Digital Signal Processing

The z-Transform and Its Application to the Analysis of LTI Systems

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The Direct z-Transform

- z-transform of $x(n)$ is defined as power series:

$$X(z) \equiv \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

  where z is a complex variable
- For convenience

$$X(z) \equiv Z\{x(n)\}$$
$$x(n) \stackrel{z}{\longleftrightarrow} X(z)$$

- Since z-transform is an infinite power series, it exists only for those values of z for which this series converges
  - **Region of convergence** (ROC) of $X(z)$ is set of all values of z for which $X(z)$ attains a finite value
  - Any time we cite a z-transform, we should also indicate its ROC
- ROC of a finite-duration signal is entire z-plane except possibly points $z = 0$ and/or $z = \infty$

The Direct z-Transform

Example

- Determine z-transforms of following finite-duration signals
- $x(n) = \{\underset{\uparrow}{1}, 2, 5, 7, 0, 1\}$
  $X(z) = 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5}$
  ROC: entire z-plane except $z = 0$
- $x(n) = \{1, 2, \underset{\uparrow}{5}, 7, 0, 1\}$
  $X(z) = z^2 + 2z + 5 + 7z^{-1} + z^{-3}$
  ROC: entire z-plane except $z = 0$ and $z = \infty$
- $x(n) = \delta(n)$
  $X(z) = 1$ [i.e., $\delta(n) \overset{z}{\longleftrightarrow} 1$], ROC: entire z-plane
- $x(n) = \delta(n - k), k > 0$
  $X(z) = z^{-k}$ [i.e., $\delta(n-k) \overset{z}{\longleftrightarrow} z^{-k}$], ROC: entire z-plane except $z = 0$
- $x(n) = \delta(n + k), k > 0$
  $X(z) = z^{k}$ [i.e., $\delta(n+k) \overset{z}{\longleftrightarrow} z^{k}$], ROC: entire z-plane except $z = \infty$

Example

- Determine z-transform of

$$x(n) = (\tfrac{1}{2})^n u(n)$$

- z-transform of $x(n)$

$$\begin{aligned} X(z) &= 1 + \frac{1}{2}z^{-1} + (\frac{1}{2})^2 z^{-2} + (\frac{1}{2})^n z^{-n} + \cdots \\ &= \sum_{n=0}^{\infty} (\frac{1}{2})^n z^{-n} = \sum_{n=0}^{\infty} (\frac{1}{2}z^{-1})^n \end{aligned}$$

For $|\frac{1}{2}z^{-1}| < 1$ or $|z| > \frac{1}{2}$, $X(z)$ converges to

$$X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}, \quad \text{ROC: } |z| > \frac{1}{2}$$

The Direct z-Transform

- Expressing complex variable z in polar form

$$z = re^{j\theta}$$

where $r = |z|$ and $\theta = \measuredangle z$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) r^{-n} e^{-j\theta n}$$

- In ROC of $X(z)$, $|X(z)| < \infty$

$$|X(z)| = \left| \sum_{n=-\infty}^{\infty} x(n) r^{-n} e^{-j\theta n} \right| \leq \sum_{n=-\infty}^{\infty} |x(n) r^{-n} e^{-j\theta n}| = \sum_{n=-\infty}^{\infty} |x(n) r^{-n}|$$

$$|X(z)| \leq \sum_{n=-\infty}^{-1} |x(n) r^{-n}| + \sum_{n=0}^{\infty} \left| \frac{x(n)}{r^n} \right| \leq \sum_{n=1}^{\infty} |x(-n) r^n| + \sum_{n=0}^{\infty} \left| \frac{x(n)}{r^n} \right|$$

If $X(z)$ converges in some region of complex plane, both sums must be finite in that region

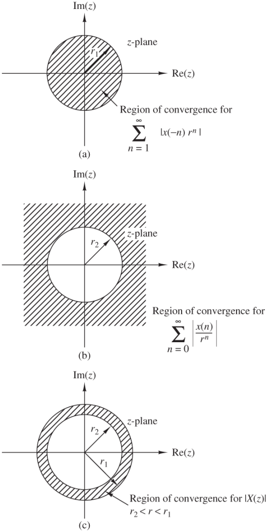


Figure 1: Region of convergence for $X(z)$ and its corresponding causal and anticausal components.

The Direct z-Transform

Example

- Determine z-transform of

$$x(n) = \alpha^n u(n) = \begin{cases} \alpha^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

- We have

$$X(z) = \sum_{n=0}^{\infty} \alpha^n z^{-n} = \sum_{n=0}^{\infty} (\alpha z^{-1})^n$$

$$x(n) = \alpha^n u(n) \overset{z}{\longleftrightarrow} X(z) = \frac{1}{1 - \alpha z^{-1}}, \quad \text{ROC: } |z| > |\alpha|$$

The Direct z-Transform

Example (continued)

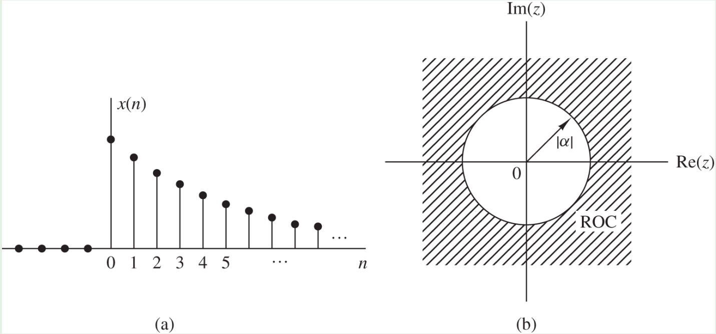


Figure 2: The exponential signal $x(n) = \alpha^n u(n)$ (a), and the ROC of its z-transform (b).

