A General Theory of Time Preferences*

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Abstract

The main result of this paper is a representation theorem for time preferences (on the prize-time space) that cover a variety of time preference models considered in the experimental and theoretical literature on intertemporal choice. In particular, along with many models induced by similarity relations on time and outcomes, exponential, quasi-hyperbolic, hyperbolic, subadditive, and intransitive time preference models are special cases of this representation. One major advantage of this result is, therefore, to identify certain factors that are common to seemingly very different time preference structures, and provide a tractable mathematical format that allows for investigating certain economic environments without subscribing to a particular time preference model. To illustrate, we study the possibility of dynamically consistent decision making with the general class of time preferences we derive axiomatically here, and show that many economic problems (such as preemptive investment timing models and sequential bargaining games) can be analyzed in a unified manner with this class.

JEL Classification: D11, D90

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1 Introduction

Intertemporal choice theory has two basic components. The first component concerns the structure of time preferences, and examines how one may evaluate the trade-offs between various alternatives that are obtained at different times. The second component concerns how plans made at an initial date on the basis of a given time preference would be carried out through time. While this component is studied extensively in the literature on dynamic consistency, there are only a few theoretical studies on the structure of time preferences. Indeed, until recently, it did not seem like there is reason to view this part of intertemporal choice theory as an interesting field of study, presumably because the canonical model of time preferences, the so-called *exponential* (*utility*) *discounting model*, was viewed as "the" appropriate formulation of rational time preferences.[1]

Over the last two decades, however, there has been an immense amount of experimental work, carried out both by economists and psychologists, which yielded certain empirical regularities that are inconsistent with the exponential discounting model.[2] The most pressing of these stem from experiments in which subjects solve intertemporal choice problems of the following form:

(a) Would you prefer to have \$100 now, or \$110 tomorrow?

(b) Would you prefer to have \$100 in one year from now, or \$110 one year and one day from now?

Most people seem to prefer the first alternative in (a) and the second in (b), which is, of course, inconsistent with the exponential discounting model, regardless of the choice of the discount factor and/or the instantaneous utility function. This anomaly, which is sometimes called the *time preference reversal* (*TPR*) phenomenon, has led to the formulation of the so-called *hyperbolic* (*utility*) *discounting model*, which keeps the basic structure of the exponential model, but takes the discount rate as a decreasing function of time. Versions of this model have recently been applied to a variety of dynamic economic problems, especially after coupled with a suitable time consistency hypothesis.[3]

[1] The exponential (utility) discounting model was first formulated by Samuelson (1937). The axiomatic foundations of this model is explored by the definitive works of Koopmans (1960), and Koopmans, Diamond and Williamson (1964) in the case of time preferences over consumption streams, and of Fishburn and Rubinstein (1982) in the case of time preferences over the prize-time space.

[2] See Frederick, Loewenstein and O'Donoghue (2002) for a brilliant survey that documents these anomalies, and provides a detailed examination of the time discounting models proposed in the literature to cope with them.

[3] The term "hyperbolic discounting" is used in the literature for time discounting models in which the per-period discount rate is an everywhere strictly decreasing function of time delay. An alternative, and more tractable model, is the so-called *quasi-hyperbolic* discounting model of Phelps and Pollak (1968) and Laibson (1997), where per-period discount rate between now and the next period is strictly smaller than the per-period discount rate for any two future periods which remain constant. All of these models, along with the classical exponential discounting model, belong to the general class of *multiplicative* (*utility*) *discounting* model (see Examples 1-3 below).

The TPR phenomenon is well-documented in a plethora of experiments, thereby pointing, convincingly, to the fact that the exponential discounting model is hardly an unexceptionable model of time preferences. It appears that there is reason to carefully think about the notion of "time preferences" from a foundational point of view, and develop a general axiomatic framework in which one may identify the common and distinguishing factors of various time discounting models.[4] In particular, despite its imminent popularity, it seems rather premature to conclude that the experimental regularities point, unequivocally, in the direction of the hyperbolic discounting model. In fact, recently various other time preference models have been proposed in the literature, which depart from this model in radical ways, and yet are consistent with the TPR phenomenon. Here are the two major examples:

(1) *Rubinstein's similarity-induced time preferences*: Rubinstein (2002) suggests a procedural model in which an individual compares two dated prizes, considering *only* the prize attribute if she deems the receival times "similar," and according to *only* the receival times if she deems the prizes "similar." This model, discussed formally in Example 6 below, is consistent with the TPR phenomenon, provided that we agree that one may deem today and tomorrow not "similar," and one-year-from-now and one-year-and-one-day-from-now "similar." In fact, Rubinstein (2002) provides experimental evidence in favor of this model relative to the hyperbolic discounting model.

(2) *Read's subadditive time preferences*: Read (2001) observes that the TPR phenomenon may, in fact, be a consequence of the total discounting of an individual over a given time period being greater than the sum of discounting when the period is divided into two parts (see Example 7). Read also provides experimental evidence in favor of this model relative to the hyperbolic discounting model.

Both of these models yield intransitive time preferences, and thus, they yield intertemporal choice models that are fundamentally different than the hyperbolic discounting model. For instance, by contrast to that model, either of these models allows for a situation like

$$(\$100, \text{today}) \succ (\$250, \text{two weeks from now}) \succ (\$200, \text{next week}) \succ (\$100, \text{today}),$$

where $\succ$ stands for the strict preference of the individual. There is really nothing "too odd" with this situation; the individual in question views \$50 good enough of a price for a delay of one week,

[4]The situation is quite reminiscent of that of the theory of choice under risk and uncertainty. The "anomalies" observed in the individual choice experiments of 60s and 70s, namely the Allais and Ellsberg paradoxes and their variations, led decision theorists to question the axiomatic work of von Neumann and Morgenstern, and Savage, thereby giving rise to a richer decision making frameworks that include reference-dependent choice models, non-expected utilities, multiple priors, and/or Knightian uncertainty. By contrast, the experimentally documented anomalies in the case of dynamic choice theory have not yet yielded substantial work on the axiomatic foundations of time preferences.

but thinks of \$150 too high a price for a delay of two weeks. Similarly, in a face-to-face bargaining situation, one may view the time delay between two consecutive offers as completely insignificant, and act as if she is indifferent between two offers like $(\$10, t)$ and $(\$10, t+1)$ for any offer period t. Yet this individual would not be indifferent between the offers $(\$10, t)$ and $(\$10, t+\tau)$ for large τ, unless she is completely insensitive to time delay. All in all, it seems clear that transitivity of time preferences, which is a precondition for the hyperbolic discounting model, is not really an innocuous assumption, especially from a descriptive angle.[5]

Motivated by these observations, our goal in this paper is to provide a general theory of time preferences that would allow for intransitivities that may arise due to the passage of time, and would admit, as special cases, the time preference models of Rubinstein and Read, along with all the standard hyperbolic discounting models proposed in the literature. Our approach is axiomatic, and hence identifies certain properties common to all such time preference models. Moreover, it yields a general representation theorem that depends on only two functions, and hence provides a model suitable not only for economic applications, but also for experimental testing. This is important, for the large number of experimental studies that focus on the "estimation" of discount rates assume particular parametric functional forms, and thus are vulnerable to functional misspecification errors. Having a general functional representation (that includes several rival theories within) may, then, help improving the accuracy of empirical work on time preferences. Finally, in certain economic environments, it may turn out that a particular result is independent of the precise structure of the adopted time preference model. For instance, it may be the case that both the quasi-hyperbolic discounting model and Rubinstein's time preference model would, in fact, yield the same predictions in a sequential bargaining game. Availability of a general model may be helpful in this regard, for such a model would allow us to study certain economic problems by using a rich class of time preferences simultaneously. (We will provide two major examples that illustrate this point later in the paper.)

Before moving to the formal analysis, let us briefly (and informally) outline the basic content of the present work. We begin with a major simplification: we consider here only those time preferences that are defined on the prize-time space. Consequently, our work falls short of saying anything about time preferences over consumption streams. It rather parallels the work of Fishburn and Rubinstein (1982), and is suitable for applications to bargaining and/or certain timing games

[5]Surprisingly, there are almost no studies on *intransitive time preferences*, despite the fact that this property arises rather naturally in the context of intertemporal choice. One exception is the recent work of Roelofsma and Read (2000) who provide quite convincing empirical evidence in favor of the intransitivity of time preferences. Another notable exception is supplied by Manzini and Mariotti (2002), who propose to model time preferences as interval orders, and allow for intransitivities to arise even within a fixed time period. (See Example 8.)

(as opposed to, say, capital accumulation problems). Objects of choice in the analysis are tuples like (x, t), where x corresponds to an (undated) outcome, and t to the receival time. We consider time preferences $\succsim$ over such prize-time tuples that are increasing in prizes (relative to a time-independent preference relation over prizes), and decreasing in time delay. We impose six axioms on $\succsim$. While some of these are standard, some rather novel, and one quite technical, all of them are satisfied by a great majority of time preference models considered in the literature. These axioms yield the following representation for $\succsim$:

$$(x, t) \succsim (y, s) \quad \text{if and only if} \quad U(x) \geq U(y) + \varphi(s, t), \tag{1}$$

where U is a real function on the prize space, and φ is a real function on date tuples, each satisfying certain properties that will be explored later. This representation provides a separation of "prize" and "time" aspects of prize-time tuples; U corresponds to the instantaneous utility function of the individual (for we in fact have $\varphi(t, t) = 0$ for each t), and $\varphi(s, t)$ captures the significance of time delay (for we have $\varphi(s, t) \leq 0$ iff $t \leq s$). It turns out that this representation embodies all the time preference models mentioned above, and it certainly allows for intransitive choices that may arise due to the passage of at least two time intervals. (See Examples 1-7.) Conversely, models in which time and prizes are not separable do not admit a representation as in (1). (See Examples 8-9.)

After stating the precise nature of the notion of representation derived here, and going through some of its properties (such as uniqueness, non-stationarity, present bias, etc.), we show in Section 3 that the resulting time preference model becomes quite powerful when coupled with a suitable time consistency property. In particular, we prove a general existence theorem on time consistent choices made on the basis of such preferences, and consider two applications, one in terms of the classical tree-cutting problem and the other in terms of the Rubinstein bargaining game. In both of these applications, one is able to analyze the problems by using an arbitrary member of the class of time preferences proposed here, without adhering to a special time preference model. In particular, it is shown that the equilibrium of the standard alternating-offers bargaining game remains unaltered if we consider alternative time preference models, so long as these models belong to our general class, they have the same per-period discount rate between now and the next period, and finally, decisions are made in a time consistent manner.

We conclude in Section 4 by discussing how the present model of time preferences fare with some other anomalies noted in the experimental literature (such as the magnitude effect), and sketching several avenues of research that should be pursued for a fuller development of the theory of intertemporal choice. The proofs of the major results appear in Section 5.

2 Time Preferences

2.1 Basic Nomenclature

Let A be a nonempty set. A **binary relation** R on A is any subset of $A \times A$, but by convention, we write $a\,R\,b$ instead of $(a,b) \in R$, and $a\,R\,b\,R\,c$ instead of $(a,b),(b,c) \in R$. R is said to be **reflexive** if $a\,R\,a$ for all $a \in A$, and **symmetric** if $a\,R\,b$ implies $b\,R\,a$ for all $a,b \in A$. If R is both reflexive and symmetric, we say that it is a **similarity relation**.

The **symmetric part** of R is the binary relation I_R on A defined by $a\,R\,b\,R\,a$. The **asymmetric part** of R is the binary relation $P_R := R \backslash I_R$. We say that R is **complete** if [not $a\,R\,b$] implies $b\,P_R\,a$, and note that every complete binary relation is reflexive.

The binary relation R is said to be **transitive** if $a\,R\,b\,R\,c$ implies $a\,R\,c$. If R is reflexive and transitive, we say that it is a **preorder** on A. In this case, any set of the form $\{o \in A : b\,P_R\,o\,P_R\,a\}$ (for some $a,b \in A$) is called an **open order-interval** in X. If R is a preorder and is antisymmetric (i.e. $a\,R\,b\,R\,a$ implies $a = b$, for any $a,b \in A$), then we say that R is a **partial order**. Finally, a complete partial order is called a **linear order**.

There are various relaxations of the transitivity property that have been studied in the literature. Perhaps the weakest such relaxation is the notion of quasitransitivity: R is said to be **quasitransitive** if P_R is transitive. It is obvious that every transitive binary relation on A is quasitransitive, but the converse is not true (unless $|A| \leq 2$). If R is reflexive and quasitransitive, we say that it is a **quasiorder** on A.

Let A be a metric space. We say that R is **upper semicontinuous** if $\{\omega \in A : \omega\,R\,a\}$ is a closed subset of A for each $a \in A$, and that R is **lower semicontinuous** if $\{\omega \in A : a\,R\,\omega\}$ is closed for all $a \in A$. R is said to be **continuous** if it is both upper and lower semicontinuous. We say that R is **completely continuous** if it is continuous, and for every nonempty open set O in A and $o \in O$, there exist $a_o, b_o \in O$ such that

(i) $b_o\,P_R\,o\,P_R\,a_o$,

(ii) for every $\omega \in A$ with $b_o\,P_R\,\omega\,P_R\,a_o$, there exists a $\omega_o \in O$ such that $\omega\,I_R\,\omega_o$.

Under suitable topological conditions (such as the separability of A), one can show that a completely continuous preorder admits a representation by a real function which is both continuous and open, whereas a continuous preorder does not in general admit a representation by an open real map. Of course, complete continuity of a continuous linear order is none other than the requirement that in every open set that contains an element o there is an open order-interval that also contains o.

2.2 Time Preferences

Let X be a metric space, and interpret this space as the (undated) outcome space. We model time discretely, and distinguish the members of X from each other according to when they are received. More precisely, we let

$$\mathcal{T} := \{T : \text{either } T = \{0, 1, ..., \tau\} \text{ for some } \tau \in \mathbb{N} \text{ or } T = \mathbb{Z}_+\},$$

and designate an arbitrary member T of $\mathcal{T}$ as the set of time periods that is relevant to the decision problem, a formulation which allows for both finite and infinite-horizon choice problems. The preferences of the decision maker are, then, defined over the product space $X \times T$. (As usual, we metrize T by the discrete metric, and $X \times T$ by the product metric.) An element (x, t) of $X \times T$ is interpreted as the situation in which the agent receives the outcome x in period t.

Formally speaking, then, by a **preference relation** in this paper we mean a binary relation $\succsim$ on $X \times T$. The symmetric part of this relation is denoted by $\sim$, and the asymmetric part of it by $\succ$. For each $t \in T$, by the **tth time projection** of $\succsim$, we mean the binary relation $\succsim_t$ on X defined as $x \succsim_t y$ iff $(x, t) \succsim (y, t)$.

The following definition introduces the main objects of the present analysis.

Definitions. *Let X be a metric space and $T \in \mathcal{T}$. A binary relation $\succsim$ on $X \times T$ is said to be a* **time preference** *if it satisfies the following conditions:*

(i) $\succsim$ *is complete and continuous,*

(ii) $\succsim_0$ *is transitive and completely continuous,*

(iii) $\succsim_0 = \succsim_1 = \succsim_2 \cdots$.

If $\succsim$ is a time preference such that $\succsim_0$ is a linear order, then we say that $\succsim$ is a **one-dimensional time preference**.

The completeness and continuity of $\succsim$ are, of course, standard requirements. By contrast to the majority of the literature on time preferences, however, we do not require that $\succsim$ be transitive. As discussed in Section 1, this is necessary for a general time preference theory that would be able to account for phenomena like subadditive discounting and/or time preferences induced by similarity relations. However, we wish to allow for intransitivities that may arise solely due to the passage of time, so each tth time projection of $\succsim$ is assumed to be transitive. In fact, not only that we require each $\succsim_t$ to be fairly well-behaved in the sense of being complete preorders on X, we further posit that each $\succsim_t$ be completely continuous. Last but not least, the final requirement of being a time preference forces us to concentrate on the case where material tastes of the agent are *unchanging* through time.

We refer to a time preference $\succsim$ as *one-dimensional* when $\succsim$ declares no two distinct outcomes that are received in the same period indifferent to each other. (Formally speaking, this means that the order-dimension of X as rendered by $\succsim_0$ is one.) For instance, if $\succsim$ is a time preference on $\mathbb{R}^d \times T$ and $\succsim_0$ is representable by a continuous and strictly increasing utility function, then $\succsim$ is one-dimensional iff $d = 1$. Most applications work with time preferences that are one-dimensional in this sense. However, as we shall show in Section 2.9, many of our results extend to the case where $\succsim$ need not be one-dimensional.

Two comments on the nonstandard aspects of our formal definition of time preferences are in order. First, we emphasize again that this definition imposes very little restraint on how intransitive $\succsim$ may be. While we will introduce some restrictions in this regard shortly, our general theory leaves room for time preferences that cannot be represented by a utility function defined on $X \times T$, due to the potential intransitivity of $\succsim$.[6] Second, assuming the complete continuity of $\succsim_0$ is slightly more demanding than what is usual. While it is needed for some of the technical arguments in the proofs of our main results, it must be nevertheless noted that this property is only a mild strengthening of the standard continuity requirement. We see this property only as a useful regularity condition, the only conceptual implication of which is to disallow X to contain either a worst or a best outcome. So, for instance, if X is a space of monetary outcomes, that is, $X \subseteq \mathbb{R}$, and $\succsim_0$ is represented by some continuous and strictly increasing utility function $u \in \mathbb{R}^X$, then $\succsim_0$ is completely continuous if X is any open interval in $\mathbb{R}$ (e.g. $X = \mathbb{R}$), but $\succsim_0$ is not completely continuous if $X = [0,1)$ or $X = [0,1]$. Similarly, if $X = \mathbb{R}^d_{++}$ (for $1 \leq d \leq \infty$), then $\succsim_0$ is completely continuous, but if $X = \mathbb{R}^d_{+}$, then it is not completely continuous (because the origin of $\mathbb{R}^d$ is then the worst element in X with respect to $\succsim_0$).[7]

2.3 Axioms for Time Preferences

In this section we introduce six axioms with which we shall obtain a general representation theorem for time preferences. We view the first three of these axioms (labelled (A1)-(A3)) as intuitive prop-

[6] In the form of interval orders and/or semiorders, non-transitive preference relations are, of course, studied extensively in the literature on static choice theory. In fact, the recent work of Manzini and Mariotti (2001) applies some of this body of work to the theory of intertemporal choice. However, the present study is quite different from these works. For, while any interval order is quasitransitive, our definition of time preference allows for non-quasitransitive relations.

[7] Of course, this does not mean that our representation of time preferences cannot be used in the case where the set of feasible outcomes that one is interested in is a set like $[0,1]$. After all, the representation will apply on any open interval that contains $[0,1]$, and hence applies also on the set $[0,1]$. What our results cannot do is to yield this representation (on $[0,1] \times T$) using the axioms only on $[0,1] \times T$. (A similar comment also applies to situations where the feasible set of outcomes is a compact set in $\mathbb{R}^d$.)

erties that are quite attractive for the general theory. The final three of them (labelled (B1)-(B3)), on the other hand, are less intuitive and somewhat technical requirements which are, nonetheless, satisfied by most time preference models used in economic applications.

