How (Not) to Raise Money

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We show that standard winner-pay auctions are inept fund-raising mechanisms because of the positive externality bidders forgo if they top another's high bid. Revenues are suppressed as a result and remain finite even when bidders value a dollar donated the same as a dollar kept. This problem does not occur in lotteries and all-pay auctions, where bidders pay irrespective of whether they win. We introduce a general class of all-pay auctions, rank their revenues, and illustrate how they dominate lotteries and winner-pay formats. The optimal fund-raising mechanism is an all-pay auction augmented with an entry fee and reserve price.

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I. Introduction

It is well known that mechanisms used to finance public goods may yield disappointing revenues because they suffer from a free-rider problem. For example, simply asking for voluntary contributions generally results in underprovision of the public good (e.g., Bergstrom, Blume, and Varian 1986). From a theoretical viewpoint, Groves and Ledyard (1977) solved the decentralized public-goods provision problem by identifying an optimal tax mechanism that overcomes the free-rider problem. This theoretical mechanism contrasts with practical ways to raise money for a public good such as lotteries and auctions. Even the voluntary contribution method is commonly observed in practice, despite its inferior theoretical properties. The coexistence of these alternative formats raises the obvious question: Which method is superior at raising money?

Morgan's (2000) work constitutes an important first step in answering this question. He studies the fund-raising properties of lotteries and makes the point that the public-good free-rider problem is mitigated by the negative externality present in lotteries. This negative externality occurs because an increase in the number of lottery tickets that one person buys lowers others' chances. As a result, lotteries have a net positive effect on the amount of money raised vis-à-vis voluntary contributions. A similar negative externality emerges in auctions, where a bidder's probability of winning is negatively affected by more aggressive bidding behavior of others.

A priori, most economists would probably expect that auctions are superior to lotteries in terms of raising money. Unlike lotteries, auctions are efficient; in equilibrium, the bidder with the highest value for the object places the highest bid and wins. This efficiency property promotes aggressive bidding and boosts revenue, suggesting that lotteries are suboptimal. However, fund-raisers that use lotteries, or "raffles," are quite prevalent, which casts doubt on the empirical validity of this conclusion.

The flaw in the above argument stems from a separate problem that emerges in auctions in which only the winner pays. When a bidder tops the highest bid of others, she wins the object but concurrently eliminates the benefit she would have derived from free-riding off that (previously highest) bid. The possible elimination of positive externalities associated with others' high bids exerts downward pressure on equilibrium bids in winner-pay auctions. Notice that this feature does not occur in lotteries in which all nonwinning tickets are paid.

In this paper we determine the extent to which bids are suppressed in winner-pay auctions and find that these formats yield dramatically low revenues. Even when bidders value \$1.00 given to the public good the same as \$1.00 for themselves, revenues are finite. In contrast, lotteries generate arbitrarily large revenue in this case, notwithstanding their inefficiency. Though extreme, this example suggests that it may make sense to use lotteries instead of winner-pay auctions to raise money.

The main virtue of lotteries in the above example, that is, that all tickets are paid, can be incorporated into an efficient mechanism. "All-pay" auctions, where everyone pays irrespective of whether they win or lose, avoid the problems inherent in winner-pay auctions. Since they are also efficient, they are prime candidates for superior fund-raising mechanisms. In this paper, we prove this intuition correct. We introduce a general class of all-pay auctions, rank their revenues, and illustrate the extent to which they dominate winner-pay auctions and lotteries. Furthermore, we show that the optimal fund-raising mechanism is among the all-pay formats we consider.

Adding an all-pay element to fund-raisers seems very natural. Indeed, the popularity of lotteries as means to finance public goods indicates that people are willing to accept the obligation to pay even though they may lose. Presumably, the costs of losing the lottery are softened because they benefit a good cause. In some cases, it may even be awkward to not collect all bids. Suppose, for instance, that a group of parents submit sealed bids for a set of prizes that are auctioned, knowing that the proceeds benefit their children's school. Some parents may be offended when told they contributed nothing because they lost the auction or, in other words, because their contributions were not high enough.

This paper is organized as follows. In Section II, we consider winner-pay auctions in which bidders derive utility from the revenue they generate. We build on the work of Engelbrecht-Wiggans (1994), who studies such auctions for the two-bidder case. We extend his finding that second-price auctions revenue-dominate first-price auctions by showing that both auctions may be dominated by a third-price auction. The main point of Section II, however, is punctuated by a novel revenue equivalence result for the case in which people are indifferent between a dollar donated and a dollar kept. We show that the amount of money generated in this case is identical for all winner-pay formats and surprisingly low.

In Section III we introduce a general class of all-pay auctions. We show how these formats avoid the shortcomings of winner-pay auctions and we rank their revenues.[1] We demonstrate that an increase in the number of bidders may decrease revenues as low bids more and more resemble voluntary contributions. Fund-raisers can therefore benefit from limiting the number of contestants. In Section IV we derive the

[1] A related paper is that of Krishna and Morgan (1997), who study first-price and second-price all-pay auctions. They show that when bidders' values are affiliated, revenue equivalence does not hold. Baye, Kovenock, and de Vries (1998, 2000) also study these all-pay formats with affiliated values and consider their applications in a wide variety of two-person contests, including patent races, lobbying, and litigation.

optimal fund-raising mechanism, which involves both an entry fee and a reserve price.

Our work is related to several papers that consider auctions in which losing bidders gain by driving up the winner's price. In takeover situations, for example, losing bidders who own some of the target's shares ("toeholds") receive payoffs proportional to the sales price (e.g., Singh 1998; Bulow, Huang, and Klemperer 1999). A related topic is the dissolution of a partnership, as analyzed by Cramton, Gibbons, and Klemperer (1987). Graham and Marshall (1987) and McAfee and McMillan (1992) study "knockout auctions," where every member of a bidding ring receives a payment proportional to the winning bid. Other examples include creditors bidding in bankruptcy auctions (Burkart 1995) and heirs bidding for a family estate (Engelbrecht-Wiggans 1994). These papers restrict attention to standard winner-pay auctions, that is, first-price, second-price, and English auctions. Another important difference is our assumption of a public-good setting: one bidder's benefit from the auction's revenue does not diminish its value to others.

The paper most closely related to ours is the one by Engers and McManus (2003), who consider "charity auctions."[2] They consider first-price and second-price auctions and extend Engelbrecht-Wiggans's (1994) ranking to the n-bidder case. Our results, however, demonstrate that (i) there exist other winner-pay formats that revenue-dominate the second-price auction, and (ii) all winner-pay formats are poor fund-raisers. Engers and McManus find that a first-price all-pay auction yields a higher revenue than a first-price auction, but that its revenue may be more or less than that of a second-price auction. Our paper provides a framework to explain these results and gives a more general ranking of all-pay revenues. In addition, we prove that the lowest-price all-pay auction augmented with an entry fee and reserve price is the optimal fund-raising mechanism.

Finally, our work is related to that of Jehiel, Moldovanu, and Stacchetti (1996), who consider auctions in which the winning bidder imposes an individual-specific negative externality on the losers. One important difference is that the magnitudes of the externalities that occur in fund-raisers are endogenously determined, whereas those considered by Jehiel et al. are fixed.

II. Winner-Pay Auctions

In this section we consider "standard" auctions in which only the winner has to pay. We start with a simple three-bidder example to illustrate and

[2] See Ledyard (1978) for an early evaluation of the use of auctions to raise money for a public good.

extend previous results in the literature and, more important, to demonstrate that winner-pay auctions are poor at raising money. We underscore our point by proving a novel revenue equivalence result: when bidders value \$1.00 given to the public good the same as \$1.00 for themselves, the revenue generated is identical for all winner-pay auctions. Most important, however, revenue in this case is only the expected value of the highest order statistic.

