INTRODUCTION

- An auction, the way we understand it here, is a mechanism in which a singe seller is trying to sell the object(s) to one of the participating bidders. The idea is that each participant (seller and bidders) has a **valuation** for the object of sale, and bidders typically have higher valuations that the seller. Each participant's valuation is a private information in that it is not known to anyone else. An auction design may attempt to achieve one of several possible objectives, such as maximizing revenue or efficiency (allocating the object(s) to the highest valuation bidder).

- There are many different kinds of auctions, depending on the nature of the object to be sold and the mechanism used for selling.

- Depending on the nature of the object, we can categorize auctions as:

  - single-unit auctions (e.g., a painting)
  - multi-unit, homogeneous good auctions (e.g., treasury bills)
  - multi-unit, heterogeneous good auctions (e.g., broadband spectra in different areas)

- For single-unit auctions, often-used auction mechanisms are:

  - English (ascending bid) (clock) auction
  - Dutch (descending bid) (clock) auction
  - first-price sealed-bid auction
  - second-price sealed-bid auction

- In multi-unit, homogeneous good auctions, bidders submit their demand schedules. Often-used mechanisms are:

  - uniform price auction
  - discriminatory price auction

- In case of multi-unit, heterogeneous good auctions, one may implement independent bidding on individual items. However, bidder valuations may depend on the whole package of auctioned items, such as having the broadband spectrum in two adjacent areas. The newest designs respect that and try to make auctions more efficient by introducing bidding on packages. These designs are called **combinatorial auctions**.

- In this course, we will focus on single-unit auctions.

- These can be further categorized depending on how bidder valuations are related:

  1. **Independent private value auction (IPV):** bidder valuations can be thought of as being random draws from some distribution. Each participant knows his own valuation but does not know the valuation of the other participants. This would be the case for auctions of consumption items without the possibility of resale, such as concert tickets just before the show, for example.
  2. **Common value auction (CV)**: bidder valuations coincide but are not completely known to the bidders. This would be the case for auctions of items purely for resale or public procurement contracts (with known revenue but unknown costs), for example.

3. **Affiliated value auction:** bidder valuations can be thought of as being correlated draws from some distribution. This is probably the most realistic case as it combines features of IV and CV. It is the CV part that accounts for the correlation and it the IV part that accounts for the correlation to be less than 1.

- In this course, we will focus on IPV and CV auctions.

- Also, we will focus on auctions without reserve prices. In many real-world auctions, however, the seller may set a reserve price (public or secret), which specifies the minimum selling price for the object. If none of the bids at least achieve this price, the object is not sold.

THEORY OF EQUILIBRIUM BIDDING IN IPV AUCTIONS

- **Theorem 1:** The English auction is strategically equivalent to the second-price sealed-bid auction.

- **Theorem 2:** The Dutch auction is strategically equivalent to the first-price sealed-bid auction.

- As a result, in the rest of this section we will only analyze first- and second-price sealed-bid auctions.

- **Theorem 3:** In the second-price sealed-bid auction, it is a dominant strategy for each bidder to bid his own valuation.

- **Theorem 4:** Under risk-neutrality of the bidders, all four types of auction generate the same expected revenue. With risk aversion, the first-price and the Dutch auctions generate more revenue that the second-price and the English auctions.

- Deriving equilibrium behavior in the first-price auction is more involved. To begin, consider the simple case of two bidders, one human and one computerized. The valuation of the human bidder is v and the computer is going to draw its bid b_c randomly from the uniform distribution on $[0, \bar{b}]$. How should the bidder bid if he wants to maximize his expected payoff?

- It never makes sense to bid more than $\bar{b}$. So, without loss of generality, consider only bids in the interval $[0, \bar{b}]$. If bidding b, the probability of winning is $b/\bar{b}$, in which case the net payoff to the human bidder is $v - b$. In case of losing, the net payoff is 0. Hence the expected payoff is given by
$$(b/\bar{b})(v - b).$$
This is maximized by choosing
$$b(v) \equiv v/2.$$

- Now generalize from this example: suppose that there are two human bidders and their valuations are random draws from the uniform distribution on $[0, 1]$. This case is much more difficult because, when computing the bidding function for one of the bidders, we do not know what strategy is used by the other bidder. We are going to look for a symmetric bidding equilibrium in that both bidders are going to be using the same bidding strategy and this strategy will be a best response to itself.

