# LECTURE 3

Introduction to Econometrics

# INTRODUCTION TO LINEAR REGRESSION ANALYSIS II

October 6, 2017

# Revision: the previous lecture

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- (Desired) properties of an estimator:
  - An estimator is **unbiased** if the mean of its distribution is equal to the value of the parameter it is estimating

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- We found the formulae for the estimates:

$$\widehat{\beta}_1 = \frac{\sum_{i=1}^{n} (x_i - \overline{x}_n)\left(y_i - \overline{y}_n\right)}{\sum_{i=1}^{n} (x_i - \overline{x}_n)^2} \qquad \widehat{\beta}_0 = \overline{y}_n - \widehat{\beta}_1 \overline{x}_n$$

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- The properties of the error term determine the properties of the estimates

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1. Interpret the meaning of the coefficient of $exper_i$.
2. Use the estimates to determine the average wage of a person with 1, 5, 20, and 40 years of experience.
3. Do the predicted wages seem realistic? Explain your answer.

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- Readings:
  - Studenmund - chapter 4
  - Wooldridge - chapters 5, 8, 9, 12

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- For observations $1, 2, \ldots, n$, we have:

$$\begin{aligned}
y_1 &= \beta_0 + \beta_1 x_{11} + \beta_2 x_{12} + \ldots + \beta_k x_{1k} + \varepsilon_1 \\
y_2 &= \beta_0 + \beta_1 x_{21} + \beta_2 x_{22} + \ldots + \beta_k x_{2k} + \varepsilon_2 \\
\vdots & \quad \vdots \\
y_n &= \beta_0 + \beta_1 x_{n1} + \beta_2 x_{n2} + \ldots + \beta_k x_{nk} + \varepsilon_n
\end{aligned}$$

# Matrix notation

- We can write in matrix form:

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} 1 & x_{11} & x_{12} & \cdots & x_{1n} \\ 1 & x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{n1} & x_{n2} & \cdots & x_{nk} \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_k \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{pmatrix}$$

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or in a simplified notation:

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$$

# OLS - DERIVATION UNDER MATRIX NOTATION

- We have to find

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# OLS - derivation under matrix notation

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- This gives us

$$\widehat{\boldsymbol{\beta}} = \left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{y}$$

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- Meaning of $\beta_1$ is the impact of a one unit increase in $P$ on the dependent variable $Q$, **holding constant the other included independent variables** $P_s$ and $Y$
- When price increases by 1 unit (and price of substitute good and income remain the same), quantity demanded decreases by 0.73 units

# Exercise

- Remember the unique dataset that includes wages of all citizens of Brno as well as their experience (number of years spent working).
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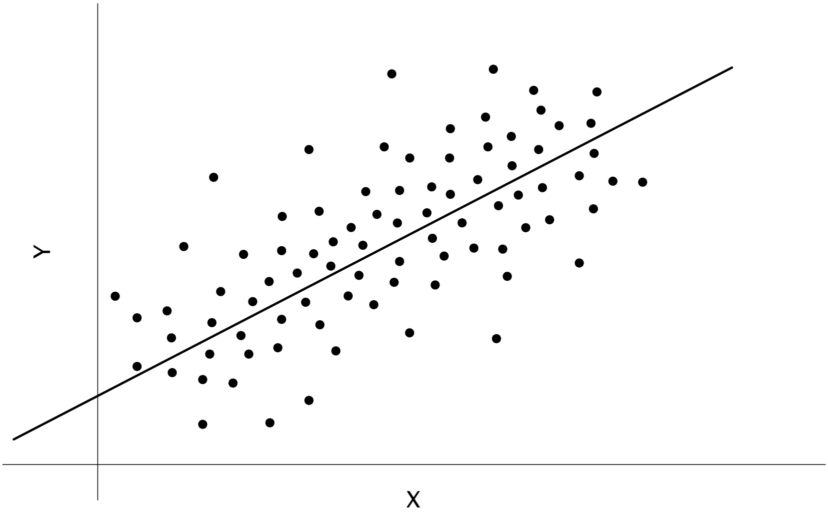
6. No explanatory variable is a perfect linear function of any other explanatory variable(s)
7. The error term is normally distributed

# Graphical representation



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  and the linearity in coefficients is restored
- Note that it is the linearity in coefficients that allows us to rewrite the general regression model in matrix form

# Exercise

Which of the following models is/are linear?

