# Theory of economic growth

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# 7 Endogenous saving, the Ramsey model

Required reading: BSiM: 2 (except 2.6.7 and 2.7)
Supplementary reading on theory of optimal control: Obstfeld (1992)
The classics: Ramsey (1928), Cass (1965), Koopmans (1965)

## 7.1 Why make savings behavior endogenous?

- Before we start our analysis of how to make savings behavior in the Solow-model endogenous, it is worth to reconsider the reasons for doing this.
  - A natural generalization
  - It gives richer comparative statistics. I.e. facilitates analysis of how the economy reacts to changes in e.g. interests rates, tax rates.
  - It gives a richer description of the transitional dynamics.
  - Allows for normative assessments of policy.
- In particular, we should remember that the Solow-model primarily tells us something interesting about growth in the short run (i.e. the phase with transitional dynamics). In addition, the neo-classical revival seem to suggest that the short run is 'long', meaning that adjustment to the steady-state takes a long time.
- Both of these factors calls for more detailed modeling of this dynamics.

## 7.2 Profit-maximizing firms and utility-maximizing households

### 7.2.1 Households

- $H$ identical households. That is, each household is of size $L(t)/H$.
- Each household grows with exogenous rate $n$ (identical to the growth rate of population).

- Households are extended across generations (dynasties).
- Since all household are identical we can normalize $H = 1$, i.e. consider a representative household. For convenience we set $L(0) = 1$, so

$$L(t) = e^{nt}$$

**Household preferences**

- Current household members (acting as one entity) seek to maximize:

$$U = \int_{t=0}^{\infty} u(c(t))e^{nt}e^{-\rho t}dt \tag{1}$$

  Where $c(t)$ equals the consumption level of *each* household member (assumed identical across household members).
- The so-called felicity function $u(c)$ characterizes the utility of a given household member of consuming $c$ at any given point in time.
- Since $L(t)$ is the number of members in the household, and $c(t)$ is the consumption of each member, the household's utility function is additive in the utility of the members (both present and future members). Further, it is additively separable over time for each individual.
- The $\rho$ equals the constant (subjective) discount rate, i.e. it reflects the impatience of the household members. It is *also* identical to the factor of discounting of felicity across generations, i.e. that current household members discount the felicity of future generations relative to their own.
- Note that this formulation both implies a strong form of altruism within the family dynasty, and a limited form of cardinal utility.
- Households have perfect foresight. I.e. they can deduce what the future will look like, and there is no uncertainty.
- We assume that $\rho > n$, so that $U$ is well defined (finite if $c$ is constant over time).
- The felicity function $u(c)$ satisfies $u'(c) > 0$ and $u''(c) < 0$. The concavity reflects a desire for smooth consumption over time.

**The household budget**

- Households hold assets in the form of capital or as loans. Each household member supplies inelastically one unit of labor.
- Household are price-takers, i.e. take as given the path $\{r(t), w(t)\}$ of prices .
- In equilibrium all markets clear, so there is no idle labor or capital.
- We hence have

$$\frac{d(\text{Assets})}{dt} = r \cdot (\text{Assets}) + wL - cL$$

  or with $a = \text{Assets}/L$ (assets per household member)

$$\dot{a}(t) = w(t) + r(t)a(t) - c(t) - na(t) = w(t) - c(t) + a(t)(r(t) - n) \quad (2)$$

- Note that this is a *dynamic* budget constraint, i.e. one that holds continuously for all $t$.
- We must rule out that households can accumulate debt forever. If this is not the case, it will obviously be optimal to pay for both consumption and interest on present debt by borrowing yet more money. We therefore require that

$$\lim_{t\to\infty} a(t)e^{-\int_0^t [r(v)-n]dv} \geq 0 \quad (3)$$

  i.e. that the net present value of assets is asymptotically non-negative. This is often referred to as a condition ruling out Ponzi-games. In reality, we would expect stricter conditions on credit.

**Utility maximization**

- The household faces the problem of optimizing the utility (1) subject to the dynamic budget constraint (2) and the Non-Ponzi game condition (3). (Formally it is also subject to $c(t) \geq 0$ and $a(0) = a_0$).
- This problem falls within a class of problems best studied by the theory of optimal control. It is also possible to solve it without using this theory. The alternative approach is, however, cumbersome and makes it difficult to reformulate the problem to cover modified situations.

- In this course we therefore make use of the main results from the theory of optimal control. It is quite straightforward to learn to apply the procedure for solving the type of problems we encounter, and nothing more will be expected of you. Even if you find it hard to really understand the method fully, you should find comfort in the fact that the results we end up with are quite easy to interpret directly.