Example

- Determine z-transform of

$$x(n) = -\alpha^n u(-n-1) = \begin{cases} 0, & n \geq 0 \\ -\alpha^n, & n \leq -1 \end{cases}$$

- We have

$$X(z) = \sum_{n=-\infty}^{-1} (-\alpha^n) z^{-n} = -\sum_{l=1}^{\infty} (\alpha^{-1} z)^l$$

where $l = -n$

$$x(n) = -\alpha^n u(-n-1) \overset{z}{\longleftrightarrow} X(z) = -\frac{\alpha^{-1} z}{1 - \alpha^{-1} z} = \frac{1}{1 - \alpha z^{-1}}$$

$$\text{ROC: } |z| < |\alpha|$$

Example (continued)

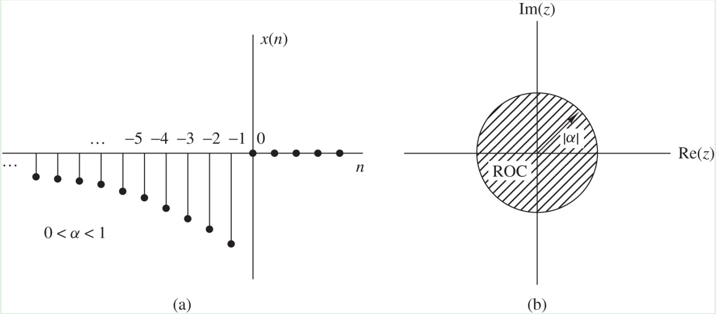


Figure 3: Anticausal signal $x(n) = -\alpha^n u(-n-1)$ (a), and the ROC of its z-transform (b).

- From two preceding examples

$$Z\{\alpha^n u(n)\} = Z\{-\alpha^n u(-n-1)\} = \frac{1}{1-\alpha z^{-1}}$$

  - This implies that a closed-form expression for z-transform does not uniquely specify the signal in time domain
  - Ambiguity can be resolved if ROC is also specified
  - *A signal $x(n)$ is uniquely determined by its z-transform $X(z)$ and region of convergence of $X(z)$*
  - *ROC of a causal signal is exterior of a circle of some radius r_2*
  - *ROC of an anticausal signal is interior of a circle of some radius r_1*

Example

- Determine z-transform of

$$x(n) = \alpha^n u(n) + b^n u(-n-1)$$

- We have

$$X(z) = \sum_{n=0}^{\infty} \alpha^n z^{-n} + \sum_{n=-\infty}^{-1} b^n z^{-n} = \sum_{n=0}^{\infty} (\alpha z^{-1})^n + \sum_{l=1}^{\infty} (b^{-1} z)^l$$

  First sum converges if $|z| > |\alpha|$, second sum converges if $|z| < |b|$

- If $|b| < |\alpha|$, $X(z)$ does not exist
- If $|b| > |\alpha|$

$$X(z) = \frac{1}{1-\alpha z^{-1}} - \frac{1}{1-bz^{-1}} = \frac{b-\alpha}{\alpha + b - z - \alpha b z^{-1}}$$

  ROC: $|\alpha| < |z| < |b|$

Example (continued)

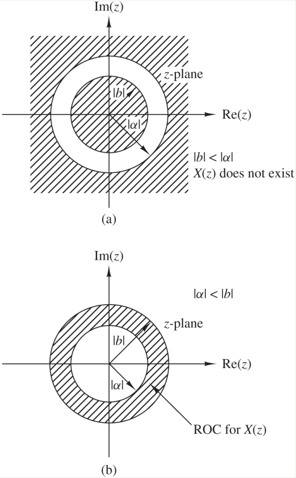
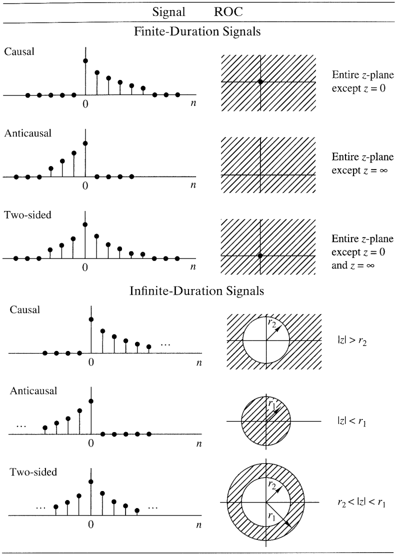


Figure 4: ROC for z-transform in the example.

Table 1: Characteristic families of signals with their corresponding ROCs



Properties of the z-Transform

- Combining several z-transforms, ROC of overall transform is, at least, intersection of ROCs of individual transforms
- **Linearity**
  - If
    $$x_1(n) \overset{z}{\longleftrightarrow} X_1(z)$$
    and
    $$x_2(n) \overset{z}{\longleftrightarrow} X_2(z)$$
    then
    $$x(n) = a_1 x_1(n) + a_2 x_2(n) \overset{z}{\longleftrightarrow} X(z) = a_1 X_1(z) + a_2 X_2(z)$$
    for any constants a_1 and a_2
  - To find z-transform of a signal, express it as a sum of elementary signals whose z-transforms are already known

Properties of the z-Transform

Example

- Determine z-transform and ROC of

$$x(n) = [3(2^n) - 4(3^n)]u(n)$$

- Defining signals $x_1(n)$ and $x_2(n)$

$$x_1(n) = 2^n u(n) \quad \text{and} \quad x_2(n) = 3^n u(n)$$
$$x(n) = 3x_1(n) - 4x_2(n)$$

According to linearity property

$$X(z) = 3X_1(z) - 4X_2(z)$$

Recall that

$$\alpha^n u(n) \overset{z}{\longleftrightarrow} \frac{1}{1-\alpha z^{-1}}, \quad \text{ROC: } |z| > |\alpha|$$

Setting $\alpha = 2$ and $\alpha = 3$

$X_1(z) = \frac{1}{1-2z^{-1}}$, ROC: $|z| > 2$ and $X_2(z) = \frac{1}{1-3z^{-1}}$, ROC: $|z| > 3$

Intersecting ROCs, overall transform is

$$X(z) = \frac{3}{1-2z^{-1}} - \frac{4}{1-3z^{-1}}, \quad \text{ROC: } |z| > 3$$

Properties of the z-Transform

- **Time shifting**
  - If
$$x(n) \stackrel{z}{\longleftrightarrow} X(z)$$
    then
$$x(n-k) \stackrel{z}{\longleftrightarrow} z^{-k}X(z)$$
  - ROC of $z^{-k}X(z)$ is same as that of $X(z)$ except for $z = 0$ if $k > 0$ and $z = \infty$ if $k < 0$

