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Expected Revenue from Open and Sealed Bid Auctions

John G. Riley

In his preceding piece, Paul Milgrom has neatly laid out many of the core results of the recent literature on the theory of competitive bidding. My immediate goal here is to provide the simplest possible derivation of three basic revenue ranking theorems. Each of these compares the expected revenue obtained from open bidding, as in the typical art auction, with the expected revenue obtained when bids are sealed and the winner pays his bid.

The first of the ranking theorems is the revenue equivalence theorem which establishes conditions under which these two seemingly very different auctions generate the same expected revenue for the seller. The other two theorems establish the effect of relaxing one of the key assumptions. When buyers are risk averse, the seller gains if buyers submit sealed bids and the winner pays his bid. On the other hand, if valuations of different buyers are positively correlated, rather than unrelated, the seller extracts more revenue in an open ascending bid auction.

In reducing these theorems to their basic elements, my deeper goal is to provide the reader with a better intuitive understanding of the results.

Revenue Equivalence

Over 25 years ago, William Vickrey (1961) showed that, under strong simplifying assumptions, competitive bidding in an open auction (where bidders call out ever

■ *John G. Riley is Professor of Economics and Chairman of the Department of Economics at the University of California at Los Angeles, Los Angeles, California.*

higher bids until only the highest bidder remains) would produce the same expected revenue as in a sealed high bid auction (where bidders submit one sealed bid and the highest bid wins). Essentially, Vickrey assumed the following:[1] there are two risk neutral bidders for an object, Alfie and Berta. Alfie and Berta both know their own valuations of the item but neither is certain how the other values it. The set of possible valuations they have for the object is $(V_1, V_2, \ldots, V_T)$ where $V_T > V_{T-1} > \cdots > V_1 = 0$. They give each of the possible valuations a probability, representing how likely that valuation is to be true.

Most important to the analysis, it is assumed that each buyer has the same beliefs about his opponent and that this is common knowledge. It is then possible to define f_t to be the commonly believed probability that an opponent's valuation is V_t. Of course the probabilities must sum to one. To simplify the exposition, F_t is also defined to be the cumulative probability that an opponent's valuation is V_t or less, that is

$$F_t = \sum_{j=1}^{t} f_j = \text{Prob}\{\text{opponent's valuation is } V_t \text{ or less}\}$$

I begin by characterizing the equilibrium of the open ascending bid auction and then turn to the sealed high bid auction. As a preliminary note that, in either auction, a buyer with a valuation $V_1 = 0$ has no incentive to participate. His expected return is therefore zero.

With open bidding, any buyer with a positive valuation has an incentive to remain in the auction as long as the bidding is no greater than his valuation. Suppose then that both Alfie and Berta behave in this way. If Alfie has a valuation V_t and Berta has a lower valuation V_j the bidding will rise to V_j, at which point Berta will exit. Alfie's gain is then the difference between his valuation and his bid, that is, $V_t - V_j$. Since this outcome occurs with probability f_j, Alfie's expected gain U_t can be written as follows.

$$(1) \qquad U_t = f_1(V_t - V_1) + f_2(V_t - V_2) + \cdots + f_t(V_t - V_t) = \sum_{j=1}^{t} f_j(V_t - V_j)$$

It should be noted that if Berta also has a valuation V_t, the bidding stops at V_t and one of the bidders is selected as the winner. Since the winner pays the full valuation, the net return is zero so the buyers are indifferent between winning and losing.

[1] Vickrey actually considered the case in which valuations were continuously and uniformly distributed over the unit interval.

Using equation 1, the expected return to an individual with valuation V_{t+1} can be written down immediately. It is

$$U_{t+1} = \sum_{j=1}^{t+1} f_j(V_{t+1} - V_j)$$

$$= \sum_{j=1}^{t} f_j(V_{t+1} - V_j), \quad \text{since the } t + 1\text{th term is zero}$$