Consider three bidders who compete for a single indivisible object. Suppose that bidders' values are independently and uniformly distributed on [0, 1] and the auction's proceeds accrue to a public good that benefits the bidders. We assume a particularly simple linear "production technology" in which every bidder receives $\$\alpha$ from \$1.00 spent on the public good. Hence, bidders in the auction receive αR in addition to their usual payoffs, where R is the auction's revenue. Engelbrecht-Wiggans (1994) first studied auctions in which bidders benefit from the auction's revenue. He derived the optimal bids for the first-price and second-price auctions when there are two bidders. His answers, however, can easily be extrapolated to our three-bidder example. In the first-price auction, equilibrium bids are[3]

$$B_{1,3}(v) = \frac{2v}{3 - \alpha}, \tag{1}$$

where the first subscript indicates the auction format and the second the number of bidders. Similarly, equilibrium bids in the second-price auction are

$$B_{2,3}(v) = \frac{v + \alpha}{1 + \alpha}. \tag{2}$$

Since the bidding functions are linear, revenues follow by evaluating (1) and (2) at the expected value of the highest and second-highest of three draws: $R_{1,3} = 3/(6 - 2\alpha)$ and $R_{2,3} = (1 + 2\alpha)/(2 + 2\alpha)$. Note that $R_{1,3} = R_{2,3} = \frac{1}{2}$ when $\alpha = 0$, which is the usual revenue equivalence result, and $R_{1,3} = R_{2,3} = \frac{3}{4}$ when $\alpha = 1$. For intermediate values of α we have $R_{2,3} > R_{1,3}$, a result first shown by Engelbrecht-Wiggans for the case of two bidders.

This suggests that the second-price auction should be preferred for fund-raising. The result is of limited interest, however, since it is easy

[3] Consider a bidder with value v who bids as though he has value w and faces rivals who bid according to $B_{1,3}(\cdot)$. The expected payoff is

$$\pi^e(B_{1,3}(w) \mid v) = [v - (1 - \alpha)B_{1,3}(w)]w^2 + \alpha \int_w^1 B_{1,3}(z)\, dz^2.$$

It is easy to verify that the first-order condition for profit maximization is $\partial_w \pi^e(B_{1,3}(w) \mid v) = 2(v - w)w$, so it is optimal for a bidder with value v to bid $B_{1,3}(v)$.

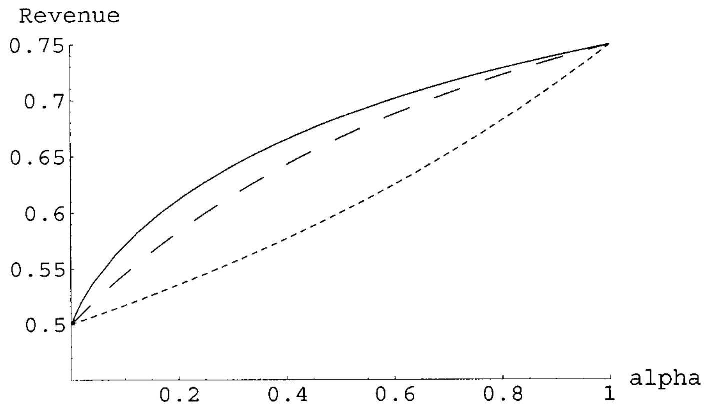

FIG. 1.—Revenues from a first-price (short dashes), second-price (long dashes), and third-price (solid line) auction with three bidders for $0 \leq \alpha \leq 1$.

to find other formats that revenue-dominate the second-price auction. Consider, for instance, a third-price auction in which the winner has to pay the third-highest price. Equilibrium bids for this format are given by

$$B_{3,3}(v) = \frac{2(v-\alpha)}{1-\alpha} + \frac{\alpha}{2(1-\alpha)}[1+\sqrt{1+(8/\alpha)}](1-v)^{(1/2)[\sqrt{1+(8/\alpha)}-1]} \tag{3}$$

with corresponding revenue

$$R_{3,3} = \frac{1-\alpha+3\alpha^2[3-\sqrt{1+(8/\alpha)}]}{2(1-\alpha)(1-3\alpha)}. \tag{4}$$

Also, the third-price auction yields revenue of one-half when $\alpha = 0$ as dictated by the revenue equivalence theorem and three-fourths when $\alpha = 1$. For intermediate values of α, the third-price auction results in higher revenues than the other two formats, as shown in figure 1.

The revenue equivalence result for $\alpha = 1$ holds quite generally. Consider a setting with n bidders whose values are identically and independently distributed on $[0, 1]$ according to a distribution $F(\cdot)$.[4] To derive the amount of money raised when $\alpha = 1$, we focus on the first-price auction, for which it is a weakly dominant strategy to bid one's value. To verify this claim, consider bidder 1 and let $b_{-1} = \max_{i=2,\ldots,n}\{b_i\}$ denote the highest of the others' bids. When $v_1 \geq b_{-1}$, bidder 1's expected payoff

[4] Throughout this paper we assume that the corresponding probability density function $f(\cdot)$ is positive and continuous.

when she bids her value is v_1, and she gets the same payoff for all bids with which she wins. When she bids too low and loses the auction, however, her expected payoff becomes $b_{-1} < v_1$. In other words, bidder 1 never gains but may lose when choosing a bid different from her value. Similarly, when $v_1 < b_{-1}$, bidder 1's expected payoff when she bids her value is b_{-1}. This payoff is the same for all bids with which she loses, but a bid that would lead her to win the auction yields a lower expected payoff equal to v_1. So it is optimal to bid one's value, and the auction's revenue is simply the expected value of the highest order statistic. We next show that other winner-pay formats yield the same revenue (see the Appendix for a proof). Let Y_k^n denote the kth-highest order statistic from n value draws.

PROPOSITION 1. The revenue of any winner-pay auction is $E(Y_2^n)$ for $\alpha = 0$ and $E(Y_1^n)$ for $\alpha = 1$.

This revenue equivalence result is somewhat interesting in its own right, but the main point is that winner-pay auctions are ineffective at raising money. Revenues are increasing with α (see fig. 1), so the highest revenue should be expected for $\alpha = 1$. In this extreme case bidders are indifferent between keeping $1.00 for themselves and giving it to the public good, yet revenues are only $E(Y_1^n)$ in a winner-pay auction. We show below that bidders would spend their entire budgets if a lottery or all-pay auction were used.

III. All-Pay Auctions

The problem with winner-pay auctions is one of opportunity costs. A high bid by one bidder imposes a positive externality on all others, who forgo this positive externality if they top the high bid. Bids are suppressed as a result, and so are revenues. This would not occur in situations in which every bidder pays, regardless of whether they win or lose.[5] In this section, we introduce kth-price all-pay auctions in which the highest bidder wins, the $n - k$ lowest bidders pay their bids, and the k highest bidders pay the kth-highest bid.

[5] Morgan (2000) considers lotteries as ways to fund public goods. Lotteries have an "all-pay" element in that losing tickets are not reimbursed. A major difference is that lotteries are not, in general, efficient; i.e., they do not necessarily assign the object for sale to the bidder who values it the most. Indeed, even in symmetric complete information environments in which efficiency plays no role, lotteries tend to generate lower revenues because the highest bidder is not necessarily the winner. To see this, suppose that the prize is worth V to all bidders. In a lottery the optimal number of tickets to buy is $(n-1)V/[n^2(1-\alpha)]$, resulting in a revenue of $(n-1)V/[n(1-\alpha)]$. In the first-price all-pay auction, the symmetric Nash equilibrium is in mixed strategies. The equilibrium distribution of bids is $F(b) = \{b/[(1-\alpha)V]\}^{1/(n-1)}$, and the resulting revenue is $V/(1-\alpha)$, which exceeds that of a lottery for all n. Note, however, that the revenue of a lottery may exceed that of a first-price winner-pay auction, for instance, where the unique symmetric equilibrium entails bidding V, and hence revenue is V, for all $\alpha \leq 1$.