- **Scaling in z-domain**
  - If
    $$x(n) \stackrel{z}{\longleftrightarrow} X(z), \quad \text{ROC: } r_1 < |z| < r_2$$
    then
    $$a^n x(n) \stackrel{z}{\longleftrightarrow} X(a^{-1}z), \quad \text{ROC: } |a|r_1 < |z| < |a|r_2$$
    for any constant a, real or complex
  - Proof
    $$Z\{a^n x(n)\} = \sum_{n=-\infty}^{\infty} a^n x(n) z^{-n} = \sum_{n=-\infty}^{\infty} x(n)(a^{-1}z)^{-n} = X(a^{-1}z)$$
    Since ROC of $X(z)$ is $r_1 < |z| < r_2$, ROC of $X(a^{-1}z)$ is
    $$r_1 < |a^{-1}z| < r_2$$
    or
    $$|a|r_1 < |z| < |a|r_2$$

- **Time reversal**
  - If

    $$x(n) \overset{z}{\longleftrightarrow} X(z), \quad \text{ROC: } r_1 < |z| < r_2$$

    then

    $$x(-n) \overset{z}{\longleftrightarrow} X(z^{-1}), \quad \text{ROC: } \frac{1}{r_2} < |z| < \frac{1}{r_1}$$

  - Proof

    $$Z\{x(-n)\} = \sum_{n=-\infty}^{\infty} x(-n)z^{-n} = \sum_{l=-\infty}^{\infty} x(l)(z^{-1})^{-l} = X(z^{-1})$$

    where $l = -n$
    ROC of $X(z^{-1})$

    $$r_1 < |z^{-1}| < r_2 \quad \text{or} \quad \frac{1}{r_2} < |z| < \frac{1}{r_1}$$

- **Differentiation in z-domain**
  - If

$$x(n) \overset{z}{\longleftrightarrow} X(z)$$

    then

$$nx(n) \overset{z}{\longleftrightarrow} -z\frac{dX(z)}{dz}$$

  - Proof

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

    differentiating both sides

$$\frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} x(n)(-n)z^{-n-1} = -z^{-1}\sum_{n=-\infty}^{\infty} [nx(n)]z^{-n}$$
$$= -z^{-1}Z\{nx(n)\}$$

  - Both transforms have same ROC

- **Convolution of two sequences**
  - If
    $$x_1(n) \overset{z}{\longleftrightarrow} X_1(z)$$
    $$x_2(n) \overset{z}{\longleftrightarrow} X_2(z)$$
    then
    $$x(n) = x_1(n) * x_2(n) \overset{z}{\longleftrightarrow} X(z) = X_1(z)X_2(z)$$
    ROC of $X(z)$ is, at least, intersection of that for $X_1(z)$ and $X_2(z)$
  - Proof: convolution of $x_1(n)$ and $x_2(n)$
    $$x(n) = \sum_{k=-\infty}^{\infty} x_1(k)x_2(n-k)$$
    z-transform of $x(n)$
    $$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} = \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x_1(k)x_2(n-k) \right] z^{-n}$$
    $$= \sum_{k=-\infty}^{\infty} x_1(k) \left[\sum_{n=-\infty}^{\infty} x_2(n-k)z^{-n} \right] = X_2(z) \sum_{k=-\infty}^{\infty} x_1(k)z^{-k}$$
    $$= X_2(z)X_1(z)$$

Properties of the z-Transform

Example

- Compute convolution $x(n)$ of signals

$$x_1(n) = \{\underset{\uparrow}{1}, -2, 1\} \quad \text{and} \quad x_2(n) = \begin{cases} 1, & 0 \leq n \leq 5 \\ 0, & \text{elsewhere} \end{cases}$$

- z-transforms of these signals

$$X_1(z) = 1 - 2z^{-1} + z^{-2}$$
$$X_2(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5}$$

- Convolution of two signals is equal to multiplication of their transforms

$$X(z) = X_1(z)X_2(z) = 1 - z^{-1} - z^{-6} + z^{-7}$$

Hence

$$x(n) = \{\underset{\uparrow}{1}, -1, 0, 0, 0, 0, -1, 1\}$$

Properties of the z-Transform

- Convolution property is one of most powerful properties of z-transform
  - It converts convolution of two signals (time domain) to multiplication of their transforms
- Computation of convolution of two signals using z-transform
  1. Compute z-transforms of signals to be convolved
  $$X_1(z) = Z\{x_1(n)\}$$
  $$X_2(z) = Z\{x_2(n)\}$$
  2. Multiply the two z-transforms
  $$X(z) = X_1(z)X_2(z)$$
  3. Find inverse z-transform of $X(z)$
  $$x(n) = Z^{-1}\{X(z)\}$$

- **Correlation of two sequences**
  - If

$$x_1(n) \stackrel{z}{\longleftrightarrow} X_1(z)$$
$$x_2(n) \stackrel{z}{\longleftrightarrow} X_2(z)$$

then

$$r_{x_1x_2}(l) = \sum_{n=-\infty}^{\infty} x_1(n)x_2(n-l) \stackrel{z}{\longleftrightarrow} R_{x_1x_2}(z) = X_1(z)X_2(z^{-1})$$

  - Proof

$$r_{x_1x_2}(l) = x_1(l) * x_2(-l)$$

Using convolution and time-reversal properties

$$R_{x_1x_2}(z) = Z\{x_1(l)\}Z\{x_2(-l)\} = X_1(z)X_2(z^{-1})$$

ROC of $R_{x_1x_2}(z)$ is at least intersection of that for $X_1(z)$ and $X_2(z^{-1})$

Properties of the z-Transform

- **The initial value theorem**
  - If $x(n)$ is causal ($x(n) = 0$ for $n < 0$), then

$$x(0) = \lim_{z \to \infty} X(z)$$

  - Proof: since $x(n)$ is causal

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n} = x(0) + x(1)z^{-1} + x(2)z^{-2} + \cdots$$

  as $z \to \infty, z^{-n} \to 0$ and hence $X(z) = x(0)$

Table 2: Some common z-transform pairs

Signal, $x(n)$	z-Transform, $X(z)$	ROC
$\delta(n)$	1	All z
$u(n)$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
$a^n u(n)$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
$na^n u(n)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
$-a^n u(-n-1)$	$\frac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
$-na^n u(-n-1)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
$(\cos \omega_0 n)u(n)$	$\frac{1-z^{-1}\cos \omega_0}{1-2z^{-1}\cos \omega_0+z^{-2}}$	$\|z\| > 1$
$(\sin \omega_0 n)u(n)$	$\frac{z^{-1}\sin \omega_0}{1-2z^{-1}\cos \omega_0+z^{-2}}$	$\|z\| > 1$
$(a^n \cos \omega_0 n)u(n)$	$\frac{1-az^{-1}\cos \omega_0}{1-2az^{-1}\cos \omega_0+a^2z^{-2}}$	$\|z\| > \|a\|$
$(a^n \sin \omega_0 n)u(n)$	$\frac{az^{-1}\sin \omega_0}{1-2az^{-1}\cos \omega_0+a^2z^{-2}}$	$\|z\| > \|a\|$

