where $0 < a < 1$, $0 < b < 1$, $\psi < 1$. Output can be used on a one-for-one basis for consumption and for investment in K and H. The depreciation rate for each type of capital is δ. Households have the usual infinite-horizon preferences, as in the Ramsey model. Assume initially that there are no irreversibility constraints on K and H, so that gross investment in either form of capital can be negative.

a. Set up the Hamiltonian and find the first-order conditions.

b. What is the optimal relation between K and H? Substitute this relation into the given production function to get a relation between Y and K. What does this "reduced-form" production function look like?

c. What is the steady-state value of the ratio of physical to human capital, $(K/H)^*$?

d. Describe the behavior of the economy over time if the initial condition is such that $K(0)/H(0) < (K/H)^*$. What are the instantaneous rates of investment in each type of capital at time 0?

e. Suppose that the inequality restrictions $I_K \geq 0$ and $I_H \geq 0$ apply. How do these constraints affect the dynamics if the economy begins with $K(0)/H(0) < (K/H)^*$?

5.2 Adjustment costs for human and physical capital. Consider the model from section 5.1 in which consumables and physical and human capital are produced by the same technology. Imagine, however, that there are adjustment costs for changes in the two types of capital. The unit adjustment costs, analogous to the formulation discussed in section 3.3, are $(b_K/2) \cdot (I_K/K)$ for K and $(b_H/2) \cdot (I_H/H)$ for H. Assume that the depreciation rates for each types of capital are 0.

a. Discuss the parameters b_K and b_H. Which one would likely be larger?

b. Suppose that $b_K = b_H$. Discuss the short-run dynamics if the economy begins with $K(0)/H(0) < (K/H)^*$. What if $K(0)/H(0) > (K/H)^*$?

c. Suppose now that $b_K < b_H$. Redo part b, and comment on the main differences in the results.

5.3 Externalities in human capital (based on Lucas, 1988). The production function for the ith producer of goods is

$$Y_i = A \cdot (K_i)^{\alpha} \cdot (H_i)^{\lambda} \cdot H^{\epsilon}$$

where $0 < \alpha < 1$, $0 < \lambda < 1$, $0 \leq \epsilon < 1$. The variables K_i and H_i are the inputs of physical and human capital used by firm i to produce goods, Y_i. The variable H is the economy's average level of human capital; the parameter ϵ represents the strength of the external effect from average human capital to each firm's productivity. Output from the goods sector can be used as consumables, C, or as gross investment in physical capital, I_K. Physical capital depreciates at the rate δ. The production function for human capital is

$$(I_H)_j = BH_j$$

where H_j is the human capital employed by the jth producer of human capital. Human capital also depreciates at the rate δ. Households have the usual infinite-horizon preferences, as in the Ramsey model, with rate of time preference ρ and intertemporal-substitution parameter θ. Consider, first, a competitive equilibrium in which producers of Y and H act as perfect competitors.

a. What is the steady-state growth rate of C, Y, and K? How does the answer depend on the size of the human-capital externality, that is, the parameter ϵ?

b. What is the steady-state growth rate of H? Under what circumstances does H grow at the same rate as K in the steady state?

c. How would the social planner's solution differ from the competitive one?