Econometrics - Lecture 4

Heteroskedasticity and Autocorrelation

Contents

- Violations of $V\{\varepsilon\} = \sigma^2 I_N$
- Heteroskedasticity
- GLS Estimation
- Autocorrelation

Gauss-Markov Assumptions

Observation y_i is a linear function

$$y_i = x_i'\beta + \varepsilon_i$$

of observations x_{ik}, $k = 1, \ldots, K$, of the regressor variables and the error term ε_i

for $i = 1, \ldots, N$; $x_i' = (x_{i1}, \ldots, x_{iK})$; $X = (x_{ik})$

A1	$E\{\varepsilon_i\} = 0$ for all i
A2	all ε_i are independent of all x_i (exogeneous x_i)
A3	$V\{\varepsilon_i\} = \sigma^2$ for all i (homoskedasticity)
A4	$Cov\{\varepsilon_i, \varepsilon_j\} = 0$ for all i and j with $i \neq j$ (no autocorrelation)

In matrix notation: $E\{\varepsilon\} = 0$, $V\{\varepsilon\} = \sigma^2 I_N$

OLS Estimator: Properties

Under assumptions (A1) and (A2):

1. The OLS estimator b is unbiased: $E\{b\} = \beta$

Under assumptions (A1), (A2), (A3) and (A4):

2. The variance of the OLS estimator is given by

$$V\{b\} = \sigma^2(\Sigma_i\, x_i\, x_i')^{-1} = \sigma^2(X'\, X)^{-1}$$

3. The sampling variance s^2 of the error terms ε_i,

$$s^2 = (N - K)^{-1}\, \Sigma_i\, e_i^2$$

is unbiased for σ^2

4. The OLS estimator b is BLUE (best linear unbiased estimator)

Violations of $V\{\varepsilon\} = \sigma^2 I_N$

Implications of the Gauss-Markov assumptions for ε:

$$V\{\varepsilon\} = \sigma^2 I_N$$

Violations:

- Heteroskedasticity: $V\{\varepsilon\} = \text{diag}(\sigma_1^2, \ldots, \sigma_N^2)$ or

  $$V\{\varepsilon\} = \sigma^2\Psi = \sigma^2 \text{diag}(h_1^2, \ldots, h_N^2)$$

- Autocorrelation: $V\{\varepsilon_i, \varepsilon_j\} \neq 0$ for at least one pair $i \neq j$ or

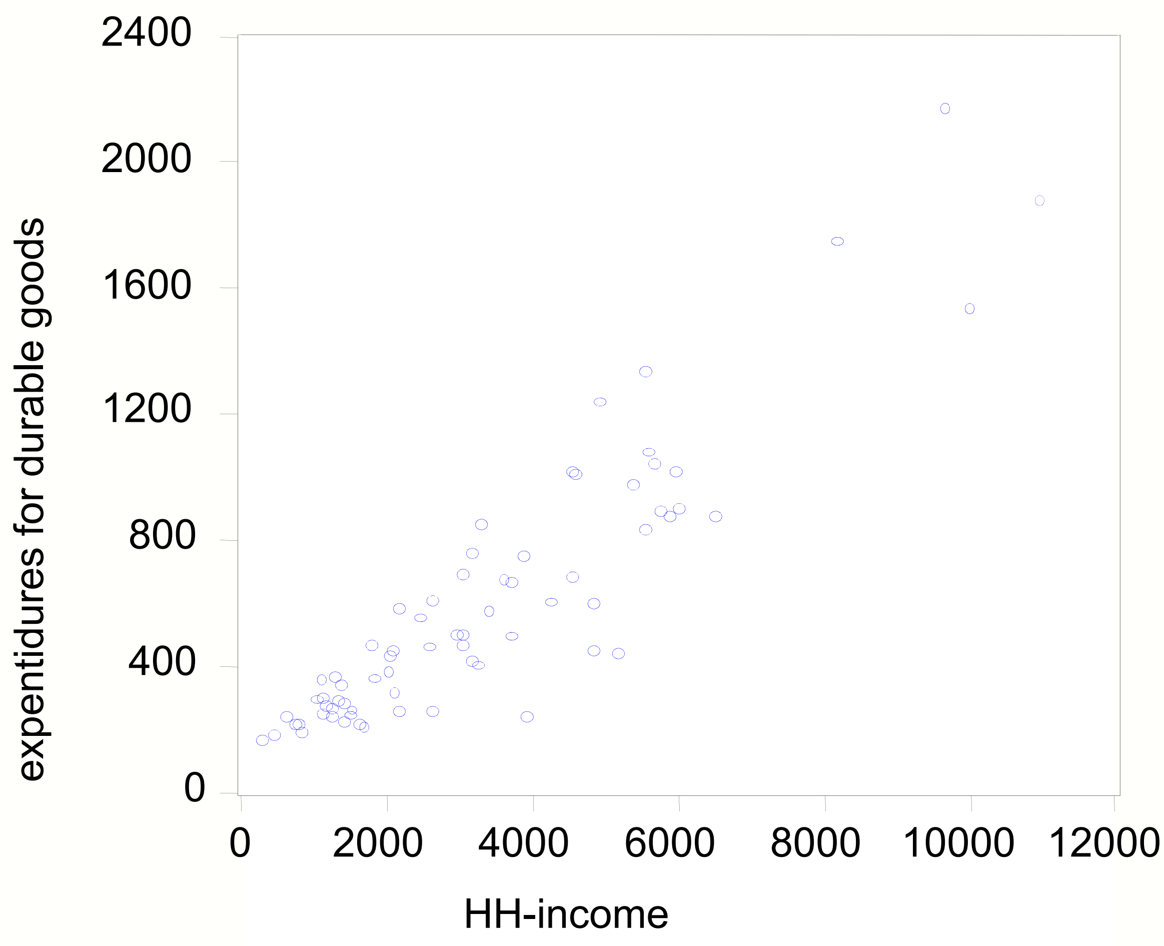
  $$V\{\varepsilon\} = \sigma^2\Psi$$

  with non-diagonal elements different from zero

Example: Household Income and Expenditures

70 households (HH):

monthly HH-income and expenditures for durable goods



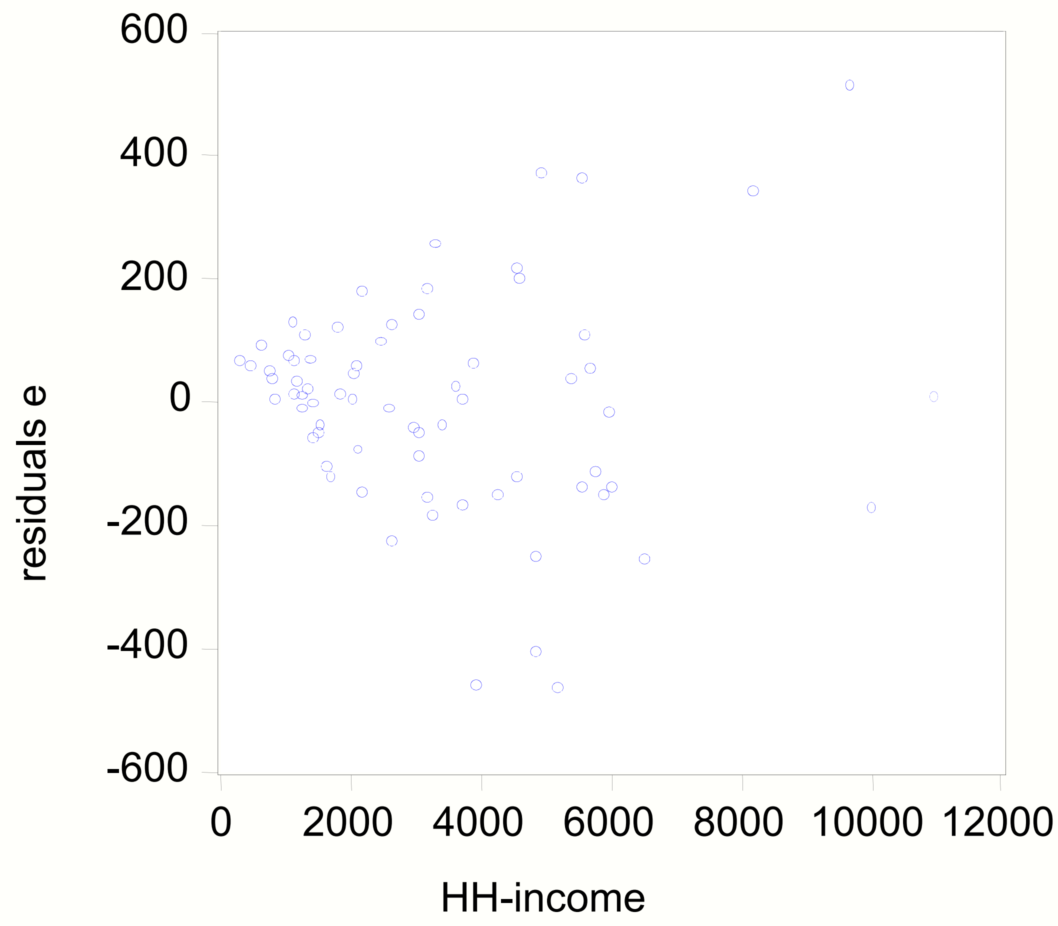
Household Income and Expenditures, cont`d

Residuals e = y- ŷ from

$$\hat{Y} = 44.18 + 0.17\,X$$

X: monthly HH-income
Y: expenditures for durable goods

the larger the income, the more scattered are the residuals



Typical Situations for Heteroskedasticity

Heteroskedasticity is typically observed

- In data from cross-sectional surveys, e.g., in households or regions
- Data with variance which depends of one or several explanatory variables, e.g., firm size
- Data from financial markets, e.g., exchange rates, stock returns