To derive the bidding functions, consider the marginal benefits and costs of raising one's bid. There are two positive effects of increasing one's bid from $B(v)$ to $B(v+\epsilon) \approx B(v) + \epsilon B'(v)$. First, it might lead one to win the auction that otherwise would have been lost. This occurs when the highest of the others' values falls between v and $v+\epsilon$, which happens with probability $(n-1)\epsilon f(v)F(v)^{n-2}$. Second, an increase in one's bid raises revenue by $\epsilon B'(v)$ if there are at least $k-1$ higher bids and by an additional $\epsilon(k-1)B'(v)$ if there are exactly $k-1$ higher bids. Let $F_{Y_{k-1}^{n-1}}$ denote the distribution function of the $(k-1)$th order statistic from $n-1$ draws with the convention $F_{Y_0^{n-1}}(v) = 0$ and $F_{Y_n^{n-1}}(v) = 1$. The probability that there are at least $k-1$ bidders with values higher than v is $1 - F_{Y_{k-1}^{n-1}}(v)$. Similarly, the probability that there are exactly $k-1$ such bidders is

$$[1 - F_{Y_{k-1}^{n-1}}(v)] - [1 - F_{Y_k^{n-1}}(v)] = F_{Y_k^{n-1}}(v) - F_{Y_{k-1}^{n-1}}(v).$$

Combining the different terms, we can write the expected marginal benefit as ϵ times

$$(n-1)vf(v)F(v)^{n-2} + \alpha B'(v)\{[1 - F_{Y_{k-1}^{n-1}}(v)] + (k-1)[F_{Y_k^{n-1}}(v) - F_{Y_{k-1}^{n-1}}(v)]\}. \tag{5}$$

Likewise, the marginal cost is $\epsilon B'(v)$ when there are at least $k-1$ higher bids, and the expected marginal cost is therefore ϵ times

$$B'(v)[1 - F_{Y_{k-1}^{n-1}}(v)]. \tag{6}$$

The optimal bids can be derived by equating benefits to costs. The resulting differential equation has a well-defined solution for $\alpha < 1/k$. This case is studied in the next proposition, which also compares the resulting revenues to that of a lottery (R^{LOT}).

PROPOSITION 2. When $\alpha < 1/k$, the equilibrium bids of the kth-price all-pay auction are

$$B_{k,n}^{\mathrm{AP}}(v) = \int_0^v \frac{(n-1)zf(z)F(z)^{n-2}}{(1-k\alpha)[1 - F_{Y_{k-1}^{n-1}}(z)] + \alpha(k-1)[1 - F_{Y_k^{n-1}}(z)]}\,dz, \tag{7}$$

and revenues are

$$R_{k,n}^{\mathrm{AP}} = \int_0^1 \frac{z[1 - F_{Y_{k-1}^{n-1}}(z)]}{(1-k\alpha)[1 - F_{Y_{k-1}^{n-1}}(z)] + \alpha(k-1)[1 - F_{Y_k^{n-1}}(z)]}\,dF_{Y_2^n}(z). \tag{8}$$

Revenues of the kth-price all-pay auction are increasing in α but may decrease with n, and $R^{\mathrm{LOT}} < R_{k-1,n}^{\mathrm{AP}} < R_{k,n}^{\mathrm{AP}}$ for $2 \le k \le n$ and $\alpha > 0$.

Thus the all-pay formats revenue-dominate the lottery, and, most important, the lowest-price all-pay auction revenue-dominates all other all-pay formats. Not surprisingly, it also revenue-dominates all winner-pay auctions. This latter result, which we prove in the next section, is fore-

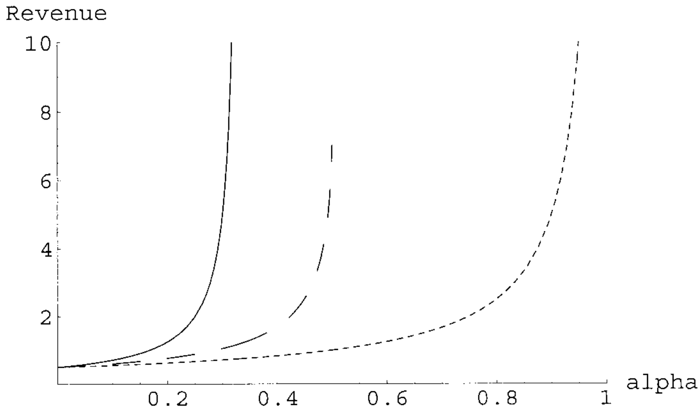

FIG. 2.—Revenues from a first-price (short dashes), second-price (long dashes), and third-price (solid line) all-pay auction with three bidders for $0 \leq \alpha \leq 1$.

shadowed by figure 2. This figure shows the revenues of a first-price, second-price, and third-price all-pay auction in which there are three bidders whose values are uniformly distributed. Comparing figures 1 and 2 illustrates clearly the extent to which revenues are suppressed in winner-pay auctions.

Unlike winner-pay formats in which revenues are increasing in both α and n,[6] all-pay formats may yield lower revenues when there are more bidders. The intuition behind this result can be made clear by considering the second-price all-pay auction. With two bidders, the loser knows that her bid determines the price paid by the winner, which provides the loser with an incentive to drive up the price. This is not true with three or more bidders, however, in which case the $n - 2$ lowest bids are paid only by the losers. Hence there are no positive externalities associated with such bids, which become like voluntary contributions to the public good. This suppresses bids of low-value bidders, who free-ride on the revenues generated by the bidders with higher values. Fund-

[6] Equilibrium bids are $B_{1,n}(v) = \int_0^v z dF_\alpha^n(z|v)$, where $F_\alpha^n(z|v) \equiv [F(z)/F(v)]^{(n-1)/(1-\alpha)}$, for the first-price auction. Note that F_α^n first-order stochastically dominates $F_{\alpha'}^n$ for all $\alpha \geq \alpha'$, and F_α^n first-order stochastically dominates $F_\alpha^{n'}$ for all $n \geq n'$. Hence an increase in α or n raises bids and revenues. Equilibrium bids for the second-price auction are $B_{2,n}(v) = \int_v^1 z dG_\alpha(z|v)$, where

$$G_\alpha(z|v) \equiv 1 - \left[\frac{1 - F(z)}{1 - F(v)}\right]^{1/\alpha},$$

independent of n. The term G_α first-order stochastically dominates $G_{\alpha'}$ for all $\alpha \geq \alpha'$, and an increase in α results in higher bids and higher revenues. Bids in the second-price auction are independent of the number of bidders, but the expected value of the second-highest order statistic increases with n and, hence, so does revenue.

raisers may thus benefit from limiting competition and restricting access to "a happy few."

When $\alpha > 1/k$, the equilibrium bidding function in (7) breaks down and revenues diverge. This divergence is, of course, a consequence of our assumption of a linear production technology for the public good. If the marginal benefit of the public good is sufficiently decreasing (instead of being constant), revenues are finite. We keep the constant marginal benefit assumption because it provides a tractable model to show how much worse winner-pay auctions are in terms of raising money than all-pay formats.

To deal with the case $\alpha > 1/k$, we assume that bidders have a finite budget M, where M is much larger than one. Recall from proposition 1 that revenues of winner-pay auctions are bounded above by one whenever $\alpha \leq 1$, and they are bounded by M when $\alpha > 1$ since only a single bidder pays. In contrast, we next show that the lowest-price all-pay auction raises the maximum amount nM when $\alpha > 1/n$.

