Integrace racionální lomené funkce

I. $\int \frac{A}{x-x_0}\,dx = A\int\frac{dx}{x-x_0} = A\cdot\ln|x-x_0| + C$

Př. $\int \frac{x}{(x-1)(x^2-4)}\,dx = \int \frac{x}{(x-1)(x-2)(x+2)}\,dx = \int\left(\frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x+2}\right)dx =$

$= A\ln|x-1| + B\ln|x-2| + C\ln|x+2| + D$

II. $\int \frac{A}{(x-x_0)^n}\,dx = A\int\frac{dx}{(x-x_0)^n} = \left|\begin{array}{l} \text{S:} \\ x-x_0=t \\ dx=dt\end{array}\right| = A\int\frac{1}{t^n}\,dt = A\cdot\frac{t^{-n+1}}{-n+1} + C =$

$= \frac{A}{1-n}\cdot\frac{1}{(x-x_0)^{n-1}} + C$

Př. $\int\frac{1}{x(x+1)^3}\,dx = \int\left(\frac{A}{x} + \frac{B}{(x+1)^3} + \frac{C}{(x+1)^2} + \frac{D}{x+1}\right)dx = A\cdot\ln|x| + D\cdot\ln|x+1| - \frac{C}{x+1} - \frac{B}{2(x+1)^2} + E$

III. $\int\frac{Bx+C}{(x-x_0)^2+a^2}\,dx = \int\frac{\frac{B}{2}(2x-2x_0) + Bx_0 + C}{(x-x_0)^2+a^2}\,dx = \frac{B}{2}\int\frac{2x-2x_0}{(x-x_0)^2+a^2}\,dx + (Bx_0+C)\int\frac{dx}{(x-x_0)^2+a^2}\,dx =$

$= \frac{B}{2}\ln\left((x-x_0)^2+a^2\right) + \frac{Bx_0+C}{a}\,\mathrm{arctg}\,\frac{x-x_0}{a} + D$

$\int\frac{dx}{(x-x_0)^2+a^2} = \frac{1}{a^2}\int\frac{dx}{\left(\frac{x-x_0}{a}\right)^2+1} = \left|\begin{array}{l}\frac{x-x_0}{a}=t \\ \frac{1}{a}dx=dt\end{array}\right| = \frac{1}{a}\int\frac{dt}{t^2+1} = \frac{1}{a}\,\mathrm{arctg}\,t + D =$

$= \frac{1}{a}\,\mathrm{arctg}\,\frac{x-x_0}{a} + D$

Př

$$\int \frac{x+1}{(x-1)(x^2-2x+5)}\,dx = \int\left(\frac{A}{x-1} + \frac{Bx+C}{x^2-2x+5}\right)dx =$$

$$= A\cdot \ln|x-1| + \frac{B}{2}\ln\left(x^2-2x+5\right) + \frac{B+C}{2}\operatorname{arctg}\frac{x-1}{2} + D.$$

$$\int \frac{Bx+C}{x^2-2x+5}\,dx = \int \frac{\frac{B}{2}(2x-2) + B + C}{x^2-2x+5}\,dx = \frac{B}{2}\ln\left(x^2-2x+5\right) + (B+C)\cdot\int\frac{dx}{(x-1)^2+4} =$$

$$= \frac{B}{2}\ln\left(x^2-2x+5\right) + \frac{B+C}{2}\operatorname{arctg}\frac{x-1}{2} + D$$

IV

$$\int \frac{Bx+C}{\left((x-x_0)^2+a^2\right)^n}\,dx = \int \frac{\frac{B}{2}(2x-2x_0) + Bx_0 + C}{\left((x-x_0)^2+a^2\right)^n}\,dx = \left|\begin{matrix}(x-x_0)^2+a^2 = t\\ (2x-2x_0)\,dx = dt\end{matrix}\right| =$$

$$= \frac{B}{2}\int\frac{dt}{t^n} + (Bx_0+C)\int\frac{dx}{\left((x-x_0)^2+a^2\right)^n} =$$

