skupina A

1

$f(x) = x e^{\frac{1}{x}}$

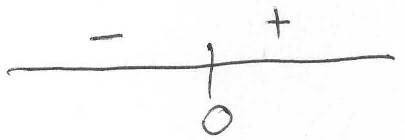
$D(f) = \mathbb{R} \setminus \{0\}$

ani sudá, ani lichá, ani periodická není

$\lim_{x\to\infty} f(x) = \infty \qquad \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{e^{\frac{1}{x}}}{\frac{1}{x}} \overset{L.P.}{=} \frac{e^{\frac{1}{x}} \cdot \left(\frac{1}{x}\right)'}{\left(\frac{1}{x}\right)'} = e^{[\infty]} = \infty$

$\lim_{x\to -\infty} f(x) = -\infty \qquad \lim_{x\to 0^-} f(x) = \ldots = e^{[-\infty]} = 0$

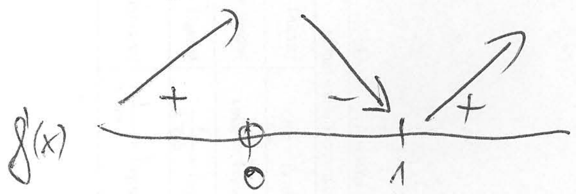
nulové body $x = 0$



první derivace:

$f'(x) = e^{\frac{1}{x}} + x e^{\frac{1}{x}} \cdot (-1) \frac{1}{x^2} = e^{\frac{1}{x}} \cdot \frac{x-1}{x}$

nulové body $f'(x)$



$f(1) = e$

druhá derivace:

$f''(x) = e^{\frac{1}{x}} (-1) \frac{1}{x^2} \left(1 - \frac{1}{x}\right) + e^{\frac{1}{x}} \cdot \frac{1}{x^2} = e^{\frac{1}{x}} \cdot \frac{1}{x^3}$

skupina A |

nulové body $f''(x)$

$f''(x)$ $\quad (-) \quad \underset{0}{\circ} \quad (+)$

asymptoty:

bez směrnice: $p: x=0$

je asymptota, neboť $\lim\limits_{x \to 0^+} f(x) = \infty$

se směrnicí: $q: y = ax+b$

$$a_+ = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} e^{\frac{1}{x}} = e^0 = 1$$

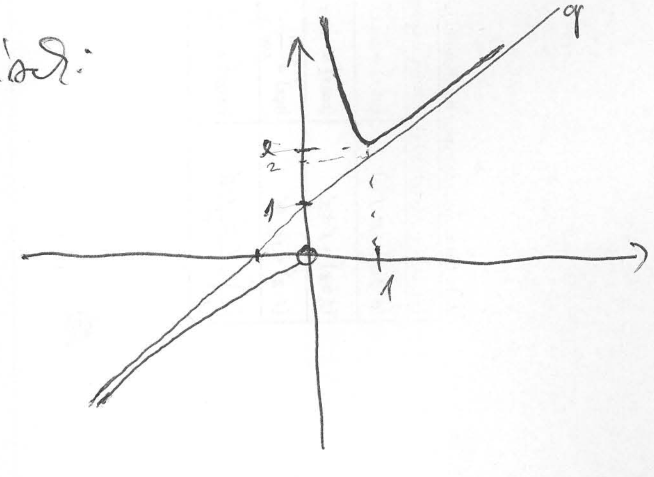
$$b_+ = \lim_{x\to\infty} \left(f(x) - x\right) = \lim_{x\to\infty} \left(x e^{\frac{1}{x}} - x\right) = \lim_{x\to\infty} x\left(e^{\frac{1}{x}} - 1\right) =$$

$$= \lim_{x\to\infty} \frac{e^{\frac{1}{x}} - 1}{\frac{1}{x}} \overset{L.P.}{=} \lim_{x\to\infty} \frac{e^{\frac{1}{x}} \cdot \left(\frac{1}{x}\right)'}{\left(\frac{1}{x}\right)'} = \lim_{x\to\infty} e^{\frac{1}{x}} = e^0 = 1$$

$\left.\begin{matrix} a_+ = a_- \\ b_+ = b_- \end{matrix}\right\} \Rightarrow$ jediná asymptota se směrnicí