- Consider bidder 1. We start by guessing that bidder 2 uses the bidding strategy $b_2(v) = \alpha_2 v$, with $\alpha_2 \in [0,1]$. What is the best response of bidder 1? Note that the bidding strategy of bidder 2 implies that his bids will be distributed uniformly on $[0, \alpha_2]$. We then know from the previous simplified scenario that $b_1(v) = v/2$ is the unique best response. Hence if we set $\alpha_2 = 1/2$, then we have arrived at a symmetric equilibrium bidding function given by
$$b(v) \equiv v/2.$$

- Now generalize this scenario by assuming that, instead of just 2, there are $N > 2$ bidders. Consider the behavior of bidder 1 and guess that all the other bidders use the bidding strategy $b_{-1}(v) = \alpha_{-1}v$, with $\alpha_{-1} \in [0,1]$. What is the best response of bidder 1? Note that the bidding strategy of the other bidders implies that a bid of b by bidder 1 will be the highest of all bids if and only if $\alpha_{-1}v_i < b$, or, equivalently, $v_i < b/\alpha_{-1}$ for all $i = 2, .., N$, which happens with probability $(b/\alpha_{-1})^{N-1}$. Therefore the optimal bid of bidder 1 with valuation v is obtained by solving
$$\max_b (b/\alpha_{-1})^{N-1}(v-b),$$
which is equivalent to solving
$$\max_b b^{N-1}(v-b).$$
The first-order condition is
$$b^{N-2}[(N-1)v - Nb] = 0.$$
Note that the first derivative is zero at $b = 0$, is positive for $b \in (0, (N-1)v/N)$, is zero at $b = (N-1)v/N$ and is negative for $b > (N-1)v/N$. As a result, the global maximum is achieved at
$$b_1(v) \equiv \frac{N-1}{N}v.$$
Hence if we set $\alpha_{-1} = (N-1)/N$, then we have arrived at a symmetric equilibrium bidding function given by
$$b(v) \equiv \frac{N-1}{N}v.$$
Note that the case of $N = 2$ we analyzed before fits this formula.

- Now we can further generalize the equilibrium bidding result by assuming that valuations are drawn from a general continuous distribution F on $[0,1]$. We can retrace the procedure we followed in the previous step. Consider the behavior of bidder 1 and guess that all the other bidders use the bidding strategy $\widetilde{b}(v)$ that is continuous, strictly increasing, differentiable, and satisfies $0 \leq \widetilde{b}(v) \leq v$. Note that this implies $\widetilde{b}(0) = 0$. What is the best response of bidder 1? Note that the bidding strategy of the other bidders implies that a bid of b by bidder 1 will be the highest of all bids if and only if $\widetilde{b}(v_i) < b$, or, equivalently, $v_i < \widetilde{b}^{-1}(b)$ for all $i = 2, .., N$, which happens with probability $\left\{F\left[\widetilde{b}^{-1}(b)\right]\right\}^{N-1}$. Therefore the optimal bid of bidder 1 with valuation v is obtained by solving
$$\max_b \left\{F\left[\widetilde{b}^{-1}(b)\right]\right\}^{N-1}(v-b).$$
The first-order condition is
$$\left\{F\left[\widetilde{b}^{-1}(b)\right]\right\}^{N-2}\left\{\frac{(N-1)f\left[\widetilde{b}^{-1}(b)\right](v-b)}{\widetilde{b}'\left[\widetilde{b}^{-1}(b)\right]} - F\left[\widetilde{b}^{-1}(b)\right]\right\} = 0.$$

The global maximum is achieved at $b_1(v)$ that is implicitly characterized by

$$\frac{(N-1)f\left[\widetilde{b}^{-1}(b_1(v))\right][v-b_1(v)]}{\widetilde{b}'\left[\widetilde{b}^{-1}(b_1(v))\right]} = F\left[\widetilde{b}^{-1}(b_1(v))\right].$$