Rational z-Transforms

- An important family of z-transforms are those for which $X(z)$ is a rational function
  - $X(z)$ is a ratio of two polynomials in z^{-1} (or z)
  - Some important issues of rational z-transforms are discussed here

Poles and Zeros

- **Zeros** of a z-transform $X(z)$ are values of z for which $X(z) = 0$
- **Poles** of a z-transform are values of z for which $X(z) = \infty$
- If $X(z)$ is a rational function (and if $a_0 \neq 0$ and $b_0 \neq 0$)

$$\begin{aligned}
X(z) = \frac{B(z)}{A(z)} &= \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \\
&= \frac{b_0 z^{-M}}{a_0 z^{-N}} \frac{z^M + (b_1/b_0) z^{M-1} + \cdots + b_M/b_0}{z^N + (a_1/a_0) z^{N-1} + \cdots + a_N/a_0} \\
&= \frac{b_0}{a_0} z^{-M+N} \frac{(z - z_1)(z - z_2)\cdots(z - z_M)}{(z - p_1)(z - p_2)\cdots(z - p_N)} \\
&= G z^{N-M} \frac{\prod_{k=1}^{M} (z - z_k)}{\prod_{k=1}^{N} (z - p_k)}
\end{aligned}$$

  - $X(z)$ has M finite zeros at $z = z_1, z_2, \ldots, z_M$
  - N finite poles at $z = p_1, p_2, \ldots, p_N$
  - $|N - M|$ zeros if $N > M$ or poles if $N < M$ at $z = 0$
  - $X(z)$ has exactly same number of poles as zeros

Example

- Determine pole-zero plot for signal

$$x(n) = a^n u(n), \qquad a > 0$$

- From Table 2

$$X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, \qquad \text{ROC: } |z| > a$$

$X(z)$ has one zero at $z_1 = 0$ and one pole at $p_1 = a$

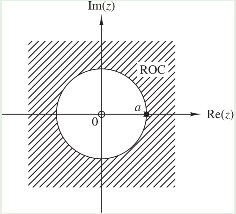


Figure 5: Pole-zero plot for the causal exponential signal $x(n) = a^n u(n)$.

Example

- Determine pole-zero plot for signal

$$x(n) = \begin{cases} a^n, & 0 \leq n \leq M-1 \\ 0, & \text{elsewhere} \end{cases}$$

  where $a > 0$

- z-transform of $x(n)$

$$X(z) = \sum_{n=0}^{M-1} a^n z^{-n} = \sum_{n=0}^{M-1} (az^{-1})^n = \frac{1-(az^{-1})^M}{1-az^{-1}} = \frac{z^M - a^M}{z^{M-1}(z-a)}$$

  Since $a > 0$, $z^M = a^M$ has M roots at

$$z_k = ae^{j2\pi k/M}, \qquad k = 0, 1, \ldots, M-1$$

Example (continued)

- Zero $z_0 = a$ cancels pole at $z = a$. Thus

$$X(z) = \frac{(z - z_1)(z - z_2)\cdots(z - z_{M-1})}{z^{M-1}}$$

which has $M - 1$ zeros and $M - 1$ poles

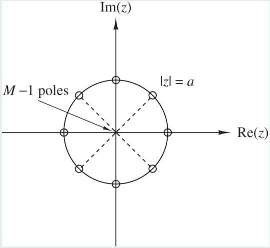


Figure 6: Pole-zero pattern for the finite-duration signal $x(n) = a^n$, $0 \leq n \leq M - 1$ $(a > 0)$, for $M = 8$.

Example

- Determine z-transform and signal corresponding to following pole-zero plot

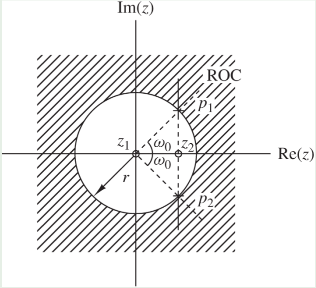


Figure 7: Pole-zero pattern.

Poles and Zeros

Example (continued)

- We use

$$X(z) = Gz^{N-M}\frac{\prod_{k=1}^{M}(z - z_k)}{\prod_{k=1}^{N}(z - p_k)}$$

There are two zeros ($M = 2$) at $z_1 = 0$, $z_2 = r\cos\omega_0$
There are two poles ($N = 2$) at $p_1 = re^{j\omega_0}$, $p_2 = re^{-j\omega_0}$

$$X(z) = G\frac{(z - z_1)(z - z_2)}{(z - p_1)(z - p_2)} = G\frac{z(z - r\cos\omega_0)}{(z - re^{j\omega_0})(z - re^{-j\omega_0})}$$

$$= G\frac{1 - rz^{-1}\cos\omega_0}{1 - 2rz^{-1}\cos\omega_0 + r^2z^{-2}}, \qquad \text{ROC: } |z| > r$$

From Table 2 we find that

$$x(n) = G(r^n\cos\omega_0 n)u(n)$$

Poles and Zeros

- z-transform $X(z)$ is a complex function of complex variable z
- $|X(z)|$ is a real and positive function of z
- Since z represents a point in complex plane, $|X(z)|$ is a surface
- z-transform

$$X(z) = \frac{z^{-1} - z^{-2}}{1 - 1.2732z^{-1} + 0.81z^{-2}}$$

has one zero at $z_1 = 1$ and two poles at $p_1, p_2 = 0.9e^{\pm j\pi/4}$

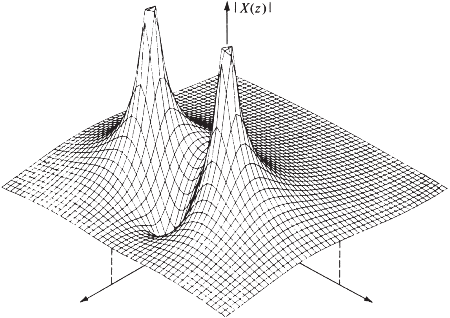


Figure 8: Graph of $|X(z)|$ for the above z-transform.