- We will cover the method in more detail on the next seminar. See also e.g. BSiM A.3.

- We start by constructing the (present value) Hamiltonian for the problem:

$$J = u(c(t))e^{-(\rho-n)t} + \nu(t)[w(t) + (r(t) - n)a(t) - c(t)] \tag{4}$$

  This is pretty much like the Lagrangian in static optimization, and $\nu(t)$ takes the role of the multiplier. Note that the first part of the expression is the instantaneous utility (at time $t$) and the second term is the 'multiplier' times the dynamic constraint. Changes in either the payoff-function or the budget are then easy to include in the Hamiltonian.

- The solution to the problem is then characterized by the following three conditions:

$$\frac{\partial J}{\partial c(t)} = 0 \tag{5}$$

$$\dot{\nu}(t) = -\frac{\partial J}{\partial a(t)} \tag{6}$$

$$\lim_{t \Rightarrow \infty} [\nu(t) \cdot a(t)] = 0 \tag{7}$$

  The first one is familiar and basically the same as the Maximum principle for the Lagrangian. The second, the Euler equation, is perhaps less intuitive and characterizes the dynamic dimension. From the envelope theorem we know that the derivative of the value function with respect to a parameter is the derivative of the Lagrangian with respect to this parameter. Thus we can think of $\frac{\partial J}{\partial a(t)}$ as the shadow price on increasing the budget. The third condition is called the transversality condition. It basically says that it will not be optimal to end up ('at infinity') with keeping net assets if these are valuable. The transversality condition can often be ignored in 'sloppy' applications of the method.

- The two first conditions are in many ways most central for characterizing the solution. The transversality condition is also crucial, but in a more formal way that is not really that interesting in our applications. We will therefore be somewhat sloppy and often ignore this aspect of the solution (in a loose analogy this is the same as ignoring economically implausible corner solutions in static optimization). You should, however, study the discussion of the transversality condition in BSiM.

- We therefore get

$$\frac{\partial J}{\partial c(t)} = 0 \Rightarrow \nu = u'(c)e^{-(\rho-n)t} \tag{8}$$

$$\dot{\nu}(t) = -\frac{\partial J}{\partial a(t)} \Rightarrow \dot{\nu} = -(r-n)\nu \tag{9}$$

$$\tag{10}$$

  or by differentiating the first condition and inserting for $\nu$ we get the Euler equation

$$r = \rho + \left[-\frac{u''(c)\cdot c}{u'(c)}\right]\cdot\frac{\dot{c}}{c}$$

  where the term in the brackets is the inverse of the intertemporal elasticity of substitution.

- The equation tells us the following: Households choose consumption so that they equate the rate of return of saving ($r$) to the rate of return to consuming today instead of later (equal to the discount factor $\rho$ offset by the rate of decrease of marginal utility of consumption due to growing $c$).

- It is common practice to work with the special felicity function of the form

$$u(c) = \frac{c^{1-\theta}}{1-\theta}, \quad \theta > 0 \tag{11}$$

  For this function the intertemporal elasticity of substitution is $1/\theta$, for this reason BSiM refer to it as the *constant intertemporal elasticity of substitution* (CIES) utility function. However, it is more frequently referred to as the CRRA (Constant Relative Risk Aversion) utility function, since it also has constant relative risk aversion (equal to $\theta$).

- In this case the Euler-equation becomes

$$\frac{\dot{c}(t)}{c(t)} = \frac{r(t)-\rho}{\theta} \tag{12}$$

- The intuition now becomes more transparent: Consumption (per $A$) is increasing when the real rate of return is higher than the rate at which the household discounts future consumption (in this situation one is saving more and hence consumption *grows* faster). But the smaller is $\theta$ the more one is willing to substitute consumption intertemporally to exploit the difference between $r(t)$ and $\rho$.

### 7.2.2 Firms

- Many equal firms with technology characterized by a neo-classical production function $Y = F(K, TL)$.
- Again we consider a representative firm.
- The technology parameter $T$ follows an exogenous path, growing at rate $x$, we normalize $T(0) = 1$.
- Firms rent capital and hire labor from households.
- Firms are price-takers in the markets for the inputs to production, that is, they adjust to given paths $\{r(t), w(t)\}$.
- Firms have perfect foresight. Profit-maximizers.
- Since capital and loans are perfect substitutes as stores of value, we must have $r = R - \delta$ or $R = r + \delta$.
- A representative firm's profit at any given point in time is

  $$\pi = F(K, TL) - (r + \delta)K - wL$$

  or transforming to units per efficient worker

  $$\pi = TL[f(\hat{k}) - (r + \delta)\hat{k} - we^{-xt}]$$

- Since there are no adjustments costs (i.e. what it does today does affect what it can do/costs later) the firm will seek to maximize profits at any given point in time.
- We then have the usual first order conditions

  $$f'(\hat{k}) = r + \delta \tag{13}$$
  $$w = [f(\hat{k}) - \hat{k}f'(\hat{k})]e^{xt} \tag{14}$$

  and there is no profit in equilibrium (for all values of $TL$).