In a symmetric equilibrium, $b_1(\cdot) = \widetilde{b}(\cdot) \equiv b(\cdot)$, and hence the equilibrium bidding function is characterized by

$$\frac{(N-1)f(v)[v-b(v)]}{b'(v)} = F(v),$$

which is equivalent to

$$b'(v) = (N-1)\frac{f(v)}{F(v)}[v-b(v)]. \tag{1}$$

The boundary condition is given by $b(0) = 0$. This can be rewritten as

$$b'(v)F(v) + (N-1)b(v)f(v) = (N-1)f(v)v.$$

Multiply both sides by $[F(v)]^{N-2}$:

$$b'(v)[F(v)]^{N-1} + b(v)(N-1)[F(v)]^{N-2}f(v) = (N-1)[F(v)]^{N-2}f(v)v.$$

Now note that the LHS is the derivative of the expression $b(v)[F(v)]^{N-1}$. So we obtain the differential equation

$$\frac{d}{dv}b(v)[F(v)]^{N-1} = (N-1)[F(v)]^{N-2}f(v)v.$$

Integrating, we obtain

$$b(v)[F(v)]^{N-1} = b(0)[F(0)]^{N-1} + \int_0^v (N-1)[F(x)]^{N-2}f(x)x\,dx.$$

Now first note that $F(0) = 0$, so the first term on the RHS drops out. Second, using integration *per partes*, we have that

$$\begin{aligned}\int_0^v (N-1)[F(x)]^{N-2}f(x)x\,dx &= [F(x)]^{N-1}x\big|_0^v - \int_0^v [F(x)]^{N-1}dx \\ &= [F(v)]^{N-1}v - \int_0^v [F(x)]^{N-1}dx.\end{aligned}$$

As a result, the equilibrium bidding function can be expressed explicitly as

$$b(v) = v - \int_0^v [F(x)/F(v)]^{N-1}dx. \tag{2}$$

Note that with the uniform distribution of valuations we have that $F(x) = x$, which gives us the formula we obtained in the previous case. Also note that the second expression on the RHS gives an equilibrium bidding discount relative to one's own valuation.

- We can do the final generalization by allowing bidders to be risk-averse instead of risk-neutral. Hence suppose that each bidder's utility function of net gain y is given by $u(y)$ with $u(0) = 0$ and $u(\cdot)$ being strictly concave. Then, following the analogous steps as in the previous case, Therefore the optimal bid of bidder 1 with valuation v is obtained by solving

$$\max_b \left\{F\left[\widetilde{b}^{-1}(b)\right]\right\}^{N-1} u(v-b).$$

Following analogous manipulations, the symmetric equilibrium bidding function $b(\cdot)$ is implicitly characterized by

$$b'(v) = (N-1)\frac{f(v)}{F(v)}\frac{u[v-b(v)]}{u'[v-b(v)]} \tag{3}$$

with the boundary condition being $b(0) = 0$. In this case it is no longer possible to obtain a closed-form solution for the bidding function in general. However, we can obtain an explicit solution for a commonly used types of risk-aversion utility function, namely the case of **constant relative risk aversion**. In this case, $u(y) = y^{1-r}$, where $r \in [0, 1)$ is the *coefficient of relative risk aversion*. Under these preferences, (**??**) becomes

$$b'(v) = (N-1)\frac{f(v)}{F(v)}\frac{y}{1-r}. \tag{4}$$

The boundary condition is given by $b(0) = 0$. We then can retrace the same steps as before with $N-1$ replaced by $(N-1)/(1-r)$ to arrive at

$$b(v) = v - \int_0^v [F(x)/F(v)]^{\frac{N-1}{1-r}}\,dx. \tag{5}$$

Note that since $0 < F(x)/F(v) < 1$ for $x \in (0, v)$ and $(N-1)/(1-r) > N-1$ for $r > (0, 1)$, the bidding discount relative to own valuation is, *ceteris paribus*, lower, and therefore bids are higher, under risk aversion. As $r \to 1$, bids are approaching valuations. For example, with the uniform distribution of valuations ($F(x) = x$), we have that

$$b(v) = \frac{N-1}{N-r}v,$$

which clearly illustrated the points.