Throughout this subsection $\succsim$ stands for a binary relation on $X \times T$, where X is any metric space and $T \in \mathcal{T}$. All axioms are imposed on $\succsim$.

(A1) *For any $x \in X$ and $s, t \in T$, there exist $y, z \in X$ such that $(z, s) \succsim (x, t) \succsim (y, s)$.*

Roughly speaking, this axiom means that "time" can always be compensated by outcomes, or that it can always be "priced." For example, if $X = (0, \infty)$ and $s > t = 0$, then (A1) says that the agent prefers getting a large "enough" sum of money at date s to receiving x dollars today, but prefers getting x now to receiving a small "enough" sum of money at date s. (Of course, what is "enough" depends on the size of x and the delay s.)

(A2) *For any $x, y, z \in X$ and $s, t, r \in T$, if $r \leq t$ and $z \succsim_0 x$, then*

$$(x,t) \begin{Bmatrix} \succ \\ \succsim \end{Bmatrix} (y,s) \quad \textit{implies} \quad (z,r) \begin{Bmatrix} \succ \\ \succsim \end{Bmatrix} (y,s).$$

This property is an ordinal formulation of the idea that $\succsim$ is "increasing" in outcomes (i.e. over X) and "decreasing" in time (i.e. over T). It is thus unexceptionable for a theory of *positive* time preferences in which, by definition, delay is undesirable (so contents of X are "good.") While we will work with this form of the axiom, our entire analysis would modify in the obvious way in the case of *negative* time preferences in which, by definition, "later is better" (so contents of X are "bad.") All one has to do is to replace "$r \leq t$" with "$r \geq t$" in (A2) to capture this case; we will comment below how our main representation theorem changes with this modification of (A2).

(A3) *For any $x, y, z \in X$ and $s, t \in T$,*

$$\begin{array}{c} (x,t) \succsim (y,s) \succsim (z,s) \\ \textit{or (inclusive)} \\ (x,t) \succsim (y,t) \succsim (z,s) \end{array} \qquad \textit{implies} \qquad (x,t) \succsim (z,s).$$

Moreover, if any of the $\succsim$s in the antecedent holds strictly, then $(x, t) \succ (z, s)$.

This property is a restricted transitivity condition, and is trivially satisfied by any transitive $\succsim$. In words, (A3) assumes away cycles that may not be caused by the passing of at least two time *intervals*. For instance, a time preference that satisfies (A3) may allow for the cycle $(x, 0) \succ (z, 2) \succ$

$(y,1) \succ (x,0)$. Indeed, as noted earlier, we wish to allow for such cycles, because, for instance, they may arise from the preferences of a person who feels the effect of time delay only after two periods have past. Moreover, as discussed in Section 1, subadditive discounting and/or time preferences induced by similarity relations may well give rise to such cycles. But these considerations can never justify a cycle like $(x,0) \succ (z,1) \succ (y,1) \succ (x,0)$, because this cycle involves only *two* time periods (and hence a single time interval). Conceptually speaking, this sort of a cycle is not the result of passage of time, but rather reflects an irrationality on how the agent aggregates her preferences over time and (undated) outcomes. (A3) avoids the occurrence of this phenomenon, and is thus a desirable property for a rational theory of time preferences.

The next two axioms can be viewed as weak separability properties that, in an ordinal sense, ensure that the disutility of time delay are independent of the size of outcomes.

(B1) *For any* $x, y, z, w \in X$ *and* $s_1, s_2, t_1, t_2 \in T$, *if*

$$\begin{array}{l}(x,t_1) \sim (y,s_1) \\ (z,t_1) \sim (w,s_1)\end{array} \quad \textit{and} \quad (w,t_2) \sim (y,s_2) \quad \textit{then} \quad (z,t_2) \sim (x,s_2). \tag{2}$$

To illustrate the basic content of (B1), suppose that $X = (0,\infty)$ and the decision maker is indifferent between receiving \$10 now ($t_1 = 0$) and receiving \$20 tomorrow ($s_1 = 1$), and similarly, she is indifferent between getting \$30 now and getting \$70 tomorrow. If we assume, for illustrative purposes, that her time preferences are stationary, then for this individual

\$10 is equivalent to \$20 with 1-period delay

and

\$30 is equivalent to \$70 with 1-period delay.

(B1) is a separability condition which says in this case that the through-time ranking of \$10 and \$30 should be consistent with the through-time ranking of \$20 and \$70. For instance, if the subject individual is indifferent between getting \$20 at date s_2 and getting \$70 at date $t_2 > s_2$, that is, (given stationarity) if

\$20 is equivalent to \$70 with $(t_2 - s_2)$-period delay,

then (B1) says that this person would also hold that

\$10 is equivalent to \$30 with $(t_2 - s_2)$-period delay.

In this sense, one may think of (B1) as a consistency requirement that brings a certain discipline to how a person may discount various outcomes through time.

It is difficult to argue that (B1) is an unexceptionable property for time preferences. However, this axiom is easily testable, and have a reasonably clear interpretation. In fact, properties that are closely related to (B1) are used routinely in the literature on the additive representation of preferences defined on a product space. In particular, the so-called *hexagon condition* (cf. Debreu (1960), Wakker (1988), and Karni and Safra (1998)) is very close to (B1) in spirit (but (B1) is in fact stronger than the hexagon condition).[8] Figure 1 compares the structure of indifference curves that are entailed by these properties in the prize-time space.

More importantly, the class of time preferences that satisfy (B1) is quite rich. For instance, the exponential and/or hyperbolic discounting models would certainly satisfy (B1). More generally, any discounting model which envisages that "outcomes" and "time" are separable in such a manner that all outcomes are discounted the same way would satisfy this axiom. Conversely, time preferences that exhibit the so-called *magnitude effect* (the decrease of the rate of discountning with the utility value of the outcomes) would in general violate (B1). (More on this matter in Section 4.1.)

To state the final two axioms that we will work with, we need to introduce some terminology. Let $n \in \{2, 3, ...\}$, and denote the generic element of $T^{2n} := T^2 \times \cdots \times T^2$ (n times) by $(s_i, t_i)_{i=1}^n$. For any binary relation $\approx$ on $X \times T$, we say that $y \in X$ is **$\approx$-reachable from x through** $(s_i, t_i)_{i=1}^n$ if there exist $y_1, ..., y_{n-1} \in X$ such that

$$(x, t_1) \approx (y_1, s_1),\ (y_1, t_2) \approx (y_2, s_2),\ ...,\ (y_{n-1}, t_n) \approx (y, s_n).$$

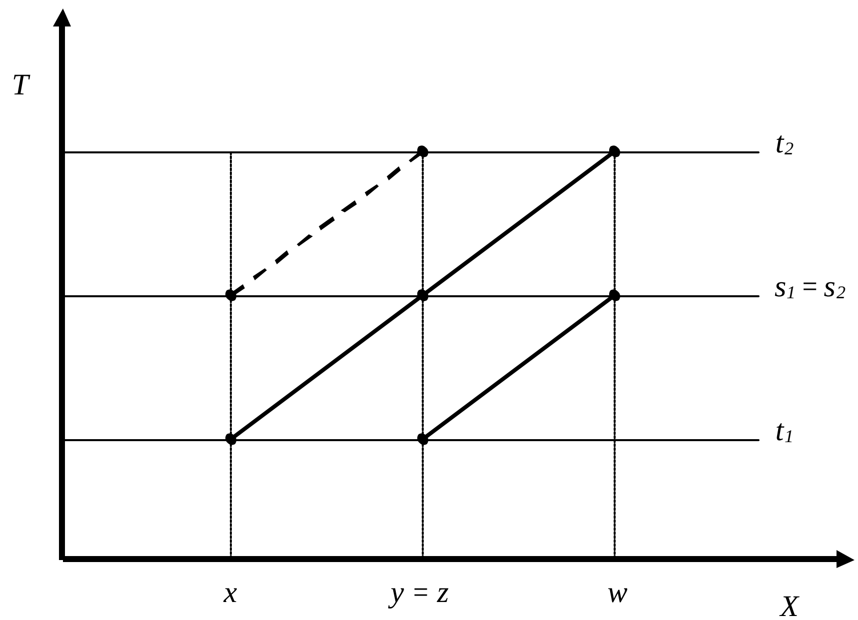
We say that y is **$\approx$-reachable from** x if $y \in X$ is $\approx$-reachable from x through some $(s_i, t_i)_{i=1}^n \in T^{2n}$ and $n = 2, 3, ...$. We denote the set of all $y \in X$ that is reachable from x as $O^{\approx}(x)$, and refer to it as the **$\approx$-orbit** of x.

The following assumption regulates the behavior of the indifference part of our $\succsim$:

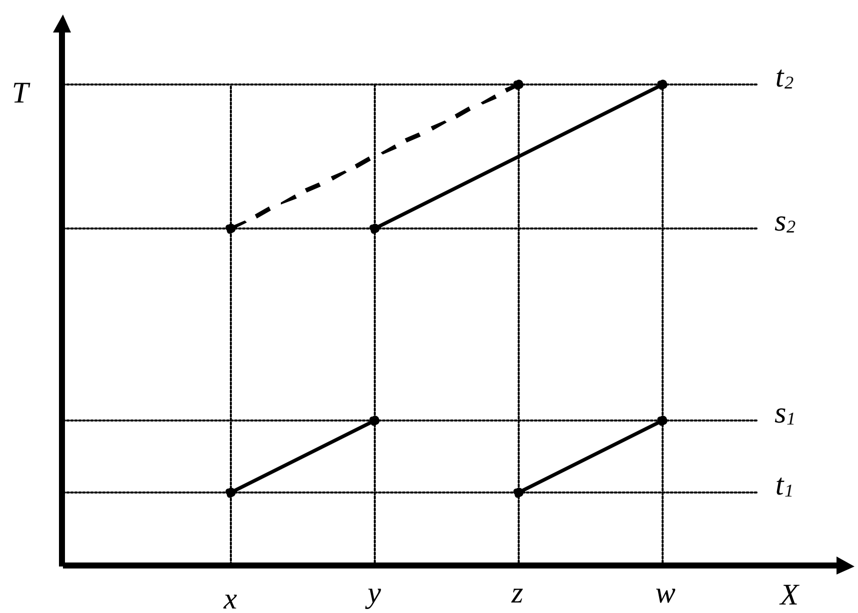
(B2) *For any $n = 2, 3, ...$, if $x \in X$ is $\sim$-reachable from x through $(s_i, t_i)_{i=1}^n \in T^{2n}$, then any $z \in X$ is $\sim$-reachable from z through $(s_i, t_i)_{i=1}^n$.*

This property is a way of asking (in an ordinal sense) that time discounting is applied to all (undated) outcomes in a symmetric way. (See Figure 2.) Like (B1), this property, too, should thus be viewed as a weak separability condition. In fact, both (B1) and (B2) are necessary properties for time preferences that views outcomes and time "separable" (in a sense that will be clarified in Theorem 1). While its statement is a bit mouthful, we note that (B2) is satisfied by almost all time

[8] Another close relative of (B1) is the so-called *Thomsen condition*, which is also used routinely in axiomatic utility theory. In fact, to be able to capture the notion of "time separability," Fishburn and Rubinstein (1982) use this condition in the context of time preferences. We note that (B1) and the Thomsen condition are logically independent.

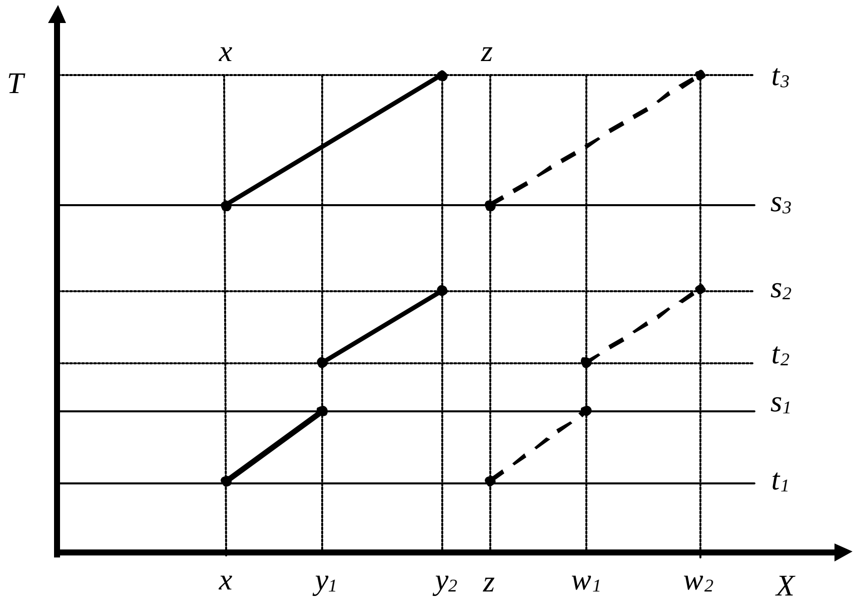


(a) The Hexagon Condition: (*) with $s_1 = s_2$ and $y = z$.



(b) The Condition (B1)

Figure 1



The Condition (B2)

Figure 2

discounting models used in practice. For instance, any multiplicative discounting model, may it be exponential, hyperbolic or qusi-hyperbolic, would surely satisfy this property. (See Example 3.)

Our final axiom is purely technical. It preconditions the set of all outcomes that can be reached by $\sim$ from some $x \in X$ to have a well-behaved topological structure.

(B3) *There exists some $x \in X$ such that $O^{\sim}(x)$ is either somewhere dense or contains at least one isolated point.*[9]

The nowhere dense sets that contain only non-isolated points are obviously geometric oddities; it is not even possible to visualize such sets. While they exist even within the real line (such as Cantor's famous "middle-thirds" set), such sets arise, indeed, only by means of quite intricate constructions. (B3) is a technical condition that asks for the $\sim$-orbit of at least one undated outcome x not to be of this form. While we do not see any conceptual argument favoring this requirement, it is quite clear that all that (B3) does is to disallow certain time preferences that have quite irregular mathematical features. Indeed, we will see below that all of the standard time preference models conform with (B3). In particular, the $\sim$-orbits of any outcome x under exponential discounting is a countable set of isolated points. The hyperbolic and/or quasi-hyperbolic models also satisfy (B3), but the $\sim$-orbit of an outcome x under such models may well be dense in X.

2.4 Time Preferences with One-Dimensional Outcome Space

The six axioms we proposed in the previous section range from extremely intuitive properties to quite technical requirements. A unifying feature of all these axioms is that they are "weak," in the sense that they are satisfied by a great variety of time preference models, even those models which appear in the literature as "rivals" (such as the hyperbolic discounting model and some time preference models induced by similarity relations). The main result of this paper gives a complete characterization of the structure of time preferences that satisfy these axioms, and hence produces a general model that admits these time preference models as special cases.

Theorem 1. *Let X be a connected and separable metric space, $T \in \mathcal{T}$, and $\succsim$ a binary relation on $X \times T$. Then $\succsim$ is a one-dimensional time preference that satisfies the properties (A1)-(A3) and (B1)-(B3) if, and only if, there exist two functions $U : X \to \mathbb{R}$ and $\varphi : T^2 \to \mathbb{R}$ such that*

$$(x,t) \left\{ \begin{array}{c} \succ \\ \sim \end{array} \right\} (y,s) \quad \textit{iff} \quad U(x) \left\{ \begin{array}{c} > \\ = \end{array} \right\} U(y) + \varphi(s,t) \tag{3}$$

[9]A nonempty set A in X is said to be *somewhere dense* if it is not nowhere dense, that is, if its closure contains a nonempty open subset of X. On the other hand, $a \in A$ is said to be an *isolated point* of A, if there exists an open set O in X such that $O \cap A = \{a\}$.

for all $(x,t),(y,s) \in X \times T$, *and*

(i) U *is a homeomorphism,*

(ii) $\varphi(\cdot,t)$ *is decreasing and* $\varphi(s,\cdot)$ *is increasing for any* $s,t \in T$,

(iii) $\varphi(s,t)+\varphi(t,s)=0$ *for any* $s,t \in T$.

Theorem 1 provides a representation for one-dimensional time preferences by means of *separating* the "delay" and "prize" aspects of dated outcomes. In this representation U simply serves as a utility function for undated outcomes; we have $(x,t) \succsim (y,t)$ iff $U(x) \geq U(y)$ for all $x,y \in X$ and $t \in T$ (since, by (iii), $\varphi(t,t)=0$ for each t). In turn, φ captures the importance of time for the individual in question. The model is one of positive time preference; we have $(x,t) \succsim (x,s)$ iff $t \leq s$ for all $x \in X$ and $s,t \in T$ (since, by (ii) and (iii), $\varphi(s,t) \leq 0$ iff $t \leq s$). More generally, the representation tells us how the agent would "aggregate" the time and outcome aspects when comparing two dated outcomes (x,t) and (y,s). If $U(x) \geq U(y)$ (that is, x is a more desirable outcome than y) and if $t \leq s$, then we have $U(x) \geq U(y) \geq U(y)+\varphi(s,t)$; so the representation accords fully with intuition. A nontrivial case is when $U(x) \geq U(y)$ and $t > s$. In this case the agent has to decide whether she should receive the better outcome later, or the worse outcome sooner. According to our representation, this is done by comparing the outcome value $U(x)$ of x with the outcome value $U(y)$ of y *plus* the "time-kick" $\varphi(s,t) \geq 0$. If $U(x) > U(y)+\varphi(s,t)$, then we understand that the individual views that the time difference does not compensate for the value difference in outcomes, so goes for (x,t) as opposed to (y,s). If, on the other hand, $U(x) < U(y)+\varphi(s,t)$, then the time aspect of the comparison outweighs the prize aspect, and the individual chooses (y,s) over (x,t). The case $U(x)=U(y)+\varphi(s,t)$ corresponds to the situation where the agent is indifferent between the two alternatives.

The condition (i) in Theorem 1 says not only that the outcome-utility function U is a continuous bijection, but also that U^{-1} is continuous. While the latter requirement is not standard, it hardly seems unacceptable; it is a fair price to pay for assuming the complete continuity of $\succsim_0$ (which is, in turn, essential for the "only if" part of Theorem 1). The conditions (ii) and (iii) are, on the other hand, easily interpreted. Condition (ii) simply says that the agent prefers getting an outcome earlier than later.[10] Condition (iii) ensures that "time delay" is evaluated symmetrically back and forth through time. For instance, if $s > t$, then $\varphi(s,t)$ is the disutility of waiting from date t until s, and (iii) says that in this case the utility from receiving an outcome at the earlier date t as opposed to s equals $-\varphi(s,t)$. We should also note that conditions (ii) and (iii) together entail that time is

[10] If we modified (A2) to obtain a model of negative time preference (that is, replace "$r \leq t$" in the statment of (A2) with "$r \geq t$," then the statement of Theorem 1 would remain valid with (ii) being replaced with the condition: $\varphi(\cdot,t)$ *is increasing and* $\varphi(s,\cdot)$ *is decreasing for any* $s,t \in T$.

valued positively by the individual, that is, $\varphi(s,t) \geq 0$ whenever $t \geq s$, and $\varphi(s,t) \leq 0$ whenever $t \leq s$.