$q: y = x + 1$

obrázek:



skupina B

$$f(x) = \frac{x^3-1}{(x-1)^3} = \frac{(x-1)(x^2+x+1)}{(x-1)(x-1)^2} \overset{\text{protože } 1\notin D(f)}{=} \frac{x^2+x+1}{(x-1)^2}$$

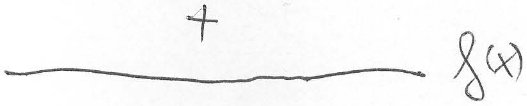
$$D(f) = \mathbb{R}\setminus\{1\}$$

ani sudá, ani lichá, ani periodická není

$$\lim_{x\to\infty} f(x) = 1 \qquad \lim_{x\to 1^+} f(x) = \left[\frac{3}{0^+}\right] = \infty$$

$$\lim_{x\to-\infty} f(x) = 1 \qquad \lim_{x\to 1^-} f(x) = \left[\frac{3}{0^+}\right] = \infty$$

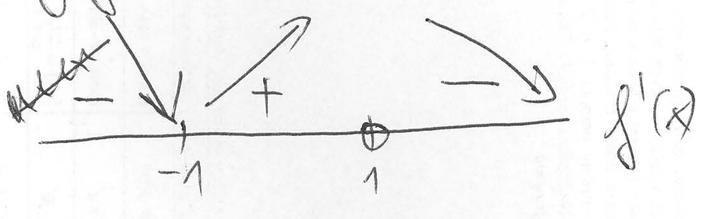
nulové body: $f(x) > 0$



první derivace:

$$f'(x) = \frac{(2x+1)(x-1)^2 - (x^2+x+1)\,2(x-1)}{(x-1)^4} = \frac{-3(x+1)}{(x-1)^3}$$

nulové body $f'(x)$: $x=1$, $x=-1$



$$f(-1) = \frac{1}{4}$$

skupina B

druhá derivace:

$$f''(x) = \frac{-3(x-1)^3 + 3(x+1)\,3(x-1)^2}{(x-1)^6} = \frac{6(x+2)}{(x-1)^4}$$

nulové body $f''(x)$

$$\underset{(-)}{} \quad -2 \quad \underset{(+)}{} \qquad f''(x)$$

$$f(-2) = \frac{1}{3}$$

asymptoty:

$p: x = 1$ je asymptota, neboť $\lim\limits_{x \to 1^+} f(x) = \infty$

(bez směrnice)

asymptoty se směrnicí: $q: y = ax + b$

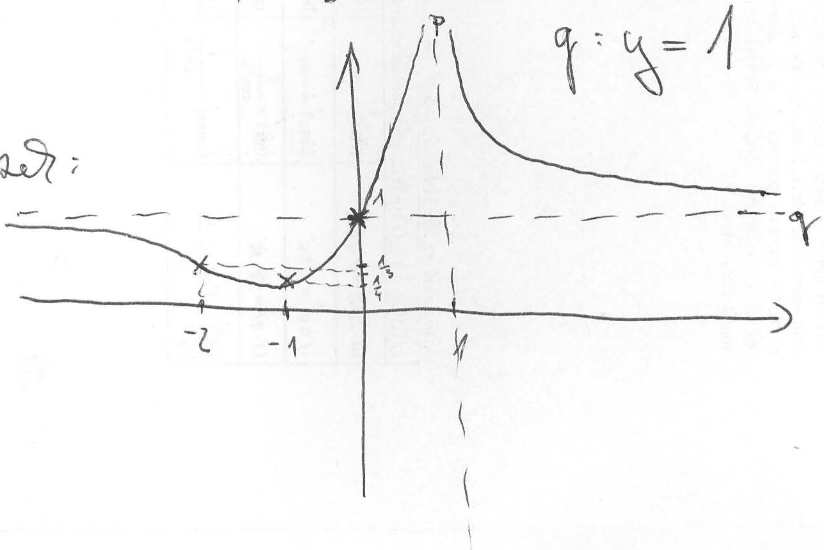
$$a_+ = \lim_{x \to \infty} \frac{f(x)}{x} = 0$$

$$b_+ = \lim_{x \to \infty} \left(f(x) - 0 \cdot x\right) = \lim_{x \to \infty} f(x) = 1$$

$$\left.\begin{array}{l} a_- = a_+ \\ b_- = b_+ \end{array}\right\} \text{jediná asymptota se směrnicí}$$