$$= \sum_{j=1}^{t} f_j(V_{t+1} - V_t) + \sum_{j=1}^{t} f_j(V_t - V_j)$$

$$= \sum_{j=1}^{t} f_j(V_{t+1} - V_t) + U_t \quad \text{using (1).}$$

Finally, since $F_t = f_1 + f_2 + \cdots + f_t$,

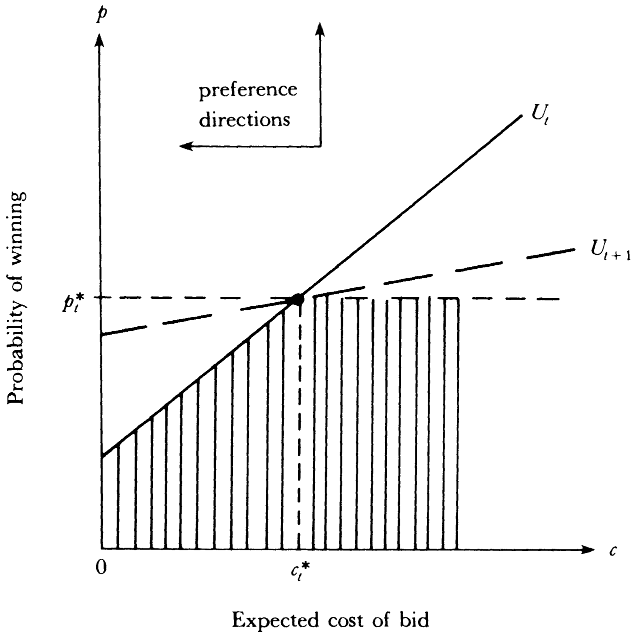
$$U_{t+1} = F_t(V_{t+1} - V_t) + U_t. \tag{2}$$

The expected return for a buyer with valuation V_1 is zero and so $U_1 = 0$. Therefore this simple difference equation completely describes equilibrium expected returns for all the possible buyer valuations in terms of the underlying data (valuations and probabilities). Below I shall show that this same difference equation describes the equilibrium expected returns in the sealed high bid auction. Thus the two auctions are equivalent from the viewpoint of the buyers. Moreover, in both auctions it is the individual with the higher valuation who wins. Therefore the social value generated by the two auctions is the same. But the expected social value of the auction is the sum of the expected gain to the buyers and the expected seller revenue. Therefore, establishing the equivalence of the two auctions for the buyers immediately implies revenue equivalence.

After establishing equivalence, I then show that the two auctions should be seen as limiting special cases of a class of auctions, all of which generate the same equilibrium expected revenue. It is this more general result which is the key to understanding the revenue equivalence of the two common auctions.

The first step in the comparison of the different auctions is to show that in a sealed bid auction it is also the case that the greater an individual's valuation, the higher he will bid. Intuitively, the greater a person's valuation, the more he is willing to pay to increase his probability of winning. More precisely, associated with any bid is a probability of winning, p, and an expected payment c. A buyer with valuation V_t

Figure 1
Buyer Indifference Curves



then has an expected return of

$$\text{(3)} \quad \underset{\text{expected return}}{U_t} = \underset{\begin{bmatrix}\text{probability}\\ \text{of winning}\end{bmatrix}}{p} \underset{[\text{valuation}]}{V_t} - \underset{\begin{bmatrix}\text{expected}\\ \text{cost}\end{bmatrix}}{c}$$

A buyer can then be viewed as having linear preferences in (c, p) space. Suppose that a buyer with valuation V_t has a best bid b_t which yields a win probability of p_t^* and an expected payment of c_t^*. This point is depicted in (c, p) space in Figure 1. The heavy line is the indifference line of this buyer through his optimum (c_t^*, p_t^*). From equation 3, the steepness of this indifference line is $dp/dc|_{U_t} = -\partial U_t/\partial c \partial U_t/\partial p = 1/V_t$. It follows immediately that an individual with a higher valuation must have a flatter indifference curve and so is willing to pay more in terms of expected cost for a given increment in probability of winning. In Figure 1 the indifference line through (c_t^*, p_t^*) for type $t + 1$ is also depicted as a heavy dashed line.

Since the point (c_t^*, p_t^*) is the best alternative for a buyer with a valuation V_t, any other feasible alternative with a lower win probability must lie in the shaded region. From the figure it is clear that no point in this region will be preferred over (c_t^*, p_t^*) by any buyer with a higher valuation and hence flatter indifference curve. Therefore the best alternative for a buyer with valuation V_{t+1} must satisfy $p_{t+1}^* \geq p_t^*$. Since the probability of winning increases with a buyer's bid, it follows that a buyer's bid is (at least weakly) increasing in his valuation.

