Digital Signal Processing

Frequency Analysis of Signals (2)

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April, 2012

- **Low-frequency signal**
  - A power signal (or energy signal) whose power density spectrum (or energy density spectrum) is concentrated about zero frequency

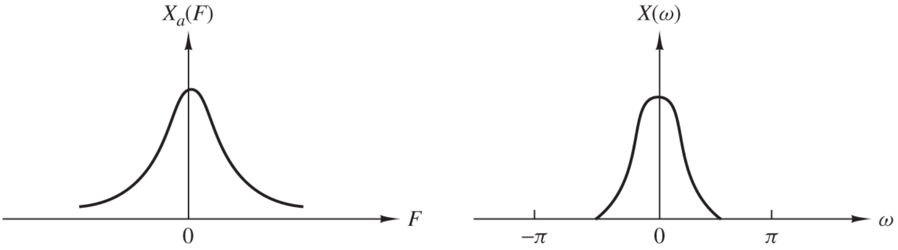

Figure 1: Low-frequency signal.

- **High-frequency signal**
  - A power signal (or energy signal) whose power density spectrum (or energy density spectrum) is concentrated at high frequencies

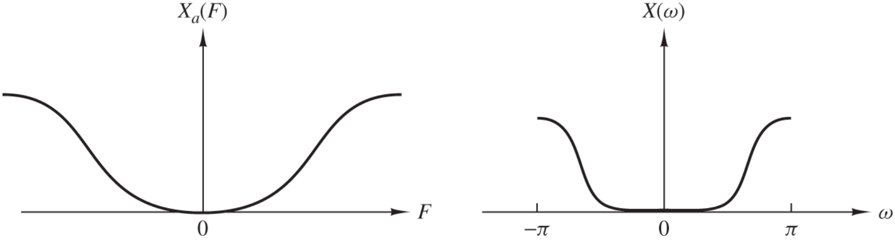

Figure 2: High-frequency signal.

- **Medium-frequency signal** or **bandpass signal**
  - A power signal (or energy signal) whose power density spectrum (or energy density spectrum) is concentrated somewhere in broad frequency range between low frequencies and high frequencies

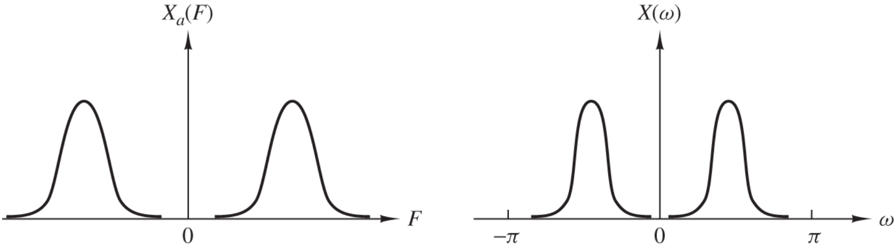

Figure 3: Medium-frequency signal.

The Concept of Bandwidth

- **Bandwidth** of a signal
  - To express quantitatively the range of frequencies over which power or energy density spectrum is concentrated
  - E.g., if a continuous-time signal has 95% of its power (or energy) density spectrum concentrated in $F_1 \leq F \leq F_2$, then 95% bandwidth of signal is $F_2 - F_1$
- A bandpass signal is **narrowband** if its bandwidth $F_2 - F_1$ is much smaller than median frequency $(F_2 + F_1)/2$
  - Otherwise, it is **wideband**
- A signal is **bandlimited** if its spectrum is zero outside frequency range $|F| \geq B$
  - A periodic continuous-time signal $x_p(t)$ is bandlimited if its Fourier coefficients $c_k = 0$ for $|k| > M$ (M is some positive integer)
  - A continuous-time finite-energy signal $x(t)$ is bandlimited if its Fourier transform $X(F) = 0$ for $|F| > B$
  - A periodic discrete-time signal with fundamental period N is **periodically bandlimited** if Fourier coefficients $c_k = 0$ for $k_0 < |k| < N$
  - A discrete-time finite-energy signal $x(n)$ is (periodically) bandlimited if $|X(\omega)| = 0$ for $\omega_0 < |\omega| < \pi$