$$= \frac{B}{2(1-n)}\,\frac{1}{t^{n-1}} + (Bx_0+C)\int\frac{dx}{\left((x-x_0)^2+a^2\right)^n} =$$

$$= \frac{B}{2(1-n)}\,\frac{1}{\left((x-x_0)^2+a^2\right)^{n-1}} +$$

$$\int \frac{dx}{\left((x-x_0)^2+a^2\right)^n} =: K_n$$

$$K_{n+1} = \frac{1}{a^2}\left(\frac{2n-1}{2n} K_n + \frac{1}{2n}\,\frac{x-x_0}{\left((x-x_0)^2+a^2\right)^n}\right)$$

$$K_1 = \int \frac{dx}{(x-x_0)^2+a^2} = \frac{1}{a}\operatorname{arctg}\frac{x-x_0}{a}$$

Pr

$$\int \frac{dx}{(x+1)(x^2+x+1)^2} = \int \frac{A}{x+1}\,dx + \int \frac{Bx+C}{(x^2+x+1)^2}\,dx + \int \frac{Dx+E}{x^2+x+1}\,dx =$$

$$= A\cdot \ln|x+1| + \int \frac{\frac{D}{2}(2x+1)-\frac{D}{2}+E}{x^2+x+1}\,dx + \int \frac{\frac{B}{2}(2x+1)-\frac{B}{2}+C}{(x^2+x+1)^2}\,dx$$

$$\int \frac{\frac{D}{2}(2x+1)-\frac{D}{2}+E}{x^2+x+1}\,dx = \frac{D}{2}\ln(x^2+x+1) - \left(\frac{D}{2}-E\right)\int \frac{dx}{(x+\frac{1}{2})^2+\frac{3}{4}} =$$

$$= \frac{D}{2}\ln(x^2+x+1) - \left(\frac{D}{2}-E\right)\cdot\frac{2}{\sqrt{3}}\operatorname{arctg}\frac{x-\frac{1}{2}}{\frac{2}{\sqrt{3}}}$$

$$\int \frac{\frac{B}{2}(2x+1)-\frac{B}{2}+C}{(x^2+x+1)^2}\,dx = \frac{B}{2}\int\frac{dt}{t^2} - \left(\frac{B}{2}-C\right)\int\frac{dx}{\left((x+\frac{1}{2})^2+\frac{3}{4}\right)^2} =$$

$$= \frac{B}{2}\cdot\frac{-1}{x^2+x+1} - \left(\frac{B}{2}-C\right)\cdot K_2$$

$$K_2 = \frac{1}{\frac{3}{4}}\left(\frac{2\cdot1-1}{2\cdot1}\cdot K_1 + \frac{1}{2\cdot1}\cdot\frac{x+\frac{1}{2}}{\left((x+\frac{1}{2})^2+\frac{3}{4}\right)^1}\right) = \frac{4}{3}\left(\frac{1}{2}\,\frac{2}{\sqrt{3}}\operatorname{arctg}\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}} + \frac{1}{2}\,\frac{x+\frac{1}{2}}{x^2+x+1}\right)$$

Typické substituce

$R(a^x)$ $\quad S: a^x = t$

Př $\displaystyle\int \frac{dx}{e^x+e^{-x}} = \left| \begin{array}{l} e^x = t \\ e^x dx = dt \\ \Downarrow \\ dx = \frac{dt}{t} \end{array} \right| = \int \frac{dt}{t(t+t^{-1})} = \int \frac{dt}{t^2+1} = \operatorname{arctg} t + C = \operatorname{arctg} e^x + C$

$R\left(x, \sqrt[n]{\frac{ax+b}{cx+d}}\right)$ $\quad S: \frac{ax+b}{cx+d} = t^n$

Př $\displaystyle\int \sqrt[3]{\frac{x-1}{x+1}} \cdot \frac{1}{x-1}\, dx = \left| \begin{array}{l} \frac{x-1}{x+1} = t^3 \Rightarrow \quad x-1 = t^3 x + t^3 \\ \quad x = \frac{t^3+1}{1-t^3} \Rightarrow x-1 = \frac{2t^3}{1+t^3} \\ \quad dx = \frac{6t^2}{(1-t^3)^2}\, dt \end{array} \right| =$

$\displaystyle = \int t \cdot \frac{1+t^3}{2t^3} \cdot \frac{6t^2}{(1-t^3)^2}\, dt = 3\int \frac{dt}{1-t^3} = 3\int \frac{dt}{(1-t)(1+t+t^2)} = \dots$

RRLF.

