$$f(x) = e^{-3x} \qquad f(0) = 1$$

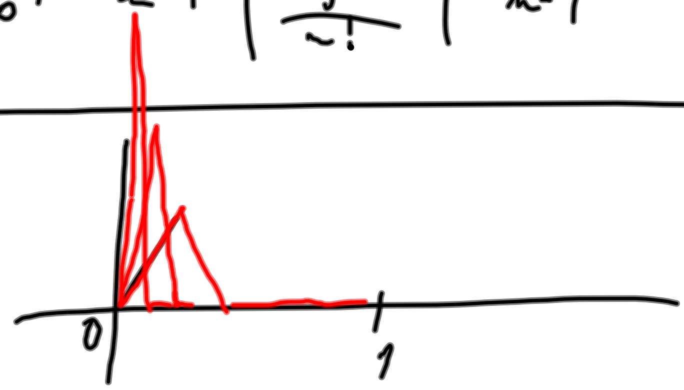
$$f'(x) = -3\, e^{-3x} \qquad f'(0) = -3$$

$$f^{(2)}(x) = 9\, e^{-3x}$$

$$\vdots \qquad\qquad \vdots$$

$$f^{(n)}(x) = (-1)^n 3^n e^{-3x} \qquad f^{(n)}(0) = (-1)^n 3^n$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{\frac{3^{n+1}}{(n+1)!}}{\frac{3^n}{n!}} \right| = \frac{3}{n+1} = 0$$



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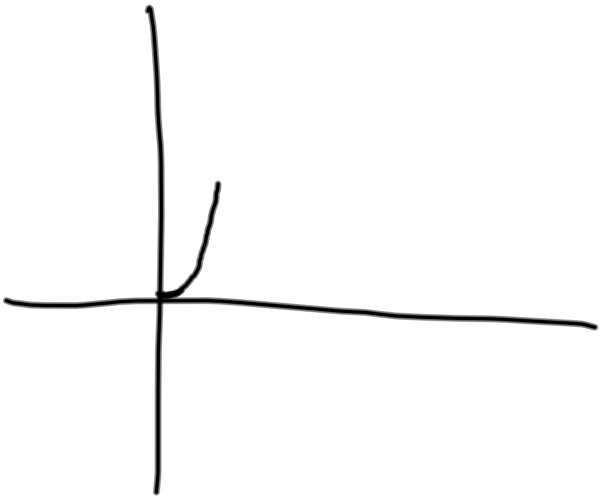


1kr.1

Ex. funkce, která by měla všechny derivace v nule nulové a byla by nenulovou?

$e^{-\frac{1}{x}}$ splňuje tyto požadavky, dodefinujeme

-ke $f(0) = 0$

$f'(x) = \frac{1}{x^2} e^{-\frac{1}{x}}$

$f^{(2)}(x) = -\frac{2}{(x^3)} \cdot e^{-\frac{1}{x}} + \frac{1}{x^2} \cdot \left(\frac{1}{x^2} e^{-\frac{1}{x}}\right) =$

$= e^{-\frac{1}{x}} \left(\frac{1}{x^4} - \frac{2}{x^3} \right)$

$\lim_{x \to 0^+} f^{(n)}(x) = 0$

$$\lim_{x\to 0^+} \frac{e^{-\frac{1}{x}}}{x^k} = \lim_{s\to\infty} \frac{e^{-s}}{\left(\frac{1}{s}\right)^k} = \lim_{s\to\infty} e^{-s}\cdot s^k =$$

$$= \lim_{s\to\infty} \frac{s^k}{e^s} = \lim_{s\to\infty} \frac{k\, s^{k-1}}{e^s} = \dots$$