$q: y = 1$

obrázek:



skupina C

$f(x) = x^2 e^{-\frac{1}{x}}$

ani lichá, ani sudá, ani periodická není

$D(f) = \mathbb{R} \setminus \{0\}$

$\lim\limits_{x \to \infty} f(x) = [\infty \cdot 1] = \infty \qquad \lim\limits_{x \to 0^+} f(x) = [0 \cdot e^{-\infty}] = 0 \cdot 0 = 0$

$\lim\limits_{x \to -\infty} f(x) = [\infty \cdot 1] = \infty \qquad \lim\limits_{x \to 0^-} f(x) = [0 \cdot \infty] = \lim\limits_{x \to 0^-} \dfrac{e^{-\frac{1}{x}}}{\frac{1}{x^2}} = \left[\dfrac{\infty}{\infty}\right] \overset{L.P.}{=}$

$= \lim\limits_{x \to 0^-} \dfrac{e^{-\frac{1}{x}} \cdot \frac{1}{x^2}}{-\frac{2}{x^3}} = \lim\limits_{x \to 0^-} \dfrac{e^{-\frac{1}{x}}}{-\frac{2}{x}} \overset{L.P.}{=} \lim\limits_{x \to 0^-} \dfrac{e^{-\frac{1}{x}} \cdot \frac{1}{x^2}}{\frac{2}{x^2}} =$

$= \lim\limits_{x \to 0^-} \frac{1}{2} e^{-\frac{1}{x}} = \left[\frac{1}{2} \cdot \infty\right] = \infty$

nulové body: $f(x) > 0$

$+$ $f(x)$

první derivace:

$f'(x) = 2x \cdot e^{-\frac{1}{x}} + x^2 e^{-\frac{1}{x}} \cdot \frac{+1}{x^2} = e^{-\frac{1}{x}}(2x+1)$

nulové body f':

$f'(x)$: $-$ ↘ $-\frac{1}{2}$ ↗ $+$

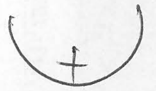
$f(-\frac{1}{2}) = \frac{e^2}{4} \sim 1{,}2$

druhá derivace:

$f''(x) = e^{-\frac{1}{x}} \cdot \frac{+1}{x^2}(2x+1) + e^{-\frac{1}{x}} \cdot 2 = e^{-\frac{1}{x}}\left(2 + \frac{2}{x} + \frac{1}{x^2}\right) = e^{-\frac{1}{x}} \cdot \frac{2x^2+2x+1}{x^2}$

Skupina C

nulové body $f''(x)$: $f''(x) > 0$



asymptoty:

bez směrnice: $p: x=0$ je asymptota, neboť $\lim\limits_{x \to 0^-} f(x) = \infty$.

se směrnicí: $q: y = ax + b$

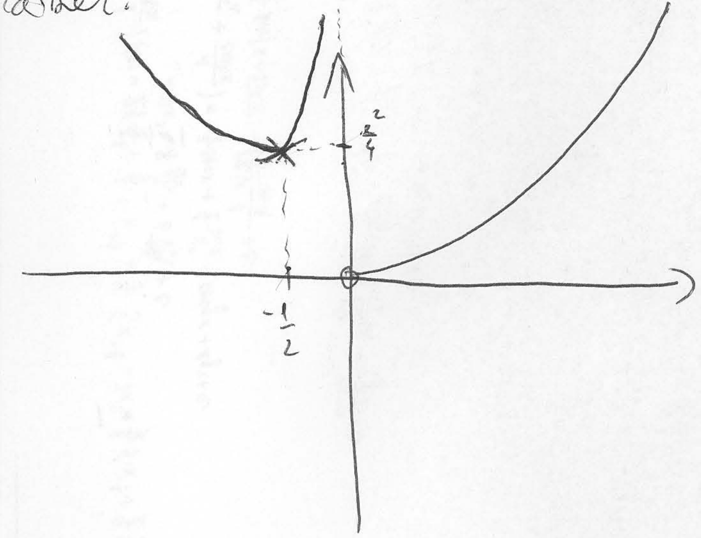
$$a_+ = \lim_{x \to \infty} \frac{f(x)}{x} = \lim_{x \to \infty} x e^{-\frac{1}{x}} = [\infty \cdot e^0] = \infty,$$ nelze počítat b_+

a v $+\infty$ není asymptota

$$a_- = \lim_{x \to -\infty} \frac{f(x)}{x} = [-\infty \cdot e^0] = -\infty$$

v $-\infty$ není asymptota.

obrázek:



skupina D

$f(x) = x e^{-\frac{1}{x}}$

$D(f) = \mathbb{R} \setminus \{0\}$

ani lichá, ani sudá, ani periodická není.

