsilon,k})} |\zeta_\varepsilon(t) - \zeta_\varepsilon(\tau_{\varepsilon,k-1})| = 0$ for $k = 1, 2, \ldots$.

Since the random variables $\varsigma_{\varepsilon,k} = 0$, the accumulation process can be represented in the form $\zeta_\varepsilon(t) = \sum_{1 \leq k \leq \mu_\varepsilon(t)-1} \xi_{\varepsilon,k}, t \geq 0$, where $\mu_\varepsilon(t) = \min(n : \tau_{\varepsilon,n} > t) = \max(n : \tau_{\varepsilon,n} \leq t) + 1, t \geq 0$. Actually, the process $\zeta_\varepsilon(t), t \geq 0$ is a sum-process with renewal type stopping. More precisely, it is a modification of the sum-process with renewal stopping described in Subsection 4.5.1.

Since the random variables $\varsigma_{\varepsilon,k} = 0$ for all $k = 1, 2, \ldots$, conditions $\mathcal{K}_{11}$ and $\mathcal{K}_{12}$ can be omitted, respectively, in the formulations of Theorems 4.6.1 and 4.6.2. Note also that, in this case, Theorem 4.6.3 (if $m = 1$) is a slight modification of Theorem 4.5.5.

Situation is different if the centralisation is involved. In this case, the processes $\zeta_\varepsilon(t) - \mathbf{c}_\varepsilon t, t \geq 0$ are not sum-process with renewal stopping of the type considered in Sections 4.3 and 4.5.

The process $\mathbf{c}_\varepsilon t - \zeta_\varepsilon(t), t \geq 0$ can be considered as a multidimensional *risk process*. Here $\mathbf{c}_\varepsilon$ should be interpreted as the premium rate. The process $\mu_\varepsilon(t) - 1, t \geq 0$ counts the number of claims in the interval $[0, t]$, whereas the random variables $\xi_{\varepsilon,k}, k = 1, 2, \ldots$ should be interpreted as values of the claims.

4.6.6. Accumulation processes with improper renewal cycles. In applications to non-recurrent Markov type processes, it can occur that the renewal moments $\tau_{\varepsilon,n}$ are improper random variables that take the value $+\infty$ with positive probabilities. A generalisation of the results presented above to this case can be found in Silvestrov (1972c, 1972d, 1972e, 1974).

4.6.7. Accumulation processes with embedded regeneration cycles in a scale-location mode. For simplicity, we assume that $m = 1$. Consider a scale-location model in which the corresponding accumulation process $\zeta(t), t \geq 0$ and the random variables $\kappa_k, k = 1, 2, \ldots$ do not depend on the series parameter $\varepsilon > 0$.

Obviously in this case, the random variables $\tau_k = \kappa_1 + \ldots + \kappa_k$, $\xi_k = \zeta(\tau_k) - \zeta(\tau_{k-1})$ and $\varsigma_k = \sup_{t\in[\tau_{k-1},\tau_k)} |\zeta(t) - \zeta(\tau_{k-1})|$ for $k \geq 1$ also do not depend on $\varepsilon > 0$, as well as the random variables $\tau_0 = \kappa_0 = 0$ and $\xi_0 = \zeta(0)$.

Condition $\mathcal{T}_5$ takes in this case the following form:

$\mathcal{T}_6$: $(\kappa_k, \xi_k, \varsigma_k), k = 1, 2, \ldots$ is a sequence of i.i.d. random variables taking values in $[0, \infty) \times \mathbb{R}_1 \times [0, \infty)$.

Let us first consider the case when the limiting process in condition $\mathcal{A}_{70}$ degenerates to a non-random linear function. To simplify formulations, we restrict consideration to the case where $\mathsf{E}\kappa_1 < \infty$.

Bellow, $0 < t_\varepsilon \to \infty$ as $\varepsilon \to 0$. Denote $\mathsf{E}\xi_1 = a$, $\mathsf{E}\kappa_1 = d$ and $c = a/d$.

Theorem 4.6.5. *Let* **(α)** $\mathsf{E}|\xi_1| < \infty$*,* **(β)** $\mathsf{E}\kappa_1 < \infty$*,* **(γ)** $\mathsf{E}\varsigma_1 < \infty$*. Then*

$$\frac{\zeta(tt_\varepsilon)}{t_\varepsilon}, t \geq 0 \overset{\mathbf{U}}{\longrightarrow} ct, t \geq 0 \text{ as } \varepsilon \to 0.$$

Let us denote $\mathsf{Var}\,\xi_1 = b^2$ and $f^2 = b^2/d$. Let also $w(t), t \geq 0$ be a standard Wiener process.

Theorem 4.6.6. *Let* **(α)** $\mathsf{E}\xi_1^2 < \infty, \mathsf{E}\xi_1 = 0$*,* **(β)** $\mathsf{E}\kappa_1 < \infty$*,* **(γ)** $\mathsf{E}\varsigma_1^2 < \infty$*. Then*

$$\frac{\zeta(tt_\varepsilon)}{\sqrt{t_\varepsilon}}, t \geq 0 \overset{\mathbf{U}}{\longrightarrow} fw(t), t \geq 0 \text{ as } \varepsilon \to 0.$$

Denote $\mathsf{Var}(\xi_1 - c\kappa_1) = g^2$ and $h^2 = g^2/d$.

Theorem 4.6.7. *Let* **(α)** $\mathsf{E}\xi_1^2 < \infty$*,* **(β)** $\mathsf{E}\kappa_1^2 < \infty$*,* **(γ)** $\mathsf{E}\varsigma_1^2 < \infty$*. Then*

$$\frac{\zeta(tt_\varepsilon) - ctt_\varepsilon}{\sqrt{t_\varepsilon}}, t \geq 0 \overset{\mathbf{U}}{\longrightarrow} hw(t), t \geq 0 \text{ as } \varepsilon \to 0.$$

Proof of Theorems 4.6.5, 4.6.6, and 4.6.7. Due to the weak law of large numbers, if $\mathsf{E}\kappa_1 < \infty$, then

$$\kappa_\varepsilon(t) = \sum_{k=1}^{[tt_\varepsilon]} \frac{\kappa_k}{t_\varepsilon}, t \geq 0 \Rightarrow dt, t \geq 0. \tag{4.6.13}$$

So, condition $\mathcal{A}_{70}$ holds with the limiting process $\kappa_0(t) = dt, t \geq 0$, and the functions $n_\varepsilon = t_\varepsilon$. In this case, the process $\nu_0(t) = d^{-1}t, t \geq 0$.

Also, by the same weak law of large number, if $\mathsf{E}|\xi_1| < \infty$, then

$$\xi_\varepsilon(t) = \sum_{k=1}^{[tt_\varepsilon]} \frac{\xi_k}{t_\varepsilon}, t \geq 0 \Rightarrow at, t \geq 0. \tag{4.6.14}$$

Hence, condition $\mathcal{A}_{69}$ holds with the limiting process $\xi_0(t) = at, t \geq 0$ and the functions $n_\varepsilon = u_\varepsilon = t_\varepsilon$.

Condition $\mathcal{K}_{11}$ also holds, since the condition $\mathsf{E}\varsigma_1 < \infty$ implies that

$$t_\varepsilon \mathsf{P}\{\varsigma_1 > \delta t_\varepsilon\} \to 0 \text{ as } \varepsilon \to 0, \ \delta > 0. \tag{4.6.15}$$

To complete the proof, we apply Theorem 4.6.3. The corresponding limiting process $\xi_0(\nu_0(t)) = ad^{-1}t = ct, t \geq 0$.

To prove Theorem 4.6.6, we need to replace (4.6.14) by a relation which is a variant of the standard central limit theorem. Thus, if $\mathsf{E}\xi_1^2 < \infty, \mathsf{E}\xi_1 = 0$, then

$$\xi_\varepsilon(t) = \sum_{k=1}^{[tt_\varepsilon]} \frac{\xi_k}{\sqrt{t_\varepsilon}}, t \geq 0 \Rightarrow bw(t), t \geq 0. \tag{4.6.16}$$

Therefore, condition $\mathcal{A}_{69}$ holds with the limiting process $\xi_0(t) = bw(t), t \geq 0$ and the functions $n_\varepsilon = t_\varepsilon, u_\varepsilon = \sqrt{t_\varepsilon}$.

Condition $\mathcal{K}_{11}$ also holds, since the condition $\mathsf{E}\varsigma_1^2 < \infty$ implies that

$$t_\varepsilon \mathsf{P}\{\varsigma_1 > \delta \sqrt{t_\varepsilon}\} \to 0 \text{ as } \varepsilon \to 0, \ \delta > 0. \tag{4.6.17}$$

To complete the proof, it remains to apply Theorem 4.6.3. The corresponding limiting process $\xi_0(\nu_0(t)) = bw(d^{-1}t) = fw(t), t \geq 0$.

To prove Theorem 4.6.7, we can apply Theorem 4.6.4.

Due to the same central limit theorem, if $\mathsf{E}\xi_1^2 < \infty$ and $\mathsf{E}\kappa_1^2 < \infty$, then $\mathsf{E}(\xi_1 - c\kappa_1) = 0$ and

$$\xi_\varepsilon(t) = \sum_{k=1}^{[tt_\varepsilon]} \frac{\xi_k - c\kappa_k}{\sqrt{t_\varepsilon}}, t \geq 0 \Rightarrow gw(t), t \geq 0. \tag{4.6.18}$$

Therefore, condition $\mathcal{A}_{71}$ holds with the limiting process $\xi_0(t) = gw(t), t \geq 0$ and the functions $n_\varepsilon = t_\varepsilon, u_\varepsilon = \sqrt{t_\varepsilon}$.

Condition $\mathcal{K}_{11}$ holds, as well as condition $\mathcal{K}_{12}$, since the condition $\mathsf{E}\kappa_1^2 < \infty$ implies that

$$t_\varepsilon \mathsf{P}\{|c|\kappa_1 > \delta \sqrt{t_\varepsilon}\} \to 0 \text{ as } \varepsilon \to 0, \delta > 0. \tag{4.6.19}$$

To complete the proof, we apply Theorem 4.6.4. The corresponding limiting process $\xi_0(\nu_0(t)) = gw(d^{-1}t) = hw(t), t \geq 0$. □

Remark 4.6.1. It should be noted that the results of Theorems 4.6.5, 4.6.6, and 4.6.7 are valid in the case when **(a)** the limiting process in condition $\mathcal{A}_{70}$ degenerates to a non-random linear function and also **(b)** the limiting process in condition $\mathcal{A}_{69}$ or $\mathcal{A}_{71}$ is either a non-random linear function or a Wiener process. These conditions actually can be provided by assumptions weaker than the corresponding moment conditions used in the theorems mentioned above. Also note that the moment conditions imposed on the random variable ς_1 can be weakened. These conditions can be replaced by relations (4.6.15), (4.6.17), or (4.6.19).

Let us now consider the case when the limiting process in condition $\mathcal{A}_{70}$ is a stable process.

Let $\alpha \in (0,1)$ and denote by $\kappa^{(\alpha)}(t), t \geq 0$ a non-negative càdlàg homogeneous *stable process with independent increments* whose Laplace transform is $\mathsf{E}\exp\{-s\kappa^{(\alpha)}(t)\} = e^{-s^{\alpha}t}, s,t \geq 0$. Let us also define the process $\nu^{(\alpha)}(t) = \sup(s : \kappa^{(\alpha)}(s) \leq t), t \geq 0$. As is known, the stable process $\kappa^{(\alpha)}(t), t \geq 0$ is a.s. strictly increasing and, therefore, the process $\nu^{(\alpha)}(t), t \geq 0$ is a.s. continuous.

Denote $n_\varepsilon = t_\varepsilon^{\alpha}/\Gamma(1-\alpha)h(t_\varepsilon)$, where $\Gamma(\lambda) = \int_0^\infty x^{\lambda-1}e^{-x}dx$ and $h(x)$ is a slowly varying function.

Theorem 4.6.8. *Let* **(α)** $\mathsf{E}|\xi_1| < \infty$, **(β)** $\mathsf{P}\{\kappa_1 > x\} \sim x^{-\alpha}h(x)$ *as* $x \to \infty$, **(γ)** $\mathsf{E}\varsigma_1 < \infty$. *Then*

$$\frac{\zeta(tt_\varepsilon)}{n_\varepsilon}, t \geq 0 \overset{\mathbf{U}}{\longrightarrow} a\nu^{(\alpha)}(t), t \geq 0 \text{ as } \varepsilon \to 0.$$

Denote $\mathsf{Var}\,\xi_1 = b^2$ and introduce the process $\zeta_0(t) = bw(\nu^{(\alpha)}(t)), t \geq 0$, where the processes $w(t), t \geq 0$ and $\nu^{(\alpha)}(t), t \geq 0$ are independent.

Theorem 4.6.9. *Let* **(α)** $\mathsf{E}\xi_1^2 < \infty, \mathsf{E}\xi_1 = 0$, **(β)** $\mathsf{P}\{\kappa_1 > x\} \sim x^{-\alpha}h(x)$ *as* $x \to \infty$, **(γ)** $\mathsf{E}\varsigma_1^2 < \infty$. *Then*

$$\frac{\zeta(tt_\varepsilon)}{\sqrt{n_\varepsilon}}, t \geq 0 \overset{\mathbf{U}}{\longrightarrow} \zeta_0(t) = bw(\nu^{(\alpha)}(t)), t \geq 0 \text{ as } \varepsilon \to 0.$$

Proof of Theorems 4.6.8 and 4.6.9. As is known (see, for example, Feller (1971)) condition **(β)** implies that

$$\kappa_\varepsilon(t) = \sum_{k=1}^{[tn_\varepsilon]} \frac{\kappa_k}{t_\varepsilon}, t \geq 0 \Rightarrow \kappa^{(\alpha)}(t), t \geq 0. \tag{4.6.20}$$

So, condition $\mathcal{A}_{70}$ holds with the limiting process $\kappa^{(\alpha)}(t), t \geq 0$.

Condition $\mathcal{A}_{69}$ holds, as was pointed out in the proofs of Theorems 4.6.5, 4.6.6, and 4.6.7. If $\mathsf{E}|\xi_1| < \infty$, then the corresponding limiting process $\xi_0(t) = at, t \geq 0$, and the function $u_\varepsilon = n_\varepsilon$. If $\mathsf{E}\xi_1^2 < \infty, \mathsf{E}\xi_1 = 0$, then the corresponding limiting process $\xi_0(t) = bw(t), t \geq 0$, and the function $u_\varepsilon = \sqrt{n_\varepsilon}$.

Condition $\mathcal{K}_{11}$ also holds either with the function $u_\varepsilon = n_\varepsilon$, if $\mathsf{E}\varsigma_1 < \infty$, or with the function $u_\varepsilon = \sqrt{n_\varepsilon}$, if $\mathsf{E}\varsigma_1^2 < \infty$.

To complete the proof, we apply Theorem 4.6.3. The corresponding limiting process is either $a\nu^{(\alpha)}(t), t \geq 0$ or $bw(\nu^{(\alpha)}(t)), t \geq 0$, where the processes $w(t), t \geq 0$ and $\nu^{(\alpha)}(t), t \geq 0$ are independent. □

4.7 Extremes with random sample size

In this section we derive a number of limit theorems for extremal processes constructed from samples with a random sample size. Extremal processes of such type naturally appear in various applications related to models with sample variables associated to stochastic flows.

4.7.1. Extremal processes with random sample size indices. Let, for every $\varepsilon > 0$, $\rho_{\varepsilon,n}, n = 1, 2, \ldots$ be a sequence of real-valued random variables and μ_ε a positive random variable. Further, we need a non-random function $n_\varepsilon > 0$ of parameter ε such that $n_\varepsilon \to \infty$ as $\varepsilon \to 0$.

If we are interested in extremal processes with non-random sample size indices, then we will deal with

$$\rho_\varepsilon(t) = \max_{k \leq 1 \vee t n_\varepsilon} \rho_{\varepsilon,k}, \ t \geq 0.$$

Our interest lies in the relevant analogues of these processes when the sample size indices are random as well. So, define

$$\zeta_\varepsilon(t) = \max_{k \leq 1 \vee t \mu_\varepsilon} \rho_{\varepsilon,k}, \ t \geq 0.$$

Let us denote by $\nu_\varepsilon = \mu_\varepsilon / n_\varepsilon$ the normalised random sample size index. Then the process $\zeta_\varepsilon(t) = \rho_\varepsilon(\nu_\varepsilon(t)), t \geq 0$ can be represented in the form of the composition of the two processes $\rho_\varepsilon(t), t \geq 0$ and $\nu_\varepsilon(t) = t\nu_\varepsilon, t \geq 0$.

The pre-limiting extremal processes $\rho_\varepsilon(t)$ and $\zeta_\varepsilon(t)$ are defined on the interval $[0, \infty)$. However, it is natural to study weak and **J**-convergence of these extremal processes on the open interval $(0, \infty)$. The reason for this is the fact that, in some cases, extremal processes $\rho_\varepsilon(t)$ may weakly converge on the interval $(0, \infty)$ to a monotone process $\rho_0(t)$ but they do not weakly converge at the point 0. Moreover, we also admit the case when the random variable $\rho_0(0) = \lim_{0<t\to\infty} \rho_0(t)$ can be improper (this limit exists with probability 1), that is, it takes the value $-\infty$ with a positive probability.

Let us introduce the following weak convergence condition:

$\mathcal{A}_{72}$: $(\nu_\varepsilon, \rho_\varepsilon(t)), t \in U \Rightarrow (\nu_0, \rho_0(t)), t \in U$ as $\varepsilon \to 0$, where (a) $\rho_0(t), t > 0$ is a nondecreasing càdlàg process, (b) ν_0 is a non-negative random variable, (c) U is a set of points everywhere dense in $(0, \infty)$ and containing 0.

Let us also assume that the following condition of **J**-compactness holds:

$\mathcal{J}_{26}$: $\lim_{c\to 0} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\Delta_J(\rho_\varepsilon(\cdot), c, T', T'') > \delta\} = 0, \ \delta > 0, 0 < T' < T'' < \infty$.

Note that it is assumed that $0 < \varepsilon \to 0$. However, under $\mathcal{A}_{72}$, condition $\mathcal{J}_{26}$ is equivalent to $\mathcal{J}_{24}$, since the limiting process $\rho_0(t), t > 0$ is a càdlàg process and, therefore, the asymptotic relation in $\mathcal{J}_{26}$ automatically holds for $\varepsilon = 0$.

Also, recall the condition

$\mathfrak{I}_4$: $\nu_0 > 0$ with probability 1.

There is a question of why the max-component is not defined in a simpler way by $\rho'_\varepsilon(t) = \max_{k \le tn_\varepsilon} \rho_{\varepsilon,k}, t \ge 0$, where the maximum over the empty set should be understood as zero. As a matter of fact, the max-process $\rho_\varepsilon(t), t \ge 0$, introduced according to the initial definition, is a monotone process. But the max-process $\rho'_\varepsilon(t), t \ge 0$ has some side effect at zero. The process $\rho'_\varepsilon(t), t > 0$ has step trajectories, is continuous from the right, and can have jumps only at the points $k/n_\varepsilon, k \ge 1$. All jump with $k \ge 2$ are positive and so the resulting process is a.s. non-decreasing on the interval $[1/n_\varepsilon, \infty)$. However, on the interval $[0, 1/n_\varepsilon)$, the process takes the value zero and the first jump can be negative if the random variable $\rho_{\varepsilon,1}$ takes a negative value. The obvious inequality **(a)** $|\rho'_\varepsilon(t) - \rho_\varepsilon(t)| \le |\rho_{\varepsilon,1}|\chi(t \le 1/n_\varepsilon)$ implies that **(b)** $\sup_{0 \le t \le T} |\rho'_\varepsilon(\nu_\varepsilon t) - \rho_\varepsilon(\nu_\varepsilon t)| \le |\rho_{\varepsilon,1}|\chi(T\nu_\varepsilon \le 1/n_\varepsilon)$. Since $n_\varepsilon \to \infty$ as $\varepsilon \to 0$ and $\nu_0 > 0$ with probability 1 by condition $\mathfrak{I}_4$, we have **(c)** $\sup_{0 \le t \le T} |\rho'_\varepsilon(\nu_\varepsilon t) - \rho_\varepsilon(\nu_\varepsilon t)| \xrightarrow{\mathsf{P}} 0$ as $\varepsilon \to 0$ for $T > 0$. So, the extremal processes $\zeta_\varepsilon(t) = \rho_\varepsilon(\nu_\varepsilon t), t > 0$ and $\zeta'_\varepsilon(t) = \rho'_\varepsilon(\nu_\varepsilon t), t > 0$ converge weakly or **J**-converge simultaneously and have the limiting process.

Let us denote by V_0 the set of $t > 0$ for which $\mathsf{P}\{\tau_{knr}/\nu_0 = t\} = 0$ for all $k, n, r = 1, 2, \ldots$, where τ_{knr}, $k = 1, 2, \ldots$ are the successive moments of jumps of the process $\rho_0(t), t \ge r^{-1}$, with absolute values of the jumps lying in the interval $[\frac{1}{n}, \frac{1}{n-1})$. Obviously, the set V_0 is $(0, \infty)$ except for at most a countable set. Note that V_0 is the set of point of stochastic continuity of the process $\rho_0(t\nu_0), t > 0$.

The following theorems are direct corollaries of the translation Theorems 2.8.2 and 3.4.4. These theorems must be applied to the compositions $\zeta_\varepsilon(t) = \rho_\varepsilon(t\nu_\varepsilon), t \ge 0$, in the case where the constant $\alpha = 0$ and the slowly varying functions $h(x) \equiv 1$. Remark 2.8.3, which describes a modification of conditions of these theorems in the case of the interval $(0, \infty)$, must also be taken in account.

Theorem 4.7.1. *Let conditions $\mathcal{A}_{72}$, $\mathfrak{J}_{26}$, and $\mathfrak{I}_4$ hold. Then*

$$\zeta_\varepsilon(t) = \rho_\varepsilon(t\nu_\varepsilon), t \in V_0 \Rightarrow \zeta_0(t) = \rho_0(t\nu_0), t \in V_0 \text{ as } \varepsilon \to 0.$$

Theorem 4.7.2. *Let conditions $\mathcal{A}_{72}$, $\mathfrak{J}_{26}$, and $\mathfrak{I}_4$ hold. Then*

$$\zeta_\varepsilon(t), t > 0 \xrightarrow{\mathbf{J}} \zeta_0(t), t > 0 \text{ as } \varepsilon \to 0.$$

Note that, in Theorems 4.7.1 and 4.7.2, we do not impose any independence conditions on the random variables $\rho_{\varepsilon,k}, k = 1, 2, \ldots$.

Note also that it makes sense to formulate Theorems 4.7.1 and 4.7.2 separately since Theorem 4.7.1 gives additional information about the set of weak convergence of the corresponding extremal processes.

4.7.2. Extremal processes based on i.i.d. random variables. Let us now consider the case when the following condition holds:

$\mathcal{T}_7$: $\rho_{\varepsilon,k}, k = 1, 2, \ldots$ is (for every $\varepsilon > 0$) a sequence of real-valued i.i.d. random variables.

The following condition is standard in limit theorems for extremes:

$\mathcal{S}_{11}$: $n_\varepsilon \mathsf{P}\{\rho_{\varepsilon,1} > w\} \to \pi_3(w)$ as $\varepsilon \to 0$ for all $w \in R_1$ which are points of continuity of the limiting function $\pi_3(w)$.

The function $\pi_3(w)$ satisfies a number of conditions: **(a)** $\pi_3(w)$ is a non-increasing function acting from $(-\infty, \infty)$ into $[0, \infty]$ and is continuous from the right (if $\pi_3(w) = \infty$, continuity from the right is interpreted as $\pi_3(x) \uparrow \infty$ as $x \downarrow w$); **(b)** $\pi_3(-\infty) = \infty$ and $\pi_3(\infty) = 0$.

As such, these conditions imply that the function $e^{-\pi_3(w)}$ is a distribution function. If we define $\upsilon = \sup(u : \pi_3(w) = \infty) \geq -\infty$, then $e^{-\pi_3(w)}$ takes positive values for $w > \upsilon$ and $e^{-\pi_3(w)} = 0$ for $w < \upsilon$.

As is known (see, for example, Loève (1955)), condition $\mathcal{S}_{10}$ holds if and only if the random variables $\rho_\varepsilon(1) \Rightarrow \rho_0(1)$ as $\varepsilon \to 0$, where $\rho_0(1)$ is a random variable with the distribution function $e^{-\pi_3(w)}$.

Note that the classical extreme value theory deals with the *scale-location model*. Here, the random variables $\rho_{\varepsilon,n}$ are represented in the form $\rho_{\varepsilon,n} = (\rho_n - a_\varepsilon)/b_\varepsilon$, where $\rho_n, n = 1, 2, \ldots$ are i.i.d. random variables and $a_\varepsilon, b_\varepsilon$ are some non-random centralisation and normalisation constants. In this case, the distribution function $e^{-\pi_3(w)}$ belongs to one of three families of the classical extremal distributions. See, for instance, books by Galambos (1978), Leadbetter, Lindgren and Rootzén (1983), Resnick (1987), and Berman (1992).

This one-dimensional weak convergence result can be extended. Denote by $\mathbf{D}_0$ the space of step functions on $(0, \infty)$ continuous from the right and with a finite number of only positive jumps in every finite sub-interval of $(0, \infty)$. It is known (see, for example, Serfozo (1982), Leadbetter, Lindgren and Rootzén (1983), Resnick (1987), and Berman (1992)) that $\mathcal{S}_{11}$ is necessary and sufficient for the following condition of weak convergence to hold:

$\mathcal{A}_{73}$: $\rho_\varepsilon(t), t > 0 \Rightarrow \rho_0(t), t > 0$ as $\varepsilon \to 0$, where $\rho_0(t), t > 0$ is a non-decreasing càdlàg process described below.