It is important to stress that the representation (3) allows for intransitive time preferences; we will consider concrete examples below. In fact, it is easily shown that a time preference that is represented as in Theorem 1 is transitive if and only if

$$\varphi(r,t) = \varphi(r,s) + \varphi(s,t) \quad \text{for all } r,s,t \in T. \tag{4}$$

It is also noteworthy that Theorem 1 covers even those time preferences $\succsim$ with $\succ$ being intransitive. So, despite its initial appearance, the representation provided by this result is not an interval order representation. In fact, one can show that a time preference that satisfies (A1)-(A3) and (B1)-(B3) is an interval order iff it is transitive.

In what follows, we shall consider a few refinements of the representation given by Theorem 1 by imposing on $\succsim$ some additional properties, such as stationary and present bias. These results will be illustrated in Section 2.7 by means of several examples.

2.5 Stationary Time Preferences

One of the main features of the classical exponential discounting model is that of its stationarity. That is, in that model, the effect of time enters into the comparison of two dated outcomes only through the *difference* between the receival times of these outcomes. In the present framework, this property is defined as follows.

Definition. *Let X be a metric space and $T \in \mathcal{T}$. A time preference $\succsim$ on $X \times T$ is said to be* **stationary** *if*

$$(x,t) \succ (y,s) \quad \text{iff} \quad (x,t+\tau) \succ (y,s+\tau)$$

for all $x,y \in X$, $s,t \in T$, and $\tau \in \mathbb{Z}$ such that $s+\tau, t+\tau \in T$.

The following immediate corollary of Theorem 1 demonstrates the structure of the stationary time preferences that satisfy the basic set of axioms introduced above.

Corollary 1. *Let X be a connected and separable metric space, $T \in \mathcal{T}$, and $\succsim$ a binary relation on $X \times T$. Then $\succsim$ is a stationary one-dimensional time preference that satisfies the properties (A1)-(A3) and (B1)-(B3) if, and only if, there exist a homeomorphism $U : X \to \mathbb{R}$ and a decreasing odd function $\eta : T - T \to \mathbb{R}$ such that*

$$(x,t) \left\{ \begin{array}{c} \succ \\ \sim \end{array} \right\} (y,s) \quad \text{iff} \quad U(x) \left\{ \begin{array}{c} > \\ = \end{array} \right\} U(y) + \eta(s-t) \tag{5}$$

for all $(x,t),(y,s) \in X \times T$.

Proof. Let $\succsim$ be a stationary one-dimensional time preference that satisfies the properties (A1)-(A3) and (B1)-(B3), and let U and φ be as found by Theorem 1. Define $\eta : T - T \to \mathbb{R}$ as $\eta(k) := \varphi(k,0)$ if $k \geq 0$, and $\eta(k) := \varphi(0,-k)$ if $k < 0$. Then η is a decreasing function, because φ satisfies condition (ii) of Theorem 1. Moreover, $\eta(-k) = -\eta(k)$ for every $k \in T - T$, because φ satisfies condition (iii) of Theorem 1. Finally, it is readily checked that (5) follows from (3) and stationarity of $\succsim$. The "if" part of the claim is an obvious consequence of Theorem 1. $\square$

The time preferences represented as in Corollary 1 are transitive if and only if

$$\eta(r-t) = \eta(r-s) + \eta(s-t) \quad \text{for all } r,s,t \in T. \tag{6}$$

In fact, the next result shows that such preferences correspond to none other than the standard exponential discounting model.

Corollary 2. *Let X be a connected and separable metric space, and $\succsim$ a binary relation on $X \times \mathbb{Z}_+$. Then $\succsim$ is a transitive and stationary one-dimensional time preference that satisfies the properties (A1)-(A3) and (B1)-(B3) if, and only if, there exist a homeomorphism $u : X \to \mathbb{R}_+$ and a $\delta \in (0,1)$ such that*

$$(x,t) \succsim (y,s) \quad \text{iff} \quad \delta^t u(x) \geq \delta^s u(y)$$

for all $(x,t),(y,s) \in X \times \mathbb{Z}_+$.

Proof. Let $\succsim$ be a transitive and stationary one-dimensional time preference on $X \times \mathbb{Z}_+$ that satisfies the properties (A1)-(A3) and (B1)-(B3), and let U and η be as found by Corollary 1. By transitivity of $\succsim$, (6) holds, and hence $\eta(a+b) = \eta(a) + \eta(b)$ for all $a,b \in \mathbb{Z}$. Thus, $\eta(t) = \alpha t$ for all $t \in \mathbb{Z}$, where $\alpha := \eta(1) \leq 0$. Then, $(x,t) \succsim (y,s)$ iff $u(x)+\alpha t \geq u(y)+\alpha s$ for all $(x,t),(y,s) \in X \times \mathbb{Z}_+$. The "only if" part of the corollary is then established upon letting $\delta := e^{\alpha}$ and $u = e^U$. The "if" part follows from Theorem 1. $\square$

While it is illuminating to see how the exponential discounting model is embedded in the class of time preferences characterized by Theorem 1, Corollary 2 should not be considered as a new finding. Indeed, apart from a few technical details, this result can be thought of as a special case of the well-known characterization of exponential discounting by Fishburn and Rubinstein (1982).[11]

[11] The difference stems from the fact that Fishburn and Rubinstein (1982) work with *strictly* positive time preferences, the time projections of which need not be completely continuous. We note that, in the case of strictly positive time preferences (i.e. when $(x,0) \succ (x,1)$ for all $x \in X$), one would only need axioms (A1) and (A2) in the statement of Corollary 2. (The proof of this auxiliary fact is somewhat tedious, and hence omitted.)

2.6 Present Bias

In view of the ample experimental evidence that shows that most individuals attach a special significance to receiving an outcome "today" (that is, at date 0) as opposed to some later date, it is of interest to identify which sorts of time preferences that we consider here exhibit such a present bias. We first need to formulate the idea of "present bias" within our ordinal framework.

Definition. *Let X be a metric space and $T \in \mathcal{T}$. A time preference $\succsim$ on $X \times T$ is said to have* **present bias** *if*

$$(x,t) \succsim (y,s) \quad \textit{implies} \quad (x,0) \succsim (y,s-t),$$

for all $x, y \in X$, and all $s, t \in T$ with $s > t \geq 0$. It is said to have **strong present bias** *if it has present bias, and if, for any $s, t \in T$ with $s > t > 0$, there exist $x, y \in X$ such that*

$$(x,t) \sim (y,s) \quad \textit{and} \quad (x,0) \succ (y,s-t). \tag{7}$$

The interpretation of this definition is straightforward. We note that, according to this definition, any stationary time preference has present bias, but not strong present bias. The following result identifies the structure of time preferences that belong to the class characterized by Theorem 1 and that have (strong) present bias.

Corollary 3. *Let X be a connected and separable metric space, $T \in \mathcal{T}$, and $\succsim$ a binary relation on $X \times T$. Then $\succsim$ is a one-dimensional time preference that satisfies the properties (A1)-(A3) and (B1)-(B3), and that have (strong) present bias if, and only if, there exist two functions $U : X \to \mathbb{R}$ and $\varphi : T^2 \to \mathbb{R}$ such that (3) holds on $X \times T$, and in addition to the properties (i)-(iii) listed in Theorem 1, we have*

(iv) $\varphi(s,t)$ $(>) \geq \varphi(s-t,0)$ whenever $s > t > 0$.

Proof. The "if" part of the claim is an easy consequence of Theorem 1 and the definition of (strong) present bias. To see the "only if" part, let $\succsim$ be a one-dimensional time preference that satisfies the properties (A1)-(A3) and (B1)-(B3), and that have present bias. Let U and φ be as found by Theorem 1, and pick any $y \in X$ and $s, t \in T$ with $s > t > 0$. Let $x := U^{-1}(U(y) + \varphi(s,t))$, which is well-defined since U is a homeomorphism between X and $\mathbb{R}$. Then, $(x,t) \sim (y,s)$, so since $\succsim$ has present bias, we have $(x,0) \succsim (y,s-t)$. By (3), therefore, we find

$$U(y) + \varphi(s,t) = U(x) \geq U(y) + \varphi(s-t,0),$$

so $\varphi(s,t) \geq \varphi(s-t,0)$. Moreover, if $\succsim$ has strong present bias, and x and y satisfy (7), then $U(x) > U(y) + \varphi(s-t,0)$, so in this case we get $\varphi(s,t) > \varphi(s-t,0)$. $\square$

2.7 Examples

In this section we shall go through various models of time preferences proposed in the literature, and show that a good number of these models fit in the general framework we have developed in the previous sections. Mainly for simplicity, and because these models are commonly studied in the case where the outcome space consists of monetary gains and time horizon is infinite, we take $X := (0, \infty)$ and $T := \mathbb{Z}_+$ in what follows. We also let u be an arbitrary strictly increasing and continuous real function on X such that $u(0-) > -\infty$ and $u(\infty) = \infty$. We interpret u as the utility function of the agent over the monetary outcomes in X. Clearly, it is without loss of generality to assume that $u(0-) = 0$ (for otherwise we would work with $u - u(0-)$). In applications, one often imposes risk neutrality and take u to be the identity function on X, but we do not have to assume this form here.

Example 1. (*Exponential Discounting*) The standard model of time preferences defines $\succsim$ on $X \times \mathbb{Z}_+$ by $(x, t) \succsim (y, s)$ iff $\delta^t u(x) \geq \delta^s u(y)$, where $\delta \in (0, 1)$ is the discount factor. As already noted in Corollary 2, such time preferences belong to the class characterized by Corollary 1. Indeed, if we let $U := \ln u$ and define the real function η on $\mathbb{Z}$ by $\eta(k) := (\ln \delta)k$, then $\succsim$ satisfies (5), and U and η satisfy the properties (i)-(iii) envisaged in Corollary 1. Moreover, being stationary, $\succsim$ has present bias, but not strong present bias.

We note that in this example we have $O^{\sim}(x) = (\ln \delta)\mathbb{Z}$ (for any $x \in X$), so the standard model satisfies (B3) because the $\sim$-orbit of any outcome x in this model is a set that consists only of isolated points. □

Before we move on to more interesting examples of time preferences, let us mention that in applications it is common to stipulate the existence of a time-independent "worst outcome" (to serve, for instance, as the disagreement point in bargaining games). If we denote this outcome by x_*, then one requires that

$$(x_*, t) \sim (x_*, s) \quad \text{and} \quad (x, t) \succ (x_*, s) \text{ for all } x \in X \text{ and } s, t \in T. \tag{8}$$

Formally speaking, our present framework does not admit such an outcome x_* in X (due to complete continuity of $\succsim_0$). Yet, this is only a minor issue. To incorporate such a "worst outcome" to the model, all one has to do is to add an object x_* to X (i.e. make the outcome space $X \cup \{x_*\}$), and extend the time preference $\succsim$ at hand to $(X \cup \{x_*\}) \times \mathbb{Z}_+$ by means of (8). For instance, in Example 1, we could take the outcome space as $[0, \infty)$ and let $u(0) = 0$ which makes 0 as the time-independent "worst outcome;" this formulation is indeed used commonly in sequential bargaining models. Obviously, this extended formulation, too, is of the form characterized in

Corollary 1, provided that we extend U to $[0,\infty)$ by setting $U(0) := -\infty$, and viewing η as defined on $[0,\infty)\times\mathbb{Z}$ as $\eta(0,\cdot) := 0$ and $\eta(y,k) := (\ln\delta)k$ for all $y>0$ and $k\in\mathbb{Z}$. This type of an extension can obviously be adopted in any of the examples we consider below.[12]

Example 2. (*Quasi-Hyperbolic Discounting*) A recently popular model of time preferences is that of quasi-hyperbolic discounting, according to which the time discounting of a present alternative and a future one is $\beta\delta$ (with $0<\beta<1$) whereas that of two future outcomes is $\delta\in(0,1)$ (cf. Phelps and Pollak (1968) and Laibson (1997).) This induces a time preference $\succsim$ that is captured by Theorem 1 (but not by Corollary 1). Indeed, if we let $U := \ln u$ and define φ on $\mathbb{Z}_+^2$ instead by $\varphi(0,0) := 0$, and

$$\varphi(s,t) := \begin{cases} s\ln\delta+\ln\beta, & \text{if } s>t=0 \\ -(t\ln\delta+\ln\beta), & \text{if } t>s=0 \\ (s-t)\ln\delta, & \text{if } s,t>0 \end{cases},$$

then $\succsim$ satisfies (3), where U and φ satisfy the properties (i)-(iii) noted in Theorem 1. Moreover, we have $\varphi(s,t)=(s-t)\ln\delta>(s-t)\ln\delta+\ln\beta=\varphi(s-t,0)$ whenever $s>t>0$, so it follows from Corollary 3 that $\succsim$ has strong present bias.

Curiously, the quasi-hyperbolic discounting model may be sharply different from the exponential discounting model with respect to the behavior of its $\sim$-orbits. Indeed, in contrast with the standard exponential model, it turns out that here $\succsim$ may satisfy (B3), because $O^{\sim}(x)$ is a dense subset of the reals (for any $x\in X$).[13] $\square$

Example 3. (*Multiplicative Discounting*) One thing that the exponential and quasi-hyperbolic discounting models have in common is that they both discount the instantaneous utility function u multiplicatively by means of a discount function. This prompts one consider time preferences $\succsim_D$ on $X\times\mathbb{Z}_+$ such that $(x,t)\succsim_D(y,s)$ iff $D(t)u(x)\geq D(s)u(y)$, where $D:\mathbb{Z}_+\to(0,1]$ is a decreasing (discount) function with $D(t)<D(0)=1$ for all $t\neq 0$. Any such time preference belongs to the

[12] In fact, one could trivially incorporate the existence of a time-independent worst outcome in our axiomatic development by letting $x_*\in\partial X$ and adopting (8) as an axiom. To do this, we would work first with $X\backslash\{x_*\}$ (which would be connected since $x_*\in\partial X$) to derive Theorem 1, and then extend the characterization by letting $U(x_*) := -\infty$, and defining φ on $X\times T^2$ by letting $\varphi(x_*,\cdot,\cdot) := 0$ and specifying $\varphi(\cdot,s,t)$ as in Theorem 1. We did not follow this route in our formal development only to increase the clarity of exposition.

[13] Here is a sketch of proof for this claim. To simplify, let u be the identity function (but this is not essential to the argument), and set $\delta=1/e$. Let G_β denote the additive group generated by the set $\{t-s : s,t\in\mathbb{N}\}\cup\{-s+\ln\beta : s\in\mathbb{N}\}$. One can use (3) to show that $O^{\sim}(x)$ must include $x+G_\beta$. But since G_β is an additive subgroup of $\mathbb{R}$, it is either dense in $\mathbb{R}$ or $G_\beta=\theta\mathbb{Z}$ for some real θ. We claim that $G_\beta\neq\theta\mathbb{Z}$ for any real θ, provided that β is rational. For, $G_\beta=\theta\mathbb{Z}$ implies that $1\in\theta\mathbb{Z}$ so that θ is rational. But it also implies that $-1+\ln\beta\in\theta\mathbb{Z}$. Thus, since a classic result of number theory says that e^q is irrational for any rational $q>0$, we find that θ is irrational, a contradiction.

class characterized by Theorem 1.[14] Indeed, letting $U := \ln u$, and defining φ on $\mathbb{Z}_+^2$ by

$$\varphi(s,t) := \ln\left(\frac{D(s)}{D(t)}\right),$$

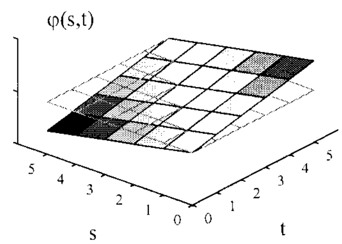
we see readily that $\succsim_D$ satisfies (3), where U and φ satisfy the properties (i)-(iii) of Theorem 1. (Figure 3 exhibits the comparative structures of the "time-kick" factor φ for the three most popular multiplicative discounting models, namely, the models in which the discounting is exponential, quasi-hyperbolic, and hyperbolic.) Moreover, it follows from Corollary 2 that $\succsim_D$ is stationary if and only if there exists a $\delta \in (0,1)$ such that $D(k) = \delta^k$, $k = 0, 1, ...$. This highlights the special structure of the exponential discounting model: this model is the only *stationary* multiplicative discounting model. As for the present bias of multiplicative discounting models, Corollary 3 shows that $\succsim_D$ has (strong) present bias if and only if $D(s)/D(t)(>) \geq D(s-t)$ whenever $s > t > 0$. □

Example 4. (*Transitive Time Preferences*) A model of time preferences which is substantially more general than the previous one obtains if we simply consider the continuous preorders on $X \times \mathbb{Z}_+$. Any such preorder would admit a functional representation, and under some straightforward conditions, would be contained in the class characterized by Theorem 1. In particular, let V be a continuous real function on $X \times \mathbb{Z}_+$ which is $\succsim_0$-increasing in the first component and decreasing in the second. Assume further that $V(x,0) \geq V(y,0)$ iff $V(x,t) \geq V(y,t)$ for all $x, y \in X$ and $t \in \mathbb{Z}_+$ (the material preferences are unchanging through time), and that $V(\cdot, 0)$ is a homeomorphism from X onto $\mathbb{R}$. Now consider the binary relation $\succsim_V$ on $X \times \mathbb{Z}_+$ defined by $(x,t) \succsim_V (y,s)$ iff $V(x,t) \geq V(y,s)$. Obviously, any one of the time preferences considered in Examples 1-3 is of this form. Moreover, it is also quite easy to show that $\succsim_V$ is a time preference that satisfies (A1)-(A3), but $\succsim_V$ need not satisfy (B1)-(B3). To formulate a necessary and sufficient condition on V that would guarantee that $\succsim_V$ satisfies (B1)-(B3), define $U := V(\cdot, 0)$, and notice that $V(\cdot, t)$ must be a strictly increasing transformation of U for each t. Therefore, there exists a strictly increasing real function f_t on $U(X)$ such that $V(\cdot, t) = f_t \circ U$ for each t. It is not difficult to show that $\succsim_V$ satisfies (B1)-(B3) (and hence is a time preference of the form characterized in Theorem 1) if and only if $f_t^{-1} \circ f_s \circ U - U$ is a constant map on X for each s and t. □

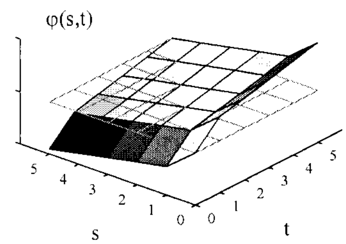
All of the examples we have considered above are transitive time preferences. One advantageous element of Theorem 1, however, is that it allows for intransitive time preferences. In the next three examples we consider a number of interesting forms of such preferences.

[14]This shows, at one stroke, that all hyperbolic discounting models in which, by definition, D is strictly decreasing everywhere (such as those of Loewenstein and Prelec (1992), and Benhabib, Bisin and Schotter (2002)) are also captured by Theorem 1.

