1, $\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$

$$\varepsilon := \frac{1}{2n(n+1)}$$

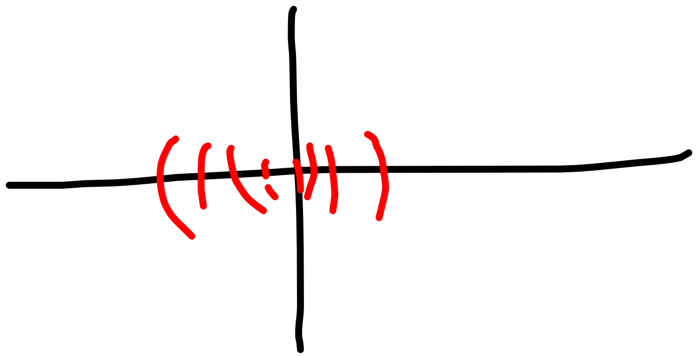
V intervalu $\left(\frac{1}{n}-\varepsilon, \frac{1}{n}+\varepsilon\right)$ leží z dané množiny pouze prvek $\frac{1}{n}$, tedy je to izolovaný bod.

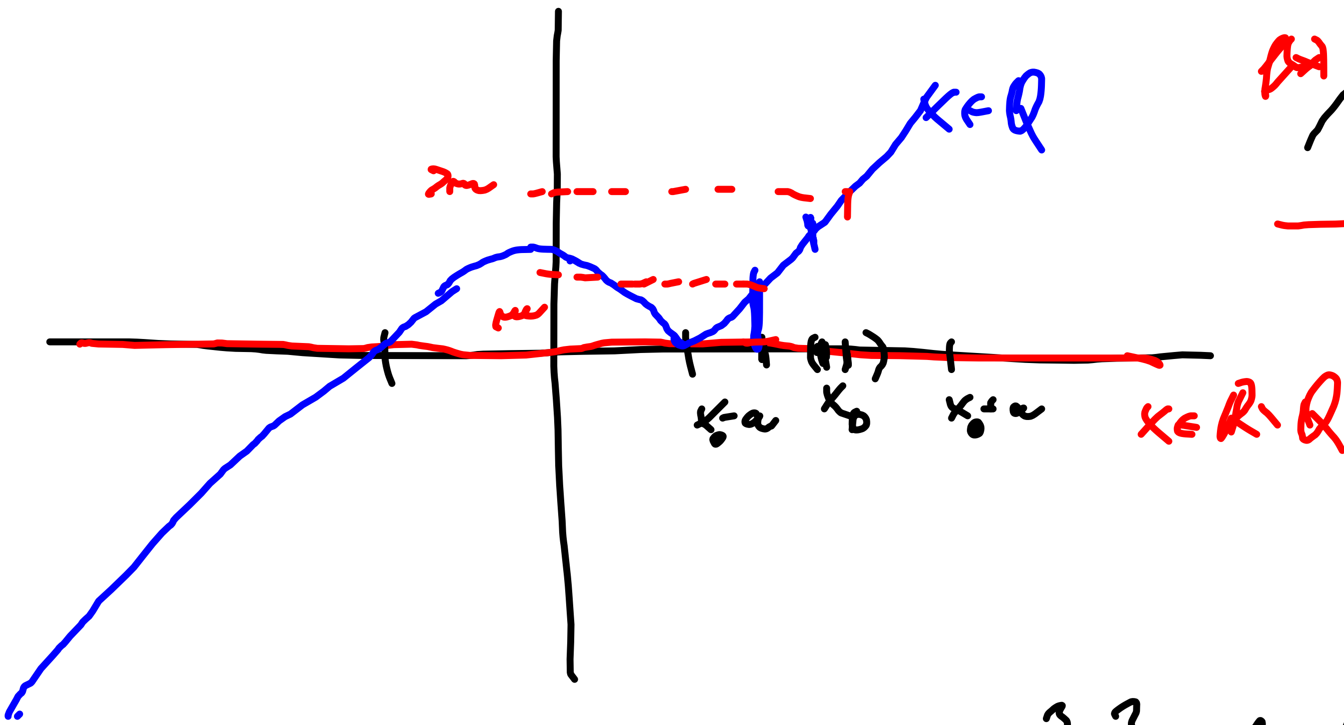
2, 1, $(0, 1\rangle$

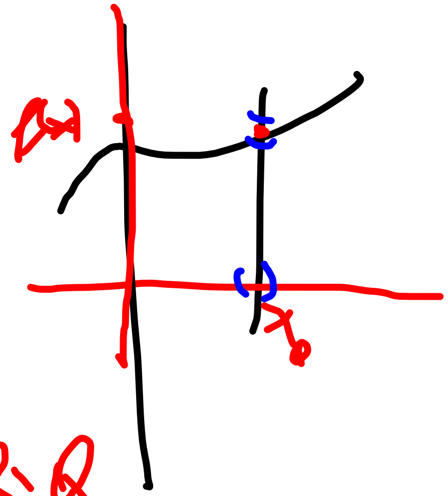
2, $\mathbb{R}$

3, $\bigcup_{n=3}^{\infty} \left\langle 1+\frac{1}{n}, 2-\frac{1}{n}\right\rangle = (1, 2)$

4, $\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$







$$x^3-x^2-x+1=(x-1)^2(x+1)$$

	1	-1	-1	1
1	1	0	-1	0
1	1	1	0	

Pro $x>1$ je daná fce v x nespojitá:

$a := x-1$, pak na intervalu $\left(x-\frac{a}{2}, x+\frac{a}{2}\right)$ je daná fce v rac. číslech zespodu ohraničená hodnotou $m = f\left(x-\frac{a}{2}\right)$. Pro lib. $\varepsilon>0$ $\left(<\frac{a}{2}\right)$, pak v intervalu $(x-\varepsilon, x+\varepsilon)$ leží nějaké rac. x_1, nějaké iracionální x_2 a potom

$$f(x_1)-f(x_2) \geq m - 0 = m$$

V bodech -1 a 1 je fce spojitá

Ze spojitosti fce x^3-x^2-x+1 v bodech -1 a 1 plyne $\lim\limits_{\substack{x\to -1\\ x\in\mathbb{Q}}} f(x)=0$. Zřejmě $\lim\limits_{\substack{x\to -1\\ x\in\mathbb{R}\setminus\mathbb{Q}}} f(x)=0$

Tedy $\lim\limits_{x\to -1} f(x)=0$

$$\lim_{n\to\infty} \frac{1}{n^3} = 0 \qquad \frac{1}{\infty}$$

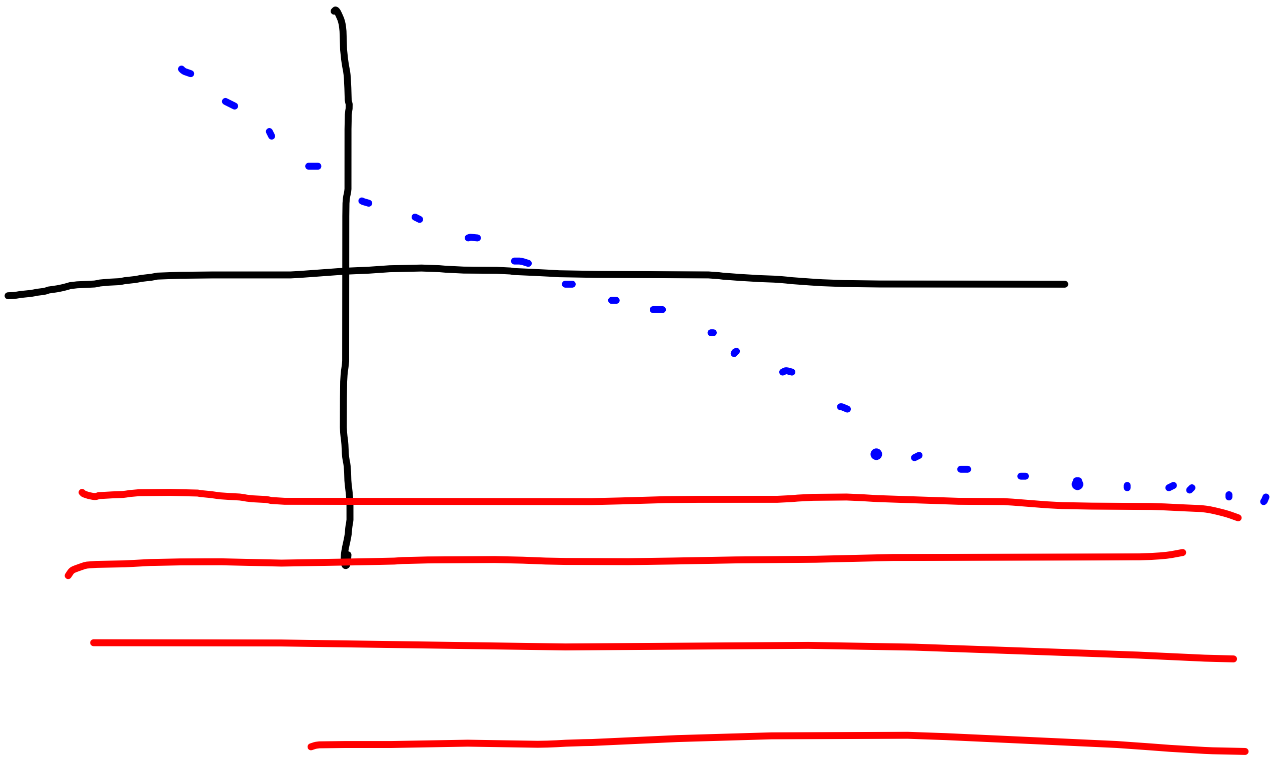
$$\lim_{n\to\infty} \frac{4n^3 - 2n^2 + 1}{2n^3 - n^2 + 3} = \lim_{n\to\infty} \frac{4 - 2\frac{1}{n} + \frac{1}{n^3}}{2 - \frac{1}{n} + \frac{3}{n^3}} =$$

