① Vyjádřit parametricky průnik rovin $\rho: x-2y+5=0$ a $\sigma: 2x+3y-z+1=0$ v $\mathbb{R}^3$.

Jsou to body $[a,b,c]\in\mathbb{R}^3$ splňující $a-2b+5=0$ a zároveň $2a+3b-c+1=0$

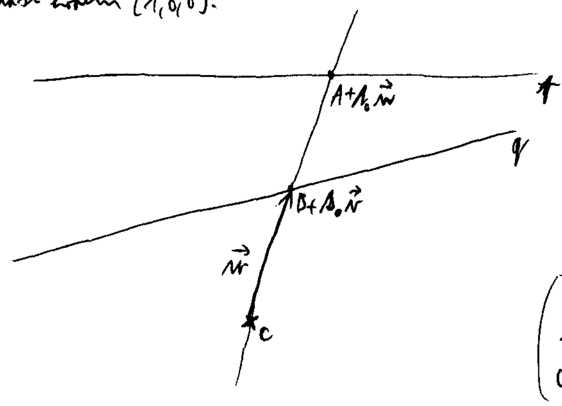
$$\left(\begin{array}{ccc|c} 1 & -2 & 0 & -5 \\ 2 & 3 & -1 & -1 \end{array}\right) \sim \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \left(\begin{array}{ccc|c} 1 & -2 & 0 & -5 \\ 0 & 7 & -1 & 9 \end{array}\right) \end{array}$$

položíme $b=t$, pak $\to c=7t-9$

$\to a=-5+2t$

Je to množina $\{[5+2t, t, 7t-9],\ t\in\mathbb{R}\}$, což $\Rightarrow$ je to přímka $[-5,0,-9]+(2,1,7)t$.

② Najděte příčku mimoběžek $p: \overset{A}{[1,1,1]}+t\overset{\vec{u}}{(2,1,0)}$ a $q: \overset{B}{[2,2,0]}+s\overset{\vec{v}}{(1,1,1)}$ takovou, že přímka jí určená prochází bodem $\overset{C}{[1,0,0]}$.



$A+t_0\vec{u} = C+r\cdot\vec{w} = C+r\cdot(B-C+s_0\vec{v}) \Rightarrow$

$\Rightarrow t_0\vec{u} - r(B-C) - rs_0\vec{v} = C-A$

$t_0\cdot(2,1,0)+r(-1,-2,0)+rs_0(-1,-1,-1)=(0,-1,-1)$

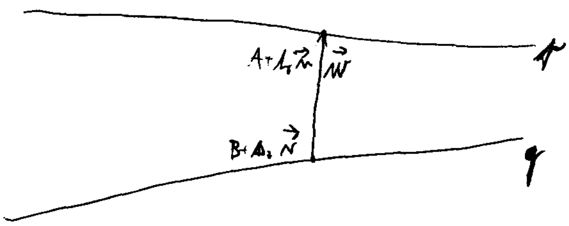
$$\left(\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 1 & -2 & -1 & -1 \\ 0 & 0 & -1 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -2 & -1 & -1 \\ 0 & 3 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right) \Rightarrow \begin{array}{l} rs_0=1 \to s_0=3 \\ 3r=1 \Rightarrow r=\frac{1}{3} \end{array}$$

$t_0=-1+\frac{2}{3}+1=\frac{2}{3}$

Pak $A+t_0\vec{u}=[1,1,1]+\left(\frac{4}{3},\frac{2}{3},0\right)=\left[\frac{7}{3},\frac{5}{3},1\right]$

$B+s_0\vec{v}=[2,2,0]+(3,3,3)=[5,5,3]$ $\Rightarrow$ Je to úsečka s krajními body $\left[\frac{7}{3},\frac{5}{3},1\right]$ a $[5,5,3]$.

③ Určete osu mimoběžek $p: \overset{A}{[3,0,3]}+\overset{\vec{u}}{(0,1,2)}t$, $q: \overset{B}{[0,-1,-2]}+\overset{\vec{v}}{(1,2,3)}s$.