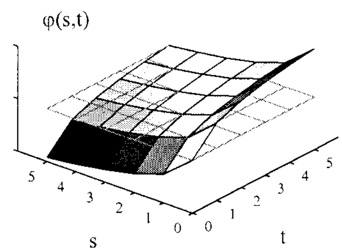
Example 1 Exponential Discounting



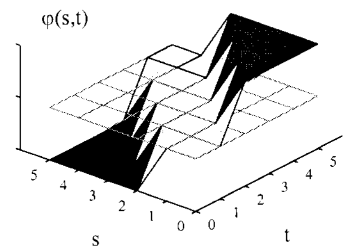
Example 2 Quasi-Hyperbolic Discounting



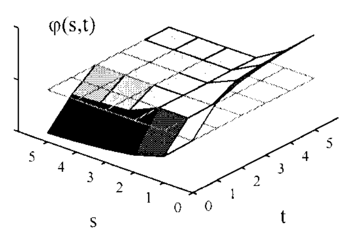
Example 3 Hyperbolic Discounting
with D(t)=1/(1+t)



Example 5 A Non-quasitransitive Time Preference



Example 6 Rubinstein Time Preference



Example 7 Subadditive Discounting

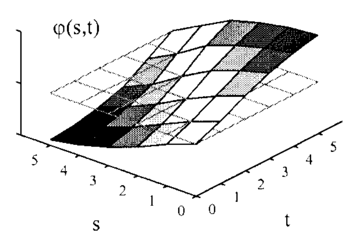


Figure 3

Example 5. (*A Non-quasitransitive Time Preference*) Define φ on $\mathbb{Z}_+^2$ by

$$\varphi(s,t) := \begin{cases} 1/2, & \text{if } t-s \geq 2 \\ -1/2, & \text{if } s-t \geq 2 \\ 0, & \text{otherwise} \end{cases}$$

(see Figure 3), and consider the binary relation $\succsim$ on $X \times \mathbb{Z}_+$ defined by $(x,t) \succ (\sim)\ (y,s)$ iff $\ln x > (=) \ln y + \varphi(s,t)$. It is readily checked that $\succsim$ is a time preference that belongs to the class characterized by Corollary 1. Yet $\succsim$ is not quasitransitive. For instance, $(e^{\cdot 6}, 2) \succ (e^{\cdot 5}, 1) \succ (e^{\cdot 4}, 0)$ but $(e^{\cdot 4}, 0) \succ (e^{\cdot 6}, 2)$. By Corollary 3, $\succsim$ has present bias, but not strong present bias. $\square$

Example 6. (*The Rubinstein Time-Preference Model*) In his critique of hyperbolic discounting, Rubinstein (2002) points to the fact that decision-making procedures based on similarity relations may better explain the experimental findings. While he does not provide a formal analysis, Rubinstein suggests that the analysis similar to the one developed in Rubinstein (1988) may be applied here. One way of formalizing this suggestion in the present context is as follows.

Let $\approx_X$ and $\approx_T$ be similarity relations on X and $\mathbb{Z}_+$, respectively. (We interpret $x \approx_X y$ as the agent viewing the alternatives x and y as "similar." The expression $t \approx_T s$ is interpreted similarly.) Let $\succsim$ be a binary relation on $X \times \mathbb{Z}_+$, and consider the following postulates:

(R1) If $x \succ_0 y$ and $t \leq s$, or $x \succsim_0 y$ and $t < s$, then $(x,t) \succ (y,s)$.

(R2) If $t \approx_T s$ but not $x \approx_X y$, then $(x,t) \succsim (y,s)$ iff $x \succsim_0 y$.

(R3) If $x \approx_X y$ but not $t \approx_T s$, then $(x,t) \succsim (y,s)$ iff $t \leq s$.

Rubinstein (1988, 2002) views the above properties as a procedure in which the agent first checks if (R1) applies, then (R2) and then (R3), but leaves unspecified how the decision is made when none of these properties apply. To complete the model, we will assume here that the agent uses a stationary rule to aggregate the utility of the outcome and the disutility of the delay. Formally, we posit that there exists a continuous function $V : X \times \mathbb{Z}_+ \to \mathbb{R}$ such that $(x,t) \succsim (y,s)$ iff $V(x,t) \geq V(y,s)$ for any (x,t) and (y,s) that fail to satisfy any of the antecedents of (R1)-(R3). Thus, the binary relation $\succsim$ is completely identified by the list $(\succsim_0, \approx_X, \approx_T, V)$, which we will refer to as a **Rubinstein time-preference model**.

It turns out that many Rubinstein time-preference models are captured by Theorem 1. To illustrate, let u be any homeomorphism that maps X onto $\mathbb{R}$, and define $\succsim_0$ as the preference relation on X that is represented by u. Also assume that $\approx_X$ is any similarity relation on X such that there exists an $\varepsilon > 0$ such that $|u(x) - u(y)| \leq \varepsilon$ whenever $x \approx_X y$. Now take any strictly decreasing $f : \mathbb{Z}_+ \to \mathbb{R}$ and any similarity relation $\approx_T$ on T such that $|f(s) - f(t)| > \varepsilon$ whenever

$s \approx_t t$ does not hold. Finally, define $V : X \times \mathbb{Z}_+ \to \mathbb{R}$ by $V(x,t) := u(x) + f(t)$. It is easily seen that the resulting Rubinstein time-preference model $(\succsim_0, \approx_X, \approx_T, V)$ admits a representation of the form characterized by Theorem 1. Indeed, the binary relation $\succsim$ identified by this model satisfies (3), where $U := u$ and $\varphi(s,t) := 0$ if $t \approx_T s$, and $\varphi(s,t) := f(s) - f(t)$ otherwise. (See Figure 3.) Depending on the structure of $\approx_T$, $\succsim$ may or may not have strong present bias. For instance, $\succsim$ has strong present bias if $\varepsilon \in (0, 1/2)$, $f(t) := 1/(1+t)$, and $t \approx_T s$ iff $|f(s) - f(t)| \leq \varepsilon$. On the other hand, $\succsim$ does not have strong present bias if $f(t) := -t$, and $t \approx_T s$ iff $t = s$. $\square$

Example 7. (*Subadditive Discounting*) In an interesting study, Read (2001) observes that phenomena like present bias and declining impatience may be explained by means of subadditive discounting (as opposed to hyperbolic discounting), that is, the idea that the total discounting over a given period may be greater than the sum of discounting when that period is divided into two parts. In the language of the present model, this means that if $(x, t+2) \sim (y, t)$ and $(x, t+2) \sim (z, t+1) \sim (w, t)$, then $w \succ_0 y$ (which is false for any multiplicative (in fact, for any transitive) time preference model). While Read (2001) discusses possible causes of subadditive discounting, and provides experimental evidence in support of its presence, he does not provide a concrete time preference model that incorporates subadditive discounting. Theorem 1, however, can be readily used to obtain various specific subadditive discounting models. Indeed, if $\succsim$ is as characterized by this theorem, and if

$$\varphi(t, t+2) > \varphi(t, t+1) + \varphi(t+1, t+2),$$

then $\succsim$ corresponds to a subadditive discounting model.[15] Of course, one can also use Theorem 1 in a similar fashion to obtain hybrid time preference models where, for instance, short time intervals are subject to multiplicative discounting, and longer ones are subject to subadditive discounting. $\square$

Finally, we consider two models of time preferences that do not fit within the framework provided by Theorem 1.

Example 8. (*Semiorders vs. Time Preferences*) The σ-δ model of Manzini and Mariotti (2002) considers semiorders $\succsim$ on $X \times \mathbb{Z}_+$ the asymmetric parts of which are represented as

$$(x,t) \succ_{\sigma\text{-}\delta} (y,s) \quad \text{iff} \quad \delta^t u(x) > \delta^s u(y) + \sigma,$$

[15] For example, if $f : \mathbb{R}_+ \to \mathbb{R}_+$ is any convex and strictly increasing function with $f(0) = 0$, and $\varphi(s,t) := f(|t-s|)$ if $s \leq t$, and $\varphi(s,t) := -f(|t-s|)$ if $s > t$, then the associated (stationary) time preference (given by Theorem 1) entails subadditive discounting. Figure 3 depicts how this particular "time-kick" factor compares with those of the other time preference models considered so far.

where $\delta \in (0,1)$ and $\sigma \in (0,\infty)$. It is worth noting that such binary relations are not admitted in the axiomatic framework we have provided above. In particular, the semiorder $\succsim_{\sigma\text{-}\delta}$ is not a time preference (as we define the term here), for its tth time projection is not a transitive relation. Thus, even when the time element is fixed, the σ-δ model presumes that the decision maker does not have the standard rational preferences. $\square$

Example 9. (*Time Preferences with Contemplation Costs*) A version of the time preference model considered by Benhabib, Bisin and Schotter (2002) posits that $\succsim$ is represented on $X \times \mathbb{Z}_+$ as follows:

$$(x,t) \succsim (y,s) \quad \text{iff} \quad D(t)u(x) - c(t) > D(s)u(y) - c(s).$$

Here D is a discount function of the form considered in Example 3, and $c : \mathbb{Z}_+ \to \mathbb{R}_+$ is an increasing function. (One may interpret $c(t)$ as the contemplation cost associated with planning about t periods ahead.) It is easily checked that $\succsim$ is a one-dimensional time preference, which reduces to the multiplicative discounting model when $c(t) = 0$ for all t. However, in general, $\succsim$ does not satisfy (A1), (B1) and (B2), and hence lies outside of the class of time preferences characterized by Theorem 1. $\square$

2.8 Uniqueness of the Representation

The uniqueness properties of the representation given in Theorem 1 depend crucially on which of the conditions of (B3) is satisfied by the time preference at hand. If the $\sim$-orbit of some outcome x contains an isolated point, then there is not much one can say in the way of uniqueness of this representation. In this case, one can show that the $\sim$-orbit of every outcome is a set that contains only isolated points, and consequently, for certain choices for φ, the set of homeomorphisms U such that the representation (3) holds is very large.[16] The exponential discounting model provides a case in point. For this model, given any $\delta \in (0,1)$, there exists a $U : X \to \mathbb{R}$ such that (3) holds with $\varphi(s,t) = (\ln \delta)(s-t)$ for all s and t. Moreover, this U is defined *arbitrarily* on a nondegenerate interval (as the proof of Theorem 2.(i) in Fishburn and Rubinstein (1982) demonstrates).[17]

[16] More precisely, there exists a real φ on T^2 that satisfies (ii) and (iii) of Theorem 1, and a nondegenerate interval I, such that *any* continuous and strictly increasing function on I can be (uniquely) extended to a homeomorphism $U : X \to \mathbb{R}$ such that (3) holds. In the language of the theory of functional equations, one says that, given this φ, U in the representation *depends on an arbitrary function.* (See Kuczma, Choczewski and Ger (1990), p. 4).

[17] We note that the situation is quite different if one considers the representation given in Theorem 1 within a continuous time model in which T is an interval in $[0,\infty)$. In this case each $\sim$-orbit would necessarily be dense, and one would thus obtain a satisfactory uniqueness result (along the lines of the forthcoming proposition) even for the exponential discounting model.

For time preferences for which $\sim$-orbits of outcomes are dense, however, the situation is markedly different. In this case, one can show that the outcome-utility function U is unique up to positive affine transformations, and the delay factor φ is unique up to positive linear transformations, and moreover, these transformation must be applied *simultaneously* (that is, the multiplicative constant in these transformations must be same for both U and φ). To state this result in precise terms, we introduce the following bit of terminology that we shall use in the next section as well.

Definition. Let X be a metric space, and $T \in \mathcal{T}$. We say that a time preference on $X \times T$ is **represented by** (U, φ) if the functions $U : X \to \mathbb{R}$ and $\varphi : T^2 \to \mathbb{R}$ satisfy all the properties that are required of them by Theorem 1.

The main result of this subsection is the following uniqueness theorem.

Proposition 1. *Let X be a metric space, $T \in \mathcal{T}$, and $\succsim$ a one-dimensional time preference on $X \times T$, which is represented by (U, φ). If $\varphi(s,t)/\varphi(s',t')$ is an irrational number for some $s, t, s', t' \in T$, then (V, ϕ) represents $\succsim$ if, and only if, $V = aU + b$ and $\phi = a\varphi$ for some $a > 0$ and $b \in \mathbb{R}$.*

If (U, φ) represents the time preference $\succsim$, and the ratio of two values of φ is irrational, then one can show that all $\sim$-orbits are dense subsets of the real line. Intuitively speaking, in this case the indifference relation $\sim$ "spreads" on the outcome space X well, allowing little degree of freedom for the choice of representing functions. Proposition 1 makes this intuition precise.

2.9 Time Preferences with Multidimensional Outcome Space

So far we have studied only one-dimensional time preferences in which, by definition, an agent is not indifferent between two distinct (undated) outcomes. In the case where the outcome space consists of, say, the consumption bundles that consist of several goods, this formulation is clearly unsatisfactory. Unfortunately, extending Theorem 1 to multidimensional time preferences $\succsim$ does not seem to be a trivial matter.[18] However, at least in the case where the outcome space is $\mathbb{R}^d_{++}$ (or an open convex subset of $\mathbb{R}^d_{++}$ which includes the diagonal), we will show next that one can in fact achieve this sort of an extension relatively easily.

[18] The "natural" strategy is to pass to the quotient space, and work with the induced relation $\succeq$ on $X/_\sim \times T$. If one can show that Theorem 1 applies to this binary relation, we would be done. However, it is not at all clear how to metrize the quotient space $X/_\sim$ in a way that would make this space connected and separable, and the time projections of the induced relation $\succeq$ completely continuous.

Let $\succsim$ be a time preference on $\mathbb{R}^d_{++} \times T$, where $d \in \mathbb{N}$ and $T \in \mathcal{T}$. We say that $\succsim$ is **strongly monotone**, if $\succsim_0$ is strictly monotonic on $\mathbb{R}^d_{++}$, that is, $x > y$ implies $x \succ_0 y$ for any $x, y \in \mathbb{R}^d_{++}$.[19] Such time preferences enjoy a representation which is analogous to that of one-dimensional time preferences.

Theorem 2. *Let $\succsim$ be a binary relation on $\mathbb{R}^d_{++} \times T$, where $T \in \mathcal{T}$. Then $\succsim$ is a strongly monotone time preference that satisfies the properties (A1)-(A3) and (B1)-(B3) if, and only if, there exist two functions $U : \mathbb{R}^d_{++} \to \mathbb{R}$ and $\varphi : T^2 \to \mathbb{R}$ such that (3) holds on $\mathbb{R}^d_{++} \times T$, and*

(i) U is a strictly increasing, continuous and open surjection,

(ii) $\varphi(\cdot, t)$ is decreasing and $\varphi(s, \cdot)$ is increasing for any $s, t \in T$,

(iii) $\varphi(s, t) + \varphi(t, s) = 0$ for any $s, t \in T$.

Certain generalizations of this result can be obrained by modifying the proof given in Section 5 in trivial ways. For example, $\mathbb{R}^d_{++}$ can be replaced with any produc space Y^d, where Y is a partially ordered connected and separable metric space, and strong monotonicity is defined with respect to the product order on Y^d. However, in the general case where X is an arbitrary connected and separable metric space, we do not know if (3) is valid for any time preference on $X \times T$ that satisfies the properties (A1)-(A3) and (B1)-(B3).

3 Time Consistency

A major advantage of the quasi-hyperbolic discounting model is that, when combined with the time consistency principle, it turns in a rather tractable model of intertemporal choice which portaits a good deal of predictive power. It turns out that, at least in the case of certain economic problems, the same is true for any time preference that is captured by Theorems 1 and/or 2. In particular, we show in this section that a large class of infinite-horizon intertemporal choice problems has time consistent solutions with this sort of a time preference. Moreover, as we shall demonstrate by means of the classical Wicksell tree-cutting problem and the Rubinstein bargaining model, sometimes one can learn quite a bit about the solution relative to such preferences by studying, instead, an "equivalent" exponential discounting model.

Throughout this section we concentrate on one-dimensional time preferences. This is, however, only for convenience; all results go through for an arbitrary time preference that is represented as in Theorem 2.

[19] For any $x, y \in \mathbb{R}^d_{++}$, by $x \geq y$ we mean $x_i \geq y_i$ for all $i = 1, ..., d$, and by $x > y$, we mean $x \geq y$ and $x \neq y$.

3.1 Existence of Time Consistent Choices

Let X be a metric space, $T \in \mathcal{T}$, and $\succsim$ a one-dimensional time preference on $X \times T$. We refer to the list

$$(\succsim, x_*, (x_t)_{t \in T})$$

as an **intertemporal choice problem**, where x_* is an object (not contained in X) such that (8) holds, and $x_t \in X$ for each $t \in T$. The idea simply is that the agent at date 0 faces the problem of consuming x_0 at period 0, or waiting until some period $t \in T$ to receive x_t at that date. We refer to $(\succsim, x_*, (x_t)_{t \in T})$ as a **compact** problem, if $\{x_t : t \in T\}$ is a relatively compact subset of X (that is, $\{x_t : t \in T\}$ is contained in a compact subset of X). In addition, we say that it is **regular** if X is connected and separable, and $\succsim$ satisfies (A1)-(A3) and (B1)-(B3). Compactness of the problem outrules the possibility of unbounded returns, and its regularity allows us to study the problem by using our representation theorem. Finally, we say that $(\succsim, x_*, (x_t)_{t \in T})$ is **non-trivial** if $\succsim$ is non-trivial, that is, if there is at least one $(x, t) \in X \times T$ such that $(x, 0) \succ (x, t)$. Obviously, trivial choice problems are not interesting for the present analysis, for "time" plays no role in solving such problems.

The standard way of analyzing time consistent solutions for such a decision problem is to treat the decision maker at time t as a different individual than the one that makes a decision at time s, $s \neq t$, and view these "selves" of the agent as playing noncooperatively an extensive-form game.[20] The relevant game for the present scenario is defined as follows. Person 0 moves first, and chooses either to consume x_0 (which ends the game) or to wait for the next period. In the latter case, person 1 moves, and decides either to consume x_1 (which ends the game) or to wait for the next period. In the latter case, person 2 moves, and so on. (See Figure 4.) The preference relation of person t, denoted $\succsim^t$, is defined on $X \times T_t$, as

$$(x, \tau) \succsim^t (y, \tau') \quad \text{if and only if} \quad (x, \tau - t) \succsim (y, \tau' - t),$$

where $T_t := \{s \in T : s \geq t\}$ for each $t \in T$.[21] (So, for instance, $\succsim^0 = \succsim$ and the time preference of person 5 is identical to that of person 0, except that person 5 views period 6 as period 1 for himself.) If $T = \mathbb{Z}_+$ and the game does not end in finite time, then we assume that every player

[20] The idea goes back to Strotz (1956) and Pollak (1968), and is adopted routinely in applications. In this sort of a formulation, time consistency is defined as (possibly a refinement) of the subgame perfect equilibria of the induced game played by the "selves" of the agent. See, for instance, Peleg and Yaari (1973), Goldman (1980), Kocherlakota (1996), and Geir (1997).