$$= \frac{4 - 2\lim\limits_{n\to\infty}\frac{1}{n} + \lim\limits_{n\to\infty}\frac{1}{n^3}}{2 - \lim\limits_{n\to\infty}\frac{1}{n} + \lim\limits_{n\to\infty}\frac{3}{n^3}} = \frac{4}{2} = 2$$

$$\lim_{n\to\infty} \frac{2n^3 + 2n^2 + 1}{n^2 - n + 3} = \lim_{n\to\infty} \frac{2n + 2 + \frac{1}{n}}{1 - \frac{1}{n} + 3} = \infty$$

Nechť nejprve $c > 1$.

Obecný fakt: nechť $\{a_n\}_{n=0}^{\infty}$ je klesající zdola omezená posloupnost, pak $\lim\limits_{n\to\infty} a_n = \inf\limits_{n\in\mathbb{N}} a_n$



Posloupnost $\{\sqrt[n]{c}\}_{n=1}^{\infty}$ je klesající (pro $c>1$),

tedy má limitu (své infimum), ta musí být ≥ 1.

Kdyby to nebylo číslo 1, tak nechť je to nějaké $1+a$, $a>0$

podle definice limity pro $\varepsilon = \frac{a^2}{4}$ musí existovat n_0

tak, že $\forall n \geq n_0 : \left|\sqrt[n]{c} - (1+a)\right| < \varepsilon$, zejména

$$\sqrt[n]{c} < (1+a) + \varepsilon = \left(1+a+\frac{a^2}{4}\right).$$

Speciálně pro $n = 2n_0$

$$\sqrt[2n_0]{c} = \sqrt{\sqrt[n_0]{c}} < \sqrt{1+a+\frac{a^2}{4}} = \sqrt{\left(1+\frac{a}{2}\right)^2} = 1+\frac{a}{2},$$

tedy $\left|\sqrt[n]{c} - (1+a)\right| > \frac{a}{2}$ ↯

Pro $0<c<1$ uvážíme posloupnost převrácených hodnot

tj. odmocnin z $\left(\frac{1}{c}\right)$.

$$n! \geq \frac{k!}{k^k} \cdot k^n = k!\, k^{n-k}$$

$$k!\underbrace{(k+1)(k+2)\cdots n}_{n-k} \geq k!\,\underbrace{k \cdot k \cdot \ldots k}_{n-k}$$

$$\lim_{n\to\infty} \sqrt[n]{n!} = ?$$

Pro lib. $k \in \mathbb{N}$

$$\lim_{n\to\infty} \sqrt[n]{n!} \geq \lim_{n\to\infty} \sqrt[n]{\frac{k!}{k^k} \cdot k^n} = \lim_{n\to\infty} k \cdot \sqrt[n]{\frac{k!}{k^k}} =$$