Denote $G_\varepsilon(u) = \mathsf{P}\{\rho_{\varepsilon,1} \leq u\}$. The process $\rho_\varepsilon(t), t > 0$ has the following finite-dimensional distributions for $0 = t_0 < t_1 < \ldots < t_n$, $-\infty < u_1 \leq \ldots u_n < \infty$, $n \geq 1$ and ε such that $t_1 n_\varepsilon \geq 1$:

$$\mathsf{P}\{\rho_\varepsilon(t_1) \leq u_1, \ldots, \rho_\varepsilon(t_n) \leq u_n\} = \prod_{k=1}^{n} G_\varepsilon(u_k)^{[t_k n_\varepsilon] - [t_{k-1} n_\varepsilon]}. \tag{4.7.1}$$

It follows in an obvious way from relation (4.7.1) that the limiting process $\rho_0(t), t > 0$, in condition $\mathcal{A}_{73}$, has the following finite-dimensional distributions for $0 = t_0 < t_1 < \ldots < t_n$, $-\infty < u_1 \leq \ldots u_n < \infty$, $n \geq 1$:

$$\mathsf{P}\{\rho_0(t_1) \leq u_1, \ldots, \rho_0(t_n) \leq u_n\} = \prod_{k=1}^{n} e^{-\pi_3(u_k)(t_k - t_{k-1})}. \tag{4.7.2}$$

The limiting process $\rho_0(t), t > 0$, in condition $\mathcal{A}_{73}$, is a called an *extremal process*.

Adjoining to the notation υ above, we let $\upsilon' = \inf(w : \pi_3(w) = 0) \leq \infty$. Then the distribution function $e^{-\pi_3(w)}$ is concentrated on the interval $[\upsilon, \upsilon']$, $\rho_0(t) \stackrel{\text{a.s.}}{\longrightarrow} \upsilon$ as $t \to 0$, and $\rho_0(t) \stackrel{\text{a.s.}}{\longrightarrow} \upsilon'$ as $t \to \infty$.

Note that, in the case where $\upsilon = \upsilon'$, the extremal process degenerates, namely $\rho_0(t) = \upsilon, t > 0$, with probability 1.

The extremal process $\rho_0(t), t > 0$ is a stochastically continuous homogeneous Markov jump process whose trajectories belong to the space $\mathbf{D}_0$ with probability 1. This process has the following one-dimensional distribution function for $s > 0$,

$$\mathsf{P}\{\rho_0(s) \leq v\} = e^{-s\pi_3(v)}, \ v \in \mathbb{R}_1, \tag{4.7.3}$$

and the following transition probabilities for $0 < s < t < \infty$,

$$\mathsf{P}\{\rho_0(s+t) \leq w \,|\, \rho_0(s) = v\} = \chi(v \leq w) e^{-t\pi_3(w)}, v, w \in \mathbb{R}_1. \tag{4.7.4}$$

Let us denote by Υ the interval (υ, υ') if $\pi_3(\upsilon) = \infty, \pi_3(\upsilon' - 0) = 0$; the interval $[\upsilon, \upsilon')$ if $\pi_3(\upsilon) < \infty, \pi_3(\upsilon' - 0) = 0$; the interval $(\upsilon, \upsilon']$ if $\pi_3(\upsilon) = \infty, \pi_3(\upsilon' - 0) > 0$; and the interval $[\upsilon, \upsilon']$ if $\pi_3(\upsilon) < \infty, \pi_3(\upsilon' - 0) > 0$.

As follows from the remarks above, **(a)** $\mathsf{P}\{\rho_0(t) \in \Upsilon, t > 0\} = 1$. This is consistent with formulas (4.7.3) and (4.7.4). It follows from these formulas that the distribution function of the random variable $\rho_0(s)$ is concentrated on the interval Υ for every $s > 0$ and the transition probability given in (4.7.4) is a distribution function in w concentrated on the interval Υ for every $v \in \Upsilon$ and $0 < s \leq t < \infty$.

A more refined representation of the extremal process is as follows. Let $s > 0$. Denote by $s < \tau_1^{(s)} < \tau_2^{(s)} < \ldots$ successive moments of jumps of the process $\rho_0(t)$ in the interval $[s, \infty)$. Write $\eta_n^{(s)} = \rho_0(\tau_n^{(s)})$ for the heights at the moments of jumps and $\kappa_n^{(s)} = \tau_n^{(s)} - \tau_{n-1}^{(s)}$ for the inter-jump times. For convenience, we put $\tau_0^{(s)} = \tau_{-1}^{(s)} = s$. Then the bivariate random sequence $(\kappa_n^{(s)}, \eta_n^{(s)}), n = 0, 1, \ldots$ is a homogeneous Markov chain with the phase space $[0, \infty) \times \mathbb{R}_1$ and the transition probabilities

$$\begin{aligned} &P\{\kappa_{n+1}^{(s)} \leq t, \eta_{n+1}^{(s)} \leq w \,|\, \kappa_n^{(s)} = t', \eta_n^{(s)} = v\} \\ &\quad = (1 - e^{-\pi_3(v)t})\chi(v \leq w)(1 - \pi_3(w)/\pi_3(v)). \end{aligned} \tag{4.7.5}$$

It follows from **(a)** that **(c)** $\mathsf{P}\{(\kappa_n^{(s)}, \eta_n^{(s)}) \in [0, \infty) \times \Upsilon, n = 0, 1, \ldots\} = 1$. This is consistent with formulas (4.7.3) and (4.7.5). It follows from these formulas that, for every $s > 0$, the two-dimensional distribution function of the random variable $(\kappa_0^{(s)}, \eta_0^{(s)}) =$

$(0, \rho_0(s))$ is concentrated on the set $[0, \infty) \times \Upsilon$ and the transition probability given in (4.7.5) is a two-dimensional distribution function in (t, w) concentrated on the set $[0, \infty) \times \Upsilon$ for every $(t', v) \in [0, \infty) \times \Upsilon$.

There are three cases when the expression in the right-hand side of (4.7.5) is not well defined. The simplest way is to set this expression equal, for every $t \geq 0$, to **(d)** 0 if $v > w$, or **(e)** 1 if $v \leq w$ and $\pi_3(v) = \pi_3(w) = \infty$, or **(f)** 1 if $v \leq w$ and $\pi_3(v) = \pi_3(w) = 0$.

With this convention, **(g)** the expression in the right-hand side of (4.7.5) is always a two-dimensional distribution function in (t, w). Note that any other admissible interpretation (possessing property **(g)**) of this expression in uncertain situations would not change the finite-dimensional distributions of the Markov chain $(\kappa_n^{(s)}, \eta_n^{(s)}), n = 0, 1, \ldots$. This follows from **(c)**.

We refer to Serfozo (1982) and Resnick (1987) for the proof that condition $\mathbf{S_{11}}$, without any additional assumptions, implies that

$$\zeta_\varepsilon(t), t > 0 \xrightarrow{\mathbf{J}} \zeta_0(t), t > 0 \text{ as } \varepsilon \to 0. \tag{4.7.6}$$

Note that relation of **J**-convergence (4.7.6) follows also from the more general relation of **J**-convergence given for mixed sum-max processes in Theorem 4.8.2.

4.7.3. Extremes with random sample size based on i.i.d. random variables. We now generalise the asymptotic results given in condition $\mathcal{A}_{73}$ and relation (4.7.6) to extremal processes with random sample size indices.

Of course, we have to assume some condition concerning the asymptotic behaviour of the random stopping indices. Such a minimal condition is $\mathcal{A}_{53}$, which states that the random variables $\nu_\varepsilon = \mu_\varepsilon / n_\varepsilon \Rightarrow \nu_0$ as $\varepsilon \to 0$, where ν_0 is an a.s. positive random variable.

Conditions $\mathcal{A}_{53}$ and $\mathcal{A}_{73}$ are sufficient to provide weak convergence of max-processes with random stopping indices in the case when the max-process $\rho_\varepsilon(t)$, $t \geq 0$ and the random stopping index ν_ε are independent. However, it is clear that in the case of dependence, conditions $\mathcal{A}_{53}$ and $\mathcal{A}_{73}$ should be replaced by a stronger condition in terms of the joint distribution of the random variable ν_ε and the process $\rho_\varepsilon(t), t \geq 0$. The following condition plays a key role in further considerations:

$\mathcal{A}_{74}$: $(\nu_\varepsilon, \rho_\varepsilon(t)), t > 0 \Rightarrow (\nu_0, \rho_0(t)), t > 0$ as $\varepsilon \to 0$, where (a) ν_0 is an a.s. non-negative random variable, and (b) $\rho_0(t)$, $t \geq 0$ is an extremal process described in (4.7.2) - (4.7.5).

We also assume that the positivity condition $\mathbf{J_4}$ holds.

Denote by V_0 the set of points $t > 0$ for which $P\{\tau_k^{(s_n)} = t\nu_0\} = 0$ for all $k, n = 1, 2, \ldots$, where $s_n = n^{-1}$. The set $\overline{V}_0$ contains no more than a countable number of points, since it coincides with the set of atoms for the distribution functions of the random variables $\tau_k^{(s_n)}/\nu_0, k, n = 1, 2, \ldots$. Therefore, the set V_0 is $(0, \infty)$ except for at most a countable set.

The set V_0 coincides with $(0, \infty)$ if the random variables $\tau_k^{(s_n)}/\nu_0, k, n = 1, 2, \ldots$ have continuous distributions. Since the random variables $\tau_k^{(s_n)}$ have continuous distribution functions, $\tau_k^{(s_n)}/\nu_0$ also have continuous distribution functions if the random variables $\tau_k^{(s_n)}$ and ν_0 are independent, for every $k, n = 1, 2, \ldots$. Hence, the set $V_0 = (0, \infty)$ if the process $\rho_0(t), t > 0$ and the random variable ν_0 are independent or, at any rate, the random variables $\tau_k^{(s_n)}$ and ν_0 are independent for every $k, n = 1, 2, \ldots$. In the latter case, the process $\rho_0(t), t > 0$ and the random variable ν_0 can be dependent.

Condition $\mathcal{A}_{74}$ implies condition $\mathcal{A}_{72}$. Also relation (4.7.6) implies that condition $\mathfrak{J}_{26}$ holds. So, by applying Theorems 4.7.1 and 4.7.2, one can formulate the following two theorems. These theorems are given in Silvestrov and Teugels (1998a).

Theorem 4.7.3. *Let conditions $\mathfrak{T}_7$, $\mathcal{A}_{74}$, and $\mathfrak{I}_4$ hold. Then*

$$\zeta_\varepsilon(t) = \rho_\varepsilon(t\nu_\varepsilon), t \in V_0 \Rightarrow \zeta_0(t) = \rho_0(t\nu_0), t \in V_0 \text{ as } \varepsilon \to 0.$$

Theorem 4.7.4. *Let conditions $\mathfrak{T}_7$, $\mathcal{A}_{74}$, and $\mathfrak{I}_4$ hold. Then*

$$\zeta_\varepsilon(t), t > 0 \xrightarrow{\mathbf{J}} \zeta_0(t), t > 0 \text{ as } \varepsilon \to 0.$$

We remark that, in Theorems 4.7.3 and 4.7.4, the external max-processes and the random stopping indices can be dependent in an arbitrary way. Only the condition of joint weak convergence is required. No independence or asymptotic independence conditions for these external max-processes and random stopping indices are involved.

4.7.4. Extremes with random sample size indices converging in probability. Let us now consider a model with random sample size indices converging in probability.

It is natural to assume in this case that the random variables $\rho_{\varepsilon,n}, n \geq 1$ and μ_ε are defined on the same probability space for all $\varepsilon > 0$. We also assume that the independence condition for the random variables $\rho_{\varepsilon,n}$ holds in the following stronger form:

$\mathfrak{T}_8$: The sets of the random variables $\{\rho_{\varepsilon,n}, \varepsilon > 0\}$ are mutually independent for $n \geq 1$.

Obviously, conditions $\mathfrak{T}_7$ and $\mathfrak{T}_8$ hold for the scale-location model. In this case, the random variables $\rho_{\varepsilon,n}$ are represented in the form $\rho_{\varepsilon,n} = (\rho_n - a_\varepsilon)/b_\varepsilon$, where $\rho_n, n \geq 1$ are i.i.d. random variables and a_ε and b_ε are some non-random centralisation and normalisation constants. It also holds for a more general model with the random variables $\rho_{\varepsilon,n} = h_\varepsilon(\rho_n), n \geq 1$, where $h_\varepsilon(\cdot)$ are non-random measurable real-valued functions.

Let us recall condition $\mathcal{P}_1$ introduced in Subsection 4.2.5,

$\mathcal{P}_1$: $\nu_\varepsilon = \mu_\varepsilon/n_\varepsilon \xrightarrow{\mathsf{P}} \nu_0$ as $\varepsilon \to 0$, where ν_0 is a non-negative random variable.

The following lemma shows that the model with normalised stopping indices converging in probability is a particular case of a model where the assumption of joint weak convergence of external sum-processes and stopping indices is made.

Lemma 4.7.1. *Let conditions $\mathcal{T}_7$, $\mathcal{T}_8$, $\mathcal{A}_{73}$ (or $\mathcal{S}_{11}$) and $\mathcal{P}_1$ hold. Then condition $\mathcal{A}_{74}$ holds, moreover,* **(α)** *the limiting process $\rho_0(t)$, $t > 0$ and the limiting random variable ν_0 are independent*; **(β)** *$\rho_0(t)$, $t \geq 0$ is a càdlàg extremal process which has the same finite-dimensional distribution as the corresponding process in condition $\mathcal{A}_{73}$*; **(γ)** *ν_0 is a random variable which has the same distribution as the the corresponding random variable in condition $\mathcal{P}_1$.*

Proof of Lemma 4.7.1. Take some subsequence $\varepsilon_n \to 0$ as $n \to \infty$ and choose some $0 < t_1 < \ldots < t_m < \infty$ and $\upsilon < s_1 \leq \ldots \leq s_m < \infty$. Define

$$A_n = \{\max_{k \leq t_l n_{\varepsilon_n}} \rho_{\varepsilon_n,k} \leq s_l, l = 1, \ldots m\}, \; A = \{\rho_0(t_l) \leq s_l, l = 1, \ldots m\}.$$

We are going first to prove that the sequence of events $A_n, n = 0, 1, \ldots$ is mixing in the sense of Rényi (1958), that is, for any $r \geq 1$,

$$\lim_{n\to\infty} \mathsf{P}(A_n \cap A_r) = \mathsf{P}(A)\mathsf{P}(A_r). \tag{4.7.7}$$

Obviously, the event $A_n = A^+_{nr} \cap A^-_{nr}$, where

$$A^-_{nr} = \{\max_{k \leq t_m n_{\varepsilon_r}} \rho_{\varepsilon_n,k} \leq s_1\}, \; A^+_{nr} = \{\max_{t_m n_{\varepsilon_r} < k \leq t_l n_{\varepsilon_n}} \rho_{\varepsilon_n,k} \leq s_l, l = 1, \ldots m\}.$$

It follows from conditions $\mathcal{A}_{73}$ and $\mathcal{T}_7$ that

$$\lim_{n\to\infty} \mathsf{P}(A^-_{nr}) = \lim_{n\to\infty} (\mathsf{P}\{\rho_{\varepsilon_n,1} \leq s_1\})^{[t_m n_{\varepsilon_r}]} = 1. \tag{4.7.8}$$

Now, by taking into account that for n large enough, $t_m n_{\varepsilon_r} < t_1 n_{\varepsilon_n}$, and using $\mathcal{T}_7$, $\mathcal{T}_8$, $\mathcal{A}_{73}$, and (4.7.8), we get

$$\begin{aligned} \lim_{n\to\infty} P(A_n \cap A_r) &= \lim_{n\to\infty} P(A^+_{nr} \cap A^-_{nr} \cap A_r) \\ &= \lim_{n\to\infty} \mathsf{P}(A^+_{nr})\mathsf{P}(A^-_{nr} \cap A_r) = \lim_{n\to\infty} \mathsf{P}(A^+_{nr})\mathsf{P}(A_r) \\ &= \lim_{n\to\infty} \mathsf{P}(A^+_{nr} \cap A^-_{nr})\mathsf{P}(A_r) = \lim_{n\to\infty} \mathsf{P}(A_n)\mathsf{P}(A_r) = P(A)P(A_r). \end{aligned} \tag{4.7.9}$$

Since the sequence $A_n, n = 1, 2 \ldots$ is mixing, $\mathsf{P}(A_n \cap B) \to \mathsf{P}(A)\mathsf{P}(B)$ as $n \to \infty$ for an arbitrary random event B. We can choose this event to be $B_z = \{\nu_0 \leq z\}$. Let also $B_{z,n} = \{\nu_{\varepsilon_n} \leq z\}$. Condition $\mathcal{P}_1$ implies that $P(B_z \Delta B_{z,n}) \to 0$ as $n \to \infty$ for any z which is a point of continuity of the distribution function of ν_0. Using these asymptotic relations we finally get

$$\lim_{n\to\infty} \mathsf{P}(A_n \cap B_{z,n}) = \lim_{n\to\infty} \mathsf{P}(A_n \cap B_z) = \mathsf{P}(A)\mathsf{P}(B_z). \tag{4.7.10}$$

Since the choice of the subsequence ε_n, the points $0 < t_1 < \ldots < t_m < \infty$, and $\upsilon < s_1 \leq \ldots \leq s_m < \infty$ was arbitrary, relation (4.7.10) is equivalent to the statement of the lemma. Note that, in the case where $s_1 \leq \upsilon$, the asymptotic independence of the events A_n and $B_{z,n}$ is obvious since, in this case, $\mathsf{P}(A_n) \to 0$ as $n \to \infty$. □

The following theorems, due to Lemma 4.7.1, are direct corollaries of Theorems 4.7.3 and 4.7.4. Note that we use that the set $V_0 = (0, \infty)$ in the case of independent limiting external process and limiting stopping index.

Theorem 4.7.5. *Let conditions* $\mathcal{T}_7$, $\mathcal{T}_8$, $\mathcal{A}_{73}$ (*or* $\mathcal{S}_{11}$), $\mathcal{P}_1$, and $\mathcal{I}_4$ hold. Then condition $\mathcal{A}_{74}$ holds with the process $\rho_0(t)$, $t \geq 0$ and the random variable ν_0 which are independent, and

$$\zeta_\varepsilon(t) = \rho_\varepsilon(t\nu_\varepsilon), t > 0 \Rightarrow \zeta_0(t) = \rho_0(t\nu_0), t > 0 \text{ as } \varepsilon \to 0.$$

Theorem 4.7.6. *Let conditions* $\mathcal{T}_7$, $\mathcal{T}_8$, $\mathcal{A}_{73}$ (*or* $\mathcal{S}_{11}$), $\mathcal{P}_1$, and $\mathcal{I}_4$ hold. Then

$$\zeta_\varepsilon(t) = \rho_\varepsilon(t\nu_\varepsilon), t > 0 \xrightarrow{\mathbf{J}} \zeta_0(t) = \rho_0(t\nu_0), t > 0 \text{ as } \varepsilon \to 0.$$

4.8 Mixed sum-max processes

In this section, we give general conditions for weak and **J**-convergence of mixed max-sum processes. Such processes are constructed from a sequence of two-dimensional i.i.d. random vectors. Note that no conditions are imposed on possible dependencies between the components of these random vectors. The first component of this sequence is used to construct a traditional real-valued sum-process of i.i.d. random variables. The second one is used to construct an extremal max-process of i.i.d. random variables.

4.8.1. Weak convergence of mixed sum-max processes. Let, for every $\varepsilon > 0$, $(\xi_{\varepsilon,n}, \rho_{\varepsilon,n})$, $n = 1, 2, \ldots$ be a sequence of random vectors taking values in $\mathbb{R}_1 \times \mathbb{R}_1$. We assume that the following condition holds:

$\mathcal{T}_9$: $(\xi_{\varepsilon,n}, \rho_{\varepsilon,n}), k = 1, 2, \ldots$ is (for every $\varepsilon > 0$) a sequence of i.i.d. random vectors.

Let the non-random functions $0 < n_\varepsilon \to \infty$ as $\varepsilon \to 0$. We introduce a *mixed sum-max process* $(\xi_\varepsilon(t), \rho_\varepsilon(t)), t \geq 0$, where

$$\xi_\varepsilon(t) = \sum_{k \leq tn_\varepsilon} \xi_{\varepsilon,k}, \ t \geq 0,$$

and

$$\rho_\varepsilon(t) = \max_{k \leq 1 \vee tn_\varepsilon} \rho_{\varepsilon,k}, \ t \geq 0.$$

Conditions that provide marginal weak convergence of the corresponding sum-processes and max-processes were formulated, respectively, in Subsections 4.2.2 and 4.7.2. These are conditions $\mathcal{S}_1$ - $\mathcal{S}_3$ and $\mathcal{S}_{11}$. Let us now formulate conditions, which should be added to these conditions, in order to imply joint weak convergence of the mixed max-sum processes.

Let us introduce the following natural condition:

$\mathbf{S_{12}}$: (a) $n_\varepsilon \mathsf{P}\{\xi_{\varepsilon,1} > v, \rho_{\varepsilon,1} > w\} \to \pi_{2,3}(v,w)$ as $\varepsilon \to 0$ for all $v > 0, w > \upsilon$, which are points of continuity of the limiting function $\pi_{2,3}(v,w)$.

(b) $n_\varepsilon \mathsf{P}\{\xi_{\varepsilon,1} \le v, \rho_{\varepsilon,1} > w\} \to \pi_{2,3}(v,w)$ as $\varepsilon \to 0$ for all $v < 0, w > \upsilon$, which are points of continuity of the limiting function $\pi_{2,3}(v,w)$.

The function $\pi_{2,3}(v,w)$ satisfies a number of conditions: **(a)** $\pi_{2,3}(v,w)$ is non-negative, non-increasing and right-continuous in every argument for $v > 0, w > \upsilon$ and such that $\pi_{2,3}(v,\infty) = \pi_{2,3}(\infty,w) = 0, v > 0, w > \upsilon$; **(b)** $\pi_{2,3}(v,w)$ is also non-negative, non-decreasing in $v < 0$, and non-increasing in $w > \upsilon$, as well as right-continuous in every argument for $v < 0, w > \upsilon$ and such that $\pi_{2,3}(v,\infty) = \pi_{2,3}(-\infty,w) = 0, v < 0, w > \upsilon$; **(c)** it defines, for every for $w > \upsilon$, a measure on the Borel σ-algebra of subsets of $(0,\infty))$ such that $\Pi^{(w)}_{2,3}((v_1,v_2]) = \pi_{2,3}(v_1,w) - \pi_{2,3}(v_2,w)$ for $0 < v_1 \le v_2 < \infty$; **(d)** it also defines, for every for $w > \upsilon$, a measure on the Borel σ-algebra of subsets of $(-\infty,0)$ such that $\Pi^{(w)}_{2,3}((v_1,v_2]) = \pi_{2,3}(v_2,w) - \pi_{2,3}(v_1,w)$ for $-\infty < v_1 \le v_2 < 0$; **(e)** the measure $\Pi^{(w)}_{2,3}(A)$ can be extended, in an obvious way, to the σ-algebra $\tilde{\mathfrak{B}}_1$ (the Borel σ-algebra of subsets of $(-\infty,0)\cup(0,\infty)$), and it is a non-increasing and continuous from the right function in $w > \upsilon$ for every $A \in \tilde{\mathfrak{B}}_1$; **(f)** the following estimates are valid: $\Pi^{(w_1)}_{2,3}(A) - \Pi^{(w_2)}_{2,3}(A) \le (\pi_3(w_1) - \pi_3(w_2)) \wedge \Pi_2(A)$, in particular, $\Pi^{(w_1)}_{2,3}(A) \le \pi_3(w_1) \wedge \Pi_2(A)$, for $\upsilon < w_1 \le w_2 < \infty$ and $A \in \tilde{\mathfrak{B}}_1$; **(g)** the function $\pi_3(w)$ possesses properties **(a)** – **(b)** listed in connection with condition $\mathbf{S_{11}}$ (in Subsection 4.7.2); **(h)** the measure $\Pi_2(A)$ possesses properties **(a)** – **(f)** listed in connection with conditions $\mathbf{S_1}$ – $\mathbf{S_3}$ (in Subsection 4.2.2).

Let denote C_{π_3} the set of points $w > \upsilon$ that are points of continuity of the function $\pi_3(w)$.

It is useful to note that the properties **(a)** and **(b)** can initially be required only for $w \in C_{\pi_3}$. Obviously, **(c)** and **(d)** follow from **(a)** and **(b)**. The properties and the estimates in **(e)** and **(f)** can be obtained, for $w \in C_{\pi_3}$ and intervals $(v_1,v_2] \subset (-\infty,0)\cup(0,\infty)$, by the limiting transition in the corresponding estimates for the functions in the left-hand side of the asymptotic relations in $\mathbf{S_{12}}$ (a) and (b), and then be extended, in an obvious way, to sets from the σ-algebra $\tilde{\mathfrak{B}}_1$.

Due to monotonicity of $\Pi^{(w)}_{2,3}(A)$ in $w \in C_{\pi_3}$, **(i)** there exist $\lim_{w' \in C_{\pi_3}, w < w' \to w} \Pi^{(w')}_{2,3}(A) = \Pi^{(w)}_{2,3}(A)$, for every $A \in \tilde{\mathfrak{B}}_1$, and $w > \upsilon, w \notin C_{\pi_3}$ and also for $w = \upsilon$ if $\upsilon > -\infty, \pi_3(\upsilon) < \infty$. The estimates in **(f)** can be verified by similar limiting transition for any $A \in \tilde{\mathfrak{B}}_1$, and $\upsilon < w_1 \le w_2 < \infty, w_1 \notin C_{\pi_3}$ and also for $\upsilon = w_1 \le w_2 < \infty$ if $\upsilon > -\infty, \pi_3(\upsilon) < \infty$. It follows from these estimates that convergence in **(i)** is uniform with respect to $A \in \tilde{\mathfrak{B}}_1$. This implies that $\Pi^{(w)}_{2,3}(A)$ is a measure, for every $w > \upsilon, w \notin C_{\pi_3}$ and for $w = \upsilon$ if $\upsilon > -\infty, \pi_3(\upsilon) < \infty$. These estimates also imply that $\Pi^{(w)}_{2,3}(A)$, as a function in w, is non-decreasing and right-continuous at any point $w > \upsilon, w \notin C_{\pi_3}$ and $w = \upsilon$ if $\upsilon > -\infty, \pi_3(\upsilon) < \infty$, for every $A \in \tilde{\mathfrak{B}}_1$. Thus, $\Pi^{(w)}_{2,3}(A)$ defined in this way satisfies all properties described in **(e)** – **(h)**.