[21] We leave unspecified the preferences of person $t \in \{1, 2, ...\}$ for alternatives like $(x, t-1)$, but of course, this does not impede the subsequent equilibrium analysis. For concreteness, one may choose to specify that the player $t > 0$ receives x_* (or any given alternative in X), if the game ends before period t is reached.

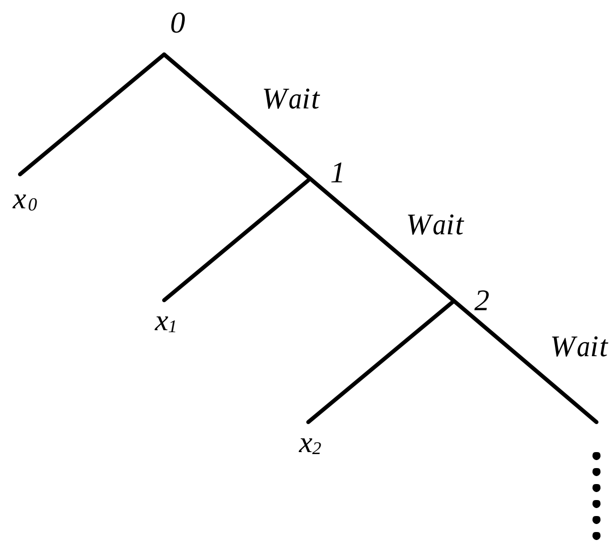


Figure 4

in the game receives the time-independent worst outcome x_*. (Note that x_* does not play a role in this formulation unless $|T| = \infty$.) This completes the description of the extensive-form game at hand, which we denote by $\mathfrak{g}(\succsim, x_*, (x_t)_{t \in T})$. Clearly, this game is of perfect information.

Definition. Let $(\succsim, x_*, (x_t)_{t \in T})$ be an intertemporal choice problem. We say that x_t is a **time consistent** choice for this problem if x_t is obtained in a subgame perfect equilibrium of the game $\mathfrak{g}(\succsim, x_*, (x_t)_{t \in T})$.

The main result of this section is the following existence theorem, which shows that there exists a time consistent choice for any compact and regular intertemporal decision problem.

Theorem 3. *For every compact and regular non-trivial intertemporal decision problem* $(\succsim, x_*, (x_t)_{t \in T})$, *there exists at least one time consistent choice. Moreover, the agent* (*at date* 0) *is indifferent between any two time consistent choices.*

This result shows that time consistency property complements the class of time preferences we have derived in Theorem 1 quite well. In particular, the possibility of time consistent decision making is guaranteed for this class in the case of a large class of intertemporal decision problems. Moreover, time consistent outcomes are (essentially) unique from the viewpoint of the agent at period 0.

In passing, we note that Theorem 3 is not obtained here as a direct corollary of the standard theorems on the existence of subgame perfect equilibria in extensive-form games of perfect information. For instance, the existence theorems of Fudenberg and Levine (1983) and Harris (1985) do not apply to the present context, for the preferences of the players in a game like $\mathfrak{g}(\succsim, x_*, (x_t)_{t \in T})$ need not be representable by a utility function defined on the terminal histories (since these preferences need not be transitive).

3.2 Application: The Tree-Cutting Problem

The classical tree-cutting (or wine-aging) problem is a particularly simple preemptive investment model. It envisages a tree of initial size $x_0 > 0$, whose growth is described by a given strictly increasing and strictly concave (production) function f. (So, if the size of the tree at date t is x_t, then its size in period $t+1$ is $x_{t+1} = f(x_t)$, $t = 0, 1, \ldots$.) The problem of the decision maker is to decide when to cut down the tree, and utilize its lumber, given that delay is undesirable.

To study this problem with respect to the class of time preferences we derived in Theorem 1, we first formalize the model as an intertemporal choice problem. Let $X := (0, \infty)$ and $T := \mathbb{Z}_+$, and write in what follows (x_t) for the sequence $(x_0, x_1, ...)$. If we denote the size of the tree at date t as x_t, then the model has it that

$$0 < x_0 < x_1 < \cdots \quad \text{and} \quad x_{t+1} - x_t < x_t - x_{t-1}, \quad t = 1, 2, ...$$

The former requirement is a consequence of the monotonicity of the production function, and the latter of its strict concavity. As is common, we assume that the production function falls below the identity function at a point $x > x_0$. This ensures that $\{x_0, x_1, ...\}$ is a relatively compact subset of X. Finally, we posit that the preferences of the decision maker $\succsim$ is represented by (U, φ), where U is strictly concave and $\varphi(0,1) > 0$. We extend U to $\mathbf{R}_+$ by letting $U(0) := -\infty$, and $\succsim$ to $\mathbf{R}_+ \times T$ by (3); this ensures that 0 serve as a time-independent worst outcome, that is, (8) holds here for $x_* = 0$. We will refer to the resulting intertemporal choice problem $(\succsim, 0, (x_t))$ as the **generalized tree-cutting problem**. If $\succsim$ corresponds to an exponential time discounting model, then we call this problem a **standard tree-cutting problem**.

It is obvious that a generalized tree-cutting problem is a compact and regular non-trivial intertemporal decision problem. Therefore, by Theorem 3, it has a time consistent solution. It is easy to find this choice in the case of a standard tree-cutting problem. Indeed, a mere inspection indicates that an exponential time discounter (i.e., when $\varphi(s,t) = (\ln \delta)(s-t)$ for all s and t, with $\delta \in (0,1)$ being the discount factor) would cut the tree precisely at date

$$t^*(U, \delta) := \min\{t \in \mathbb{Z}_+ : U(x_{t+1}) < U(x_t) - \ln \delta\}.$$

Moreover, it turns out that this date is the *latest* that the tree can be cut with respect to any time preference that is represented by (U, φ) with $\varphi(0,1) = -\ln \delta$ (even though the agent may not be discounting delay for more than one period, i.e., $\varphi(0,2) = \varphi(0,3) = \cdots = -\ln \delta$). More precisely, we have:

Proposition 2. *Let $(\succsim, 0, (x_t))$ be a generalized tree-cutting problem, where $\succsim$ is represented by (U, φ). Any time consistent choice for this problem takes place earlier than the date $t^*(U, e^{-\varphi(0,1)})$.*

This result provides a good illustration of the fact that a lot can be learned about the time consistent choices of an individual with (seemingly complicated) time preferences that is represented by some pair (U, φ) with $\varphi(0,1) > 0$, by studying the choices that this individual would have made, had his preferences were such that $(x,t) \succsim (y,s)$ iff $\delta^t u(x) \geq \delta^s u(y)$, where $u := e^U$ and $\delta := e^{-\varphi(0,1)}$. In particular, in the context of the generalized tree-cutting problem, this method tells us that the time consistent choice of the individual can be found by applying backward induction starting from the period $t^*(U, e^{-\varphi(0,1)})$.

We note that the time consistent choice in Proposition 2 may obtain strictly earlier than the date $t^*(U, e^{-\varphi(0,1)})$. For example, consider any generalized tree-cutting problem $(\succsim, 0, (x_t))$ where $\succsim$ is represented by some (U, φ) such that $U(x_0) = 1$, $U(x_1) = 5$, $U(x_2) = 7$, $U(x_3) = 8$, $(8, U(x_4), U(x_5), ...)$ is a sequence that increases with decreasing increments and that is bounded above by 8.4, $\varphi(0,1) = 1/2$, and $\varphi(0,t) = 4$, $t = 2, 3, ...$. In this case, we have $t^*(U, e^{-\varphi(0,1)}) = 4$, but the time consistent choice for $(\succsim, 0, (x_t))$ takes place at time 2.

3.3 Application: The Rubinstein Bargaining Game

We consider in this section the classical infinite-horizon alternating-offers bargaining game played among two individuals who have identical time preferences that belong to the class characterized by Theorem 1. As in the previous section, let $X := (0, \infty)$ and $T := \mathbb{Z}_+$, and let $\succsim$ be a time preference on $X \times T$ that is represented by a pair (U, φ) with U being strictly increasing and strictly concave, and with $\varphi(0,1) > 0$. Again, we extend U to $\mathbf{R}_+$ by letting $U(0) := -\infty$, and $\succsim$ to $\mathbf{R}_+ \times T$ by (3); this makes 0 the time-independent worst outcome, that is, (8) holds here for $x_* = 0$.

The players engage in the standard Rubinstein model in which the set A of all agreements is the set of all divisions of a cake of size 1: $A := \{(a, 1-a) : 0 \leq a \leq 1\}$. The preferences of the players are extended to $A \times T$ in the obvious way: Player 1, for instance, prefers $((a, 1-a), t)$ to $((b, 1-b), s)$ iff $U(a) \geq U(b) + \varphi(s,t)$. If no agreement is ever reached, each player receives the worst outcome 0. We refer to the resulting game as the **generalized Rubinstein (U, φ)-bargaining game**, or for short, *(U, φ)-bargaining game.* If $\succsim$ corresponds to an exponential time discounting model (i.e., when $\varphi(s,t) = (\ln \delta)(s-t)$ for all s and t, with $\delta \in (0,1)$ being the discount factor), then we refer to this game as the **standard Rubinstein (U, δ)-bargaining game**. It is a classical result that this game has a unique subgame perfect equilibrium.

We are interested in the equilibria of the (U, φ)-bargaining game that would obtain when the players determine their strategies in a time consistent manner. In line with the notion of time consistency advanced in Section 3.1, therefore, we treat each player at time t as a different individual. The preferences of these "selves" of the players are defined in the obvious way. For instance, the tth period "self" of player 1 prefers $((a, 1-a), \tau)$ to $((b, 1-b), \tau')$ iff $U(a) \geq U(b) + \varphi(\tau - t, \tau' - t)$, where $\tau, \tau' \geq t$.[22] We refer to any subgame perfect equilibrium of the resulting (infinite-player) game as a **time consistent subgame perfect equilibrium** of the (U, φ)-bargaining game.[23]

Clearly, the subgame perfect equilibrium of the standard Rubinstein (U, δ)-bargaining game

[22] The preferences of this individual over agreements reached *before* time t are irrelevant; for concreteness we may assume that this individual receives 0 (or any $a \in [0,1]$) if the game ends before period t is reached.

[23] Of course, here we identify the equilibrium action of the period t "self" of player i in the former game with the action of the player i in period t in the (U, φ)-bargaining game, $i = 1, 2$.

is the unique time consistent subgame perfect equilibrium of the (U,φ)-bargaining game, where $\varphi(s,t) = (\ln\delta)(s-t)$ for all $s,t \in T$. This, again, points to the very special structure of the exponential time discounting model which ascertains time consistent choices by its very nature. Moreover, we have:

Proposition 3. *There is a unique time consistent subgame perfect equilibrium of the generalized Rubinstein (U,φ)-bargaining game. This equilibrium equals the unique subgame perfect equilibrium of the standard Rubinstein $(U, e^{-\varphi(0,1)})$-bargaining game.*

Therefore, one can study a generalized Rubinstein (U,φ)-bargaining game simply by examining the "equivalent" standard game in which player 1 prefers $((a,1-a),t)$ to $((b,1-b),s)$ iff $\delta^t u(a) \geq \delta^s u(b)$, where $u := e^U$ and $\delta := e^{-\varphi(0,1)}$, *under the proviso of time consistency.* For example, if $U(x) := \ln x$ for all $x \geq 0$ and φ is an *arbitrary* function that satisfies conditions (ii) and (iii) of Theorem 1, then in the time consistent subgame perfect equilibrium of the (U,φ)-bargaining game the agreement $\left(\frac{1}{1+\delta}, \frac{\delta}{1+\delta}\right)$ obtains without delay, where $\delta := e^{-\varphi(0,1)}$.[24]

We conclude this section by noting that the time consistency postulate is a very powerful one, which, at least in some models, takes away from the richness of the class of preferences derived in Theorem 1. For the proponents of this postulate, this is good news, since the results above illustrate that in some important economic instances it may not at all matter what sort of a time discounting model one should adopt, all one has to know is the utility discount for one-period forward from the present (i.e. the magnitude of $\varphi(0,1)$). A more boundedly rational model, however, is likely to change this conclusion dramatically, and bring to the forefront the differences between various members of the class of time preferences studied here. With this sort of a decision making model, one would expect the generalized Rubinstein (U,φ)-bargaining game to have nonstationary equilibria, possibly with significant time delay (despite the complete information structure of the game).

4 Conclusions

4.1 Other Anomalies and More General Time Preference Models

In this paper we have derived a time preference model for intertemporal choices that involve single delayed events. The advantage of this model is to incorporate many models that are proposed in the literature in order to conform with the experimental regularity that discount rates decline with

[24]This result draws a close parallel with the main result of Volij (2002) that says that the behavior of time consistent players in the alternating-offers bargaining game with exogenous risk of breakdown cannot be distinguished from the behavior of the standard agents who maximize their expected utilities.

the length of delay. It thereby allows one study certain intertemporal choice models in a way that conforms with this regularity, but without subscribing to a particular time discounting formula. One may, however, not readily satisfied with the generality of the present time preference model, and ask how it fares with other regularities that are observed in the experiments, in particular, with the notions of the delay-speedup asymmetry, the magnitude effect, and the sign effect. We take up each of these "anomalies" in turn.

The Delay-Speedup Asymmetry. Demonstrated first by Loewenstein (1988), this effect corresponds to the observation that inferred discount rates are greater when decision makers are confronted with decisions that involve delaying anticipated rewards than for decisions that involve expediting rewards. For instance, Loewenstein (1988) shows that one may be willing to pay \$54 to receive a VCR of value a now as opposed to one year, and at the same time, ask for \$126 to delay the receipt of the same VCR for one year. In the language of this paper, this amounts to saying that $(a-54, 0) \sim (a, 1)$ and $(a, 0) \sim (a+126, 1)$. The time preference model advanced here is easily checked to be consistent with this sort of an observation. In fact, this observation is consistent with even the standard exponential discounting model. Loewenstein (1988) reports that, on average, the individuals valued the VCR in question at about \$270, so let us set $a = 270$. Taking the interest rate to be .05, then we have $\delta \simeq .95$, so, if $u : \mathbb{R}_{++} \to \mathbb{R}$ is a concave and strictly increasing utility function (for money) such that $u(216)/u(270) = .95 = u(270)/u(396)$, then the "anomaly" is seen to be fully consistent with the standard discounting model.

The Magnitude Effect. Demonstrated first by Thaler (1981), and then studied by several other experimental researchers (such as Benzion, Rapoport, and Yagil (1989), and Green, Myerson, and McFadden (1997)), this effect pertains to the observation that small outcomes are discounted more than large ones. For instance, Thaler (1981) reports that his subjects were, on average, indifferent between \$15 immediately and \$60 in a year, and \$3000 immediately and \$4000 a year. In the language of this paper, this amounts to saying that $(15, 0) \sim (60, 1)$ and $(3000, 0) \sim (4000, 1)$. Once again, the time preference model advanced here, in fact the exponential discounting model, is easily checked to be consistent with this sort of an observation. This model would indeed yield these indifferences if $\delta := .95$ and $u(x) := x^{0.42} + 45.9$ for all $x > 0$.

The Sign Effect. Demonstrated first by Thaler (1981), this effect pertains to gains being discounted more heavily than losses. Since the present work focuses only on problems with gains (or with losses, but not with both), one cannot talk about how our model fares with this empirical regularity. It is an interesting open problem if, and how, one may extend the present model to one that incorporates situations that involve gains and losses simultaneously, and that conforms with the sign effect.

4.2 Open Problems

Multidimensional Time Preferences. The representation notion that is axiomatized by Theorem 1 extends in the natural way to the case where the preferences allow for two distinct undated outcomes to be indifferent. Indeed, in the special case where the outcome space is $\mathbb{R}^d_{++}$, Theorem 2 provides an axiomatization of such multidimensional time preferences. A more complete theory, however, would require obtaining a similar characterization when the outcome space is an arbitrary connected and separable metric space. This problem stands open at present.

Time Preferences and Risky Outcomes. The time preference theory presented here treats the preferences over undated outcomes in an ordinal way, and hence it is not suitable for intertemporal choice models in which current and/or future outcomes may obtain in risky environments. An important item in the related future research agenda should thus concern how to extend the present theory to the case where (1) the time projections admit an expected (or non-expected) utility representation, and/or (2) time preferences are defined over lotteries on the entire prize-time space.

Time Preferences over Consumption Streams. We have considered here only the time preferences that are defined on the prize-time space. While this structure is sufficient for some interesting economic applications, it does not apply to numerous dynamic situations, such as the capital accumulation problem, search models, repeated games, etc.. An important next step is, therefore, to extend the present analysis to the case of time preferences defined over consumption streams through time. This extension is likely to be highly nontrivial, for it is not even clear what is, if any, the natural generalization of the present model to this case.

Time Preferences and Time Inconsistency. There are two major aspects of the theory of intertemporal choice. The first is the structure of time discounting, which is the topic studied here, and the second is how the choices are made on the basis of a given time preference. One answer is obtained by postulating time consistent behavior, as we have done in Section 3. This is, however, not the only reasonable postulate. Agents, after all, may well be only boundedly rational, and violate time consistency in systematic ways. There is room for investigating what sorts of choice models would be obtained on the basis of the time preference model developed here and of some semi-myopic decision making postulate. In contrast to the results reported in Section 3, the latter postulate is likely to induce vastly different behavior for exponential, hyperbolic and intransitive time preferences.

5 Proofs

5.1 An Observation on Systems of Abel Functional Equations

The Abel functional equation is much studied in the literature on iterative functional equations (see Kuczma, et al. 1990).[25] There are relatively fewer studies on systems of such equations (but see Neumann (1982), Zdun (1989) and Jarczyk, et al. (1994)). We state below the main existence result obtained for such systems in the literature.

For any topological spaces X and Y, we denote the set of all homeomorphisms that map X onto Y by $Hom(X, Y)$, but we write $Hom(X)$ for $Hom(X, X)$. If f is a self-map on X (that is, $f \in X^X$), then $Fix(f)$ denotes the set of all fixed points of f, that is, $Fix(f) := \{x \in X : x = f(x)\}$. We denote the identity function on X by Id_X.

Let G be any group, and $S \subseteq G$. The smallest subgroup of G that contains S is called the **group generated** by S, and denoted as $\langle S \rangle$. It easy to show that $s \in \langle S \rangle$ if and only if there are finitely many $s_1, ..., s_n \in G$ such that $s = s_1 \cdots s_n$ and either $s_i \in S$ or $s_i^{-1} \in S$ for each $i = 1, ..., n$. In passing, we note the following well-known result of number theory, which we will invoke subsequently.

