Příklad 0.6 *Vypočítejte limity posloupností:*

- $\lim\limits_{n\to\infty} \frac{2n^2+3}{1+2n-n^2}$

$a_1 = \frac{2\cdot 1^2+3}{1+2\cdot 1-1^2} = \frac{5}{2}$

$a_2 = \frac{2\cdot 2^2+3}{1+2\cdot 2-2^2} = 11$

$a_3 = \frac{2\cdot 3^2+3}{1+2\cdot 3-3^2} = \frac{21}{-2} = -\frac{21}{2}$

$a_4 = \frac{2\cdot 4^2+3}{1+2\cdot 4-4^2} = \frac{35}{-7} = -5$

$a_5 = \frac{2\cdot 5^2+3}{1+2\cdot 5-5^2} = \frac{53}{-16} = -3\frac{5}{16}$

$\vdots$

$$\boxed{\lim_{n\to\infty} \frac{a n^k + \dots}{b n^l + \dots}} = 0, \text{ je-li } k<l$$

$$= \infty, \text{ je-li } k>l$$

$$= \frac{a}{b}, \text{ je-li } k=l$$

$$\lim_{n\to\infty} \frac{2n^2+3}{1+2n-n^2} = \lim_{n\to\infty} \frac{2n^2}{-n^2} = -2$$

$$\lim_{n\to\infty} \frac{2^n+n^2}{n^2+2n+1} = \lim_{n\to\infty} \frac{2^n}{n^2} = \infty$$

Funkce z $\mathbb{R}\to\mathbb{R}$, $k<l$, $\alpha<\beta$

$x^2 \ll x^3$

$x^k \ll x^l \ll \alpha^x \ll \beta^x \ll x^x$

$\ll$ porovnání růstu funkcí

$f(x) \ll g(x)$, g roste výrazně rychleji než f

Příklad: $x^{1000} \ll 2^x$

- $\lim\limits_{n\to\infty} \frac{n^4+n}{n^3+3n+1} = \infty$

protože stupeň pol. n^4+n > stupeň polynomu n^3+3n+1

člen s nejvyšší mocninou ve jmenovateli

$$\lim_{n\to\infty} \frac{\frac{\cancel{n^4}\, n}{\cancel{n^3}} + \frac{\cancel{n}}{\cancel{n^3}}\cdot\frac{1}{n^2}}{\frac{\cancel{n^3}\, 1}{\cancel{n^3}} + \frac{3n}{n^{\cancel{3}2}} + \frac{1}{n^3}} = \lim_{n\to\infty} \frac{n+\overbrace{\frac{1}{n^2}}^{\to 0}}{1+\underbrace{\frac{3}{n^2}}_{\to 0}+\underbrace{\frac{1}{n^3}}_{\to 0}} =$$

$$= \lim_{n\to\infty} \frac{n}{1} = \lim_{n\to\infty} n = \infty$$

- $\lim\limits_{n\to\infty}\left(n-\sqrt{n(n-1)}\right) = [\infty - \infty] \longrightarrow$ nedá se vyčíslit

$n \to \infty$, $\sqrt{n(n-1)} \approx \sqrt{n^2} = n \longrightarrow \infty$

$$\lim_{n\to\infty}\underbrace{\left(n-\sqrt{n(n-1)}\right)}_{a-b}\cdot\frac{\overbrace{n+\sqrt{n(n-1)}}^{a+b}}{n+\sqrt{n(n-1)}} =$$

$$= \lim_{n\to\infty}\frac{n^2-\left(\sqrt{n(n-1)}\right)^2}{n+\sqrt{n(n-1)}} = \lim_{n\to\infty}\frac{n^2-n\cdot(n-1)}{n+\sqrt{n(n-1)}}$$

$$= \lim_{n\to\infty}\frac{n}{n+\sqrt{n(n-1)}} = \lim_{n\to\infty}\frac{n}{n+[n\cdot(n-1)]^{\frac{1}{2}}}$$

$$= \lim_{n\to\infty}\frac{n}{n+\underbrace{[n^2-n]}_{(n^2)^{\frac{1}{2}}}{}^{\frac{1}{2}}} = \lim_{n\to\infty}\frac{n}{n+(n^2)^{\frac{1}{2}}} = \lim_{n\to\infty}\frac{n}{n+n}$$

$$= \lim_{n\to\infty}\frac{n}{2n} = \frac{1}{2}$$

$[n\cdot(n-1)]^{\frac{1}{2}}$

$f(x) = \sqrt{x} = x^{\frac{1}{2}}$

$g(x) = \sqrt[m]{x^n} = x^{\frac{n}{m}}$

- $$\lim_{n\to\infty}\frac{1+\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2^{n-1}}}{1+\frac{1}{3}+\frac{1}{9}+\ldots+\frac{1}{3^{n-1}}} = \frac{\lim\limits_{n\to\infty}\left(1\cdot\frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}}\right)}{\lim\limits_{n\to\infty}\left(1\cdot\frac{1-\left(\frac{1}{3}\right)^n}{1-\frac{1}{3}}\right)} = \frac{\frac{1}{\left(1-\frac{1}{2}\right)}}{\frac{1}{1-\frac{1}{3}}} = \frac{2}{\frac{3}{2}} = \frac{4}{3}$$