For $w > \upsilon$ and for $w = \upsilon$ if $\upsilon > -\infty, \pi_3(\upsilon) < \infty$, we define a measure on the σ-algebra $\tilde{\mathfrak{B}}_1$ by the following formula:

$$\hat{\Pi}_{2,3}^{(w)}(A) = \Pi_2(A) - \Pi_{2,3}^{(w)}(A). \tag{4.8.1}$$

This measure plays the role of the jump measure in the Lévy-Khintchine representation of infinitely divisible characteristic functions, for $t \geq 0$,

$$\phi_{2,3}^{(w)}(t, z) = \exp\{t\psi_{2,3}^{(w)}(z)\}, \ z \in \mathbb{R}_1, \tag{4.8.2}$$

where

$$\psi_{2,3}^{(w)}(z) = \mathrm{i}a^{(w)}z - \frac{1}{2}b^2z^2 + \int_{\mathbb{R}_1}(e^{\mathrm{i}zs} - 1 - \frac{\mathrm{i}zs}{1+s^2})\hat{\Pi}_{2,3}^{(w)}(ds), \tag{4.8.3}$$

and

$$a^{(w)} = a - \int_{\mathbb{R}_1} \frac{s}{1+s^2}\Pi_{2,3}^{(w)}(ds). \tag{4.8.4}$$

We additionally define $\hat{\Pi}_{2,3}^{(w)}(A) \equiv 0$ and $\phi_{2,3}^{(w)}(t, z) = \exp\{t(\mathrm{i}az - \frac{1}{2}b^2z^2)\}$, for $w < \upsilon$ if $\upsilon > -\infty, \pi_3(\upsilon) < \infty$.

It follows from **(e)** and **(f)** that $a^{(w)}$ is a right-continuous function, and, therefore, the function $\phi_{2,3}^{(w)}(t, z)$ is right-continuous in w for every $z \in \mathbb{R}_1, t \geq 0$.

Recall also that the constants $a, a^{(w)}, b$ and the measures $\Pi_2(A), \Pi_{2,3}^{(w)}(A)$ in (4.8.1), (4.8.3), and (4.8.4) are determined by conditions $\mathcal{S}_1 - \mathcal{S}_3$ and $\mathcal{S}_{11} - \mathcal{S}_{12}$.

Recall the space $\mathbf{D}_0$ of step functions on $(0, \infty)$ continuous from the right and with a finite number of only positive jumps in every finite sub-interval of $(0, \infty)$.

Let us introduce a càdlàg homogeneous *mixed Markov process* $(\xi_0(t), \rho_0(t)), t > 0$, whose trajectories belong to the space $\mathbf{D}_{(0,\infty)}^{(1)} \times \mathbf{D}_0$ with probability 1. and the transition probabilities have the following *hybrid characteristic-distribution form*:

$$\begin{aligned} &\mathsf{E}\left\{e^{\mathrm{i}z(\xi_0(t+s)-\xi_0(s))} \cdot \chi(\rho_0(t+s) \leq w) \,|\, \xi_0(s) = v', \rho_0(s) = w'\right\} \\ &\quad = \chi(w' \leq w)e^{-t\pi_3(w)}\phi_{2,3}^{(w)}(t, z). \end{aligned} \tag{4.8.5}$$

It should be remarked that the second component, $\rho_0(t), t > 0$, of this process is an extremal process while the first one, $\xi_0(t), t > 0$, is a càdlàg homogeneous process with independent increments and the characteristics determined by the second component. Note, however, that the corresponding Gaussian sub-component of $\xi_0(t), t > 0$ is independent of $\rho_0(t), t > 0$.

As follows from the remarks in Subsection 4.7.2, **(e)** $\mathsf{P}\{(\xi_0(t), \rho_0(t) \in \mathbb{R}_1 \times \Upsilon, t > 0\} = 1$, where the interval Υ was defined in this subsection. This is consistent with formula (4.8.5).

The following theorem is a particular case of the corresponding result given in Silvestrov and Teugels (2001).

Theorem 4.8.1. *Let the conditions* $\mathcal{T}_9$, $\mathcal{S}_1 - \mathcal{S}_3$, *and* $\mathcal{S}_{11} - \mathcal{S}_{12}$ *hold. Then*

$$(\xi_\varepsilon(t), \rho_\varepsilon(t)), t > 0 \Rightarrow (\xi_0(t), \rho_0(t)), t > 0 \text{ as } \varepsilon \to 0.$$

Proof of Theorem 4.8.1. By the definition of the processes $\xi_\varepsilon(t)$ and $\rho_\varepsilon(t)$, for any $0 = t_0 < t_1 < \dots t_m < \infty$, $z_1, \dots, z_m \in \mathbb{R}_1$, $-\infty < w_1 < \dots < w_m < \infty$, $m \geq 1$ and ε such that $t_1 n_\varepsilon \geq 1$, we have

$$\begin{aligned} &\mathsf{E} \exp\{i \sum_{k=1}^{m} z_k \xi_\varepsilon(t_k)\} \cdot \prod_{k=1}^{m} \chi(\rho_\varepsilon(t_k) \leq w_k) \\ &\quad = \prod_{k=1}^{m} (\mathsf{E} \exp\{i z_{k,m} \xi_{\varepsilon,1}\} \cdot \chi(\rho_{\varepsilon,1} \leq w_k))^{[t_k n_\varepsilon] - [t_{k-1} n_\varepsilon]}, \end{aligned} \tag{4.8.6}$$

where $z_{k,m} = z_k + \dots z_m, k = 1, \dots, m$.

It follows from (4.8.6) that the statement of weak convergence given in Theorem 4.8.1 will be proved if we show that the following relation holds for every $z \in \mathbb{R}_1$ and every w, which is a continuity point of $\pi_3(w)$:

$$(\mathsf{E} \exp\{iz\xi_{\varepsilon,1}\} \cdot \chi(\rho_{\varepsilon,1} \leq w))^{n_\varepsilon} \to e^{-\pi_3(w)} \phi_{2,3}^{(w)}(1, z) \text{ as } \varepsilon \to 0. \tag{4.8.7}$$

This relation is obvious in the case where $w < \upsilon$, since, in this case, the expression in the left-hand side of (4.8.7) tends to zero due to condition $\mathcal{S}_{11}$ and the expression in the right-hand side of (4.8.7) is also equal to zero, as implied by the same condition.

If υ is a point of continuity of the function $\pi_3(w)$, then $\pi_3(\upsilon) = \infty$. In this case, again the expression in the left-hand side of (4.8.7), taken for $w = \upsilon$, tends to zero due to condition $\mathcal{S}_{11}$ and the expression in the right-hand side of (4.8.7), taken for $w = \upsilon$, is also equal to zero.

So, the only case that needs to be considered is when $w > \upsilon$. Obviously,

$$\begin{aligned} &(\mathsf{E} \exp\{iz\xi_{\varepsilon,1}\} \cdot \chi(\rho_{\varepsilon,1} \leq w))^{n_\varepsilon} \\ &\quad = (\mathsf{P}\{\rho_{\varepsilon,1} \leq w\})^{n_\varepsilon} (\mathsf{E}\{\exp\{iz\xi_{\varepsilon,1}\} \,|\, \rho_{\varepsilon 1} \leq w\})^{n_\varepsilon}. \end{aligned} \tag{4.8.8}$$

By condition $\mathcal{S}_{11}$ for any $w > \upsilon, w \in C_{\pi_3}$, we have

$$(\mathsf{P}\{\rho_{\varepsilon,1} \leq w\})^{n_\varepsilon} \to e^{-\pi_3(w)} \text{ as } \varepsilon \to 0. \tag{4.8.9}$$

It follows from (4.8.8) and (4.8.9) that (4.8.7) will be proved if we show that, for every $z \in R_1$ end every $w > \upsilon, w \in C_{\pi_3}$,

$$(\mathsf{E}\{\exp\{iz\xi_{\varepsilon,1}\} \,|\, \rho_{\varepsilon,1} \leq w\})^{n_\varepsilon} \to \phi_{2,3}^{(w)}(1, z) \text{ as } \varepsilon \to 0. \tag{4.8.10}$$

For every $\varepsilon > 0$ and $w > \upsilon$, define sequences of i.i.d. random variables $\xi_{\varepsilon,n}^{(w)}, n = 1, 2, \dots$ such that for $v \in \mathbb{R}_1$,

$$\mathsf{P}\{\xi_{\varepsilon,1}^{(w)} \leq v\} = \mathsf{P}\{\xi_{\varepsilon,1} \leq v \,|\, \rho_{\varepsilon,1} \leq w\}. \tag{4.8.11}$$

Using these sequences we can define sum-processes

$$\xi_\varepsilon^{(w)}(t) = \sum_{k \leq tn_\varepsilon} \xi_{\varepsilon,k}^{(w)}, \ t \geq 0. \tag{4.8.12}$$

For a given $w > \upsilon$, relation (4.8.10) is actually equivalent to

$$\xi_\varepsilon^{(w)}(t), t \geq 0 \Rightarrow \xi_0^{(w)}(t), t \geq 0 \text{ as } \varepsilon \to 0. \tag{4.8.13}$$

It has been pointed out in Subsection 4.2.2 that conditions $\mathcal{S}_1 - \mathcal{S}_3$ are necessary and sufficient for relation (4.8.13) to hold. Of course, all of these conditions should be checked for the random variables $\xi_{\varepsilon,1}^{(w)}$, instead of the random variables $\xi_{\varepsilon,1}$. These conditions should also be checked for every point $w > \upsilon, w \in C_{\pi_3}$. Comparison of formulas (4.2.1) and (4.8.2) shows that we need that the constants a, b and the measures $\Pi_2(A)$ be replaced in these conditions by the constants $a^{(w)}, b$ and the measures $\hat{\Pi}_{2,3}^{(w)}(A)$. This will be done in separate steps.

(i) The asymptotic relations (a) and (b) in condition $\mathcal{S}_1$ have the same structure. Thus, we give a proof of only one of them.

Note, first, that condition $\mathcal{S}_{11}$ implies that for all $w > \upsilon$,

$$\mathsf{P}\{\rho_{\varepsilon,1} \leq w\} \to 1 \text{ as } \varepsilon \to 0. \tag{4.8.14}$$

Using conditions $\mathcal{S}_1$ (a), $\mathcal{S}_{12}$, and relation (4.8.14) we have, for every $w > \upsilon, w \in C_{\pi_3}$, and $v > 0$, which are points of continuity of the limiting function (considered as a function of v for a given w), that

$$\begin{aligned} n_\varepsilon \mathsf{P}\{\xi_{\varepsilon,1}^{(w)} > v\} &= n_\varepsilon \frac{\mathsf{P}\{\xi_{\varepsilon,1} > v, \rho_{\varepsilon,1} \leq w\}}{\mathsf{P}\{\rho_{\varepsilon,1} \leq w\}} \\ &= n_\varepsilon \frac{\mathsf{P}\{\xi_{\varepsilon,1} > v\} - \mathsf{P}\{\xi_{\varepsilon,1} > v, \rho_{\varepsilon,1} > w\}}{\mathsf{P}\{\rho_{\varepsilon,1} \leq w\}} \\ &\to \pi_2(v) - \pi_{2,3}(v, w) = \hat{\Pi}_{2,3}^{(w)}((v, \infty)) \text{ as } \varepsilon \to 0. \end{aligned} \tag{4.8.15}$$

(ii) Consider the asymptotic relation given in condition $\mathcal{S}_2$. Use conditions $\mathcal{S}_1 - \mathcal{S}_2$ and $\mathcal{S}_{11} - \mathcal{S}_{12}$. For every $w > \upsilon, w \in C_{\pi_3}$, and $v_k > 0$, points of continuity of the limiting function (regarded as a function of v_k for a given w) and such that $0 < v_k \to 0$ as $k \to \infty$, we have

$$\begin{aligned} &\overline{\lim}_{\varepsilon\to 0} |n_\varepsilon \mathsf{E}\xi_{\varepsilon,1}\chi(|\xi_{\varepsilon,1}| \leq v_k, \rho_{\varepsilon,1} > w)| \\ &\quad \leq \overline{\lim}_{\varepsilon\to 0} n_\varepsilon \mathsf{E}|\xi_{\varepsilon,1}|\chi(|\xi_{\varepsilon,1}| \leq v_k, \rho_{\varepsilon,1} > w) \\ &\quad \leq \overline{\lim}_{\varepsilon\to 0} v_k n_\varepsilon \mathsf{P}\{\rho_{\varepsilon,1} > w\} = v_k \pi_3(w) \to 0 \text{ as } k \to \infty. \end{aligned} \tag{4.8.16}$$

We use (4.8.15) again, together with (4.8.16), to see that, for every appropriate $w > \upsilon$ and $v > 0$,

$$\begin{aligned}\lim_{\varepsilon\to 0} n_\varepsilon \mathsf{E}\xi_{\varepsilon,1}\chi(|\xi_{\varepsilon,1}| \le v, \rho_{\varepsilon,1} > w) &= \lim_{k\to\infty}\lim_{\varepsilon\to 0} n_\varepsilon \mathsf{E}\xi_{\varepsilon,1}\chi(v_k < |\xi_{\varepsilon,1}| \le v, \rho_{\varepsilon,1} > w) \\ &= \lim_{k\to\infty}\int_{v_k<|s|\le v} s\Pi_{2,3}^{(w)}(ds) = \fint_{|s|\le v} s\Pi_{2,3}^{(w)}(ds).\end{aligned} \tag{4.8.17}$$

By (4.8.17) and condition $\mathcal{S}_2$, we get, for appropriate $w > \upsilon$ and $v > 0$, that

$$\begin{aligned} n_\varepsilon \mathsf{E}\xi_{\varepsilon,1}^{(w)}\chi(|\xi_{\varepsilon,1}^{(w)}| \le v) &= n_\varepsilon \frac{\mathsf{E}\xi_{\varepsilon,1}\chi(|\xi_{\varepsilon,1}| \le v, \rho_{\varepsilon,1} \le w)}{\mathsf{P}\{\rho_{\varepsilon,1} \le w\}} \\ &= n_\varepsilon \frac{\mathsf{E}\xi_{\varepsilon,1}\chi(|\xi_{\varepsilon,1}| \le v) - \mathsf{E}\xi_{\varepsilon,1}\chi(|\xi_{\varepsilon,1}| \le v, \rho_{\varepsilon,1} > w)}{\mathsf{P}\{\rho_{\varepsilon,1} \le w\}} \\ &\to a^{(w)}(v) = a(v) - \fint_{|s|\le v} s\Pi_{2,3}^{(w)}(ds) \text{ as } \varepsilon \to 0.\end{aligned} \tag{4.8.18}$$

Relation (4.8.18) enables us to calculate the corresponding constant $a^{(w)}$ in (4.8.4) that replaces a in (4.2.2). Indeed,

$$\begin{aligned} a^{(w)} &= a^{(w)}(v) - \fint_{|s|<v} \frac{s^3}{1+s^2}\hat{\Pi}_{2,3}^{(w)}(ds) + \int_{|s|>v} \frac{s}{1+s^2}\hat{\Pi}_{2,3}^{(w)}(ds) \\ &= a(v) - \fint_{|s|<v} s\Pi_{2,3}^{(w)}(ds) - \fint_{|s|<v} \frac{s^3}{1+s^2}[\Pi_{2,3}(ds) - \Pi_{2,3}^{(w)}(ds)] \\ &\quad + \int_{|s|>v} \frac{s}{1+s^2}[\Pi_{2,3}(ds) - \Pi_{2,3}^{(w)}(ds)] \\ &= a - \fint_{|s|<v} s\Pi_{2,3}^{(w)}(ds) + \fint_{|s|<v} \frac{s^3}{1+s^2}\Pi_{2,3}^{(w)}(ds) \\ &\quad - \int_{|s|>v} \frac{s}{1+s^2}\Pi_{2,3}^{(w)}(ds) = a - \fint_{\mathbb{R}_1} \frac{s}{1+s^2}\Pi_{2,3}^{(w)}(ds).\end{aligned} \tag{4.8.19}$$

(iii) Finally, we must check condition $\mathcal{S}_3$ for the random variables $\xi_{\varepsilon,1}^{(w)}$. Note that relation (4.8.18) implies, in an obvious way, that for $w > \upsilon$ and $v > 0$,

$$\overline{\lim}_{\varepsilon\to 0} n_\varepsilon (\mathsf{E}\xi_{\varepsilon,1}^{(w)}\chi(|\xi_{\varepsilon,1}^{(w)}| \le v))^2 = 0. \tag{4.8.20}$$

Let again $w > \upsilon, w \in C_{\pi_3}$ and $v > 0$ a point of continuity of the limiting function, regarded as a function of v for a given w. Then, using conditions $\mathcal{S}_3$ and $\mathcal{S}_{11} - \mathcal{S}_{12}$ we get

$$\begin{aligned}
&\overline{\lim}_{\varepsilon\to 0} n_\varepsilon \mathsf{E}\xi^2_{\varepsilon,1}\chi(|\xi_{\varepsilon,1}| \le v, \rho_{\varepsilon,1} > w) \\
&\quad \le \overline{\lim}_{\varepsilon\to 0} \sqrt{n_\varepsilon \mathsf{E}\xi^4_{\varepsilon,1}\chi(|\xi_{\varepsilon,1}| \le v)} \cdot \sqrt{n_\varepsilon \mathsf{P}\{\rho_{\varepsilon,1} > w\}} \\
&\quad \le \overline{\lim}_{\varepsilon\to 0} \sqrt{v^2 n_\varepsilon \mathsf{E}\xi^2_{\varepsilon,1}\chi(|\xi_{\varepsilon,1}| \le v)} \cdot \overline{\lim}_{\varepsilon\to 0} \sqrt{n_\varepsilon \mathsf{P}\{\rho_{\varepsilon,1} > w\}} \\
&\quad \le \sqrt{v^2 \overline{\lim}_{\varepsilon\to 0} \mathsf{E}\xi^2_{\varepsilon,1}\chi(|\xi_{\varepsilon,1}| \le v)} \cdot \sqrt{\pi_3(w)} \to 0 \text{ as } 0 < v \to 0.
\end{aligned} \tag{4.8.21}$$

Using (4.8.14), (4.8.20), (4.8.21), and conditions $\mathcal{S}_2$ and $\mathcal{S}_3$ we get, for appropriate $w > \upsilon$ and $v > 0$, that

$$\begin{aligned}
&\lim_{0<v\to 0} \overline{\lim}_{\varepsilon\to 0} n_\varepsilon \operatorname{Var} \xi^{(w)}_{\varepsilon,1}\chi(|\xi^{(u)}_{\varepsilon,1}| \le v) \\
&\quad = \lim_{0<v\to 0} \overline{\lim}_{\varepsilon\to 0} n_\varepsilon \mathsf{E}(\xi^{(w)}_{\varepsilon,1})^2\chi(|\xi^{(w)}_{\varepsilon,1}| \le v) \\
&\quad = \lim_{0<v\to 0} \overline{\lim}_{\varepsilon\to 0} \Big(n_\varepsilon \frac{\mathsf{E}\xi^2_{\varepsilon,1}\chi(|\xi_{\varepsilon,1}| \le v)}{\mathsf{P}\{\rho_{\varepsilon,1} \le w\}} - n_\varepsilon \frac{\mathsf{E}\xi^2_{\varepsilon,1}\chi(|\xi_{\varepsilon,1}| \le v, \rho_{\varepsilon,1} > w)}{\mathsf{P}\{\rho_{\varepsilon,1} \le w\}}\Big) \\
&\quad = \lim_{0<v\to 0} \overline{\lim}_{\varepsilon\to 0} n_\varepsilon \mathsf{E}\xi^2_{\varepsilon,1}\chi(|\xi_{\varepsilon,1}| \le v) \\
&\quad = \lim_{0<v\to 0} \overline{\lim}_{\varepsilon\to 0} n_\varepsilon \operatorname{Var} \xi_{\varepsilon,1}\chi(|\xi_{\varepsilon,1}| \le v) = b^2.
\end{aligned} \tag{4.8.22}$$

Note that the constant b does not depend on $w > \upsilon$. Combining the above we complete the proof. □

Remark 4.8.1. Conditions $\mathcal{S}_1 - \mathcal{S}_3$, and $\mathcal{S}_{11} - \mathcal{S}_{12}$ are not only sufficient but also necessary for the relation of weak convergence given in Theorem 4.8.1 to hold.

Let us just give a sketch of the proof. As far as conditions $\mathcal{S}_1 - \mathcal{S}_3$ are concerned, the necessity statement is a part of the central criterion of convergence and the marginal weak convergence of the sum-processes $\xi_\varepsilon(t), t > 0$, which follows from the relation of weak convergence given in Theorem 4.8.1. Condition $\mathcal{S}_{11}$ follows from marginal weak convergence of the processes $\rho_\varepsilon(t), t > 0$, which also follows from the relation of weak convergence given in Theorem 4.8.1. This was mentioned in Subsection 4.7.2. Thus, only condition $\mathcal{S}_{12}$ requires a proof in the necessity statement. By using condition $\mathcal{S}_{11}$ and relations (4.8.2) - (4.8.10), one can show that the relation of weak convergence given in Theorem 4.8.1 implies relation (4.8.13) to hold. Then, by using $\mathcal{S}_1 - \mathcal{S}_3$ and applying the necessity statement in the central criterion of convergence to the sum-processes $\xi^{(w)}_\varepsilon(t), t > 0$, for $w > \upsilon$ one can prove that condition $\mathcal{S}_{12}$ holds.

4.8.2. Examples. The sum-process $\xi_\varepsilon(t), t \ge 0$ and the max-process $\rho_\varepsilon(t), t \ge 0$ are asymptotically independent if and only if the function $\pi_{2,3}(v, w) \equiv 0$ for $w > \upsilon$, $v \ne 0$ in condition $\mathcal{S}_{12}$. In this case, $a^{(w)} \equiv a$ and the measure $\hat{\Pi}^{(w)}_{2,3}(A) \equiv \Pi_2(A)$ for all $w \ge \upsilon$. Therefore, $\phi^{(w)}_{2,3}(t, z) = \phi_2(t, z)$.

Let us also consider the special case when the sum- and max- variables coincide, i.e. $\xi_{\varepsilon,k} \equiv \rho_{\varepsilon,k}, k = 1, 2, \ldots$. In this case, conditions $\mathcal{S}_{\mathbf{11}}$ - $\mathcal{S}_{\mathbf{12}}$ are implied by condition $\mathcal{S}_{\mathbf{1}}$.

Moreover, in this case, condition $\mathcal{S}_{\mathbf{1}}$ (a) implies that $\pi_2(v) = \pi_3(v)$ for $v > 0$. Also $\upsilon = 0$ as follows from the condition $\mathcal{S}_{\mathbf{1}}$, which implies that $n_\varepsilon \mathsf{P}\{\xi_{\varepsilon,1} > v\} \to \infty$ as $\varepsilon \to 0$ for any $v < 0$. Further, $\pi_{2,3}(v, w) = \pi_2(v \vee w)$ for $v, w > 0$, while $\pi_{2,3}(v, w) = 0$ for $v < 0, w > 0$.

Looking at the limiting processes, it is obvious that, in this case for ε such that $n_\varepsilon t \geq 1$, the random variable $\rho_\varepsilon(t) = f_t(\xi_\varepsilon(\cdot))$, where $f_t(x(\cdot)) = \max_{s\in(0,t]} \Delta_s(x(\cdot))$ and $\Delta_s(x(\cdot)) = x(s) - x(s - 0)$. So, $\rho_\varepsilon(t)$ is the maximal jump of the process $\xi_\varepsilon(s)$ in the interval $(0, t]$. This functional is a.s. $\mathbf{J}$-continuous with respect to measure generated by the limiting process $\xi_0(s), s > 0$ on Borel σ-algebra of the space $\mathbf{D}^{(1)}_{(0,\infty)}$ for every $t > 0$. Therefore, $\rho_0(t) = f_t(\xi_0(\cdot)), t > 0$.

4.8.3. J-convergence of mixed sum-max processes. We now turn to $\mathbf{J}$-convergence of mixed sum-max processes. We will be dealing with the process $\boldsymbol{\gamma}_\varepsilon(t) = (\xi_\varepsilon(t), \rho_\varepsilon(t))$, $t > 0$, which has the phase space $\mathbb{R}_1 \times \mathbb{R}_1$ and whose trajectories, by the definition, belong to the space $\mathbf{D}^{(1)}_{(0,\infty)} \times \mathbf{D}_0$ with probability 1. It is a Markov process. We denote the transition probabilities of this process by $\mathsf{P}_\varepsilon((v, w), t, t + s, A)$.

The following theorem is a variant of the corresponding result given in Silvestrov and Teugels (2001).