Lemma 1. *If S is a nonempty subset of the additive group of real numbers, then either $\langle S \rangle = \theta\mathbb{Z}$ for some $\theta \in \mathbb{R}$ or $\langle S \rangle$ is dense in $\mathbb{R}$.*

Consider the following set of real maps

$$\mathcal{A} := \{f \in Hom(\mathbb{R}) : Fix(f) = \emptyset\},$$

and let $\mathcal{F}$ be any nonempty subset of $\mathcal{A}$. The subgroup generated by $\mathcal{F}$ under the composition operation is denoted by $\langle \mathcal{F} \rangle$. The $\mathcal{F}$**-orbit** of $a \in \mathbb{R}$ is defined as

$$Q_{\mathcal{F}}(a) := \{(f_1 \circ \cdots \circ f_n)(a) : n \in \mathbb{N} \text{ and } f_1, ..., f_n \in \mathcal{F}\}.$$

We let $L_{\mathcal{F}}(a)$ denote the set of limit points of $Q_{\mathcal{F}}(a)$, that is, $b \in L_{\mathcal{F}}(a)$ iff there is a sequence (a_m) in $Q_{\mathcal{F}}(a) \backslash \{b\}$ such that $a_m \to b$.

We are now ready to state the following fundamental existence theorem for simultaneous Abel functional equations.

[25] The general form of this equation is $\psi(f(x)) = \psi(x) + 1$ for all $x \in X$, where X is a cone in a Banach space, and the "known" function f is a self-map on X. The "unknown" of the equation is the function $\psi \in \mathbb{R}^X$.

Theorem A. (Jarczyk, Loskot, Zdun, 1994) *Let $\mathcal{F}$ be any nonempty subset of $\mathcal{A}$ which contains a map $g \in \mathcal{A}$ with $g > Id_{\mathbb{R}}$. If $\langle \mathcal{F} \rangle$ is Abelian, then*

$$\nu_g(f) := \sup\left\{\frac{m}{n} : (m,n) \in \mathbb{Z} \times \mathbb{N} \text{and } g^m < f^n\right\} \neq 0 \quad \text{for all } f \in \langle \mathcal{F} \rangle .$$

Moreover, if

(i) *$\langle \mathcal{F} \rangle$ is Abelian and the only member of $\langle \mathcal{F} \rangle$ that has a fixed point is $Id_{\mathbb{R}}$,*

and

(ii) *there is some $a \in \mathbb{R}$ such that either $L_{\mathcal{F}}(a) = \{-\infty, \infty\}$ or $L_{\mathcal{F}}(a) = \overline{\mathbb{R}}$,*

then there exists a continuous bijection $F : \mathbb{R} \to \mathbb{R}$ such that

$$F \circ f - F = \nu_g(f) \quad \text{for all } f \in \mathcal{F}.$$

Proof. This statement obtains upon combining Propositions 1 and 2, and Theorem 1 of Jarczyk, et al. (1994). □

5.2 Proof of Theorem 1

[⇒] Let $\succsim$ be a one-dimensional time preference on $X \times T$ that satisfies the properties (A1)-(A3) and (B1)-(B3).

Claim 1. There exists a $u \in Hom(X, \mathbb{R})$ that represents $\succsim_0$.

Proof. Since X is separable metric space, it is second countable, so given that $\succsim_0$ is a continuous linear order on X, we may apply the classic Debreu Representation Theorem to find a continuous injection $v : X \to \mathbb{R}$ that represents $\succsim_0$. We define $u : X \to \mathbb{R}$ by

$$u(x) := \log\left(\frac{v(x) - \inf v(X)}{\sup v(X) - v(x)}\right)$$

if $\inf v(X) > -\infty$ and $\sup v(X) < \infty$,

$$u(x) := -\log\left(\sup v(X) - v(x)\right)$$

if $\inf v(X) = -\infty$ and $\sup v(X) < \infty$,

$$u(x) := \log\left(v(x) - \inf v(X)\right)$$

if $\inf v(X) > -\infty$ and $\sup v(X) = \infty$, and $u(x) := v(x)$ otherwise. Since $\succsim_0$ is completely continuous, neither $\inf v(X)$ nor $\sup v(X)$ belongs to $v(X)$, so u is well-defined. Moreover, it follows from the properties of v that u is a continuous injection that represents $\succsim_0$. Since X is connected and u is continuous, $u(X)$ is connected, which means that $u(X)$ is an interval in $\mathbb{R}$. But, by definition, we have $\inf u(X) = -\infty$ and $\sup u(X) = \infty$, so it follows that u is a continuous bijection that maps X onto $\mathbb{R}$.

Now let O be any nonempty open subset of X. Since $\succsim_0$ is completely continuous and linear, there exist $y_x, z_x \in O$ such that (i) $z_x \succ_0 x \succ_0 y_x$, and (ii) $\{\omega \in X : z_x \succ_0 \omega \succ_0 y_x\} \subseteq O$. Thus, letting $O_x := \{\omega : z_x \succ_0 \omega \succ_0 y_x\}$ for each $x \in O$, we find that $O = \bigcup\{O_x : x \in O\}$, so

$$u(O) = u\left(\bigcup\{O_x : x \in O\}\right) = \bigcup\{u(O_x) : x \in O\} = \bigcup_{x \in O}(u(y_x), u(z_x)),$$

which shows that $u(O)$ is an open subset of $\mathbb{R}$. Thus u is an open mapping, and is thus a homeomorphism. $\square$

Claim 2. For any $x, y \in X$ and $s, t \in T$ such that $(x,t) \succ (y,s)$, there exist $z, w \in X$ such that $(x,t) \succ (z,t) \succ (y,s)$ and $(x,t) \succ (w,s) \succ (y,s)$.

Proof. Define $A := \{\omega \in X : (x,t) \succ (\omega,t)\}$ and $B := \{\omega \in X : (\omega,t) \succ (y,s)\}$. While $A \neq \emptyset$ since $\succsim_t = \succsim_0$ is completely continuous, we have $B \neq \emptyset$ because $x \in B$ by hypothesis. Since $\succsim_t$ is upper semicontinuous, A is open in X, and since $\succsim$ is lower semicontinuous, B is open in X. We next claim that $X \subseteq A \cup B$, that is, for any $\omega \in X$, either $(x,t) \succ (\omega,t)$ or $(\omega,t) \succ (y,s)$. This is true, because if $(\omega,t) \succsim (x,t)$ is the case, then given that $(x,t) \succ (y,s)$, (A3) implies $(\omega,t) \succ (y,s)$. Thus $X = A \cup B$. Since X is connected, therefore, we must have $A \cap B \neq \emptyset$. The second claim is proved analogously. $\square$

Claim 3. For any $y \in X$ and $s, t \in T$, there exists a unique $x \in X$ such that $(x,t) \sim (y,s)$.

Proof. By (A1), there exists an $\omega \in X$ such that $(\omega,t) \succsim (y,s)$, so

$$A := \{\omega \in X : (\omega,t) \succsim (y,s)\} \neq \emptyset.$$

Take any $u \in \mathit{Hom}(X,\mathbb{R})$ that represents $\succsim_0$ (Claim 1). If $\inf u(A) = -\infty$, then for any $z \in X$ we can find an $\omega \in A$ with $u(z) > u(\omega)$, that is, $(z,t) \succ (\omega,t) \succsim (y,s)$. Thus in this case (A3) implies that $(z,t) \succ (y,s)$ for all $z \in X$, but this violates (A1). It follows that $\inf u(A)$ is a real number, so by surjectivity of u, there exists a $z \in X$ such that $u(x) = \inf u(A)$.

Now if $(y,s) \succ (x,t)$, then there exists a $w \in X$ such that $(y,s) \succ (w,t) \succ (x,t)$ by Claim 2. Since $x \notin A$ and u is injective, we have $\inf u(A) = u(x) \notin u(A)$, so we can find a sequence (ω_n) in A such that $u(\omega_1) > u(\omega_2) > \cdots$ and $u(\omega_n) \to u(x)$. Since u represents $\succsim_0 = \succsim_t$, we have $u(w) > u(x)$, so there exists an $n_0 \in \mathbb{N}$ such that $u(w) > u(\omega_{n_0}) > u(x)$. Then $(y,s) \succ (w,t) \succ (\omega_{n_0},t)$, and it follows from (A3) that $\omega_{n_0} \notin A$, a contradiction. Since $\succsim$ is complete, we thus obtain $(x,t) \succsim (y,s)$. However, if $(x,t) \succ (y,s)$, then there exists a $z \in X$ such that $(x,t) \succ (z,t) \succ (y,s)$ by Claim 2, and this contradicts the fact that $u(x) = \inf u(A)$. Therefore, $(x,t) \sim (y,s)$, which settles the existence part of the claim.

To prove the uniqueness part, assume that $(x', t) \sim (y, s)$ holds for some $x' \in X$. If $(x', t) \sim (x, t)$ is false, then (A3) implies that either $(x, t) \sim (y, s)$ or $(x', t) \sim (y, s)$ is false. Thus we must have $(x', t) \sim (x, t)$, that is $x' \sim_t x$. Since $\succsim_t = \succsim_0$ is antisymmetric, it follows that $x' = x$. □

For any $s, t \in T$, we define the self-map $\chi_{s,t} : X \to X$ by the statement

$$(y, s) \sim (\chi_{s,t}(y), t), \qquad y \in X.$$

By Claim 3, $\chi_{s,t}$ is well-defined for any $s, t \in T$. The following two claims report further properties of these self-maps.

Claim 4. $\chi_{s,t}$ is a bijection and $\chi_{s,t}^{-1} = \chi_{t,s}$ for any $s, t \in T$.

Proof. Apply Claim 3 and the definitions of $\chi_{s,t}$ and $\chi_{t,s}$. □

Claim 5. For any $s, t \in T$ and $u \in \mathit{Hom}(X, \mathbb{R})$ that represents $\succsim_0$, $u \circ \chi_{s,t}$ is a continuous bijection that maps X onto $\mathbb{R}$.

Proof. We only need to establish the continuity of $u \circ \chi_{s,t}$. Take any $y \in X$ and any sequence (y_n) in X with $y_n \to y$. Towards deriving a contradiction, we assume that $\limsup u(\chi_{s,t}(y_n)) > u(\chi_{s,t}(y))$. Then by continuity of u and connectedness of X, there exists a $z \in X$, a strictly increasing sequence (n_k) of natural numbers, and a natural number K such that

$$u(\chi_{s,t}(y_{n_k})) > u(z) > u(\chi_{s,t}(y)) \quad \text{for all } k \geq K,$$

where we applied the Intermediate Value Theorem. Since u represents $\succsim_0 = \succsim_t$, we then find

$$(y_{n_k}, s) \sim (\chi_{s,t}(y_{n_k}), t) \succ (z, t) \succ (\chi_{s,t}(y), t) \sim (y, s) \quad \text{for all } k \geq K.$$

Applying (A3) twice, we then get $(y_{n_k}, s) \succ (z, t) \succ (y, s)$ for all $k \geq K$. Since $\succsim$ is upper semicontinuous, letting $k \to \infty$ yields $(y, s) \succsim (z, t) \succ (y, s)$, a contradiction. This proves that $u \circ \chi_{s,t}$ is upper semicontinuous. The lower semicontinuity of $u \circ \chi_{s,t}$ is verified in the analogous way. □

From now on we will work with a fixed $u \in \mathit{Hom}(X, \mathbb{R})$ that represents $\succsim_0$ (Claim 1). For any $s, t \in T$, we define

$$f_{s,t} := u \circ \chi_{s,t} \circ u^{-1}.$$

Claim 6. $f_{s,t} \in \mathit{Hom}(\mathbb{R})$ and $f_{s,t}^{-1} = f_{t,s}$ for any $s, t \in T$.

Proof. $f_{s,t}$ is a bijection by Claim 4, and since u^{-1} is continuous, it is continuous by Claim 5, for any $s, t \in T$. Moreover, by Claim 4,

$$f_{s,t}^{-1} = u \circ \chi_{s,t}^{-1} \circ u^{-1} = u \circ \chi_{t,s} \circ u^{-1} = f_{t,s},$$

so since $f_{t,s}$ is continuous, $f_{s,t} \in Hom(\mathbb{R})$, for any $s, t \in T$. $\square$

Now define

$$\Gamma := \{(s,t) \in T^2 : \chi_{s,t} \neq Id_X\},$$

and note that $(t,t) \notin \Gamma$ for any $t \in T$. If $\Gamma = \emptyset$, then the proof is completed by letting $U := u$ and $\varphi := 0$, so we assume in what follows that $\Gamma \neq \emptyset$. Observe that if $(x,s) \sim (x,t)$ for some $x \in X$ and $s, t \in T$, then (B2) ensures that $(z,s) \sim (z,t)$ for all $z \in X$, that is, $(s,t) \notin \Gamma$. It follows that

$$\Gamma = \{(s,t) \in T^2 : Fix(\chi_{s,t}) = \emptyset\}, \tag{9}$$

and hence

$$\mathcal{F} := \{f_{s,t} : (s,t) \in \Gamma\} \subseteq \{f \in Hom(\mathbb{R}) : Fix(f) = \emptyset\}$$

by Claim 6. We shall demonstrate below that $\mathcal{F}$ satisfies the other requirements of Theorem A as well.

Fix an arbitrary $(s^*, t^*) \in \Gamma$ with $s^* < t^*$, and define

$$g := f_{s^*,t^*}.$$

By (A2) we have $(y, s^*) \succsim (y, t^*)$. Since $(s^*, t^*) \in \Gamma$, we must then have $(y, s^*) \succ (y, t^*)$, for otherwise $\chi_{s^*,t^*} = Id_X$. Thus $(\chi_{s^*,t^*}(y), t^*) \sim (y, s^*) \succ (y, t^*)$. If $(y, t^*) \succsim (\chi_{s^*,t^*}(y), t^*)$, then (A3) yields the contradiction $(y, t^*) \succsim (y, s^*)$, so we must have $(\chi_{s^*,t^*}(y), t^*) \succ (y, t^*)$ by completeness of $\succsim$. Since $\succsim_{t^*} = \succsim_0$, it follows that

$$g(u(y)) = f_{s^*,t^*}(y) = u(\chi_{s^*,t^*}(y)) > u(y).$$

Since y is arbitrary in X, and $u(X) = \mathbb{R}$, this proves that $g > Id_{\mathbb{R}}$.

Now take any $(s_1, t_1), (s_2, t_2) \in \Gamma$, and any $y \in X$. Let $w := \chi_{s_2,t_2}(y)$, $z := \chi_{s_1,t_1}(w)$ and $x := \chi_{s_1,t_1}(y)$. Then by definition of χ_{s_2,t_2} and (B1), we have

$$\chi_{s_1,t_1}(\chi_{s_2,t_2}(y)) = \chi_{s_1,t_1}(w) = z = \chi_{s_2,t_2}(x) = \chi_{s_2,t_2}(\chi_{s_1,t_1}(y)).$$

Since y is arbitrary in X, this shows that χ_{s_1,t_1} and χ_{s_2,t_2} commute, and it follows easily from this observation that so do f_{s_1,t_1} and f_{s_2,t_2}. Since (s_1, t_1) and (s_2, t_2) are arbitrary in Γ, it follows that $\langle \mathcal{F} \rangle$ must be an Abelian group.

We next claim that if $f \in \langle \mathcal{F} \rangle$ and $Fix(f) \neq \emptyset$, then $f = Id_{\mathbb{R}}$. For, $f \in \langle \mathcal{F} \rangle$ means that there exist finitely many $(s_1, t_1), ..., (s_n, t_n) \in \Gamma$ such that $f = f_{s_1,t_1} \circ \cdots \circ f_{s_n,t_n}$. So if $a = f(a)$ for some real a, then

$$a = \left(u \circ \chi_{s_1,t_1} \circ \cdots \circ \chi_{s_n,t_n} \circ u^{-1}\right)(a).$$

Letting $x := u^{-1}(a) \in X$, we then get $x = \left(\chi_{s_1,t_1} \circ \cdots \circ \chi_{s_n,t_n}\right)(x)$. But by (B2), this implies that $\chi_{s_1,t_1} \circ \cdots \circ \chi_{s_n,t_n} = Id_X$, so it follows that $f = u \circ Id_X \circ u^{-1} = Id_{\mathbb{R}}$, as is sought.

Now take any $x \in X$ for which (B3) holds and let $a := u(x)$. Clearly,

$$\begin{aligned} Q_{\mathcal{F}}(a) &:= \{(f_{s_1,t_1} \circ \cdots \circ f_{s_n,t_n})(a) : n \in \mathbb{N} \text{ and } (s_1,t_1),...,(s_n,t_n) \in \Gamma\} \\ &= \{u((\chi_{s_1,t_1} \circ \cdots \circ \chi_{s_n,t_n})(x)) : n \in \mathbb{N} \text{ and } (s_1,t_1),...,(s_n,t_n) \in \Gamma\} \\ &= u(O^{\sim}(x)). \end{aligned}$$

Claim 7. If $Q_{\mathcal{F}}(a)$ has an isolated point, then all points of $Q_{\mathcal{F}}(a)$ are isolated, and if $Q_{\mathcal{F}}(a)$ is somewhere dense, then every real number is a limit point of $Q_{\mathcal{F}}(a)$.

Proof. Suppose that $b \in \mathbb{R}$ is an isolated point of $Q_{\mathcal{F}}(a)$, but $c \in Q_{\mathcal{F}}(a)$ is not isolated. Then there exists a sequence (c_m) in $Q_{\mathcal{F}}(a)\backslash\{c\}$ such that $c_m \to c$. Clearly,

$$b = (f_{s_1,t_1} \circ \cdots \circ f_{s_n,t_n})(a) \quad \text{and} \quad c = (f_{s'_1,t'_1} \circ \cdots \circ f_{s'_k,t'_k})(a)$$

for some $(s_1,t_1),...,(s_n,t_n),(s'_1,t'_1),...,(s'_k,t'_k) \in \Gamma$ and $n,k \in \mathbb{N}$. Let

$$b_m := (f_{s_1,t_1} \circ \cdots \circ f_{s_n,t_n} \circ f^{-1}_{s'_k,t'_k} \circ \cdots \circ f^{-1}_{s'_1,t'_1})(c_m), \quad m = 1,2,... .$$

By Claim 4 and (9) Γ is a symmetric set (that is, $(t,s) \in \Gamma$ for all $(s,t) \in \Gamma$), so it follows from Claim 6 that $b_m \in Q_{\mathcal{F}}(a)$ for each m, whereas it is obvious that $b_m \to b$. Moreover, if $b_m = b$ for some m, then $a = (f^{-1}_{s'_k,t'_k} \circ \cdots \circ f^{-1}_{s'_1,t'_1})(c_m)$, so $c = c_m$, a contradiction. Thus (b_m) is a sequence in $Q_{\mathcal{F}}(a)\backslash\{b\}$ that converges to b, contradicting that b is an isolated member of $Q_{\mathcal{F}}(a)$, thereby establishing the first part of the claim.