Example: Household Expenditures

With growing income increasing variation of expenditures; from Verbeek, Fig. 4.1

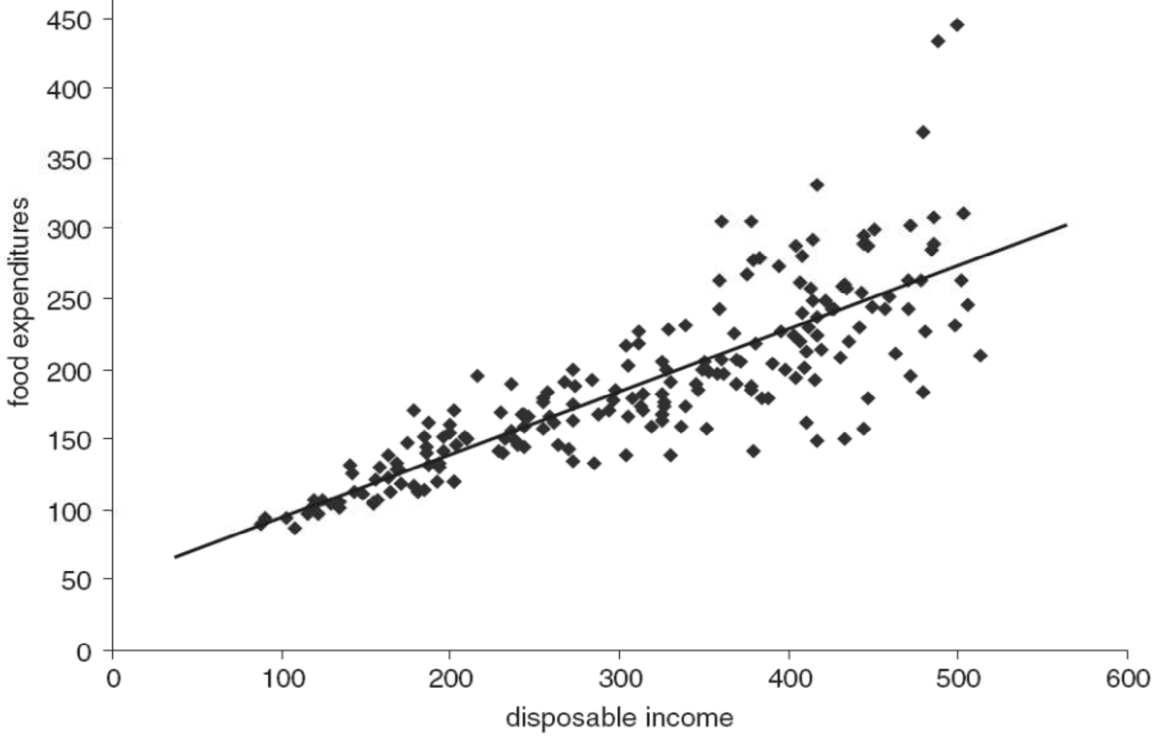


Figure 4.1 An Engel curve with heteroskedasticity

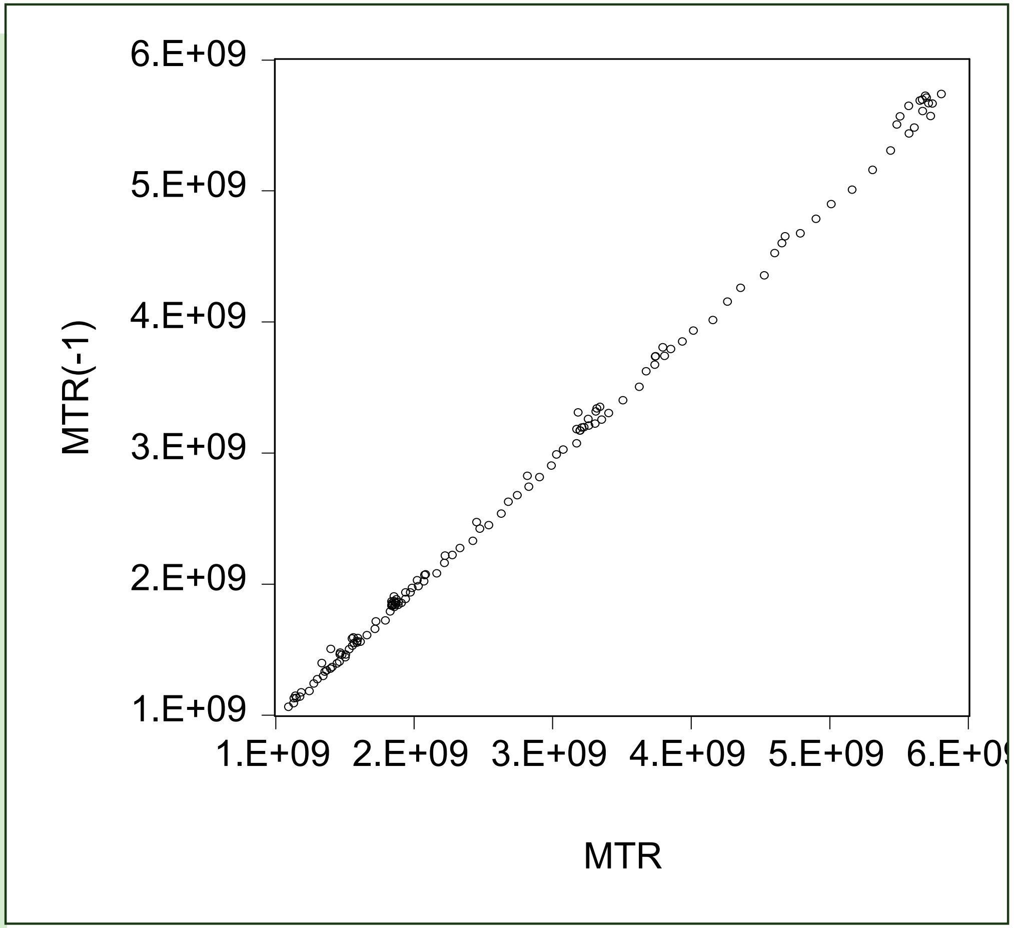
Autocorrelation of Economic Time-series

- Consumption in actual period is similar to that of the preceding period; the actual consumption „depends“ on the consumption of the preceding period
- Consumption, production, investments, etc.: it is to be expected that successive observations of economic variables correlate positively
- Seasonal adjustment: application of smoothing and filtering algorithms induces correlation

Example: Imports

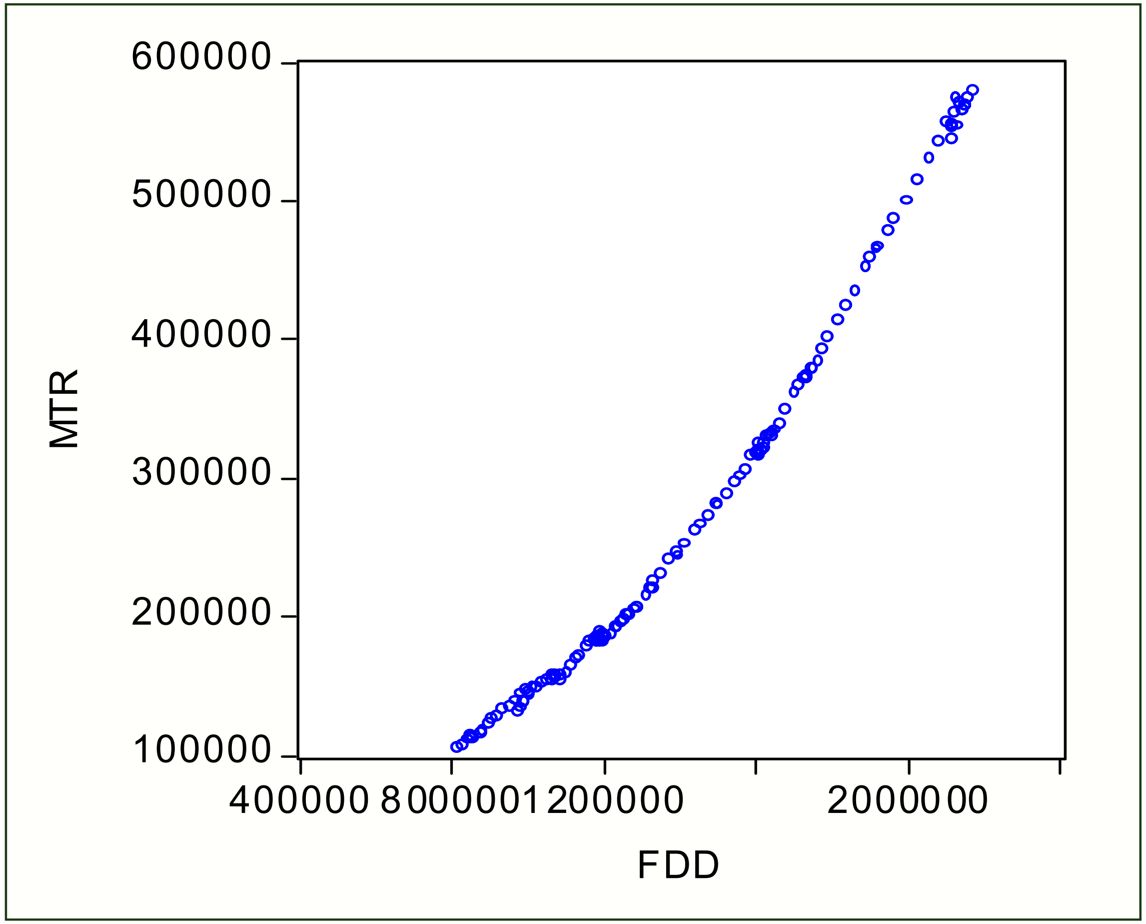
Scatter-diagram of by one period lagged imports [MTR(-1)] against actual imports [MTR]

Correlation coefficient between MTR und MTR(-1): 0.9994



Example: Import Function

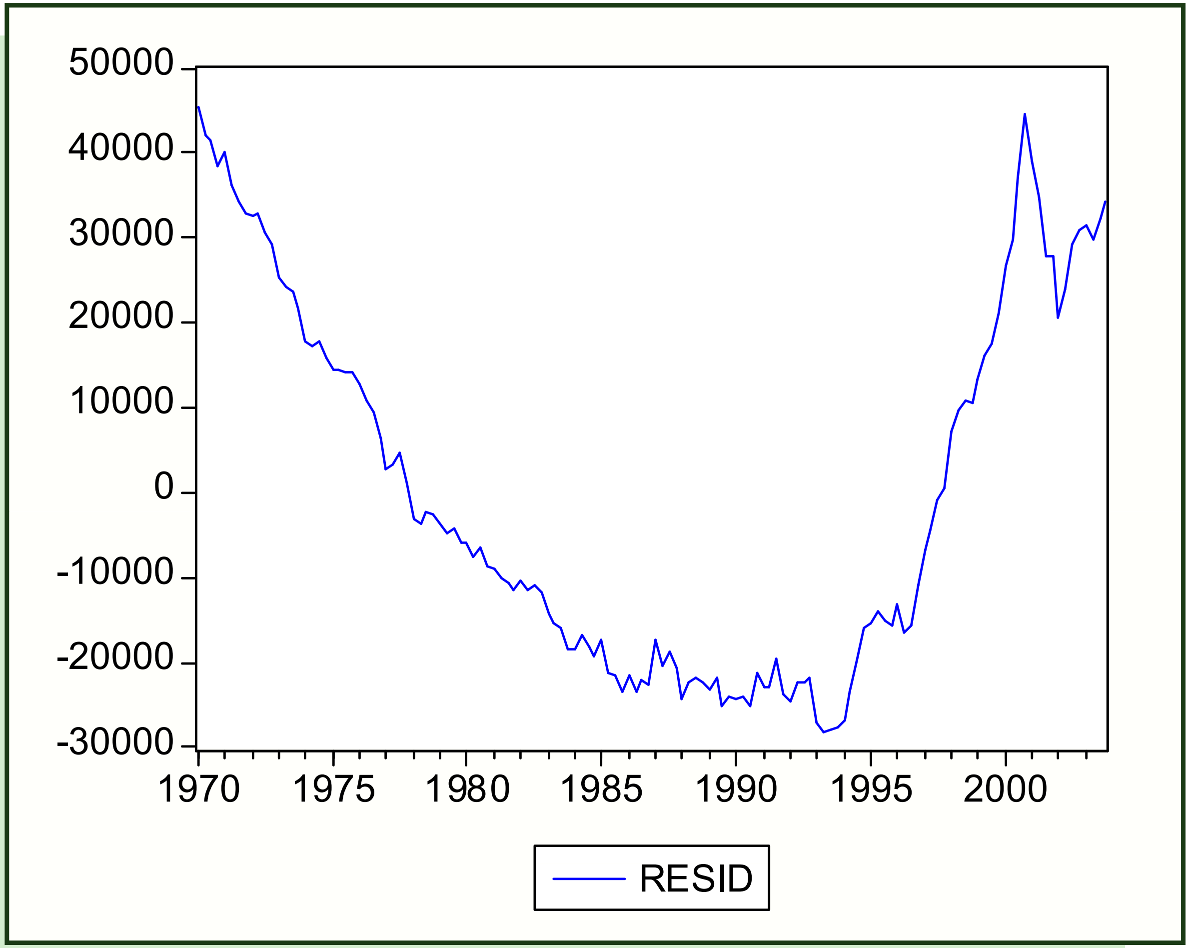
MTR: Imports
FDD: Demand
(from AWM-database)