$$= \dots = 0$$

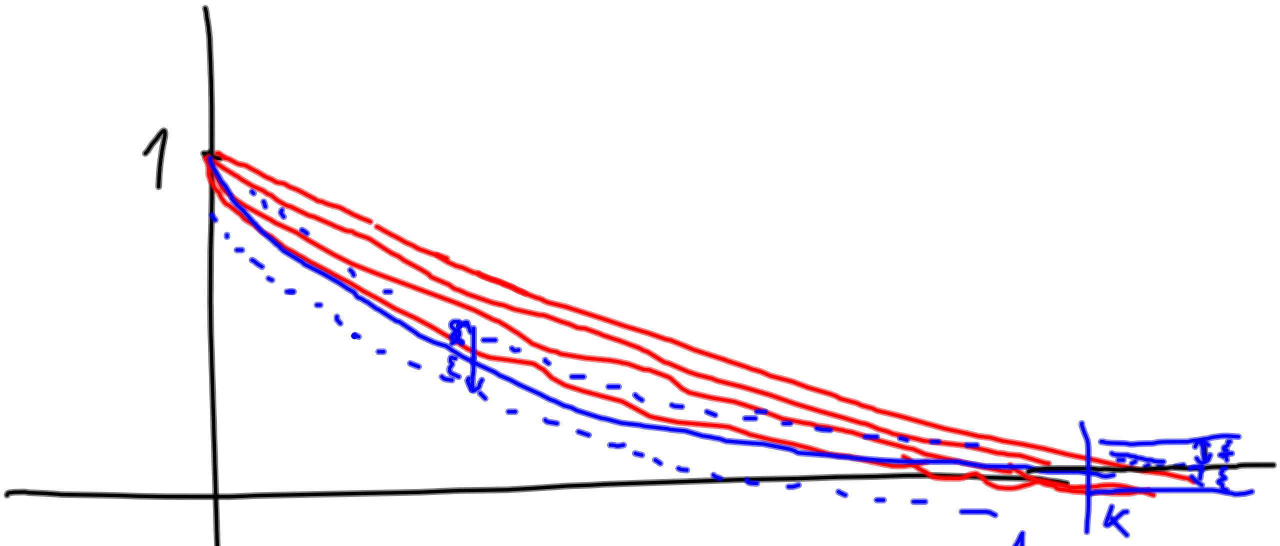
$n=1$ $$e^{-\frac{1}{x-x^2}} = e^{-\frac{1}{x(x-1)}} = e^{\frac{1}{x}-\frac{1}{x-1}} = \frac{e^{\frac{1}{x}}}{e^{\frac{1}{x-1}}}$$

$$\lim_{n \to \infty} \int_0^\infty \frac{\cos\left(\frac{x}{n}\right)}{\left(1 + \frac{x}{n}\right)^n} \, dx =$$

$$= \int_0^\infty \left(\lim_{n \to \infty} \frac{\cos\left(\frac{x}{n}\right)}{\left(1 + \frac{x}{n}\right)^n} \right) dx = \int_0^\infty \frac{1}{e^x} \, dx =$$

$$= \left[-e^{-x} \right]_0^\infty = 1$$

Záměnu limity a integrace je možno provést, pokud $f_n := \frac{\cos\left(\frac{x}{n}\right)}{\left(1 + \frac{x}{n}\right)^n} \rightrightarrows \frac{1}{e^x}$



najdeme K takové, že $\frac{1}{1+K} < \frac{\varepsilon}{2}$

$\Rightarrow |f_n(x)| < \frac{\varepsilon}{2}$ pro $x \geq K$

Na intervalu $\langle 0, K \rangle$ ukážeme, že

$\left(1+\frac{x}{n}\right)^n \rightrightarrows e^x$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots + \frac{x^n}{n!}$$

Tato řada konverguje stejnoměrně na $\langle 0, K \rangle$. $\quad \sum_{n=0}^{\infty} \frac{K^n}{n!}$ konverguje

$$g_n := \sum_{i=0}^{n} \frac{x^i}{i!}. \quad g_n \rightrightarrows e^x \Rightarrow \frac{n}{n-1} g_n \rightrightarrows e^x$$

$$\Rightarrow \frac{n-1}{n} g_n \rightrightarrows e^x$$

$$h_n := \left(1 + \frac{x}{n}\right)^n.$$

$$h_n(x) \geq \frac{n-1}{n} g_n(x)$$

$$h_n = 1 + x + \binom{n}{2}\frac{x^2}{n^2} + \binom{n}{3}\frac{x^3}{n^3} + \dots$$

$$1 + x + \frac{n(n-1)}{n^2} \cdot \frac{x^2}{2!} + \frac{n(n-1)(n-2)}{n^3} \frac{x^3}{3!} + \dots$$

f, g na $\langle a, b\rangle$

$$\langle f, g\rangle = \int_a^b f \cdot g \, dx$$

$$\langle f+h, g\rangle = \langle f, g\rangle + \langle h, g\rangle$$

$$\int_a^b (f+h)g \, dx = \int_a^b (fg + hg)\, dx = \int_a^b fg \, dx + \int_a^b hg \, dx$$

