ECON4310: Macroeconomics

Espen Henriksen

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General and partial equilibrium

General equilibrium:

1. Decision makers (indviduals, firms, ...) take prices as given and decide on quantities
2. Markets clear, i.e. "for given decisions over quantities what must prices be for supply to equal demand."

Partial equilibrium:

- Decision makers (indviduals, firms, ...) take prices as given and decide on quantities

State variables:

- In general equilibrium, the state is fully described by the quantities – i.e. prices are never state variables
- In partial equilibrium, prices might be state variables

Indirect utility and profit functions

- The logic of the indirect utility function or a reduced form profit function generated by an optimizing firm underlies dynamic programming.

- This static problem ignores dynamics – savings, spending on durable goods, uncertainty about the future
  These are all important when individuals make decisions – and material we will study in this class

- We will start by reviewing the indirect utility function

Indirect utility function

$$v(p, m) = \max u(x)$$

subject to

$$px = m$$

where p is a vector of prices, x is a vector of consumption goods, and m is income.

The first order condition is given by

$$\frac{u_j(x)}{p_j} = \lambda \qquad \text{for all } j$$

where λ is the shadow price of the budget constraint and $u_j(x)$ is the marginal utility from good j.

$v(p, m)$ is the maximized utility *from the current state* (p, m).

Someone in this state can be predicted to attain this utility

One does not need to know what that person will do with his income; it's enough to know what his income is and what the prices he will face.

This powerful logic underlies dynamic programming.

Indirect profit function

$$\pi(w,p,k) = \max_l pf(k,l) - wl$$

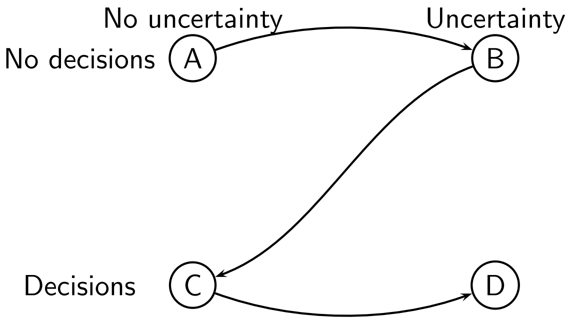
A labor demand function depends on (w,p,k)

As with $v(p,m)$, $\pi(w,p,k)$ summarizes the value of the firm given factor prices w, the product price p and the stock of capital k

Not specified where k,p,w,m come from. *In general equilibrium*:

- k control (decision) variable and endogenous state variable
- p and w are never state variables in general equilibrium – prices clear markets.
- m might follow some endogenous or exogenous process

Building the toolbox



A : Solow growth model
B : Value iteration on a Markov process
C : Ramsey growth model
D : Stochastic neoclassical growth model

First time

Product approach to output: $y_t = \bar{z} f\left(k_t, \bar{n}_t\right).$

Income approach to output: $y_t = r_t k_t + w_t \bar{n}_t.$

Expenditure approach to output: $y_t = c_t + x_t.$

Law of motion for capital: $k_{t+1} = (1-\delta) k_t + x_t.$

Behavioral assumption:

$$i_t = \sigma y_t$$

This time

$$\max_{\{c_t, x_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u\left(c_t, 1 - \bar{n}_t\right)$$

subject to

Product approach to output: $y_t = \bar{z} f\left(k_t, \bar{n}_t\right).$

Income approach to output: $y_t = r_t k_t + w_t \bar{n}_t.$

Expenditure approach to output: $y_t = c_t + x_t.$

Law of motion for capital: $k_{t+1} = (1 - \delta) k_t + x_t.$

Next time

$$\max_{\{c_t, x_t\}_{t=0}^{\infty}} \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t u\left(c_t, 1 - \bar{n}_t\right)$$

subject to

Product approach to output: $y_t = z_t f\left(k_t, \bar{n}_t\right).$

Income approach to output: $y_t = r_t k_t + w_t \bar{n}_t.$

Expenditure approach to output: $y_t = c_t + x_t.$

Law of motion for capital: $k_{t+1} = (1-\delta) k_t + x_t.$

Total factor productivity: $z_t = \rho z_{t-1} + \varepsilon_t, \qquad \varepsilon \sim \mathcal{N}(0, \sigma^2),$

estimated from $\{z_t\}_{t=0}^{T}$ $\ln z_t = \ln y_t - \alpha \ln k_t - (1-\alpha) \ln n_t.$