Theorem 4.8.2. *Let the conditions* $\mathcal{T}_{\mathbf{9}}$, $\mathcal{S}_{\mathbf{1}} - \mathcal{S}_{\mathbf{3}}$, *and* $\mathcal{S}_{\mathbf{11}} - \mathcal{S}_{\mathbf{12}}$ *hold. Then*

$$\boldsymbol{\gamma}_\varepsilon(t), t > 0 \xrightarrow{\mathbf{J}} \boldsymbol{\gamma}_0(t), t > 0 \text{ as } \varepsilon \to 0.$$

Proof of Theorem 4.8.2. The weak convergence of the processes $\boldsymbol{\gamma}_\varepsilon(t), t > 0$ has been proved in Theorem 4.8.1. So, Theorem 4.8.2 will follow if we can show that, for every $\delta > 0$ and $0 < T < T' < \infty$,

$$\lim_{c\to 0} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\Delta_J(\boldsymbol{\gamma}_\varepsilon(\cdot), c, T, T') \geq \delta\} = 0. \tag{4.8.23}$$

Note that the second component $\rho_\varepsilon(t), t > 0$ is a non-decreasing process with probability 1. We use this property to reduce the phase space of the second component to the interval $[h, \infty)$. This is an essential part in the proof of (4.8.23).

We choose **(a)** $h > -\infty$ to be a point of continuity of the function $\pi_3(w)$ if $\upsilon = -\infty$, and **(b)** $h = \upsilon$ if $\upsilon > -\infty$.

Introduce the truncated random variables $\hat{\rho}^{(h)}_{\varepsilon,k} = \rho_{\varepsilon,k} \vee h$, $k = 1, 2, \ldots$, and the corresponding max-processes

$$\hat{\rho}^{(h)}_\varepsilon(t) = \max_{k\leq 1\vee tn_\varepsilon} \hat{\rho}^{(h)}_{\varepsilon,k} = \rho_\varepsilon(t) \vee h, \ t \geq 0. \tag{4.8.24}$$

The bivariate process $\hat{\boldsymbol{\gamma}}^{(h)}_\varepsilon(t) = (\xi_\varepsilon(t), \hat{\rho}^{(h)}_\varepsilon(t)), t > 0$ has the phase space $\mathbb{R}_1 \times [h, \infty)$ and its trajectories belong to the space $\mathbf{D}^{(1)}_{(0,\infty)} \times \mathbf{D}_0$ with probability 1. It is a Markov processes which has, for $(v, w) \in \mathbb{R}_1 \times [h, \infty)$, the same transition probabilities, $P_\varepsilon((v, w), t, t+s, A)$, as the process $\gamma_\varepsilon(t), t > 0$.

Note that Theorem 4.8.1 can be applied to the max-sum processes $\hat{\gamma}_\varepsilon^{(h)}(t), t > 0$. All conditions of Theorem 4.8.1 are satisfied.

The only difference is that, in the case under consideration, the corresponding limiting functions $\pi_2(v)$, $\pi_3(w)$ and $\pi_{2,3}(v, w)$ in conditions $\mathbf{S_1} - \mathbf{S_3}$ and $\mathbf{S_{11}} - \mathbf{S_{12}}$ should be changed. We introduce new functions indexed with an upper index (h) as follows: **(c)** $\pi_2^{(h)}(v) = \pi_2(v)$ for $v \neq 0$; **(d)** $\pi_3^{(h)}(w) = \pi_3(w)$ for $w \geq h$, and $\pi_3^{(h)}(w) = \infty$ for $w < h$; **(e)** $\pi_{2,3}^{(h)}(v, w) = \pi_{2,3}(v, w)$ for $w \geq h, v \neq 0$, and $\pi_{2,3}^{(h)}(v, w) = \pi_2(v)$ for $w < h, v \neq 0$. The corresponding changes should also be introduced in the constants a and b.

Note that only in the case **(a)**, the changes are actually made, whereas in the case **(b)**, the new functions coincide with the old ones.

According to Theorem 4.8.1, the following relation holds:

$$\hat{\gamma}_\varepsilon^{(h)}(t), t > 0 \Rightarrow \hat{\gamma}_0^{(h)}(t), t \geq 0 \text{ as } \varepsilon \to 0. \tag{4.8.25}$$

This relation also follows directly from the statement of Theorem 4.8.1, since according to (4.8.24), the random vector $\hat{\gamma}_\varepsilon^{(h)}(t)$ is a continuous function of the random vector $\gamma_\varepsilon(t)$ for every $t > 0$ and $h \in \mathbb{R}_1$.

The limiting process $\hat{\gamma}_0^{(h)}(t), t > 0$ is completely similar to the process described in Theorem 4.8.1 with the following modification. Its characteristic $\pi_{2,3}^{(h)}(v, w)$ is defined above in **(d)** - **(e)**. Moreover, it is easily can be shown that the process $\hat{\gamma}_0^{(h)}(t), t > 0$, can be constructed from the process $\gamma_0(t), t > 0$ by simply truncating the second component of this process, that is, $\hat{\gamma}_0^{(h)}(t) = (\xi_0(t), \hat{\rho}_0^{(h)}(t)), t > 0$, where $\hat{\rho}_0^{(h)}(t) = \rho_0(t) \vee h, t > 0$.

Note that in the case **(a)**, the truncation does takes place as opposed to the case **(b)**, where the process $\hat{\rho}_0^{(h)}(t) \equiv \rho_0(t), t > 0$.

Let us now use the following inequality that holds for any $\delta > 0$ and $0 < T' < T'' < \infty$:

$$\begin{aligned} &\mathsf{P}\{\Delta_J(\gamma_\varepsilon(\cdot), c, T, T') \geq 2\delta\} \\ &\quad \leq \mathsf{P}\{\Delta_J(\hat{\gamma}_\varepsilon^{(h)}(\cdot), c, T, T') \geq \delta\} + \mathsf{P}\{\sup_{T \leq t \leq T'} |\hat{\rho}_\varepsilon^{(h)}(t) - \rho_\varepsilon(t)| \geq \delta\}. \end{aligned} \tag{4.8.26}$$

Obviously,

$$P\{\sup_{T \leq t \leq T'} |\hat{\rho}_\varepsilon^{(h)}(t) - \rho_\varepsilon(t)| \geq \delta\} \leq \mathsf{P}\{\rho_\varepsilon(T) \leq h - \delta\}. \tag{4.8.27}$$

In the case **(a)**, one can always choose $\delta/2 \leq \delta_h \leq \delta$ in such a way that the point $h - \delta_h$ is also a point of continuity of the function $\pi_3(w)$. In the case **(b)**, the point $\upsilon - \delta_h$ is automatically such a point. In both cases, for ε such that $n_\varepsilon T \geq 1$, we have

$$\mathsf{P}\{\rho_\varepsilon(T) \leq h - \delta\} = (\mathsf{P}\{\rho_{\varepsilon,1} \leq h - \delta_h\})^{[n_\varepsilon T]} \to e^{-\pi_3(h-\delta_h)T} \text{ as } \varepsilon \to 0. \tag{4.8.28}$$

In the case **(a)**, for every $0 < T < \infty$

$$\lim_{h \to -\infty} \overline{\lim}_{\varepsilon \to 0} \mathsf{P}\{\rho_\varepsilon(T) \leq h - \delta_h\} = \lim_{h \to -\infty} e^{-\pi_3(h-\delta_h)T} = 0. \tag{4.8.29}$$

Inequalities (4.8.27) and relations (4.8.28) and (4.8.29) imply that in the case **(a)**, for every $\delta > 0$ and $0 < T < T' < \infty$,

$$\lim_{h\to-\infty} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\sup_{T\le t\le T'} |\hat{\rho}^{(h)}_\varepsilon(t) - \rho_\varepsilon(t)| \ge \delta\} = 0. \tag{4.8.30}$$

In the case **(b)**, the limiting expression in the left-hand side of (4.8.28) is equal to zero. Thus, the additional limit transition given in (4.8.29) is not required. This shows that inequality (4.8.27) and relation (4.8.28) imply that, in the case **(b)**, for every $\delta > 0$ and $0 < T < T' < \infty$,

$$\overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\sup_{T\le t\le T'} |\hat{\rho}^{(h)}_\varepsilon(t) - \rho_\varepsilon(t)| \ge \delta\} = 0. \tag{4.8.31}$$

Relations (4.8.26), (4.8.30), and (4.8.31) imply that relation (4.8.23) will follow if we show that, for every h chosen according **(a)** or **(b)**, and $\delta > 0$, $0 < T < T' < \infty$,

$$\lim_{c\to 0} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\Delta_J(\hat{\gamma}^{(h)}_\varepsilon(\cdot), c, T, T') \ge \delta\} = 0. \tag{4.8.32}$$

Define

$$\alpha_\varepsilon(h, c, T, T', \delta) = \sup_{-\infty<v<\infty, w\ge h} \sup_{T\le t\le t+s\le t+c\le T'} \mathsf{P}_\varepsilon((v, w), t, t+s, S_\delta((v, w))),$$

where $S_\delta((v, w)) = \{(v', w') : (|v - v'|^2 + |w - w'|^2)^{1/2} > \delta\}$.

We showed in (4.8.25) that the processes $\hat{\gamma}^{(h)}_\varepsilon(t), t > 0$ weakly converge. As is known (see, for example, Skorokhod (1958) or Gikhman and Skorokhod (1971)), relation (4.8.32) follows in this case from the following relation that should be proved for every $\delta > 0$ and $0 < T < T' < \infty$:

$$\lim_{c\to 0} \overline{\lim}_{\varepsilon\to 0} \alpha_\varepsilon(h, c, T, T', \delta) = 0. \tag{4.8.33}$$

We now use the fact that the process $\rho_\varepsilon(t), t > 0$ is non-decreasing and that $\xi_\varepsilon(t), t > 0$ is a càdlàg process with independent increments. We get the following estimate:

$$\begin{aligned}
&\alpha_\varepsilon(h, c, T, T', 2\delta) \\
&\quad \le \sup_{T\le t\le t+s\le t+c\le T'} \mathsf{P}\{|\xi_\varepsilon(t+s) - \xi_\varepsilon(t)| > \delta\} \\
&\quad + \sup_{w\ge h} \sup_{T\le t\le t+s\le t+c\le T'} P\{\rho_\varepsilon(t+s) - \rho_\varepsilon(t) > \delta | \rho_\varepsilon(t) = w\} \\
&\quad \le \sup_{T\le t\le t+s\le t+c\le T'} ([n_\varepsilon(t+s)] - [n_\varepsilon t])(\mathsf{P}\{|\xi_{\varepsilon,1}| > \delta\} \\
&\quad + |\mathsf{E}\xi_{\varepsilon,1}\chi(|\xi_{\varepsilon,1}| \le \delta)| + \mathsf{Var}\,\xi_{\varepsilon,1}\chi(|\xi_{\varepsilon,1}| \le \delta)) \\
&\quad + \sup_{w\ge h} \sup_{T\le t\le t+s\le t+c\le T'} (1 - (\mathsf{P}\{\rho_{\varepsilon,1} \le w + \delta\})^{[n_\varepsilon(t+s)]-[n_\varepsilon t]}) \\
&\quad \le c n_\varepsilon(\mathsf{P}\{|\xi_{\varepsilon,1}| > \delta\} + |\mathsf{E}\xi_{\varepsilon,1}\chi(|\xi_{\varepsilon,1}| \le \delta)| \\
&\quad + \mathsf{Var}\,\xi_{\varepsilon,1}\chi(|\xi_{\varepsilon,1}| \le \delta)) + 1 - (\mathsf{P}\{\rho_{\varepsilon,1} \le h + \delta\})^{n_\varepsilon c}.
\end{aligned} \tag{4.8.34}$$

We are now in a position to use the truncation of the phase space described above, and conditions $\mathcal{S}_1$ –$\mathcal{S}_3$ and $\mathcal{S}_{11}$.

Choose h according **(a)** or **(b)**, and then $\delta/2 \le \delta_h \le \delta$ in such a way that the point $h + \delta_h$ is also a point of continuity of the function $\pi_3(w)$. In both cases, the quantity

$$\pi_3(h + \delta_h) < \infty. \tag{4.8.35}$$

Conditions $\mathcal{S}_1$ –$\mathcal{S}_3$ and $\mathcal{S}_{11}$, applied to (4.8.34), and (4.8.35) yield

$$\begin{aligned}
&\lim_{c\to 0} \overline{\lim}_{\varepsilon\to 0} \alpha_\varepsilon(h, c, T, T', 2\delta) \\
&\quad \le \lim_{c\to 0} c \cdot \overline{\lim}_{\varepsilon\to 0} n_\varepsilon(\mathsf{P}\{|\xi_{\varepsilon,1}| > \delta\} + |\mathsf{E}\xi_{\varepsilon,1}\chi(|\xi_{\varepsilon,1}| \le \delta)| \\
&\quad + \mathsf{Var}\,\xi_{\varepsilon,1}\chi(|\xi_{\varepsilon,1}| \le \delta)) + \lim_{c\to 0} \overline{\lim}_{\varepsilon\to 0}(1 - (\mathsf{P}\{\rho_{\varepsilon,1} \le h + \delta_h\})^{n_\varepsilon c}) \\
&\quad = \lim_{c\to 0}(1 - e^{-\pi_3(h+\delta_h)c}) = 0.
\end{aligned} \tag{4.8.36}$$

The proof is completed. □

Remark 4.8.2. Theorem 4.8.2 yields **J**-convergence of the mixed sum-max processes $\boldsymbol{\gamma}_\varepsilon(t)$, $t > 0$ on the open interval $(0, \infty)$. At the same time, conditions $\mathcal{S}_1$ – $\mathcal{S}_3$ imply that the first component of these processes $\xi_\varepsilon(t), t \ge 0 \xrightarrow{\mathbf{J}} \xi_0(t), t \ge 0$ as $\varepsilon \to 0$. This means that these processes satisfy condition $\mathcal{O}_{13}$.

4.8.4. Transformed mixed sum-max processes. Let $f(t, x)$ be a continuous function defined on $[0, \infty) \times \mathbb{R}_2$ and taking values in $\mathbb{R}_1$. The transformed stochastic process $f(t, \boldsymbol{\gamma}_\varepsilon(t)), t > 0$ has trajectories that belong to the space $\mathbf{D}^{(1)}_{(0,\infty)}$ with probability 1.

Theorems 1.6.12 and 4.8.2 imply **J**-convergence of the transformed processes

$$f(t, \boldsymbol{\gamma}_\varepsilon(t)), t > 0 \xrightarrow{\mathbf{J}} f(t, \boldsymbol{\gamma}_0(t)), t > 0 \text{ as } \varepsilon \to 0. \tag{4.8.37}$$

Below, several examples illustrate (4.8.37). We can apply this relation to a number of processes that represent modifications of the original mixed sum-max processes.

As the first example, take

$$\rho_\varepsilon(t) - \xi_\varepsilon(t), t > 0 \xrightarrow{\mathbf{J}} \rho_0(t) - \xi_0(t), t > 0 \text{ as } \varepsilon \to 0. \tag{4.8.38}$$

Other examples would be

$$\frac{\rho_\varepsilon(t)}{a + |\xi_\varepsilon(t)|}, t > 0 \xrightarrow{\mathbf{J}} \frac{\rho_0(t)}{a + |\xi_0(t)|}, t > 0 \text{ as } \varepsilon \to 0 \tag{4.8.39}$$

and

$$\frac{\rho_\varepsilon(t)}{at + |\xi_\varepsilon(t)|}, t > 0 \xrightarrow{\mathbf{J}} \frac{\rho_0(t)}{at + |\xi_0(t)|}, t > 0 \text{ as } \varepsilon \to 0. \tag{4.8.40}$$

Here $a > 0$ is a regularisation parameter that prevents the denominator in the last two relations to take the value zero.

Relations (4.8.38)-(4.8.40) establish weak convergence of the functionals that describe deviations of max- and sum-processes.

Since the process $\gamma_0(t), t > 0$ and the transformed process $f(t, \gamma_0(t)),\ t > 0$ are stochastically continuous, (4.8.37) implies that for any $0 < T_1 < T_2 < \infty$,

$$\sup_{t\in[T_1,T_2]} f(t, \gamma_\varepsilon(t)) \Rightarrow \sup_{t\in[T_1,T_2]} f(t, \gamma_0(t)) \text{ as } \varepsilon \to 0. \tag{4.8.41}$$

This relation, if applied to the modified processes (4.8.38) – (4.8.40), establishes weak convergence of the functional that describes the maximal deviations given by the corresponding processes.

4.8.5. Mixed sum-max processes with random stopping indices. Theorems 4.8.1 and 4.8.2 can be generalised to a model with random stopping indices.

Let us assume the following weak convergence condition:

$\mathcal{A}_{75}$: $(\nu_\varepsilon, \gamma_\varepsilon(t)), t > 0 \Rightarrow (\nu_0, \gamma_0(t)), t > 0$ as $\varepsilon \to 0$, where (a) ν_0 is an a.s. non-negative random variable, and (b) $\gamma_0(t) = (\xi_0(t), \rho_0(t),\ t \geq 0$ is a càdlàg homogeneous Markov process described in (4.8.5).

As above, we also assume that condition $\mathfrak{I}_4$ holds.

The following theorem generalises Theorems 4.2.4 and 4.7.4. It is a direct corollary of the translation Theorem 3.4.4 that must be applied to the compositions $\gamma_\varepsilon(t\nu_\varepsilon), t > 0$, in the case where the constant $\alpha = 0$ and the slowly varying functions $h(x) \equiv 1$. Remark 2.8.3 that describes a modification of the conditions for the case of the interval $(0, \infty)$ must also be used. Conditions $\mathcal{A}_{75}$ and $\mathfrak{I}_4$, together with Theorem 4.8.2, imply that conditions of Theorem 3.4.4 hold.

Theorem 4.8.3. *Let conditions $\mathcal{T}_9$, $\mathcal{A}_{75}$, and $\mathfrak{I}_4$ hold. Then*

$$\zeta_\varepsilon(t) = \gamma_\varepsilon(t\nu_\varepsilon), t > 0 \xrightarrow{\mathbf{J}} \zeta_0(t) = \gamma_0(t\nu_0), t > 0 \textit{ as } \varepsilon \to 0.$$

Note that the external mixed sum-max processes and the random stopping indices can be dependent in an arbitrary way in Theorem 4.8.3. The only condition of joint weak convergence is required. No independence or asymptotic independence conditions for external sum-max processes and random stopping indices are needed.

Let us impose the following condition:

$\mathcal{T}_{10}$: The sets of random vectors $\{(\xi_{\varepsilon,n}, \rho_{\varepsilon,n}), \varepsilon > 0\}$ are mutually independent for $n \geq 1$.

Also, the following lemma can be useful in the case when the normalised stopping indices converge in probability. We give it without a proof, since its proof is analogous to those of Lemmas 4.2.1 and 4.7.1.

Lemma 4.8.1. *Let conditions* $\mathcal{T}_9$, $\mathcal{T}_{10}$, $\mathcal{S}_1 - \mathcal{S}_3$, $\mathcal{S}_{11} - \mathcal{S}_{12}$, *and* $\mathcal{P}_1$ *hold. Then condition* $\mathcal{A}_{75}$ *holds, moreover,* **(α)** *the limiting process* $\gamma_0(t)$, $t > 0$ *and the limiting random variable* ν_0 *are independent*; **(β)** $\gamma_0(t)$, $t \geq 0$ *is a càdlàg homogeneous Markov process which has the same finite-dimensional distribution as the corresponding process described in* (4.8.5); **(γ)** ν_0 *is a random variable which has the same distribution as the the corresponding random variable in condition* $\mathcal{P}_1$.

The following theorem follows from Theorem 4.8.3 and Lemma 4.8.1.

Theorem 4.8.4. *Let conditions* $\mathcal{T}_9$, $\mathcal{T}_{10}$, $\mathcal{S}_1 - \mathcal{S}_3$, $\mathcal{S}_{11} - \mathcal{S}_{12}$, $\mathcal{P}_1$, *and* $\mathcal{I}_4$ *hold. Then condition* $\mathcal{A}_{75}$ *holds with the process* $\gamma_0(t)$, $t \geq 0$ *and the random variable* ν_0 *which are independent, and*

$$\zeta_\varepsilon(t) = \gamma_\varepsilon(t\nu_\varepsilon), t > 0 \xrightarrow{\mathbf{J}} \zeta_0(t) = \gamma_0(t\nu_0), t > 0 \text{ as } \varepsilon \to 0.$$

In conclusion, we would like to refer to some works concerning studies of joint asymptotic behaviour of maxima and sums of i.i.d. random variables. Conditions for their quotients to tend to 1 were studied by Arov and Bobrov (1960), O'Brien (1980), Maller and Resnick (1984), and Pruitt (1987). Related results can also be found in Darling (1952), Smirnov (1952), and Aebi, Embrechts and Mikosch (1992). Joint asymptotic distributions of maxima and sums of i.i.d. random variables were studied for the scale-location model by Breiman (1965), Chow and Teugels (1979), Resnick (1986), and Haas (1992). Related results can also be found in Lamperty (1964), Anderson and Turkman (1991), Kesten and Maller (1994), Hsing Tailen (1995), Ho Hwai-Chung and Hsing Tailen (1996), and the book edited by Hahn, Mason and Weiner (1991).

4.9 Max-processes with renewal stopping

In this section, we study weak and **J**-convergence limit theorems for the so-called max-processes with renewal stopping. These processes give another example of the generalised exceeding processes.

4.9.1. Max-processes with renewal stopping. Let, for every $\varepsilon > 0$, $(\kappa_{\varepsilon,n}, \rho_{\varepsilon,n})$, $n = 1, 2, \ldots$ be a sequence of random variables taking values in $[0, \infty) \times \mathbb{R}_1$. Further, let $n_\varepsilon > 0$ be a non-random function of parameter ε such that $n_\varepsilon \to \infty$ as $\varepsilon \to 0$.

We first introduce a mixed sum-max process with non-random stopping index,

$$\boldsymbol{\beta}_\varepsilon(t) = (\kappa_\varepsilon(t), \rho_\varepsilon(t)) = (\sum_{k \leq tn_\varepsilon} \kappa_{\varepsilon,k}, \max_{k \leq 1 \vee tn_\varepsilon} \rho_{\varepsilon,k}), \ t \geq 0.$$

In this case, the following process is usually referred to as a *renewal process*:

$$\nu_\varepsilon(t) = \sup(s : \kappa_\varepsilon(s) \leq t), t > 0,$$

and

$$\zeta_\varepsilon(t) = \rho_\varepsilon(\nu_\varepsilon(t)), t > 0$$

is called a *max-process with renewal stopping*.

The max-processes with renewal stopping $\zeta_\varepsilon(t), t \geq 0$ is another example of the generalised exceeding processes considered in Sections 4.3 and 4.4. In this case, the process $\boldsymbol{\beta}_\varepsilon(t) = (\kappa_\varepsilon(t), \rho_\varepsilon(t), t \geq 0$ replaces the process $\boldsymbol{\alpha}_\varepsilon(t) = (\kappa_\varepsilon(t), \xi_\varepsilon(t)), t \geq 0$ in the definition of the generalised exceeding process.

Theorems formulated in Sections 4.3 – 4.4 can be directly translated to max-processes with renewal stopping. However, the extremal process $\rho_\varepsilon(t), t \geq 0$ may not weakly converge at the point 0. So, it is necessary to apply those variants of these theorems, which relate to the case when the external processes do not converge at the point 0. These modifications are given in Subsections 4.3.8 and 4.4.2.

Condition $\mathcal{A}_{58}$ takes in this case the following form:

$\mathcal{A}_{76}$: $(\kappa_\varepsilon(t), \rho_\varepsilon(t)), t \in V \times U \Rightarrow (\kappa_0(t), \rho_0(t)), t \in V \times U$ as $\varepsilon \to 0$, where: (a) V and U are subsets of $(0, \infty)$, dense in this interval, (b) $\kappa_0(t), t \geq 0$ is a non-negative and non-decreasing càdlàg process, (c) $\rho_0(t), t > 0$ is a non-decreasing càdlàg process.

Note that, by the definition, $\kappa_\varepsilon(t), t > 0$ is a non-negative and non-decreasing càdlàg process for all $\varepsilon > 0$. The relation of weak convergence given in condition $\mathcal{A}_{76}$ imply in this case that the limiting càdlàg process $\kappa_0(t), t > 0$ is a.s. non-negative and non-decreasing. Note that the process $\kappa_\varepsilon(t)$ is actually defined on the interval $[0, \infty)$. Since the càdlàg process $\kappa_0(t), t > 0$ is a.s. non-negative and non-decreasing, there exists a proper random variable $\kappa(0)$ such that $\kappa_0(t) \xrightarrow{\text{a.s.}} \kappa(0)$ as $0 < t \to 0$. So, we can assume that the process $\kappa_0(t)$ is also defined on the interval $[0, \infty)$. Finally, this process can be replaced, in condition $\mathcal{A}_{76}$, by some stochastically equivalent càdlàg modification.

Since the limiting process $\kappa_0(t), t \geq 0$ is a non-negative process, condition $\mathfrak{I}_6$ holds.

However, condition $\mathcal{A}_{76}$ does not require weak convergence of the processes $\kappa_\varepsilon(t), t \geq 0$ at the point 0. So, despite that $\kappa_\varepsilon(0) = 0$ with probability 1 for every $\varepsilon > 0$, it is not guaranteed that the random variable $\kappa_0(0) = 0$ with probability 1. Since this assumption usually holds max-processes with renewal stopping based on i.i.d. random variables, we adopt the following condition:

$\mathfrak{I}_{23}$: $\kappa_0(0) = 0$ with probability1.

Note that this condition implies that condition $\mathfrak{I}_{13}$ used in Subsection 4.3.8 holds.

Consider first the case when the limiting stopping renewal process $\nu_0(t), t \geq 0$ is an a.s. continuous process.

Condition $\mathfrak{K}_5$ introduced in Subsection 4.3.1 remains with no change. This condition permits to avoid considering the case when the random variables $\nu_\varepsilon(t)$ can be improper. So, we have that **(a)** the random variable $\nu_\varepsilon(t)$ is finite with probability 1 for every $t \geq 0$.

Condition $\mathfrak{I}_9$ introduced in Subsection 4.3.1 also stays the same. Recall that this condition requires for $\kappa_0(t), t \geq 0$ to be an a.s. strictly increasing càdlàg process. Condition $\mathfrak{I}_9$ implies that **(b)** $\nu_0(t), t \geq 0$ is an a.s. continuous process.