$R\left(x, \sqrt{ax^2+bx+c}\right)$

Př $\displaystyle\int \frac{x\, dx}{(x+1)\sqrt{-x^2+x+2}} = \int \frac{x\, dx}{(x+1)\sqrt{(2-x)(x+1)}} = \left| \begin{array}{l} S: \frac{2-x}{x+1} = t^2 \\ x = \frac{2-t^2}{t^2+1} \qquad dx = \frac{-6t}{(t^2+1)^2}\, dt \\ x+1 = \frac{3}{t^2+1}, \quad \sqrt{(2-x)(x+1)} = \sqrt{\frac{2-x}{x+1}(x+1)^2} \end{array} \right.$

$\displaystyle = \int \frac{\frac{2-t^2}{t^2+1} \cdot \frac{-6t}{(t^2+1)^2}}{\frac{3}{t^2+1} \cdot t \cdot \frac{3}{t^2+1}}\, dt = \frac{2}{3}\int \frac{t^2-2}{t^2+1}\, dt = RLF \dots$

(Př)
$$\int \frac{dx}{x+\sqrt{x^2+x+1}} = \left| \begin{aligned} \sqrt{x^2+x+1} &= -x+t \;/^2 \\ \cancel{x^2}+x+1 &= \cancel{x^2}-2xt+t^2 \\ x &= \frac{t^2-1}{1+2t} \\ dx &= \frac{2t^2+2t+2}{(1+2t)^2}\,dt \end{aligned} \right| =$$

$$= \int \frac{2t^2+2t+2}{(1+2t)^2}\cdot\frac{1}{t}\,dt = \text{RRLF} \dots$$

$$\int \sin^m x \cos^n x \, dx \quad , m,n \in \mathbb{Z}$$

m liché $\rightarrow$ S: $\cos x = t$

n liché $\rightarrow$ $\sin x = t$

m,n sudá $\rightarrow$ $\operatorname{tg} x = t$

(Př)
$$\int \sin^5 x \underbrace{\cos^5 x}_{(\cos^2 x)^2\cdot\cos x} dx = \left| \begin{aligned} \sin x &= t \\ \cos x\,dx &= dt \end{aligned} \right| = \int t^5\cdot(1-t^2)^2\,dt = \dots$$

(Př)
$$\int \frac{\sin^2 x}{\cos^6 x}\,dx = \left| \begin{aligned} \operatorname{tg} x &= t \\ \frac{1}{\cos^2 x}\,dx &= dt \end{aligned} \right| = \int t^2\cdot(t^2+1)\,dt = \dots$$

$$\frac{1}{\cos^2 x} = ?$$

$$t^2 = \operatorname{tg}^2 x = \frac{\sin^2 x}{\cos^2 x} = \frac{1-\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} - 1$$

$R(\sin x, \cos x)$:

$$R(\sin x)\cdot\cos x \to \sin x = t$$

$$R(\cos x)\cdot\sin x \to \cos x = t$$

$$R(\operatorname{tg} x) \longrightarrow \operatorname{tg} x = t \quad \Leftrightarrow R(\sin x, \cos x) = R(-\sin x, -\cos x)$$