- Characteristic behavior of causal signals depends on whether poles of transform are contained in region
  - $|z| < 1$
  - or $|z| > 1$
  - or on circle $|z| = 1$
- Circle $|z| = 1$ is called **unit circle**
- If a real signal has a z-transform with one pole, this pole has to be real
  - The only such signal is the real exponential

$$x(n) = a^n u(n) \overset{z}{\longleftrightarrow} X(z) = \frac{1}{1 - az^{-1}}, \quad \text{ROC: } |z| > |a|$$

having one zero at $z_1 = 0$ and one pole at $p_1 = a$ on real axis

Pole Location & Time-Domain Behavior for Causal Signals

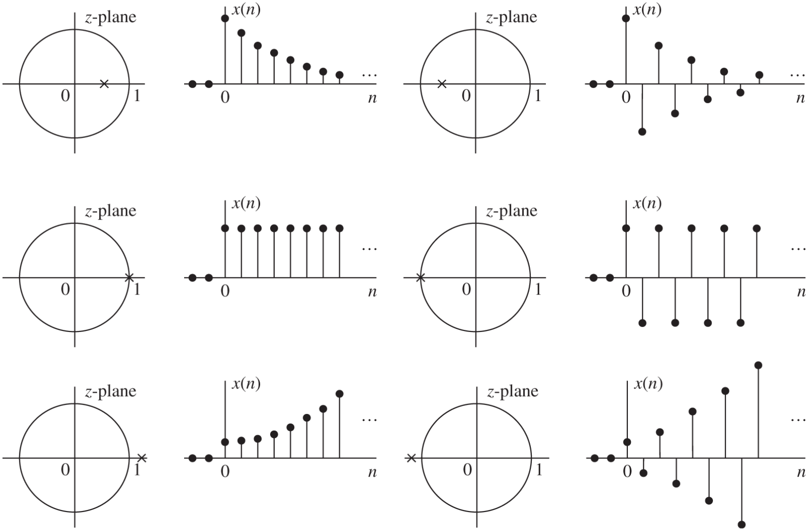


Figure 9: Time-domain behavior of a single-real-pole causal signal as a function of the location of the pole with respect to the unit circle.

Pole Location & Time-Domain Behavior for Causal Signals

- A causal real signal with a double real pole has the form

$$x(n) = na^n u(n) \stackrel{z}{\longleftrightarrow} X(z) = \frac{az^{-1}}{(1 - az^{-1})^2}, \quad \text{ROC: } |z| > |a|$$

  - In contrast to single-pole signal, a double real pole on unit circle results in an unbounded signal

Pole Location & Time-Domain Behavior for Causal Signals

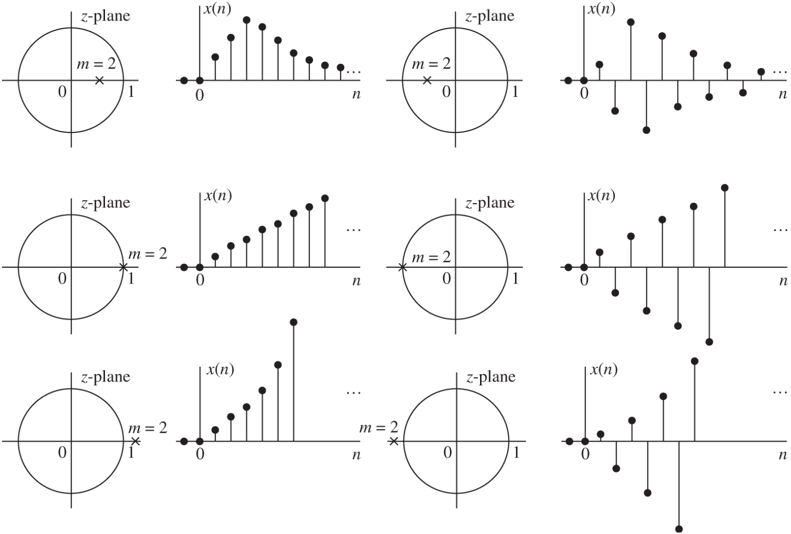


Figure 10: Time-domain behavior of causal signals corresponding to a double ($m = 2$) real pole, as a function of the pole location.

Pole Location & Time-Domain Behavior for Causal Signals

- Configuration of poles as a pair of complex-conjugates results in an exponentially weighted sinusoidal signal

$$x(n) = (a^n \cos \omega_0 n) u(n) \overset{z}{\longleftrightarrow} X(z) = \frac{1 - az^{-1} \cos \omega_0}{1 - 2az^{-1} \cos \omega_0 + a^2 z^{-2}}$$

$$\text{ROC: } |z| > |a|$$

$$x(n) = (a^n \sin \omega_0 n) u(n) \overset{z}{\longleftrightarrow} X(z) = \frac{az^{-1} \sin \omega_0}{1 - 2az^{-1} \cos \omega_0 + a^2 z^{-2}}$$

$$\text{ROC: } |z| > |a|$$

  - Distance r of poles from origin determines envelope of sinusoidal signal
  - Angle ω_0 with real positive axis determines relative frequency

Pole Location & Time-Domain Behavior for Causal Signals

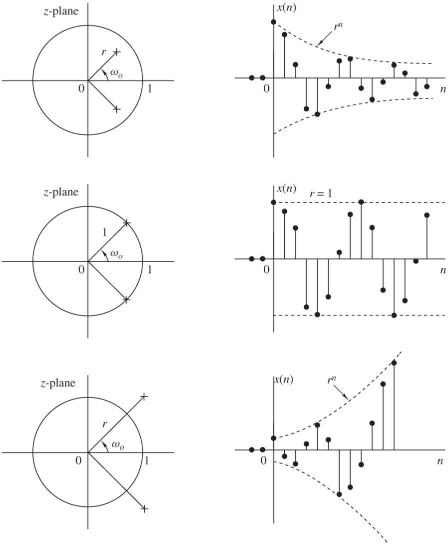


Figure 11: A pair of complex-conjugate poles corresponds to causal signals with oscillatory behavior.