- $y = \beta_0 + \beta_1 x + \varepsilon$
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- $y = x^{\beta_1} + \varepsilon$ is NOT a linear model
- Regression models are **linear in parameters**, but they do not need to be linear in variables

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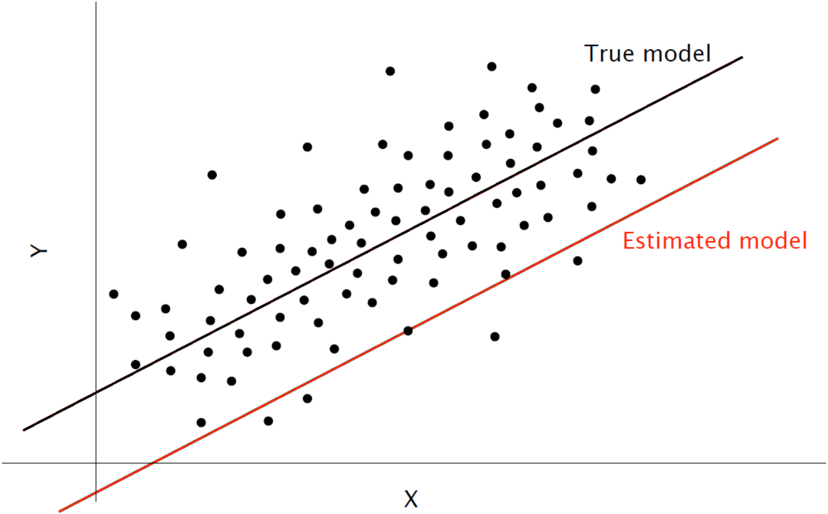
- Idea: observations are distributed around the regression line, the average of deviations is zero
- In fact, the mean of $\varepsilon_i$ is forced to be zero by the existence of the intercept ($\beta_0$) in the equation
- Hence, this assumption is satisfied as long as there is an **intercept** included in the equation

# Graphical representation



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- We will solve this problem using Generalized Least Squares estimator

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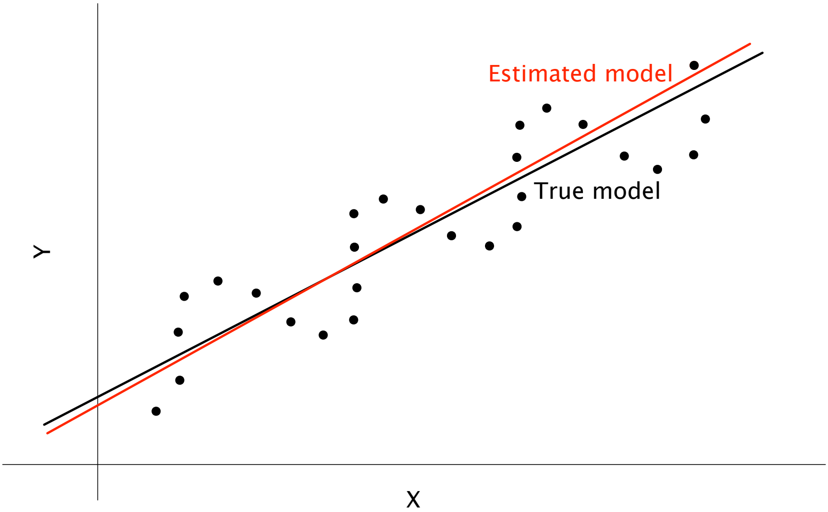

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- We will solve this problem using Hull-White robust standard errors

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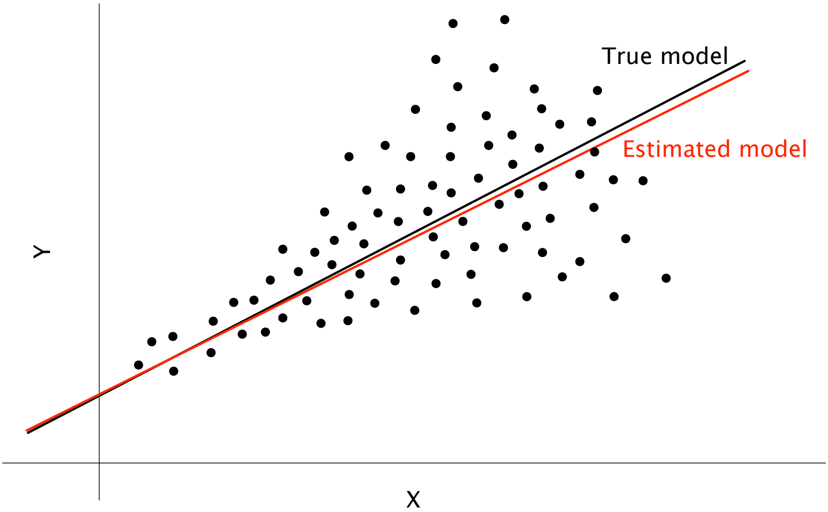