The time after that

$$\max_{\{c_t,x_t\}_{t=0}^{\infty}} \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t u\left(c_t, 1-n_t\right)$$

subject to

Product approach to output:	$y_t = z_t f\left(k_t, n_t\right).$
Income approach to output:	$y_t = r_t k_t + w_t n_t.$
Expenditure approach to output:	$y_t = c_t + x_t.$
Law of motion for capital:	$k_{t+1} = (1-\delta)k_t + x_t.$
Total factor productivity:	$z_t = \rho z_{t-1} + \varepsilon_t, \qquad \varepsilon \sim \mathcal{N}(0,\sigma^2),$
estimated from $\{z_t\}_{t=0}^{T}$	$\ln z_t = \ln y_t - \alpha \ln k_t - (1-\alpha)\ln n_t.$

Class objectives today

- Solve the Ramsey-Cass-Koopmans model (decisions, but no uncertainty)

Developed by Ramsey (1928), Cass (1965), Koopmans (1965). Avoids all market imperfections and all issues raised by heterogenous households and links between generations.

More specifically, we will solve

- The sequential problem, abstracting from labor-leisure choice and uncertainty.
- First in finite time, then with infinite horizon.
- Give a recursive formulation of the same problem

The Ramsey optimal savings problem

Mathematical strategy:

- Maximizing a function (total utility) of *an infinity of variables* (consumption and capital at each date) subject to technological constraints.
- Set up the problem in continuous time (we'll solve it in discrete time) and characterized the utility-maximizing dynamics.
- Under suitable assumptions, capital stock should converge monotonically to the level that, if sustained, maximizes consumption per unit of time.

Key: Solve for a sequence of *allocations*.

Finite time, social planner's problem

$$\max_{\{c_t, x_t\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t u\left(c_t, 1 - \bar{n}_t\right)$$

subject to

Product approach to output:	$y_t = \bar{z} f\left(k_t, \bar{n}_t\right).$
Expenditure approach to output:	$y_t = c_t + x_t.$
Law of motion for capital:	$k_{t+1} = (1 - \delta) k_t + x_t.$

with

$$k_0 > 0, \quad k_{t+1} \geq 0, \quad c_t > 0.$$

Simplifying

$$\max_{\{c_t, x_t\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t u(c_t)$$

subject to

$$c_t + k_{t+1} \leq f(k_t) + (1-\delta)k_t$$

with

$$k_0 > 0, \quad k_{t+1} \geq 0, \quad c_t > 0.$$

Lagrangian / Kuhn-Tucker

Solve:

$$\mathcal{L} = \sum_{t=0}^{T} \beta^t \left[u(c_t) - \lambda_t \left[c_t + k_{t+1} - f(k_t) - (1-\delta) k_t \right] + \mu_t k_{t+1} \right]$$

First-order conditions are:

$$\frac{\partial \mathcal{L}}{\partial c_t} = \beta^t \left[u'(c_t) - \lambda_t \right] = 0, \qquad t = 0, \ldots, T$$

$$\frac{\partial \mathcal{L}}{\partial k_{t+1}} = -\beta^t \lambda_t + \beta^t \mu_t + \beta^{t+1} \lambda_{t+1} \left[1 + f'(k_{t+1}) - \delta \right] = 0, \qquad t = 0, \ldots, T-1$$

$$\frac{\partial \mathcal{L}}{\partial k_{T+1}} = -\beta^T \lambda_T + \beta^T \mu_T = 0, \qquad t = T$$

Simplifying

First-order conditions are:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial c_t} &= u'(c_t) - \lambda_t = 0, & t &= 0, \ldots, T \\
\frac{\partial \mathcal{L}}{\partial k_{t+1}} &= -\lambda_t + \mu_t + \beta \lambda_{t+1} \left[1 + f'(k_{t+1}) - \delta\right] = 0, & t &= 0, \ldots, T-1 \\
\frac{\partial \mathcal{L}}{\partial k_{T+1}} &= -\lambda_T + \mu_T = 0, & t &= T
\end{aligned}$$