Import function: MTR = -227320 + 0.36 FDD
R^2 = 0.977, t_{FFD} = 74.8

Import Function, cont'd

MTR: Imports
FDD: Demand
(from AWM-database)

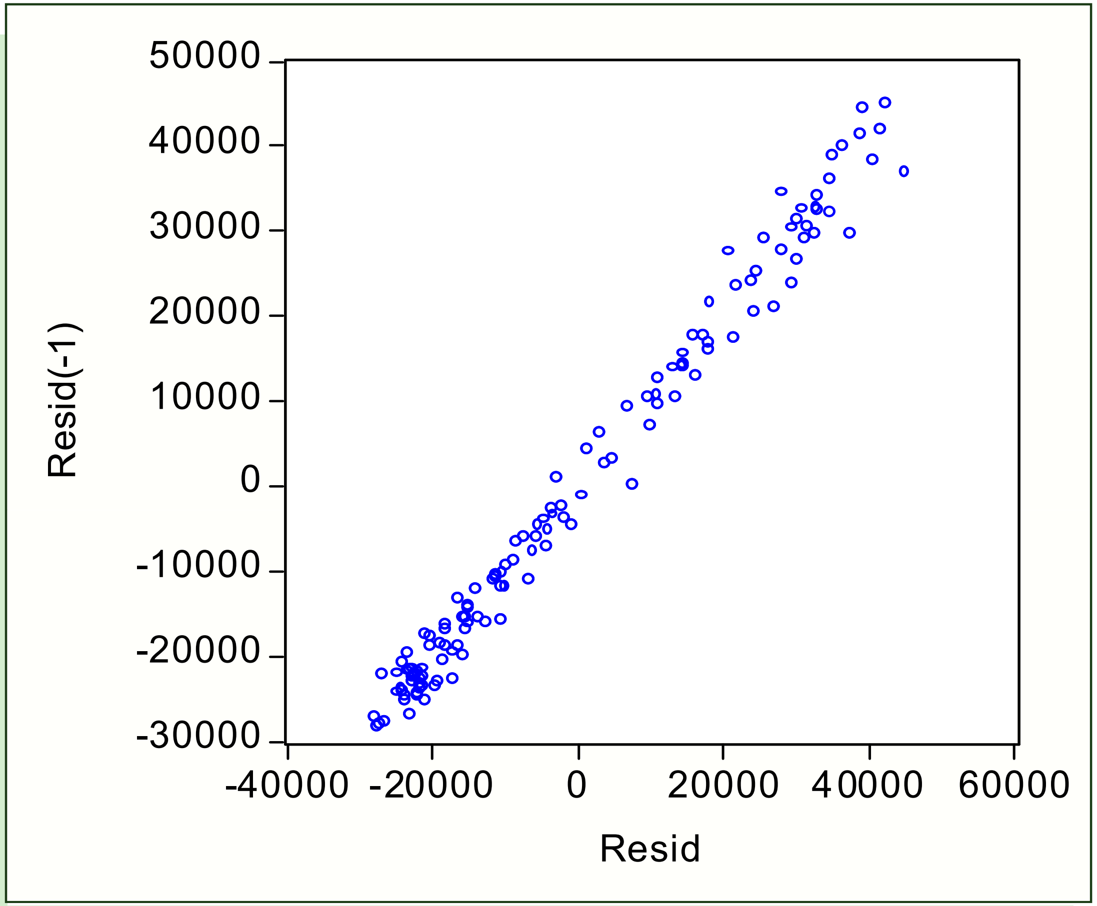


RESID: e_t = MTR - (-227320 + 0.36 FDD)

Import Function, cont'd

Scatter-diagram of by one period lagged residuals [Resid(-1)] against actual residuals [Resid]

Serial correlation!



Typical Situations for Autocorrelation

Autocorrelation is typically observed if

- a relevant regressor with trend or seasonal pattern is not included in the model: miss-specified model
- the functional form of a regressor is incorrectly specified
- the dependent variable is correlated in a way that is not appropriately represented in the systematic part of the model

Warning! Omission of a relevant regressor with trend implies autocorrelation of the error terms; in econometric analyses autocorrelation of the error terms is always possible!

- Autocorrelation of the error terms indicates deficiencies of the model specification
- Tests for autocorrelation are the most frequently used tool for diagnostic checking the model specification

Import Functions

- Regression of imports (MTR) on demand (FDD)

  $MTR = -2.27x10^9 + 0.357\ FDD$, $t_{FDD} = 74.9$, $R^2 = 0.977$

  Autocorrelation (order 1) of residuals:

  $Corr(e_t, e_{t-1}) = 0.993$
- Import function with trend (T)

  $MTR = -4.45x10^9 + 0.653\ FDD – 0.030x10^9\ T$

  $t_{FDD} = 45.8$, $t_T = -21.0$, $R^2 = 0.995$

  Multicollinearity? Corr(FDD, T) = 0.987!
- Import function with lagged imports as regressor

  $MTR = -0.124x10^9 + 0.020\ FDD + 0.956\ MTR_{-1}$

  $t_{FDD} = 2.89$, $t_{MTR(-1)} = 50.1$, $R^2 = 0.999$

Consequences of $V\{\varepsilon\} \neq \sigma^2 I_N$

OLS estimators *b* for β

- are unbiased
- are consistent
- have the covariance-matrix

$$V\{b\} = \sigma^2 (X'X)^{-1} X'\Psi X (X'X)^{-1}$$

- are not efficient estimators, not BLUE
- follow – under general conditions – asymptotically the normal distribution

The estimator $s^2 = e'e/(N-K)$ for σ^2 is biased

and

Consequences of $V\{\varepsilon\} \neq \sigma^2 I_N$ for Applications

- OLS estimators *b* for β are still unbiased
- Routinely computed standard errors are biased; the bias can be positive or negative
- *t*- and *F*-tests may be misleading

Remedies

- Alternative estimators
- Corrected standard errors
- Modification of the model

Tests for identification of

- heteroskedasticity
- autocorrelation

are important tools

Contents

- Violations of $V\{\varepsilon\} = \sigma^2 I_N$
- Heteroskedasticity
- GLS Estimation
- Autocorrelation

Inference under Heteroskedasticity

Covariance matrix of *b*:

$$V\{b\} = \sigma^2 (X'X)^{-1} X'\Psi X (X'X)^{-1}$$

Use of $\sigma^2 (X'X)^{-1}$ (the standard output of econometric software) instead of $V\{b\}$ for inference on β may be misleading

Remedies

- Use of correct variances and standard errors
- Transformation of the model so that the error terms are homoskedastic

The Correct Variances

- $V\{\varepsilon_i\} = \sigma_i^2 = \sigma^2 h_i^2$: each observation has its own unknown parameter h_i
- N observation for estimating N unknown parameters?

To estimate σ^2_i – and $V\{b\}$

- Known form of the heteroskedasticity, specific correction
  - E.g., $h_i^2 = z_i'\alpha$ for some variables z_i
  - Requires estimation of α
- White's heteroskedasticity-consistent covariance matrix estimator (HCCME)

$$\tilde{V}\{b\} = \sigma^2(X'X)^{-1}(\Sigma_i \hat{h}_i^2 x_i x_i') (X'X)^{-1}$$

with $\hat{h}_i^2 = e_i^2$

  - Denoted as HC_0
  - Inference based on HC_0: heteroskedasticity-robust inference

White's Standard Errors

White's standard errors for *b*

- Square roots of diagonal elements of HCCME
- Underestimate the true standard errors
- Various refinements, e.g., $HC_1 = HC_0[N/(N-K)]$

In **GRETL**: HC_0 is the default HCCME, HC_1 and other refinements are optionally available

An Alternative Estimator for *b*

Idea of the estimator

- Transform the model so that it satisfies the Gauss-Markov assumptions
- Apply OLS to the transformed model
- Should result in a BLUE

Transformation often depends upon unknown parameters that characterizing heteroskedasticity: two-step procedure

1. Estimate the parameters that characterize heteroskedasticity and transform the model
2. Estimate the transformed model

The procedure results in an approximately BLUE

An Example

Model:

$y_i = x_i'\beta + \varepsilon_i$ with $V\{\varepsilon_i\} = \sigma_i^2 = \sigma^2 h_i^2$

Division by h_i results in

$y_i / h_i = (x_i / h_i)'\beta + \varepsilon_i / h_i$

with a homoskedastic error term

$V\{\varepsilon_i / h_i\} = \sigma_i^2 / h_i^2 = \sigma^2$

OLS applied to the transformed model gives

$$\hat{\beta} = \left(\sum_i h_i^{-2} x_i x_i' \right)^{-1} \sum_i h_i^{-2} x_i y_i$$

It is called a generalized least squares (GLS) or weighted least squares (WLS) estimator

Weighted Least Squares Estimator

- A GLS or WLS estimator is a least squares estimator where each observation is weighted by a non-negative factor $w_i > 0$:

$$\hat{\beta}_w = \left(\sum_i w_i x_i' x_i \right)^{-1} \sum_i w_i x_i' y_i$$

- Weights proportional to the inverse of the error term variance: Observations with a higher error term variance have a lower weight; they provide less accurate information on β
- Needs knowledge of the h_i
  - Is seldom available
  - Is mostly provided by estimates of h_i based on assumptions on the form of h_i
  - E.g., $h_i^2 = z_i'\alpha$ for some variables z_i
- Analogous with general weights w_i

Example: Labor Demand

Verbeek's data set "labour2": Sample of 569 Belgian companies (data from 1996)

- Variables
  - *labour*: total employment (number of employees)
  - *capital*: total fixed assets
  - *wage*: total wage costs per employee (in 1000 EUR)
  - *output*: value added (in million EUR)
- Labour demand function

$$labour = \beta_1 + \beta_2 * wage + \beta_3 * output + \beta_4 * capital$$

Labor Demand Function

For Belgian companies, 1996; Verbeek

Table 4.1 OLS results linear model

Dependent variable: *labour*

Variable	Estimate	Standard error	t-ratio
constant	287.72	19.64	14.648
wage	−6.742	0.501	−13.446
output	15.40	0.356	43.304
capital	−4.590	0.269	−17.067

$s = 156.26$ $R^2 = 0.9352$ $\bar{R}^2 = 0.9348$ $F = 2716.02$

Labor Demand Function, cont'd

Can the error terms be assumed to be homoskedastic?