The Concept of Bandwidth

- Exploiting duality between frequency and time domains, a signal $x(t)$ is **time-limited** if

$$x(t) = 0, \qquad |t| > \tau$$

  - If a signal is periodic with period T_p, it is **periodically time-limited** if

$$x_p(t) = 0, \qquad \tau < |t| < T_p/2$$

  - A discrete-time signal $x(n)$ of finite duration ($x(n) = 0, \; |n| > N$) is also time-limited
  - When a signal is periodic with fundamental period N, it is periodically time-limited if

$$x(n) = 0, \qquad n_0 < |n| < N$$

- *No signal can be time-limited and bandlimited simultaneously*
- A reciprocal relationship exists between time duration and frequency duration of a signal
  - The narrower the pulse becomes in time domain, the larger the bandwidth of signal becomes
  - Consequently, product of time duration and bandwidth of a signal is fixed

Frequency-Domain and Time-Domain Signal Properties

- There are two time-domain characteristics that determine type of signal spectrum we obtain
  1. Whether time variable is continuous or discrete
  2. Whether signal is periodic or aperiodic
- Summary of results of previous sections
  - **Continuous-time signals have aperiodic spectra**
    - Because complex exponential $\exp(j2\pi Ft)$ is a function of continuous variable t, and hence it is not periodic in F
    - Thus frequency range extends from $F = 0$ to $F = \infty$
  - **Discrete-time signals have periodic spectra**
    - Both Fourier series and transform are periodic here with period $\omega = 2\pi$
    - Frequency range is finite and extends from $\omega = -\pi$ to $\omega = \pi$, where $\omega = \pi$ corresponds to the highest possible rate of oscillation
  - **Periodic signals have discrete spectra**
    - Periodic signals are described by means of Fourier series
    - Fourier series coefficients provide *lines* that constitute discrete spectrum
    - Line spacing is $\Delta F = 1/T_p$ for continuous-time periodic signals and $\Delta f = 1/N$ for discrete-time signals
  - **Aperiodic finite energy signals have continuous spectra**
    - Because $X(F)$ and $X(\omega)$ are functions of $\exp(j2\pi Ft)$ and $\exp(j\omega n)$, respectively, which are continuous functions of F and ω

Frequency-Domain and Time-Domain Signal Properties

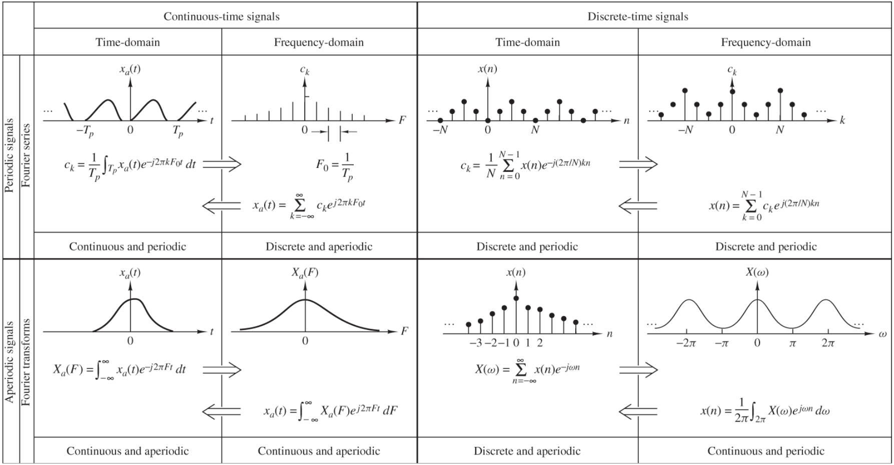

Figure 4: Summary of analysis and synthesis formulas.