### 7.3 Equilibrium

- In equilibrium all debt must cancel, so $a = k$.
- Inserting $a = k$ and the factor prices in the household's budget constraint then gives us (after converting to units per efficient worker)

$$\dot{\hat{k}} = f(\hat{k}) - \hat{c} - (x + n + \delta)\hat{k} \tag{15}$$

- This is basically the same fundamental equation for the capital intensity as in the Solow model (the reason we end up with this equation even in the presence of profit-maximizing firms is the same as before).
- The difference is that the term for gross saving is now $f(\hat{k}) - \hat{c}$ instead of $sf(\hat{k})$ (note that $f(\hat{k}) - \hat{c}$ implies $s(\hat{k})f(\hat{k})$, i.e. a savings rate that varies with $\hat{k}$).
- Thus we must also characterize the behavior of $\hat{c}$ in order to have a full description.
- This follows from the Euler-equation, which after inserting for $r(t)$ and converting to units per efficient worker, reads

$$\frac{\dot{\hat{c}}(t)}{\hat{c}(t)} = \frac{\dot{c}(t)}{c(t)} - x = \frac{f'(\hat{k}) - \delta - \rho - \theta x}{\theta} \tag{16}$$

- The two equations (15) and (16), together with the initial condition $\hat{k}(0)$ and the transversality condition determines the time paths of $\hat{k}$ and $\hat{c}$, which is what we need to fully characterize the economy.

- We draw in the $\dot{\hat{k}} = 0$ and $\dot{\hat{c}} = 0$ lines in a phase-diagram in $(\hat{k}, \hat{c})$. This shows the existence of a saddle-path and a steady state.

- We can use the phase diagram to do comparative statistics. Note that we can not get only get gradual adjustments along the $\hat{k}$ axis (no jumps). (Unless the exogenous shifts affect $\hat{k}(t)$ directly).

## 7.4 Long run and the Solow-model

- The main conclusion concerning the long run is the same as before: We reach a steady state with $\dot{\hat{k}} = 0$ and $\dot{\hat{c}} = 0$.

- Thus, from a descriptive point of view the Ramsey model has not taught us much more about what happens in the long run.

## 7.5 The transversality condition

- When solving the model with consumer optimization we have been somewhat relaxed about the transversality condition, i.e. that

$$\lim_{t \to \infty} [\nu(t) \cdot a(t)] = 0$$

- Remember its interpretation: with $\nu$ the (present value) shadow price/value of $a(t)$ it tells us that the value of the household's assets $(\nu(t) \cdot a(t))$ asymptotically must approach 0. Or loosely speaking: Households should not plan to be stuck with any valuable assets at the end of their horizon (i.e. infinity).

- Neglecting this condition is usually OK in our applications. But we should realize that the transversality condition is crucial for the formal solution, and can make a major difference in some cases.
- To see this, consider the case where we have a finite horizon. Then household's should end up having consumed all their capital in the final period. This gives a solution off the saddle-path, ending up on the vertical axis.
- Remember that

$$\dot{\nu}(t) = -\frac{\partial J}{\partial a(t)} \Rightarrow \dot{\nu} = -(r-n)\nu$$

or

$$\nu(t) = \nu(0) \cdot \exp\left\{-\int_0^t [r(v) - n]dv\right\}$$

which gives the transversality condition

$$\lim_{t\to\infty}\left[a(t) \cdot \exp\left\{-\int_0^t [r(v) - n]dv\right\}\right] = 0 \tag{17}$$

- Inserting for $a = k$, $\hat{k} = ke^{-xt}$, and $r(t) = f'(\hat{k}) - \delta)$ we have

$$\lim_{t\to\infty}\left[\hat{k}(t) \cdot \exp\left\{-\int_0^t [f'(\hat{k}) - \delta - x - n]dv\right\}\right] = 0 \tag{18}$$

- When the solution entails reaching a steady state $\hat{k}^*$ this requires that

$$f'(\hat{k}^*) - \delta > x + n$$

i.e. that the steady-state rate of return exceeds $n + x$ the steady state growth rate of $K$.