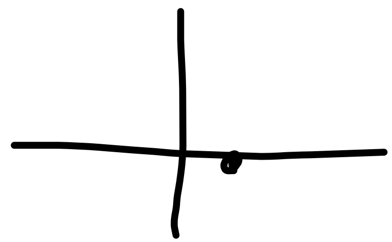
$$= k \cdot \lim_{n\to\infty} \sqrt[n]{\frac{k!}{k^k}} = k$$

$$\Rightarrow \lim_{n\to\infty} \sqrt[n]{n!} = \infty$$

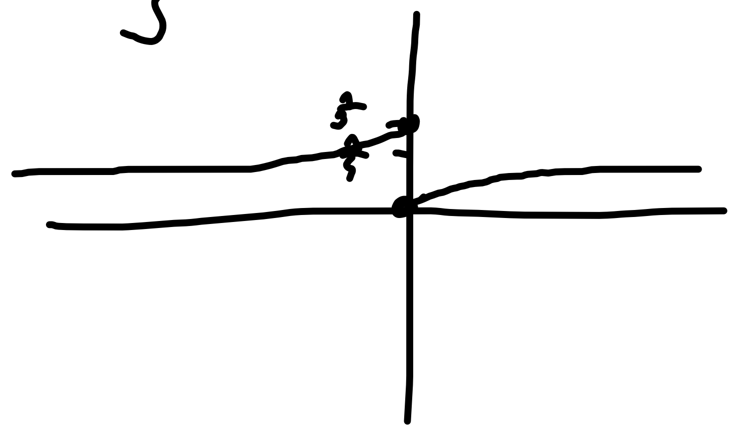
$$\lim_{x \to 4} \frac{x-4}{x^2-x-12} = \lim_{x \to 4} \frac{x-4}{(x-4)(x+3)} = \lim_{x \to 4} \frac{1}{(x+3)} = \frac{1}{7}$$

$$\lim_{x \to 1} \frac{\sqrt{x}-1}{x-1} = \lim_{x \to 1} \frac{\sqrt{x}-1}{(\sqrt{x}-1)(\sqrt{x}+1)} = \lim_{x \to 1} \frac{1}{\sqrt{x}+1} = \frac{1}{2}$$

$$\lim_{x \to 0^+} \frac{1}{3+2^{\frac{1}{x}}} = \frac{1}{3+2^{\infty}} = 0$$



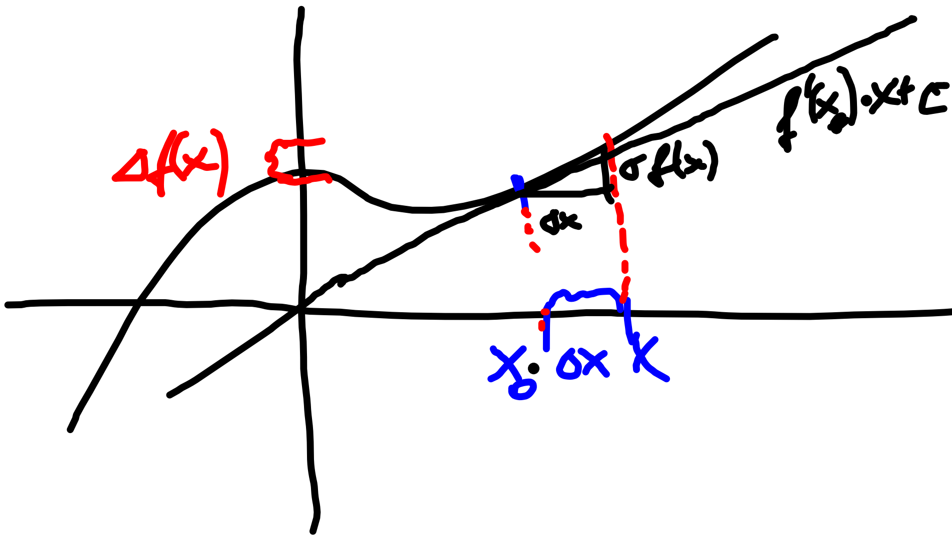
$$\lim_{x \to 0^-} \frac{1}{3+2^{\frac{1}{x}}} = \frac{1}{3+2^{-\infty}} = \frac{1}{3}$$



$$\lim_{x\to 2} \frac{x-2}{\sqrt{x^2-4}} = \lim_{x\to 2} \frac{x^2-4}{\sqrt{x^2-4}\,(x+2)} = \lim_{x\to 2} \frac{\sqrt{x^2-4}}{x+2} = 0$$

$$\lim_{x\to 1} \frac{x-1}{\sqrt{x^2+3}-2} = \lim_{x\to 1} \frac{(x-1)\left(\sqrt{x^2+3}+2\right)}{x^2+3-4} =$$

$$= \lim_{x\to 1} \frac{(x-1)\left(\sqrt{x^2+3}+2\right)}{(x-1)(x+1)} = \lim_{x\to 1} \frac{\left(\sqrt{x^2+3}+2\right)}{(x+1)} = 2$$



