Econ 4310
Fall 2008
Chen

# Problem Set 4

Due on October 14th/16th, 2008

1. **Theory: Intertemporal Labor Supply**

   Consider the following problem

$$
\begin{aligned}
& \max_{\{c_t, h_t, A_t\}_{t=0}^{T}} E \sum_{t=0}^{T} \beta^t \left\{ \log(c_t) + \psi \frac{(1-h_t)^{1-\sigma} - 1}{1-\sigma} \right\} \\
& s.t. \\
A_{t+1} &= R(A_t + w_t h_t - c_t), \; A_0 \; given
\end{aligned}
$$

   where $c_t$ is consumption, $h_t$ is labor supply, $A_0$ is initial wealth, $R = 1 + r$ is the gross rate of return for saving. Here only $w_t$, the wage rate, is stochastic.

   (a) Derive and interpret the first order conditions for $c_t, h_t$ and $A_{t+1}$. What are the intertemporal and intratemporal optimality conditions?

   (b) Judging from the first order conditions, which variables link $c_t$ and $h_t$ in our model? What is the economic interpretation of this variable?

   (c) Define the Frisch elasticity of labor supply as

$$\eta = \frac{\partial \ln h_t}{\partial \ln w_t} \mid u'(c) \text{ constant}$$

   That is, Frisch elasticity of labor supply capture the elasticity of hours worked to wage rate, given the marginal utility of consumption unchanged. In other words, Frisch elasticity measures the substitution effects of a change in wage rate on labor supply. (Why is that? Find out what marginal utility of consumption in the first order condition is equal to and you will get the answer.) Compute $\eta$ in this model.

   (d) Use the Focs and discuss the effect of the following situations on $c_t$, and $h_t$.

i. Unexpected temporary shocks to wages
ii. Unexpected permanent shocks to wages

2. **Computation**

Consider the following version of the real business cycle model. The economy is populated by a large number of identical agents with total measure one. Each agents has a time endowment of 1 per period. Agents rank life time consumption and leisure according to

$$E\sum_{t=0}^{\infty}\beta^t u\left(c_t, h_t\right), \ \beta \in (0,1)$$

There is a large number of identical firms with aggregate technology given by

$$y_t = e^{z_t} F\left(k_t, h_t\right)$$

where $z_t$ is a discrete state Markov process with transaction probability matrix $\Pi$ and stationary distribution $P$. The function $F$ is assumed to be strictly increasing in each of its arguments, strictly concave, constant returns to scale and it also satisfies the INADA conditions. The technology shock has an unconditional mean of zero and it follows a two$-$state Markov process; the state space of this Markov chain is $Z = \{-\varepsilon, \varepsilon\}$. Denote the transition matrix $\Pi$ for this Markov process as

$$\Pi = \begin{pmatrix} \kappa & 1-\kappa \\ 1-\kappa & \kappa \end{pmatrix}$$

where the $(i,j)$ element $\pi_{i,j} = prob\left\{z_{t+1} = z_j \mid z_t = z_i\right\}$. The law of motion for the aggregate capital stock is given by

$$k_{t+1} = (1-\delta)\, k_t + i_t$$

where $\delta$ is the depreciation rate of capital. The resource constraint is given by $c_t + i_t = y_t$

(a) Write the Bellman's equation for the social planner's problem. State clearly what are the endogenous state variables, the exogenous state variables and the control variables in this problem

(b) Now assume that

$$
\begin{aligned}
u(c_t, h_t) &= \log(c_t) - \psi h_t \\
F(k_t, h_t) &= k_t^{\alpha} h_t^{1-\alpha}
\end{aligned}
$$

Compute the deterministic steady-state for the social planner's problem. A satisfactory answer should consist of three equations that can be used to solve for steady state value of $k, c, h$.

(c) Calibrate this model economy, so that its competitive allocation matches the long run observations

- $c/y = 0.85$
- $k/y = 3.0$,
- labor share of income is equal to 70%,
- and the fraction of leisure in total nonsleeping hours is equal to 80%.

In particular you need to choose values for $\alpha$, $\delta$, $\beta$, $\psi$ to match the above targets.

(d) We now would like to solve the Bellman's equation in part (a) by the method of value function iterations. To simply the problem, in this exercise we shut down the fluctuation of technology shock and let $z_t = 0$ for all $t$. (Note without aggregate uncertainty, the model setup is very close to that in our term paper without labor supply restriction except the utility specification).

i. Without aggregate uncertainty, what are the state variables and control variables now?

ii. **Discretize the state space** into $S = \{k_1, k_2, .., k_{kg}\}$. Use $k_1 = 0.85\overline{k}$ and $k_{kg} = 1.15\overline{k}$, where $\overline{k}$ is the steady state capital you calculated in part (b). Discretize the choice set of $h$ into grid space $\{h_1, h_2, .., h_{hg}\}$, use $h_1 = 0.8\overline{h}$ and $h_{hg} = 1.2\overline{h}$. Use $kg = 101$ and $hg = 51$.

iii. **Construct consumption and welfare matrix**
Compute the $(hg \times kg \times kg)$ dimension consumption matrix $C$, where element $C(i, j, n)$ denote the value of consumption when $h = h_i, k' = k_j$, and $k = k_n$. Next compute the $(hg \times kg \times kg)$ dimensional matrix for the current period with element $U(i, j, n)$ denoting the utility value when $h = h_i, k' = k_j$, and $k = k_n$.

iv. **Defined the initial guess for value function and compute the value function by value function iteration** Define an initial $(kg \times 1)$ dimensional vector (of zeros) for the initial guess of value function $v_0$. Compute the $(kg \times 1)$ dimensional vector of one-period value functions:
The operator $T$ maps bounded continuous functions into bounded continuous functions (contracting mapping)

$$(Tv)(k) = \max_{k',h} \left\{ U + \beta v_0 \left(k'\right) \right\}$$

whereby we have computed a closely approximated true value function by solving Bellman equation

$$(Tv)(k) = \max_{k',h} \left\{ U + \beta \mathbf{1} \left[ v_0 \left(k'\right) \right]^T \right\}$$

(Hint: when you write the code, you may define $objfun(:,:,n) = U(:,:,n) + \beta \mathbf{1} v_0^T$ for each $k = k_n$, where the superscript $T$ on $v$ denote the transpose of vector, $\mathbf{1}$ denotes a column vector of 1with length $hg$. Then use the max operator (twice) to find the optimal $k'$ and $h$ in a two-dimensional grid space. In other words, for each $k = k_n$ we are now searching on a two dimensional grid space defined on $h$ and $k'$ to find the maximum value).

v. Update the guess for the value function by setting $v_0 = Tv$ and continue by iterating on Bellman equation until the distance between $Tv$ and $v_0$ is very small.

(e) Plot the decision rules for $k'$, $h$ and $c$.