PROPOSITION 3. When $\alpha > 1/k$ and bidders face a budget constraint M, the equilibrium bids of the kth-price all-pay auction are[7]

$$B^{AP}_{k,n}(v,\ M) = \begin{cases} B^{AP}_{k,n}(v) & \text{for } v < v^* \\ M & \text{for } v \geq v^*, \end{cases} \tag{9}$$

with $B^{AP}_{k,n}(v)$ given by (7). The cut point, v^*, satisfies $0 < v^* < 1$ when $k < n$ and $1/n < \alpha < 1$, in which case $R^{LOT} < R^{AP}_{1,n} < nM$ and $R^{AP}_{k,n} < R^{AP}_{n,n} = nM$. When $\alpha \geq 1$, $v^* = 0$ and $R^{LOT} = R^{AP}_{k,n} = nM$ for all k.

In particular, when bidders value \$1.00 for the public good the same as \$1.00 kept, revenues of a lottery or any of the all-pay auctions are equal to the sum of the bidders' budgets, nM. This maximum possible revenue contrasts with the expected revenue of a winner-pay auction, $E(Y^n_1) < 1$ (see proposition 1). When $\alpha < 1$ and $k < n$, bidders with sufficiently small values continue to bid according to (7) in the kth-price all-pay auction because the value from possibly winning the item is too small to justify the increased cost of a jump to M. Revenues are strictly smaller than nM in this case unless the lowest-price auction is used for which this maximum amount is guaranteed whenever $\alpha > 1/n$.

IV. Optimal Fund-Raising Mechanisms

In the previous section we showed that when $\alpha > 1/n$, the lowest-price all-pay auction raises the maximum possible revenue. Here we prove that the lowest-price all-pay auction is the optimal fund-raising mechanism for $\alpha < 1/n$ and, hence, is optimal generally. Consider first the case in which the seller cannot commit to keeping the good, so that he

[7] See Gavious, Moldovanu, and Sela (2002) for a similar analysis and results.

cannot use an entry fee or a reserve price. Note that this assumption is closely related to the assumption in the Coase conjecture that a seller of a durable good cannot commit to selling the good for the monopoly price (Coase 1972).

PROPOSITION 4. When the seller cannot commit to keeping the good, the lowest-price all-pay auction is revenue maximizing. The total amount raised is increasing in α.

The intuition for this result is as follows. The total surplus generated by the auction is maximized when the auction outcome is efficient. This surplus is divided between the bidders and the seller: the bidders' share is minimized, and, hence, revenues are maximized, when the lowest-value bidder has zero expected payoffs (see the proof in the Appendix). The lowest-price all-pay auction maximizes total surplus because it assigns the object to the highest-value bidder. In addition, the zero-value bidder who loses for sure also determines the price paid in the auction. Hence, the zero-value bidder's expected payoff is $n\alpha B_{n,n}^{\mathrm{AP}}(0)$, which is zero by (7) for all $\alpha < 1/n$.[8]

For the standard case without a public good ($\alpha = 0$), it is well known that the seller can obtain higher revenues by screening out low-value bidders. Myerson (1981) and Riley and Samuelson (1981) prove that it is revenue maximizing to screen out all bidders with values less than the cut-off value, $\hat{v}$, that satisfies[9]

$$\hat{v} - \frac{1 - F(\hat{v})}{f(\hat{v})} = 0. \tag{10}$$

Screening can be implemented, for instance, by imposing a "minimum bid" or reserve price. By using a reserve price, the seller lowers the expected payoffs of bidders with values between zero and $\hat{v}$ to zero, thus capturing part of the bidders' rents.

When $\alpha > 0$, however, the optimal mechanism cannot be implemented with a reserve price only since low-value bidders who abstain from bidding would still get utility from the amount raised for the public good. Consider instead the following two-stage auction mechanism, $\Gamma(r, \varphi)$, that involves both a reserve price, r, and an entry fee, φ. In the first stage, bidders are asked whether or not they want to participate. If at least one of the bidders refuses to participate, the game ends and the seller keeps the object. Otherwise, each bidder pays the seller the entry fee φ. Then bidders enter the second stage and play the lowest-

[8] Indeed, a strictly positive bid by the zero-value bidder implies that the expected lowest bid is strictly positive, and since the zero-value bidder's expected profit is $n\alpha - 1 < 0$ times the expected lowest bid, she is better off bidding zero.

[9] We make the common assumption that the marginal revenue $\mathrm{MR}(v) \equiv v - \{[1 - F(v)]/f(v)\}$ is strictly increasing in v (see Myerson 1981). Under this assumption there is a unique solution to (10).

price all-pay auction with reserve price r. In this auction, each bidder either submits a bid of at least r or abstains from bidding. If all bidders abstain, the object remains in the hands of the seller; otherwise it will be sold to the bidder with the highest bid. All bidders who submit a bid pay the auction price, which equals the lowest bid when all bidders submit a bid and equals r otherwise.

The equilibrium strategy for the lowest-price all-pay auction in the presence of a reserve price r changes as follows:

$$B_r(v) \equiv \begin{cases} B(v, \hat{v}) & \text{for } v \geq \hat{v} \\ \text{no bid} & \text{for } v < \hat{v}, \end{cases} \tag{11}$$

where

$$B(v, \hat{v}) \equiv r + \frac{n-1}{1-n\alpha}\int_{\hat{v}}^{v} \frac{zf(z)F(z)^{n-2}}{1-F(z)^{n-1}}\,dz, \tag{12}$$

the threshold $\hat{v}$ satisfies (10), and r is the unique solution to $\hat{v}F(\hat{v})^{n-1} = (1-\alpha)r$.[10] Note that (12) has a structure similar to the solution derived in proposition 2 for $k = n$. The reason is that for bidders with values $v > \hat{v}$, the optimal bids again follow by equating the expected marginal benefits and costs in (5) and (6), respectively. The only difference is that the boundary condition is now given by $B_r(\hat{v}) = r$ instead of $B(0) = 0$.

PROPOSITION 5. The optimal fund-raising mechanism is given by the two-stage mechanism $\Gamma(r, \varphi)$, in which bidders first decide whether or not to pay an entry fee φ and then compete in a lowest-price all-pay auction with reserve price r, where

$$(1-\alpha)r = \hat{v}F(\hat{v})^{n-1},$$

$$(1-n\alpha)\varphi = \alpha r(n-1)[1-F(\hat{v})],$$

and $\hat{v}$ satisfies (10). In equilibrium, all bidders participate and play according to (11). The total amount raised is increasing in α.

In practice, fund-raising events frequently employ structures with some of the characteristics of Γ. Under one common structure, attendees are charged a fee for dinner and drinks and then are allowed to bid in auctions later in the event. However, these fund-raisers usually use winner-pay auctions and thus do not maximize revenue. Indeed, an

[10] The reserve price r can be derived by noting that, in equilibrium, a bidder with value $\hat{v}$ must be indifferent between abstaining and bidding r. Hence

$$\alpha(n-1)r[1-F(\hat{v})] = -r + \alpha r\{1 + (n-1)[1-F(\hat{v})]\} + \hat{v}F(\hat{v})^{n-1},$$

where $(n-1)[1-F(\hat{v})]$ is the expected number of other bidders who submit a bid in excess of r.

easy corollary to proposition 5 is that the formats most commonly employed in practice are nonoptimal.

COROLLARY. Lotteries and winner-pay auctions (with or without reserve prices) are nonoptimal.