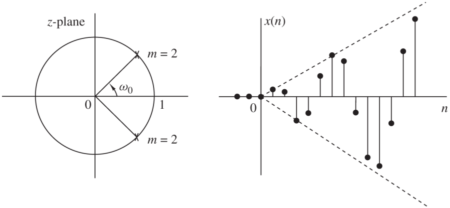


Figure 12: Causal signal corresponding to a double pair of complex-conjugate poles on the unit circle.

- In summary
  - Causal real signals with simple real poles or simple complex-conjugate pairs of poles, inside or on unit circle, are always bounded in amplitude
  - A signal with a pole, or a complex-conjugate pair of poles, near origin decays more rapidly than one near (but inside) unit circle
    - Thus, time behavior of a signal depends strongly on location of its poles relative to unit circle
  - Zeros also affect behavior of a signal but not as strongly as poles
    - E.g., for sinusoidal signals, presence and location of zeros affects only their phase

Inversion of the z-Transform

- There are three methods for evaluation of inverse z-transform
  1. Direct evaluation by contour integration
  2. Expansion into a series of terms, in variables z and z^{-1}
  3. Partial-fraction expansion and table lookup
- **Inverse z-transform by contour integration**

$$x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

  - Integral is a contour integral over a closed path C
  - C encloses origin and lies within ROC of $X(z)$

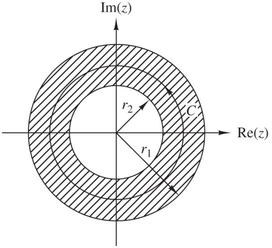


Figure 13: Contour C.

The Inverse z-Transform by Power Series Expansion

- Given $X(z)$ with its ROC, expand it into a power series of form

$$X(z) = \sum_{n=-\infty}^{\infty} c_n z^{-n}$$

  - By uniqueness of z-transform, $x(n) = c_n$ for all n
  - When $X(z)$ is rational, expansion can be performed by long division
    - Long division method becomes tedious when n is large
    - Although this method provides a direct evaluation of $x(n)$, a closed-form solution is not possible
    - Hence this method is used only for determining values of first few samples of signal

Example

- Determine inverse z-transform of

$$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}$$

1. When ROC: $|z| > 1$
2. When ROC: $|z| < 0.5$

- ROC: $|z| > 1$
  Since ROC is exterior of a circle, $x(n)$ is a causal signal. Thus we seek negative powers of z by dividing numerator of $X(z)$ by its denominator

$$X(z) = \frac{1}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}} = 1 + \frac{3}{2}z^{-1} + \frac{7}{4}z^{-2} + \frac{15}{8}z^{-3} + \frac{31}{16}z^{-4} + \cdots$$

$$x(n) = \{\underset{\uparrow}{1}, \frac{3}{2}, \frac{7}{4}, \frac{15}{8}, \frac{31}{16}, \ldots\}$$

The Inverse z-Transform by Power Series Expansion

Example (continued)

- ROC: $|z| < 0.5$

  Since ROC is interior of a circle, $x(n)$ is anticausal. To obtain positive powers of z, write the two polynomials in reverse order and then divide

$$
\begin{array}{r|l}
 & 2z^2 + 6z^3 + 14z^4 + 30z^5 + 62z^6 + \cdots \\
\frac{1}{2}z^{-2} - \frac{3}{2}z^{-1} + 1 & 1 \\
 & \underline{1 - 3z + 2z^2} \\
 & \quad 3z - 2z^2 \\
 & \quad \underline{3z - 9z^2 + 6z^3} \\
 & \qquad 7z^2 - 6z^3 \\
 & \qquad \underline{7z^2 - 21z^3 + 14z^4} \\
 & \qquad\quad 15z^3 - 14z^4 \\
 & \qquad\quad \underline{15z^3 - 45z^4 + 30z^5} \\
 & \qquad\qquad 31z^4 - 30z^5
\end{array}
$$

$$X(z) = \frac{1}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}} = 2z^2 + 6z^3 + 14z^4 + 30z^5 + 62z^6 + \cdots$$

$$x(n) = \{\cdots 62, 30, 14, 6, 2, 0, \underset{\uparrow}{0}\}$$

The Inverse z-Transform by Partial-Fraction Expansion

- In table lookup method, express $X(z)$ as a linear combination

$$X(z) = \alpha_1 X_1(z) + \alpha_2 X_2(z) + \cdots + \alpha_K X_K(z)$$

  - $X_1(z), \ldots, X_K(z)$ are expressions with inverse transforms $x_1(n), \ldots, x_K(n)$ available in a table of z-transform pairs
  - Using linearity property

$$x(n) = \alpha_1 x_1(n) + \alpha_2 x_2(n) + \cdots + \alpha_K x_K(n)$$

- If $X(z)$ is a rational function

$$X(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

  dividing both numerator and denominator by a_0

$$X(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

  - This form of rational function is called **proper** if $a_N \neq 0$ and $M < N$

The Inverse z-Transform by Partial-Fraction Expansion

- An improper rational function ($M \geq N$) can always be written as sum of a polynomial and a proper rational function

Example

- Express improper rational function

$$X(z) = \frac{1 + 3z^{-1} + \frac{11}{6}z^{-2} + \frac{1}{3}z^{-3}}{1 + \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}}$$

in terms of a polynomial and a proper function

- Terms z^{-2} and z^{-3} should be eliminated from numerator
Do long division with the two polynomials written in reverse order
Stop division when order of remainder becomes z^{-1}

$$X(z) = 1 + 2z^{-1} + \frac{\frac{1}{6}z^{-1}}{1 + \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}}$$

The Inverse z-Transform by Partial-Fraction Expansion

- Any improper rational function ($M \geq N$) can be expressed as

$$X(z) = \frac{B(z)}{A(z)} = c_0 + c_1 z^{-1} + \cdots + c_{M-N} z^{-(M-N)} + \frac{B_1(z)}{A(z)}$$

  - Inverse z-transform of the polynomial can easily be found by inspection
  - We focus our attention on inversion of proper rational transforms
    1. Perform a partial fraction expansion of proper rational function
    2. Invert each of the terms