with complementary slackness conditions

$$\begin{aligned}
\mu_t k_{t+1} &= 0 \\
\lambda_t \left[c_t + k_{t+1} - f(k_t) - (1-\delta) k_t\right] &= 0 \\
\lambda_t,\ k_{t+1},\ \mu_t &\geq 0
\end{aligned}$$

$$
\begin{aligned}
& & u'(c_t) > 0 & \qquad \forall t \\
\Rightarrow & & \lambda_t > 0 & \qquad \forall t \\
\Rightarrow & & \mu_T > 0 & \qquad t = T \\
\Rightarrow & & k_{T+1} = 0 & \qquad t = T
\end{aligned}
$$

$$
\begin{aligned}
& & k_0 > 0, & \quad t = 0 \\
\Rightarrow & & k_{t+1} > 0, & \quad t = 0, \ldots, T-1 \\
\Rightarrow & & \mu_t = 0, & \quad t = 0, \ldots, T-1 \\
\Rightarrow & & -\lambda_t + \beta \lambda_{t+1} \left[1 + f'(k_{t+1}) - \delta\right] = 0, & \quad t = 0, \ldots, T-1 \\
\Rightarrow & & -u'(c_t) + \beta u'(c_{t+1}) \left[1 + f'(k_{t+1}) - \delta\right] = 0, & \quad t = 0, \ldots, T-1 \\
\Rightarrow & & \beta \frac{u'(c_{t+1})}{u'(c_t)} \left[1 + f'(k_{t+1}) - \delta\right] = 1, & \quad t = 0, \ldots, T-1
\end{aligned}
$$

Identified system of equations

$T + 2$ unknown variables

$$\{k_t\}_{t=0}^{T+1}$$

$T + 2$ equations

$$k_0 \quad \text{given}$$

$$\beta \frac{u'\left(k_{t+2} - f\left(k_{t+1}\right) - (1-\delta) k_{t+1}\right)}{u'\left(k_{t+1} - f\left(k_t\right) - (1-\delta) k_t\right)} \left[1 + f'\left(k_{t+1}\right) - \delta\right] = 1$$

$$k_{T+1} = 0$$

Since the number of unknowns is equal to the number of equations, the difference equation system will

- typically have a solution (existence),

and under appropriate assumptions on primitives,

- there will be only one such solution (uniqueness).

Conditions

A brief look at the conditions under which there is only one solution to the first-order conditions – or, alternatively, under which the first-order conditions are sufficient.

- u is concave.
- Since the sum of concave functions is concave it follows that $U = \sum_{t=0}^{T} u(c_t)$ is concave in the vector $\{c_t\}_{t=0}^{T}$.
- f is concave.
- From the definitions of a convex set and a concave function it follows that the constraint set is convex in $\{c_t, k_{t+1}\}_{t=0}^{T}$.
- So, concavity of the functions u and f makes the overall objective concave and the choice set convex, and thus the first-order conditions are sufficient.

The Euler Equation

$$\beta R_{t+1} \frac{u'(c_{t+1})}{u'(c_t)} = 1$$

where for the neoclassical growth model

$$R_{t+1} = 1 + f'(k_{t+1}) - \delta.$$

$$\underbrace{u'(c_t)}_{\text{marginal disutility of saving one more unit}} = \underbrace{\beta}_{\text{discount factor between two periods}} \text{rate of ret}$$

Determinants of saving

1. The the concavity of utility / consumption "smoothing": if the utility function is strictly concave, the individual prefers a smooth consumption stream.
2. Discounting / Impatience: via β, we see that a low β (a low discount factor, or a high discount rate) will tend to be associated with low c_{t+1}'s and high c_t's.
3. The return to savings: $f'(k_{t+1}) - \delta$ clearly also affects behavior, but its effect on consumption cannot be signed unless we make more specific assumptions. Moreover, k_{t+1} is endogenous, so when f' nontrivially depends on it, we cannot vary the return independently.

Infinite horizon

- Models with an infinite time horizon demand *more advanced mathematical tools*.
- Consumers in our models are now choosing infinite sequences. These are *no longer elements of Euclidean space* R^n, which was used for our finite-horizon case.
- A basic question is *when solutions to a given problem exist*. Suppose we are seeking to maximize a function
- Several issues arise.
    - How do we define continuity in this setup?
    - What is an open set?
    - What does compactness mean?
- We will not answer these questions here.