- They may vary depending of the company size, measured by, e.g., size of output or capital
- Regression of squared residuals on appropriate regressors will indicate heteroskedasticity

Labor Demand Function, cont'd

Auxiliary regression of squared residuals, Verbeek

Table 4.2 Auxiliary regression Breusch–Pagan test

Dependent variable: e_i^2

Variable	Estimate	Standard error	t-ratio
constant	−22719.51	11838.88	−1.919
wage	228.86	302.22	0.757
output	5362.21	214.35	25.015
capital	−3543.51	162.12	−21.858

$s = 94182$ $R^2 = 0.5818$ $\bar{R}^2 = 0.5796$ $F = 262.05$

Indicates dependence of error terms on *output*, *capital*, not on *wage*

Labor Demand Function, cont'd

Estimated function

labour = β_1 + β_2**wage* + β_3**output* + β_4**capital*

OLS estimates without (s.e.) and with White standard errors (White s.e.), and GLS estimates with $w_i = 1/e_i$

	β_1	β_2	β_3	β_4
Coeff OLS	287.19	-6.742	15.400	-4.590
s.e.	19.642	0.501	0.356	0.269
White s.e.	64.877	1.852	2.482	1.713
Coeff GLS	282.06	-6.609	15.235	-4.197
s.e.	1.808	0.042	0.094	0.141

The standard errors are inflated by factors 3.7 (wage), 6.4 (*capital*), 7.0 (*output*) wrt the White s.e.

Labor Demand Function, cont'd

With White standard errors: Output from **GRETL**

Dependent variable : LABOR
Heteroskedastic-robust standard errors, variant HC0,

	coefficient	std. error	t-ratio	p-value
const	287,719	64,8770	4,435	1,11e-05 ***
WAGE	-6,7419	1,8516	-3,641	0,0003 ***
CAPITAL	-4,59049	1,7133	-2,679	0,0076 ***
OUTPUT	15,4005	2,4820	6,205	1,06e-09 ***

Mean dependent var	201,024911	S.D. dependent var	611,9959
Sum squared resid	13795027	S.E. of regression	156,2561
R- squared	0,935155	Adjusted R-squared	0,934811
F(2, 129)	225,5597	P-value (F)	3,49e-96
Log-likelihood	455,9302	Akaike criterion	7367,341
Schwarz criterion	-3679,670	Hannan-Quinn	7374,121

Tests against Heteroskedasticity

Due to unbiasedness of *b*, residuals are expected to indicate heteroskedasticity

Graphical displays of residuals may give useful hints

Residual-based tests:

- Breusch-Pagan test
- Koenker test
- Goldfeld-Quandt test
- White test

Breusch-Pagan Test

For testing whether the error term variance is a function of $Z_2, \ldots, Z_p$

Model for heteroskedasticity

$$\sigma_i^2/\sigma^2 = h(z_i'\alpha)$$

with function h with $h(0)=1$, p-vectors z_i und α, an intercept and p-1 variables $Z_2, \ldots, Z_p$

Null hypothesis

$$H_0: \alpha = 0$$

implies $\sigma_i^2 = \sigma^2$ for all i, i.e., homoskedasticity

Auxiliary regression of the standardized squared OLS residuals $g_i = e_i^2/s^2 - 1$, $s^2 = e'e/N$, on z_i (and squares of z_i)

Test statistic: BP = N*ESS with the explained sum of squares ESS = N*V($\hat{g}$), of the auxiliary regression; $\hat{g}$ are the fitted values for g. BP follows approximately the Chi-squared distribution with p d.f.

Breusch-Pagan Test, cont'd

Typical functions h for $h(z_i'\alpha)$

- Linear regression: $h(z_i'\alpha) = z_i'\alpha$
- Exponential function $h(z_i'\alpha) = \exp\{z_i'\alpha\}$
  - Auxiliary regression of the log (e_i^2) upon z_i
  - "Multiplicative heteroskedasticity"
  - Variances are non-negative
- Koenker test: variant of the BP test which is robust against non-normality of the error terms
- **GRETL**: The output window of OLS estimation allows the execution of the Breusch-Pagan test with $h(z_i'\alpha) = z_i'\alpha$
  - OLS output => Tests => Heteroskedasticity => Breusch-Pagan
  - Koenker test: OLS output => Tests => Heteroskedasticity => Koenker

Labor Demand Function, cont'd

Auxiliary regression of squared residuals, Verbeek

Table 4.2 Auxiliary regression Breusch–Pagan test

Dependent variable: e_i^2

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constant	−22719.51	11838.88	−1.919
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capital	−3543.51	162.12	−21.858

$s = 94182 \quad R^2 = 0.5818 \quad \bar{R}^2 = 0.5796 \quad F = 262.05$

NR^2 = 331.04, p-value = 2.17E-70; reject null hypothesis of homoskedasticity

Goldfeld-Quandt Test

For testing whether the error term variance has values σ_A^2 and σ_B^2 for observations from regime A and B, respectively, $\sigma_A^2 \neq \sigma_B^2$ regimes can be urban vs rural area, economic prosperity vs stagnation, etc.

Example (in matrix notation):

$y_A = X_A\beta_A + \varepsilon_A$, $V\{\varepsilon_A\} = \sigma_A^2 I_{NA}$ (regime A)

$y_B = X_B\beta_B + \varepsilon_B$, $V\{\varepsilon_B\} = \sigma_B^2 I_{NB}$ (regime B)

Null hypothesis: $\sigma_A^2 = \sigma_B^2$

Test statistic:

$$F = \frac{S_A}{S_B} \frac{N_B - K}{N_A - K}$$

with S_i: sum of squared residuals for i-th regime; follows under H_0 exactly or approximately the F-distribution with N_A-K and N_B-K d.f.

Goldfeld-Quandt Test, cont'd

Test procedure in three steps:

1. Sort the observations with respect to the regimes
2. Separate fittings of the model to the N_A and N_B observations; sum of squared residuals S_A and S_B
3. Calculation of test statistic F

White Test

For testing whether the error term variance is a function of the model regressors, their squares and their cross-products

Auxiliary regression of the squared OLS residuals upon x_i's, squares of x_i's and cross-products

Test statistic: NR^2 with R^2 of the auxiliary regression; follows the Chi-squared distribution with the number of coefficients in the auxiliary regression as d.f.

The number of coefficients in the auxiliary regression may become large, maybe conflicting with size of N, resulting in low power of the White test

Labor Demand Function, cont'd

White's test for heteroskedasticity
OLS, using observations 1-569
Dependent variable: uhat^2

	coefficient	std. error	t-ratio	p-value	
const	-260,910	18478,5	-0,01412	0,9887	
WAGE	554,352	833,028	0,6655	0,5060	
CAPITAL	2810,43	663,073	4,238	2,63e-05	***
OUTPUT	-2573,29	512,179	-5,024	6,81e-07	***
sq_WAGE	-10,0719	9,29022	-1,084	0,2788	
X2_X3	-48,2457	14,0199	-3,441	0,0006	***
X2_X4	58,5385	8,11748	7,211	1,81e-012	***
sq_CAPITAL	14,4176	2,01005	7,173	2,34e-012	***
X3_X4	-40,0294	3,74634	-10,68	2,24e-024	***
sq_OUTPUT	27,5945	1,83633	15,03	4,09e-043	***

Unadjusted R-squared = 0,818136

Test statistic: TR^2 = 465,519295,
with p-value = P(Chi-square(9) > 465,519295) = 0,000000

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Generalized Least Squares Estimator

- A GLS or WLS estimator is a least squares estimator where each observation is weighted by a non-negative factor $w_i > 0$
- Example:

  $y_i = x_i'\beta + \varepsilon_i$ with $V\{\varepsilon_i\} = \sigma_i^2 = \sigma^2 h_i^2$

  - Division by h_i results in a model with homoskedastic error terms

    $V\{\varepsilon_i / h_i\} = \sigma_i^2 / h_i^2 = \sigma^2$

  - OLS applied to the transformed model results in the weighted least squares (GLS) estimator with $w_i = h_i^{-2}$:

$$\hat{\beta} = \left(\sum_i h_i^{-2} x_i x_i' \right)^{-1} \sum_i h_i^{-2} x_i y_i$$

- The concept of transforming the model so that Gauss-Markov assumptions are fulfilled is used also in more general situations, e.g., for autocorrelated error terms

Properties of GLS Estimators

The GLS estimator

$$\hat{\beta} = \left(\sum\nolimits_i h_i^{-2} x_i x_i'\right)^{-1} \sum\nolimits_i h_i^{-2} x_i y_i$$

is a least squares estimator; standard properties of OLS estimator apply

- The covariance matrix of the GLS estimator is

$$V\left\{\hat{\beta}\right\} = \sigma^2 \left(\sum\nolimits_i h_i^{-2} x_i x_i'\right)^{-1}$$

- Unbiased estimator of the error term variance

$$\hat{\sigma}^2 = \frac{1}{N-K} \sum\nolimits_i h_i^{-2} \left(y_i - x_i'\hat{\beta}\right)^2$$

- Under the assumption of normality of errors, *t*- and *F*-tests can be used; for large *N*, these properties apply approximately without normality assumption

Feasible GLS Estimator

Is a GLS estimator with estimated weights w_i

- Substitution of the weights $w_i = h_i^{-2}$ by estimates $\hat{h}_i^{-2}$

$$\hat{\beta}^* = \left(\sum_i \hat{h}_i^{-2} x_i x_i'\right)^{-1} \sum_i \hat{h}_i^{-2} x_i y_i$$

- Feasible (or estimated) GLS or FGLS or EGLS estimator
- For consistent estimates $\hat{h}_i$, the FGLS and GLS estimators are asymptotically equivalent
- For small values of N, FGLS estimators are in general not BLUE
- For consistently estimated $\hat{h}_i$, the FGLS estimator is consistent and asymptotically efficient with covariance matrix (estimate for σ^2: based on FGLS residuals)

$$V\{\hat{\beta}^*\} = \hat{\sigma}^2 \left(\sum_i \hat{h}_i^{-2} x_i x_i'\right)^{-1}$$

- Warning: the transformed model is uncentered

Multiplicative Heteroskedasticity

Assume $V\{\varepsilon_i\} = \sigma_i^2 = \sigma^2 h_i^2 = \sigma^2 \exp\{z_i'\alpha\}$

- The auxiliary regression

  $$\log e_i^2 = \log \sigma^2 + z_i'\alpha + v_i \quad \text{with } v_i = \log(e_i^2/\sigma_i^2)$$

  provides a consistent estimator a for α
- Transform the model $y_i = x_i'\beta + \varepsilon_i$ with $V\{\varepsilon_i\} = \sigma_i^2 = \sigma^2 h_i^2$ by dividing through $\hat{h}_i$ from $\hat{h}_i^2 = \exp\{z_i'a\}$
- Error term in this model is (approximately) homoskedastic
- Applying OLS to the transformed model gives the FGLS estimator for β

FGLS Estimation

In the following steps:

1. Calculate the OLS estimates b for β
2. Compute the OLS residuals $e_i = y_i – x_i'b$
3. Regress $\log(e_i^2)$ on z_i and a constant, obtaining estimates a for α

   $$\log e_i^2 = \log \sigma^2 + z_i'\alpha + v_i$$

4. Compute $\hat{h}_i^2 = \exp\{z_i'a\}$, transform all variables and estimate the transformed model to obtain the FGLS estimators:

   $$y_i / \hat{h}_i = (x_i / \hat{h}_i)'\beta + \varepsilon_i / \hat{h}_i$$

5. The consistent estimate s^2 for σ^2, based on the FGLS-residuals, and the consistently estimated covariance matrix

   $$\hat{V}\{\hat{\beta}^*\} = s^2 \left(\sum_i \hat{h}_i^{-2} x_i x_i' \right)^{-1}$$

   are part of the standard output when regressing the transformed model

Labor Demand Function

For Belgian companies, 1996; Verbeek

Table 4.5 OLS results loglinear model with White standard errors

Dependent variable: log(*labour*)