$\vec{w}=A-B+t_0\vec{u}-s_0\vec{v}$

$\vec{w}\perp\vec{u}: \langle (A-B)+t_0\vec{u}-s_0\vec{v}, \vec{u}\rangle = 0 \Rightarrow$

$\Rightarrow \langle (A-B),\vec{u}\rangle + t_0\langle\vec{u},\vec{u}\rangle - s_0\langle\vec{v},\vec{u}\rangle = 0$

$t_0\langle\vec{u},\vec{u}\rangle - s_0\langle\vec{v},\vec{u}\rangle = -\langle (A-B),\vec{u}\rangle$

Obdobně $\vec{w}\perp\vec{v} \Rightarrow t_0\langle\vec{u},\vec{v}\rangle - s_0\langle\vec{v},\vec{v}\rangle = -\langle (A-B),\vec{v}\rangle$

$5t_0-8s_0 = -\langle(3,1,5),(0,1,2)\rangle = -11$

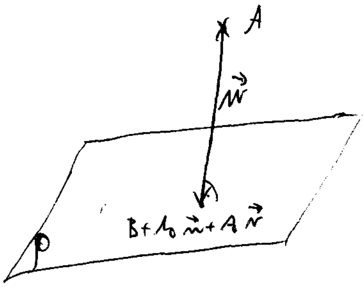
$8t_0-14s_0 = -(3+2+15) = -20$

$$\left(\begin{array}{cc|c} 5 & -8 & -11 \\ 8 & -14 & -20 \end{array}\right) \sim \left(\begin{array}{cc|c} 5 & -8 & -11 \\ -2 & 2 & 2 \end{array}\right) \sim \left(\begin{array}{cc|c} 5 & -8 & -11 \\ -1 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{cc|c} 0 & -3 & -6 \\ -1 & 1 & 1 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{cc|c} 0 & 1 & 2 \\ -1 & 1 & 1 \end{array}\right) \Rightarrow \begin{array}{l} s_0=2 \\ t_0=1 \end{array}$$

Pak $A+t_0\vec{u}=[3,1,5]$, $B+s_0\vec{v}=[2,3,4]$ $\Rightarrow$ Je to úsečka s krajními body $[3,1,5]$ a $[2,3,4]$.

4) Určete patu kolmice spuštěné z bodu $\overset{A}{[0,0,7]}$ na rovinu $\rho: \overset{B}{[0,5,3]} + \overset{\vec{u}}{(1,2,1)}t + \overset{\vec{v}}{(-2,1,1)}s$.



$\vec{w} = (B-A) + t_0\vec{u} + s_0\vec{v}$, platí: $\vec{w} \perp \vec{u}$, $\vec{w} \perp \vec{v}$, tj. $\langle \vec{w},\vec{u}\rangle = \langle \vec{w},\vec{v}\rangle = 0$

$$\Rightarrow t_0\langle \vec{u},\vec{u}\rangle + s_0\langle \vec{v},\vec{u}\rangle = -\langle (B-A),\vec{u}\rangle$$
$$t_0\langle \vec{u},\vec{v}\rangle + s_0\langle \vec{v},\vec{v}\rangle = -\langle (B-A),\vec{v}\rangle$$

$$6t_0 + s_0 = -\langle (0,5,-4),(1,2,1)\rangle = -6 \qquad \Rightarrow t_0 = -1,\ s_0 = 0$$
$$t_0 + 6s_0 = -\langle (0,5,-4),(-2,1,1)\rangle = -1$$

Pata $B + t_0\vec{u} + s_0\vec{v} = \underline{[-1,3,2]}$.

5) Zjistěte, zda jsou podprostory $U = \left\langle \overset{\vec{u}}{\begin{pmatrix}2\\1\\2\\2\end{pmatrix}} \right\rangle$, $V = \left\langle \overset{\vec{v}_1}{\begin{pmatrix}-1\\0\\-1\\2\end{pmatrix}}, \overset{\vec{v}_2}{\begin{pmatrix}-1\\0\\1\\0\end{pmatrix}}, \overset{\vec{v}_3}{\begin{pmatrix}0\\0\\1\\-1\end{pmatrix}} \right\rangle$ prostoru $\mathbb{R}^4$ na sebe kolmé. Pokud ano, je $\mathbb{R}^4 = U \oplus V$, tj. je $U^\perp = V$?