a_1, $1-q$

$|q|<1$

$\frac{3-1}{3} = \frac{2}{3}$

$a_n = \left(\frac{1}{2}\right)^{n-1}$

$a_1 = \left(\frac{1}{2}\right)^{1-1} = \left(\frac{1}{2}\right)^0 = 1$

$a_2 = \left(\frac{1}{2}\right)^{2-1} = \frac{1}{2}$

$\vdots$

geometrická posloupnost s $q = \frac{1}{2}$, $a_1 = 1$

$$\boxed{s_n = a_1\cdot\frac{1-q^n}{1-q}} = 1\cdot\frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}} =$$

$$= \frac{1-\left(\frac{1}{2}\right)^n}{\frac{1}{2}} = \ldots \text{ neznáme } n$$

$$= \frac{1-\frac{1^n}{2^n}}{\frac{1}{2}} = \frac{1-\frac{1}{2^n}}{\frac{1}{2}}$$

$1^n \longrightarrow \infty$

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\frac{1-\frac{1}{2^n}}{\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$$

$\frac{1}{2^n} \to 0$

$$\boxed{\lim_{n\to\infty} s_n = \frac{a_1}{1-q}}$$, kde a_n je geom. posl.

a_1 je první člen

q kvocient: $|q|<1$

- $\lim\limits_{n\to\infty}\frac{2^n-1}{2^n+1}$ $= \lim\limits_{n\to\infty}\frac{1\cdot 2^n}{1\cdot 2^n} = 1$

$$\lim_{n\to\infty}\frac{2^n+\cancel{n^2}-\cancel{1}}{2^n-\cancel{3n^3}+\cancel{1}} = \lim_{n\to\infty}\frac{2^n}{2^n} = 1$$

$$\lim_{n\to\infty}\frac{3\cdot 3^n-n^5+\sqrt{n}+1}{10\cdot 2^n+n^{10}-3} = \lim_{n\to\infty}\frac{3\cdot 3^n}{10\cdot 2^n} = \infty$$

$$\lim_{n\to\infty}\frac{2^n+\cancel{n^{100}-\sqrt{n^{50}}+1000000}}{3^n+\cancel{n^3-1}} =$$

$$= \lim_{n\to\infty}\frac{2^n}{3^n} = 0$$

- $\lim\limits_{n\to\infty}\left(\frac{1+4+7+\ldots+(3n-2)}{n+1}-\frac{3n}{2}\right)$
- $\lim\limits_{n\to\infty}\frac{n!+(n+1)!}{(n+2)!}$

$$\lim_{n\to\infty}\frac{\frac{n}{2}\cdot(3n-1)}{n+1}-\frac{3n}{2} \overset{(*)}{=}$$

$$s_n = \overbrace{1+4+7+\cdots+(3n-2)}^{\text{kolik členů}\to n} = \boxed{\frac{n}{2}\cdot(a_1+a_n)} = \frac{n}{2}(1+3n-2)$$

$$a_n = a_1+(n-1)\cdot d = 1+(n-1)\cdot 3 = \boxed{3n-2}$$

$a_1 = 1,\ d = 3$

$$\lim_{n\to\infty}\frac{\frac{n}{2}(3n-1)}{\underbrace{n+1}_{2\cdot(n+1)}}-\frac{3n}{2} = \lim_{n\to\infty}\frac{3n-1-3n\cdot(n+1)}{2(n+1)}$$

$$= \lim_{n\to\infty}\frac{\cancel{3n}-1-3n^2-\cancel{3n}}{2n+2} = \lim_{n\to\infty}\frac{-1-3n^2}{2n+2} =$$

$$= \lim_{n\to\infty}\frac{-3n^2}{2n} = -\infty$$

- $$\lim_{n\to\infty} \frac{n!+(n+1)!}{(n+2)!} = \lim_{n\to\infty} \frac{n! + (n+1)\cdot n!}{(n+2)\cdot(n+1)\cdot n!}$$

$$= \lim_{n\to\infty} \frac{\cancel{n!}\,[1+n+1]}{\cancel{n!}\,(n+2)(n+1)} = \lim_{n\to\infty} \frac{n+2}{n^2+3n+2} =$$

$$= \lim_{n\to\infty} \frac{\frac{n+2}{n^2}}{\frac{n^2+3n+2}{n^2}} = \lim_{n\to\infty} \frac{\frac{1}{n}\cancel{\frac{n}{n^2}} + \frac{2}{n^2}}{1\cancel{\frac{n^2}{n^2}} + \frac{3n}{n^2} + \frac{2}{n^2}}$$

0, 0, 0, 0

$$= \frac{0}{1} = \underline{\underline{0}}$$

vezmu sčítanec s nejvyšší mocninou ve jmenovateli a podělím jím čitatele a jmenovatele