- Remember also that (from $\dot{\hat{c}} = 0$) that

$$f'(\hat{k}^*) = \delta + \rho + \theta x$$

- It follows that the transversality condition can only be satisfied if

$$\rho > n + (1-\theta)x \tag{19}$$

i.e. if the households are sufficiently impatient (in relation to population growth and technological growth).

- **Exercise:** Show that (19) must be satisfied for life-time utility to be finite when $c$ grows at rate $x$.

- We will only consider problems where (19) is satisfied.

## 7.6 Convergence revisited

- We have seen that also in the model with optimizing household we get convergence to a steady state $\hat{k}^*$.

- In the transition to the steady state $\hat{k}$ will grow (assuming we start out below). Hence growth will be more rapid in the short run than in the long run, just as in the Solow-model.

- It is, however, now far less obvious that $\gamma_{\hat{k}}$ will decline monotonously along the transition as it did in the Solow-model (we do not have the clear phase-diagram in $(\hat{k}, \gamma_{\hat{k}})$ space that we are used to in the Solow-model).

- But it can be shown (you should read the proof in BSiM 2.11) that $\gamma_{\hat{k}}$ will also now fall monotonously. Hence, we still get a clear prediction of conditional $\beta$-convergence.

- What then about the speed of convergence?

- It is harder to log-linearize the path of $\gamma_y$ now, because we have dynamics in both $k$ and $c$ (see BSiM 2.8). The $\beta$ is now a rather complex function of the parameters, and it is harder to pin down plausible values than in the Solow-model.

- An alternative technique is to assess the speed of convergence from looking at numerical solutions of the system when calibrated by plausible parameter values (see BSiM pp. 113–118).

- The crucial new parameter influencing the speed of convergence is now $\theta$, i.e. the inverse of the intertemporal elasticity of substitution.

- The higher is $\theta$, the stronger is the desire to smooth consumption. Hence, the harder will households try to shift future consumption (high) to present consumption (low). They hence invest little, and the stable arm will lie close to the $\dot{\hat{k}} = 0$ path. Low levels of investments means a slow transition, and a low speed of convergence.

- The intertemporal elasticity of substitution is also crucial for how the savings rate evolves with $\hat{k}$. In the CD-case, it can be shown (you should read BSiM 2.6.4) that it either falls monotonously when $s^* < 1/\theta$ or rises monotonously when $s^* > 1/\theta$. As discussed in connection with Exercise BSiM 1.3 in the seminar, this implies respectively higher and lower speed of convergence than in the Solow-model.

- Interestingly, the Solow-model transpires as a special case of the model when $s^* = 1/\theta$, which holds true for certain combinations of rather plausible parameter values. (Note that the equivalence holds only for the Solow model with this particular $s^*$.)

# 8 Policy in the neo-classical model

Required reading: BSiM: 3.1, 4.1

## 8.1 Optimal growth and the golden rule

- Turning back to the fixed savings rate in the Solow model we have consumption given as:

$$\hat{c}^* = f(\hat{k}^*) - (n + x + \delta)\hat{k}^*$$

where $\hat{k}^* = \hat{k}^*(s, n, x, \delta)$. We then have:

$$\frac{\partial \hat{c}^*}{\partial s} = [f'(\hat{k}^*(s, n, x, \delta)) - (n + x + \delta)]\frac{\partial \hat{k}^*(s, n, x, \delta)}{\partial s} \qquad (20)$$

Maximization of $\hat{c}^*$ thus requires that the capital level is at $\hat{k}_{GR}$ which satisfies:

$$f'(\hat{k}_{GR}) = n + x + \delta \qquad (21)$$

The capital stock $\hat{k}_{GR}$ is called the *golden-rule* level of the capital stock.

- In the Ramsey model we saw that the steady state level of capital was given by

$$f'(\hat{k}^*) = \delta + \rho + \theta x$$

but due to (19) we have $\rho + \theta x > n + x$ and since $f'' < 0$ we have

$$\hat{k}^* < \hat{k}_{GR}$$

- Why? Households are impatient, and have preferences for also having high consumption during the transition to the steady state.

- Modified golden-rule.
- Notice that this is the reason why we always draw the $\dot{\hat{c}} = 0$ line to the left of the peak of the $\dot{\hat{k}} = 0$ curve.
- Notice in particular that we can never get dynamic inefficiency (oversaving).
- But is this socially optimal?