Protože $\langle \vec{u},\vec{v}_1\rangle = \langle \vec{u},\vec{v}_2\rangle = \langle \vec{u},\vec{v}_3\rangle = 0$, je $U \perp V$. Zjistíme, jestli jsou $\vec{u}, \vec{v}_1, \vec{v}_2, \vec{v}_3$ lin. nezávislé:

$$\begin{pmatrix}2&1&2&2\\-1&0&-1&2\\-1&0&1&0\\0&0&1&-1\end{pmatrix} \sim \begin{pmatrix}2&1&2&2\\-1&0&-1&2\\0&0&2&-2\\0&0&1&-1\end{pmatrix} \sim \begin{pmatrix}2&1&2&2\\-1&0&-1&2\\0&0&1&-1\\0&0&0&0\end{pmatrix} \Rightarrow$$ jsou lin. závislé $\Rightarrow$ $\mathbb{R}^4 \neq U \oplus V$, $U^\perp \neq V$.

6) Najděte ortogonální doplněk $U^\perp$ podprostoru $U = \{(x_1,x_2,x_3,x_4)^T;\ x_1 = x_3,\ x_2 = x_3 + 6x_4\} \subset \mathbb{R}^4$.

$U^\perp = \{v \in \mathbb{R}^4;\ \langle v, u\rangle = 0$ pro všechna $u \in U\}$, dále $U = \{(x_1, x_1+6x_4, x_1, x_4)^T;\ x_1,x_4 \in \mathbb{R}\} = \left\langle \overset{u_1}{\begin{pmatrix}1\\1\\1\\0\end{pmatrix}}, \overset{u_2}{\begin{pmatrix}0\\6\\0\\1\end{pmatrix}} \right\rangle$.

Tedy $U^\perp = \{v \in \mathbb{R}^4;\ \langle v,u_1\rangle = \langle v,u_2\rangle = 0\} = \left\{\begin{pmatrix}v_1\\v_2\\v_3\\v_4\end{pmatrix} \in \mathbb{R}^4;\ v_1+v_2+v_3 = 0,\ 6v_2+v_4 = 0\right\}$.

Volíme $v_2 = t$, $v_1 = s$, pak $v_3 = -s-t$, $v_4 = -6t$

$$\Rightarrow U^\perp = \left\{\begin{pmatrix}s\\t\\-s-t\\-6t\end{pmatrix};\ s,t \in \mathbb{R}\right\} = \left\langle \begin{pmatrix}1\\0\\-1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\-1\\-6\end{pmatrix} \right\rangle$$

7) Určete vzdálenost mimoběžek $p: \overset{A}{[1,-1,0]} + t\overset{\vec{u}}{(-1,2,3)}$ a $q: \overset{B}{[2,5,-1]} + s\overset{\vec{v}}{(-1,-2,1)}$ v $\mathbb{R}^3$.

1. způsob: spočte se velikost osy p a q.

$\langle \vec{u},\vec{v}\rangle^\perp$ ← podprostor, vzhl. skal. součinu

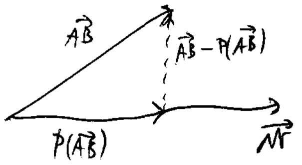
2. způsob: je to velikost kolmého průmětu libovolné příčky (spojnice) p a q do $\langle \vec{u},\vec{v}\rangle^\perp$. Tohle je nejsn.