Condition $\mathfrak{I}_9$ and $\mathfrak{I}_{23}$ imply that **(c)** $\nu_0(0) = 0$ with probability 1 and also that **(d)** $\nu_0(t) > 0$ with probability 1 for every $t > 0$.

Condition $\mathcal{J}_{24}$ takes in this case the following form:

$\mathcal{J}_{27}$: $\lim_{c\to 0} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\Delta_J(\boldsymbol{\beta}_\varepsilon(\cdot), c, T', T'') > \delta\} = 0,\ \delta > 0, 0 < T' < T'' < \infty.$

Note that it is assumed that $0 < \varepsilon \to 0$. However, under $\mathcal{A}_{76}$ condition $\mathcal{J}_{27}$ is equivalent to $\mathcal{J}_{24}$, since the limiting process $\boldsymbol{\beta}_0(t), t > 0$ is a càdlàg process and, therefore, the asymptotic relation in $\mathcal{J}_{27}$ automatically holds for $\varepsilon = 0$.

Let us restate here Theorem 4.3.11 applying this theorem to max-processes with renewal stopping and taking into account the remarks made in Subsection 4.3.8.

Theorem 4.9.1. *Let conditions $\bar{\mathcal{K}}_5$, $\mathcal{A}_{76}$, $\mathcal{J}_{27}$, $\mathfrak{I}_9$, and $\mathfrak{I}_{23}$ hold for the max-processes $\boldsymbol{\beta}_\varepsilon(t)$, $t \geq 0$. Then*

$$\zeta_\varepsilon(t), t > 0 \xrightarrow{\mathbf{J}} \zeta_0(t), t > 0 \text{ as } \varepsilon \to 0.$$

We now consider the case when the limiting renewal process $\nu_0(t), t \geq 0$ is a step càdlàg process.

In this case, the random variables $\tau_{\varepsilon n}, n = 0, 1, \ldots$ should be defined in the same way as in Subsection 4.4.1, i.e., as successive moments of positive jumps of the process $\kappa_\varepsilon(t), t \geq 0$, for $\varepsilon > 0$ and $\varepsilon = 0$.

Conditions $\mathfrak{I}_{16}$, $\bar{\mathcal{K}}_7$, $\bar{\mathcal{K}}_8$, and $\mathcal{A}_{76}$ remain with no changes. These conditions should be required to hold.

Since $\kappa_\varepsilon(t), t \geq 0$ is a non-negative and non-decreasing process for $\varepsilon > 0$ and $\varepsilon = 0$, condition $\mathfrak{I}_{17}$ holds. By Remark 4.4.2, conditions $\mathfrak{I}_{17}$, $\mathfrak{I}_{23}$, and $\mathfrak{N}_2$ imply that $\mathfrak{R}_1$ holds. Conditions $\mathfrak{I}_{16}$ and $\mathfrak{I}_{23}$ also imply that $\nu_0(0) > 0$ with probability 1, that is, condition $\mathfrak{I}_{14}$ holds.

Theorem 4.4.4, applied to the max-processes with renewal stopping, takes in this case the following form.

Theorem 4.9.2. *Let conditions $\mathfrak{I}_{16}$, $\bar{\mathcal{K}}_7$, $\bar{\mathcal{K}}_8$, $\mathcal{A}_{76}$, $\mathcal{J}_{27}$, $\mathfrak{N}_2$, and $\mathfrak{I}_{23}$ hold for the max-processes $\boldsymbol{\beta}_\varepsilon(t), t \geq 0$. Then*

$$\zeta_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{J}} \zeta_0(t), t \geq 0 \text{ as } \varepsilon \to 0.$$

4.9.2. Max-processes with renewal stopping based on i.i.d. random variables. Let, for every $\varepsilon > 0$, $(\kappa_{\varepsilon,n}, \rho_{\varepsilon,n})$, $n = 1, 2, \ldots$ be a sequence of random vectors taking values in $[0, \infty) \times \mathbb{R}_1$. We assume that the following condition holds:

$\mathcal{T}_{11}$: $(\kappa_{\varepsilon,n}, \rho_{\varepsilon,n}), k = 1, 2, \ldots$ is (for every $\varepsilon > 0$) a sequence of i.i.d. random vectors.

The conditions that imply weak convergence of mixed sum-max processes were given in Section 4.8. These are conditions $\mathcal{S}_1 - \mathcal{S}_3$ and $\mathcal{S}_{11} - \mathcal{S}_{12}$.

Since the random variables $\kappa_{\varepsilon,n}$ are non-negative, these conditions can be transformed in the following way. Conditions $\mathcal{S}_1 - \mathcal{S}_3$ can be replaced by conditions $\mathcal{S}_4 - \mathcal{S}_5$. Condition $\mathcal{S}_{11}$ does not require any change. Condition $\mathcal{S}_{12}$ should be modified as follows:

$\mathcal{S}_{13}$: $n_\varepsilon \mathsf{P}\{\kappa_{\varepsilon,1} > u, \rho_{\varepsilon,1} > w\} \to \pi_{1,3}(u, w)$ as $\varepsilon \to 0$ for all $u > 0, w > \upsilon$, which are points of continuity of the limiting function $\pi_{1,3}(u, w)$.

Here, properties of the limiting function are: **(a)** the function $\pi_{1,3}(u, w)$ is non-negative, non-increasing, and right-continuous in every argument for $u > 0, w > \upsilon$ such that $\pi_{2,3}(v, \infty) = \pi_{2,3}(\infty, w) = 0$ for $v > 0, w > \upsilon$; **(b)** it defines, for every $w > \upsilon$, a measure on the σ-algebra of subsets of $\mathfrak{B}_1^+$ such that $\Pi_{1,3}^{(w)}((u_1, u_2]) = \pi_{1,3}(u_1, w) - \pi_{1,3}(u_2, w)$ for $0 < u_1 \leq u_2 < \infty$; **(c)** $\Pi_{1,3}^{(w)}(A)$ is a non-increasing and continuous from the right function in $w > \upsilon$ for every $A \in \mathfrak{B}_1^+$; **(d)** the following estimates are valid: $\Pi_{1,3}^{(w_1)}(A) - \Pi_{1,3}^{(w_2)}(A) \leq (\pi_3(w_1) - \pi_3(w_2)) \wedge \Pi_1(A)$, in particular, $\Pi_{1,3}^{(w_1)}(A) \leq \pi_3(w_1) \wedge \Pi_1(A)$, for $\upsilon < w_1 \leq w_2 < \infty$ and $A \in \mathfrak{B}_1^+$; **(e)** the function $\pi_3(w)$ possesses properties **(a)** – **(b)** listed in connection with condition $\mathcal{S}_{11}$ (in Subsection 4.7.2); **(f)** the measure $\Pi_1(A)$ possesses properties **(a)** – **(d)** listed in connection with conditions $\mathcal{S}_4 - \mathcal{S}_5$ (in Subsection 4.5.2).

For $w > \upsilon$ and for $w = \upsilon$ if $\upsilon > -\infty, \pi_3(\upsilon) < \infty$, we define a measure on the σ-algebra $\mathfrak{B}_1^+$ by the following formula:

$$\hat{\Pi}_{1,3}^{(w)}(A) = \Pi_1(A) - \Pi_{1,3}^{(w)}(A). \tag{4.9.1}$$

This measure plays the role of the jump measure in the Lévy-Khintchine representation of infinitely divisible characteristic functions, for $t \geq 0$,

$$\phi_{1,3}^{(w)}(t, y) = \exp\{t\psi_{1,3}^{(w)}(y)\},\ y \in \mathbb{R}_1, \tag{4.9.2}$$

where

$$\psi_{1,3}^{(w)}(y) = \mathrm{i}d^{(w)}y + \int_0^\infty (e^{\mathrm{i}ys} - 1 - \frac{\mathrm{i}ys}{1 + s^2})\hat{\Pi}_{1,3}^{(w)}(ds), \tag{4.9.3}$$

and

$$d^{(w)} = d - \int_0^\infty \frac{s}{1 + s^2}\Pi_{1,3}^{(w)}(ds). \tag{4.9.4}$$

We additionally define $\hat{\Pi}_{1,3}^{(w)}(A) \equiv 0$ and $\phi_{1,3}^{(w)}(t, y) = e^{itdy}$ for $w < \upsilon$ if $\upsilon > -\infty, \pi_3(\upsilon) < \infty$.

It follows from **(c)** and **(d)** that $d^{(w)}$ is a right-continuous function, and, therefore, the function $\phi_{1,3}^{(w)}(t, y)$ is right-continuous in w for every $y \in \mathbb{R}_1, t \geq 0$.

Also recall that the constants $d, d^{(w)}$ and the measures $\Pi_1(A), \Pi_{1,3}^{(w)}(A)$ in (4.9.1), (4.9.3), and (4.9.4) are determined by conditions $\mathcal{S}_4 - \mathcal{S}_5$ and $\mathcal{S}_{11}, \mathcal{S}_{13}$.

Let us also introduce a càdlàg homogeneous *mixed Markov process* $(\kappa_0(t), \rho_0(t))$, $t > 0$, such that its trajectories belong to the space $\mathbf{D}^{(1)}_{(0,\infty)} \times \mathbf{D}_0$ with probability 1 and the transition probabilities have the following *hybrid characteristic-distribution form*:

$$\begin{aligned} \mathsf{E}\left\{e^{iy(\kappa_0(t+s)-\kappa_0(s))} \cdot \chi(\rho_0(t+s) \le w) \,|\, \xi_0(s) = u', \rho_0(s) = w'\right\} \\ = \chi(w' \le w) e^{-t\pi_3(w)} \phi^{(w)}_{1,3}(t, y). \end{aligned} \tag{4.9.5}$$

It is worth remarking that the second component, $\rho_0(t), t > 0$, of this limiting process is an extremal process, while the first one, $\kappa_0(t), t > 0$, is a càdlàg non-negative homogeneous process with independent increments and the characteristics determined by the second component.

As follows from the remarks in Subsection 4.7.2, $\mathsf{P}\{(\kappa_0(t), \rho_0(t) \in [0,\infty)\times\Upsilon, t > 0\} = 1$, where the interval Υ was defined in this subsection. This is consistent with formula (4.9.5).

As it follows from Theorem 4.8.1, conditions $\mathcal{S}_4 - \mathcal{S}_5$ and $\mathcal{S}_{11}$, $\mathcal{S}_{13}$ imply the following condition:

$\mathcal{A}_{77}$: $(\kappa_\varepsilon(t), \rho_\varepsilon(t)), t > 0 \Rightarrow (\kappa_0(t), \rho_0(t)), t > 0$ as $\varepsilon \to 0$, where $(\kappa_0(t), \rho_0(t))$, $t > 0$ is a càdlàg homogeneous Markov process described in (4.9.5).

Also, by Theorem 4.8.2, the same conditions $\mathcal{S}_4 - \mathcal{S}_5$ and $\mathcal{S}_{11}$, $\mathcal{S}_{13}$, without any additional assumptions, also imply that

$$(\kappa_\varepsilon(t), \rho_\varepsilon(t)), t > 0 \xrightarrow{\mathbf{J}} (\kappa_0(t), \rho_0(t)), t > 0 \text{ as } \varepsilon \to 0. \tag{4.9.6}$$

Let us first consider the case when condition $\mathcal{I}_{21}$, introduced in Subsection 4.5.4, holds. This condition imply condition $\mathcal{I}_9$. Therefore, the process $\nu_0(t), t \ge 0$ is an a.s. continuous process.

It is obvious in this case that $\kappa_0(0) = \kappa_0(0+0) = 0$ with probability 1, i.e., condition $\mathcal{I}_{23}$ holds. So, $\nu_0(0) = 0$ with probability 1 and $\nu_0(t) > 0$ with probability 1 for every $t > 0$.

The next two theorems are from Silvestrov and Teugels (2001).

Theorem 4.9.3. *Let conditions* $\mathcal{T}_{11}$, $\mathcal{S}_4$, $\mathcal{S}_5$, $\mathcal{S}_{11}$, $\mathcal{S}_{13}$, *and* $\mathcal{I}_{21}$ *hold. Then*

$$\zeta_\varepsilon(t) = \rho_\varepsilon(\nu_\varepsilon(t)), t > 0 \xrightarrow{\mathbf{J}} \zeta_0(t) = \rho_0(\nu_0(t)), t > 0 \text{ as } \varepsilon \to 0.$$

Proof of Theorem 4.9.3. Condition $\mathcal{A}_{77}$ implies that condition $\mathcal{A}_{76}$ holds. Also, the relation of $\mathbf{J}$-convergence (4.9.6) implies that condition $\mathcal{J}_{27}$ holds. Conditions $\mathcal{S}_4 - \mathcal{S}_5$ and $\mathcal{I}_{21}$ also imply that $\mathsf{P}\{\kappa_{\varepsilon,1} > 0\} > 0$ for all ε small enough. Without loss of generality, one can assume that this holds for all $\varepsilon \ge 0$ and, therefore, $\kappa_\varepsilon(t) \xrightarrow{\mathsf{P}} \infty$ as $t \to \infty$, i.e., condition $\bar{\mathcal{K}}_5$ holds. Also, condition $\mathcal{I}_{21}$ implies that condition $\mathcal{I}_9$ holds. Finally, as was pointed out above, condition $\mathcal{I}_{23}$ also holds. Therefore, Theorem 4.9.1 can be used and this gives the statement of Theorem 4.9.3. □

Let Y_0 be the set of points of stochastic continuity of the limiting process $\zeta_0(t) = \rho_0(\nu_0(t)), t > 0$. This set is $(0, \infty)$ except for at most countable set. It follows from Lemma 1.6.5 that the processes $\zeta_\varepsilon(t), t \geq 0$ weakly converge on the set Y_0.

The structure of the set Y_0 needs a special study, as well as the question whether condition $\mathfrak{I}_{21}$ implies that $Y_0 = [0, \infty)$ or not.

Let us also consider the case when condition $\mathfrak{I}_{22}$ holds, i.e., the limiting renewal stopping process $\nu_0(t), t \geq 0$ is a step càdlàg process.

Theorem 4.9.4. *Let conditions $\mathcal{T}_{11}$, $\mathcal{S}_4 - \mathcal{S}_6$, $\mathcal{S}_{11}$, $\mathcal{S}_{13}$, and $\mathfrak{I}_{22}$ hold. Then*

$$\zeta_\varepsilon(t), t \geq 0 \stackrel{\mathbf{J}}{\longrightarrow} \zeta_0(t), t \geq 0 \text{ as } \varepsilon \to 0.$$

Proof of Theorems 4.9.4. We use Theorem 4.9.2. Condition $\mathcal{A}_{77}$ implies that condition $\mathcal{A}_{76}$ holds. Also, the relation of $\mathbf{J}$-convergence (4.9.6) implies that condition $\mathcal{J}_{27}$ holds. The step processes $\kappa_\varepsilon(t), t \geq 0$ have the structure described in the proof of Lemma 4.5.3. It is readily seen that relations **(d)** - **(l)** given in this proof imply that conditions $\mathfrak{I}_{16}$, $\bar{\mathfrak{K}}_7$, $\bar{\mathfrak{K}}_8$ hold (at least for all ε small enough) and that also condition $\mathfrak{N}_2$ holds. Therefore, Theorem 4.9.2 can be applied and this yields the statement of the Theorem 4.9.4. □

Let Y_0 be the set of all points of stochastic continuity of the process $\zeta_0(t), t > 0$. Recall that the set V_0, which is the set of points of stochastic continuity of the process $\nu_0(t), t > 0$ coincides with $(0, \infty)$ except for at most a countable set. Actually, this set was described in Subsection 4.5.3. Obviously, the process $\zeta_0(t) = \xi_0(\nu_0(t)), t > 0$ is stochastically continuous at points of the set V_0, i.e., $V_0 \subseteq Y_0$. It follows from Lemma 1.6.5 that the processes $\zeta_\varepsilon(t), t > 0$ weakly converge on the set Y_0.

The structure of the set Y_0 needs a special study. In particular, it would be interesting to verify whether $Y_0 = V_0$ if condition $\mathfrak{I}_{22}$ holds and the external limiting process $\rho_0(t), t \geq 0$ does not degenerate to a constant.

4.9.3. Extremes for regenerative processes. One of natural applications of the results concerning max-processes with renewal stopping relates to extremes for regenerative processes. Let, for every $\varepsilon > 0$, $\eta_\varepsilon(t), t \geq 0$ be a real-valued regenerative càdlàg process with regenerative moments $0 = \tau_{\varepsilon,0} \leq \tau_{\varepsilon,1} \leq \ldots$.

We are interested in limit theorems for the extremal processes

$$\varrho_\varepsilon(t) = \sup_{s \leq tn_\varepsilon} \eta_\varepsilon(s), \ t > 0.$$

Let us introduce some functionals which play an essential role in further consideration. First of all, $\kappa_{\varepsilon,k} = \tau_{\varepsilon,k} - \tau_{\varepsilon,k-1}$ is the time between two successive regenerations, $\kappa_\varepsilon(t) = \tau_{\varepsilon,[tn_\varepsilon]} = \sum_{k \leq tn_\varepsilon} \kappa_{\varepsilon,k}$ is the moment of the last regeneration before the moment tn_ε, and $\nu_\varepsilon(t) = \sup(s : \kappa_\varepsilon(s) \leq t)$. By the definition, $n_\varepsilon \nu_\varepsilon(t) - 1$ is the number of regenerations in the interval $[0, tn_\varepsilon]$. Further, $\rho_{\varepsilon,k} = \sup_{\tau_{\varepsilon,k-1} \leq s < \tau_{\varepsilon,k}} \eta_\varepsilon(s)$ is the maximum of the process $\eta_\varepsilon(t)$ in the corresponding regeneration period, and $\rho_\varepsilon(t) = \max_{k \leq tn_\varepsilon} \rho_{\varepsilon,k}$ is the maximum of the process $\eta_\varepsilon(t)$ on the interval $[0, \kappa_\varepsilon(t))$. Finally, $\varsigma_\varepsilon(t) = \sup_{\kappa_\varepsilon(t) \leq s \leq tn_\varepsilon} \eta_\varepsilon(s)$ is the maximum of the process $\eta_\varepsilon(t)$ on the interval $[\kappa_\varepsilon(t), tn_\varepsilon]$. Here $k = 0, 1, \ldots$ and $t > 0$.

Note that $(\kappa_{\varepsilon,k}, \rho_{\varepsilon,k}), k \geq 1$ is a sequence of i.i.d. random vectors, since $\eta_\varepsilon(t), t \geq 0$ is a regenerative process.

Applications of results about renewal type extremal processes to the processes $\varrho_\varepsilon(t), t > 0$ are based on the following representation:

$$\varrho_\varepsilon(t) = \max(\max_{k \leq n_\varepsilon \nu_\varepsilon(t) - 1} \rho_{\varepsilon,k}, \ \varsigma_\varepsilon(t)), t > 0. \tag{4.9.7}$$

From this representation, it follows that the extremal process $\varrho_\varepsilon(t), t > 0$ can be approximated from below and above by two max-processes with renewal type stopping,

$$\zeta'_\varepsilon(t) \leq \varrho_\varepsilon(t) \leq \zeta_\varepsilon(t) \text{ for } t > 0, \tag{4.9.8}$$

where

$$\zeta_\varepsilon(t) = \max_{k \leq n_\varepsilon \nu_\varepsilon(t)} \rho_{\varepsilon,k}, \ \zeta'_\varepsilon(t) = \max_{k \leq n_\varepsilon \nu_\varepsilon(t) - 1} \rho_{\varepsilon,k}, \ t > 0. \tag{4.9.9}$$

By the definition, $\zeta_\varepsilon(t) = \rho_\varepsilon(\nu_\varepsilon(t)), t > 0$ and $\zeta'_\varepsilon(t) = \rho_\varepsilon(\nu_\varepsilon(t) - 1/n_\varepsilon), t > 0$.

The process $\zeta_\varepsilon(t), t > 0$ is a max-process with renewal stopping. Such processes were considered in Subsections 4.9.1. and 4.9.2. In the theorem formulated below, we impose conditions on the distributions of the random vectors $(\kappa_{\varepsilon,1}, \rho_{\varepsilon,1})$. We identify these random vectors with the ones defined in Subsection 4.9.2 and the corresponding limiting process with those in Theorem 4.9.3.

The process $\zeta'_\varepsilon(t)$ is, however, a slight modification of a renewal type extremal process with the internal stopping process $\nu_\varepsilon(t)$ replaced by the process $\nu_\varepsilon(t) - 1/n_\varepsilon$. Under the conditions of Theorem 4.9.5 formulated below, both approximation processes, as we shall see, converge weakly to the same limiting process and so do the extremal processes $\varrho_\varepsilon(t), t > 0$.

Let Y_0 be the set of points of stochastic continuity of the process $\zeta_0(t) = \rho_0(\nu_0(t)$, $t > 0$.

Theorem 4.9.5. *Let conditions $\mathcal{S}_4$, $\mathcal{S}_5$, $\mathcal{S}_{11}$, $\mathcal{S}_{13}$, and $\mathfrak{I}_{21}$ hold. Then*

$$\varrho_\varepsilon(t), t \in Y_0 \Rightarrow \varrho_0(t) = \rho_0(\nu_0(t)), t \in Y_0 \text{ as } \varepsilon \to 0. \tag{4.9.10}$$

Proof of Theorem 4.9.5. Theorem 4.9.3 implies that

$$\zeta_\varepsilon(t) = \rho_\varepsilon(\nu_\varepsilon(t)), t > 0 \xrightarrow{\mathbf{J}} \zeta_0(t) = \rho_0(\nu_0(t)), t > 0 \text{ as } \varepsilon \to 0. \tag{4.9.11}$$

A similar relation can be obtained for the processes $\zeta'_\varepsilon(t) = \rho_\varepsilon(\nu_\varepsilon(t) - 1/n_\varepsilon), t > 0$. Indeed, as was pointed out in the proof of Theorem 4.9.3, condition $\mathcal{A}_{77}$ and $\mathfrak{I}_{21}$ imply that condition $\mathcal{A}_{76}$, which is a variant of condition $\mathcal{A}_{58}$, and condition $\bar{\mathcal{K}}_5$ hold. Also, relation (4.9.11) implies that condition $\mathcal{J}_{27}$, which a version of condition $\mathcal{J}_{24}$, holds.

In this case, **(a)** $\nu_\varepsilon(0) \stackrel{\mathsf{P}}{\longrightarrow} \nu_0(0) = 0$ as $\varepsilon \to 0$. Note also that $(\nu_0(s), \rho_0(t)), t > 0$ is a stochastically continuous process in this case. So, it follows from **(a)** and Lemma 4.3.1 that

$$\begin{aligned}&(\nu_\varepsilon(s), \rho_\varepsilon(t)), (s, t) \in [0, \infty) \times (0, \infty)\\ &\qquad \Rightarrow (\nu_0(s), \rho_0(t)), (s, t) \in [0, \infty) \times (0, \infty) \text{ as } \varepsilon \to 0.\end{aligned} \tag{4.9.12}$$

It is obvious that the process $\nu_\varepsilon(t), t > 0$ can be replaced by the process $\nu_\varepsilon(t) - 1/n_\varepsilon, t > 0$, in (4.9.12).

As was pointed out in the proofs of Theorems 4.3.10 and 4.3.11, conditions $\mathcal{A}_{58}$, $\bar{\mathcal{K}}_5$, and $\mathcal{J}_{24}$ imply that condition $\mathcal{F}_6$ holds for the processes $\rho_\varepsilon(t), t \geq 0$, which replace, in this case, the processes $\xi_\varepsilon(t), t \geq 0$, and the processes $\nu_\varepsilon(t), t \geq 0$. Obviously, condition $\mathcal{F}_6$ holds also for the processes $\rho_\varepsilon(t), t \geq 0$ and the processes $\nu_\varepsilon(t) - 1/n_\varepsilon, t \geq 0$, with the same set W_0''.

Relation of weak convergence (4.9.11) was obtained by applying Theorem 3.4.3 to the compositions $\zeta_\varepsilon(t) = \rho_\varepsilon(\nu_\varepsilon(t)), t > 0$. But it follows from the remarks made above that Theorem 3.4.3 can also be applied to the compositions $\zeta'_\varepsilon(t) = \rho_\varepsilon(\nu_\varepsilon(t) - 1/n_\varepsilon), t > 0$. So, the following relation holds:

$$\zeta'_\varepsilon(t) = \rho_\varepsilon(\nu_\varepsilon(t) - 1/n_\varepsilon), t > 0 \stackrel{\mathbf{J}}{\longrightarrow} \zeta_0(t) = \rho_0(\nu_0(t)), t > 0 \text{ as } \varepsilon \to 0. \tag{4.9.13}$$

Relations (4.9.11) and (4.9.13) imply weak convergence of the processes $\zeta_\varepsilon(t)$ and $\zeta'_\varepsilon(t)$ on the set Y_0, i.e.,

$$\zeta_\varepsilon(t), t \in Y_0 \Rightarrow \zeta_0(t), t \in Y_0 \text{ as } \varepsilon \to 0. \tag{4.9.14}$$

and

$$\zeta'_\varepsilon(t), t \in Y_0 \Rightarrow \zeta_0(t), t \in Y_0 \text{ as } \varepsilon \to 0. \tag{4.9.15}$$

Inequality (4.9.8), relations (4.9.14) and (4.9.15), and Lemma 1.2.6 imply the asymptotic relation given in Theorem 4.9.5. □

The question about the **J**-convergence of the processes $\varrho_\varepsilon(t), t > 0$ is more complicated. The procedure used to prove relations (4.9.14) and (4.9.15) also allows to prove that

$$(\zeta_\varepsilon(t), \zeta'_\varepsilon(t)), t \in Y_0 \Rightarrow (\zeta_0(t), \zeta_0(t)), t \in Y_0 \text{ as } \varepsilon \to 0. \tag{4.9.16}$$

From (4.9.16), it follows that, for $t \in Y_0$,

$$0 \leq \max(\varrho_\varepsilon(t) - \zeta'_\varepsilon(t), \zeta_\varepsilon(t) - \varrho_\varepsilon(t)) \leq \zeta_\varepsilon(t) - \zeta'_\varepsilon(t) \stackrel{\mathsf{P}}{\longrightarrow} 0 \text{ as } \varepsilon \to 0. \tag{4.9.17}$$

However, (4.9.11), (4.9.13), and (4.9.17) do not necessarily guarantee **J**-convergence of the processes $\varrho_\varepsilon(t), t > 0$.