THEORY OF EQUILIBRIUM BIDDING IN CV AUCTIONS

- In this case, we will focus only on the first-price sealed bid auction.
- There is a true common value v of the object and each bidder observes a conditionally independent signal of it, which we will call v_i. Suppose that each individual signal is unbiased in that $E(v_i) = v$.
- These types of auctions are prone to bidder behavior known as the **winner's curse**. The story goes as follows. Each individual signal is an unbiased estimate of v, and hence it appears that one is going to do the best by bidding based on this signal in a fashion similar to IPV. However, the only time the bid of particular bidder has any consequence is if the bid is the winning bid. But conditionally on the bid being the winning bid, this bidder's signal is an upward-biased estimate of v, and hence the bidder is likely to overpay for the object.

- We ran a Takeover Game experiment, which is a simple illustration of the tendency of the buyers to overbid in common value environments. Consider the following setup: a seller is selling an asset worth v to him, but the only thing that a buyer knows is that v comes from a uniform distribution on $[0, 1]$. The (unknown) value to the buyer is αv, where $\alpha \in (1, 2)$. The buyer makes a price offer and the seller decides whether to sell or not. Note that conditional on agreeing to sell for an offered price $b > 0$ (bid), the expected value of the asset to the seller is $b/2$, and hence the expected value of the asset to the buyer is $b\alpha/2 < b$. As a result, a buyer is hurting himself in expectation and he should not make any positive bids. Therefore there is a unique equilibrium in which buyers don't make any positive bids an no positive value assets ever change hands, even though there would be a lot of efficiency gain from such trades if it was not for the asymmetry of information. This is also an example of **Akerlof's lemon market** in which in equilibrium all transactions vanish except for the ones with the worst lemons. In experiments, however, bidders typically offer positive prices and there is a very slow convergence toward bidding 0.

- More generally, ignoring the fact that, conditional on winning, the value of the firm increases with one's bid much more steeply than what pertains to own v_i may lead to both overbidding and underbidding, as we will later see.

- Now coming back to the common value auction. Rational bidders will foresee the danger of the winner's curse and will adjust their bids accordingly. Characterization of a symmetric equilibrium bidding function in general depends on the information structure of the problem (the distribution from which v is drawn and how signals are drawn conditional on v). Let's consider a simple example to illustrate equilibrium bidding. Suppose that there are two risk-neutral bidders for a company, and each bidder gets to audit one of the two subsidiaries, but not the other one. Let the revealed values of the two subsidiaries be v_1 and v_2, respectively, and let the true value of the company be given by the average of v_1 and v_2. That is,

$$v = \frac{v_1 + v_2}{2}.$$

  Further, suppose that v_1 and v_2 are drawn independently from the uniform distribution on $[0, 1]$. What are the equilibrium bidding functions?

- Again, think about the decision process of bidder 1. We are going to guess that bidder 2 uses a linear bidding function $b_2(v_2) \equiv \alpha_2 v_2$. What is the optimal response of bidder 1? When bidding b, the probability of winning is equal to the probability that $\alpha_2 v_2 < b$, or $v_2 < b/\alpha_2$, which is b/α_2. What is the expected value of the firm conditional on winning with the bid b? It is

$$\begin{aligned} \frac{v_1 + E(v_2 | v_2 < b/\alpha_2)}{2} &= \frac{v_1 + b/2\alpha_2}{2} \\ &= \frac{v_1}{2} + \frac{b}{4\alpha_2}. \end{aligned} \tag{6}$$

  Hence the expected payoff of bidder 1 when bidding b is

$$\frac{b}{\alpha_2}\left(\frac{v_1}{2} + \frac{1 - 4\alpha_2}{4\alpha_2} b\right).$$

  This is a negative quadratic function that is optimized with respect to b at

$$b_1(v_1) \equiv \frac{\alpha_2}{4\alpha_2 - 1} v_1.$$

In a symmetric bidding equilibrium, we need to have

$$\frac{\alpha_2}{4\alpha_2 - 1} = \alpha_2,$$

giving

$$\alpha_2 = \frac{1}{2}.$$

Therefore the symmetric equilibrium bidding function is given by

$$b(v_i) = \frac{v_i}{2}.$$

- Now think about a bidder who plays against an equilibrium bidder who ignores conditioning of the expectation of the firm value on the fact that he is winning. The probability of winning with a bid of b is $2b$, but now the expected value of the firm conditional only on the observation of v_1 is