- Let $X(z)$ be a proper rational function ($a_N \neq 0$ and $M < N$)

$$X(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

  Eliminating negative powers of z

$$X(z) = \frac{b_0 z^N + b_1 z^{N-1} + \cdots + b_M z^{N-M}}{z^N + a_1 z^{N-1} + \cdots + a_N}$$

  Since $N > M$, the function

$$\frac{X(z)}{z} = \frac{b_0 z^{N-1} + b_1 z^{N-2} + \cdots + b_M z^{N-M-1}}{z^N + a_1 z^{N-1} + \cdots + a_N}$$

  is also proper

  - To perform a partial-fraction expansion, this function should be expressed as a sum of simple fractions
  - First factor denominator polynomial into factors that contain poles $p_1, p_2, \ldots, p_N$ of $X(z)$

- **Distinct poles**
  - Suppose poles $p_1, p_2, \ldots, p_N$ are all different. We seek expansion

$$\frac{X(z)}{z} = \frac{A_1}{z - p_1} + \frac{A_2}{z - p_2} + \cdots + \frac{A_N}{z - p_N}$$

To determine coefficients $A_1, A_2, \ldots, A_N$, multiply both sides by each of terms $(z - p_k), k = 1, 2, \ldots, N$, and evaluate resulting expressions at corresponding pole positions, $p_1, p_2, \ldots, p_N$

$$\frac{(z - p_k)X(z)}{z} = \frac{(z - p_k)A_1}{z - p_1} + \cdots + A_k + \cdots + \frac{(z - p_k)A_N}{z - p_N}$$

$$A_k = \left.\frac{(z - p_k)X(z)}{z}\right|_{z=p_k}, \qquad k = 1, 2, \ldots, N$$

Example

- Determine partial-fraction expansion of

$$X(z) = \frac{1 + z^{-1}}{1 - z^{-1} + 0.5z^{-2}}$$

- Eliminate negative powers by multiplying by z^2

$$\frac{X(z)}{z} = \frac{z+1}{z^2 - z + 0.5} \longrightarrow p_1 = \frac{1}{2} + j\frac{1}{2} \text{ and } p_2 = \frac{1}{2} - j\frac{1}{2}$$

$$\xrightarrow{p_1 \neq p_2} \frac{X(z)}{z} = \frac{z+1}{(z-p_1)(z-p_2)} = \frac{A_1}{z-p_1} + \frac{A_2}{z-p_2}$$

$$A_1 = \left.\frac{(z-p_1)X(z)}{z}\right|_{z=p_1} = \left.\frac{z+1}{z-p_2}\right|_{z=p_1} = \frac{\frac{1}{2} + j\frac{1}{2} + 1}{\frac{1}{2} + j\frac{1}{2} - \frac{1}{2} + j\frac{1}{2}} = \frac{1}{2} - j\frac{3}{2}$$

$$A_2 = \left.\frac{(z-p_2)X(z)}{z}\right|_{z=p_2} = \left.\frac{z+1}{z-p_1}\right|_{z=p_2} = \frac{\frac{1}{2} - j\frac{1}{2} + 1}{\frac{1}{2} - j\frac{1}{2} - \frac{1}{2} - j\frac{1}{2}} = \frac{1}{2} + j\frac{3}{2}$$

- *Complex-conjugate poles result in complex-conjugate coefficients*

- **Multiple-order poles**
  - If $X(z)$ has a pole of multiplicity m (there is factor $(z - p_k)^m$ in denominator), partial-fraction expansion must contain the terms

$$\frac{A_{1k}}{z - p_k} + \frac{A_{2k}}{(z - p_k)^2} + \cdots + \frac{A_{mk}}{(z - p_k)^m}$$

Coefficients $\{A_{ik}\}$ can be evaluated through differentiation

Example

- Determine partial-fraction expansion of

$$X(z) = \frac{1}{(1 + z^{-1})(1 - z^{-1})^2}$$

- Expressing in terms of positive powers of z

$$\frac{X(z)}{z} = \frac{z^2}{(z+1)(z-1)^2}$$

$X(z)$ has a simple pole at $p_1 = -1$ and a double pole at $p_2 = p_3 = 1$

$$\frac{X(z)}{z} = \frac{z^2}{(z+1)(z-1)^2} = \frac{A_1}{z+1} + \frac{A_2}{z-1} + \frac{A_3}{(z-1)^2}$$

$$A_1 = \left.\frac{(z+1)X(z)}{z}\right|_{z=-1} = \frac{1}{4}$$

Example (continued)

$$A_3 = \left.\frac{(z-1)^2X(z)}{z}\right|_{z=1} = \frac{1}{2}$$

- To obtain A_2

$$\frac{(z-1)^2X(z)}{z} = \frac{(z-1)^2}{z+1}A_1 + (z-1)A_2 + A_3$$

Differentiating both sides and evaluating at $z = 1$, A_2 is obtained

$$A_2 = \frac{d}{dz}\left[\frac{(z-1)^2X(z)}{z}\right]_{z=1} = \frac{3}{4}$$

The Inverse z-Transform by Partial-Fraction Expansion

- Having performed partial-fraction expansion, final step in inversion is as follows
  - If poles are distinct

$$\frac{X(z)}{z} = \frac{A_1}{z-p_1} + \frac{A_2}{z-p_2} + \cdots + \frac{A_N}{z-p_N}$$

$$X(z) = A_1 \frac{1}{1-p_1 z^{-1}} + A_2 \frac{1}{1-p_2 z^{-1}} + \cdots + A_N \frac{1}{1-p_N z^{-1}}$$

    $x(n) = Z^{-1}\{X(z)\}$ is obtained by inverting each term and taking the corresponding linear combination
  - From table 2

$$Z^{-1}\left\{\frac{1}{1-p_k z^{-1}}\right\} = \begin{cases} (p_k)^n u(n), & \text{if ROC:}|z| > |p_k| \text{ (causal)} \\ -(p_k)^n u(-n-1), & \text{if ROC:}|z| < |p_k| \text{ (anticausal)} \end{cases}$$

  - If $x(n)$ is causal, ROC is $|z| > p_{max}$, where

$$p_{max} = \max\{|p_1|, |p_2|, \ldots, |p_N|\}$$

    In this case all terms in $X(z)$ result in causal signal components

$$x(n) = (A_1 p_1^n + A_2 p_2^n + \cdots + A_N p_N^n) u(n)$$

The Inverse z-Transform by Partial-Fraction Expansion

- If all poles are distinct but some of them are complex, and if signal $x(n)$ is real, complex terms can be reduced into real components
  - If p_j is a pole, its complex conjugate p_j^* is also a pole
    - If $x(n)$ is real, the polynomials in $X(z)$ have real coefficients
    - If a polynomial has real coefficients, its roots are either real or occur in complex-conjugate pairs
  - Their corresponding coefficients in partial-fraction expansion are also complex-conjugates