$\vec{AB} = (1,6,-1)$. Dále $\langle \vec{u},\vec{v}\rangle^\perp = \{(x_1,x_2,x_3) \in \mathbb{R}^3;\ -x_1+2x_2+3x_3 = 0,\ -x_1-2x_2+x_3 = 0\}$

$$\left(\begin{array}{ccc|c}-1&2&3&0\\-1&-2&1&0\end{array}\right) \sim \left(\begin{array}{ccc|c}-1&2&3&0\\0&4&2&0\end{array}\right) \sim \left(\begin{array}{ccc|c}-1&2&3&0\\0&2&1&0\end{array}\right) \quad x_2 = t,\ x_3 = -2t,\ x_1 = -4t$$

$$\Rightarrow \langle \vec{u},\vec{v}\rangle^\perp = \{(-4t, t, -2t);\ t \in \mathbb{R}\} = \langle \overset{\vec{w}}{(-4, 1, -2)}\rangle$$

⑦ Alternativně:



Jistě $P(\vec{AB}) = k\cdot\vec{w}$ pro vhodné $k\in\mathbb{R}$. Dále $\vec{AB}-P(\vec{AB}) \perp \vec{w} \Rightarrow$

$\Rightarrow \langle \vec{AB} - k\cdot\vec{w}, \vec{w}\rangle = 0 \Rightarrow \langle \vec{AB}, \vec{w}\rangle - k\langle \vec{w}, \vec{w}\rangle = 0 \Rightarrow$

skal. součin $\Rightarrow k = \dfrac{\langle \vec{AB}, \vec{w}\rangle}{\langle \vec{w}, \vec{w}\rangle}$

velikost $k\cdot\vec{w}$ je $\|k\cdot\vec{w}\| = |k|\cdot\|\vec{w}\| = \left|\dfrac{\langle \vec{AB}, \vec{w}\rangle}{\langle \vec{w}, \vec{w}\rangle}\right|\cdot\|\vec{w}\| = \dfrac{|\langle \vec{AB}, \vec{w}\rangle|}{\|\vec{w}\|^2}\|\vec{w}\| = \dfrac{|\langle \vec{AB}, \vec{w}\rangle|}{\|\vec{w}\|} = \dfrac{|-4+6+2|}{\sqrt{16+1+4}} = \dfrac{4}{\sqrt{21}}$

⑧ Pomocí Gram-Schmidtova ortogonalizačního procesu najděte ~~ota~~ ortogonální bázi podprostoru

$$W = \{(x_1,x_2,x_3,x_4)^T \in \mathbb{R}^4;\ x_1+x_2+x_3+x_4=0\}$$

$x_2 = r,\ x_3 = s,\ x_4 = t \Rightarrow x_1 = -r-s-t \Rightarrow W = \left\langle \underbrace{\begin{pmatrix}-1\\1\\0\\0\end{pmatrix}}_{\vec{v}_1}, \underbrace{\begin{pmatrix}-1\\0\\1\\0\end{pmatrix}}_{\vec{v}_2}, \underbrace{\begin{pmatrix}-1\\0\\0\\1\end{pmatrix}}_{\vec{v}_3} \right\rangle$

$k = -\dfrac{\langle \vec{v}_2, \vec{u}_1\rangle}{\langle \vec{u}_1, \vec{u}_1\rangle} = -\dfrac{1}{2}$

Ortogonální báze bude $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$.

Položme $\boxed{\vec{u}_1 := \vec{v}_1}$. Dále $\vec{u}_2 = k\vec{u}_1 + \vec{v}_2 \perp \vec{u}_1 \Rightarrow \langle k\vec{u}_1 + \vec{v}_2, \vec{u}_1\rangle = 0 \Rightarrow$ ~~$k = -\dfrac{\langle \vec{u}_1, \vec{u}_1\rangle}{\langle \vec{v}_2, \vec{u}_1\rangle} = -\dfrac{2}{1} = -2$~~

$\Rightarrow \vec{u}_2 = \begin{pmatrix}-\frac{1}{2}\\-\frac{1}{2}\\1\\0\end{pmatrix}$, můžeme vzít $\boxed{\vec{u}_2 = \begin{pmatrix}-1\\-1\\2\\0\end{pmatrix}}$.