Henceforth I shall refer to this result as the monotonicity property.

There is a second important property of equilibrium bidding in the sealed high bid auction. This continuity property says that for any bids b' and b'' less than an individual's maximum bid, the probability that he will bid in the interval $[b', b'']$ is strictly positive. Moreover as this interval is made small the probability goes to zero.

Essentially, the continuity property tells us that there are no gaps or points of strictly positive probability in the distribution of bids. But if there are no points of strictly positive probability, equilibrium strategies must be *mixed*. Given the monotonicity property of equilibrium bids, this implies that if Alfie has a valuation V_2 he must bid over some interval $[0, \bar{b}_2]$, if he has a valuation V_3 he bids over some interval $[\bar{b}_2, \bar{b}_3]$ and so on.[2]

To see why there can be no mass points, suppose Berta bids 8 with a probability of 0.2 when her valuation is V_t. Since Berta has a chance of winning with a very small bid, she will never bid her full valuation since this would produce a zero expected gain. Then V_t is greater than 8. Now consider Alfie's decision. For arbitrarily small ε, he can increase his probability of winning by at least $(0.2) \times f_t$ when raising his bid from $8 - \varepsilon$ to $8 + \varepsilon$. As long as Alfie's valuation exceeds 8, he is definitely better off doing so for small enough ε. Therefore Alfie will never bid in the interval $[8 - \varepsilon, 8]$. But then Berta's bid of 8 cannot, after all, be an equilibrium bid since she could also reduce her bid below 8 without changing her probability of winning.

The argument as to why there can be no gaps in the equilibrium bid distribution is similar. Suppose to the contrary that Berta's smallest bid above 5 is 8. Then Alfie would never make a bid like 7 since he could reduce his bid to just above 5 without reducing his probability of winning. But then Berta can reduce her bid from 8 to 7 without reducing her win probability so 8 is not, after all, an equilibrium bid.

By appealing to these two properties—monotonicity and continuity—equilibrium buyer returns in the sealed high bid auction are readily determined. First of all, as we have already noted the bidding strategy of each buyer must be a *mixed* strategy. That is, for each valuation Alfie bids probabilistically. By monotonicity the set of possible bids in the mixed strategy when his valuation is V_{t+1} must be at least as great as the set of possible bids when his valuation is V_t. Let $\bar{b}_t$ be Alfie's largest possible bid when his valuation is V_t. By continuity there can be no gap in the bid distribution so $\bar{b}_t$ must also be the smallest possible bid by Alfie when his valuation is V_{t+1}.

A mixed strategy can only be an equilibrium strategy if Alfie is indifferent between all the possible bids associated with a particular valuation. Given the symmetry of the model, the goal is to characterize a symmetric equilibrium.[3] That is,

[2]It may seem a bit unrealistic for bidders to adopt mixed strategies. However, this is an implication of our simplifying assumption that each bidder has one of a finite set of possible valuations $\{V_1, V_2, \ldots, V_T\}$. As the number of valuations is made large and $V_{t+1} - V_t$ is made small, the mixed strategies approximate more and more closely a pure strategy. In the limit, with a continuous distribution of types, the equilibrium strategy is a pure strategy. That is, for each valuation V a buyer makes a bid $b(v)$ where $b(\cdot)$ is an increasing function.

[3]Given the symmetric position of the bidders it seems natural to consider an equilibrium in which a buyer's bidding strategy depends upon his type but not his identity. Actually, it can be shown that, given the assumptions made above, there are no asymmetric equilibria (Maskin and Riley, 1986).

for each valuation V_t, both Alfie and Berta bid over some interval $[\underline{b}_{t-1}, \underline{b}_t]$ according to the same mixed strategy.

