Assignment 7

(Solution)

Financial Investments
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1. Consider a risk-free rate of 1% and an economy where the feasible future state of the price are $S_T = \{90, 100, 110, 120\}$, each one with equal probability. Given this, what is the fair value of a Call option with strike 100?

   Ans.

   - In each feasible future scenario (time T), the payoff of the Call option (C) is:

   $$\mathrm{C}_T = \begin{cases} \max(90 - 100, 0) = 0 & \text{with probability } 1/4 \\ \max(100 - 100, 0) = 0 & \text{with probability } 1/4 \\ \max(110 - 100, 0) = 10 & \text{with probability } 1/4 \\ \max(120 - 100, 0) = 20 & \text{with probability } 1/4 \end{cases}$$

   - The expected value of the Call option at time T (fair value) is given by

   $$\langle C_T \rangle = \mathbb{E}(C_T) = \frac{(0 + 0 + 10 + 20)}{4} = 7.5$$

   - The value of the derivative today (fair value at time 0) is computed as the discounted expected future value:

   $$C_0 = \frac{\langle C_T \rangle}{1 + r} = \frac{7.5}{1.01} \approx 7.43$$

2. Consider a risk-free rate of 1% and an economy where the feasible future state of the price are $S_T = \{90, 100, 110, 120\}$, each scenario with the following probabilities $\mathrm{P}^{\mathbb{Q}} = \{1/3, 1/3, 1/6, 1/6\}$. Given this, what is the fair value of a Call option with strike 100?

   Ans.

- We have the same feasible future scenario (time T) as in the previous exercise, but the probabilities differ:

$$\mathrm{C}_T = \begin{cases} \max(90-100,0)=0 & \text{with probability } 1/3 \\ \max(100-100,0)=0 & \text{with probability } 1/3 \\ \max(110-100,0)=10 & \text{with probability } 1/6 \\ \max(120-100,0)=20 & \text{with probability } 1/6 \end{cases}$$

- The expected value of the Call option at time T (fair value) is given by

$$\langle C_T \rangle = \mathbb{E}(C_T) = 0 \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} + 10 \cdot \frac{1}{6} + 20 \cdot \frac{1}{6} = 5$$

- The value of the derivative today (fair value at time 0) is computed as the discounted expected future value:

$$C_0 = \frac{\langle C_T \rangle}{1+r} = \frac{6}{1.01} \approx 5.94$$

- The fair premium differs than the computed one in the previous exercise (cheaper). It's because the probability to be in-the-money is lower now.

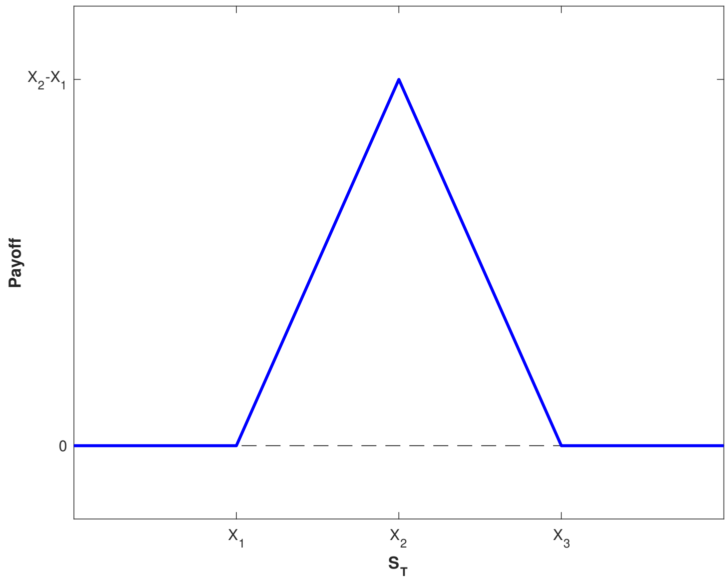
3. (a) A butterfly spread is the purchase of one call at exercise price X_1, the sale of two calls at exercise price X_2, and the purchase of one call at exercise price X_3. X_1 is less than X_2 and X_2 is less than X_3 by equal amounts, and all Calls have the same expiration date. Graph the payoff diagram to this strategy.

Ans.

- First, we compute the payoff of the strategy, i.e., the payoffs for each Call at time T, according the value of the underlying at maturity (S_T).
- You can give any value to the the set $X = \{X_1, X_2, X_3\}$ while it fulfills the conditions imposed in the problem: $X_1 < X_2 < X_3$ and $X_2 - X_1 = X_3 - X_2$. For example, $X_1 = 90$, $X_2 = 100$, and $X_3 = 110$.
- Remember that a short position in a European Call (selling the option) with strike K pays at maturity $-\max(S_T - K)$.

	$S_T < X_1$	$X_1 \leq S_T \leq X_2$	$X_2 < S_T < X_3$	$S_T \geq X_3$
$C_T^1(X_1)$	0	$S_T - X_1$	$S_T - X_1$	$S_T - X_1$
$-2C_T^2(X_2)$	0	0	$-2(S_T - X_2)$	$-2(S_T - X_2)$
$C_T^3(X_3)$	0	0	0	$S_T - X_3$
Total	0	$S_T - X_1$	$2X_2 - S_T - X_1$	$2X_2 - X_3 - X_1$

- Using the Total's row, we can draw the payoff diagram:



- In this strategy, the maximum profit is achieved if at maturity the underlying worths X_2.
- This strategy is designed with the aim the future price will be around X_2.