$\lim\limits_{x \to \infty} f(x) = [\infty \cdot e^0] = \infty$

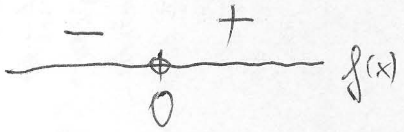
$\lim\limits_{x \to -\infty} f(x) = [-\infty \cdot e^0] = -\infty$

$\lim\limits_{x \to 0^+} f(x) = [0 \cdot e^{-\infty}] = 0 \cdot 0 = 0$

$\lim\limits_{x \to 0^-} f(x) = [0 \cdot e^{\infty}] = [0 \cdot \infty] = \lim\limits_{x \to 0^-} \dfrac{e^{-\frac{1}{x}}}{\frac{1}{x}} = \left[\dfrac{\infty}{\infty}\right] \overset{L.P.}{=}$

$\overset{L.P.}{=} \lim\limits_{x \to 0^-} \dfrac{e^{-\frac{1}{x}} \left(-\frac{1}{x}\right)'}{\left(\frac{1}{x}\right)'} = \lim\limits_{x \to 0^-} -e^{-\frac{1}{x}} = [-e^{\infty}] = -\infty$

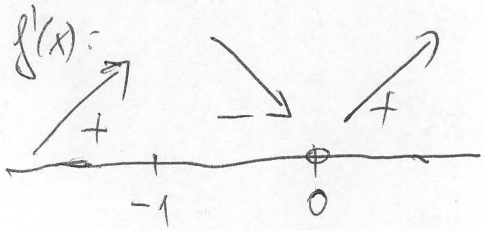
nulové body $f(x)$: $x = 0$



první derivace $f'(x)$:

$f'(x) = e^{-\frac{1}{x}} + x e^{-\frac{1}{x}} \cdot \dfrac{1}{x^2} = e^{-\frac{1}{x}} \cdot \dfrac{x+1}{x}$

nulové body $f'(x)$:

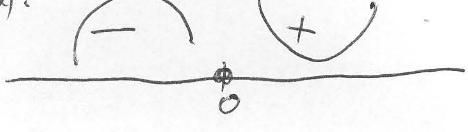


$f(-1) = -e$

druhá derivace:

$f''(x) = e^{-\frac{1}{x}} \cdot \dfrac{1}{x^2} \cdot \dfrac{x+1}{x} + e^{-\frac{1}{x}} \cdot \dfrac{x-(x+1)}{x^2} = e^{-\frac{1}{x}} \cdot \dfrac{1}{x^3}$

nulové body $f''(x)$:



asymptoty:

bez směrnice $\boxed{p: x=0}$ je asymptota, neboť $\lim\limits_{x\to 0^-} f(x) = -\infty$

se směrnicí: $q: y = ax+b$

$$a_+ = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} e^{-\frac{1}{x}} = e^0 = 1$$