For example, consider a regenerative process with regeneration moments $\tau_{\varepsilon,k} = k/n_\varepsilon, k = 0, 1, \ldots$ and $\eta_\varepsilon(t) = \xi_k \cdot (t - (k-1)/n_\varepsilon)$, for $(k-1)/n_\varepsilon \leq t < k/n_\varepsilon, k = 1, 2, \ldots$. Here, $\xi_k, k \geq 1$ is a sequence of i.i.d. random variables with the distribution function $G(u) = \chi_{[1,\infty)}(u)(1 - 1/u)$.

Conditions $\mathbf{\mathcal{S}_4} - \mathbf{\mathcal{S}_5}$ and $\mathbf{\mathcal{S}_{11}}$, $\mathbf{\mathcal{S}_{13}}$ hold. The process $\kappa_0(t) = \nu_0(t) = t, t > 0$, and $\rho_0(t), t > 0$ is an extremal process with the function $\pi_3(u) = u^{-1}$.

In this case, the processes $\zeta'_\varepsilon(t) = \max_{k \leq [tn_\varepsilon]} \xi_k/n_\varepsilon$. Let us take some $\delta > 0$. It is not difficult to show that $\underline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\sup_{0\leq t\leq T} \Delta_t(\zeta'_\varepsilon(\cdot)) > \delta\} > 0$ for $T > 0$. For any point $t_{\varepsilon,\delta}$ where the process $\zeta'_\varepsilon(t)$ has a jump greater than or equal to δ, there exist points $t_{\varepsilon,\delta} - 1/n_\varepsilon \leq t' < t < t'' \leq t_{\varepsilon,\delta}$ such that $\varrho_\varepsilon(t) - \varrho_\varepsilon(t') \geq \delta/2$ and $\varrho_\varepsilon(t'') - \varrho_\varepsilon(t) \geq \delta/2$. This implies that the processes $\varrho_\varepsilon(t), t > 0$ can not be compact in the topology **J**.

In connection with Theorem 4.9.5, we would like to mentioned that some related results concerning exceedances of ergodic regenerative processes with discrete time can be found in papers of Serfozo (1980), Rootzén (1988), and Leadbetter and Rootzén (1988). For the case of asymptotically independent external processes and internal stopping processes, Theorem 4.9.5 was proved in Silvestrov and Teugels (1998a).

4.10 Shock processes

In this section we consider the class of so-called shock processes. This class of processes is also an example of generalised exceeding processes considered in Sections 4.3 and 4.4.

4.10.1. General shock processes. Let, for every $\varepsilon > 0$, $(\xi_{\varepsilon,n}, \rho_{\varepsilon,n})$, $n = 1, 2, \ldots$ be a sequence of random vectors taking values in $\mathbb{R}_1 \times [0, \infty)$. In what follows, we let $n_\varepsilon > 0$ be a non-random function such that $n_\varepsilon \to \infty$ as $\varepsilon \to 0$.

We introduce a mixed sum-max process,

$$\gamma_\varepsilon(t) = (\xi_\varepsilon(t), \rho_\varepsilon(t)) = (\sum_{k \leq tn_\varepsilon} \xi_{\varepsilon,k}, \max_{k \leq 1 \vee tn_\varepsilon} \rho_{\varepsilon,k}), \ t \geq 0.$$

Let us now introduce a *max-renewal process*,

$$\nu_\varepsilon(t) = \sup(s : \rho_\varepsilon(s) \leq t), t \geq 0,$$

and a process which can be called a *shock process*,

$$\zeta_\varepsilon(t) = \xi_\varepsilon(\nu_\varepsilon(t)), t \geq 0.$$

Shock processes give another example of generalised exceeding processes. However, in this case the sum-process $\xi_\varepsilon(t), t \geq 0$ plays the role of an external process, while the max-process $\rho_\varepsilon(t), t \geq 0$ is used to construct the internal stopping process $\nu_\varepsilon(t), t \geq 0$.

Therefore, the basic process $(\kappa_\varepsilon(t), \xi_\varepsilon(t))$, $t \geq 0$ should be replaced in this case by the process $(\rho_\varepsilon(t), \xi_\varepsilon(t))$, $t \geq 0$. In order to keep notations consistent with those introduced in Section 4.8, we exchange components of this process and use the notation $\boldsymbol{\gamma}_\varepsilon(t) = (\xi_\varepsilon(t), \rho_\varepsilon(t))$, $t \geq 0$.

Let us note that the assumption that the random variables $\rho_{\varepsilon,n}, n = 1, 2, \ldots$ are non-negative, is not important. Indeed, if these random variables are real-valued, they can be replaced by the non-negative random variables $\rho^+_{\varepsilon,n} = \max(\rho_{\varepsilon,n}, 0)$, $n = 1, 2, \ldots$. Obviously, the process $\rho^+_\varepsilon(t) = \max_{k \leq 1 \vee t m_\varepsilon} \rho^+_{\varepsilon,k} = \max(\rho_\varepsilon(t), 0), t \geq 0$. But the process $\nu_\varepsilon(t) = \sup(s : \rho_\varepsilon(s) \leq t) = \sup(s : \rho^+_\varepsilon(s) \leq t), t \geq 0$.

By the definition, the max-process $\rho_\varepsilon(t), t > 0$ is a non-decreasing process. Moreover, it is a step càdlàg process with only positive jumps. Let $s > 0$ and $s = \tau^{(s)}_{\varepsilon,0} < \tau^{(s)}_{\varepsilon,1} < \tau^{(s)}_{\varepsilon,2} < \ldots$ be a sequence of successive moments of positive jumps of this process on the interval $[s, \infty)$. This sequence is a.s. strictly monotone. Then $\rho_\varepsilon(s) = \rho_\varepsilon(\tau^{(s)}_{\varepsilon,0}) < \rho_\varepsilon(\tau^{(s)}_{\varepsilon,1}) < \rho_\varepsilon(\tau^{(s)}_{\varepsilon,n}) < \ldots$ is a sequence of values of this process at the successive moments of positive jumps. This sequence is also a.s. strictly monotone. It follows from the definition of these sequences that $\rho_\varepsilon(t) = \rho_\varepsilon(\tau^{(s)}_{\varepsilon,k-1})$ for $t \in [\tau^{(s)}_{\varepsilon k-1}, \tau^{(s)}_{\varepsilon k}), k \geq 1$.

In Subsection 4.10.2, we consider the basic case when the shock processes are constructed from a sequences of i.i.d. random variables. Let us adjust the conditions to this situation. In particular, we restrict the consideration to the case when the corresponding limiting extremal process is an a.s. step càdlàg process with only positive jumps.

The weak convergence condition $\mathcal{A}_{56}$ takes, in this case, the following form:

$\mathcal{A}_{78}$: $(\xi_\varepsilon(t), \rho_\varepsilon(t)), t \in U \times V \Rightarrow (\xi_0(t), \rho_0(t)), t \in U \times V$ as $\varepsilon \to 0$, where (a) U is a subset of $[0, \infty)$ that is dense in this interval and contains the point 0, (b) V is a subset of $(0, \infty)$, dense in this interval, (c) $\xi_0(t), t \geq 0$ is an a.s. càdlàg process, (d) $\rho_0(t), t > 0$ is an a.s. step càdlàg process with only positive jumps and a finite number of jumps in any finite sub-interval of the interval $(0, \infty)$.

The assumption (d) in this condition means that, for every $s > 0$, **(a)** $s = \tau^{(s)}_{0,0} < \tau^{(s)}_{0,1} < \tau^{(s)}_{0,2} < \ldots$ is an a.s. strictly monotone sequence of successive moments of positive jumps of the process $\rho_0(t), t \in [s, \infty)$, **(b)** $\rho_0(s) = \rho_0(\tau^{(s)}_{0,0}) < \rho_0(\tau^{(s)}_{0,1}) < \rho_0(\tau^{(s)}_{0,n}) < \ldots$ is an a.s. strictly monotone sequence of values of this process at the successive moments of positive jumps, **(c)** $\rho_0(t) = \rho_0(\tau^{(s)}_{0,k-1})$ for $t \in [\tau^{(s)}_{0k-1}, \tau^{(s)}_{0k}), k \geq 1$.

The process $\rho_\varepsilon(t), t \geq 0$ should replace the process $\kappa_\varepsilon(t), t \geq 0$ in conditions $\mathfrak{I}_{18}$, $\bar{\mathcal{K}}_9$, $\bar{\mathcal{K}}_{10}$. These conditions should be required to hold.

Condition $\mathcal{J}_{24}$ takes, in this case, the following form:

$\mathcal{J}_{28}$: $\lim_{c \to 0} \overline{\lim}_{\varepsilon \to 0} \mathsf{P}\{\Delta_J(\boldsymbol{\gamma}_\varepsilon(\cdot), c, T', T'') > \delta\} = 0, \ \delta > 0, 0 < T' < T'' < \infty.$

It follows from the definition of the sum-processes $\xi_\varepsilon(t), t \geq 0$ that the random variable $\xi_\varepsilon(0) = 0$ with probability 1 for every $\varepsilon > 0$. Due to condition $\mathcal{A}_{78}$, we also have $\xi_0(0) = 0$ with probability 1.

Condition $\mathfrak{O}_{13}$ takes, in this case, the following form:

$\mathfrak{O}_{15}$: $\lim_{0<c\to 0} \overline{\lim}_{\varepsilon\to 0} \mathsf{P}\{\sup_{0\leq t\leq c} |\xi_\varepsilon(t)| > \delta\} = 0, \ \delta > 0.$

Note that it is assumed that $0 < \varepsilon \to 0$. However, under $\mathcal{A}_{78}$, condition $\mathcal{J}_{28}$ is equivalent to $\mathcal{J}_{24}$ and condition $\mathfrak{O}_{15}$ is equivalent to $\mathfrak{O}_{13}$. Indeed, the limiting processes $\gamma_0(t), t > 0$ and $\xi_0(t), t \geq 0$ are càdlàg processes and, therefore, the asymptotic relations in $\mathcal{J}_{28}$ and $\mathfrak{O}_{15}$ automatically hold for $\varepsilon = 0$.

The process $\rho_\varepsilon(t), t \geq 0$ should replace the process $\kappa_\varepsilon(t), t \geq 0$ in conditions $\mathfrak{N}_3$, $\mathfrak{R}_3$ and $\mathfrak{R}_4$. These conditions should be required to hold. Moreover, since $\rho_\varepsilon(t), t \geq 0$ is a non-negative process for every $\varepsilon \geq 0$, condition $\mathfrak{I}_{17}$ holds in this case. Conditions $\mathfrak{I}_{17}$ and $\mathfrak{R}_4$ imply that condition $\mathfrak{R}_3$ holds.

Remarks made above let us reformulate Theorem 4.4.6 in the following form.

Theorem 4.10.1. *Let conditions* $\mathfrak{I}_{18}$, $\bar{\mathcal{K}}_9$, $\bar{\mathcal{K}}_{10}$, $\mathcal{A}_{78}$, $\mathcal{J}_{28}$, $\mathfrak{O}_{15}$, $\mathfrak{N}_3$, *and* $\mathfrak{R}_4$ *hold. Then*

$$\zeta_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{J}} \zeta_0(t), t \geq 0 \text{ as } \varepsilon \to 0.$$

4.10.2. Shock processes based on i.i.d. random variables. Let, for every $\varepsilon > 0$, $(\xi_{\varepsilon,n}, \rho_{\varepsilon,n}), n = 1, 2, \ldots$ be a sequence of random vectors taking values in $\mathbb{R}_1 \times [0, \infty)$.

We assume that condition $\mathcal{T}_9$ is satisfied. This means that $(\xi_{\varepsilon,n}, \rho_{\varepsilon,n}), k = 1, 2, \ldots$ is a sequence of i.i.d. random vectors.

We also assume that conditions $\mathcal{S}_1 - \mathcal{S}_3$ and $\mathcal{S}_{11} - \mathcal{S}_{12}$ hold. According to Theorem 4.8.1, these conditions imply the following relation of weak convergence:

$$\gamma_\varepsilon(t) = (\xi_\varepsilon(t), \rho_\varepsilon(t)), t > 0 \Rightarrow \gamma_0(t) = (\xi_0(t), \rho_0(t)), t > 0 \text{ as } \varepsilon \to 0, \tag{4.10.1}$$

where the process $(\xi_0(t), \rho_0(t)), t > 0$ is described in Theorem 4.8.1.

Note that, in this case, the random variable $\xi_\varepsilon(0) = 0$ with probability 1 for every $\varepsilon > 0$. Also, since $\xi_0(t), t > 0$ is a homogeneous càdlàg process with independent increments, the random variables $\xi_0(t) \xrightarrow{\mathsf{P1}} 0$ as $0 < t \to 0$. Thus, we can always define $\xi_0(0) = 0$ and replace relation (4.10.1) by the following condition:

$\mathcal{A}_{79}$: $(\xi_\varepsilon(t), \rho_\varepsilon(s)), (t, s) \in [0, \infty) \times (0, \infty) \Rightarrow (\xi_0(t), \rho_0(t)), (t, s) \in [0, \infty) \times (0, \infty)$ as $\varepsilon \to 0$, where the process $(\xi_0(t), \rho_0(t)), t > 0$ is described in Theorem 4.8.1.

As was shown in Theorem 4.8.2, conditions $\mathcal{S}_1 - \mathcal{S}_3$ and $\mathcal{S}_{11} - \mathcal{S}_{12}$ imply also that

$$\gamma_\varepsilon(t), t > 0 \xrightarrow{\mathbf{J}} \gamma_0(t), t > 0 \text{ as } \varepsilon \to 0. \tag{4.10.2}$$

Relation (4.10.2) implies that the condition of **J**-compactness $\mathcal{J}_{28}$ holds.

Conditions $\mathcal{S}_1$ - $\mathcal{S}_3$ imply also that

$$\xi_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{J}} \xi_0(t), t \geq 0 \text{ as } \varepsilon \to 0. \tag{4.10.3}$$

By Remark 4.8.2, relation (4.10.3) implies that condition $\mathbf{O_{15}}$ holds.

Denote $G_\varepsilon(u) = \mathsf{P}\{\rho_{\varepsilon,1} \leq u\}$. This distribution function is concentrated on the interval $[0, \infty)$, i.e., $G_\varepsilon(u) = 0$ for $u < 0$.

To avoid the case when $\nu_\varepsilon(t), t \geq 0$ is an improper process, we assume that the following condition holds:

$\mathbf{J_{24}}$: $1 - G_\varepsilon(v) > 0$ for $v \geq 0$ and every $\varepsilon > 0$.

Under condition $\mathbf{J_{24}}$, the random variable $\nu_\varepsilon(t) < \infty$ with probability 1 for every $t \geq 0$ and $\varepsilon > 0$.

Let us define, for every $\varepsilon \geq 0$, the random variables $\kappa^{(s)}_{\varepsilon,n} = \tau^{(s)}_{\varepsilon,n} - \tau^{(s)}_{\varepsilon,n-1}, n = 0, 1, \ldots$. Here $\tau_{\varepsilon,-1} = s$ and, therefore, $\kappa_{\varepsilon,0} = 0$. For $\varepsilon > 0$, the sequence of random variables $(\kappa^{(s)}_{\varepsilon,n}, \rho_\varepsilon(\tau^{(s)}_{\varepsilon,n})), n = 0, 1, \ldots$ is a homogeneous Markov chain with the phase space $[0, \infty) \times [0, \infty)$, the initial distribution

$$\mathsf{P}\{\kappa^{(s)}_{\varepsilon,0} \leq t, \rho_\varepsilon(\tau^{(s)}_{\varepsilon,0}) \leq v\} = \chi(0 \leq t) G_\varepsilon(v)^{[sn_\varepsilon]}, \tag{4.10.4}$$

and the transition probabilities

$$\begin{aligned} &\mathsf{P}\{\kappa^{(s)}_{\varepsilon,n+1} \leq t, \rho_\varepsilon(\tau^{(s)}_{\varepsilon,n+1}) \leq w \,|\, \kappa^{(s)}_{\varepsilon,n} = t', \rho_\varepsilon(\tau^{(s)}_{\varepsilon,n}) = v\} \\ &\quad = (1 - G_\varepsilon(v)^{[tn_\varepsilon]}) \chi(v \leq w)(1 - (1 - G_\varepsilon(w))/(1 - G_\varepsilon(v))). \end{aligned} \tag{4.10.5}$$

The corresponding limiting sequence $(\kappa^{(s)}_{0,n}, \rho_0(\tau^{(s)}_{0,n})), n = 0, 1, \ldots$ is also a homogeneous Markov chain with the phase space $[0, \infty) \times [0, \infty)$, the initial distribution

$$\mathsf{P}\{\kappa^{(s)}_{0,0} \leq t, \rho_0(\tau^{(s)}_{0,0}) \leq v\} = \chi(0 \leq t) e^{-\pi_3(v)s}, \tag{4.10.6}$$

and the transition probabilities

$$\begin{aligned} &\mathsf{P}\{\kappa^{(s)}_{0,n+1} \leq t, \rho_0(\tau^{(s)}_{0,n+1}) \leq w \,|\, \kappa^{(s)}_{0,n} = t', \rho_0(\tau^{(s)}_{0,n}) = v\} \\ &\quad = (1 - e^{-\pi_3(v)t}) \chi(v \leq w)(1 - \pi_3(w)/\pi_3(v)). \end{aligned} \tag{4.10.7}$$

Note that, because the random variables $\rho_{\varepsilon,k}, k \geq 1$ are non-negative for every $\varepsilon > 0$, the functional $\upsilon = \sup(w : \pi_3(w) = \infty) \geq 0$.

As follows from the remarks given in Subsection 4.7.2, **(a)** $\mathsf{P}\{(\kappa^{(s)}_{0,n}, \rho_0(\tau^{(s)}_{0,n})) \in [0, \infty) \times \Upsilon, n = 0, 1, \ldots\} = 1$, where the interval Υ was defined in this subsection. This is consistent with formulas (4.10.6) and (4.10.7). It follows from these formulas that, for every $s > 0$, the two-dimensional distribution function of the random variable $(\kappa^{(s)}_{0,n}, \rho_0(\tau^{(s)}_{0,n})) = (0, \rho_0(s))$ is concentrated on the set $[0, \infty) \times \Upsilon$ and the transition probability given in (4.10.7) is a two-dimensional distribution function in (t, w) concentrated on the set $[0, \infty) \times \Upsilon$ for every $(t', v) \in [0, \infty) \times \Upsilon$.

So, we need to use formula (4.10.7) only if **(b)** $(t', v), (t, w) \in [0, \infty) \times \Upsilon$. In this case, the expression in the right-hand side of (4.10.7) is well defined.

The way to deal with this formula when the expression in the right-hand side of (4.10.7) is not well-defined is described in Subsection 4.7.2.

To avoid the case when $\nu_0(t), t \geq 0$ is an improper process, we assume that the following condition holds:

$\mathfrak{I}_{25}$: $\pi_3(w) > 0$ for $w \geq 0$.

Under condition $\mathfrak{I}_{25}$, the random variable $\nu_0(t) < \infty$ with probability 1 for every $t \geq 0$.

Note also that, under condition $\mathfrak{I}_{26}$, the interval Υ is either (υ, ∞), if $\pi_3(\upsilon) = \infty$, or $[\upsilon, \infty)$, if $\pi_3(\upsilon) < \infty$.

Conditions $\mathfrak{I}_{24}$ and $\mathfrak{I}_{25}$ imply that conditions $\mathfrak{I}_{18}$, $\bar{\mathfrak{K}}_9$, and $\bar{\mathfrak{K}}_{10}$ formulated in Subsection 4.4.3 hold.

In order to simplify the consideration, we assume that the following two conditions hold:

$\mathfrak{S}_{14}$: $n_\varepsilon(1 - G_\varepsilon(\upsilon)) \to \pi_3(\upsilon)$ as $\varepsilon \to 0$,

and

$\mathfrak{I}_{26}$: $\pi_3(w)$ is a continuous function for $w > \upsilon$.

Note that is the case where $\pi_3(\upsilon) = \infty$, condition $\mathfrak{S}_{14}$ is implied by condition $\mathfrak{S}_{11}$.

Theorem 4.10.2. *Let conditions $\mathfrak{T}_9$, $\mathfrak{S}_1 - \mathfrak{S}_3$, $\mathfrak{S}_{11}$, $\mathfrak{S}_{12}$, $\mathfrak{S}_{14}$, and $\mathfrak{I}_{24} - \mathfrak{I}_{26}$ hold. Then*

$$\zeta_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{J}} \zeta_0(t), t \geq 0 \text{ as } \varepsilon \to 0.$$

Proof of Theorem 4.10.2. The only conditions $\mathfrak{N}_3$ and $\mathfrak{R}_4$ need to be proved.

Take an arbitrary $s > 0$. Note that the limiting distribution function $e^{-\pi_3(v)s}$ is continuous on the interval $[\upsilon, \infty)$, due to condition $\mathfrak{I}_{26}$. At the point υ, this distribution function is either continuous, if $\pi_3(\upsilon) = \infty$, or has the jump $e^{-\pi_3(\upsilon)s}$, if $\pi_3(\upsilon) < \infty$. However, conditions $\mathfrak{S}_{11}$, $\mathfrak{S}_{14}$, and $\mathfrak{I}_{26}$ imply that

$$G_\varepsilon(v)^{[sn_\varepsilon]} \to e^{-\pi_3(v)s} \text{ as } \varepsilon \to 0, \ v \geq 0. \tag{4.10.8}$$

It follows from relation (4.10.8) that

$$\mathsf{P}\{\rho_\varepsilon(\tau^{(s)}_{\varepsilon,0}) \in \overline{\Upsilon}\} \to 0 \text{ as } \varepsilon \to 0. \tag{4.10.9}$$

Due to monotonicity of the random sequence $\rho_\varepsilon(\tau^{(s)}_{\varepsilon,n}), n = 0, 1, \ldots$, **(c)** a relation similar to (4.10.9) also holds for the random variables $\rho_\varepsilon(\tau^{(s)}_{\varepsilon,n})$ for every $n = 0, 1, \ldots$.

Also, conditions $\mathfrak{S}_{11}$, $\mathfrak{S}_{14}$, and $\mathfrak{I}_{26}$ imply that, for all $v, w \in \Upsilon$,

$$\begin{aligned} &\chi(v \leq w)(1 - (1 - G_\varepsilon(w))/(1 - G_\varepsilon(v))) \\ &\quad \to \chi(v \leq w)(1 - \pi_3(w)/\pi_3(v))) \text{ as } \varepsilon \to 0. \end{aligned} \tag{4.10.10}$$

Take an arbitrary $w \in \Upsilon$. Relation (4.10.10) can be supplemented with the following facts. Firstly, by condition $\mathbf{S_{14}}$, **(d)** we have that relation (4.10.10) also holds for $u = \upsilon$. Secondly, by the definition, **(e)** the functions $\chi(v \leq w)(1 - (1 - G_\varepsilon(w))/(1 - G_\varepsilon(v)))$ and $\chi(v \leq w)(1 - \pi_3(w)/\pi_3(v)))$ are non-negative and no-increasing in v on the interval $[\upsilon, \infty)$. Thirdly, by condition $\mathbf{J_{26}}$, **(f)** the function $\chi(v \leq w)(1 - \pi_3(w)/\pi_3(v)))$ is continuous on the interval $[\upsilon, \infty)$.

Taking into account **(d)** – **(f)** and using relations (4.10.8), (4.10.9), and (4.10.10) we can get, in an obvious way, that for any $w_0, w_1 \in \Upsilon$,

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \mathsf{P}\{\rho_\varepsilon(\tau^{(s)}_{\varepsilon,0}) \leq w_0, \rho_\varepsilon(\tau^{(s)}_{\varepsilon,1}) \leq w_1\} \\
&\quad = \lim_{\varepsilon\to 0} \int_{[0,w_0]} \chi(v \leq w_1)\frac{G_\varepsilon(w_1) - G_\varepsilon(v)}{1 - G_\varepsilon(v)} d_v G_\varepsilon(v)^{[sn_\varepsilon]} \\
&\quad = \lim_{\varepsilon\to 0} \int_{\overline{\Upsilon}\cap[0,w_0]} \chi(v \leq w_1)\frac{G_\varepsilon(w_1) - G_\varepsilon(v)}{1 - G_\varepsilon(v)} d_v G_\varepsilon(v)^{[sn_\varepsilon]} \\
&\quad + \lim_{\varepsilon\to 0} \int_{\Upsilon\cap[0,w_0]} \chi(v \leq w_1)\frac{G_\varepsilon(w_1) - G_\varepsilon(v)}{1 - G_\varepsilon(v)} d_v G_\varepsilon(v)^{[sn_\varepsilon]} \\
&\quad = \int_{\Upsilon\cap[0,w_0]} \chi(v \leq w_1)\frac{\pi_3(v) - \pi_3(w_1)}{\pi_3(v)} d_v \exp^{-\pi_3(v)s} \\
&\quad = \int_{[0,w_0]} \chi(v \leq w_1)\frac{\pi_3(v) - \pi_3(w_1)}{\pi_3(v)} d_v \exp^{-\pi_3(v)s} \\
&\quad = \mathsf{P}\{\rho_0(\tau^{(s)}_{0,0}) \leq w_0, \rho_0(\tau^{(s)}_{0,1}) \leq w_1\}.
\end{aligned} \tag{4.10.11}$$

Note that, by (4.10.9) and **(c)**, the expression in the left-hand side of (4.10.11) tends to 0 if at least one of the points w_0, w_1 does not belong to the interval Υ. So, relation (4.10.11) imply that **(g)** the random vectors $(\rho_\varepsilon(\tau^{(s)}_{\varepsilon,0}), \rho_\varepsilon(\tau^{(s)}_{\varepsilon,1}))$ weakly converge as $\varepsilon \to 0$.