Frequency-Domain and Time-Domain Signal Properties

- *Periodicity with "period" α in one domain automatically implies discretization with "spacing" of $1/\alpha$ in the other domain, and vice versa*
  - "Period" in frequency domain means frequency range
  - "Spacing" in time domain is sampling period T
  - Line spacing in frequency domain is ΔF

$$\alpha = T_p \longrightarrow 1/\alpha = 1/T_p = \Delta F$$
$$\alpha = N \longrightarrow \Delta f = 1/N$$
$$\alpha = F_s \longrightarrow T = 1/F_s$$

- Direct transform (analysis equation)

$$X(\omega) \equiv F\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

- Inverse transform (synthesis equation)

$$x(n) \equiv F^{-1}\{X(\omega)\} = \frac{1}{2\pi}\int_{2\pi} X(\omega)e^{j\omega n}d\omega$$

- $x(n)$ and $X(\omega)$ are a *Fourier transform pair*

$$x(n) \overset{F}{\longleftrightarrow} X(\omega)$$

- Suppose both $x(n)$ and $X(\omega)$ are complex-valued

$$x(n) = x_R(n) + jx_I(n) \tag{1}$$
$$X(\omega) = X_R(\omega) + jX_I(\omega) \tag{2}$$

Putting (1) and $e^{-j\omega} = \cos\omega - j\sin\omega$ in $X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$

$$X_R(\omega) = \sum_{n=-\infty}^{\infty} [x_R(n)\cos\omega n + x_I(n)\sin\omega n] \tag{3}$$
$$X_I(\omega) = -\sum_{n=-\infty}^{\infty} [x_R(n)\sin\omega n - x_I(n)\cos\omega n] \tag{4}$$

Putting (2) and $e^{j\omega} = \cos\omega + j\sin\omega$ in $x(n) = \frac{1}{2\pi}\int_{2\pi} X(\omega)e^{j\omega n}d\omega$

$$x_R(n) = \frac{1}{2\pi}\int_{2\pi} [X_R(\omega)\cos\omega n - X_I(\omega)\sin\omega n]d\omega \tag{5}$$
$$x_I(n) = \frac{1}{2\pi}\int_{2\pi} [X_R(\omega)\sin\omega n + X_I(\omega)\cos\omega n]d\omega \tag{6}$$

- **Real signals.** If $x(n)$ is real, then $x_R(n) = x(n)$ and $x_I(n) = 0$
  - Hence (3) and (4) reduce to

$$X_R(\omega) = \sum_{n=-\infty}^{\infty} x(n) \cos \omega n \quad \text{and} \quad X_I(\omega) = -\sum_{n=-\infty}^{\infty} x(n) \sin \omega n$$

Since $\cos(-\omega n) = \cos \omega n$ and $\sin(-\omega n) = -\sin \omega n$

$$X_R(-\omega) = X_R(\omega) \quad \text{and} \quad X_I(-\omega) = -X_I(\omega)$$

$$X^*(\omega) = X(-\omega)$$

Magnitude and phase spectra for real signals

$$|X(\omega)| = \sqrt{X_R^2(\omega) + X_I^2(\omega)} \quad \text{and} \quad \measuredangle X|\omega| = \tan^{-1} \frac{X_I(\omega)}{X_R(\omega)}$$

$$|X(\omega)| = |X(-\omega)| \quad \text{and} \quad \measuredangle X(-\omega) = -\measuredangle X(\omega)$$

For inverse transform of a real-valued signal ($x(n) = x_R(n)$), (5) implies

$$x(n) = \frac{1}{2\pi} \int_{2\pi} [X_R(\omega) \cos \omega n - X_I(\omega) \sin \omega n] d\omega$$

Since both $X_R(\omega) \cos \omega n$ and $X_I(\omega) \sin \omega n$ are even functions of ω

$$x(n) = \frac{1}{\pi} \int_0^{\pi} [X_R(\omega) \cos \omega n - X_I(\omega) \sin \omega n] d\omega$$