  $$\frac{v_1}{2} + \frac{1}{4},$$

  which is an overestimate of the true conditional value $v_1/2 + b/2$ (this is because with the other bidder using $b(v_i) = v_i/2$, it never makes sense to consider bids at 1/2 or above). As a result, the bidder optimizes his (mistakenly believed) expected payoff given by

  $$\frac{b}{\alpha_2}\left(\frac{v_1 + 0.5}{2} - b\right).$$

  This is optimized with respect to b at

  $$b_1(v_1) \equiv 0.125 + \frac{v_1}{4}.$$

  Note that this bidding strategy exceeds the best response given by $b(v_i) = v_i/2$ for $v_1 \in [0, 0.5)$. So bidder 1 overbids for low values of v_1 and underbids for high values of v_1. Therefore the winner's curse in a classic sense is only present for the lower half of valuations.

- Now generalize to N bidders. Again, think about the decision process of bidder 1. We are going to guess that bidders 2 through N use a linear bidding function $\widetilde{b}(v_i) \equiv \alpha v_i$. What is the optimal response of bidder 1? When bidding b, the probability of winning is equal to the probability that $\alpha v_i < b$, or $v_i < b/\alpha$ for all $i = 2, ..., N$, which is $(b/\alpha)^{N-1}$. What is the expected value of the firm conditional on winning with the bid b? It is

  $$\begin{aligned} \frac{v_1 + E(v_2 | v_i < b/\alpha, i = 2, ..., N)}{2} &= \frac{v_1 + (N-1)b/2\alpha}{N} \\ &= \frac{v_1}{N} + \frac{N-1}{2N\alpha}b. \end{aligned} \tag{7}$$

  Hence the expected payoff of bidder 1 when bidding b is

  $$\left(\frac{b}{\alpha_2}\right)^{N-1}\left(\frac{v_1}{2} + \frac{N - 1 - 2N\alpha}{2N\alpha}b\right).$$

  This function is optimized with respect to b at

  $$b_1(v_1) \equiv \frac{(N-1)\alpha}{2N\alpha - (N-1)}v_1.$$

In a symmetric bidding equilibrium, we need to have

$$\frac{(N-1)\alpha}{2N\alpha-(N-1)}=\alpha,$$

giving

$$\alpha_2=\frac{N-1}{N}.$$

Therefore the symmetric equilibrium bidding function is given by

$$b(v_i)=\frac{N-1}{N}v_i.$$

- Now again think about a bidder who plays against a set of equilibrium bidders and who ignores conditioning of the expectation of the firm value on the fact that he is winning. The probability of winning with a bid of b is $\{[N/(N-1)]b\}^{N-1}$, but now the expected value of the firm conditional only on the observation of v_1 is

$$\frac{v_1}{N}+\frac{(N-1)0.5}{N},$$

which is an overestimate of the true conditional value $v_1/N+b/2$(this is because with the other bidders using $b(v_i)=[(N-1)/N]v_i$, it never makes sense to consider bids at $(N-1)/N$ or above). As a result, the bidder optimizes his (mistakenly believed) expected payoff given by

$$\{[N/(N-1)]b\}^{N-1}\left(\frac{v_1}{N}+\frac{(N-1)0.5}{N}-b\right).$$

$$\frac{(N-1)v_1}{N^2}+\frac{(N-1)^2 0.5}{N^2}=b$$

This is optimized with respect to b at

$$b_1(v_1)\equiv\frac{(N-1)^2 0.5}{N^2}+\frac{(N-1)v_1}{N^2}.$$

Note that this bidding strategy exceeds the best response given by $[(N-1)/N]v_1$ for $v_1\in[0,0.5)$. So bidder 1 overbids for low values of v_1 and underbids for high values of v_1. Therefore the winner's curse in a classic sense is only present for the lower half of valuations, and this conclusion is independent of the number of bidders.

SUMMARY OF EXPERIMENTAL RESULTS ON IPV AUCTIONS

- Large literature.
- Summarized by **Kagel (1995)**.
- **Typical findings for IPV auctions:**
  - Subjects behave more or less in line with theory in English (clock) auction (drop out of bidding at their valuation).