## 8.2 The first theorem of welfare

- We have investigated the Ramsey model in a decentralized economy. Let us now look at how a social planner would allocate resources.
- Since there is only one representative household, the social planner also maximizes

$$U = \int_{t=0}^{\infty} u(c(t))e^{nt}e^{-\rho t}dt \tag{22}$$

- The budget constraint for the social planner is (we rule out debt since we are in a closed economy, hence $a(t) = k(t)$)

$$\dot{\hat{k}}(t) = f(\hat{k}(t)) - c(t)e^{-xt} - \hat{k}(t)(\delta + x + n) \tag{23}$$

- Note the equality between the household constraint in the equilibrium (i.e. equation (15)) and the economy's resource constraint (23).
- The social planner maximizes (22) subject to (23). The Hamiltonian for this problem is

$$J = u(c(t))e^{-(\rho-n)t} + \nu(t)[f(\hat{k}(t)) - c(t)e^{-xt} - \hat{k}(t)(\delta + x + n)] \tag{24}$$

- The solution is then

$$\frac{\partial J}{\partial c(t)} = 0 \Rightarrow \nu = u'(c)e^{-(\rho-n-x)t} \tag{25}$$

$$\dot{\nu}(t) = -\frac{\partial J}{\partial \hat{k}(t)} \Rightarrow \dot{\nu} = -[f'(\hat{k}) - (\delta + x + n)]\nu \tag{26}$$

$$\tag{27}$$

- And we get the Euler-equation:

$$\frac{\dot{\hat{c}}(t)}{\hat{c}(t)} = \frac{\dot{c}(t)}{c(t)} - x = \frac{f'(\hat{k}) - \delta - \rho - \theta x}{\theta} \tag{28}$$

which is identical to the one we derive for the decentralized solution.

- Thus, the social planner chooses the same solution (allocation) as that realized in the decentralized economy. Hence, the competitive equilibrium is Pareto-efficient. This is an important and very strong result.

- This result follows from the first theorem of welfare, which says that the competitive equilibrium is pareto-efficient in the absence of externalities, and with complete markets.

## 8.3 Government in the Ramsey framework

- We now introduce a government who levies taxes and purchases goods and services.

- Initially we assume that households and firms do not benefit from public spending. In a sense public spending is therefore like throwing away resources.

- The governments is assumed to run a balanced budget (so there is no possibility of public debt):

$$G + V = \tau_w wL + \tau_a r \cdot (\text{assets}) + \tau_c C + + \tau_f \cdot (\text{firm's earnings})$$

where $\tau_i$ are tax-rates, $G$ public spending and $V$ transfers to households.

- The per capita budget of the household's is now

$$\dot{a} = (1 - \tau_w) \cdot w + (1 - \tau_a) \cdot ra - (1 + \tau_c) \cdot c - na + v$$

- Replacing the former budget, it is straightforward to show that the Euler equation is now:

$$\frac{\dot{c}(t)}{c(t)} = \frac{(1 - \tau_a)r - \rho}{\theta} \tag{29}$$

i.e. the tax on assets returns is the only new factor affecting households' intertemporal decision, entering as a reduction in the interest rate $r$.

- For firms we now have

$$\text{after tax profit} = (1 - \tau_f) \cdot [F(K, TL) - wL - \delta K] - rK$$

giving the first order condition

$$f'(\hat{k}) = \frac{r}{1 - \tau_f} + \delta$$

So a higher $\tau_f$ raises the required marginal product of capital.

- We still have

$$w = [f(\hat{k}) - \hat{k} f'(\hat{k})] e^{xt}$$

- The same type of argument as before gives us that the evolution of $\hat{k}$ in equilibrium is characterized by

$$\dot{\hat{k}} = f(\hat{k}) - \hat{c} - (x + n + \delta)\hat{k} - \hat{g} \tag{30}$$

i.e. it coincides with the economy's resource constraint.

- The Euler-equation characterizing the path of consumption in equilibrium is

$$\frac{\dot{\hat{c}}(t)}{\hat{c}(t)} = \frac{(1 - \tau_a) \cdot (1 - \tau_f) \cdot (f'(\hat{k}) - \delta) - \rho - \theta x}{\theta} \tag{31}$$

- Note that taxes on wages and consumption are like lump-sum taxes here, and do not affect the choice of consumption over time. This is because labor supply is fixed, and a constant proportional tax on consumption does not change relative prices of consumption at different points in time.

- Taxes on asset incomes and/or firm's earnings will affect the consumption decision by lowering the interest rate, and hence reduce incentives to save and invest (shifts the $\dot{\hat{c}} = 0$ curve to the left). Higher $\tau_a$ and/or $\tau_f$ hence gives a lower $\hat{k}^*$.

- An unanticipated and permanent increase in government consumption will shift the $\dot{\hat{k}} = 0$ curve down and result in a jump down to a new saddle-path consistent with the new steady-state. If the economy was initially in a steady-state it will jump directly down to the new steady-state simply by reducing consumption by the same amount as $\hat{g}$ increased, amounting to a full crowding-out of private consumption by government consumption.

