Matice zobrazení

Příklad: Projekce podél osy z do xy.

v bázi e_1, e_2, e_3 je dáno maticí

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = A \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = A \cdot x$$

v bázi:

$$f_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad f_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad f_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Příklad 2:

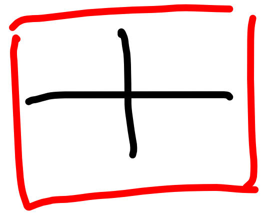
$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \qquad D = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Co je to za zobrazení?

$C: \quad e_1 \longmapsto a \cdot e_1 \qquad e_2 \longmapsto b \cdot e_2$

$\mathbb{R}^2$



$$\mathbb{R}^2 = \langle e_1 \rangle \oplus \langle e_2 \rangle$$

$C \cdot x = \lambda x$ (vlastní λ, $x \neq 0$)

$\iff (C - \lambda E) \cdot x = 0 \iff C - \lambda E$ není invert.

$\iff |C - \lambda E| = 0$

$$\begin{vmatrix} a-\lambda & 0 \\ 0 & b-\lambda \end{vmatrix} = (a-\lambda)\cdot(b-\lambda) = 0$$

řešení $|A-\lambda E| = 0$ .. vlastní čísla

řeší $(A-\lambda E)\cdot x = 0$, $x \neq 0$, tj. vlastní čísla .. vlast. vektory

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pro $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ speciální případ C ✓

$B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad |B-\lambda E| = \begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = \lambda^2$

$\Rightarrow \lambda_{1,2} = 0 \quad 0 = (B-\lambda E)\cdot x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot x$

$\Longleftrightarrow \begin{pmatrix} x_1 \\ 0 \end{pmatrix}$

tj. $e_1 \mapsto 0$, $e_2 \mapsto e_1$

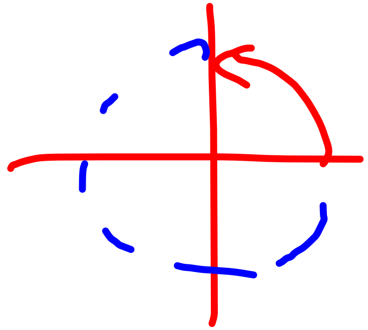
$A^k = 0 \qquad A\cdot x = \lambda x$

$\Rightarrow A^k \cdot x = \lambda^k \cdot x = 0$

$\Rightarrow \lambda = 0$

$$D = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad e_1 \mapsto e_2 \qquad e_2 \mapsto -e_1$$

$$|D - \lambda E| = \left| \begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} \right| = \lambda^2 + 1$$

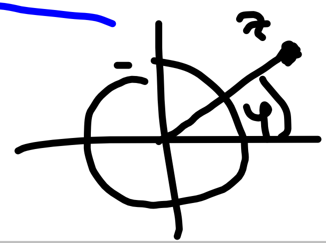


$\varphi = 90°$

$$\lambda_{1,2} = \pm i$$

$\overline{|D - \lambda E|}$ ... charakteristický polynom

VŽDY: komplexně sdruženým kořenům odpovídá rotace v rovině o argument a dilatace absolutní hodnotou kořene $x \pm iy$.



$|z|$

φ

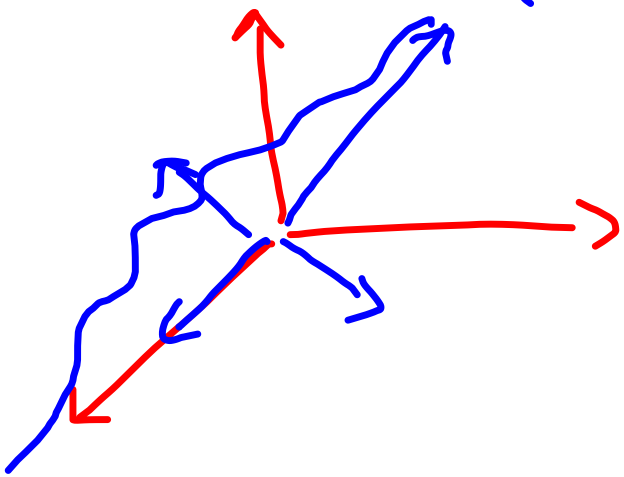
Příklad 3:

$f: \mathbb{R}^3 \to \mathbb{R}^3$, e_1, e_2, e_3

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad e_1 \mapsto e_3,\; e_2 \mapsto e_2,\; e_3 \to e_1$$

$$|A - \lambda E| = \begin{vmatrix} -\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = \lambda^2(1-\lambda) - (1-\lambda)$$

$$= (1-\lambda)(\lambda^2 - 1) \Rightarrow \begin{aligned} \lambda_1 &= 1 \\ \lambda_2 &= 1 \\ \lambda_3 &= -1 \end{aligned}$$



$\lambda = 1$:

$$(A - E) = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$ 2 dim řešení $x_1 = x_3$, např.

$$\left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle$$

$\lambda = -1$:

$$(A + E) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$ $x_2 = 0$ $x_1 = -x_3$ např.

$$\left\langle \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\rangle$$

Příklad 4 $V = \mathbb{R}_2[x]$ derivace

$$a_0 + a_1 x + a_2 x^2 \longmapsto a_1 + 2a_2 x$$

v bázi $e_1 = 1$, $e_2 = x$, $e_3 = x^2$

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \qquad |A - \lambda E| = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 2 \\ 0 & 0 & -\lambda \end{vmatrix}$$

$$A^2 = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad = -\lambda^3$$

$$A^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$$

Příklad 5 $\mathbb{R}^3 \to \mathbb{R}^3$ ve standardní bázi:

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix}.$$

Najděte matici v bázi

$$\boxed{f_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad f_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad f_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}}$$

a zjistěte vlastní čísla a vlastní vektory.

$$\underline{e} = \{e_1, e_2, e_3\} \longmapsto \underline{f} = \{f_1, f_2, f_3\}$$

$$A \longmapsto T^{-1} \cdot A \cdot T = B$$

$$|B - \lambda E| = |T^{-1} \cdot A \cdot T - \lambda E|$$

$$= |T^{-1} \cdot (A - \lambda E) \cdot T| = |\cancel{T^{-1}}| \cdot |\cancel{T}| \cdot |A - \lambda E|$$

vlastní čísla:

$$|A-\lambda E| = \begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 2-\lambda & 1 \\ 1 & 2 & 1-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)(1-\lambda)$$

$$+1-2(1-\lambda)-(1-\lambda) = 0-2\lambda+4\lambda^2-\lambda^3$$

$$= -\lambda(\lambda^2-4\lambda+2)$$

$\lambda_1 = 0$

$\lambda_{2,3} = \dfrac{4 \pm \sqrt{8}}{2} = 2 \pm \sqrt{2}$

vlastní vektory:

$\lambda = 0$: $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow \left\langle \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\rangle$

$\lambda = 2+\sqrt{2}$

$$(A-\lambda E) = \begin{pmatrix} -1-\sqrt{2} & 1 & 0 \\ 1 & -\sqrt{2} & 1 \\ 1 & 2 & -1-\sqrt{2} \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} -1-\sqrt{2} & 1 & 0 \\ 1 & -\sqrt{2} & 1 \\ 0 & 2+\sqrt{2} & -1-\sqrt{2} \end{pmatrix}$$

. . . .

řeš!

$$\left\langle \begin{pmatrix} 1 \\ 1+\sqrt{2} \\ 1+\sqrt{2} \end{pmatrix} \right\rangle$$

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shrnutí:

$$\left\langle \begin{pmatrix} 1 \\ 1-\sqrt{2} \\ 1-\sqrt{2} \end{pmatrix} \right\rangle$$

$f_1 = 1\cdot e_1 - 1\cdot e_2 + 1\cdot e_3\,,\quad f_2 = \dots$

$$T^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \xmapsto{\;T\;} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \xmapsto{\;A\;} A\cdot x \xmapsto{\;T^{-1}\;} B\cdot y$$

souřadnice v $\underline{f}$ souřadnice v $\underline{e}$ souřadnice v $\underline{f}$

$$B = T^{-1}\cdot A\cdot T$$

$$T = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & -1 & 1 \\ 2 & 1 & -1 \\ -2 & 1 & 3 \end{pmatrix} \cdot \underbrace{|T|^{-1}}_{= 1/4}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 1 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 1 & 1 & 0 \\ 0 & 0 & 4 & -2 & 1 & 3 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1/3 & 1/3 & 1/3 & 0 \\ 0 & 0 & 1 & -1/2 & 1/4 & 3/4 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/4 & 1/4 \\ 0 & 1 & 0 & 1/2 & 1/4 & -1/4 \\ 0 & 0 & 1 & -1/2 & 1/4 & 3/4 \end{array}\right)$$