$$\frac{\Delta f(x)}{\Delta x} \approx f'(x_0)$$

↑

směrnice tečny ke grafu v bodě x_0

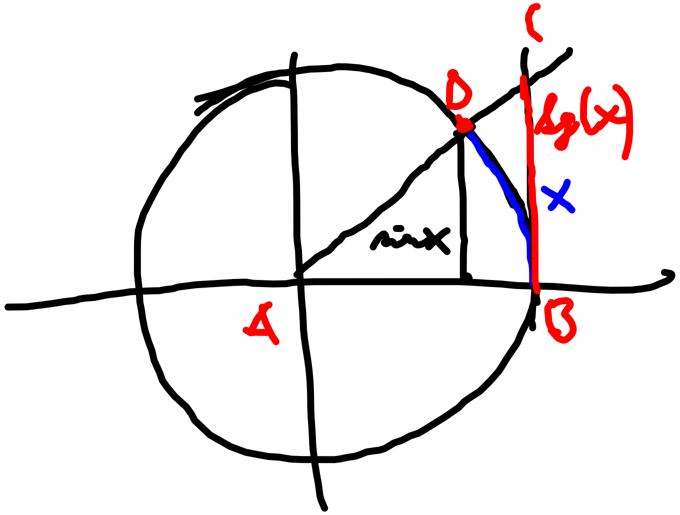
$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

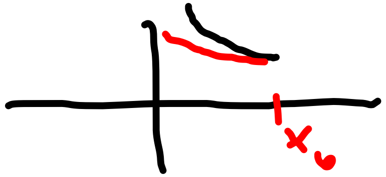
$$(x^2)' = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} =$$

$$= \lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0} 2x + h = 2x$$

$$\left(\frac{1}{x}\right)' = \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h \to 0} \frac{\frac{\overbrace{x - x - h}^{-h}}{x(x+h)}}{h} =$$

$$= \lim_{h \to 0} -\frac{1}{x(x+h)} = -\frac{1}{x^2}$$





$$\sin x < x$$

$$S_{ABC} = \frac{1}{2}|AB|\cdot|BC| = \frac{1}{2}\,\mathrm{tg}(x) >$$

$$> S_{ABD} = \frac{1}{2}x$$

$$\Rightarrow \quad \mathrm{tg}(x) > x \qquad (\text{pro } 0<x<\tfrac{\pi}{2})$$

$$\sin x < x < \mathrm{tg}\, x$$

$$\lim_{x\to 0} \frac{\sin x}{x} \le \lim_{x\to 0} \frac{x}{x} = \lim_{x\to 0} 1 = 1$$

$$\lim_{x\to 0} \frac{\sin x}{x} \ge \lim_{x\to 0} \frac{\sin x}{\mathrm{tg}\, x} = \lim_{x\to 0} \cos x = 1$$

$$\Rightarrow \quad \lim_{x\to 0} \frac{\sin x}{x} = 1$$

$$(\sin(x))' = \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} =$$

$\sin(x+h) =$

$$(\cos x + i\sin x)(\cos h + i\sin h) = (\cos(x+h) + i\underline{\sin(x+h)})$$

$$= \cos x\cos h - \sin x\sin h + i\underline{(\sin x\cos h + \cos x\sin h)}$$

$$= \lim_{h\to 0} \frac{\sin x\cos h + \cos x\sin h - \sin x}{h} =$$

$$\underbrace{\lim_{h\to 0} \frac{\sin x\cos h - \sin x}{h}}_{0} + \underbrace{\lim_{h\to 0} \frac{\cos x\sin h}{h}}_{\cos x} =$$

$$\boxed{\cos x}$$

$$\sin x \cos h - \sin x = \sin x(\cos h - 1) \leq \sin x \sin^2 h$$

$$\cancel{\cos} \quad \lim_{h \to 0} \frac{|\cos h - 1|}{h}$$

$$\cos h - \cos^2 h - \sin^2 h = \cos h(1 - \cos h) - \sin^2 h$$

$$1 - \cos h \leq 1 - \cos^2 h = \sin^2 h \qquad \cos^2 h \leq \cos h.$$

$$\lim_{h \to 0} \frac{|\cos h - 1|}{h} \leq \lim_{h \to 0} \frac{\sin^2 h}{h} = \overbrace{\lim_{h \to 0} \sin h}^{0} \cdot \underbrace{\lim_{h \to 0} \frac{\sin h}{h}}_{1}$$