The intuition is that a lottery does not maximize revenues because the expected payoff of the lowest-value bidder is strictly positive, and the object is not necessarily allocated to the bidder with the highest value. A winner-pay auction does not maximize revenues since the lowest-value bidder expects strictly positive utility from the winner's payment.

V. Conclusion

Large voluntary contributions, such as the recent $24 billion committed by Bill Gates to the Bill and Melinda Gates Foundation, make up a substantial part of total fund-raising revenue today.[11] Not surprisingly, such gifts garner significant attention in the popular media ("Bill's Biggest Bet Yet," *Newsweek*, February 4, 2002, 46). The vast majority of fund-raising organizations, however, seek small contributions from a large number of donors. These organizations frequently prefer lotteries and auctions over the solicitation of voluntary contributions.[12]

Moreover, as electronic commerce on the Internet has grown, Web sites offering charity auctions have proliferated. Electronic auction leaders such as eBay and Yahoo! have specific sites for charity auctions in which dozens of items are sold each day. The established fund-raising community has taken notice of these developments. In a recent report for the W. K. Kellogg Foundation, Reis and Clohesy (2000) identified auctions as one of the most important, and fastest-growing, options that fund-raisers use to leverage the power of the Internet. Given these trends, it is clear that professional fund-raisers can profit from an improved auction design.

Currently, most fund-raisers employ standard auctions in which only the winner pays. These familiar formats have long been applied in the sales of a variety of goods, and their revenue-generating virtues are well established, both in theory and in practice (e.g., Klemperer 1999). We show, however, that they are ill suited for fund-raising. The problem with winner-pay auctions in this context is one of opportunity costs. A high bid by one bidder imposes a positive externality on all others, which they forgo if they top the high bid. Bids are suppressed as a result, and so are revenues. We show that the amount raised by winner-pay

[11] Total giving was an estimated $190 billion in 1999, according to *Giving USA*.

[12] For example, in the year 2000, Ducks Unlimited raised a total of $75 million from special events organized by its 3,300 local chapters, with over 50 percent of the revenue coming from auctions.

auctions is surprisingly low even when people are indifferent between a dollar donated and a dollar kept.

The elimination of positive externalities associated with others' bids does not occur when bidders have to pay irrespective of whether they win or lose. Many fund-raisers employ lotteries, for example, in which losing tickets are not reimbursed (see Morgan 2000). Lotteries are generally not efficient, however, which negatively affects revenues. We introduce a novel class of all-pay auctions, which are efficient while avoiding the shortcomings of winner-pay formats. We rank the different all-pay formats and demonstrate their superiority in terms of raising money (see figs. 1 and 2). We prove that the lowest-price all-pay auction augmented with a reserve price and an entry fee is the optimal fund-raising mechanism.

Our findings are not just of theoretical interest. The frequent use of lotteries as fund-raisers indicates that people are willing to accept an obligation to pay even though they may lose. The all-pay formats studied here may be characterized as incorporating "voluntary contributions" into an efficient mechanism. They are easy to implement and may revolutionize the way in which money is raised.

Appendix

Proof of Proposition 1

Consider a standard auction format in which the highest bidder wins and only the winner pays. In an efficient auction, the surplus generated is $S = E(Y_1^n) + n\alpha R$, with R the auction's revenue. This surplus is divided between the seller and the bidders: $S = R + U_{\text{bidders}}$, where U_{bidders} denotes the ex ante expected payoffs for the group of bidders. Solving for R, we derive

$$R = \frac{E(Y_1^n) - U_{\text{bidders}}}{1 - n\alpha}. \tag{A1}$$

The revenue equivalence result for $\alpha = 0$ is standard. When $\alpha = 1$, the winning bidder's *net* payment is zero. A bidder with a value of one, who wins for sure, therefore has an expected payoff of one. A simple envelope theorem argument shows that the expected rents for a bidder with value v are given by

$$U(v) = U(0) + \int_0^v F^{n-1}(z)dz$$

(see also lemma A1 in the proofs of propositions 4 and 5 below), from which we derive

$$\begin{aligned} U(0) &= 1 - \int_0^1 F(z)^{n-1} dz \\ &= (n-1)\int_0^1 zf(z)F(z)^{n-2} dz \\ &= (n-1)\int_0^1 zf(z)\{F(z)^{n-1} + F(z)^{n-2}[1-F(z)]\} dz \\ &= \frac{1}{n}[(n-1)E(Y_1^n) + E(Y_2^n)]. \end{aligned}$$

Moreover $U_{\text{bidders}} = n\int_0^1 U(v)dF(v)$, so

$$\begin{aligned} U_{\text{bidders}} &= nU(0) + n\int_0^1 \int_0^v F(z)^{n-1} dz dF(v) \\ &= nU(0) + n\int_0^1 \int_z^1 dF(v)F(z)^{n-1} dz \\ &= nU(0) + E(Y_1^n) - E(Y_2^n) \\ &= nE(Y_1^n). \end{aligned}$$

From the last line and (A1) we derive

$$R = \frac{E(Y_1^n) - nE(Y_1^n)}{1-n} = E(Y_1^n),$$

which completes the proof. QED

Proof of Proposition 2

Let $B(\cdot)$ denote the bidding function given in (7). Since the denominator in (7) is bounded away from zero for all $v < 1$ when $\alpha < 1/k$, the bidding function is well defined for all $v < 1$ and possibly diverges in the limit $v \to 1$. The derivative of the expected profit of a bidder with value v who bids as though of type w and who faces rivals bidding according to $B(\cdot)$ is

$$\begin{aligned} \partial_w \pi^e(B(w)|v) &= (n-1)vf(w)F(w)^{n-2} - (1-\alpha)B'(w)[1 - F_{Y_{k-1}^n}(w)] \\ &\quad + \alpha(k-1)B'(w)[F_{Y_k^n}(w) - F_{Y_{k-1}^n}(w)]. \end{aligned}$$

Using the expression for $B(\cdot)$ given by (7), we can rewrite the marginal expected profits as

$$\partial_w \pi^e(B(w)|v) = (n-1)(v-w)f(w)F(w)^{n-2},$$

and it is therefore optimal for a bidder with value v to bid $B(v)$. The revenue of the kth-price all-pay auction equals

$$R = \sum_{i=k+1}^{n} \int_0^1 B(v)dF_{Y_i^n}(v) + k\int_0^1 B(v)dF_{Y_k^n}(v)$$

$$= n\int_0^1 B(v)dG(v),$$

where

$$G(v) \equiv \frac{1}{n}\sum_{i=k+1}^{n} F_{Y_i^n}(v) + \frac{k}{n}F_{Y_k^n}(v).$$