- **Real and even signals.** If $x(n)$ is real and even ($x(-n) = x(n)$), then $x(n)\cos\omega n$ is even and $x(n)\sin\omega n$ is odd

$$X_R(\omega) = \sum_{n=-\infty}^{\infty} x(n)\cos\omega n \longrightarrow X_R(\omega) = x(0) + 2\sum_{n=1}^{\infty} x(n)\cos\omega n \quad \text{(even)}$$

$$X_I(\omega) = -\sum_{n=-\infty}^{\infty} x(n)\sin\omega n \longrightarrow X_I(\omega) = 0$$

$$x(n) = \frac{1}{\pi}\int_0^{\pi} [X_R(\omega)\cos\omega n - X_I(\omega)\sin\omega n]d\omega \longrightarrow$$

$$x(n) = \frac{1}{\pi}\int_0^{\pi} X_R(\omega)\cos\omega n d\omega$$

  Thus spectra for real and even signals are

  - Real-valued
  - Even functions of ω

- **Real and odd signals.** If $x(n)$ is real and odd $(x(-n) = -x(n))$, then $x(n)\cos\omega n$ is odd and $x(n)\sin\omega n$ is even

$$X_R(\omega) = \sum_{n=-\infty}^{\infty} x(n)\cos\omega n \longrightarrow X_R(\omega) = 0$$

$$X_I(\omega) = -\sum_{n=-\infty}^{\infty} x(n)\sin\omega n \longrightarrow X_I(\omega) = -2\sum_{n=1}^{\infty} x(n)\sin\omega n \quad \text{(odd)}$$

$$x(n) = \frac{1}{\pi}\int_0^{\pi}[X_R(\omega)\cos\omega n - X_I(\omega)\sin\omega n]d\omega \longrightarrow$$

$$x(n) = -\frac{1}{\pi}\int_0^{\pi} X_I(\omega)\sin\omega n d\omega$$

Thus spectra for real-valued odd signals are

- Purely imaginary-valued
- Odd functions of ω

Example

- Determine and sketch $X_R(\omega)$, $X_I(\omega)$, $|X(\omega)|$, and $\measuredangle X(\omega)$ for

$$X(\omega) = \frac{1}{1 - ae^{-j\omega}}, \quad -1 < a < 1$$

- Multiplying both numerator and denominator by complex conjugate of denominator

$$X(\omega) = \frac{1 - ae^{j\omega}}{(1 - ae^{-j\omega})(1 - ae^{j\omega})} = \frac{1 - a\cos\omega - ja\sin\omega}{1 - 2a\cos\omega + a^2}$$

$$X_R(\omega) = \frac{1 - a\cos\omega}{1 - 2a\cos\omega + a^2} \quad \text{and} \quad X_I(\omega) = -\frac{a\sin\omega}{1 - 2a\cos\omega + a^2}$$

$$|X(\omega)| = \sqrt{X_R^2(\omega) + X_I^2(\omega)} = \frac{1}{\sqrt{1 - 2a\cos\omega + a^2}}$$

$$\measuredangle X|\omega| = \tan^{-1}\frac{X_I(\omega)}{X_R(\omega)} = -\tan^{-1}\frac{a\sin\omega}{1 - a\cos\omega}$$

