ECON 4310

Fall 2009

Problem set 1

1. **Theory**

   Verify the form of the true value function: Consider a model economy where the social planner chooses an infinite sequence of consumption and next period's capital stock $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ in order to

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

   subject to

$$\begin{aligned} c_t + k_{t+1} &\leq y_t, && \forall t \\ c_t, k_t &\geq 0, && \forall t \\ k_0 &> 0. && \text{given} \end{aligned}$$

   Assume the following functional forms

$$\begin{aligned} u(c_t) &= \ln c_t, && \forall t, \sigma > 0 \\ y_t = F(k_t, 1) &= \gamma k_t^{\alpha}, && \forall t, \alpha \in (0,1) \end{aligned}$$

   Reformulate this optimization problem as a dynamic programming problem and write up the Bellman equation. Verify that the value function solving the functional equation (i.e. the Bellman equation) is of the following form

$$v(k) = a + b \ln k.$$

   Find a and b as functions of the model economy's structural parameters.

2. **Computations**

Consider a model economy where the social planner chooses an infinite sequence of consumption and next period's capital stock $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ in order to

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$\begin{aligned} c_t + k_{t+1} &\leq f(k_t) + (1-\delta) k_t, && \forall t \\ c_t, k_t &\geq 0, && \forall t \\ k_0 &> 0. && \text{given} \end{aligned}$$

Assume the following functional forms

$$\begin{aligned} u(c_t) &= \frac{c_t^{1-\sigma} - 1}{1-\sigma}, && \forall t, \sigma > 0 \\ y_t &= \gamma k_t^{\alpha}, && \forall t, \alpha \in (0,1) \end{aligned}$$

where $\alpha = .35, \beta = .98, \delta = .025, \sigma = 2,$ and $\gamma = 5.$

As we have derived in class we know that we can rewrite this as a recursive problem and that the Bellman equation is

$$v(k_t) = \max_{k_{t+1}} \{u(k_t, k_{t+1}) + \beta v(k_{t+1})\}.$$

In order to compute the stationary value function you choose to use discrete value function iteration. The capital stock can take three discrete values; $k \in \{k^{(1)},\ k^{(2)},\ k^{(3)}\} = \{2.85,\ 3.00,\ 3.15\}$. That means that $v(k_t)$ and $v(k_{t+1})$ are 3×1 vectors (Figure 1) whereas $u(k_t, k_{t+1})$ is a 3×3 matrix (Figure 2).

(a) Compute a (3×3) dimensional consumption matrix $C(i, j)$ with the value of consumption for all the (3×3) values of k_t and k_{t+1}. Then compute a (3×3)-dimensional matrix with the utility of consumption for all the (3×3) values of k_t and k_{t+1} similar to the one in Figure 2.

(b) Assume

$$v(k_{t+1}) = \begin{bmatrix} 167.6 \\ 168.1 \\ 168.6 \end{bmatrix}.$$

Before you maximize $\{u(k_t, k_{t+1}) + \beta v(k_{t+1})\}$ you need to compute the sum of two elements $u(k_t, k_{t+1})$ and $\beta v(k_{t+1})$. But since $u(k_t, k_{t+1})$ is a 3×3 matrix whereas $v(k_{t+1})$ is a 3×1 vector you need to transform $v(k_{t+1})$ into a 3×3 matrix. Note that $v(k_{t+1})$ is independent of k_t. The resulting matrix should therefore be like the matrix represented in Figure 3.