Variable	Estimate	Heteroskedasticity-consistent Standard error	*t*-ratio
constant	6.177	0.294	21.019
log(*wage*)	−0.928	0.087	−10.706
log(*output*)	0.990	0.047	21.159
log(*capital*)	−0.004	0.038	−0.098

$s = 0.465 \quad R^2 = 0.8430 \quad \bar{R}^2 = 0.8421 \quad F = 544.73$

Log-tranformation is expected to reduce heteroskedasticity

Labor Demand Function, cont'd

For Belgian companies, 1996; Verbeek

Table 4.6 Auxiliary regression multiplicative heteroskedasticity

Dependent variable: $\log e_i^2$

Variable	Estimate	Standard error	t-ratio
constant	−3.254	1.185	−2.745
log(*wage*)	−0.061	0.344	−0.178
log(*output*)	0.267	0.127	2.099
log(*capital*)	−0.331	0.090	−3.659

$s = 2.241 \quad R^2 = 0.0245 \quad \bar{R}^2 = 0.0193 \quad F = 4.73$

Breusch-Pagan test: NR^2 = 66.23, p-value: 1,42E-13

Labor Demand Function, cont'd

For Belgian companies, 1996; Verbeek

Weights estimated assuming multiplicative heteroskedasticity

Table 4.7 EGLS results loglinear model

Dependent variable: log(*labour*)

Variable	Estimate	Standard error	*t*-ratio
constant	5.895	0.248	23.806
log(*wage*)	−0.856	0.072	−11.903
log(*output*)	1.035	0.027	37.890
log(*capital*)	−0.057	0.022	−2.636

$s = 2.509$ $R^2 = 0.9903$ $\bar{R}^2 = 0.9902$ $F = 14401.3$

Labor Demand Function, cont'd

Estimated function

$$\log(labour) = \beta_1 + \beta_2 * \log(wage) + \beta_3 * \log(output) + \beta_4 * \log(capital)$$

The table shows: OLS estimates without (s.e.) and with White standard errors (White s.e.) as well as FGLS estimates and standard errors

	β_1	β_2	β_3	β_4
OLS coeff	6.177	-0.928	0.990	-0.0037
s.e.	0.246	0.071	0.026	0.0188
White s.e.	0.293	0.086	0.047	0.0377
FGLS coeff	5.895	-0.856	1.035	-0.0569
s.e.	0.248	0.072	0.027	0.0216

Labor Demand Function, cont'd

Some comments:

- Reduction of standard errors in FGLS estimation as compared with heteroskedasticity-robust estimation, efficiency gains
- Comparison with OLS estimation not appropriate
- FGLS estimates differ slightly from OLS estimates; effect of capital is indicated to be relevant (*p*-value: 0.0086)
- R^2 of FGLS estimation is misleading
  - Model is uncentered, no intercept
  - Comparison with that of OLS estimation not appropriate, explained variable differ

Contents

- Violations of $V\{\varepsilon\} = \sigma^2 I_N$
- Heteroskedasticity
- GLS Estimation
- Autocorrelation

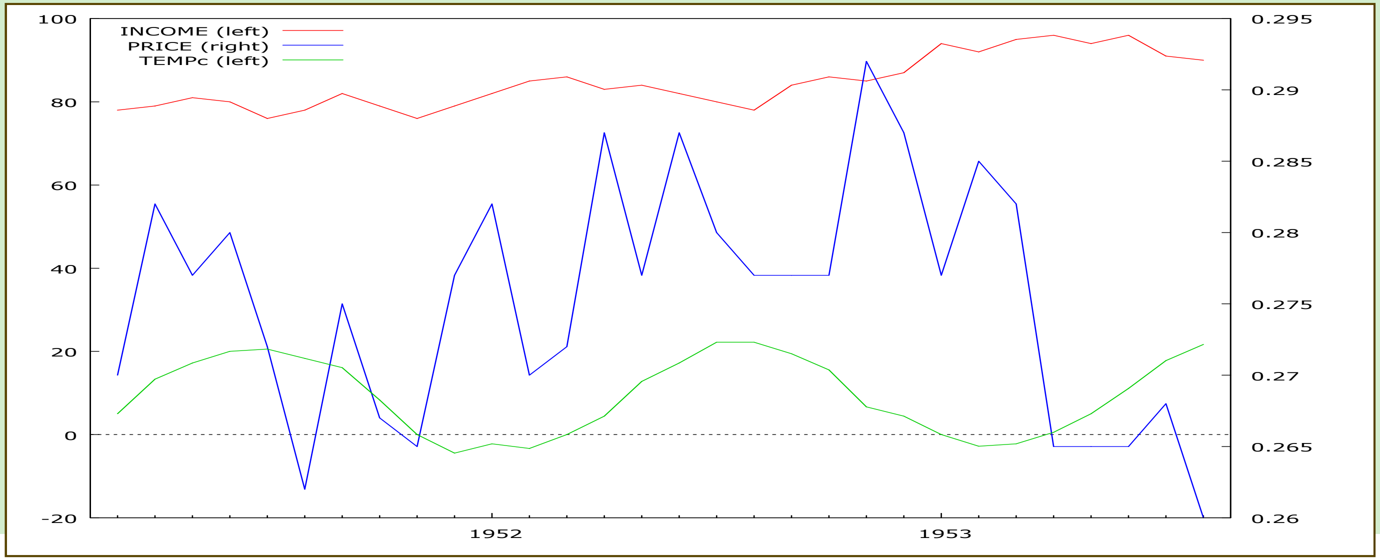
Autocorrelation

- Typical for time series data such as consumption, production, investments, etc., and models for time series data
- Autocorrelation of error terms is typically observed if
  - a relevant regressor with trend or seasonal pattern is not included in the model: miss-specified model
  - the functional form of a regressor is incorrectly specified
  - the dependent variable is correlated in a way that is not appropriately represented in the systematic part of the model
- Autocorrelation of the error terms indicates deficiencies of the model specification such as omitted regressors, incorrect functional form, incorrect dynamic
- Tests for autocorrelation are the most frequently used tool for diagnostic checking the model specification

Example: Demand for Ice Cream

Time series of 30 four weekly observations (1951-1953)

- Variables
  - *cons*: consumption of ice cream per head (in pints)
  - *income*: average family income per week (in USD, red line)
  - *price*: price of ice cream (in USD per pint, blue line)
  - *temp*: average temperature (in Fahrenheit); tempc: (green, in °C)



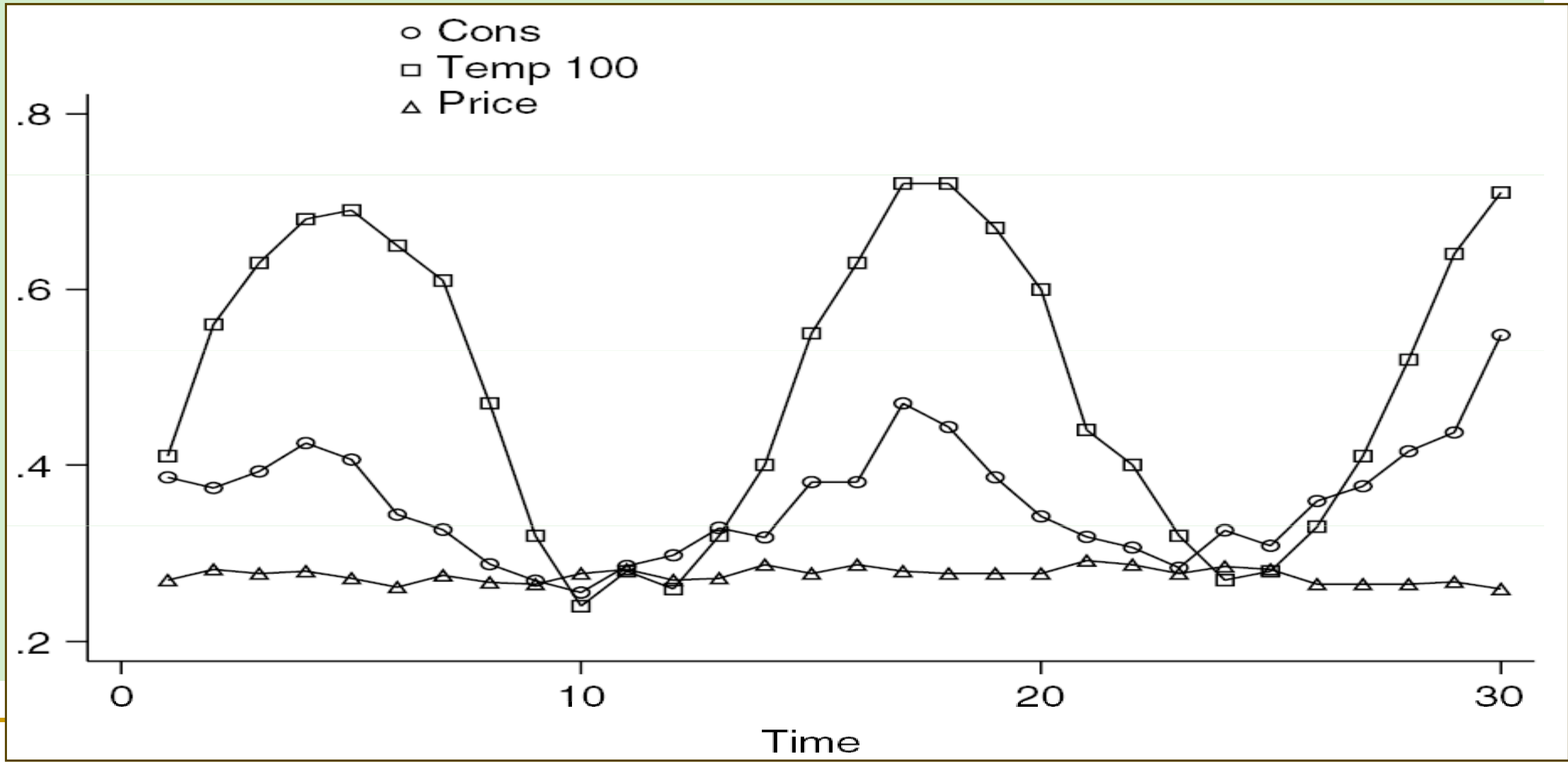
Demand for Ice Cream, cont'd

Time series plot of

Cons: consumption of ice cream per head (in pints); mean: 0.36

Temp/100: average temperature (in Fahrenheit)

Price (in USD per pint); mean: 0.275 USD



Demand for Ice Cream, cont'd

Demand for ice cream, measured by *cons*, explained by *price*, *income*, and *temp*

Table 4.9 OLS results

Dependent variable: *cons*

Variable	Estimate	Standard error	t-ratio
constant	0.197	0.270	0.730
price	−1.044	0.834	−1.252
income	0.00331	0.00117	2.824
temp	0.00345	0.00045	7.762