Suppose Alfie's valuation is V_t. If he bids his largest possible equilibrium bid $\underline{b}_t$ he wins whenever Berta has a lower valuation. He also wins with probability 1 if Berta also has a valuation V_t.[4] Therefore his probability of winning is $F_t = f_1 + f_2 + \cdots + f_t$. His equilibrium expected gain is then

$$\bar{U}_t = F_t[V_t - \bar{b}_t] \tag{4}$$

$$\begin{bmatrix}\text{expected}\\ \text{return}\end{bmatrix} = \begin{bmatrix}\text{probability}\\ \text{of winning}\end{bmatrix}[\text{valuation} - \text{bid}].$$

Finally, as just argued, $\bar{b}_t$ is also the smallest bid by Alfie if his valuation is V_{t+1}. Since the probability of winning with such a bid is F_t, Alfie's expected return, when he has valuation V_{t+1} is $\bar{U}_{t+1} = F_t[V_{t+1} - \bar{b}_t]$. Combining this with expression 4, it follows that:

$$\bar{U}_{t+1} = F_t(V_{t+1} - V_t) + \bar{U}_t. \tag{5}$$

Of course a buyer with a zero valuation will never submit a positive bid. Therefore $\bar{U}_1 = 0$. Comparing expressions 2 and 5, it follows that equilibrium expected returns are the same in the two auctions.

As we have already argued, revenue equivalence follows immediately since the social value generated by the two auctions is the same.

Despite the logic of the above argument, it hardly qualifies as an intuitive explanation. However, there is a direct extension of the analysis which greatly strengthens the intuition behind the result. Consider the following modification of the sealed bid auction. As before each person submits a sealed bid and the high bidder is the winner. However suppose the high bidder "only" has to pay a weighted average of his bid and the second highest bid.

$$\begin{matrix}\text{winner's}\\ \text{payment}\end{matrix} = (1-\lambda)\begin{bmatrix}\text{winning}\\ \text{bid}\end{bmatrix} + \lambda\begin{bmatrix}\text{second highest}\\ \text{bid}\end{bmatrix}, \quad 0 \le \lambda \le 1.$$

With $\lambda = 0$ this is just the sealed high bid auction. With $\lambda = 1$ the high bidder wins and pays the second highest bid. As Vickrey first noted, this sealed bid auction is equivalent to the open auction. The point is simply stated. Regardless of what Berta does, Alfie is happy to win as long as Berta's bid is lower than his valuation. For then he captures some surplus. It follows that Alfie's optimal strategy is to submit a sealed bid equal to his valuation. The same argument holds for Berta. Therefore, if both behave optimally, the individual with the higher valuation wins and pays the second valuation, just as in the open auction.

Consider next an intermediate case with strictly positive weights on both high and second highest bid ($0 < \lambda < 1$). Just as argued above, associated with any bid is a

[4] Remember the mixed strategy has no points of positive probability therefore, if Alfie and Berta both have valuation V_t, Alfie's probability of winning rises towards one as he increases his bid closer and closer to $\bar{b}_t$.

probability of winning p and an expected cost c. Preferences are therefore exactly as depicted in Figure 1 and so the monotonicity property continues to hold. Moreover, the argument as to why there can be no gaps or jumps in the distribution of bids extends without change. That is, the continuity property continues to hold also.

But, as has already been shown, these two properties imply that equilibrium expected returns for buyers are given by the difference equation 5. The central insight, therefore, is that *any* auction for which the monotonicity and continuity properties hold generates the same expected return to buyers. Moreover, these two properties also imply that it is the buyer with the higher valuation who wins with probability 1. Therefore all such auctions generate the same social value and hence the same expected seller revenue.

In particular, expected revenue when $\lambda = 0$ (sealed high bid auction) is the same as when λ is close to 1 and so the auction is very close to being a sealed second bid auction. While the latter auction is a rather special case (in that it does not satisfy the continuity property) it should be seen as the limit of the family of auctions in which the weight on the second bid approaches 1.

To summarize, the key insight is that the monotonicity and continuity properties by themselves determine the difference between a buyer's expected return when he has a valuation of V_{t+1} rather than V_t. Therefore, whenever these properties hold, expected returns to buyers (and hence the seller) are the same as in the sealed high bid auction.[5]

Risk Aversion

To understand the effects of introducing risk aversion it is helpful to consider another form of open auction: the Dutch auction.[6] Instead of letting the asking price rise as in the usual open (or English) auction, the seller has prices marked on the face of a clock. The clock is started and the asking price ticks down until someone calls out. The clock then stops and the buyer pays the indicated price.