By continuing the asymptotic calculations given in (4.10.8) – (4.10.11) in an obvious iterative way, it can be shown that

$$\rho_\varepsilon(\tau^{(s)}_{\varepsilon,n}), n = 0, 1, \ldots \Rightarrow \rho_0(\tau^{(s)}_{0,n}), n = 0, 1, \ldots \text{ as } \varepsilon \to 0. \tag{4.10.12}$$

Due to (4.10.12), conditions $\mathbf{N_3}$ and $\mathbf{R_4}$ follow from formulas (4.10.7) and (4.10.6), and conditions $\mathbf{J_{25}}$, $\mathbf{J_{26}}$. The first one is that **(h)** the sequence of random variables $\rho_0(\tau^{(s)}_{0,n}), n = 0, 1, \ldots$ is strictly increasing with probability 1. The second one is that **(i)** $\mathsf{P}\{0 < \rho_0(\tau^{(s)}_{0,0}) \leq t\} = e^{-\pi_3(t)s} - e^{-\pi_3(0)s} \to 0$ as $0 < t \to 0$. □

It is useful to note that the finite-dimensional distributions of the stopping process $\nu_0(t) = \sup(s : \rho_0(s) \leq t), t \geq 0$ have, in this case, the following form,

$$\begin{aligned}
\mathsf{P}\{\nu_0(t_k) \geq s_k, k = 1, \ldots, n\} &= \mathsf{P}\{\rho_0(s_k) \leq t_k, k = 1, \ldots, n\} \\
&= \prod_{k=1}^{n} \exp\{-\pi_3(t_k)(s_k - s_{k-1})\},
\end{aligned} \tag{4.10.13}$$

for $0 = t_0 \le t_1 \le \ldots \le t_n$, $0 = s_0 \le s_1 \le \ldots \le s_n$, $n \ge 1$.

In particular, the random variable $\nu_0(t)$ has an exponential distribution with parameter $\pi_3(t)$, for $t \ge 0$.

4.10.3. Examples. Let us consider the scale-location model in which the random vectors $(\xi_k, \rho_k), k \ge 1$ do not depend on the series parameter $\varepsilon > 0$. We assume that the following condition holds:

$\mathcal{T}_{12}$: $(\xi_k, \rho_k), k = 1, 2, \ldots$ is a sequence of i.i.d. random variables taking values in $\mathbb{R}_1 \times [0, \infty)$.

Let us consider the case when the limiting process $\xi_0(t), t > 0$, in condition $\mathcal{A}_{79}$, is a non-random linear function or a standard Wiener process, while $\rho_0(t), t > 0$ is a stable extremal process.

Let $\alpha \in (0, 1)$ and denote by $\rho^{(\alpha)}(t), t > 0$, the extremal process described in Subsection 4.7.2 with the function $\pi_3(w) = \infty$ for $w \le 0$ and $w^{-\alpha}/\Gamma(1-\alpha)$ for $w > 0$. Let also $\nu^{(\alpha)}(t) = \sup(s : \rho^{(\alpha)}(s) \le t), t \ge 0$. Bellow, $0 < t_\varepsilon \to \infty$ as $\varepsilon \to 0$ and $n_\varepsilon = t_\varepsilon^\alpha/\Gamma(1-\alpha)h(t_\varepsilon)$, where $h(x)$ is a slowly varying function.

Let $\mathsf{E}\xi_1 = a$, $\mathsf{Var}\,\xi_1 = b^2$, and $w(t), t \ge 0$ be a standard Wiener process independent of the process $\nu^{(\alpha)}(t), t \ge 0$.

Theorem 4.10.3. *Let* **(α)** $\mathsf{E}|\xi_1| < \infty$, **(β)** $\mathsf{P}\{\rho_1 > x\} \sim x^{-\alpha}h(x)$ *as* $x \to \infty$. *Then*

$$\frac{\zeta(tt_\varepsilon)}{n_\varepsilon}, t \ge 0 \xrightarrow{\mathbf{J}} a\nu^{(\alpha)}(t), t \ge 0 \text{ as } \varepsilon \to 0.$$

Theorem 4.10.4. *Let* **(α)** $\mathsf{E}\xi_1^2 < \infty, \mathsf{E}\xi_1 = 0$, **(β)** $\mathsf{P}\{\rho_1 > x\} \sim x^{-\alpha}h(x)$ *as* $x \to \infty$. *Then*

$$\frac{\zeta(tt_\varepsilon)}{\sqrt{n_\varepsilon}}, t \ge 0 \xrightarrow{\mathbf{J}} bw(\nu^{(\alpha)}(t)), t \ge 0 \text{ as } \varepsilon \to 0.$$

Proof of Theorems 4.10.3 and 4.10.4. Let us first prove Theorem 4.10.3. In this case, we use the normalisation functions $n_\varepsilon, t_\varepsilon$ as defined above, and the random variables $\xi_{\varepsilon,k} = \xi_k/n_\varepsilon, k \ge 1$ and $\rho_{\varepsilon,k} = \rho_k/t_\varepsilon, k \ge 1$.

Condition **(β)** implies that

$$n_\varepsilon \mathsf{P}\{\rho_1 > wt_\varepsilon\} \to w^{-\alpha}/\Gamma(1-\alpha) \text{ as } \varepsilon \to 0,\ w > 0. \tag{4.10.14}$$

Since the random variable ρ_1 is non-negative, relation (4.10.14) implies that condition $\mathcal{S}_{11}$ holds for the random variables $\rho_{\varepsilon,1} = \rho_1/t_\varepsilon$ with the function $\pi_3(w)$ described above.

Obviously, condition **(β)** implies that condition $\mathcal{I}_{24}$ holds. Also, conditions $\mathcal{I}_{25}$ and $\mathcal{I}_{26}$ hold. This follows from the explicit formula for $\pi_3(w)$ given above.

Condition $\mathcal{S}_{13}$ is implied by condition $\mathcal{S}_{11}$, since in this case $\upsilon = 0$ and $\pi_3(0) = \infty$.

Also, the condition $\mathsf{E}|\xi_1| < \infty$ implies that

$$n_\varepsilon \mathsf{P}\{|\xi_1| > vn_\varepsilon\} \to 0 \text{ as } \varepsilon \to 0, \ v > 0. \tag{4.10.15}$$

Relation (4.10.15) implies that condition $\mathcal{S}_1$ holds for the random variables $\xi_{\varepsilon,1} = \xi_1/n_\varepsilon$ with the function $\pi_2(v) = 0$ for $v \neq 0$. Note that, in this case, conditions $\mathcal{S}_2$ and $\mathcal{S}_3$ hold with the limiting constants a and 0, respectively.

Relation (4.10.15) also implies that

$$n_\varepsilon \mathsf{P}\{|\xi_1| > vn_\varepsilon, \rho_1 > wt_\varepsilon\} \to 0 \text{ as } \varepsilon \to 0, \ v, w > 0. \tag{4.10.16}$$

Hence, condition $\mathcal{S}_{12}$ also holds with the function $\pi_{2,3}(v, w) = 0$ for $v \neq 0, w > 0$.

By applying Theorem 4.8.1, we get now that condition $\mathcal{A}_{79}$ holds and the corresponding limiting process is $(\xi_0(t), \rho_0(t)) = (at, \rho^{(\alpha)}(t)), t > 0$.

Now we can complete the proof of Theorem 4.10.3 by applying Theorem 4.10.2.

The proof of Theorem 4.10.4 is analogous. In this case, we use n_ε as defined above, and the random variables $\xi_{\varepsilon,k} = \xi_k/\sqrt{n_\varepsilon}, k \geq 1$ and $\rho_{\varepsilon,k} = \rho_k/t_\varepsilon, k \geq 1$.

The condition $\mathsf{E}\xi_1^2 < \infty, \mathsf{E}\xi_1 = 0$ implies that

$$n_\varepsilon \mathsf{P}\{|\xi_1| > v\sqrt{n_\varepsilon}\} \to 0 \text{ as } \varepsilon \to 0, \ v > 0. \tag{4.10.17}$$

Relation (4.10.17) implies that condition $\mathcal{S}_1$ holds for the random variables $\xi_{\varepsilon,1} = \xi_1/\sqrt{n_\varepsilon}$ with the function $\pi_2(v) = 0$ for $v \neq 0$. Note that, in this case, conditions $\mathcal{S}_2$ and $\mathcal{S}_3$ hold with the limiting constants 0 and b^2, respectively.

Relation (4.10.17) also implies that

$$n_\varepsilon \mathsf{P}\{|\xi_1| > v\sqrt{n_\varepsilon}, \rho_1 > wt_\varepsilon\} \to 0 \text{ as } \varepsilon \to 0, \ v, w > 0. \tag{4.10.18}$$

Hence, condition $\mathcal{S}_{12}$ also holds with the function $\pi_{2,3}(v, w) = 0$ for $v \neq 0, w > 0$.

By applying Theorem 4.8.1, we get now that condition $\mathcal{A}_{79}$ holds and the corresponding limiting process is $(\xi_0(t), \rho_0(t)) = (bw(t), \rho^{(\alpha)}(t)), t > 0$, where the processes $w(t), t > 0$ and $\rho^{(\alpha)}(t), t > 0$ are independent.

Now we can complete the proof of Theorem 4.10.4 by applying Theorem 4.10.2. □

It should be noted that the corresponding limiting processes in Theorems 4.10.3 and 4.10.4 are stochastically continuous. So, the set of weak convergence in both cases is the interval $[0, \infty)$.

4.10.4. References Theorem 4.2.1, which gives conditions of weak convergence for general sum-processes with random stopping, and Theorem 4.2.3, which specify Theorem 4.2.1 for sum-processes with random stopping based on i.i.d. random variables, are direct corollaries of the limit theorems from Silvestrov (1971b, 1972a, 1972e). As far as conditions of **J**-convergence are concerned, the corresponding results, given in Theorem 4.2.2 and 4.2.4, are direct corollaries of the corresponding limit theorems from

Silvestrov (1972b, 1972e). Conditions of **J**-convergence for general sum-processes with random stopping, similar to those given in Theorem 4.2.2, can also be derived from the results of Whitt (1973, 1980). In the case of scale-location model, the related results can be also found in Durret and Resnik (1977).

Theorems 4.2.3 and 4.2.4 give the most general and natural condition of weak and **J**-convergence of such sums in the case of arbitrary dependent external sum-processes based on i.i.d. random variables and random stopping indices. These theorems cover many results related to random sums, in particular, to a model with independent external sum-processes and stopping indices as well as to a model with normalised random indices converging in probability. References to the works related to these two models are given in the bibliographical remarks. The latter case originates from classical works of Anscombe (1952) and Rényi (1957, 1958, 1960). Theorems 4.2.5 and 4.2.6 give a general triangular array version of the corresponding results. Lemma 4.2.1 is a generalisation to the triangular array mode of the well known result of Rényi (1958, 1960). Theorems 4.2.7 and 4.2.8 are from Silvestrov (1971b, 1972a, 1972b, 1972e). These theorems imply Theorems 4.2.9, 4.2.10, 4.2.11, and 4.2.12 as direct corollaries. The latter four theorems can also be considered as corollaries of the results of Billigsley (1968), since in this case the limiting external process is continuous.

Theorems 4.3.1, 4.3.2, and 4.3.4, which give general conditions for weak and **J**-convergence of generalised exceeding times, and Lemma 4.3.1 are from Silvestrov (1972e, 1974). Theorems 4.3.3, 4.3.5, 4.3.6, and 4.3.7, as well as Lemma 4.3.2, are new results. Note also that the result analogous to those given in Theorems 4.3.6 and 4.3.7 but obtained by using another method (see, remarks in Subsection 4.3.11) can be found in Silvestrov (1974, 2000a). Theorems 4.4.1 and 4.4.2 may be considered in the context of the results on **J**-convergence of step càdlàg processes given in different variants in many works. Bibliographical remarks contain additional comments and references to the works on limit theorems for generalised exceeding processes and other renewal type processes.

Theorems 4.5.5 – 4.5.7, which give general conditions for weak and **J**-convergence of sum-processes with renewal stopping based on i.i.d. random variables, are from Silvestrov (1972e, 1974). Note that the proofs given here are new. The proofs given in Silvestrov (1974) are based on the use of Markov property of stopping moments (see remarks in Subsection 4.5.6). References to numerous works related to this model are given in the bibliographical remarks.

Theorems 4.6.1 and 4.6.2, which give conditions for **U**-convergence of general accumulation processes are from Silvestrov (1971c, 1972c, 1972d) as well as Theorems 4.6.3 and 4.6.4, which cover the case of accumulation processes with embedded regeneration cycles. Theorems 4.6.5, 4.6.6, 4.6.7, 4.6.8, and 4.6.9, which specify Theorems 4.6.3 and 4.6.4 in the case of scale-location models, are direct corollaries of these theorems. Results similar to Theorems 4.6.5, 4.6.6, 4.6.7 can also be found in Borovkov (1967a) and Serfozo (1975).

Theorems 4.7.1 and 4.7.2, which give conditions for weak and **J**-convergence of general max-processes with random stopping are direct corollaries of the general limit theorems from Silvestrov (1971b, 1972a, 1972b, 1972e). Theorems 4.7.3 and 4.7.4 are from Silvestrov and Teugels (1998a), as well as Theorems 4.7.5 and 4.7.6. The latter two theorems give a general triangular array version of the results that have been given in a variety of different forms by Berman (1962), Barndorff-Nielsen (1964), Mogyoródi (1967), Sen (1972), and Galambos (1973, 1978, 1992).

Theorems 4.8.1 and 4.8.2, which give general conditions for weak and **J**-convergence of mixed sum-max processes based on i.i.d. random variables are from Silvestrov and Teugels (2001). Some preceding results can be found in Breiman (1965), Chow and Teugels (1979), and Resnick (1986), Haas (1992), Silvestrov and Teugels (1998a). Theorems 4.8.3 and 4.8.4 are new. Additional references are given in Subsection 4.8.5 and the bibliographical remarks.

Theorems 4.9.1 and 4.9.2, which give conditions for weak and **J**-convergence of general max-processes with renewal stopping are new results. Theorems 4.9.3 – 4.9.4, which specify the results of theorems listed above for max-processes with renewal stopping based on i.i.d. random variables are from Silvestrov and Teugels (2001). The case of asymptotically independent external processes and internal stopping processes was considered in Silvestrov and Teugels (1998a).

Theorem 4.10.1, which gives conditions for weak and **J**-convergence of general shock processes, is a new result. Theorems 4.10.2 and 4.10.4 that specify the result of the theorems listed above for shock processes based on i.i.d. random variables are also new results. Results analogous to those in Theorem 4.10.3 can be found in the preceding works by Shanthrikumar and Sumita (1983), Gut and Hüsler (1999), and Gut (2001). Additional references are given in the bibliographical remarks.

Bibliographical remarks

This book is devoted to a study of weak limit theorems for randomly stopped stochastic processes and functional limit theorems for compositions of stochastic processes. Below, we give short bibliographical remarks concerned the works related to the subject of the book. Although we mainly study general limit theorems for arbitrary dependent external processes and internal stopping moments or processes, the remarks also cover works on limit theorems for random sums with independent summands and random indices, renewal models, and other models with random stopping. At the same time, they do not include works related to other types of asymptotic results beyond the framework of weak convergence. In particular, we do not mention results on the rate of convergence, moment convergence, large deviation asymptotics, etc. We also give references to works dealing with various applications of randomly stopped processes if they contain results related to limit theorems. The bibliography covers more than 750 works. It would probably have double size without the restrictions mentioned above.

Chapter 1. In this chapter, we give a survey of results concerning functional limit theorems for càdlàg processes.

A general framework for the development of the theory have been created by the classical works of Khintchine (1933), Lévy (1937, 1948), Gnedenko and Kolmogorov (1949), Doob (1953), and Loève (1955).

As far as functional limit theorems is concerned, the papers of Kolmogorov (1931, 1933), Erdös and Kac (1946a, 1946b), Doob (1949), Donsker (1951, 1952), Gikhman (1953), Prokhorov (1953), and Kolmogorov and Prokhorov (1954) are considered to be the precursors to the theory. In particular, Donsker (1951) gave the first functional limit theorem called by him, an invariance principle. This theorem establishes weak convergence of **U**-continuous functionals defined on sum-processes constructed from i.i.d. random variables to the same functionals defined on the limiting Wiener process. Kolmogorov and Prokhorov (1954) connected the invariance principle with theorems about weak convergence of measures in the functional metric space of continuous functions.

Prokhorov (1956) has completed the general theory of weak convergence of measures in metric spaces and gave general conditions for convergence of continuous stochastic processes in the uniform **U**-topology. Prokhorov's basic results concerning weak convergence in metric spaces are formulated in Theorems 1.3.4 and 1.3.5. The basic result, also due to Prokhorov, concerning convergence of continuous stochastic processes in the **U**-topology is given in Theorem 1.6.4. It is formulated in the extended form given by Skorokhod (1956) for the case where the limiting process is continuous but the pre-limiting processes are allowed to be discontinuous càdlàg processes.

Skorokhod (1955a, 1955b, 1956) has invented the main topology in the space **D** of càdlàg functions, the **J**-topology, and gave conditions for **J**-convergence of càdlàg processes. Skorokhod's original conditions for **J**-convergence of càdlàg functions are given in Theorem 1.4.3 and the main result concerning **J**-convergence of càdlàg processes is formulated in Theorem 1.6.2. Skorokhod's original approach was based on his representation Theorem 1.3.6 and the method of a single probability space, presented in Theorems 1.6.14 and 1.6.15. Kolmogorov (1956) has shown that the space **D** can be equipped with an appropriate metric that makes the **J**-convergence equivalent to the convergence in this metric. The metric d_J, which makes **D** a Polish space, was constructed by Billingsley (1968). These results permitted to consider limit theorems for càdlàg processes in the framework of the general theory of weak convergence of measures in metric spaces. This approach was used in the books of Parthasarathy (1967) and Billingsley (1968). One can find historical remarks concerning the early period of the development of the theory in the recent paper of Billingsley and Wishura (2000).

In the paper of Skorokhod (1956), some other topologies, called $\mathbf{J}_2$, $\mathbf{M}_1$, and **M**-topologies, were also defined. These topologies are not so widely used since, in most cases, càdlàg processes converge in the **J**-topology. Nevertheless, they are useful in some special cases. For example, the **M**-topology is often applied to extremal processes. The book of Whitt (2002) gives a detailed account of the corresponding results. Related references also include Puhalskii and Whitt (1997) and O'Brien (2000).

The original theory of functional theorems was developed in the case when the stochastic processes are defined on a finite interval. An extension of functional limit theorems to stochastic processes that are defined on the interval $[0, \infty)$ is needed in limit theorems for randomly stopped stochastic processes. This is due to the possibility for the random stopping moments to be stochastically unbounded random variables. This extension of the theory was given by Stone (1963) and Lindvall (1973). Relevant references also include Whitt (1970), Borovkov (1972b), and Grigelionis (1973).

To complete the picture, we would also like to mention some other directions of development of the general theory of functional limit theorems. For example, LeCam (1957), Varadarajan (1958, 1961), and Dudley (1966) have generalised the main results on weak convergence from metric spaces to spaces of a more general type. Borovkov (1972a, 1976, 1984) has developed a version of the theory based on his methods of individual functionals. Stroock and Varadhan (1969a, 1969b, 1979) have developed martingale methods that cover large classes of martingale type stochastic processes. The general theory originated from this method is given in Liptser and Shiryaev (1986), and Jacod and Shiryaev (1987).

A complete theory can be found in the books of Skorokhod (1961, 1964), Gikhman and Skorokhod (1965, 1971), Parthasarathy (1967), Billingsley (1968, 1999), Pollard (1984), Ethier and Kurtz (1986), Liptser and Shiryaev (1986), Jacod and Shiryaev (1987), Davidson (1994), Borovkov, Mogul'skij and Sakhanenko (1995), and Whitt (2002). These books also contain bibliographies of works in the area as well as recent survey papers of Bloznelis and Paulauskas (2000), and Mishura (2000).

Chapter 2. The first works in the area were related to the classical model of random sums, i.e., sums of random variables with a random index (the number of summands). In this regard, we mention the works of Doeblin (1938), Wald (1945), Robbins (1948), Kolmogorov and Prokhorov (1949), and Anscombe (1952). Three lines of results have been developed in these studies.

The **first** direction in this area is related to the model of stochastic processes stopped at random moments that are independent of the external process. Here, the main role is played by various conditions that imply weak convergence of the external processes and weak convergence of the properly normalised random stopping moments. The limiting random variable is, naturally, the limiting external process stopped at the limiting stopping moment that is independent of this process. Most of the results are related to random sums, i.e., sums of random variables with a random number of summands. As a rule, the conditions contain assumptions on independence or weak independence of the summands. The methods used in these studies are chiefly based on characteristic functions. The results resemble a generalisation of the classical results concerning sums of independent random variables.

Conditions for weak convergence of random sums and randomly stopped sum-processes were studied in the works of Robbins (1948), Gnedenko (1964, 1967, 1972, 1983), Nagaev (1968), Gnedenko and Fahim (1969), Mamatov and Nematov (1971), Szász (1971a, 1971b, 1972a, 1972b, 1972c, 1975), Szász and Frayer (1971), Rychlik and Szynal (1972, 1973, 1975), Szynal (1972, 1976), Banis (1973), Pechinkin (1973), Rosinski (1975, 1976a, 1976b), Kruglov (1976, 1989, 1991, 1995, 1996, 1998), Rychlik (1976), Belov and Pechinkin (1979), Jozwiak (1980), Lin Zhengyan, Lu Chuanrong and Lu Chuanlai (1980), Grishchenko (1982), von Chossy and Rappl (1983), Shanthrikumar and Sumita (1984), Kubacki (1985), Kubacki and Szynal (1985b), Nagaev and Asadullin (1985), Azlarov, Aripov and Dzhamirzaev (1986), Finkelstein and Tucker (1989), Korolev (1989, 1993, 1994, 1995b, 1997a, 1997b, 1997c), Jankovič (1990), Kruglov and Korolev (1990), Niki Naoto, Nakagawa Shigekazu and Inoue Hideyuki (1990), Finkelstein, Tucker, and Veeh (1991, 1994), Klebanov and Melamed (1991), Umarov (1992), Fotopoulos and Wang (1993), Korolev and Kruglov (1993, 1998), Krajka and Rychlik (1993), Finkelstein, Kruglov, and Tucker (1994), Kossova (1994), Abdullaev (1995), Zhang Bo (1995a, 1995b, 1998), Gnedenko and Korolev (1996), Klebanov and Rachev (1996), Vellaisamy and Chaudhuri (1996), Griffin (1997), Kalashnikov (1997), Su Chun and Wang Yue Bao (1997), Cacoullos, Papadatos, and Papathanasiou (1998), Kozubowski and Panorska (1998), Liang Qiong and Zhang Chun Yong (1998), Kruglov and Zhang Bo (2001a, 2001b), Rychlik and Walczyński (2001a), and Bening and Korolev (2002).

Another type of models is presented by extremes with random sample size indices that are independent of the sample. Here, we refer to the works of Berman (1962, 1992), Thomas (1972), Galambos (1973, 1975, 1978, 1992, 1994), B. Gnedenko and D. Gnedenko (1982), Baumann (1991), and Beirlant and Teugels (1992). Asymptotics for maxima of random sums were studied by Rybko (1988), Kruglov and Rybko (1989), Azlarov, Dzhamirzaev, and Mamurov (1991), Kruglov (1996), Kruglov and Zhang Bo (1996), and Kowalski and Rychlik (1998). Asymptotic distributions for point type processes with independent thinning have been studied by Rényi (1955), Belyaev (1963), Kovalenko (1965), Gnedenko and Fraier (1969), Kennedy (1970), Iglehart (1974), and Wang Xiaoming (1999).

For limit theorems on general sequences of random variables and randomly stopped stochastic processes with independent random indexes, we would like to refer to the works of Dobrushin (1955), Szász (1971a), Thomas (1972), Kallenberg (1975), Grandell (1976), Kubacki and Szynal (1985a), Korolev (1993, 1995a, 1992, 1994), Korolev and Kossova (1995), Zhang Bo (1995b), Steinsaltz (1999) and Gajowiak and Rychlik (2000).

The books of Kruglov and Korolev (1990), Gnedenko and Korolev (1996), and Bening and Korolev (2002) contain an extended presentation of results related to this model, and bibliographies of works in the area.

The **second** direction of studies is related to the model of stochastic processes stopped at random moments that are asymptotically independent of the external processes. Various conditions that imply marginal weak convergence of external processes and internal stopping processes are used. The asymptotic independence of these processes, as a rule, is provided firstly, by Rényi type mixing conditions on external processes and, secondly, by conditions on convergence of the stopping moments in probability. Again, most of the works are related to studies of the model of random sums. These studies originate in the papers of Doeblin (1938), Robbins (1948), and Anscombe (1952), where all the authors considered the case with normalised random stopping indices that converge in probability to a constant. An extension to the general case with normalised random stopping indices that converge in probability to a positive random variable was first obtained by Rényi (1957, 1958, 1960, 1963), whose works gave rise to a series of papers related to this model.

The model of random sums were studied in the papers of Révész, (1958, 1959), Mogyoródi (1961, 1962, 1964, 1965, 1966, 1967b, 1971a), Billingsley (1962), Blum, Hunson and Rosenblatt (1963), Wittenberg (1964), Richter (1965a, 1965b, 1965c), Teicher (1965), Guiasu (1967a, 1967b, 1971), Sreehari (1968, 1970), Gleser (1969), Prakasa (1969), Csörgo and Fischler (1970, 1973), Fernandez (1970, 1971), Kembleton (1970a), M. Csörgo and S. Csörgo (1973), Jagers (1973), S. Csörgo (1974a), Fischer (1977), Horváth (1984a), Prakasa and Sreehari (1984), Gut (1985), Baumann (1991), Haas (1992), Fotopoulos and Wang (1993), Rybko (1994), Adler (1997), Kowalski and Rychlik (1998), and Gajowwiak and Rychlik (2000).