$s = 0.0368$ $R^2 = 0.7190$ $\bar{R}^2 = 0.6866$ $F = 22.175$
$dw = 1.0212$

Demand for Ice Cream, cont'd

Demand for ice cream explained from income and price index

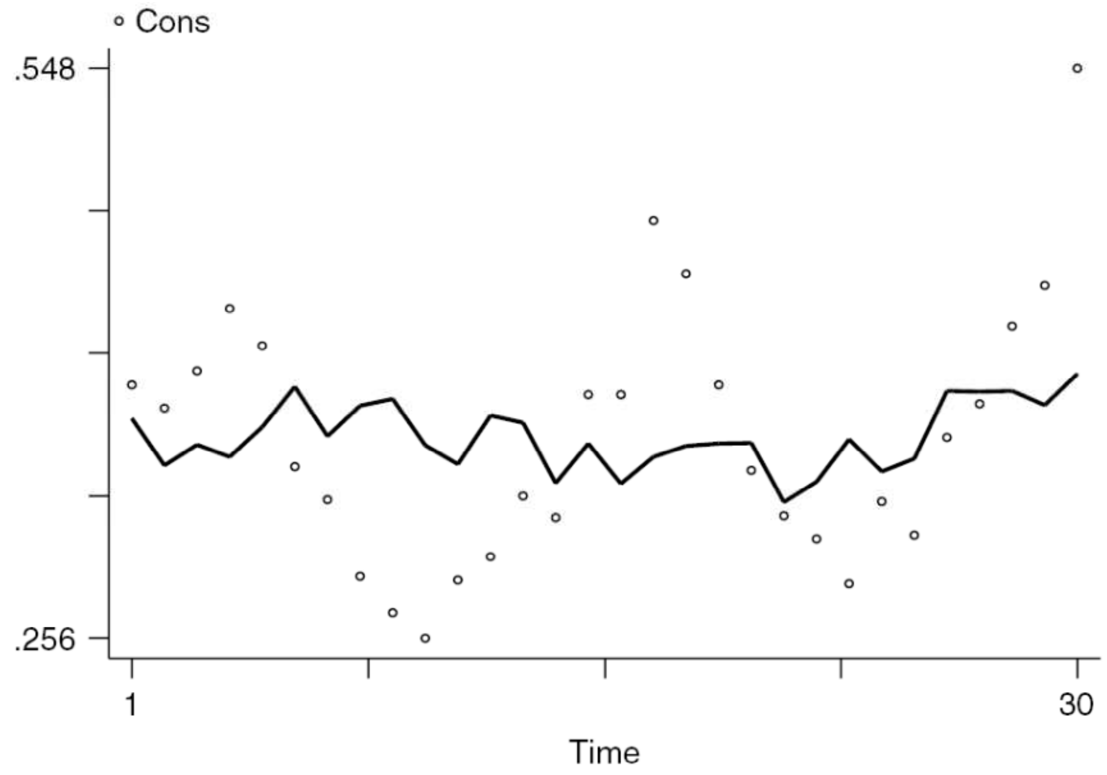
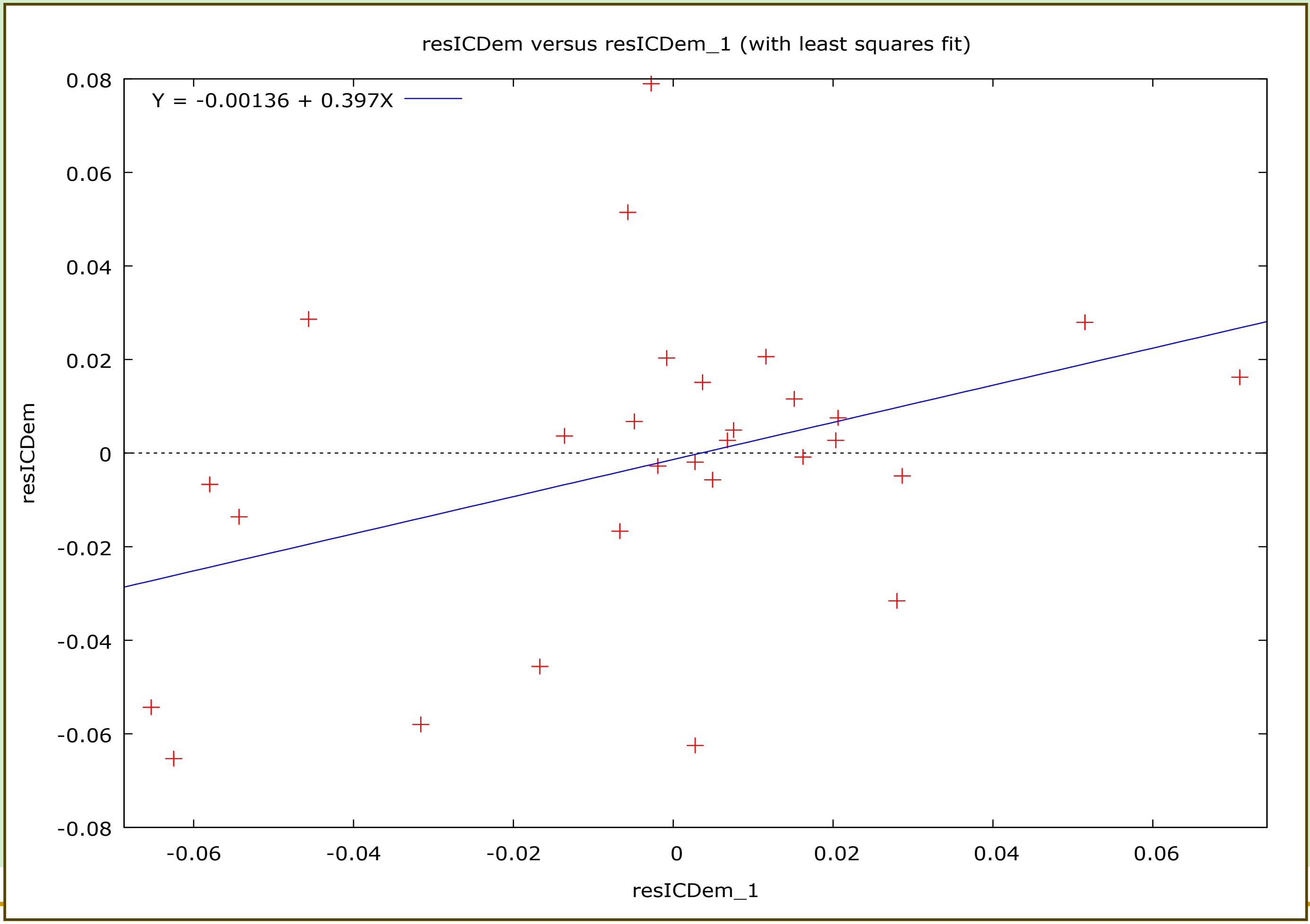


Figure 4.1 Actual and fitted consumption of ice cream, March 1951–July 1953

Demand for Ice Cream, cont'd

Ice cream model: Scatter-plot of residuals e_t vs e_{t-1} (r = 0.401)



A Model with AR(1) Errors

Linear regression

$$y_t = x_t'\beta + \varepsilon_t \text{ }^{1)}$$

with

$$\varepsilon_t = \rho\varepsilon_{t-1} + v_t \text{ with } -1 < \rho < 1 \text{ or } |\rho| < 1$$

where v_t are uncorrelated random variables with mean zero and constant variance σ_v^2

- For $\rho \neq 0$, the error terms ε_t are correlated; the Gauss-Markov assumption $V\{\varepsilon\} = \sigma_\varepsilon^2 I_N$ is violated
- The other Gauss-Markov assumptions are assumed to be fulfilled

The sequence ε_t, t = 0, 1, 2, … which follows $\varepsilon_t = \rho\varepsilon_{t-1} + v_t$ is called an autoregressive process of order 1 or AR(1) process

---

1) In the context of time series models, variables are indexed by „*t*“

Properties of AR(1) Processes

Repeated substitution of ε_{t-1}, ε_{t-2}, etc. results in

$$\varepsilon_t = \rho\varepsilon_{t-1} + v_t = v_t + \rho v_{t-1} + \rho^2 v_{t-2} + \ldots$$

with v_t being uncorrelated and having mean zero and variance σ_v^2:

- $E\{\varepsilon_t\} = 0$
- $V\{\varepsilon_t\} = \sigma_\varepsilon^2 = \sigma_v^2(1-\rho^2)^{-1}$

This results from $V\{\varepsilon_t\} = \sigma_v^2 + \rho^2\sigma_v^2 + \rho^4\sigma_v^2 + \ldots = \sigma_v^2(1-\rho^2)^{-1}$ for $|\rho|<1$ as the geometric series $1 + \rho^2 + \rho^4 + \ldots$ has the sum $(1-\rho^2)^{-1}$ given that $|\rho| < 1$

  - for $|\rho| > 1$, $V\{\varepsilon_t\}$ is undefined

- $Cov\{\varepsilon_t, \varepsilon_{t-s}\} = \rho^s \sigma_v^2 (1-\rho^2)^{-1}$ for $s > 0$

  all error terms are correlated; covariances – and correlations $Corr\{\varepsilon_t, \varepsilon_{t-s}\} = \rho^s (1-\rho^2)^{-1}$ – decrease with growing distance *s* in time

AR(1) Process, cont'd

The covariance matrix V{ε}:

$$V\{\varepsilon\} = \sigma_v^2 \Psi = \frac{\sigma_v^2}{1-\rho^2} \begin{pmatrix} 1 & \rho & \cdots & \rho^{N-1} \\ \rho & 1 & \cdots & \rho^{N-2} \\ \vdots & \vdots & \ddots & \vdots \\ \rho^{N-1} & \rho^{N-2} & \cdots & 1 \end{pmatrix}$$

- V{ε} has a band structure
- Depends only of two parameters: ρ and σ_v^2

Consequences of $V\{\varepsilon\} \neq \sigma^2 I_T$

OLS estimators b for β

- are unbiased
- are consistent
- have the covariance-matrix

$$V\{b\} = \sigma^2 (X'X)^{-1} X'\Psi X (X'X)^{-1}$$

- are not efficient estimators, not BLUE
- follow – under general conditions – asymptotically the normal distribution

The estimator $s^2 = e'e/(T-K)$ for σ^2 is biased

For an AR(1)-process ε_t with $\rho > 0$, s.e. from $\sigma^2 (X'X)^{-1}$ underestimates the true s.e.