# 9 Investments and savings in an open economy

Required reading: BSiM:3.2-3.4

## 9.1 The augmented model revisited

- We again consider the augmented production function

$$Y = K^{\alpha} H^{\eta} (TL)^{(1-\alpha-\eta)} \Rightarrow \hat{y} = f(\hat{k}, \hat{h}) = \hat{k}^{\alpha} \hat{h}^{\eta}$$

- The representative firm then have profits:

$$\pi = TL[f(\hat{k}, \hat{h}) - R_k \hat{k} - R_h \hat{h} - we^{-xt}]$$

  where $R_k$ is the rental price of physical capital and $R_h$ the rental price of human capital.

- Maximization of the present value of future cash flows is equal to profit maximization in each period when there are no adjustment costs.

- Thus the firm adjusts such that marginal returns equal the rental prices:

$$\begin{aligned} R_k &= \frac{\partial f(\hat{k}, \hat{h})}{\partial \hat{k}} = \alpha y / k & (32) \\ R_h &= \frac{\partial f(\hat{k}, \hat{h})}{\partial \hat{h}} = \eta y / h & (33) \\ w &= \left[ \hat{y} - R_k \hat{k} - R_h \hat{h} \right] e^{xt} = y - R_k k - R_h h & (34) \end{aligned}$$

- The dynamical budget constraint for the household (per capita) is

$$\dot{a} = \dot{h} + \dot{k} - \dot{d} = w + (R_k - \delta - n)k + (R_h - \delta - n)h - (r - n)d - c \quad (35)$$

  Where $d = k + h - a$ is (per capita) debt.

- We stick to the one-sector production technology where output can go on a one-to-one basis to consumption, additions to physical capital or additions to human capital. I.e. units of $k$ can immediately, and with full reversibility, be changed to units of $h$ and vice versa.

- In addition, we maintain the assumption that either of the capital stocks or financial assets (loans) are perfect substitutes as stores of value. Then we must have

$$r = R_k - \delta = R_h - \delta \tag{36}$$

- As we saw in Section 4.2.1, this implies $k/h = \alpha/\eta$. With no adjustments costs, households jump to the desired ratio, and it stays at this level.

- Thus, as previously demonstrated, we can simply transform the problem by defining a measure of broad capital $z = k + h$. Which inserting for $r = R_k - \delta = R_h - \delta$, gives the budget constraint

$$\dot{a} = \dot{z} - \dot{d} = w + (r - n)(z - d) - c = w + (r - n)a - c$$

  While the production function can be written $\hat{y} = B\hat{z}^{\alpha+\eta}$ where $B = \alpha^{\alpha}\eta^{\eta}(\alpha + \eta)^{-(\alpha+\eta)}$.

- Since $k/h = \alpha/\eta$ at all points during the transition, $k$ and $h$ grow at the same rate, and at the same rate as $z$.

- The firm's choice between $k$ and $h$ does not affect the decision made by the household, since they simply supply whatever capital they have inelastically.

- In sum the model augmented with human capital still behaves just like the original model, only replacing physical capital $k$ by a broad measure of capital $z$. Importantly, the share in income of this broad measure is larger than for physical capital ($\alpha + \eta$ and $\alpha$ respectively)

- $\alpha + \eta$ takes the role of $\alpha$ in the expression for the approximate speed of convergence. The production function is less concave, declining returns to the accumulated factor sets in more slowly, and hence convergence is slower.

## 9.2 Convergence to steady state in an open economy

- Remember that what drives convergence is declining marginal returns to the accumulated factor. That is, since the interest rate in a closed economy falls during the transition, this feeds back on the choice of $h$ and $k$ giving convergence to a steady state.

- In a small open economy, the world interest rate $r^w$ pegs the domestic interest rate $r$.

- For simplicity and as a point of reference, assume that $r^w$ is constant. (E.g. because the 'world' is in its steady state.)

- But with $r^w = r = R_k - \delta = R_h - \delta$ it follows from (32) and (33) that also $h$ and $k$ must be constant. Thus, the open economy will jump instantaneously to the level of $h$ and $k$ that is accordance with this interest rate. That is, it will jump directly to its steady state.

- This instantaneous adjustment can be achieved by international borrowing or lending, i.e. by adjusting $d$. That is, in the open economy investments are not determined by domestic saving and changes in $k$ and $h$ are no longer constrained by the available resources from domestic production after consumption.

- Thus, the model predicts that in a small open economy, we should observe an infinite speed of convergence.