Limit theorems for extremes with a random sample size were studied in the papers of Berman (1962), Barndorff-Nielsen (1964), Mogyoródi (1967a), Pickands (1971), Sen (1972), Galambos (1973, 1978, 1992), Matsunawa and Ikeda (1976), Gut (1985), Barakat (1987), Barakat and El-Shandidy (1990), Haas (1992), Zhang Guo Sheng (1993), Silvestrov and Teugels (1998a, 1998b), and Zhang Jian (1998).

The case of general sequences and processes was studied in Doeblin (1938), Anscombe (1952), Guiasu (1963, 1965, 1967a, 1967b, 1971), Richter (1965a, 1965b), Gleser (1969), Durret and Resnik (1977), Aldous (1978a), Csörgo and Rychlik (1980, 1981), Hall and Heyde (1980), and Kubacki and Szynal (1986).

Asymptotics of various statistics with a random sample size were studied in the papers of Anscombe (1952), Robbins (1959), Chow and Robbins (1965), Starr (1966), Zacks (1966, 1971), Mogyoródi (1967c), Nades (1967), Pyke (1968), Simons (1968), Gleser (1969), Khan (1969), Sproule (1969), M. Csörgo and S. Csörgo (1970), Kembleton (1970a), Koul (1970), M. Csörgo (1973), S. Csörgo (1974b), Nikitin (1974), Deo (1975), Silvestrov, Mirzahmedov, and Tursunov (1976, 1983), Tursunov (1976), Ahmad (1980), Csenki (1981), Csörgo and Révész (1981), Ghosh and Mukhopadhyay (1981), Sen and Ghosh (1981), Horváth (1985), Hebda-Grabowska (1987), Ghosh, Mukhopadhyay, and Sen (1997), Aras, Jammalamadaka and Zhou (1989), Basu and Bhattacharya (1990, 1992), Baumann (1991), Glynn (1992), Glynn and Whitt (1992), Haas (1992), Csörgo and Horváth (1993), Fotopoulos and Wang (1993), Dmitrienko and Govindarajulu (2000), and Scheffler and Becker-Kern (2000).

The **third** direction of studies is related to the general limit theorems for randomly stopped càdlàg stochastic processes and compositions of stochastic processes. Usually, one makes no assumptions about independence or asymptotic independence of external processes and internal stopping moments or processes. Here, the basic condition is the joint weak convergence of external processes and stopping moments. The limiting random variable is, naturally, the limiting external process stopped at the limiting stopping moment that can depend on the external process. In the process setting, the limiting process is the composition of the limiting external process and the limiting non-decreasing internal stopping process that can be dependent in an arbitrary way. The results presented in Chapters 2 and 3 are concerned with this third direction. For this reason, we will be more specific about the results.

The first general result in which the condition of the joint weak convergence of external and internal stopping processes was involved in the case of non-constant limiting stopping processes was given in Billigsley (1968). There, the author deals with the case when both the external and the internal limiting processes are continuous. These results were extended in Iglehart and Kennedy (1970), Silvestrov (1971c, 1972b, 1972e), and Whitt (1973, 1980).

General conditions for weak convergence of randomly stopped càdlàg processes, in the general situation when the limiting external process could be a discontinuous càdlàg process, were obtained in Silvestrov (1971b, 1972a). These conditions are formulated in Theorems 2.3.1 and 2.3.3. The results formulated in these theorems constitute the main results of Chapter 2 as well as and new results from Silvestrov (2002a, 2002b) formulated in Theorems 2.4.1 and 2.4.2.

Other theorems in Chapter 2 extend these results to the case of weak convergence of scalar and vector compositions of càdlàg processes and to the model with random normalisation, etc. Most of these results are from Silvestrov (1971b, 1972a, 1972b, 1972e, 1973a, 1974). The main results here are contained in Theorem 2.6.1 from Silvestrov (1972a, 1972e) and Theorem 2.7.5 from Silvestrov (1974). The latter theorem gives convenient conditions for weak convergence of càdlàg processes on a set dense in the interval $[0, \infty)$. Some results related to the case when the limiting stopping process is not only continuous but also strictly monotonic can be derived from Whitt (1973, 1980). Additional results can also be found in Anisimov (1977, 1988), Durret and Resnik (1977), and Silvestrov (1979a). Theorems 2.6.4 and 2.6.5 and Lemma 2.6.4, which are based on new weakened continuity conditions, are new results from Silvestrov (2002a, 2002b). Some additional comments are also given in the last section of Chapter 2.

We would also like to mention the results on limit theorems for random sums with dependent summands and random indices obtained in Kruglov (1996), Kowalski and Rychlik (1998), Gajowwiak and Rychlik (2000), Rychlik and Walczyński (2001b), Kruglov and Zhang Bo (1996, 2001b), Jiang Tao, Su Chun, and Tang Qi He (2001), and Zhang Bo (2002), where an alternative approach based on approximation of such sums by the associated random sums with independent summands and random indices is used.

Chapter 3. When discussing conditions for **J**-convergence of compositions of càdlàg processes, four cases should be considered: **(a)** both the limiting external and internal stopping processes are continuous; **(b)** the limiting external process is continuous; **(c)** the limiting internal stopping process is continuous; **(d)** both the limiting external process and the internal stopping processes can be discontinuous.

The simplest case **(a)** was considered in Billigsley (1968). The main result of Billigsley is given in Theorem 3.2.1. This result was extended in various directions in Iglehart and Kennedy (1970), and Silvestrov (1971b, 1972a, 1972e, 1974), Whitt (1973, 1980, 2002), and Serfozo (1973). Case **(b)** was considered in Whitt (1973, 1980) and Silvestrov (1974). Theorem 3.3.2 is a new result.

Case **(c)**, where the limiting stopping process is continuous, is important for many applications. For example, this is often the case for the model with renewal type stopping processes. Conditions for **J**-convergence of compositions of càdlàg processes for this case was given in Silvestrov (1972b, 1972e, 1973a). These results are formulated in Theorem 3.4.1. The improved version given in Theorem 3.4.2 is from Silvestrov (1974). Under the additional condition that the limiting stopping process is not only continuous but also strictly monotone, case **(c)** was also considered in Whitt (1973, 1980). Theorem 3.4.3, which generalises Theorem 3.4.2, is a new result from Silvestrov (2002a, 2002b).

The most general and difficult case is that in **(d)**, where both the limiting external and stopping processes can be discontinuous. General conditions for **J**-compactness and

J-convergence of compositions of càdlàg processes were obtained in Silvestrov (1974). These results are formulated in Theorems 3.6.1, 3.6.2, 3.8.1, and 3.8.2, which make the main results of Chapter 3. The first two theorems cover the case of scalar compositions of càdlàg processes; the last two theorems cover the more complicated case of vector compositions. We refer to the survey of Silvestrov (2000b) which contains additional bibliographical remarks concerning the results related to general conditions for **J**-convergence of compositions of càdlàg processes. Theorems 3.6.4 and 3.8.6, which generalises theorems mentioned above, are new results from Silvestrov (2002a, 2002b).

We would also like to mention some related works in the area. Conditions for convergence of compositions of càdlàg processes in Skorokhod's topologies $\mathbf{J}_2$, $\mathbf{M}_1$, and **M**-topologies, which supplement the main topologies **U** and **J**, were studied in Whitt (1973, 1980, 2002), Pomarede (1976), and Anisimov (1977, 1988). Conditions for **J**-convergence of compositions of step processes were given in Kennedy (1972) and Whitt (1973, 1980, 2002). The results in Iglehart and Whitt (1970), Silvestrov (1972e, 1974), Whitt (1973, 1980), and Serfozo (1973) are related to the model with non-random limiting stopping processes. This model leads to **J**-convergence theorems for compositions with non-random centering as well as to the so-called inverse theorems in which convergence of external processes is derived from **J**-convergence of the compositions. Also, works on thinnings of random measures and point processes that can be considered as a special class of compositions of monotone processes should also be referred to. They include the papers of Mogyoródi (1971b, 1972a, 1972b), Szantai (1971a, 1971b), Råde (1972a, 1972b), Zakusilo (1972a, 1972b), Jagers (1974), Jagers and Lindvall (1974), Tomko (1974), Kallenberg (1975) and Serfozo (1976, 1984a, 1984b), Lindvall (1978), Gasanenko (1980), Böker and Serfozo (1983). We would also like to mention the works of Silvestrov (1972e, 1973b) and Mishura (1978), where some results of the theory were extended to randomly stopped random fields and compositions of random fields. Internal stopping processes are usually monotone and, in some cases, are step or point type processes. In this context, the functional limit theorems were studied for monotone processes in Vervaat (1972), Whitt (1973, 1980, 2002), Silvestrov (1974), Walk (1975), Serfozo (1976), Jacod, Memin, and Metevier (1983), and Jacod and Shiryaev (1987), and, for point processes and random measures, in Kallenberg (1973, 1975), Jagers (1974), Serfoso (1976), and Resnick (1986, 1987).

The above grouping of works into three directions is relative. Many works placed in the first two groups, especially those dealing with functional limit theorems, can also be classified as belonging to the third group. For example, some of these works are Billingsley (1962), Iglehart (1974), Aldous (1978a), M. Csörgo and S. Csörgo (1970), Sreehari (1968), Mirzahmedov, Silvestrov and Tursunov (1976), Tursunov (1976), Rychlik and Szynal (1975), Prakasa and Sreehari (1984), Csörgo and Horváth (1993), and Silvestrov and Teugels (1998a, 1998b).

The book Silvestrov (1974) is devoted to general limit theorems for randomly stopped stochastic processes and compositions of càdlàg processes in topologies **U** and **J**. We also refer to the books Csörgo and Révész (1981), Anisimov (1988), Gut (1988), Csörgo and Horváth (1993), Rahimov (1995), Bening and Korolev (2002) and Whitt (2002), which also contain results on limit theorems for randomly stopped random processes and compositions of stochastic processes. We would also like to refer to Silvestrov (2002a), which is a preliminary report version of the current book.

Chapter 4. Natural areas of applications and examples of general distributional and functional limit theorems covered in Chapters 2 and 3 include various concrete models of randomly stopped càdlàg processes and compositions of càdlàg processes in which some specific structural assumptions about internal stopping processes are used when proving the limit theorems.

The first class of such models is represented by sum-processes with random stopping (random sums), max-processes with random stopping (extremes with random sample size), and related models that originate in the classical works of Anscombe (1952) and Rényi (1957, 1958, 1960). An essential part of distributional and functional limit theorems for random sums, extremes with random sample size relates to models with independent external processes and stopping moments converging in probability. Works related to limit theorems for these models have been cited above and we do not repeat these references. Here we would like to note that many of these results can be obtained by directly applying the general limit theorems given in Chapters 2 and 3. General limit theorems for random sums and extremes with random sample size include many of the preceding results in this area. These are, respectively, Theorems 4.2.2, 4.2.4, 4.7.2, which are direct corollaries of the limit theorems from Silvestrov (1971b, 1972a, 1972e), and Theorem 4.7.4 from Silvestrov and Teugels (1998a).

The second such class includes randomly stopped processes and compositions of stochastic processes of Markov or martingale type processes with Markov type stopping. Weak convergence and functional limit theorems have been studied for these type of processes in the works of Silvestrov (1972e, 1974, 1977), McLeich (1974, 1978), Anisimov (1975, 1977, 1988), Gikhman and Skorokhod (1975), Rychlik and Szynal (1975), Rootzén (1977, 1980), Aldous (1978b), Gänssler, Strobel, and Stute (1978), Gänssler and Häusler (1979), Rychlik (1979), Helland (1980, 1982), Beska, Klopotowski, and Slominski (1982), Butzer and Schulz (1983, 1984a, 1984b), Kubacki and Szynal (1983, 1986), Prakasa and Sreehari (1984), Kubacki (1987), and Rahimov (1987, 1995). Conditions for **J**-compactness of càdlàg processes based on stopping times were given in Aldous (1978b, 1989), Jacod, Memin and Metevier (1983), Sørensen (1983), Nikunen (1984), Joffe and Metevier (1986), and Jacod and Shiryaev (1987).

The third special and important class of models constitutes randomly stopped processes and compositions of stochastic processes with exceeding and renewal type stopping internal processes. One can find a more detailed bibliographical remarks on works related to the composition of stochastic processes with renewal type stopping in Silvestrov (2000a). Here, we give a shortened version of these remarks.

Functional limit theorems for renewal type processes based on general càdlàg processes can be found in Skorokhod (1956), Iglehart and Kennedy (1970), Iglehart and Whitt (1971), Whitt (1971a, 1972, 1973, 1980, 2002), Borovkov (1972a, 1972b, 1976), Silvestrov (1972e, 1974, 2000a), Vervaat (1972), Iglehart (1973), Anisimov (1974b, 1975, 1977, 1988), Resnick (1974), Serfozo (1976), Goldie (1977), M. Csörgo, S. Csörgo, Horváth and Revesz (1982), Csörgo, Horváth, and Steinebach (1987), Charlot and Merad (1989), Doss and Gill (1992), Puhalskii (1994), and Puhalskii and Whitt (1997). Chapter 4 contains some new results for general renewal type processes. In particular, we would like to mention Theorems 4.3.4 – 4.3.7 and Lemma 4.3.2.

There exists a huge number of works devoted to studies of weak convergence and **J**-convergence of sum-processes with renewal stopping constructed from sums of random variables (independent, weakly dependent, defined on Markov chains, etc.). There are also numerous studies on limit theorems for exceeding, extremal, and renewal types stopping processes. Here, we point out to some of them selecting works based on the use of renewal structure of stopping processes and functional limit theorems. In this context, we list works on limit theorems for sum-processes defined on Markov type processes pertaining mainly to the case of sum-processes with semi-Markov stopping.

A good bibliography of works on limit theorems for renewal type processes for the period up to the beginning of 70s can be found in Serfozo (1975). We mention here the works of Feller (1949, 1966), Dynkin (1955), Smith (1955, 1958), Lamperty (1958, 1961, 1962a), Takacs (1959), Kesten (1962), Farrell (1964, 1966a, 1966b), Pyke and Schanflie (1964), Prabhu (1965), Borovkov (1967a, 1967b), Heyde (1967), Siegmund (1968), Iglehart (1969, 1974), Silvestrov (1969c, 1970b, 1970c, 1972e, 1974), Iglehart and Kennedy (1970), Whitt (1971a, 1971b), Basu (1972), Bingham (1972), Gut (1973, 1974, 1975), Kaminskene (1973), and Serfozo (1975).

Further works are Chow and Hsiung (1976), Mohan (1976), Lindberger (1978), Mishura (1978), Pakshirajan and Mohan (1978), Gut and Ahlberg (1981), Sen (1981), Grishchenko (1982), Teicher (1982), Gut and Janson (1983), Khusanbayev (1983, 1984), Horváth (1984b, 1986), Lalley (1984), Niculescu (1984), Kasahara (1985), Angus (1986), Csörgo, Horváth, and Steinebach (1986, 1987), Murphree and Smith (1986), Csörgo, Deheuvels, and Horváth (1987), Levy and Taqqu (1987, 2000), Mason and van Zwet (1987), Shedler (1987, 1993), Steinebach (1987, 1988, 1991), Gut (1988), Niculescu (1988), Charlot and Merad (1989), Roginsky (1989, 1992, 1994), Deheuvels and Steinebach (1992), Li Deli and Wu Zhiquan (1992), Alex and Steinebach (1994), Horváth (1984b, 1986), Konstantopoulos, Papadakis, and Walrand (1994), Niculescu and Omey (1994), Sagitov (1994), Zhang Hanqin and Hsu Guang-Hui (1994), Steinebach and Eastwood (1996), Zhang Lixin (1996), Gut, Klesov, and Steinebach (1997), Klesov and Steinbach (1997), Li Linxiong (1997), Mitov, Grishechkin and Yanev (1997), Zhang Yucheng and Song Shibin (1997), Yamada Keigo (1999), Csenki (2000), Silvestrov (2000a), Borovkov and Mogul'skij (2001), Frolov, Martikainen, and Steinebach (2001), Mitov and Yanev (2001), and Silvestrov and Teugels (2001).

In this context, the works on limit theorems for risk processes and their diffusion type approximations should also be mentioned. We refer here to the works of Cramér (1955), Sparre Andersen (1957), Feller (1966), Gerber (1979), Iglehart (1969), Bohman (1972), Grandell (1972, 1977, 1991a, 1991b, 1998), Whitt (1972), Siegmund (1975), von Bahr (1975), Harrison (1977), Gerber (1979), Asmussen (1984, 1987, 1989, 1996, 2000), Beard, Pentikäinen, and Pesonen (1984), Garrido (1985), Glynn (1990), Schmidli (1992, 1997a, 1997b), Aebi, Embrechts, and Mikosch (1994), Bening and Korolev (1996a, 1996b, 1997, 1998a, 1988b, 2000a, 2000b, 2002), Embrechts, Klüpenberg and Mikosch (1997), Korolev (1998, 2000), Asmussen and Højgaard (1999), Rolski, Schmidli, Schmidt, and Teugels (1999), Bening, Korolev, and Liu Lixin (2000), Gyllenberg and Silvestrov (2000b), Silvestrov (2000a, 2000d) and Bening, Korolev, and Kudryavtsev (2001).

For works on limit theorems for sum type processes with renewal semi-Markov stopping and related models, we would like first of all to indicate that a good bibliography of works for the period up to the mid of 70s can be found in Teugels (1976). Here, we only mention the works of Silvestrov (1969a, 1969c, 1970a, 1970b, 1970c, 1971a, 1973c, 1980b), Anisimov (1970a, 1970b, 1972, 1988, 1999, 2000a, 2000b), Bingham (1971), Arjas (1972), Cheong and Teugels (1972), Lev (1972), Masol and Silvestrov (1972), Prizva (1972), Masol (1973, 1974), Szász (1974), Nummelin (1976), Cherenkov (1977), Oprişan (1977), Popescu (1977), Arsenishvili and Ezhov (1978), Athreya, McDonald, and Ney (1978), Athreya and Ney (1978), Korolyuk and Turbin (1978, 1983), Kaplan and Silvestrov (1979), Shurenkov and Eleĭko (1979), Silvestrov and Tursunov (1979), Eleĭko (1980, 1990a, 1990b, 1998), Tomko (1981), Hordijk and Schassberger (1982), V. S. Korolyuk and V. V. Korolyuk (1983, 1999), Kravets (1985), Malinovskij (1985, 1986, 1988), Radian (1985), Korolyuk (1986, 1990), Korolyuk and Svishchuk (1986, 1989a, 1989b, 1991, 1992, 2000), Grigorescu and Popescu (1987), V. S. Korolyuk, Svishchuk, and V. V. Korolyuk (1987), Anisimov and Aliev (1989), Didkovskiĭ and Silvestrov (1989), Sviridenko (1989), Hsu Guanghui and He Qiming (1990), V. V. Korolyuk (1991), Wajda (1991), Stenflo (1996, 1998), Soltani and Khorshidian (1998), Silvestrov and Stenflo (1998), Alsmeyer and Gut (1999), Ball (1999), Korolyuk and Limnios (1999a, 1999b, 2000, 2001), Pyke (1999), Stefanov (1999), Thorisson (2000), Lee and Shin (2001), and Limnios and Oprişan (2001).

The results in Borovkov (1967a, 1967b, 1969), Miller (1971), Silvestrov (1971c, 1972c, 1972d, 1972e, 1980a, 1980b, 1983a, 1990, 1991), Milyoshina (1975), Serfozo (1975), Lindberger (1978), Steinebach (1978), Kaplan and Silvestrov (1979, 1980), and Glynn and Whitt (1986, 1987, 1988, 1993, 2002) are related to limit theorems for accumulation processes with random embedded cycles.

We would also like to mention the works of Serfozo (1980), Shanthrikumar and Sumita (1983), Sumita and Shanthikumur (1985), Anderson (1987, 1988), Gut (1990, 2001), Silvestrov and Teugels (1998a, 2001), Gut and Hüsler (1999), and Mallor and Omey (2001), which deal with extremal processes with random embedded cycles and max-renewal models such as max-processes with renewal stopping and shock processes.

Works on limit theorems for additive type functionals defined on discrete Markov chains should also be mentioned. One of the main methods of studies here is based on the use of regenerative embedded cycles between the return moments in special subsets of states followed by the use of limit theorems for the corresponding embedded renewal type processes. We refer here to works of Doeblin (1937, 1938), Feller (1949), Kolmogorov (1949), Romanovskij (1949), Dobrushin (1953, 1956a, 1956b), Doob (1953), Sarimsakov (1954), Sirazhdinov (1955), Billingsley (1956), Darling and Kac (1957), Kendall (1957), Nagaev (1957, 1962), Volkonskii (1957), Meshalkin (1958), Skorokhod (1958, 1961), Orey (1959), Chung Kai Lai (1960), Ciucu and Theodorescu (1960), Kesten (1962), Hanen (1963a, 1963b, 1963c, 1963d), Gikhman and Skorokhod (1965, 1975), Borovkov (1967a, 1967b, 1969), Friedman (1967), Maigret (1968), Gikhman (1969, 1974), Iosifescu and Theodorescu (1969), Silvestrov (1969b, 1969c, 1974, 1983b), Statulyavichus (1969a, 1969b, 1969c), Kembleton (1970b), Skorokhod and Slobodenyuk (1970), Fal' (1971), Cogburn (1972, 1991), Anisimov (1973, 1974a), Grigorescu and Popescu (1973), Silvestrov and Polešč̌uk (1974), Formanov (1975, 1979), Gorostiza (1975), Milyoshina (1975), Polešč̌uk (1975, 1977), Grigorescu and Oprisan (1976), Kasahara (1976/77), Silvestrov and Tursunov (1977), Wolfson (1977), Gordin and Lifšic (1978), Kaijser (1978, 1979, 1981a, 1981b), Lifšic (1978), Siraždinov and Formanov (1979), de Dominicis (1980), Kaplan (1980), Kaplan and Silvestrov (1980), Kurtz (1981), Kaplan, Motsa, and Silvestrov (1982, 1983), Motsa (1982a, 1982b, 1982c, 1990), Bingham and Hawkes (1983), Nummelin (1984), Kipnis and Varadhan (1986), Barnsley and Elton (1988), Barnsley, Elton and Hardin (1988), Levental (1988), Elton (1990), Iosifescu and Grigorescu (1990), Silvestrov and Brusilovskiĭ (1990), Tong (1990), Gudinas (1991, 1995), Grigorescu (1992), Woodroofe (1992), Chan (1993a, 1993b), Meyn and Tweedie (1993), Moskal'tsova and Shurenkov (1993a, 1993b, 1994, 1995), Velikii, Motsa, and Silvestrov (1994), Łoskot and Rudnicki (1995), Hsiau Shoou-Ren (1997), Ruzhevich (1997), Benda (1998), Korolyuk and Limnios (1998), Stenflo (1998), Chen (1999), Gallardo (1999), Maxwell and Woodroofe (2000), Wu Wei Biao and Woodroofe (2000), Derriennic and Lin (2001), Hennion and Hervé (2001), Lee Oesook and Kim Jihyun (2001), and Steichen (2001)

We also refer to works devoted to statistical studies of Markov chains and semi-Markov processes based on limit theorems for sum of random variables defined on Markov chains. These references are Billingsley (1961), Baum and Petrie (1966), Leroux (1992), Bickel and Ritov (1996), MacDonald and Zucchini (1997), Francq and Roussignol (1998), Krishnamurthy and Rydén, (1998), Jensen and Petersen (1999), Ouhbi and Limnios (1999a, 1999b, 1999c), Le Gland and Mevel (2000), Douc and Matias (2001), and Limnios and Oprişan (2001).

Works on limit theorems for regenerative, Markov and semi-Markov sum processes with another type of random Markov type stopping moments, in particular those of the first-hitting-time types include Simon and Ando (1961), Kingman (1963), Gnedenko and Kovalenko (1964), Darroch and Seneta (1965, 1967), Keilson (1966, 1978), Schweitzer (1968), Korolyuk (1969), Silvestrov (1969a, 1969b, 1969c, 1970b, 1970c, 1971a, 1972e, 1974, 1976, 1978, 1979b, 1980b, 1981, 1981b, 1995, 2000c), Korolyuk and Turbin (1970, 1976, 1978, 1982), Anisimov (1971a, 1971b, 1975, 1988), Gusak and Korolyuk (1971), Solov'ev (1971, 1983), Turbin (1971), Kovalenko (1973, 1975, 1977, 1980, 1988, 1994), Masol (1973, 1974), Latouch and Louchard (1978), Ivchenko, Kashtanov, and Kovalenko (1979), Kaplan (1979, 1980), Voǐna (1979), Shurenkov (1980a, 1980b, 1990a, 1990b), Brodi and Bogdantsev (1981), Aldous (1982), Kalashnikov and Vsekhsvyatski (1982), D. V. Korolyuk and Silvestrov (1983, 1984), Abadov (1984), Silvestrov and Velikii (1988), Alimov and Shurenkov (1990), Pollett and Roberts (1990), Silvestrov and Abadov (1991, 1993), Barlow (1993), Todorovic (1993), Alimov (1994), Gyllenberg and Silvestrov (1994, 1999, 2000a), Eleǐko and Shurenkov (1995), Kartashov (1996), Motsa and Silvestrov (1996), Englund and Silvestrov (1997), Kijima (1997), Kovalenko, Kuznetzov, and Pegg (1997), and Englund (1999a, 1999b, 1999c, 2000, 2001).

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Index (terms)

Index (conditions)

Index (symbols)

List of Figures