Inference under Autocorrelation

Covariance matrix of *b*:

$$V\{b\} = \sigma^2 (X'X)^{-1} X'\Psi X (X'X)^{-1}$$

Use of $\sigma^2 (X'X)^{-1}$ (the standard output of econometric software) instead of $V\{b\}$ for inference on β may be misleading

Identification of autocorrelation:

- Statistical tests, e.g., Durbin-Watson test

Remedies

- Use of correct variances and standard errors
- Transformation of the model so that the error terms are uncorrelated

Estimation of ρ

Autocorrelation coefficient ρ: parameter of the AR(1) process

$$\varepsilon_t = \rho\varepsilon_{t-1} + v_t$$

Estimation of ρ

- by regressing the OLS residual e_t on the lagged residual e_{t-1}

$$r = \frac{\sum_{t=2}^{T} e_t e_{t-1}}{(T-k)s^2}$$

- estimator is
  - biased
  - but consistent under weak conditions

Autocorrelation Function

Autocorrelation of order *s*:

$$r_s = \frac{\sum_{t=s+1}^{T} e_t e_{t-s}}{(T-k)s^2}$$

- Autocorrelation function assigns r_s to *s*
- Correlogram: graphical representation of the autocorrelation function

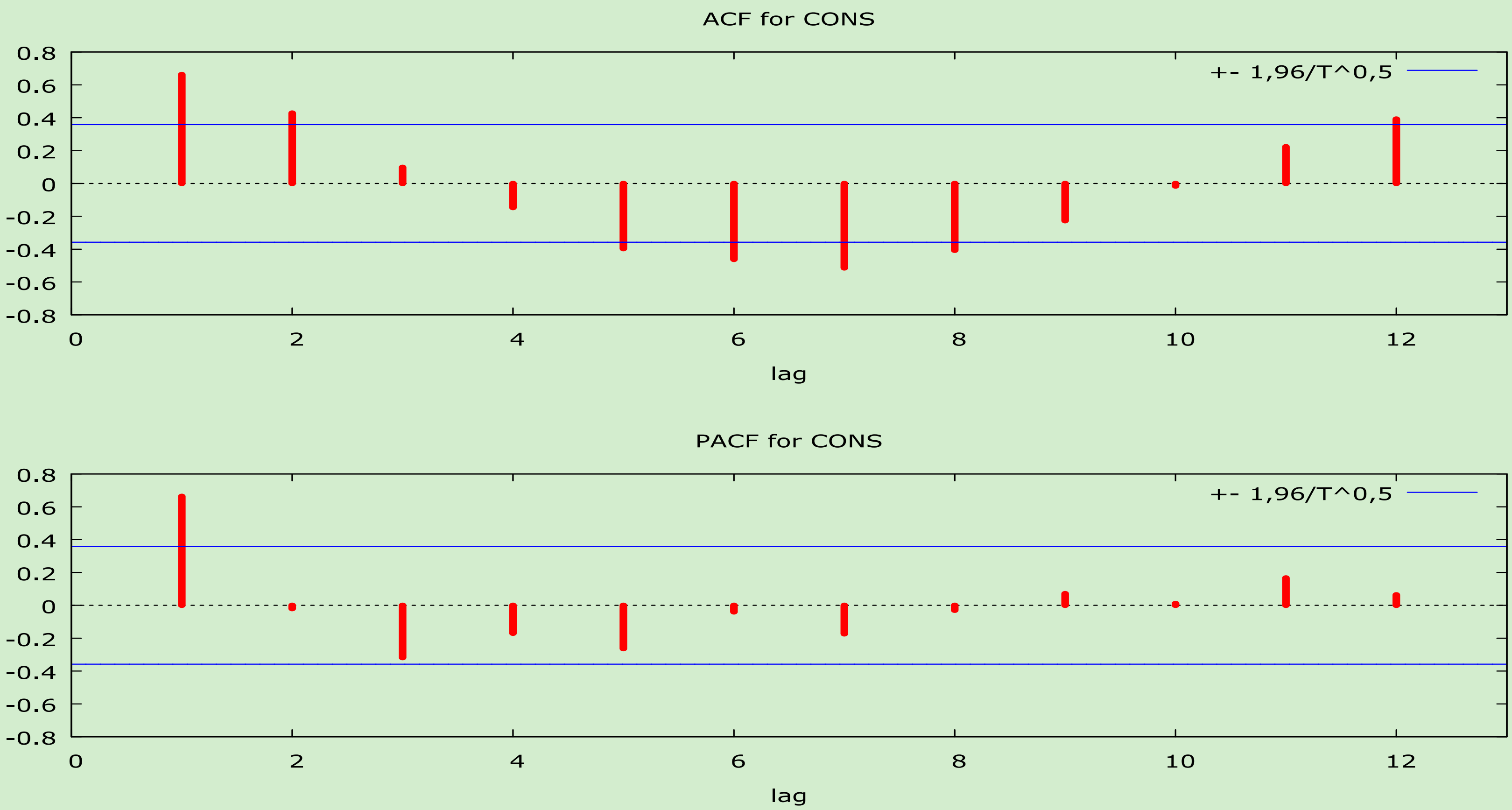
Example: Ice Cream Demand

Autocorrelation function (ACF) of *cons*

LAG	ACF		PACF		Q-stat. [p-value]
1	0,6627	***	0,6627	***	14,5389 [0,000]
2	0,4283	**	-0,0195		20,8275 [0,000]
3	0,0982		-0,3179	*	21,1706 [0,000]
4	-0,1470		-0,1701		21,9685 [0,000]
5	-0,3968	**	-0,2630		28,0152 [0,000]
6	-0,4623	**	-0,0398		36,5628 [0,000]
7	-0,5145	***	-0,1735		47,6132 [0,000]
8	-0,4068	**	-0,0299		54,8362 [0,000]
9	-0,2271		0,0711		57,1929 [0,000]
10	-0,0156		0,0117		57,2047 [0,000]
11	0,2237		0,1666		59,7335 [0,000]
12	0,3912	**	0,0645		67,8959 [0,000]

Example: Ice Cream Demand

Correlogram of *cons*



Tests for Autocorrelation of Error Terms

Due to unbiasedness of *b*, residuals are expected to indicate autocorrelation

Graphical display, correlogram of residuals may give useful hints

Residual-based tests:

- Durbin-Watson test
- Box-Pierce test
- Breusch-Godfrey test

Durbin-Watson Test

Test of H_0: $\rho = 0$ against H_1: $\rho \neq 0$

Test statistic

$$dw = \frac{\sum_{t=2}^{T}(e_t - e_{t-1})^2}{\sum_{t=1}^{T} e_t^2} \approx 2(1-r)$$

- For $\rho > 0$, *dw* is expected to have a value in (0,2)
- For $\rho < 0$, *dw* is expected to have a value in (2,4)
- *dw* close to the value 2 indicates no autocorrelation of error terms
- Critical limits of *dw*
  - depend upon x_t's
  - exact critical value is unknown, but upper and lower bounds can be derived, which depend only of the number of regression coefficients
- Test can be inconclusive

Durbin-Watson Test: Bounds for Critical Limits

Derived by Durbin and Watson

Upper (d_U) and lower (d_L) bounds for the critical limits and $\alpha = 0.05$

T	*K*=2		*K*=3		*K*=10	
	d_L	d_U	d_L	d_U	d_L	d_U
15	1.08	1.36	0.95	1.54	0.17	3.22
20	1.20	1.41	1.10	1.54	0.42	2.70
100	1.65	1.69	1.63	1.71	1.48	1.87

- $dw < d_L$: reject H_0
- $dw > d_U$: do not reject H_0
- $d_L < dw < d_U$: no decision (inconclusive region)

Durbin-Watson Test: Remarks

- Durbin-Watson test gives no reference to causes of rejection of the null hypothesis and how the model to modify
- Various types of misspecification may cause the rejection of the null hypothesis
- Durbin-Watson test is a test against first-order autocorrelation; a test against autocorrelation of other orders may be more suitable, e.g., order four if the model is for quarterly data
- Use of tables unwieldy
  - Limited number of critical bounds (K, T, α) in tables
  - Inconclusive region

Asymptotic Tests

AR(1) process for error terms

$$\varepsilon_t = \rho\varepsilon_{t-1} + v_t$$

Auxiliary regression of e_t on x_t'β and e_{t-1}: produces

- R_e^2

Test of H_0: $\rho = 0$

1. Breusch-Godfrey test (GRETL: OLS output => Tests => Autocorr.)
   - R_e^2 of the auxiliary regression: close to zero if $\rho = 0$
   - $(T\text{-}1)\ R_e^2$ follows approximately the Chi-square distribution with 1 d.f. if $\rho = 0$
   - Lagrange multiplier F (LMF) statistic: F-test for explanatory power of e_{t-1}; follows approximately the $F(1, T\text{-}K\text{-}1)$ distribution if $\rho = 0$
   - General case of the Breusch-Godfrey test: Auxiliary regression based on higher order autoregressive process

Asymptotic Tests, cont'd

2. Box-Pierce test
   - The corresponding *t*-statistic

     $t = \sqrt{(T)}\, r$

     follows approximately the *t*-distribution if $\rho = 0$
   - Test based on $\sqrt{(T)}\, r$ is a special case of the Box-Pierce test which uses the test statistic $Q_m = T\, \Sigma_s^m\, r_s^2$
   - Similar the Ljung-Box test, based on

     $$\frac{T(T-2)}{T-1} \sum_{s=1}^{m} r_s^2$$

     which follows the Chi-square distribution with *m* d.f. if $\rho = 0$
   - Ljung-Box test in **GRETL**: OLS output => Graphs => Residual correlogram

Asymptotic Tests, cont'd

Remarks

- If the model of interest contains lagged values of *y* the auxiliary regression should also include all explanatory variables (just to make sure the distribution of the test is correct)
- If heteroskedasticity is suspected, White standard errors may be used in the auxiliary regression

Demand for Ice Cream, cont'd

Demand for ice cream, measured by *cons*, explained by *price*, *income*, and *temp*

Table 4.9 OLS results

Dependent variable: *cons*

Variable	Estimate	Standard error	t-ratio
constant	0.197	0.270	0.730
price	−1.044	0.834	−1.252
income	0.00331	0.00117	2.824
temp	0.00345	0.00045	7.762

$s = 0.0368$ $R^2 = 0.7190$ $\bar{R}^2 = 0.6866$ $F = 22.175$
$dw = 1.0212$

Demand for Ice Cream, cont'd

OLS estimated demand function: Output from **GRETL**

Dependent variable : CONS

	coefficient	std. error	t-ratio	p-value
const	0.197315	0.270216	0.7302	0.4718
INCOME	0.00330776	0.00117142	2.824	0.0090 ***
PRICE	-1.04441	0.834357	-1.252	0.2218
TEMP	0.00345843	0.000445547	7.762	3.10e-08 ***

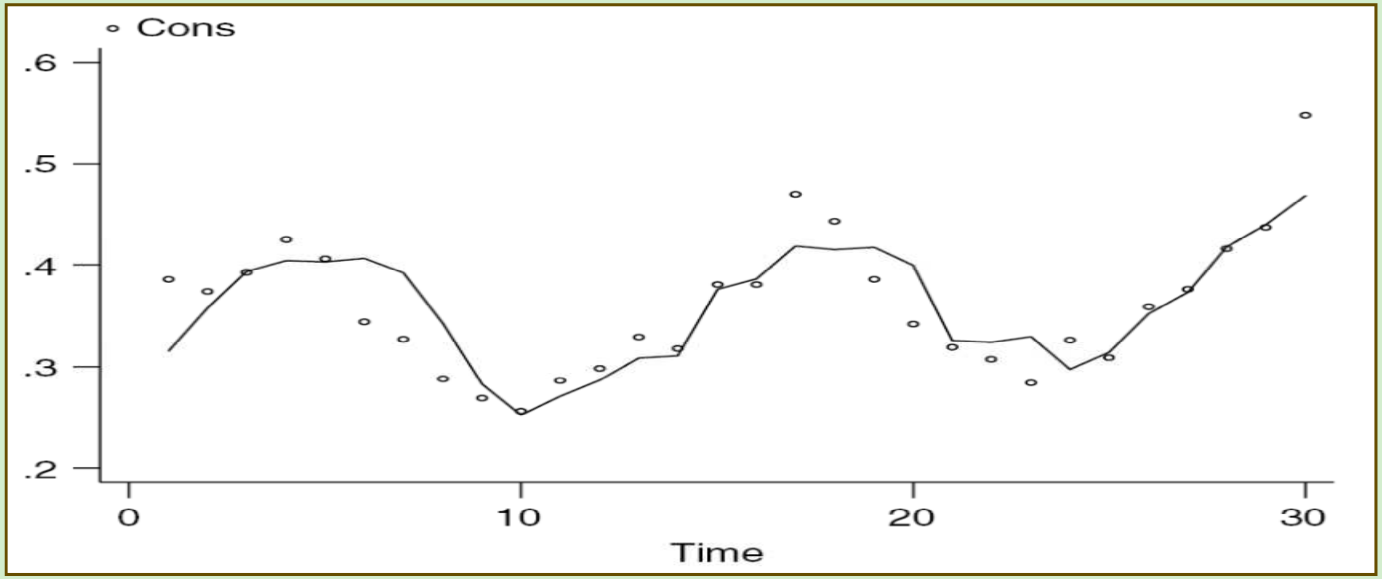
Mean dependent var	0.359433	S.D. dependent var	0,065791
Sum squared resid	0,035273	S.E. of regression	0,036833
R- squared	0,718994	Adjusted R-squared	0,686570
F(2, 129)	22,17489	P-value (F)	2,45e-07
Log-likelihood	58,61944	Akaike criterion	-109,2389
Schwarz criterion	-103,6341	Hannan-Quinn	-107,4459
rho	0,400633	Durbin-Watson	1,021170