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- Notation:
  - no correlation: $corr(\varepsilon_i \varepsilon_j) \Rightarrow E[\varepsilon_i \varepsilon_j] = 0$ for each $i, j$
  - homoskedasticity: $E[\varepsilon_i^2] = \sigma^2$ for each $i$
- Matrix notation:

$$Var[\boldsymbol{\varepsilon}] = \begin{pmatrix} \sigma^2 & 0 & 0 & \cdots & 0 \\ 0 & \sigma^2 & 0 & \cdots & 0 \\ 0 & 0 & \sigma^2 & \cdots & 0 \\ & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \sigma^2 \end{pmatrix} = \sigma^2 \mathbf{I}$$

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  - Households with lower incomes may indicate higher consumption (because of shame)
  - Negative correlation between X and error term (measurement error higher for lower incomes)

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  - Households with lower incomes may indicate higher consumption (because of shame)
  - Negative correlation between X and error term (measurement error higher for lower incomes)
- Leads to biased and inconsistent estimates

# 5. Variables uncorrelated with the error term

*All explanatory variables are uncorrelated with the error term.*

- Notation: $E[x_i \varepsilon_i] = 0$ or $E[\mathbf{X}'\boldsymbol{\varepsilon}] = \mathbf{0}$
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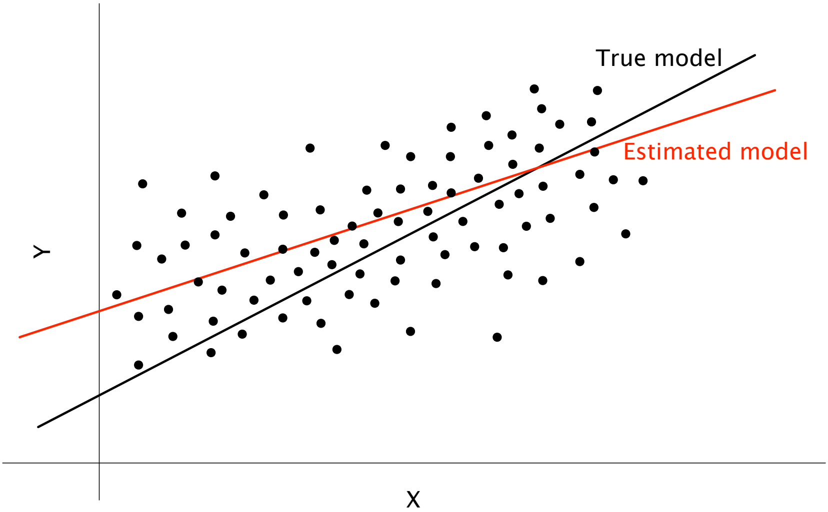
- Leads to biased and inconsistent estimates
- We will solve this problem using IV approach

# Graphical representation



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  - Example: we include dummy variables for men and women together with the intercept

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- Solution: drop one of the variables

# EXERCISE

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- Knowing the distribution of the estimate allows us to find its confidence intervals and to test hypotheses about coefficients

# Properties of the OLS estimate

## PROPERTIES OF THE OLS ESTIMATE

- OLS estimate is defined by the formula

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- The properties of $\widehat{\boldsymbol{\beta}}$ are based on the properties of $\boldsymbol{\varepsilon}$

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  - OLS has the minimum variance of all unbiased estimators (it is efficient)

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- Since $E\left[\widehat{\boldsymbol{\beta}}\right] = \boldsymbol{\beta},$ OLS is unbiased

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- Note that the normality of errors is not required for large samples, be-cause $\widehat{\boldsymbol{\beta}}$ is asymptotically normal anyway

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- **Consistency is the most important property of any estimate!!!**

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- Moreover, possible deviations from the classical model can be corrected
- As a consequence, the assumption of zero correlation between explanatory variables and the error term

$$E\left[\mathbf{X}'\boldsymbol{\varepsilon}\right] = \mathbf{0}$$

is the most important one to satisfy in regression models

# Summary

- We expressed the multivariate OLS model in matrix notation $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$ and we found the formula of the estimate:
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