- In the first-price auction, subjects tend to bid above the risk-neutral equilibrium bidding function. This can be explained by the presence of risk aversion.
- Many subjects overbid and some subjects underbid their valuation in the second-price auction. Some authors (**Levine, 1998**) have attempted to explain this finding by spite. As a result, if auction is just a part of a larger design and it is important that subjects correctly reveal their valuations, it is preferable to use the English auction rather than the second-price auction, even though the two are theoretically equivalent.
- The Dutch and first-price auctions typically generate about the same amount of revenue (or the first-price auction generates a bit more).
- The English and second-price auctions generate a similar amount of revenue, but less than the Dutch or the first-price auction (which would be consistent with risk aversion of the bidders).

- **Typical findings for CV auctions:**
  - Presence of the winner's curse, even among experienced subjects.

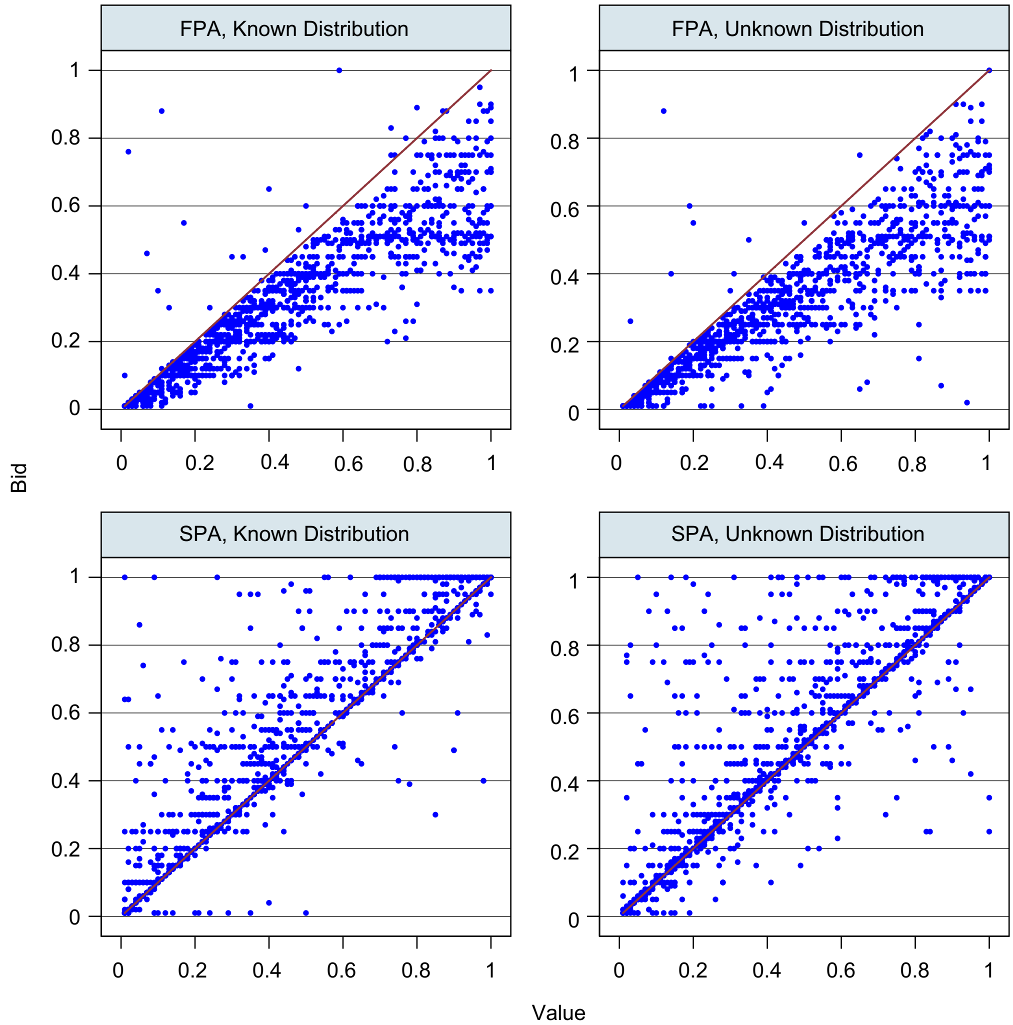

Fig. 2. Raw bids in all treatments.