I now want to argue that equilibrium behavior in this auction is exactly as in the sealed high bid auction. The key point is that both Alfie and Berta can plan their bids prior to the moment the clock is started. That is, each can ask, "If the price ticks down to such-and-so should I call out?" But if they can plan how they will bid, they can write their bids down on a piece of paper and hand them over to the auctioneer. He will then know when the clock will stop. Therefore, instead of actually starting the clock he can simply announce when it will stop, that is, announce the winning bid. Indeed neither buyer should mind if the clock is removed, the two bids are simply

[5]The interested reader might consider the following auction in which the continuity property does not hold. With probability 0.5 a buyer's valuation is 12, otherwise it is 16. Buyers must submit a sealed bid of either 11 or 14. It is not too difficult to confirm that the equilibrium strategies of Alfie and Berta are to bid 11 if they have low valuations and 14 if they have high valuations. Expected revenue for such an auction can then be shown to exceed expected revenue in the two common auctions. For discussions of optimal auctions from the seller's viewpoint see Myerson (1981) and Riley and Samuelson (1981).

[6]This has long been the method of the sale of tulips in Amsterdam. It also used to be a common way of selling fish on New England docks.

unsealed and the item is sold to the highest bidder. Of course in so doing, the Dutch auction becomes a sealed high bid auction.

In thinking about the Dutch auction, it is easy to see how risk aversion will affect bidding. The more risk averse an individual is, the more reluctant he will be to let the clock tick down below his valuation. Therefore risk averse buyers will tend to cry out more quickly and so raise expected seller revenue.

For the standard open auction, risk aversion has no such effect. Just as in the risk neutral case, buyers will stay in the auction as long as the asking price remains below their valuations. It follows that expected seller revenue is the same as when buyers are risk neutral. Since the two auctions generate the same expected revenue under risk neutrality, it is the sealed high bid auction which generates the higher expected seller revenue when buyers are risk averse. Intuitively, the sealed high bid auction exploits risk averse buyers' greater fear of loss.[7]

Correlated Beliefs

Central to the revenue equivalence theorem is the assumption that each buyer has the same beliefs about his opponent's valuation. While this is a natural first approximation, it is often the case that beliefs are positively related. This is especially important for the sale of items such as oil field leases, where each bidder gathers information prior to the auction. For then a bidder who obtains favorable information (for example, via seismic testing) will believe that his opponents are likely to have obtained favorable information as well. Conversely, a bidder who obtains unfavorable information will believe it is more likely that his opponents' information is also unfavorable.

To illustrate the implications of correlated beliefs we consider the simplest possible example in which each buyer has one of three valuations, V_1, V_2 or V_3 with $V_3 > V_2 > V_1 = 0$. Now, however, each buyer's beliefs are dependent upon his valuation.[8] Define $F_j^i \equiv$ Probability that opponent's valuation is V_j or less to a buyer with valuation V_i. For our example, the crucial assumption is that

$$\frac{F_1^2}{F_2^2} > \frac{F_1^3}{F_2^3}, \text{ that is } \frac{f_1^2}{f_1^2 + f_2^2} > \frac{f_1^3}{f_1^3 + f_2^3}.$$

This requires that a buyer will think it more likely that his opponent's valuation is V_1 rather than V_2 if his own valuation is lower.[9]

For the open auction, the analysis proceeds essentially as in the opening discussion. Each bidder stays in the auction until the asking price is equal to his valuation. His expected return is therefore given by expression 2 except that now the

[7] Maskin and Riley (1984) show that a seller could, in principle, modify the rules of the sealed high bid auction to further exploit this fear of loss and so increase expected seller revenue.

[8] The example does not capture all of the elements of bidding with correlated valuations. In particular, valuations are private so that there is no "winner's curse". However, the argument below is central to the analysis of the more general case as well.