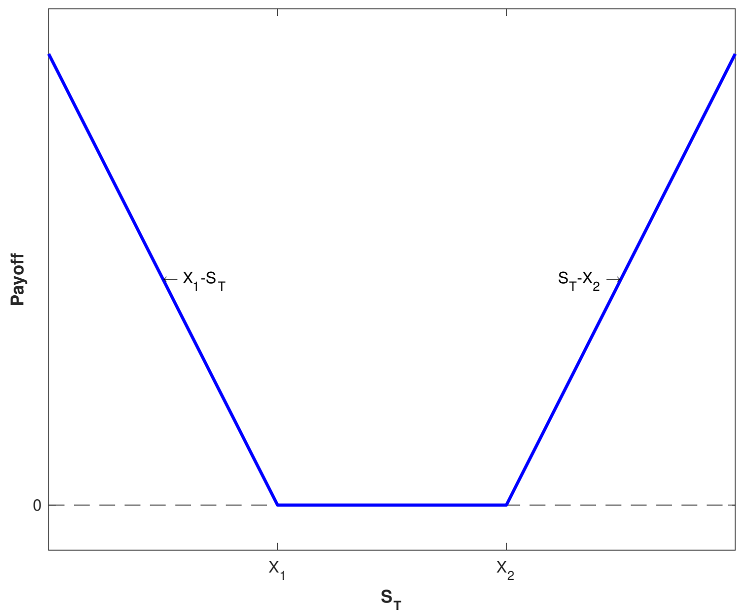
(b) A vertical combination is the purchase of a Call with exercise price X_2 and a Put with exercise price X_1, with X_2 greater than X_1. Graph the payoff of this strategy.

Ans.

- Payoff of the strategy according the value of the underlying at maturity:

	$S_T < X_1$	$X_1 \leq S_T \leq X_2$	$S_T \geq X_2$
$P_T^1(X_1)$	$X_1 - S_T$	0	0
$C_T^2(X_2)$	0	0	$S_T - X_2$
Total	$X_1 - S_T$	0	$S_T - X_2$

- Payoff diagram:



- In this strategy, the maximum profit is achieved when expected prices are outside of the interval $[X_1, X_2]$
- This strategy is designed to take advantage oh high volatility scenarios.

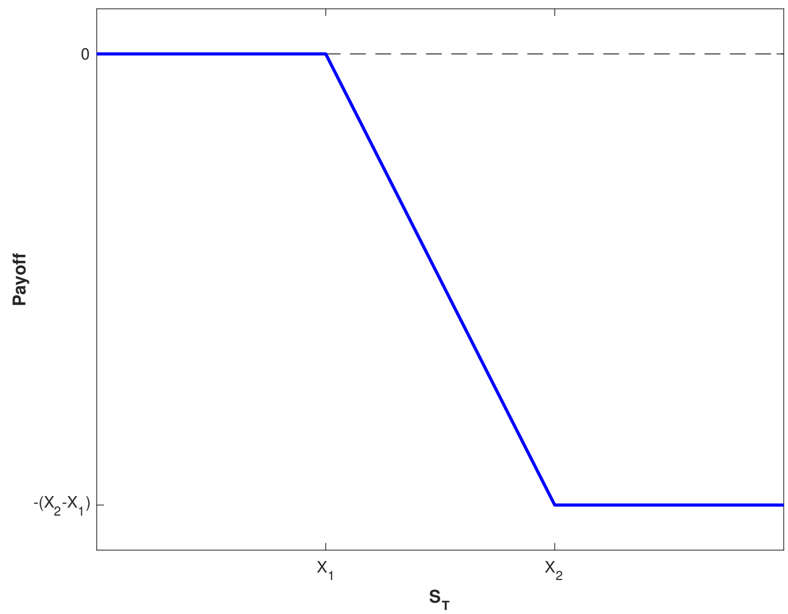
4. A bearish spread is the purchase of a Call with exercise price X_2 and the sale of a Call with exercise price X_1, with X_2 greater than X_1. Graph the payoff of this strategy.

   Ans.

   - Payoff of the strategy according the value of the underlying at maturity:

	$S_T < X_1$	$X_1 \leq S_T \leq X_2$	$S_T \geq X_2$
-$C_T^1(X_1)$	0	$-(S_T - X_1)$	$-(S_T - X_1)$
$C_T^2(X_2)$	0	0	$S_T - X_2$
Total	0	$X_1 - S_T$	$X_1 - X_2$

   - Payoff diagram:



- Why I should play this strategy if the maximum payoff is zero?
- We should consider the premium payments. Since $X_2 > X_1$, the value of a Call with strike X_1 is higher than a Call with strike X_2. Then, sale $C^1(X_1)$ and purchase $C^2\,(X_2)$ leads to a positive balance. If $S_T < X_1$ we have a positive and constant return. Otherwise, for $S_T > X_2$, the strategy is defective, but the losses are bounded (opossed to the naked Call situation). Then, the investor plays a bearish strategy (he/she expects prices goes down), but his/her losses are limited.