$$\int f^2 dx$$

$f_1 := \sin(x)$

$f_2 := \cos(x) - k \cdot \sin(x)$, $k \in \mathbb{R}$

podmínka na k

$$\int_0^{\frac{\pi}{2}} \sin x \, (\cos x - k \sin(x))\, dx = 0$$

$$\int_0^{\frac{\pi}{2}} (\sin x \cos x - k (\sin^2 x))\, dx = 0$$

$$\left[-\frac{\cos 2x}{4}\right]_0^{\frac{\pi}{2}} - k \int_0^{\frac{\pi}{2}} (\sin^2 x)\, dx =$$

$$\int \sin^2 x \, dx = -\cos x \sin x + \int \cos^2 x \, dx = -\cos x \sin x + x - \int \sin^2 x \, dx$$

$$\begin{bmatrix} u = \sin x & u' = \cos x \\ v' = \sin x & v = -\cos x \end{bmatrix}$$

$$\frac{1}{2} - k\frac{1}{2}\Big[x - \sin x\cos x\Big]_0^{\frac{\pi}{2}} =$$

$$= \frac{1}{2} - \frac{\pi}{4}k \;\Rightarrow\; k = \frac{2}{\pi} \;\Rightarrow\; f_2 = \cos x - \frac{2}{\pi}\sin x$$

$\left(\sin x, \cos x - \frac{2}{\pi}\sin x\right)$ je orthogonální basí uvažovaného vekt. prostoru

$$\|f_1\|^2 = \langle f_1, f_1\rangle = \int_0^{\frac{\pi}{2}} \sin^2 x\,dx = \frac{\pi}{4}$$

$$\|f_2\|^2 = \langle f_2, f_2\rangle = \int_0^{\frac{\pi}{2}} \left(\cos x - \frac{2}{\pi}\sin x\right)^2 dx = \int_0^{\frac{\pi}{2}} \cos^2 x\,dx - \frac{4}{\pi}\int_0^{\frac{\pi}{2}} \cos x \sin x\,dx +$$

$$+ \frac{4}{\pi^2}\int_0^{\frac{\pi}{2}} \sin^2 x\,dx = \frac{\pi}{4} - \frac{4}{\pi}\cdot\frac{1}{2} + \frac{4}{\pi^2}\cdot\frac{\pi}{4} =$$

$$= \frac{\pi}{4} - \frac{2}{\pi} + \frac{1}{\pi} = \frac{\pi}{4} - \frac{1}{\pi}$$

$$pr\,x = \frac{\langle \vec{x}, \vec{f_1} \rangle}{\langle \vec{f_1}, \vec{f_1} \rangle} \cdot \vec{f_1} + \frac{\langle \vec{x}, \vec{f_2} \rangle}{\langle \vec{f_2}, \vec{f_2} \rangle} \cdot \vec{f_2} =$$

$$=$$

$$\langle \vec{x}, \vec{f_1} \rangle = \int_0^{\frac{\pi}{2}} x(\sin x)\,dx = \left[-x\cos x\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos x\,dx =$$

$$\left[\begin{array}{ll} u = x & u' = 1 \\ v' = \sin x & v = -\cos x \end{array}\right.$$

$$= \left[\sin x\right]_0^{\frac{\pi}{2}} = 1$$

$$\langle \vec{x}, \vec{f_2} \rangle = \int_0^{\frac{\pi}{2}} x\left(\cos x - \frac{2}{\pi}\sin x\right) = \int_0^{\frac{\pi}{2}} x\cos x - \frac{2}{\pi}\int_0^{\frac{\pi}{2}} x\sin x\,d$$

$$\int x\cos x = x\sin x - \int \sin x = x\sin x + \cos x$$

$$\left[\begin{array}{ll} u = x & u' = 1 \\ v' = \cos x & v = \sin x \end{array}\right]$$

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$$\left[x\sin x+\cos x\right]_0^{\frac{\pi}{2}} - \frac{2}{\pi} = \frac{\pi}{2}-1-\frac{2}{\pi}$$

$$pr\,x = \frac{4}{\pi}\sin x + \frac{\frac{\pi}{2}-1-\frac{2}{\pi}}{\frac{\pi}{4}-\frac{1}{\pi}}\left(\cos x-\frac{2}{\pi}\sin x\right) =$$

$$= \frac{4}{\pi}\sin x + \frac{\frac{\pi^2-2\pi-4}{2\pi}}{\frac{\pi^2-4}{4\pi}}\left(\cos x-\frac{2}{\pi}\sin x\right) =$$

$$= \frac{4}{\pi}\sin x + \frac{2(\pi^2-2\pi-4)}{\pi^2-4}\left(\cos x-\frac{2}{\pi}\sin x\right) =$$