Contribution of two complex-conjugate poles is

$$x_k(n) = [A_k(p_k)^n + A_k^*(p_k^*)^n)]u(n)$$

Expressing A_j and p_j in polar form

$$A_k = |A_k|e^{j\alpha_k} \quad \text{and} \quad p_k = r_k e^{j\beta_k}$$

which gives

$$x_k(n) = |A_k|r_k^n[e^{j(\beta_k n+\alpha_k)} + e^{-j(\beta_k n+\alpha_k)}]u(n)$$
$$\text{or} \quad x_k(n) = 2|A_k|r_k^n \cos(\beta_k n + \alpha_k)u(n)$$

Thus

$$Z^{-1}\left(\frac{A_k}{1-p_k z^{-1}} + \frac{A_k^*}{1-p_k^* z^{-1}}\right) = 2|A_k|r_k^n \cos(\beta_k n + \alpha_k)u(n)$$

if ROC is $|z| > |p_k| = r_k$

The Inverse z-Transform by Partial-Fraction Expansion

- In case of multiple poles, either real or complex, inverse transform of terms of the form $A/(z-p_k)^n$ is required
  - In case of a double pole, from table 2

$$Z^{-1}\left\{\frac{pz^{-1}}{(1-pz^{-1})^2}\right\} = np^n u(n)$$

    provided that ROC is $|z| > |p|$
  - In case of poles with higher multiplicity, multiple differentiation is used

Example

- Determine inverse z-transform of

$$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}$$

1. If ROC: $|z| > 1$
2. If ROC: $|z| < 0.5$
3. If ROC: $0.5 < |z| < 1$

- Partial-fraction expansion for $X(z)$

$$X(z) = \frac{z^2}{z^2 - 1.5z + 0.5} \xrightarrow[p_2=0.5]{p_1=1} \frac{X(z)}{z} = \frac{z}{(z-1)(z-0.5)} = \frac{A_1}{z-1} + \frac{A_2}{z-0.5}$$

$$A_1 = \left.\frac{(z-1)X(z)}{z}\right|_{z=1} = 2$$

$$A_2 = \left.\frac{(z-0.5)X(z)}{z}\right|_{z=0.5} = -1$$

The Inverse z-Transform by Partial-Fraction Expansion

Example (continued)

$$X(z) = \frac{2}{1 - z^{-1}} - \frac{1}{1 - 0.5z^{-1}}$$

- When ROC is $|z| > 1$, $x(n)$ is causal and both terms in $X(z)$ are causal

$$\frac{1}{1 - p_k z^{-1}} \overset{z}{\longleftrightarrow} (p_k)^n u(n)$$

$$x(n) = 2(1)^n u(n) - (0.5)^n u(n) = (2 - 0.5^n) u(n)$$

- When ROC is $|z| < 0.5$, $x(n)$ is anticausal and both terms in $X(z)$ are anticausal

$$\frac{1}{1 - p_k z^{-1}} \overset{z}{\longleftrightarrow} -(p_k)^n u(-n-1)$$

$$x(n) = [-2 + (0.5)^n] u(-n-1)$$

Example (continued)

- When ROC is $0.5 < |z| < 1$ (ring), signal $x(n)$ is two-sided
  One of the terms corresponds to a causal signal and the other to an anticausal signal
  Since the ROC is overlapping of $|z| > 0.5$ and $|z| < 1$, pole $p_2 = 0.5$ provides causal part and pole $p_1 = 1$ anticausal

$$x(n) = -2(1)^n u(-n-1) - (0.5)^n u(n)$$

Example

- Determine causal signal $x(n)$ whose z-transform is

$$X(z) = \frac{1 + z^{-1}}{1 - z^{-1} + 0.5z^{-2}}$$

- We have already obtained partial-fraction expansion as

$$X(z) = \frac{A_1}{1 - p_1 z^{-1}} + \frac{A_2}{1 - p_2 z^{-1}} \longrightarrow A_1 = A_2^* = \frac{1}{2} - j\frac{3}{2} \text{ and } p_1 = p_2^* = \frac{1}{2} + j\frac{1}{2}$$

For a pair of complex-conjugate poles ($A_k = |A_k|e^{j\alpha_k}$ and $p_k = r_k e^{j\beta_k}$)

$$Z^{-1}\left(\frac{A_k}{1 - p_k z^{-1}} + \frac{A_k^*}{1 - p_k^* z^{-1}}\right) = 2|A_k| r_k^n \cos(\beta_k n + \alpha_k) u(n)$$

$$A_1 = (\sqrt{10}/2)e^{-j71.565} \text{ and } p_1 = (1/\sqrt{2})e^{j\pi/4}$$

$$x(n) = \sqrt{10}(1/\sqrt{2})^n \cos(\pi n/4 - 71.565^\circ)u(n)$$

Example

- Determine causal signal $x(n)$ having z-transform

$$X(z) = \frac{1}{(1+z^{-1})(1-z^{-1})^2}$$

- We have already obtained partial-fraction expansion as

$$X(z) = \frac{1}{4}\frac{1}{1+z^{-1}} + \frac{3}{4}\frac{1}{1-z^{-1}} + \frac{1}{2}\frac{z^{-1}}{(1-z^{-1})^2}$$

For causal signals

$$\frac{1}{1-pz^{-1}} \overset{z}{\longleftrightarrow} (p)^n u(n) \quad \text{and} \quad \frac{pz^{-1}}{(1-pz^{-1})^2} \overset{z}{\longleftrightarrow} np^n u(n)$$

$$x(n) = \frac{1}{4}(-1)^n u(n) + \frac{3}{4}u(n) + \frac{1}{2}nu(n) = \left[\frac{1}{4}(-1)^n + \frac{3}{4} + \frac{n}{2}\right] u(n)$$

References

John G. Proakis, Dimitris G. Manolakis, *Digital Signal Processing: Principles, Algorithms, and Applications*, Prentice Hall, 2006.