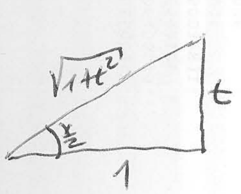
nebo univerzální substituce

$$\boxed{\operatorname{tg}\frac{x}{2} = t}$$

$$\frac{x}{2} = \operatorname{arctg} t \qquad \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$$

$$x = 2\operatorname{arctg} t \qquad \cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2}$$

$$dx = 2\cdot\frac{1}{t^2+1}\,dt$$

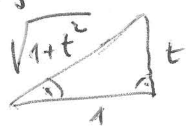


$$\sin\frac{x}{2} = \frac{t}{\sqrt{1+t^2}}, \quad \cos\frac{x}{2} = \frac{1}{\sqrt{1+t^2}}$$

$$\Downarrow \qquad \Downarrow$$

$$\sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}$$

Př: $\displaystyle\int \frac{(\sin x + 2)\cdot\cos x}{\sin^2 x - 2\sin x + 5}\,dx = \left|\begin{array}{l}\sin x = t\\ \cos x\,dx = dt\end{array}\right| = \int\frac{t+2}{t^2-2t+5}\,dt =$ RRLF...



Př: $\displaystyle\int\frac{\sin x - \cos x}{\sin x + 2\cos x}\,dx = \left|\begin{array}{l}\operatorname{tg} x = t\\ x = \operatorname{arctg} t\end{array}\right| \to dx = \frac{1}{t^2+1}\,dt, \quad \sin x = \frac{t}{\sqrt{t^2+1}}, \quad \cos x = \frac{1}{\sqrt{t^2+1}}$

$$\left(\frac{-\sin + \cos x}{-\sin x - 2\cos x} = \frac{\sin x - \cos x}{\sin x + 2\cos x}\right) \to$$

$$= \int\frac{\frac{t}{\sqrt{1+t^2}} - \frac{1}{\sqrt{1+t^2}}}{\frac{t}{\sqrt{1+t^2}} + 2\frac{1}{\sqrt{1+t^2}}}\cdot\frac{1}{1+t^2}\,dt = \int\frac{t-1}{(t+2)(t^2+1)}\,dt$$

Př $\int \frac{1-\sin x}{1+\cos x}\,dx = \left| \begin{array}{l} tg\frac{x}{2} = t \\ + \text{napočítáme} \end{array} \right| =$

$$= \int \frac{1 - \frac{2t}{1+t^2}}{1 + \frac{1-t^2}{1+t^2}} \cdot 2 \cdot \frac{1}{1+t^2}\,dt = \int \frac{t^2+1-2t}{1+t^2}\,dt = \left| \text{RLF nebo} \right| =$$

$$= \int \frac{t^2+1}{1+t^2}\,dt - \int \frac{2t}{1+t^2}\,dt = t - \ln(1+t^2) + C =$$

$$= tg\frac{x}{2} + \ln\left(1 + tg^2\frac{x}{2}\right) + C$$

$R(x, \sqrt{a^2-x^2}) \longrightarrow$ S: $x = a\cdot\sin t$ nebo $x = a\cdot\cos t$

$R(x, \sqrt{x^2-a^2}) \longrightarrow$ $x = \frac{a}{\sin t}$ nebo $x = \frac{a}{\cos t}$

$R(x, \sqrt{x^2+a^2}) \longrightarrow$ $x = a\,tg\,t$

Př $\int \frac{1}{x^2\sqrt{x^2+a^2}}\,dx = \left| \begin{array}{l} x = a\cdot tg\,t \\ dx = \frac{a}{\cos^2 t}\,dt \end{array} \right| = \int \frac{1}{a^2\frac{\sin^2 t}{\cos^2 t}} \cdot a\frac{1}{\cos t}\,dt = \int \frac{1\cdot\cos t}{a\cdot\sin^2 t}\,dt$

$$\sqrt{x^2+a^2} = \sqrt{a^2\frac{\sin^2 t}{\cos^2 t} + a^2} = a\sqrt{\frac{\sin^2 t + \cos^2 t}{\cos^2 t}} = a\sqrt{\frac{1}{\cos^2 t}}$$

$$= \left| \begin{array}{l} \sin t = u \\ \cos t\,dt = du \end{array} \right| = \cancel{\frac{1}{a}\int \frac{du}{u^2} = \frac{1}{a}\cdot\frac{-1}{u} + C = \frac{-1}{a\sin t} + C} =$$

$$\cancel{= \frac{-1}{a\sin\operatorname{arctg}\frac{x}{a}}}$$

~~a sin t = cos t~~