[9] More generally, the crucial assumption is that buyers' beliefs have the property that F_j^i/F_k^i is nondecreasing in i for all j and k. While mildly weaker, this is the essence of "affiliation" (Milgrom and Weber, 1982).

probability weights depend upon his valuation. To be precise, it follows from expression 2 that U_2 and U_3 satisfy

$$(6) \qquad U_2 = F_1^2 V_2, \quad U_3 = \hat{U}_2 + F_2^3(V_3 - V_2) \quad \text{where } \hat{U}_2 = F_1^3 V_2.$$

To contrast this with the sealed high bid auction, we first consider such an auction when beliefs are correlated but buyers are unaware of this. That is, each buyer thinks that his opponents' beliefs and his own are the same. Given this, the arguments of the opening discussion continue to hold. Therefore, expected buyers' returns will again be given by the relations in equation 6. It follows that revenue equivalence will hold for this case as well.

The question remaining is how a bidder's strategy will change if he realizes that beliefs are correlated. Since a buyer with valuation V_3 always bids higher than a buyer with valuation V_2, this information has no effect on the latter buyer. His expected return is again given by the relations in 6. However, the strategy of a buyer with valuation V_3 is affected. To understand this we must consider more closely the bidding of a buyer with valuation V_2.

$$U_2(b) = \left\{ \begin{array}{c} \text{probability of} \\ \text{winning with a} \\ \text{bid of } b \end{array} \right\} [V_2 - b]$$

By bidding very close to zero he wins if his opponent has a zero valuation. That is, with probability F_1^2. Therefore $U_2(0) = F_1^2[V_2 - 0] = F_1^2 V_2$. By bidding his maximum bid $\bar{b}_2$ he also wins with probability 1 against an opponent with valuation V_2. That is, $U_2(\bar{b}_2) = F_2^2[V_2 - \bar{b}_2]$. Since he must be indifferent between all bids on the interval $[0, \bar{b}_2]$ it follows that $F_1^2 V_2 = F_2^2[V_2 - \bar{b}_2]$ and hence $\bar{b}_2 = V_2(1 - F_1^2/F_2^2)$.

The crucial point to note is that the maximum bid by a buyer with valuation V_2 is a decreasing function of the ratio F_1^2/F_2^2. It is easy to see why this must be the case. If Alfie has a valuation V_2 and believes that Berta is very likely to have the same rather than a lower valuation ($F_1^2/F_2^2 \approx 0$) then Alfie will bid aggressively. Indeed, in the limit, if the probability of a zero valuation is zero, Alfie will bid his full valuation. At the other extreme, if Alfie believes that Berta is much more likely to have a zero valuation rather than $V_2(F_1^2/F_2^2 \approx 0)$ then he gains very little by submitting anything other than a very small bid.

The actual maximum bid by a buyer with valuation V_2 is therefore to bid $V_2(1 - F_1^2/F_2^2)$ while a high valuation buyer who ignored the asymmetry would have believed that this maximum bid was $V_2(1 - F_1^3/F_2^3)$. It follows that if high valuation buyers realize that beliefs are correlated, they will realize that they do not have to bid as aggressively. In particular the minimum bid is smaller and it is fairly straightforward to confirm that the entire distribution of equilibrium bids shifts to the left. The equilibrium expected return to high valuation buyers is therefore greater in the sealed high bid auction than it is in the open auction. Since both auctions generate the same social value, it follows that expected seller revenue is *lower* in the sealed high bid auction.

In sum, it is the linkage between the bids of higher valued buyers and the more conservative beliefs of lower valued buyers which leads to lower seller revenue under sealed bidding. The careful reader will have noted that Milgrom refers to a "linkage principle" in his introduction to bidding theory. He argues that it is the linkages in open bidding which result in higher revenue from that auction. Despite the plausibility of this "intuition" I do not believe that it really goes to the heart of the analysis. Instead, it is only by stripping the model to its bare essentials that the correct intuition is revealed.

Concluding Remarks

The first objective of this paper has been to explain why it is that expected seller revenue is the same under very different auction rules. The second objective has been to explain why revenue equivalence no longer holds once the central assumptions of risk aversion and independence of beliefs are modified. The effect of risk aversion is to make a sealed high bid auction more attractive from a seller's viewpoint. The effect of correlated beliefs is to favor open bidding.

Which of the two factors dominates under plausible assumptions about risk aversion parameters and about the degree to which beliefs are correlated remains an important open question. This is one area where a careful blend of data analysis and simulation techniques should prove enormously fruitful. With policymakers currently struggling to come up with proposals to make the procurement process more efficient via competitive bidding, such work would also be very timely.

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