Note that $G(\cdot)$ is increasing with $G(0) = 0$ and $G(1) = 1$. Using $(1/n)\sum_{i=1}^{n} F_{Y_i^n} = F$, we can rewrite the distribution $G(\cdot)$ as

$$G(v) = F(v) + \frac{1}{n}\sum_{i=1}^{k-1}[F_{Y_k^n}(v) - F_{Y_i^n}(v)]$$

$$= F(v) + \frac{1}{n}\sum_{i=1}^{k-1}\left\{\sum_{j=n+1-k}^{n}\binom{n}{j}F(v)^j[1 - F(v)]^{n-j} - \sum_{j=n+1-i}^{n}\binom{n}{j}F(v)^j[1 - F(v)]^{n-j}\right\}$$

$$= F(v) + \frac{1}{n}\sum_{i=1}^{k-1}\sum_{j=n+1-k}^{n-i}\binom{n}{j}F(v)^j[1 - F(v)]^{n-j}$$

$$= F(v) + \frac{1}{n}\sum_{j=n+1-k}^{n-1}\sum_{i=1}^{n-j}\binom{n}{j}F(v)^j[1 - F(v)]^{n-j}$$

$$= F(v) + [1 - F(v)]F_{Y_{k-1}^{n-1}}(v),$$

where we used some basic properties of order statistics (see Mood, Graybill, and Boes 1963). The revenue of the kth-price all-pay auction thus becomes

$$R = n\int_0^1\int_0^v \frac{(n-1)zf(z)F(z)^{n-2}}{(1-k\alpha)[1 - F_{Y_{k-1}^{n-1}}(z)] + \alpha(k-1)[1 - F_{Y_k^{n-1}}(z)]}dzdG(v)$$

$$= n\int_0^1\left[\int_z^1 dG(v)\right]\frac{(n-1)zf(z)F(z)^{n-2}}{(1-k\alpha)[1 - F_{Y_{k-1}^{n-1}}(z)] + \alpha(k-1)[1 - F_{Y_k^{n-1}}(z)]}dz$$

$$= \int_0^1 \frac{z[1 - F_{Y_{k-1}^{n-1}}(z)]}{(1-k\alpha)[1 - F_{Y_{k-1}^{n-1}}(z)] + \alpha(k-1)[1 - F_{Y_k^{n-1}}(z)]}dF_{Y_2^n}(z),$$

where we used $G(1) - G(z) = [1 - F(z)][1 - F_{Y_{k-1}^{n-1}}(z)]$.

The derivative of (8) with respect to α is the integral of a strictly positive function times

$$k[1 - F_{Y_{k-1}^{n-1}}(z)] - (k-1)[1 - F_{Y_k^{n-1}}(z)] =$$

$$[1 - F_{Y_{k-1}^{n-1}}(z)] + (k-1)[F_{Y_k^{n-1}}(z) - F_{Y_{k-1}^{n-1}}(z)] > 0$$

for all $z<1$. Hence revenues are increasing in α. Note that the revenue of the kth-price all-pay auction (8) can be written as

$$R_{k,n}^{\mathrm{AP}} = \int_0^1 z\left\{(1-\alpha) - (k-1)\alpha\left[\frac{F_{Y_k^{n-1}}(z) - F_{Y_{k-1}^{n-1}}(z)}{1 - F_{Y_{k-1}^{n-1}}(z)}\right]\right\}^{-1} dF_{Y_2^n}(z).$$

A sufficient condition for revenues to be increasing in k is that the term between the brackets is increasing in k for all $z \neq 0, 1$. We first make this condition somewhat more intuitive. Consider an urn filled with red and blue balls and let $q = 1 - F(z)$ be the chance of drawing a blue ball, where $0 < q < 1$. Suppose that we draw $n-1$ times with replacement. The above condition can then be rephrased as follows: the chance of drawing exactly $k-1$ blue balls, given that at least $k-1$ blue balls were drawn, is increasing in k. Hence, for all k it has to be true that

$$\frac{\binom{n-1}{k-1}q^{k-1}(1-q)^{n-k}}{\sum_{j=k-1}^{n-1}\binom{n-1}{j}q^j(1-q)^{n-j-1}} < \frac{\binom{n-1}{k}q^k(1-q)^{n-k-1}}{\sum_{j=k}^{n-1}\binom{n-1}{j}q^j(1-q)^{n-j-1}}.$$

Introducing $x \equiv q/(1-q) > 0$, we can rearrange the above inequality as

$$\left[1 - \frac{\binom{n-1}{k}}{\binom{n-1}{k-1}}x\right]\left[1 + \sum_{j=k+1}^{n-1}\frac{\binom{n-1}{j}}{\binom{n-1}{k}}x^{j-k}\right] < 1.$$

The left side of this inequality can be expanded as $1 + \sum_{i=1}^{n-k} a_i x^i$, where

$$a_i = \frac{\binom{n-1}{k+i}}{\binom{n-1}{k}} - \frac{\binom{n-1}{k+i-1}}{\binom{n-1}{k-1}} = -\frac{\binom{n-1}{k+i-1}}{\binom{n-1}{k-1}}\frac{ni}{(n-k)(k+i)} < 0,$$

which shows that revenues increase in k.

To prove that a lottery yields less revenue than all-pay auctions, it is sufficient to show that the first-price all-pay auction revenue-dominates a lottery. Let $R^{\mathrm{LOT},\alpha}$ and $R^{\mathrm{AP},\alpha}$ denote the expected revenue from a lottery and an all-pay auction, respectively, given $\alpha > 0$. In both formats, bidders' payments are equal to their bids and $\alpha > 0$ acts as a simple rebate. Hence,

$$R^{\mathrm{LOT},\alpha} = \frac{R^{\mathrm{LOT},0}}{1-\alpha}$$

since, in equilibrium, each bidder submits a bid equal to $1/(1-\alpha)$ times the equilibrium bid for $\alpha = 0$. Likewise,

$$R^{\mathrm{AP},\alpha} = \frac{R^{\mathrm{AP},0}}{1-\alpha}.$$

The first-price all-pay auction is efficient; that is, the object is always allocated to the bidder with the highest value, whereas a lottery is not. As, by assumption, the bidder with the highest value is the bidder with the highest marginal revenue, it follows from lemma A2 below (see the proofs of propositions 4 and 5) that

$$R^{\mathrm{LOT},0} < R^{\mathrm{AP},0}$$

since for $\alpha = 0$ the utility for the lowest type equals zero for both the lottery and the all-pay auction. Therefore,

$$R^{\mathrm{LOT},\alpha} < R^{\mathrm{AP},\alpha}$$

for all $0 \leq \alpha < 1$.

Finally, to prove that revenue may decrease with n, we compare the revenues of the $(n-1)$th-price all-pay auction with $n-1$ and n players when $\alpha \to 1/(n-1)$. In this limit, the revenue with $n-1$ players tends to

$$\lim_{\alpha \to 1/(n-1)} R^{\mathrm{AP}}_{n-1,n-1} = \frac{n-1}{n-2}\int_0^1 z\left[\frac{1 - F_{Y^{n-2}_{n-2}}(z)}{1 - F_{Y^{n-2}_{n-1}}(z)}\right] dF_{Y_2^{n-1}}(z),$$

which diverges to infinity as $1 - F_{Y^{n-2}_{n-1}}(z) = 0$ for all z. The revenue with n players is equal to

$$\lim_{\alpha \to 1/(n-1)} R^{\mathrm{AP}}_{n-1,n} = \frac{n-1}{n-2}\int_0^1 z\left[\frac{1 - F_{Y^{n-1}_{n-2}}(z)}{1 - F_{Y^{n-1}_{n-1}}(z)}\right] dF_{Y_2^{n}}(z),$$

which is finite. Hence, for α close to $1/(n-1)$, revenues are higher with $n-1$ bidders than with n bidders. QED

Proof of Proposition 3

First, consider the two cases (i) $\alpha \geq 1$ and (ii) $\alpha > 1/n$ and $k = n$. When $\alpha \geq 1$, any contribution to the public good returns at least as much as it costs, and it is optimal to bid M. This is also true for the zero-value bidder in the lowest-price all-pay auction when $\alpha > 1/n$. Thus, in both cases i and ii, $v^* = 0$ and revenue equals nM.