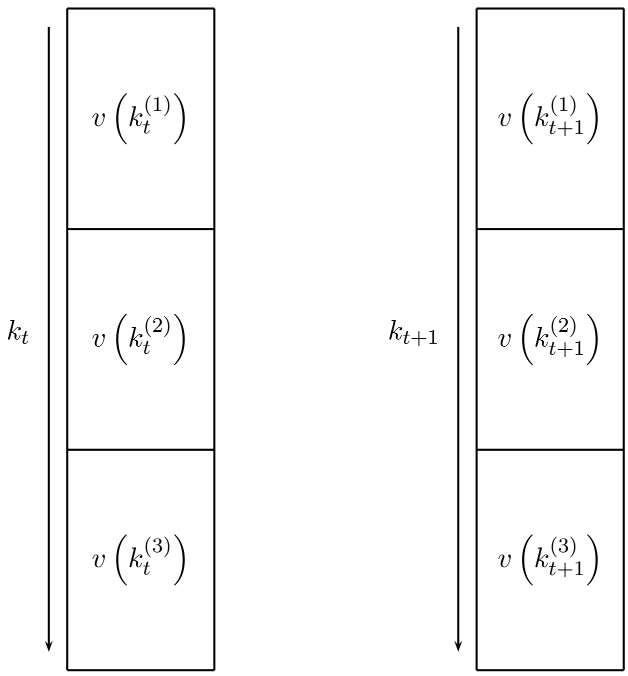
Hint: To transpose a matrix in Matlab you simply use $'$.

(c) Now that you have $\{u(k_t, k_{t+1}) + \beta v(k_{t+1})\}$ you can compute $v(k_t)$;

$$v(k_t) = \max_{k_{t+1}} \{u(k_t, k_{t+1}) + \beta v(k_{t+1})\}.$$

Hint: lookup help for `max`, i.e. type `help max`.

Appendix: Figures



(a) $v\left(k_t\right)$ (b) $v\left(k_{t+1}\right)$

Figure 1: $v\left(k_t\right)$ and $v\left(k_{t+1}\right)$

k_{t+1} →

k_t ↓			
	$u\left(k_t^{(1)}, k_{t+1}^{(1)}\right)$	$u\left(k_t^{(1)}, k_{t+1}^{(2)}\right)$	$u\left(k_t^{(1)}, k_{t+1}^{(3)}\right)$
	$u\left(k_t^{(2)}, k_{t+1}^{(1)}\right)$	$u\left(k_t^{(2)}, k_{t+1}^{(2)}\right)$	$u\left(k_t^{(2)}, k_{t+1}^{(3)}\right)$
	$u\left(k_t^{(3)}, k_{t+1}^{(1)}\right)$	$u\left(k_t^{(3)}, k_{t+1}^{(2)}\right)$	$u\left(k_t^{(3)}, k_{t+1}^{(3)}\right)$

Figure 2: $u\left(k_t, k_{t+1}\right)$

k_{t+1} →

k_t ↓

$u\left(k_t^{(1)}, k_{t+1}^{(1)}\right) + \beta v\left(k_{t+1}^{(1)}\right)$	$u\left(k_t^{(1)}, k_{t+1}^{(2)}\right) + \beta v\left(k_{t+1}^{(2)}\right)$	$u\left(k_t^{(1)}, k_{t+1}^{(3)}\right) + \beta v\left(k_{t+1}^{(3)}\right)$
$u\left(k_t^{(2)}, k_{t+1}^{(1)}\right) + \beta v\left(k_{t+1}^{(1)}\right)$	$u\left(k_t^{(2)}, k_{t+1}^{(2)}\right) + \beta v\left(k_{t+1}^{(2)}\right)$	$u\left(k_t^{(2)}, k_{t+1}^{(3)}\right) + \beta v\left(k_{t+1}^{(3)}\right)$
$u\left(k_t^{(3)}, k_{t+1}^{(1)}\right) + \beta v\left(k_{t+1}^{(1)}\right)$	$u\left(k_t^{(3)}, k_{t+1}^{(2)}\right) + \beta v\left(k_{t+1}^{(2)}\right)$	$u\left(k_t^{(3)}, k_{t+1}^{(3)}\right) + \beta v\left(k_{t+1}^{(3)}\right)$

Figure 3: $u\left(k_t, k_{t+1}\right) + \beta v\left(k_{t+1}\right)$