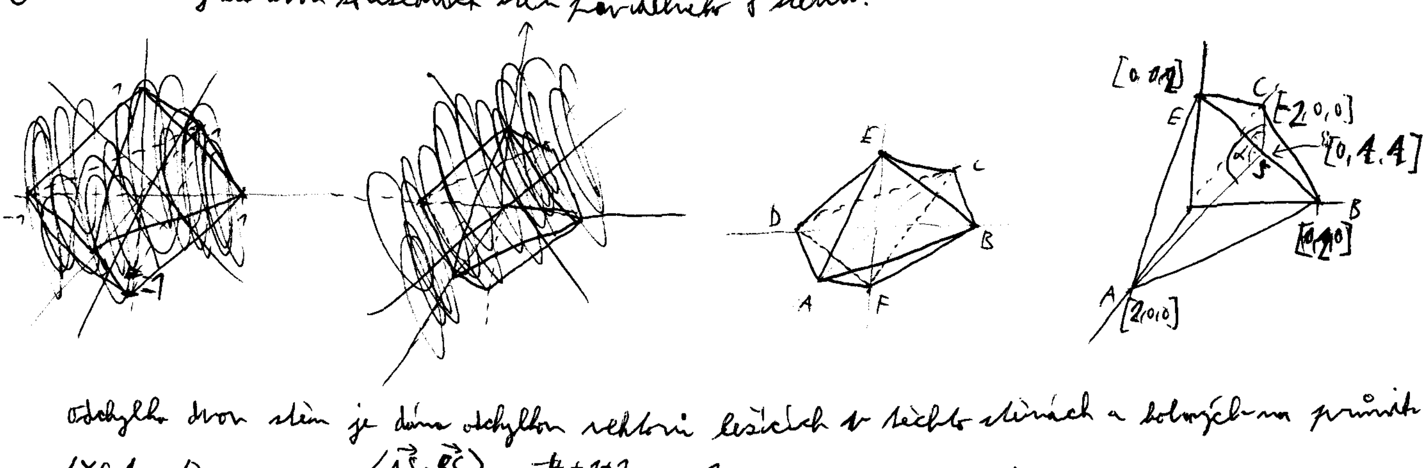
$\vec{u}_3 = k_1\vec{u}_1 + k_2\vec{u}_2 + \vec{v}_3 \perp \vec{u}_1, \vec{u}_2 \Rightarrow$ $k_1\langle \vec{u}_1, \vec{u}_1\rangle + k_2\overbrace{\langle \vec{u}_2, \vec{u}_1\rangle}^{0} + \langle \vec{v}_3, \vec{u}_1\rangle = 0$

$k_1\underbrace{\langle \vec{u}_1, \vec{u}_2\rangle}_{0} + k_2\langle \vec{u}_2, \vec{u}_2\rangle + \langle \vec{v}_3, \vec{u}_2\rangle = 0$

$\Rightarrow k_1 = -\dfrac{1}{2} = -\dfrac{1}{2}$, $k_2 = -\dfrac{1}{1+1+4} = -\dfrac{1}{6} \Rightarrow \vec{u}_3 = \begin{pmatrix}\frac{1}{2}+\frac{1}{6}-1\\-\frac{1}{2}+\frac{1}{6}+0\\0-\frac{1}{3}+0\\0+0+1\end{pmatrix} = \begin{pmatrix}-\frac{1}{3}\\-\frac{1}{3}\\-\frac{1}{3}\\1\end{pmatrix}$, můžeme vzít $\boxed{\vec{u}_3 = \begin{pmatrix}1\\1\\1\\-3\end{pmatrix}}$

Ortogonální báze W je např. $\left(\begin{pmatrix}-1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}-1\\-1\\2\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\\-3\end{pmatrix}\right)$.

⑨ Najděte $\overbrace{\text{odchylku}}^{\cos\alpha,\ \text{kde}\ \alpha\ \text{je}}$ odchylka dvou sousedních stěn pravidelného 8 stěnu.



Odchylka dvou stěn je dána odchylkou vektorů ležících v těchto stěnách a kolmých na průnik těchto stěn: $\cos\alpha = \dfrac{\langle \vec{AS}, \vec{BS}\rangle}{\|\vec{AS}\|\cdot\|\vec{BS}\|} = \dfrac{-4+1+1}{\sqrt{6}\cdot\sqrt{6}} = -\dfrac{1}{3} \Rightarrow \alpha \doteq 109{,}5°$

$\vec{AS} = (-2,1,1)$, $\vec{BS} = (2,1,1)$