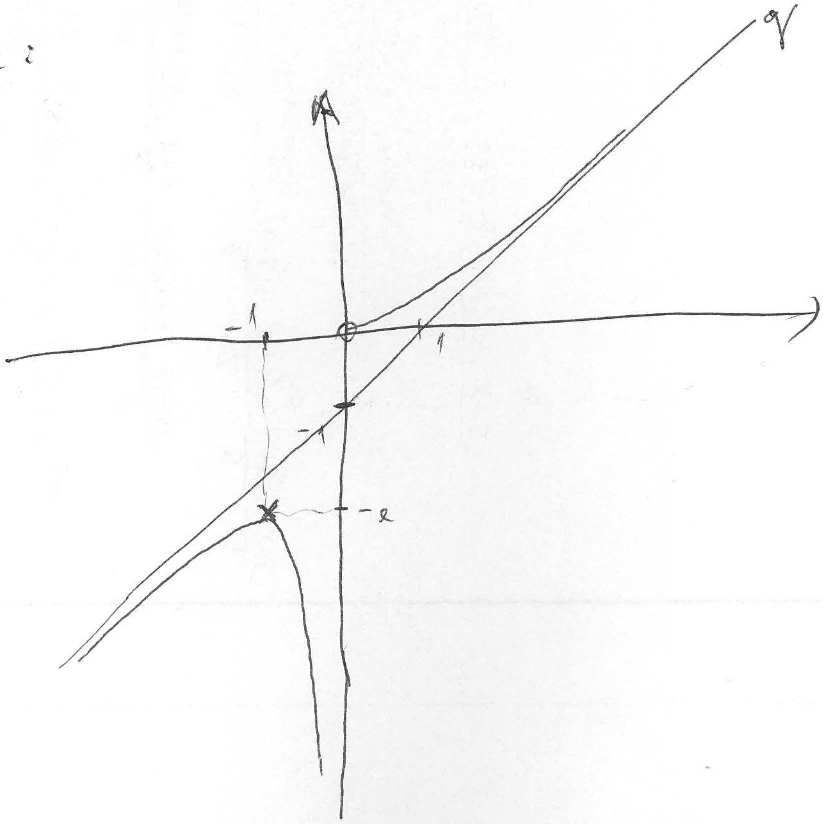
$$b_+ = \lim_{x\to\infty} \left(f(x) - x\right) = \lim_{x\to\infty} x\left(e^{-\frac{1}{x}} - 1\right) = \lim_{x\to\infty} \frac{e^{-\frac{1}{x}} - 1}{\frac{1}{x}} \overset{L.P.}{=} \left[\frac{0}{0}\right] =$$

$$\overset{L.P.}{=} \lim_{x\to\infty} \frac{e^{-\frac{1}{x}} \cdot \frac{1}{x^2}}{-\frac{1}{x^2}} = \lim_{x\to\infty} -e^{-\frac{1}{x}} = -e^0 = -1$$

$$\left.\begin{array}{l} a_- = a_+ \\ b_- = b_+ \end{array}\right\} \Rightarrow \text{jediná asymptota se směrnicí}$$

$$\boxed{q: \quad y = x - 1}$$

obrázek:



Skupina E

9

$$f(x) = \frac{2x}{x^2-1} + x = \frac{x(x^2+1)}{(x-1)(x+1)}$$

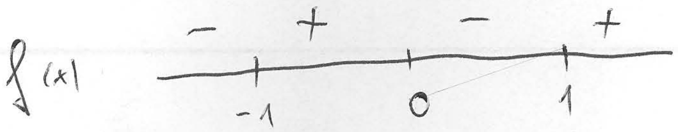
$D(f) = \mathbb{R} \setminus \{1,-1\}$

je lichá, není sudá ani periodická

$\lim_{x\to\infty} f(x) = \infty$ $\quad \lim_{x\to 1^+} f(x) = \left[\frac{2}{0^+}\right] = \infty$ $\quad \lim_{x\to -1^+} f(x) = +\infty$

$\lim_{x\to-\infty} f(x) = -\infty$ $\quad \lim_{x\to 1^-} f(x) = \left[\frac{2}{0^-}\right] = -\infty$ $\quad \lim_{x\to -1^-} f(x) = -\infty$

nulové body jsou $x=0, x=1, x=-1$



první derivace:

$$f'(x) = \frac{(3x^2+1)(x^2-1)-(x^3+x)2x}{(x^2-1)^2} = \frac{x^4-4x^2-1}{(x^2-1)^2}$$

nulové body f': $x^4-4x^2-1=0$

$S: x^2=a$

$a^2-4a-1=0 \Rightarrow a_{1,2} = 2\pm\sqrt{5}$

$S^{-1}: x^2=a$

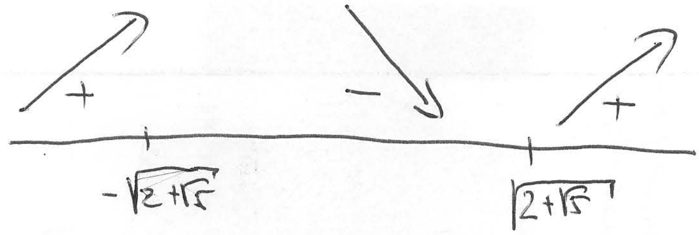
$x_{1,2,3,4} = \pm\sqrt{2\pm\sqrt{5}}$, reálné jsou jen $\pm\sqrt{2+\sqrt{5}}$