- There does not exist any transitional dynamics in the model. That is, the short run does not exist.

- Since the model only can tell us something interesting about the hows and whys of growth as transitional phenomena, does this imply that the neo-classical model is uninteresting for growth?

## 9.3 Problems of the model for an open economy

- We do not observe anything even resembling instantaneous convergence. So what is wrong?

- There are also other problematic predictions of the Ramsey model when applied to an open economy.

- The Euler equation characterizing the intertemporal decisions of households is

$$\frac{\dot{\hat{c}}(t)}{\hat{c}(t)} = \frac{r^w - \rho - \theta x}{\theta} \tag{37}$$

- Remember that for a closed economy we had $r = \rho + \theta x$ in the steady state. What then if the world interest rate deviates from this value, perhaps because preferences are different in other countries?

- If $r^w > \rho + \theta x$ the home economy would accumulate assets indefinitely. Eventually the assumption of being a *small* open economy (i.e. not affecting $r$) would be violated.

- If $r^w < \rho + \theta x$, i.e. if the home economy is sufficiently impatient, then it can be shown (see BSiM pp. 163-164) that it will plan to consume so much today that it goes fully at the expense of consumption in the far future ($c$ goes to 0 asymptotically), and it achieves this by mortgaging all of its capital and future labor incomes.

- Clearly, these predictions do not make very much sense.

- The predictions do not become any better when we consider what an international equilibrium must look like (see BSiM pp. 164-165).

- There are several reasons to question these conclusion, because the model fails to consider:

  1. Constraints on international credit, i.e. constraints on $d$
  2. Costs of adjusting the capital stock
  3. There does appear to be strong link between domestic saving and domestic investment even in open economies (Feldstein-Horioka puzzle).
  4. Agents might make decisions with a finite horizon.
  5. Preferences might be changing over time.

- We will now consider the first points in some detail, and also say something about the second and third. BSiM 3.5-3.6 considers the last two points, but this is **not** required knowledge for this course.

- Before turning to these extensions it should be said that the present state of research has not led to fully satisfactory answers to these problems.

## 9.4 Constraints on international credit

- A plausible restriction on the level of international debt is

$$d \leq k$$

- That is, physical capital can be used as collateral for international borrowing while human capital cannot.

- The other side of the same coin is that foreigners can not own domestic human capital and raw labor. Thus, the constraint is also relevant for foreign direct investment.
- Plausible assumption? Physical capital is easier to repossess and monitor.
- Besides, we can interpret the division of capital into $k$ and $h$ as respectively capital that can be and cannot be used as collateral. (Note that this has implication for the relevant magnitudes of $\alpha$ and $\eta$.)
- If $k(0)+h(0)-d(0) \geq h^*$ the constraint on credit is not binding, hence we focus on the case $k(0)+h(0)-d(0) < h^*$.
- The two forms of capital are no longer perfect substitutes, so condition (36) need no longer be satisfied.
- However, $a$ and $k$ are still perfect substitutes as store of value, so we must still have:

  $$r^w = R_k - \delta \Rightarrow k/y = \alpha/(r^w + \delta)$$

  Thus the capital-output ratio $k/y$ is constant (which is accordance with one of the empirical regularities of economic development listed in a famous article by Kaldor (1961)).
- Inserting this, the production function now can now be written:

  $$\hat{y} = \tilde{B}\hat{h}^{\epsilon} \tag{38}$$

  where $\tilde{B} = [\alpha/(r^w + \delta)]^{\alpha/(1-\alpha)}$ and $\epsilon = \eta/(1-\alpha)$. Note that since $\alpha + \eta < 1$, we have $0 < \epsilon < \alpha + \eta < 1$.
- If we insert for the wage $w$ in the budget constraint (35) we get

  $$\dot{a} = y-(r+\delta)(k+h-a)-(n+\delta)a-c = y-(r+\delta)d-(n+\delta)a-c \tag{39}$$

  or

  $$\dot{\hat{a}} = \hat{y} - (r+\delta)\hat{d} - (n+\delta+x)\hat{a} - \hat{c} \tag{40}$$
- When the credit constraint is binding we have $d = k$ (implying $a = h$) which gives

  $$\dot{\hat{a}} = \hat{y} - (r+\delta)(\hat{k}) - (n+\delta+x)\hat{h} - \hat{c} \tag{41}$$

- Finally inserting for $\hat{y}$ from (38) and using $k/y = \alpha/(r^w + \delta)$ we get

$$\dot{\hat{h}} = (1-\alpha)\tilde{B}\hat{h}^\epsilon - (\delta + n + x)\hat{h} - \hat{c} \qquad (42)$$

- Note that the subtraction $\alpha\hat{y} = \alpha\tilde{B}h^\epsilon$ is the rental payment to foreigners.