Next, consider the case $k < n$ and $\alpha < 1$. The condition determining the cut point v^* is that the difference between the expected payoff of bidding M and $B^{\mathrm{AP}}_{k,n}(v^*)$ is zero:

$$\begin{aligned} 0 = {} & \sum_{j=1}^{n-1} \frac{v^*}{j+1}\left[F_{Y^{n-1}_{j+1}}(v^*) - F_{Y^{n-1}_{j}}(v^*)\right] \\ & + [M - B^{\mathrm{AP}}_{k,n}(v^*)](\alpha k - 1)[F_{Y^{n-1}_{k}}(v^*) - F_{Y^{n-1}_{k-1}}(v^*)] \\ & - [M - B^{\mathrm{AP}}_{k,n}(v^*)](1-\alpha)[1 - F_{Y^{n-1}_{k}}(v^*)]. \end{aligned} \tag{A2}$$

To understand the right-hand side of (A2), consider a bidder with value v^* and assume that all others bid according to $B^{\mathrm{AP}}_{k,n}(v, M)$ in (9). The expression in the top line captures the bidder's increased chance of winning the object worth v^* when she raises her bid from $B^{\mathrm{AP}}_{k,n}(v^*)$ to M. The expression in the second line pertains to the case in which she is the kth-highest bidder and, by bidding M, she increases the price she and the $(k-1)$ highest bidders pay from $B^{\mathrm{AP}}_{k,n}(v^*)$ to M. The third line applies when $B^{\mathrm{AP}}_{k,n}(v^*)$ is not among the k highest bids and by bidding M the bidder increases only her own price. When these benefits and costs balance, the bidder is indifferent between bidding $B^{\mathrm{AP}}_{k,n}(v^*)$ and M.

To show that there exists an interior solution $0 < v^* < 1$ to (A2), we define

$$\zeta(v) \equiv (\alpha k - 1)[F_{Y^{n-1}_{k}}(v) - F_{Y^{n-1}_{k-1}}(v)] - (1-\alpha)[1 - F_{Y^{n-1}_{k}}(v)]$$

and

$$\psi(v) \equiv [M - B^{\mathrm{AP}}_{k,n}(v)]\zeta(v) + \sum_{j=1}^{n-1} \frac{v}{j+1} [F_{Y^{n-1}_{j+1}}(v) - F_{Y^{n-1}_{j}}(v)].$$

The indifference condition (A2) then becomes $\psi(v^*) = 0$. First note that $\psi(0) < 0$ as $\zeta(0) < 0$. Since $B^{\mathrm{AP}}_{k,n}(v)$ diverges when v tends to one (if not before), there must be some value $\tilde{v} < 1$ for which $B^{\mathrm{AP}}_{k,n}(\tilde{v}) = M$. At that value, $\psi(\tilde{v}) > 0$. Hence, continuity of $\psi(\cdot)$ implies that there exists an interior value in which $\psi(\cdot)$ vanishes. Let $0 < v^* < 1$ denote the smallest v for which $\psi(v) = 0$.

To prove that (9) constitutes an equilibrium, we need to show that (i) bidders with values $v < v^*$ submit bids according to a well-defined increasing bid function, (ii) this bid function follows from the same equilibrium differential equation as in proposition 2, and (iii) bidders with values $v \geq v^*$ strictly prefer to bid M given others' equilibrium bids. Condition ii is readily checked. Condition iii follows immediately from (A2), since for bidders with values $v > v^*$ only the potential gain in the top line changes. Hence, if a bidder with $v = v^*$ is indifferent between bidding M and $B^{\mathrm{AP}}_{k,n}(v^*)$, bidders with types $v > v^*$ strictly prefer to bid M. The only condition that remains to be checked is i.

Let v^{**} be the smallest v for which $\zeta(v) = 0$. We first show that $B^{\mathrm{AP}}_{k,n}(v)$ is well defined and increasing in v for all $v < v^{**}$ and then show that $v^* \leq v^{**}$. Recall from (7) that

$$B^{\mathrm{AP}}_{k,n}(v) = \int_0^v \frac{(n-1)zf(z)F(z)^{n-2}}{-(\alpha k - 1)[1 - F_{Y^{n-1}_{k-1}}(z)] + \alpha(k-1)[1 - F_{Y^{n-1}_{k}}(z)]}\, dz,$$

which is well defined and strictly increasing as long as the denominator is positive, that is, when $\zeta(v) < 0$. As $\zeta(\cdot)$ is continuous and $\zeta(0) < 0$, $B^{\mathrm{AP}}_{k,n}(v)$ is strictly increasing in v for all $v < v^{**}$. (Note that by the definition of v^{**}, the derivative of $B^{\mathrm{AP}}_{k,n}(v)$ with respect to v approaches infinity as v approaches v^{**}.) Clearly $\psi(v^{**}) \geq 0$, and since $\psi(0) < 0$, continuity of $\psi(\cdot)$ implies that $\psi(v) = 0$ for some $v \leq v^{**}$: the smallest v for which this holds is v^*, so $v^* \leq v^{**}$.

Finally, $R^{\mathrm{AP}}_{k,n} < nM$ for all $k < n$ when $\alpha < 1$ since the cut point satisfies $v^* > 0$ in this case. Thus there is a positive probability that at least one bidder bids less than M, and expected revenue is thus less than nM. From the proof of proposition 2 we know that $R^{\mathrm{LOT}} < R^{\mathrm{AP}}_{1,n}$ when $\alpha < 1$, so $R^{\mathrm{LOT}} < nM$ when $\alpha < 1$. QED

Proofs of Propositions 4 and 5

To prove that the lowest-price all-pay auction is optimal, we consider more general mechanisms and derive their revenue properties in lemmas A1 and A2. First, some notation:

$$V \equiv [0, 1]^n$$

and

$$V_{-i} \equiv [0, 1]^{n-1},$$

with typical elements $\boldsymbol{v} = (v_1, \ldots, v_n)$ and $\boldsymbol{v}_{-i} \equiv (v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n)$, respectively. Let

$$g(\boldsymbol{v}) \equiv \prod_j f(v_j)$$

be the joint density of $\boldsymbol{v}$, and let

$$g_{-i}(\boldsymbol{v}_{-i}) \equiv \prod_{j \neq i} f(v_j)$$

be the joint density of $\boldsymbol{v}_{-i}$. We define the marginal revenue $\mathrm{MR}_i(v_i) \equiv v_i - \{[1 - F(v_i)]/f(v_i)\}$ and assume that it is strictly increasing in v_i.

We follow Myerson (1981) closely. Using the revelation principle, we may assume, without loss of generality, that the seller considers feasible direct mechanisms only.[13] Let (p, x) denote a feasible direct mechanism, where $p: V \rightarrow [0, 1]^n$ with $\sum_j p_j(\boldsymbol{v}) \leq 1$ and $x: V \rightarrow \Re^n$. We interpret $p_i(\boldsymbol{v})$ as the probability that bidder i wins and $x_i(\boldsymbol{v})$ as the expected payments by i to the seller when the vector of values $\boldsymbol{v} = (v_1, \ldots, v_n)$ is truthfully announced. Given v_i, bidder i's interim utility under (p, x) is

$$U_i(p, x, v_i) \equiv \int_{V_{-i}} \left[v_i p_i(\boldsymbol{v}) - x_i(\boldsymbol{v}) + \alpha \sum_{j=1}^{n} x_j(\boldsymbol{v}) \right] g_{-i}(\boldsymbol{v}_{-i}) d\boldsymbol{v}_{-i}. \tag{A3}$$

Similarly, the seller's expected utility is

$$U_0(p, x) \equiv \int_V \sum_{i=1}^{n} x_i(\boldsymbol{v}) g(\boldsymbol{v}) d\boldsymbol{v}.$$

The following two lemmas will be used to solve the seller's problem.

Lemma A1. Let (p, x) be a feasible direct revelation mechanism. Then the interim utility of (p, x) for bidder i is given by

$$U_i(p, x, v_i) = U_i(p, x, 0) + \int_0^{v_i} Q_i(w) dw, \tag{A4}$$

with $Q_i(v_i) \equiv E_{v_{-i}}\{p_i(\boldsymbol{v})\}$.