Figure 1: All Rounds (1-30)

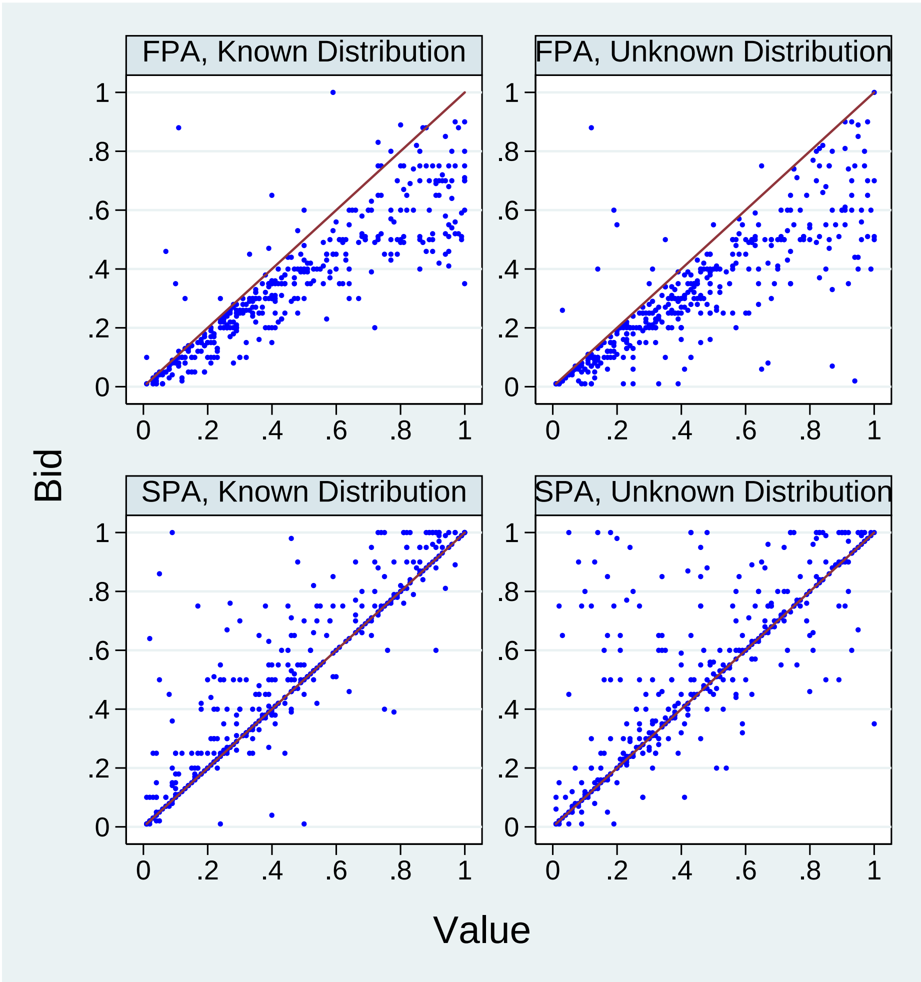

Figure 2: Early Rounds (1-10)