- If we now look at the problem within the framework of a single representative consumer-producer we see that he seeks to maximize utility constrained by (42). This is a problem very much like that we faced in the original formulation of the Ramsey model for a closed economy, both with and without human capital.

- The only differences is that $h$ take the role of the accumulated factor (instead of $k$ or $z$), the share in income of this factor is now $\epsilon$ (in place of $\alpha$ or $\alpha + \eta$), and the presence of the constant $(1-\alpha)\tilde{B}$ (in place of 1 or $B$).

- The solution to the model is therefore basically the same as that of a closed economy with a saddle path describing the transition to a steady state.

- The rate of convergence is then finite even in the open economy, but is higher than in the closed economy because $\epsilon < \alpha + \eta$.

- Note that in the closed economy $k/h$ was fixed, while here it falls during the transition, so the decreasing returns to $h$ sets in earlier.

- Calculating the approximate speed of convergence in this case shows that the difference to the closed economy is not all that large. In fact, it appears to fit reasonably well with the rate of convergence observed in the data.

## 9.5 Investments with adjustment costs

- There are several reasons why it is costly to change the capital stock.

- We will be focusing on internal adjustment costs: I.e. that new machines must be installed, and the workers trained to used them.

- For ease of exposition we return to the simplest model with only physical capital.

- The main alteration is that we now assume that the representative firm faces

$$\text{Cost of invetsment} = I \cdot [1 + \phi(I/K)]$$

where $\phi' > 0$ reflects the cost of adjusting capital, or rather a cost of adjusting to gross investments. Importantly, we assume that costs depend on the proportionate change $(I/K)$. We assume also that $\phi'' \geq 0$. With $\phi'' > 0$ it would be better to make gradual adjustments, rather than large one off changes.

- The firm seeks to maximize the present value of its net cash flow:

$$V = \int_{t=0}^{\infty} e^{-\bar{r}(t)t}\{F(K, TL) - wL - I[1 + \phi(I/K)]\}dt$$

where

$$\bar{r}(t) = 1/t \int_0^t r(v)dv$$

- The variables under the control of the firm is the level of investment, $I(t)$, and labor $L(t)$. The level of capital, $K(t)$, is determined by previous decisions and is a state variable following the law of motion:

$$\dot{K}(t) = I(t) - \delta K(t)$$

- The current value Hamiltonian for this problem is

$$\mathcal{H}(K(t), I(t), L(t)) = F(K, TL) - wL - I[1 + \phi(I/K)] + q(I - \delta K)$$

where $q(t)$ is the co-state variable measured in current values, i.e. at time $t$.

- Solving this model is left for self-study (see BSiM 3.2), we here only discuss the main results.

- Remember the interpretation of the co-state variable as a shadow value of the state variable, i.e. installed capital $K$.

- A central result is that $I/K = \Phi(q)$ where $\Phi'(q) > 0$, i.e. investments depend on $q$ only. An important aspect of $q$ is that it summarizes all information about the future that is relevant for the firm's investment decision.

- The $q$ derived as a co-state variable relates closely to Tobin's $q$, defined as the ratio of the market value of capital to its replacement value. Note that the relevant $q$ in our model is measured at the margin. Under the special conditions underlying our model the two concepts coincide.

- The solution to the model is a saddle-path for the evolution of $(\hat{k}, q)$. The most important thing for us to note is that this entails a gradual adjustment of $\hat{k}$ to any factor (such as a change in the world interest rate) that shifts the location of the steady state. I.e. we get an immediate impact on $q$ to which $\hat{k}$ gradually adjusts.

- In light of this we should expect that countries facing adjustment costs of investments to take time to adopt to the steady state level of capital implied by the world interest rate. In short: we should no longer expect instant convergence.

## 9.6 The Feldstein-Horioka puzzle

- In a famous study Feldstein and Horioka reported least squares regression results indicating a strong link between domestic savings rate and domestic investment rates, and that the relationship was surprisingly close to 1-1.

- There has been a large discussion about this empirical regularity, which seemingly is at odds with capital being quite mobile across countries.

- Possible explanations are:

  1. Governments might adjust fiscal or monetary policy to avoid large and protracted current account imbalances.
  2. Underlying variables might affect both saving and investment, i.e. taxes, demographic conditions.
  3. Corporate saving (retained earnings) might play an important role.
  4. A home-bias on the part of investors.

- In sum, these factors contribute to weaken the impact of an economy's openness on its speed of convergence.

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