To prove the second part, assume that $Q_{\mathcal{F}}(a)$ is somewhere dense, that is, $(\alpha,\beta) \subseteq cl(Q_{\mathcal{F}}(a))$ for some $-\infty < \alpha < \beta < \infty$. Take any $\gamma \in (\alpha,\beta)$. If γ is not a limit point of $Q_{\mathcal{F}}(a)$, then since $\gamma \in cl(Q_{\mathcal{F}}(a))$, γ must be an isolated point of $Q_{\mathcal{F}}(a)$. But then there exist an $\varepsilon > 0$ such that $(\gamma-\varepsilon,\gamma+\varepsilon) \cap Q_{\mathcal{F}}(a) = \{\gamma\}$, which contradicts that $(\alpha,\beta) \subseteq cl(Q_{\mathcal{F}}(a))$. Thus every point of (α,β) is a limit point of $Q_{\mathcal{F}}(a)$, that is, $(\alpha,\beta) \subseteq L_{\mathcal{F}}(a)$. Then $L_{\mathcal{F}}(a)$ is somewhere dense in $\mathbb{R}$, and thus the claim follows from Proposition 1.(c) of Jarczyk, et al. (1994). $\square$

Since $Q_{\mathcal{F}}(a) = u(O^{\sim}(x))$ and u is a homeomorphism, if $O^{\sim}(x)$ contains an isolated point, so does $Q_{\mathcal{F}}(a)$, and if $O^{\sim}(x)$ is somewhere dense, so is $Q_{\mathcal{F}}(a)$. Moreover, since $g > Id_{\mathbb{R}}$, we have

$$Q_{\mathcal{F}}(a)\backslash\{a\} \ni g^n(a) \to \infty \quad \text{as } n \to \infty,$$

where g^n stands for the nth iteration of g. Similarly,

$$Q_{\mathcal{F}}(a)\backslash\{a\} \ni g^{-n}(a) \to -\infty \quad \text{as } n \to \infty,$$

for by Claim $g^{-1} < Id_{\mathbb{R}}$. Thus, by (B3) and Claim 7, we have either $L_{\mathcal{F}}(a) = \{-\infty, \infty\}$ or $L_{\mathcal{F}}(a) = \overline{\mathbb{R}}$.

We now verified that $\mathcal{F}$ satisfies all requirements of Theorem A. Define $\varphi : \Gamma \to \mathbb{R}$ by $\varphi(s,t) := \nu_g(f_{s,t})$, where ν_g is defined as in Theorem A. By this theorem, $\varphi(s,t) \neq 0$ for all $(s,t) \in \Gamma$, and there exists a continuous bijection $F : \mathbb{R} \to \mathbb{R}$ such that

$$F(f_{s,t}(a)) - F(a) = \varphi(s,t) \quad \text{for all } (a,(s,t)) \in \mathbb{R} \times \Gamma. \tag{10}$$

Claim 8. For any $(s,t) \in \Gamma$, we have

$$\varphi(s,t) = -\varphi(t,s) \quad \text{and} \quad (s-t)\varphi(s,t) < 0.$$

Proof. Let $(s,t) \in \Gamma$, and take any $a \in \mathbb{R}$. By (10) and Claim 6, we have

$$\begin{aligned} F(f_{s,t}(a)) - F(a) &= \varphi(s,t) \\ &= F(f_{s,t}(f_{s,t}^{-1}(a))) - F(f_{s,t}^{-1}(a)) \\ &= F(a) - F(f_{t,s}(a)) \\ &= -\varphi(t,s). \end{aligned}$$

To prove the second claim, assume that $s > t$. Then we have $g > Id_{\mathbb{R}} > f_{s,t}$, so for any $(m,n) \in \mathbb{Z}_+ \times \mathbb{N}$ we have $g^m > Id_{\mathbb{R}} > f_{s,t}^n$. Thus by the first part of Theorem A, we have $\varphi(s,t) = \nu_g(f_{s,t}) < 0$, as we sought. $\square$

Take any $a \in \mathbb{R}$ and let $b := g(a)$. By Claim 8, we have $F(b) = F(a) + \varphi(s^*,t^*) > F(a)$. Since $g > Id_{\mathbb{R}}$, we have $b > a$, so it follows that F cannot be strictly decreasing. Since F is bijective, therefore, it follows that it must be strictly increasing. We now define $U : X \to \mathbb{R}$ by $U := F \circ u$, and extend φ to T^2 by setting $\varphi|_{T^2 \setminus \Gamma} := 0$. It will be shown next that U and φ satisfy the requirements made of them in Theorem 1.

We observe first that U is a continuous bijection that represents $\succsim_0$ since F is continuous, surjective and strictly increasing. It can be shown that U must be an open map in exactly the same way we have shown that u is open in Claim 1. Therefore, U represents $\succsim_0$ and $U \in Hom(X, \mathbb{R})$.

To verify (3), take any $(x,t),(y,s) \in X \times T$. Clearly, $(x,t) \succ (\sim)\ (y,s)$ holds if and only if $(x,t) \succ (\sim)\ (y,s) \sim (\chi_{s,t}(y), t)$. Thus, by completeness of $\succsim$ and (A3), we have

$$(x,t) \left\{ \begin{array}{c} \succ \\ \sim \end{array} \right\} (y,s) \quad \text{iff} \quad (x,t) \left\{ \begin{array}{c} \succ \\ \sim \end{array} \right\} (\chi_{s,t}(y), t).$$

Since U represents $\succsim_0 = \succsim_t$, we thus have

$$(x,t) \left\{ \begin{array}{c} \succ \\ \sim \end{array} \right\} (y,s) \quad \text{iff} \quad U(x) - U(y) \left\{ \begin{array}{c} > \\ = \end{array} \right\} U(\chi_{s,t}(y)) - U(y).$$

But if $a := u(y)$, then by (10),

$$\begin{aligned} U(\chi_{s,t}(y)) - U(y) &= F(u(\chi_{s,t}(u^{-1}(a)))) - F(u(y)) \\ &= F(f_{s,t}(a)) - F(a) \\ &= \varphi(s,t), \end{aligned}$$

provided that $(s,t) \in \Gamma$. If, on the other hand, $(s,t) \notin \Gamma$, then $\chi_{s,t} = Id_X$, so

$$U(\chi_{s,t}(y)) - U(y) = 0 = \varphi(s,t).$$

Combining these observations, we conclude that

$$(x,t) \left\{ \begin{array}{c} \succ \\ \sim \end{array} \right\} (y,s) \quad \text{iff} \quad U(x) \left\{ \begin{array}{c} > \\ = \end{array} \right\} U(y) + \varphi(s,t),$$

which establishes (3).

In view of Claim 8, it remains only to establish property (ii) for φ. To this end, fix any $s \in T$ and take any $r, t \in T$ with $r \leq t$. Since $(\chi_{s,t}(y), t) \sim (y,s)$, (A2) implies that $(\chi_{s,t}(y), r) \succsim (y,s)$, so by (3),

$$\varphi(s,t) = U(\chi_{s,t}(y)) - U(y) \geq (U(y) + \varphi(s,r)) - U(y) = \varphi(s,r).$$

Thus $\varphi(s,\cdot)$ is increasing for all $s \in T$. By Claim 8, it follows from this that $\varphi(\cdot,t)$ is decreasing for all $t \in T$.

[$\Leftarrow$] Assume that $\succsim$ is a binary relation on $X \times T$ such that there exist a $U \in Hom(X,\mathbb{R})$ and a $\varphi : T^2 \to \mathbb{R}$ such that (3) holds for all (x,t) and (y,s) in $X \times T$, and φ satisfies the properties (ii)-(iii) asserted in Theorem 1. It is obvious that $\succsim$ is complete. To see that it is continuous, fix any $(x,t) \in X \times T$ and let $A := \{(\omega,s) \in X \times T : (x,t) \succsim (\omega,s)\}$. Let (ω_n, s_n) be a sequence in A such that $(\omega_n, s_n) \to (\omega, s) \in X \times T$. Since the topology on T is discrete, there exists an $n_0 \in \mathbb{N}$ such that $s_n = s$ for all $n \geq n_0$. It then follows from the continuity of U that $(\omega, s) \in A$, which proves that $\succsim$ is lower semicontinuous. Upper semicontinuity of $\succsim$ is verified analogously. On the other hand, by property (iii), we have $\varphi(t,t) = 0$, so by (3), U represents $\succsim_0 = \succsim_t$ for all $t \in T$. Moreover, since U is continuous and injective, $\succsim_0$ is a continuous linear order on X. Finally, let O be a nonempty subset of X and $x \in O$. Since U is an open map, $U(O)$ is an open set in $\mathbb{R}$ with $U(x) \in U(O)$, so there exist real numbers a_x and b_x with $b_x > U(x) > a_x$ and $(a_x, b_x) \subseteq U(O)$. Let $y_x := U^{-1}(a_x)$ and $z_x := U^{-1}(b_x)$. Then $y_x, z_x \in O$, because U is injective, and $z_x \succ_0 x \succ_0 y_x$, because U represents $\succsim_0$. Furthermore, if $\omega \in X$ and $z_x \succ_0 \omega \succ_0 y_x$, then $U(\omega) \in U(O)$, so $\omega \in O$ since U is injective. This establishes that $\succsim_0$ is completely continuous, and we may thus conclude that $\succsim$ is a one-dimensional time preference.

While that $\succsim$ satisfies (A1) is an immediate consequence of (3) and surjectivity of U, that it satisfies (A2) follows from (3), property (ii) of φ, and the fact that U represents $\succsim_0$. Similarly, $\succsim$ satisfies (A3) because (3) holds, and U represents $\succsim_0$. One can also easily show that (B1) follows from (3) and property (iii) of φ. To establish (B2) and (B3), we need the following observation.

Claim 9. For any $x, y \in X$ and $n = 2, 3, ...$, y is $\sim$-reached from x through $(s_i, t_i)_{i=1}^n \in T^{2n}$ if and only if

$$U(x) = U(y) + \sum_{i=1}^{n} \varphi(s_i, t_i). \tag{11}$$

Proof. If y is $\sim$-reached from x through $(s_i, t_i)_{i=1}^n$, then there exist an $n \in \{2, 3, ...\}$, $(s_i, t_i)_{i=1}^n \in T^{2n}$, and $y_1, ..., y_{n-1} \in X$ such that

$$U(x) = U(y_1) + \varphi(s_1, t_1), \quad U(y_1) = U(y_2) + \varphi(s_2, t_2), \quad ..., \quad U(y_{n-1}) = U(y) + \varphi(s_n, t_n), \tag{12}$$

so (11) follows by successive substitutions. Conversely, assume that (11) holds for some $(s_i, t_i)_{i=1}^n \in T^{2n}$. Define $y_1 := U^{-1}(U(x) - \varphi(s_1, t_1))$, and provided that $n \geq 3$,

$$y_i := U^{-1}(U(y_{i-1}) - \varphi(s_i, t_i)), \qquad i = 2, ..., n-1.$$

It follows from the definitions that all but the last equations in (12) hold. Moreover, these equations entail that

$$U(x) = U(y_{n-1}) + \sum_{i=1}^{n-1} \varphi(s_i, t_i),$$

and combining this with (11), we get $U(y_{n-1}) = U(y) + \varphi(s_n, t_n)$. Thus all equations in (12) hold, so we may conclude that y is $\sim$-reached from x through $(s_i, t_i)_{i=1}^n$. $\square$

An immediate implication of this claim is that an $x \in X$ is $\sim$-reached from x through $(s_i, t_i)_{i=1}^n$ if and only if $\sum_{i=1}^n \varphi(s_i, t_i) = 0$, and it follows that $\succsim$ satisfies (B2). Moreover, we have

Claim 10. Either $O^{\sim}(x)$ is a dense set for any $x \in X$, or $O^{\sim}(x)$ consists only of isolated points for any $x \in X$.

Proof. Let $S := \{\varphi(s, t) : (s, t) \in T^2\}$, and denote by $\langle S \rangle$ the additive subgroup of $\mathbb{R}$ that is generated by S. By Claim 9, we have, for any $x, y \in X$,

$$y \in O^{\sim}(x) \quad \text{iff} \quad \text{(11) holds for some } (s_i, t_i)_{i=1}^n \in T^{2n} \text{ and } n = 2, 3... \ .$$

It follows readily from this observation that

$$O^{\sim}(x) = U^{-1}(U(x) + \langle S \rangle), \qquad x \in X.$$

By Lemma 1, either $\langle S \rangle = \theta\mathbb{Z}$ for some $\theta \in \mathbb{R}$ or $\langle S \rangle$ is a dense subset of $\mathbb{R}$. Since U is a homeomorphism, Claim 9 entails that in the former case $O^{\sim}(x)$ is a set of isolated points for any $x \in X$, and in the latter case $O^{\sim}(x)$ is dense in X for any $x \in X$. $\square$

Claim 10 shows that $\succsim$ satisfies (B3), and hence the proof of Theorem 1 is now complete.

5.3 Proof of Proposition 1

Let $\succsim$ be a one-dimensional time preference on $X \times T$, which is represented by both (U, φ) and (V, ϕ). Clearly, both U and V are homeomorphisms on X that represent $\succsim_0$, so there exists an $f \in$ $Hom(X)$ such that $V = f \circ U$. Applying Claim 9 of the proof of Theorem 1, therefore, we have

$$U(x) = U(y) + \sum_{i=1}^{n} \varphi(s_i, t_i) \quad \text{and} \quad f(U(x)) = f(U(y)) + \sum_{i=1}^{n} \phi(s_i, t_i)$$

whenever y is $\sim$-reached from x through $(s_i, t_i)_{i=1}^n \in T^{2n}$. Now let $y := U^{-1}(0)$, and observe that, for any $(s_i, t_i)_{i=1}^n \in T^{2n}$, there exists an x such that y is $\sim$-reached from x through $(s_i, t_i)_{i=1}^n \in T^{2n}$. (This, again, follows from Claim 9 of the proof of Theorem 1.) Thus, we must have $U(x) = \sum^n \varphi(s_i, t_i)$ and $f(U(x)) = f(0) + \sum^n \phi(s_i, t_i)$, so

$$f\left(\sum_{i=1}^{n} \varphi(s_i, t_i)\right) = f(0) + \sum_{i=1}^{n} \phi(s_i, t_i) \quad \text{for all } (s_i, t_i)_{i=1}^n \in T^{2n}. \tag{13}$$

In particular,

$$f(\varphi(s, t)) = f(0) + \phi(s, t) \quad \text{for all } (s, t) \in T^2. \tag{14}$$

Combining this with (13), we obtain

$$f\left(\sum_{i=1}^{n} \varphi(s_i, t_i)\right) - f(0) = \sum_{i=1}^{n} \left(f(\varphi(s_i, t_i)) - f(0)\right) \quad \text{for all } (s_i, t_i)_{i=1}^n \in T^{2n}.$$

Letting $g := f - f(0)$, we can rewrite this fact as

$$g\left(\sum_{i=1}^{n} \alpha_i\right) = \sum_{i=1}^{n} g(\alpha_i) \quad \text{for all } \alpha_1, ..., \alpha_n \in S,$$

where $S := \{\varphi(s, t) : (s, t) \in T^2\}$. Thus, if $\alpha, \beta \in \langle S \rangle$, then $\alpha = \sum^n \alpha_i$ and $\beta = \sum^n \beta_i$ for some $n, m \in \mathbb{N}$ and $\alpha_1, ..., \alpha_n, \beta_1, ..., \beta_m \in S$, so we get

$$g(\alpha + \beta) = g\left(\sum_{i=1}^{n} \alpha_i + \sum_{i=1}^{m} \beta_i\right) = \sum_{i=1}^{n} g(\alpha_i) + \sum_{i=1}^{m} g(\beta_i) = g(\alpha) + g(\beta).$$

That is, g satisfies the Cauchy functional equation on the additive group $\langle S \rangle$. Then, given that g is strictly increasing, there must exist an $a > 0$ such that $g(\alpha) = a\alpha$ for all $\alpha \in \langle S \rangle$. But, by hypothesis, α/β is an irrational number for some $\alpha, \beta \in \langle S \rangle$. It is easy to see that this implies that $\langle S \rangle \neq \theta\mathbb{Z}$ for any real number θ. Thus, by Lemma 1, $\langle S \rangle$ is a dense subset of $\mathbb{R}$. Since g is continuous, it follows that $g(\alpha) = a\alpha$ for all $\alpha \in \mathbb{R}$. Letting $b := -f(0)$, therefore, we obtain $f(\alpha) = a\alpha + b$ for all $\alpha \in \mathbb{R}$, which means that $V = aU + b$. Moreover, by (14), we have $a\varphi = \phi$, and the proof is complete.

5.4 Proof of Theorem 2

[$\Rightarrow$] Let $\succsim$ be a strongly monotone time preference on $\mathbb{R}^d_{++} \times T$ that satisfies the properties (A1)-(A3) and (B1)-(B3) (with $X = \mathbb{R}^d_{++}$). Let $\mathbf{1}_d := (1, ..., 1) \in \mathbb{R}^d$ and define

$$\triangle_d := \{\alpha \mathbf{1}_d : \alpha > 0\},$$

which is a connected and separable subspace of $\mathbb{R}^d_{++}$. Finally, define the binary relation $\succsim^*$ on $\triangle_d \times T$ by

$$(x,t) \succsim^* (y,s) \quad \text{iff} \quad (x,t) \succsim (y,s),$$

that is, $\succsim^*$ is the restriction of $\succsim$ to $\triangle_d \times T$. It is obvious that $\succsim^*$ is a time preference on $\triangle_d \times T$ that satisfies (A2), (A3) and (B1) (with $X = \triangle_d$). Moreover, by strong monotonicity of $\succsim$, it is a one-dimensional time preference. We will show below that $\succsim^*$ also satisfies (A1), (B2) and (B3).

By using the strong monotonicity and continuity of $\succsim$ a standard argument shows that for every $x \in \mathbb{R}^d_{++}$ there exists a unique $\alpha_x > 0$ such that $x \sim_0 \alpha_x \mathbf{1}_d$. We define $\sigma : \mathbb{R}^d_{++} \to \triangle_d$ by $\sigma(x) := \alpha_x \mathbf{1}_d$. Since $\succsim_0$ is continuous, a routine argument establishes that σ is a continuous surjection. Moreover, σ is an open map, because for any $x \in \mathbb{R}^d_{++}$ and $\varepsilon > 0$ such that the ε-ball $N_\varepsilon(x)$ around x is contained in $\mathbb{R}^d_{++}$, we have

$$\sigma(N_\varepsilon(x)) = \left(\sigma(x) - \frac{\varepsilon}{\sqrt{d}}, \sigma(x) + \frac{\varepsilon}{\sqrt{d}}\right) \subseteq \triangle_d \tag{15}$$

by strong monotonicity of $\succsim_0$ and continuity of σ.