zbylé 2 jsou komplexní

skupina E

10

$f'(x)$



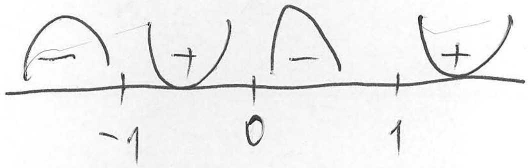
$$f\left(\sqrt{2+\sqrt{5}}\right) = \frac{10}{3}$$

$$f\left(-\sqrt{2+\sqrt{5}}\right) = -\frac{10}{3}$$

druhá derivace

$$f''(x) = \frac{(4x^3-8x)(x^2-1)^2-(x^4-4x^2-1)\,2(x^2-1)\cdot 2x}{(x^2-1)^4} = \frac{4x(x^2+3)}{(x^2-1)^3}$$

$f''(x)$



asymptoty:

$\boxed{p_1: x=1,\ p_2: x=-1}$ jsou asymptoty bez směrnice, neboť

$$\lim_{x\to 1^+} f(x) = \infty \quad \text{a} \quad \lim_{x\to -1^-} f(x) = \infty$$

se směrnicí: $y = ax+b$

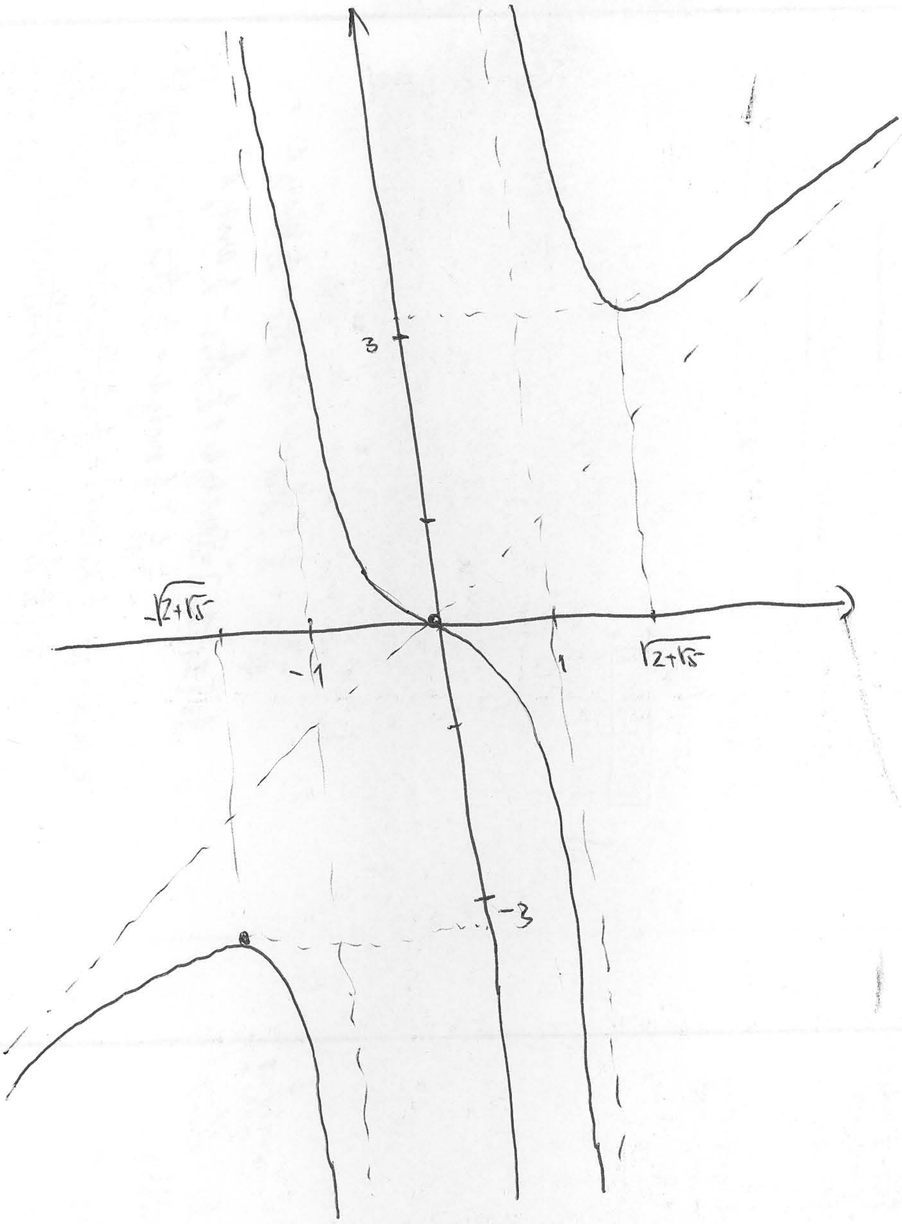
$$a_+ = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{x^2+1}{x^2-1} = 1$$