Maximization of utility under an infinite horizon

- Will mostly involve the same mathematical techniques as in the finite-horizon case.
- In particular, (Kuhn-Tucker) first-order conditions (Euler equation): barring corner constraints; choosing a path, defined for any k_0, such that the marginal effect of any choice variable on utility is zero.
- In finite time horizon, the final zero capital condition was key to determining the optimal path of capital: it provided us with a terminal condition for a difference equation system.
- In the infinite time case, there is no such final T: the economy will continue forever. Therefore, *the difference equation that characterizes the first-order condition may have an infinite number of solutions*.

Transversality condition

- Need some other way of pinning down the households' choice. Turns out that the missing condition is analogous to the requirement that the capita stock be zero at $T+1$
- The missing condition is called the transversality condition – typically, a necessary condition for an optimum.

$$\lim_{t\to\infty} \beta^t \lambda_t k_{t+1} = 0.$$

- It expresses the following idea:
  It cannot be optimal for the household to choose a capital sequence such that, in present-value utility terms, the shadow value of k_t remains positive as t goes to infinity. This could not be optimal because it would represent saving too much: a reduction in saving would still be feasible and would increase utility.

Recursive formulation

Alternative mathematical strategy:

- Seek the optimal savings/investment function directly and then to use this function to compute the optimal sequence of investments *from any initial capital stock*.
- This way of looking at the problem – decide on the immediate action to take as a function of the current situation – is called a *recursive formulation*.
- It exploits the observation that *a decision problem of the same structure recurs every period*.

Recursive formulation

$$v(k_s) \equiv \max_{\{c_t, k_t+1\}_{t=s}^{\infty}} \sum_{t=s}^{\infty} \beta^{t-s} u(c_t)$$

subject to

$$c_t + k_{t+1} \leq f(k_t) + (1-\delta) k_t \qquad \forall t \geq s$$

$$v(k_t) = \max_{k_{t+1}} \{u(c_t) + \beta v(k_{t+1})\}$$

Solution characterized by *decision rules/functions*

$$\begin{aligned} k_{t+1} &= g(k_t) \\ c_t &= f(k_t) + (1-\delta) - g(k_t) \end{aligned}$$

A few concepts

1. A **functional equation** is an equation in terms of independent variables, and also *unknown functions*, which are to be solved for.
   Usually the term functional equation is reserved for equations that are not in some simple sense reducible to algebraic equations.

   Simple examples. Which *functions* satisfy the following equations?

$$f(xy) = f(x) + f(y) \qquad ?$$

$$f(x + y) = f(x)f(y) \qquad ?$$

A few concepts cont'd

2. **State variable(s)** – the set of variables that are sufficient to summarize all information (past and present) needed to solve the forward looking optimization problem. We distinguish between *endogenous* and *exogenous state variables*.
3. **Control variable(s)** – the variable(s) that is/are chosen.
4. **Stationarity** – essential for recursive programming is that all relations can be expressed without an indication of time. In a given state the optimal choice of the agent will be the same regardless of "when" he optimizes.
5. **Structural parameters** ("the greeks") and **endogenous variables** ("the latins") – essentially, what is "the structure of the world", i.e. independent of choices, and what is determined by peoples choices.

Contraction mapping

In general, $v(\cdot)$ is unknown, but the Bellman equation can be used to find it.

In most of the cases we will deal with, the Bellman equation satisfies **a contraction mapping theorem**, which implies that

1. There is a unique function $v(\cdot)$ which satisfies the Bellman equation.
2. If we begin with any initial function $v_0(k)$ and define

$$v_{i+1}(k) = \max_{c,k'}[u(c) + v_i(k')]$$

subject to

$$c + k' = f(k) + (1-d)k$$

for $i = 0, 1, 2, \ldots$ then $\lim_{i \to \infty} v_{i+1}(k) = v(k)$.

How to find the true value function

The above two implications give us two alternative means of uncovering the value function.

1. Implication 1 above, if we are fortunate enough to correctly guess the value function $v(\cdot)$; then we can simply plug $v(k_{t+1})$ into the right side and then verify that $v(k_t)$ solves the Bellman equation. This procedure only works in a few cases.

2. Implication 2 above is useful for doing numerical work. One approach is to find an approximation to the value function and iterate.