Proof. The proof follows in a straightforward manner from the incentive compatibility constraints (see Myerson 1981). QED

Lemma A2. Let (p, x) be a feasible direct revelation mechanism. The seller's expected revenue from (p, x) is given by

$$U_0(p, x) = \frac{E_v\{\sum_{i=1}^{n} \mathrm{MR}_i(v_i) p_i(\boldsymbol{v})\} - \sum_{i=1}^{n} U_i(p, x, 0)}{1 - n\alpha}. \tag{A5}$$

Proof. Define $X_i = \int_V x_i(\boldsymbol{v}) g(\boldsymbol{v}) d\boldsymbol{v}$, $W_i = \int_V v_i p_i(\boldsymbol{v}) g(\boldsymbol{v}) d\boldsymbol{v}$, and $Y_i = \int_{V_i} U_i(p, x, v_i) f(v_i) dv_i$. By (A3), we have, for all i,

$$Y_i = W_i - X_i + \alpha \sum_{j=1}^{n} X_j. \tag{A6}$$

[13] A direct mechanism is a mechanism in which bidders are simply asked to announce their values. We say that a mechanism is feasible if it satisfies individual rationality conditions, incentive compatibility conditions, and straightforward restrictions on the allocation rule.

Summing over i in (A6) and rearranging shows that the seller's expected revenue from a feasible direct revelation mechanism (p, x) is given by

$$U_0(p, x) = \sum_{i=1}^{n} X_i = \frac{\sum_{i=1}^{n} W_i - \sum_{i=1}^{n} Y_i}{1 - n\alpha}. \tag{A7}$$

Taking the expectation of (A4) over v_i and using integration by parts, we obtain

$$Y_i = U_i(p, x, 0) + E_{v_i}\left\{\frac{1 - F(v_i)}{f(v_i)} Q_i(v_i)\right\},$$

so that (A5) follows. QED

Now, using lemma A2, we prove propositions 4 and 5. From (A5), it is clear that a feasible auction mechanism is revenue maximizing if it (1) assigns the object to the bidder with the highest marginal revenue if the highest marginal revenue is positive and leaves the object in the hands of the seller otherwise, and (2) gives the lowest type zero expected utility.

We first prove proposition 4. Under the restriction that the seller cannot commit to keeping the good, a feasible auction mechanism is revenue maximizing if it assigns the good to the bidder with the highest marginal revenue (even if negative) and guarantees the lowest-type bidder zero expected utility. It is clear that $B_{n,n}^{\mathrm{AP}}(v)$ in (7) is strictly increasing in v since the denominator of the integrand in (7) is strictly positive when $\alpha < 1/n$. So the lowest-price auction assigns the object to the bidder with the highest value and, hence, to the bidder with the highest marginal revenue. Furthermore, a zero-type bidder bids zero according to (7), which sets the auction's revenue at zero, leaving the lowest-type bidder with zero expected utility. Hence, the lowest-price all-pay auction is revenue maximizing. (We have already shown that revenue increases with α in the proof of proposition 2.)

Next we turn to proposition 5. In the equilibrium defined by (11), only bidders with values $v > \hat{v}$ submit a bid according to a strictly increasing bid function whereas bidders with values $v < \hat{v}$ abstain from bidding. Hence, Γ assigns the good only to bidders with positive marginal revenues (if at all). Moreover, the bidding function in (11) is strictly increasing in v so that the bidder with the highest marginal revenue receives the object. Finally, the expected utility of a bidder with the lowest type equals zero over both stages of Γ since

$$U(p, x, 0) = (n\alpha - 1)\varphi + \alpha r(n - 1)[1 - F(\hat{v})] = 0$$

by the definition of φ. The given strategies constitute a Bayesian Nash equilibrium, and when these are played, Γ maximizes (A5) and is thus optimal. Revenues increase with α since the denominator of (A5) is decreasing in α. QED

References

Baye, Michael R., Dan Kovenock, and Casper de Vries. 1998. "A General Linear Model of Contests." Manuscript, Dept. Econ., Indiana Univ.

———. 2000. "Comparative Analysis of Litigation Systems: An Auction-Theoretic Approach." Manuscript, Dept. Econ., Indiana Univ.

Bergstrom, Theodore, Lawrence Blume, and Hal Varian. 1986. "On the Private Provision of Public Goods." *J. Public Econ.* 29 (February): 25–49.

Bulow, Jeremy, Ming Huang, and Paul Klemperer. 1999. "Toeholds and Takeovers." *J.P.E.* 107 (June): 427–54.

Burkart, Mike. 1995. "Initial Shareholdings and Overbidding in Takeover Contests." *J. Finance* 50 (December): 1491–1515.

Coase, Ronald H. 1972. "Durability and Monopoly." *J. Law and Econ.* 15 (April): 143–49.

Cramton, Peter, Robert Gibbons, and Paul Klemperer. 1987. "Dissolving a Partnership Efficiently." *Econometrica* 55 (May): 615–32.

Engelbrecht-Wiggans, Richard. 1994. "Auctions with Price-Proportional Benefits to Bidders." *Games and Econ. Behavior* 6 (May): 339–46.

Engers, Maxim, and Brian McManus. 2003. "Charity Auctions." Manuscript, Olin School Bus., Washington Univ.

Gavious, Arieh, Benny Moldovanu, and Aner Sela. 2002. "Bid Costs and Endogenous Bid Caps." *Rand J. Econ.* 33 (Winter): 709–22.

Graham, Daniel A., and Robert C. Marshall. 1987. "Collusive Bidder Behavior at Single-Object Second-Price and English Auctions." *J.P.E.* 95 (December): 1217–39.

Groves, Theodore, and John O. Ledyard. 1977. "Optimal Allocation of Public Goods: A Solution to the 'Free Rider' Problem." *Econometrica* 45 (May): 783–809.

Jehiel, Philippe, Benny Moldovanu, and Ennio Stacchetti. 1996. "How (Not) to Sell Nuclear Weapons." *A.E.R.* 86 (September): 814–29.

Klemperer, Paul. 1999. "Auction Theory: A Guide to the Literature." *J. Econ. Surveys* 13 (July): 227–86.

Krishna, Vijay, and John Morgan. 1997. "An Analysis of the War of Attrition and the All-Pay Auction." *J. Econ. Theory* 72 (February): 343–62.

Ledyard, John O. 1978. "The Allocation of Public Goods with Sealed-Bid Auctions: Some Preliminary Evaluations." Discussion Paper no. 336 (September), Center Math. Studies Econ. and Management Sci., Northwestern Univ.

McAfee, R. Preston, and John McMillan. 1992. "Bidding Rings." *A.E.R.* 82 (June): 579–99.

Mood, Alexander M., Franklin A. Graybill, and Duane C. Boes. 1963. *Introduction to the Theory of Statistics.* 3rd ed. New York: McGraw-Hill.

Morgan, John. 2000. "Financing Public Goods by Means of Lotteries." *Rev. Econ. Studies* 67 (October): 761–84.

Myerson, Roger B. 1981. "Optimal Auction Design." *Math. Operations Res.* 6 (February): 58–73.

Reis, Thomas, and Stephanie Clohesy. 2000. "E-Philanthropy, Volunteerism, and Social Changemaking: A New Landscape of Resources, Issues and Opportunities." Manuscript, W. K. Kellogg Found., Battle Creek, MI.

Riley, John G., and William F. Samuelson. 1981. "Optimal Auctions." *A.E.R.* 71 (June): 381–92.

Singh, Rajdeep. 1998. "Takeover Bidding with Toeholds: The Case of the Owner's Curse." *Rev. Financial Studies* 11 (Winter): 679–704.