Demand for Ice Cream, cont'd

Test for autocorrelation of error terms

- H_0: ρ = 0, H_1: ρ ≠ 0
- $dw = 1.02 < 1.21 = d_L$ for $T = 30$, $K = 4$
- **GRETL** also shows the autocorrelation coefficient: $r = 0.401$

Plot of actual (o) and fitted (polygon) values



Demand for Ice Cream, cont'd

Auxiliary regression $\varepsilon_t = \rho\varepsilon_{t-1} + v_t$: OLS estimation gives

$$e_t = 0.401\ e_{t-1}$$

with s.e.(r) = 0.177, R^2 = 0.154

Test of H_0: $\rho = 0$ against H_1: $\rho > 0$

1. Box-Pierce test:
   - $t \approx \sqrt{(30)}\ 0.401 = 2.196$, p-value: 0.018
   - t-statistic: 2.258, p-value: 0.016
2. Breusch-Godfrey test
   - $(T-1)\ R^2 = 4.47$, p-value: 0.035

Both reject the null hypothesis

Inference under Autocorrelation

Covariance matrix of *b*:

$$V\{b\} = \sigma^2 (X'X)^{-1} X'\Psi X (X'X)^{-1}$$

Use of $\sigma^2 (X'X)^{-1}$ (the standard output of econometric software) instead of $V\{b\}$ for inference on β may be misleading

Remedies

- Use of correct variances and standard errors
- Transformation of the model so that the error terms are uncorrelated

HAC-estimator for V{*b*}

Substitution of Ψ in

$$V\{b\} = \sigma^2 (X'X)^{-1} X'\Psi X (X'X)^{-1}$$

by a suitable estimator

- Newey-West: substitution of $S_x = \sigma^2(X'\Omega X)/T = (\Sigma_t\Sigma_s\sigma_{ts}x_t x_s')/T$ by

$$\hat{S}_x = \frac{1}{T}\sum_t e_t^2 x_t x_t' + \frac{1}{T}\sum_{j=1}^{p}\sum_t (1-w_j)e_t e_{t-j}(x_t x_{t-j}' + x_{t-j}x_t')$$

  with $w_j = j/(p+1)$; p, the *truncation lag*, is to be chosen suitably
- The estimator

$$T\,(X'X)^{-1}\,\hat{S}_x\,(X'X)^{-1}$$

  for V{*b*} is called *heteroskedasticity and autocorrelation consistent* (HAC) estimator, the corresponding standard errors are the HAC s.e.

Demand for Ice Cream, cont'd

Demand for ice cream, measured by *cons*, explained by *price*, *income*, and *temp,* OLS and HAC standard errors

	coeff	s.e.	
		OLS	HAC
constant	0.197	0.270	0.288
price	-1.044	0.834	0.876
*income**10^{-3}	3.308	1.171	1.184
*temp**10^{-3}	3.458	0.446	0.411

Cochrane-Orcutt Estimator

GLS estimator

- With transformed variables $y_t^* = y_t - \rho y_{t-1}$ and $x_t^* = x_t - \rho x_{t-1}$, also called quasi-differences, the model $y_t = x_t'\beta + \varepsilon_t$ with $\varepsilon_t = \rho\varepsilon_{t-1} + v_t$ can be written as

$$y_t - \rho y_{t-1} = y_t^* = (x_t - \rho x_{t-1})'\beta + v_t = x_t^{*\prime}\beta + v_t \qquad \text{(A)}$$

- The model in quasi-differences has error terms which fulfill the Gauss-Markov assumptions
- Given observations for $t = 1, \ldots, T$, model (A) is defined for $t = 2, \ldots, T$
- Estimation of ρ using, e.g., the auxiliary regression $\varepsilon_t = \rho\varepsilon_{t-1} + v_t$ gives the estimate r; substitution of r in (A) for ρ results in FGLS estimators for β
- The FGLS estimator is called Cochrane-Orcutt estimator

Cochrane-Orcutt Estimation

In following steps

1. OLS estimation of b for β from $y_t = x_t'\beta + \varepsilon_t$, $t = 1, \ldots, T$
2. Estimation of r for ρ from the auxiliary regression $\varepsilon_t = \rho\varepsilon_{t-1} + v_t$
3. Calculation of quasi-differences $y_t^* = y_t - ry_{t-1}$ and $x_t^* = x_t - rx_{t-1}$
4. OLS estimation of β from

   $$y_t^* = x_t^{*\prime}\beta + v_t,\ t = 2, \ldots, T$$

   resulting in the Cochrane-Orcutt estimators

Steps 2. to 4. can be repeated: iterated Cochrane-Orcutt estimator

GRETL provides the iterated Cochrane-Orcutt estimator:

Model => Time series => Cochrane-Orcutt

Demand for Ice Cream, cont'd

Iterated Cochrane-Orcutt estimator

Table 4.10 EGLS (iterative Cochrane–Orcutt) results

Dependent variable: *cons*

Variable	Estimate	Standard error	t-ratio
constant	0.157	0.300	0.524
price	−0.892	0.830	−1.076
income	0.00320	0.00159	2.005
temp	0.00356	0.00061	5.800
$\hat{\rho}$	0.401	0.2079	1.927

$s = 0.0326^*$ $R^2 = 0.7961^*$ $\bar{R}^2 = 0.7621^*$ $F = 23.419$
$dw = 1.5486^*$

Durbin-Watson test: $dw = 1.55$; $d_L = 1.21 < dw < 1.65 = d_U$

Demand for Ice Cream, cont'd

Demand for ice cream, measured by *cons*, explained by *price*, *income*, and *temp,* OLS and HAC standard errors, and Cochrane-Orcutt estimates

	coeff	s.e.		Cochrane-Orcutt	
		OLS	HAC	coeff	se
constant	0.197	0.270	0.288	0.157	0.300
price	-1.044	0.834	0.881	-0.892	0.830
income	3.308	1.171	1.151	3.203	1.546
temp	3.458	0.446	0.449	3.558	0.555

Demand for Ice Cream, cont'd

Model extended by $temp_{-1}$

Table 4.11 OLS results extended specification

Dependent variable: *cons*

Variable	Estimate	Standard error	t-ratio
constant	0.189	0.232	0.816
price	−0.838	0.688	−1.218
income	0.00287	0.00105	2.722
temp	0.00533	0.00067	7.953
$temp_{t-1}$	−0.00220	0.00073	−3.016

$s = 0.0299$ $R^2 = 0.8285$ $\bar{R}^2 = 0.7999$ $F = 28.979$
$dw = 1.5822$

Durbin-Watson test: $dw = 1.58$; $d_L = 1.21 < dw < 1.65 = d_U$

Demand for Ice Cream, cont'd

Demand for ice cream, measured by *cons*, explained by *price*, *income*, and *temp,* OLS and HAC standard errors, Cochrane-Orcutt estimates, and OLS estimates for the extended model

	OLS		Cochrane-Orcutt		OLS	
	coeff	HAC	coeff	se	coeff	se
constant	0.197	0.288	0.157	0.300	0.189	0.232
price	-1.044	0.881	-0.892	0.830	-0.838	0.688
income	3.308	1.151	3.203	1.546	2.867	1.053
temp	3.458	0.449	3.558	0.555	5.332	0.670
$temp_{-1}$					-2.204	0.731

General Autocorrelation Structures

Generalization of model

$$y_t = x_t'\beta + \varepsilon_t$$

with $\varepsilon_t = \rho\varepsilon_{t-1} + v_t$

Alternative dependence structures of error terms

- Autocorrelation of higher order than 1
- Moving average pattern

Higher Order Autocorrelation

For quarterly data, error terms may develop according to

$$\varepsilon_t = \gamma\varepsilon_{t-4} + v_t$$

or - more generally - to

$$\varepsilon_t = \gamma_1\varepsilon_{t-1} + \ldots + \gamma_4\varepsilon_{t-4} + v_t$$

- ε_t follows an AR(4) process, an autoregressive process of order 4
- More complex structures of correlations between variables with autocorrelation of order 4 are possible than with that of order 1

Moving Average Processes

Moving average process of order 1, MA(1) process

$$\varepsilon_t = v_t + \alpha v_{t-1}$$

- ε_t is correlated with ε_{t-1}, but not with ε_{t-2}, ε_{t-3}, …
- Generalizations to higher orders

Remedies against Autocorrelation

- Change functional form, e.g., use log(y) instead of y
- Extend the model by including additional explanatory variables, e.g., seasonal dummies, or additional lags
- Use HAC standard errors for the OLS estimators
- Reformulate the model in quasi-differences (FGLS) or in differences

Your Homework

1. Use the data set "labour2" of Verbeek for the following analyses:
   a. Estimate (OLS) the model where log *labor* is explained by log *output* and log *wage*; generate a display of the residuals which may indicate heteroskedasticity of the error term
   b. Perform the Breusch-Pagan (i) with $h(z_i'\alpha) = \exp(z_i'\alpha)$ and (ii) with $h(z_i'\alpha) = z_i'\alpha$, and the White test (iii) with and (iv) without interactions; explain the tests and compare and interpret the results
   c. Compare (i) the OLS and the White standard errors with (ii) HC0 and (iii) HC1 of the estimated coefficients; interpret the results
   d. Estimate the model of a., using FGLS and weights obtained in the auxiliary regression of the Breusch-Pagan test (ii) in b.; compare the results with that of a.

Your Homework, cont'd

2. Use the data set "icecream" of Verbeek for the following analyses:
   a. Estimate the model where *cons* is explained by *income* and *temp*; generate two displays of the residuals which may indicate autocorrelation of the error terms
   b. Use the Durbin-Watson and the Breusch-Godfrey test against autocorrelation; interpret the result
   c. Repeat a., using (i) the iterative Cochrane-Orcutt estimation and (ii) OLS estimation of the model in differences; interpret the result.
3. Durbin-Watson: (a) Explain the meaning of the statement "The Durbin-Watson test is a misspecification test"; (b) show that $dw \approx 2 - 2r$; (c) which of the following tests is a generalization of the DW test? (i) Box-Pierce test; (ii) Breusch-Godfrey test; explain why.