$$b_+ = \lim_{x\to\infty} (f(x)-x) = \lim_{x\to\infty} \left(\frac{x(x^2+1)}{x^2-1} - x\right) = \lim_{x\to\infty} \frac{\cancel{x^3}+x-\cancel{x^3}+x}{x^2-1} = 0$$

$a_- = a_+,\ b_- = b_+ \Rightarrow$ jediná $\boxed{p:\ y=x}$

Skupina E

11



skupina F

12

$$f(x) = \operatorname{arctg} \frac{1+x}{2-x}$$

$$D(f) = \mathbb{R} \setminus \{2\}$$

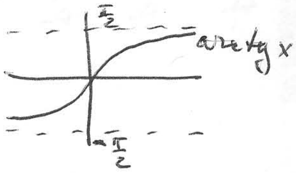
ani sudá ani lichá ani periodická není

$$\lim_{x\to\infty} f(x) = \operatorname{arctg} -1 = -\frac{\pi}{4}$$

$$\lim_{x\to 2^+} f(x) = \operatorname{arctg}\left[\frac{3}{0^-}\right] = \operatorname{arctg}[-\infty] = -\frac{\pi}{2}$$

$$\lim_{x\to-\infty} f(x) = \operatorname{arctg} -1 = -\frac{\pi}{4}$$

$$\lim_{x\to 2^-} f(x) = \operatorname{arctg}\left[\frac{3}{0^+}\right] = \operatorname{arctg}[+\infty] = \frac{\pi}{2}$$



nulové body:

$$\operatorname{arctg} \frac{1+x}{2-x} = 0$$

$\boxed{x = -1}$ + nulový bod $\boxed{x = +2 \notin D(f)!!}$

$f(x)$: $-$ | -1 | $+$ | 2 | $-$

první derivace:

$$f'(x) = \frac{1}{1+\left(\frac{1+x}{2-x}\right)^2} \cdot \left(\frac{(2-x)+(1+x)}{(2-x)^2}\right) = \frac{3}{2x^2-2x+5} > 0$$

$f'(x)$: $+$ celá rostoucí

$\boxed{\text{skupina F}}$

druhá derivace

$$f''(x) = \frac{-6(2x-1)}{(2x^2-2x+5)^2}$$

jediný nulový bod $x = \frac{1}{2}$.

$$f''(x) \qquad (+) \quad \Big|_{\frac{1}{2}} \quad (-)$$

$$\boxed{f\left(\tfrac{1}{2}\right) = \frac{\pi}{4}}$$

asymptoty:

bez směrnice: $q: x = 2$ není asymptota, neboť ani jedna z limit $\lim\limits_{x \to 2^{+/-}} f(x)$ není $\pm\infty$.

se směrnicí: $p: y = ax + b$

$$a_+ = \lim_{x \to \infty} \frac{f(x)}{x} = \left[\frac{-\frac{\pi}{4}}{\infty}\right] = 0$$

$$b_+ = \lim_{x \to \infty} (f(x) - 0 \cdot x) = \lim_{x \to \infty} f(x) = -\frac{\pi}{4}$$ (už máme spočítáno)

$$\left.\begin{aligned} a_- &= a_+ \\ b_- &= b_+ \end{aligned}\right\}$$ jediná asymptota

$$y = -\frac{\pi}{4}$$

Skupina F