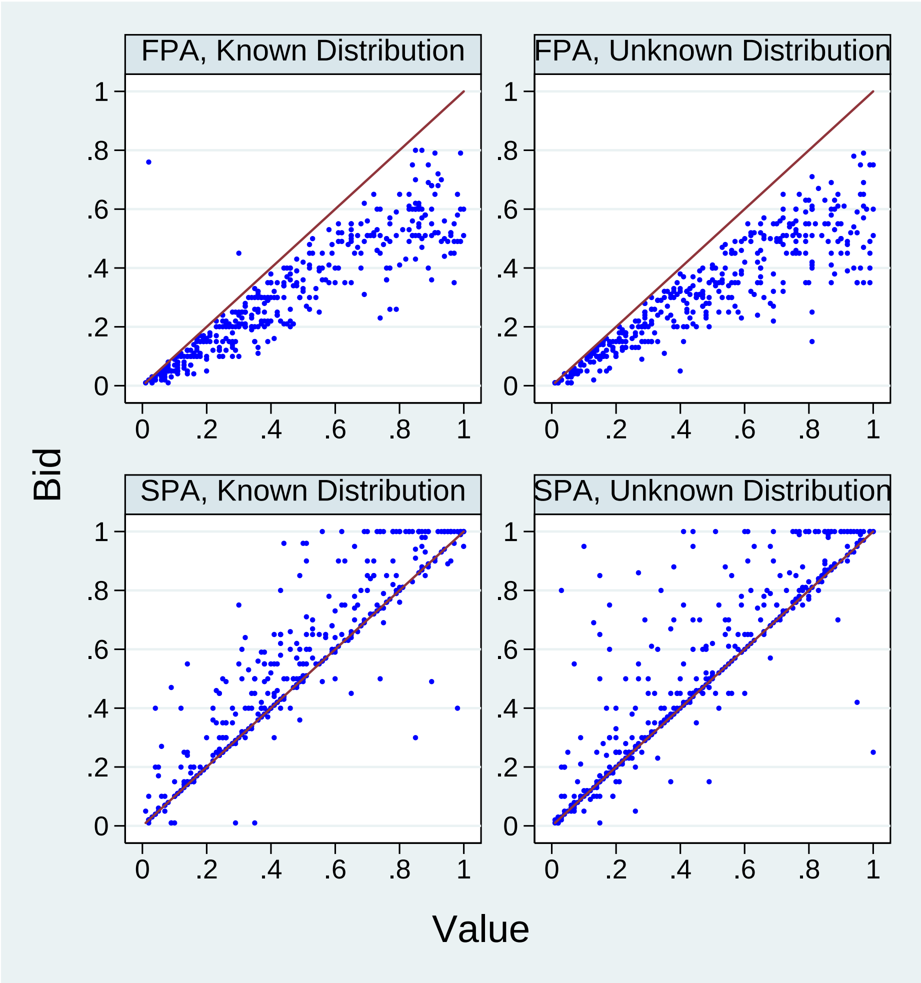

Figure 3: Late Rounds (21-30)

SELECTED PAPERS ON AUCTIONS

Thaler (1988)

- Summarizes studies that find winner's curse.
- **Study 1**: auctioning a jar filled with \$8 worth of coins (unknown to subjects). Subjects on average estimate the jar to contain only \$5.13, but the mean winning bid is \$10.01, producing an average loss to the winner of \$2.01.
- **Study 2**: takeover game. Over 90% of subjects make positive bids in the range of 50% to 75% of the maximum possible valuation of the firm. Furthermore, very little earning occurs even when subjects play the game repeatedly and get a feedback after each round.
- **Study 3**: lab studies (Kagel and coauthors): winners record positive profits in smaller groups (3-5 bidders), but record losses in larger groups (6-7 bidders).
- **Study 4**: lab study with construction firm managers - subject to the winner's curse as much as college students.
- **Field studies**: winner curse in bidding for oil and gas fields.

Lucking-Reiley (1999)

- Conducts a field experiment (selling Magic Cards) over the internet using various auction formats.
- He finds that the Dutch auction generates about 30% higher revenue than the first-price auction (contradictory to earlier results), and that the English auction and the second-price auction generate about the same amount of revenue.

Feliz-Ozbay and Ozbay (2007)

- Study the impact of providing feedback in the first-price auction.
- **Feedback 1**: telling the winner (as opposed to not telling) how much the second highest bid was - may cause a **winner's regret** in that the winner wishes he would have bid less.
- **Feedback 2**: telling the runner-up (as opposed to not telling) how much the highest bid was - may cause a **loser's regret** in that the loser wishes he would have bid more.
- Question: if subjects anticipate getting this feedback (as opposed to not getting it), does this potential anticipated regret affect their bidding behavior?
- Answer: there is little evidence of any anticipated winner's regret. On the other hand, there is significant evidence of anticipated loser's regret.
- This result underlines that bidding behavior may be sensitive to small procedural details and that past results may need to be compared and iterpreted with care depending on what the procedural details of each individual experiment were.

Chen, Katuscak and Ozdenoren (2009)

- Compare behavior of men and women in sealed bid auctions.
- In the first-price auction, women bid more than men for high valuations and record lower profits. Could potentially be explained by women being more risk-averse than men, but the Holt-Laury measure of risk aversion is not significantly different between men and women.
- In the second-price auction, there are no statistically significant differences in the probability of dominant strategy play, overbidding or underbidding.