Now since $\succsim$ satisfies (A1), for any $x \in \triangle_d$ and $s, t \in T$, there exist $y, z \in \mathbb{R}^d_{++}$ such that

$$(\sigma(z), s) \sim (z, s) \succsim (x,t) \succsim (y,s) \sim (\sigma(y), s),$$

so by (A3) we have $(\sigma(z), s) \succsim (x,t) \succsim (\sigma(y), s)$, that is, $\succsim^*$ satisfies (A1). That $\succsim^*$ satisfies (B2) is verified by a similar argument. Finally, to see that $\succsim^*$ satisfies (B3), pick any $x \in \mathbb{R}^d_{++}$ such that $O^\sim(x)$ is either somewhere dense or contains an isolated point. If $O^\sim(x)$ has an isolated point, say y, then there exists an $\varepsilon > 0$ such that $N_\varepsilon(y) \cap O^\sim(x) = \{y\}$. Now if $z \in \sigma(N_\varepsilon(y)) \cap O^{\sim^*}(\sigma(x))$ (where $\sim^*$ is the symmetric part of $\succsim^*$), then $z = \sigma(w)$ for some $w \in N_\varepsilon(y)$, and z is $\sim^*$-reached from $\sigma(x)$. But $z = \sigma(w)$ implies that w is $\sim$-reached from z, so since x is obviously $\sim$-reached from $\sigma(x)$, we find (by using (A3)) that $w \in N_\varepsilon(y) \cap O^{\sim}(x)$, that is, $w = y$, so $z = \sigma(y)$. But one can use (A3) to verify that

$$\sigma(O^{\sim}(x)) = O^{\sim^*}(\sigma(x)), \tag{16}$$

which implies that $\sigma(y)$ indeed belongs to $\sigma(N_\varepsilon(y)) \cap O^{\sim^*}(\sigma(x))$. Therefore, we have $\sigma(N_\varepsilon(y)) \cap O^{\sim^*}(\sigma(x)) = \{\sigma(y)\}$, and since $\sigma(N_\varepsilon(y))$ is an open set (for σ is an open map), this shows that $\sigma(y)$ is an isolated point of $O^{\sim^*}(\sigma(x))$. If, on the other hand, $O^{\sim}(x)$ is somewhere dense in $\mathbb{R}^d_{++}$, then there exists a nonempty open set O in $\mathbb{R}^d_{++}$ such that $O \subseteq cl(O^{\sim}(x))$. But since σ is an open map, $\sigma(O)$ is open in $\triangle_d$, and by continuity of σ and (16), we have

$$\sigma(O) \subseteq \sigma(cl(O^{\sim}(x))) \subseteq cl(\sigma(O^{\sim}(x))) = cl(O^{\sim^*}(\sigma(x))),$$

which proves that $O^{\sim^*}(\sigma(x))$ is somewhere dense in $\triangle_d$. Consequently, $\succsim^*$ satisfies (B3).

We may now apply Theorem 1 to $\succsim^*$ to find a $U^* \in Hom(\triangle_d, \mathbb{R})$ and $\varphi : T^2 \to \mathbb{R}$ such that (3) holds for all (x,t) and (y,s) in $\triangle_d \times T$, and φ satisfies the properties (ii)-(iii) asserted in Theorem 1. We define $U : \mathbb{R}^d_{++} \to \mathbb{R}$ by $U(x) := U^*(\sigma(x))$. Since σ is continuous, open and surjective, so is U. Moreover, by (A3) we have

$$(x,t)\left\{\begin{array}{c}\succ\\ \\ \sim\end{array}\right\}(y,s) \quad \text{iff} \quad (\sigma(x),t)\left\{\begin{array}{c}\succ^*\\ \\ \sim^*\end{array}\right\}(\sigma(y),s) \quad \text{iff} \quad U^*(\sigma(x))\left\{\begin{array}{c}>\\ \\ =\end{array}\right\}U^*(\sigma(y))+\varphi(s,t)$$

and it follows that (3) holds for all $(x,t),(y,s) \in \mathbb{R}^d_{++} \times T$. Since U represents the strongly monotone preorder $\succsim_0$, it must clearly be strictly increasing.

Conversely, assume that $\succsim$ is a binary relation on $\mathbb{R}^d_{++} \times T$ such that there exist a $U : \mathbb{R}^d_{++} \to \mathbb{R}$ and a $\varphi : T^2 \to \mathbb{R}$ such that (3) holds for all (x,t) and (y,s) in $\mathbb{R}^d_{++} \times T$, and U and φ satisfy the properties (i)-(iii) envisaged in Theorem 4. By straightforward modifications of the arguments given for the "if" part of Theorem 1, one can show that $\succsim$ is a strongly monotone time preference on $\mathbb{R}^d_{++} \times T$ that satisfies (A1)-(A3) and (B1)-(B2).

To verify that it also satisfies (B3), define $\succsim^*$ as above, and notice that Claim 10 applies to $\succsim^*$, so either $O^{\sim^*}(x)$ is a dense set for any $x \in X$, or $O^{\sim^*}(x)$ consists only of isolated points for any $x \in X$. Let us consider first the former case. Take any $x \in \mathbb{R}^d_{++}$, and let $y := \sigma(x)$. To derive a contradiction, assume that $O^{\sim}(x)$ is not dense in $\mathbb{R}^d_{++}$, then we can find an $\varepsilon > 0$ and a $w \in \mathbb{R}^d_{++}$ such that $N_\varepsilon(w) \cap O^{\sim}(x) = \emptyset$. But since σ is open and $O^{\sim *}(x)$ is dense in $\triangle_d$, we have $\sigma(N_\varepsilon(w)) \cap O^{\sim^*}(y) \neq \emptyset$, that is, there exists a $z \in N_\varepsilon(w)$ such that $\sigma(z)$ can be $\sim^*$-reached from y. But y can be $\sim$-reached from x, and z can be $\sim$-reached from $\sigma(z)$, so z can be $\sim$-reached from x, that is, $z \in O^{\sim}(x)$, a contradiction. Thus in this case $O^{\sim}(x)$ must be dense in $\mathbb{R}^d_{++}$.

Assume now that $O^{\sim^*}(x)$ consists only of isolated points for any $x \in X$. To derive a contradiction, suppose that there is a non-isolated point z in $O^{\sim}(x)$, so $N_\varepsilon(z) \cap O^{\sim}(x) \neq \emptyset$ for all $\varepsilon > 0$.

But we have $\sigma(N_\varepsilon(z)) \cap O^{\sim^*}(\sigma(x)) = \emptyset$ for some $\varepsilon > 0$ since $\sigma(N_\varepsilon(z))$ is an open subset of $\triangle_d$ that contains $\sigma(z)$, $\sigma(z) \in O^{\sim^*}(\sigma(x))$ by (16), and $O^{\sim^*}(\sigma(x))$ is a set of isolated points. Then

$$\emptyset \neq \sigma\left(N_\varepsilon(z) \cap O^{\sim}(x)\right) \subseteq \sigma(N_\varepsilon(z)) \cap O^{\sim^*}(\sigma(x)) = \emptyset,$$

which is absurd. Thus all points of $O^{\sim}(x)$ are in fact isolated, and the proof is complete.

5.5 Proof of Theorem 3

The claim is trivial for finite T, so we assume throughout that $T = \mathbb{Z}_+$, and recall that

$$T_t := \{t, t+1, ...\}, \qquad t = 0, 1, ... \,.$$

Since $(\succsim, x_*, (x_t)_{t \in T})$ is regular, we can find by Theorem 1 two functions $U : X \to \mathbb{R}$ and $\varphi : T^2 \to \mathbb{R}$ that satisfy (3) and the properties (i)-(iii) listed in Theorem 1. Moreover, since the problem is non-trivial, we must have $\varphi(0, t) > 0$ for some $t \in T$. Let

$$t^* := \min\{t \in T : \varphi(0, t) > 0\} \quad \text{ and } \quad \xi := \varphi(0, t^*).$$

(Clearly, $\varphi(0, t) > 0$ for all $t \geq t^*$.) We define the sequence $(\tau_1, \tau_2, ...)$ of nonnegative integers recursively as follows:

$$\tau_1 := \min\left\{ t \in T : \sup_{s \in T} U(x_s) - U(x_t) < \xi \right\}$$

and

$$\tau_{k+1} := \min\left\{ t \in T_{\tau_k + 1} : \sup_{s \in T_{\tau_k + 1}} U(x_s) - U(x_t) < \xi \right\}, \qquad k = 1, 2, ...$$

This sequence is well-defined, because U is continuous and $\{x_1, x_2, ...\}$ has compact closure.

For any $\tau \in T$, let $\mathfrak{g}^\tau$ denote the subgame of $\mathfrak{g}(\succsim, x_*, (x_t)_{t \in T})$ that starts at date τ, and for any $t \in T_\tau$, let $\mathfrak{g}^\tau_t$ denote the truncation of $\mathfrak{g}^\tau$ at date t. (That is, $\mathfrak{g}^\tau_t$ ends with person t choosing x_t.) We note that if we can show that $\mathfrak{g}^\tau$ has a subgame perfect equilibrium for any $\tau \in T$, then it will follow that so does $\mathfrak{g}(\succsim, x_*, (x_t)_{t \in T})$ by a straightforward backward induction argument. So, to prove the first claim in Theorem 3, it is enough to show that $\mathfrak{g}^{\tau_1}$ has a subgame perfect equilibrium.

Define

$$S := \{x_1, W\} \times \{x_2, W\} \times \cdots,$$

where W denotes the action of "waiting" for an extra period. For each $t \geq \tau_1 + t^*$, let $(s^t_{\tau_1}, ..., s^t_t)$ be a subgame perfect equilibrium of $\mathfrak{g}^{\tau_1}_t$ (where $s^t_i \in \{x_i, W\}$ denotes the equilibrum strategy of person $i = \tau_1, ..., t$), and let

$$s^t := (s^t_{\tau_1}, ..., s^t_t, W, W...), \qquad t \in T_{\tau_1 + t^*}.$$

Finally, define $R := \{s^t : t \in T_{\tau_1 + t^*}\}$ which is a subset of S. We now construct an equilibrium for $\mathfrak{g}^{\tau_1}$ by the diagonal argument. There must be infinitely many s^t in R in which $s^t_{\tau_1}$ equals a fixed action in $\{x_{\tau_1}, W\}$. Let a_{τ_1} be any such action, and let $R_1 := \{s^t \in R : s^t_{\tau_1} = a_{\tau_1}\}$. But there must be infinitely many elements s^t in R_1 in which $s^t_{\tau_1+1}$ equals a fixed action in $\{x_{\tau_1+1}, W\}$. Let a_{τ_1+1} be any such action, and let $R_2 := \{s^t \in R_1 : s^t_{\tau_1+1} = a_{\tau_1+1}\}$. There must be infinitely many s^t in R_2 in which $s^t_{\tau_1+2}$ equals a fixed action, and so on. Proceeding this way inductively, therefore, we obtain an action profile $a := (a_{\tau_1}, a_{\tau_1+1}, ...)$. We claim that a is a subgame perfect equilibrium for $\mathfrak{g}^{\tau_1}$.

Fix any $k \in \mathbb{N}$. We will first verify that a_{τ_k} is a best response to $(a_{\tau_1}, ..., a_{\tau_k - 1}, a_{\tau_k + 1}, ...)$. Observe first that, by construction, a_{τ_k} is a best response to $(a_{\tau_k+1}, ..., a_{\tau_k+t^*+i})$ in the game $\mathfrak{g}^{\tau_k}_{\tau_k+t^*+i}$ for infinitely many $i \in \mathbb{Z}_+$. But, a_{τ_k} is a best response to $(a_{\tau_1}, ..., a_{\tau_k-1}, a_{\tau_k+1}, ...)$ if, and only if, a_{τ_k} is a best response to $(a_{\tau_k+1}, ..., a_{\tau_k+t^*+i})$ in the game $\mathfrak{g}^{\tau_k}_{\tau_k+t^*+i}$ for some $i \in \mathbb{Z}_+$. For, the game $\mathfrak{g}^{\tau_k}$ must end in equilibrium before the period $\tau_k + t^* + 1$ is reached since, for any $t > \tau_k + t^*$, we have

$$\begin{aligned} U(x_{\tau_k}) &> \sup_{s \in T_{\tau_k+1}} U(x_s) - \xi \\ &= \sup_{s \in T_{\tau_k+1}} U(x_s) + \varphi(t^*, 0) \\ &\geq U(x_t) + \varphi(t^*, 0) \\ &\geq U(x_t) + \varphi(t - \tau_k, 0) \end{aligned}$$

where the last inequality follows from the fact that $\varphi(\cdot^*, 0)$ is decreasing. Thus waiting cannot be a best response to any action profile $(s_{\tau_k+1}, s_{\tau_k+2}, ...)$ that does not end the game before $\tau_k + t^* + 1$ is reached. It follows that a_{τ_k} is a best response to $(a_{\tau_1}, ..., a_{\tau_k-1}, a_{\tau_k+1}, ...)$, $k = 1, 2, ...$.

Now fix again any $k \in \mathbb{N}$, and assume $\tau_k + 1 \leq \tau_{k+1} - 1$ (so there is at least one person between persons τ_k and τ_{k+1}). Let $t \in \{\tau_k + 1, ..., \tau_{k+1} - 1\}$. Observe that, by construction, a_{τ_k} is a best response to $(a_{\tau_{k+1}}, ..., a_{\tau_k+t^*+i})$ in the game $\mathfrak{g}^{\tau_k}_{\tau_k+t^*+i}$ for infinitely many $i \in \mathbb{Z}_+$. But, a_t is a best response to $(a_{\tau_1}, ..., a_{t-1}, a_{t+1}, ...)$ if, and only if, a_t is a best response to $(a_t, ..., a_{\tau_k+t^*+i})$ in the game $\mathfrak{g}^{\tau_k}_{\tau_k+t^*+i}$ for some $i \in \mathbb{Z}_+$.This is because the game $\mathfrak{g}^t$ must end in equilibrium before $\tau_{k+1} + t^* + 1$ is reached (as shown in the previous paragraph, person τ_{k+1} will make sure of this). It follows that a_t is a best response to $(a_{\tau_1}, ..., a_{t-1}, a_{t+1}, ...)$, and the proof is complete.

5.6 Proof of Proposition 2

Since $\{x_0, x_1, ...\}$ has compact closure and U is continuous, $U(cl\{x_t : t = 0, 1, ...\})$ is a compact subset of $\mathbb{R}$. But since U is a homeomorphism, $cl\{U(x_t) : t = 0, 1, ...\}) = U(cl\{x_t : t = 0, 1, ...\})$, so we find that $\{U(x_t) : t = 0, 1, ...\}$ is relatively compact. Consequently, there exists a $t \in \mathbb{Z}_+$ such that

$$\sup_{s \in \mathbb{Z}_+} U(x_s) - U(x_t) < \varphi(0, 1).$$

Since $(x_0, x_1, ...)$ is an increasing sequence, this means that

$$\sup_{s \in \mathbb{Z}_+} U(x_s) - U(x_r) < \varphi(0, 1), \qquad r = t, t+1, ... \, .$$

In turn, since $\varphi(0, \cdot)$ is increasing, we find

$$U(x_{r+k}) - U(x_r) < \varphi(0, r) \quad \text{ for all } r = t, t+1, ... \text{ and } k = 1, 2, ... \, . \tag{17}$$

It follows that, in equilibrium, every player $r \geq t$ must choose to end the game $\mathfrak{g}(\succsim, 0, (x_t))$ when it is her turn to move.

Now let $\mathcal{T}$ be the set of all $t \in \mathbb{Z}_+$ such that if $(s_1, s_2, ...)$ is a subgame perfect equilibrium of $\mathfrak{g}(\succsim, 0, (x_t))$, then $s_t = s_{t+1} = \cdots = W$ (where, again, W denotes the action of "waiting" for an extra period). By the argument given in the previous paragraph, $\mathcal{T} \neq \emptyset$, so we may define $\tau := \min \mathcal{T}$. Clearly, Proposition 2 will be established if we can show that $t^*(U, e^{-\varphi(0,1)}) = \tau$.

Let $t^* := t^*(U, e^{-\varphi(0,1)})$, and suppose that $\tau > t^*$. Note that, by concavity of U, we have

$$\frac{U(x_{t+1}) - U(x_t)}{x_{t+1} - x_t} \leq \frac{U(x_t) - U(x_{t-1})}{x_t - x_{t-1}}, \qquad t = 1, 2, ...,$$

so, since $x_{t+1} - x_t < x_t - x_{t-1}$ for each $t \in \mathbb{N}$, we have

$$U(x_{t+1}) - U(x_t) \leq U(x_t) - U(x_{t-1}), \qquad t = 1, 2, ... \, .$$

Therefore, since $\tau - 1 \geq t^*$, by the definition of t^* we must have

$$U(x_\tau) - U(x_{\tau-1}) \leq U(x_{t^*+1}) - U(x_{t^*}) < \varphi(0, 1). \tag{18}$$

Given that player τ ends the game $\mathfrak{g}(\succsim, 0, (x_t))$ in any equilibrium, it follows from (18) that player $\tau - 1$ must end the game in any equilibrium as well. But this is impossible by the definition of τ. Thus, we may conclude, $\tau \leq t^*$. Assume now that $\tau < t^*$. By definition of τ, in this case players $t^* - 1$ and t^* end the game in any equilibrium. Clearly, this is possible only if $U(x_{t^*}) < U(x_{t^*-1}) + \varphi(0, 1)$, which is, however, impossible due to the definition of t^*. Thus, $\tau = t^*$, and the proof is complete.

5.7 Proof of Proposition 3

As in the standard analysis of the Rubinstein bargaining game, for any pair of agreements $(x^*, y^*) \in A^2$ such that player 1 is indifferent between $(x^*, 1)$ and $(y^*, 0)$, and player 2 is indifferent between

$(x^*, 0)$ and $(y^*, 1)$, there is a subgame perfect equilibrium of the induced infinite-player game in which all "selves" of player 1 always propose x^* and accept a proposal $x \in A$ iff $U(x_1) > U(y_1^*)$, while all "selves" of player 2 always propose y^* and accept a proposal $x \in A$ iff $U(x_2) > U(x_2^*)$. Moreover, the standard Shaked-Sutton argument applies to the present setting without alteration, and hence all one needs to verify here is that there exists a unique $(x_1^*, y_1^*) \in [0,1]^2$ such that

$$U(x_1^*) = U(y_1^*) + \varphi(0,1) \quad \text{and} \quad U(1 - y_1^*) = U(1 - x_1^*) + \varphi(0,1). \tag{19}$$

Let f be the self-map on $[-\infty, U(1)]$ defined by $f(\omega) := U(1 - U^{-1}(\omega))$. It is easy to check that (19) implies $f(U(x_1^*)) = f(U(x_1^*) - \alpha) - \alpha$, where $\alpha := \varphi(0,1) > 0$. Define $H : [-\infty, U(1)] \to [-\infty, \alpha]$ by $H(\omega) := f(\omega) - f(\omega - \alpha) - \alpha$. Since f is concave and H is the difference between two concave functions, H and F are simultaneously differentiable almost everywhere in $[-\infty, U(1)]$, and hence, since f is strictly concave, we have $H'(\omega) = f'(\omega) - f'(\omega - \alpha) < 0$ almost everywhere in $[-\infty, U(1)]$. Since H is continuous, it follows that it is strictly decreasing on $[-\infty, U(1)]$. Since $H(-\infty) = \alpha > 0$ and $H(U(1)) = -\infty$, therefore, there exists a unique zero of H, say ω^*, in $(-\infty, U(1))$. Consequently, there is a unique $(x_1^*, y_1^*) \in [0,1]^2$ that satisfies (19), and we have $x_1^* := U^{-1}(\omega^*)$ and $y_1^* := U^{-1}(U^{-1}(\omega^*) - \alpha)$.

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