Properties of Fourier Transform for Discrete-Time Signals

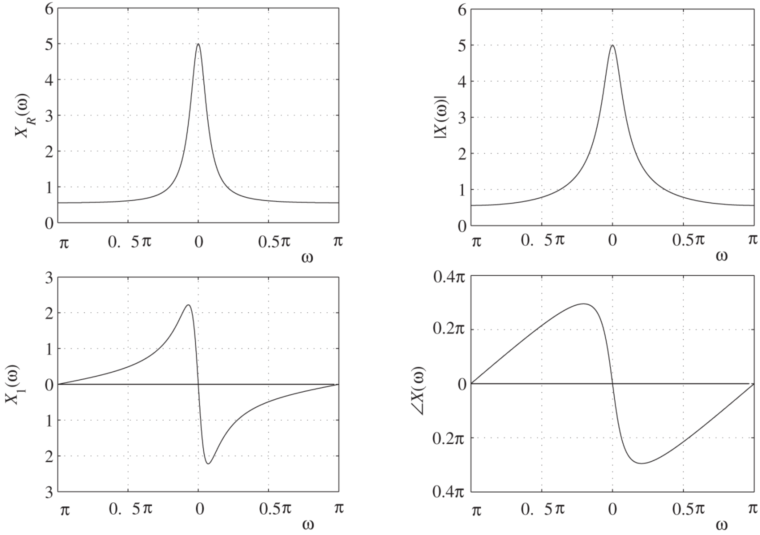

Figure 5: Spectra of the transform in Example for $a = 0.8$; all symmetry properties for the spectra of real signals apply to this case.

Example

- Determine Fourier transform of

$$x(n) = \begin{cases} A, & -M \leq n \leq M \\ 0, & \text{elsewhere} \end{cases}$$

- $x(n)$ is real and even $(x(-n) = x(n))$

$$X_R(\omega) = x(0) + 2\sum_{n=1}^{\infty} x(n)\cos\omega n \quad \text{(even)} \quad \text{and} \quad X_I(\omega) = 0$$

$$X(\omega) = X_R(\omega) = A\left(1 + 2\sum_{n=1}^{M}\cos\omega n\right) = A\frac{\sin(M+\frac{1}{2})\omega}{\sin(\omega/2)}$$

$$|X(\omega)| = \left|A\frac{\sin(M+\frac{1}{2})\omega}{\sin(\omega/2)}\right| \quad \text{and} \quad \measuredangle X(\omega) = \begin{cases} 0, & \text{if } X(\omega) > 0 \\ \pi, & \text{if } X(\omega) < 0 \end{cases}$$

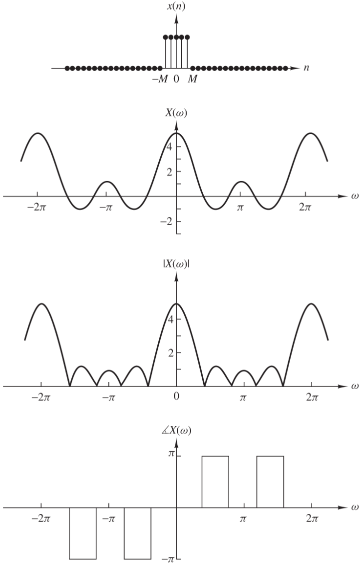

Figure 6: Spectral characteristics of rectangular pulse in Example.

Fourier Transform Theorems and Properties

- **Linearity**
  - If
    $$x_1(n) \stackrel{F}{\longleftrightarrow} X_1(\omega)$$
    and
    $$x_2(n) \stackrel{F}{\longleftrightarrow} X_2(\omega)$$
    then
    $$a_1 x_1(n) + a_2 x_2(n) \stackrel{F}{\longleftrightarrow} a_1 X_1(\omega) + a_2 X_2(\omega)$$
  - Fourier transformation, viewed as an operation on a signal $x(n)$, is a linear transformation

Example

- Determine Fourier transform of

$$x(n) = a^{|n|}, \qquad -1 < a < 1$$

- $x(n)$ can be expressed as

$$x(n) = x_1(n) + x_2(n)$$

where

$$x_1(n) = \begin{cases} a^n, & n \geq 0 \\ 0, & n < 0 \end{cases} \quad \text{and} \quad x_2(n) = \begin{cases} a^{-n}, & n < 0 \\ 0, & n \geq 0 \end{cases}$$

Fourier transform of $x_1(n)$

$$X_1(\omega) = \sum_{n=-\infty}^{\infty} x_1(n) e^{-j\omega n} = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty} (ae^{-j\omega})^n = \frac{1}{1 - ae^{-j\omega}}$$

knowing that $|ae^{-j\omega}| = |a| < 1$

Example (continued)

- Fourier transform of $x_2(n)$

$$
\begin{aligned}
X_2(\omega) &= \sum_{n=-\infty}^{\infty} x_2(n) e^{-j\omega n} = \sum_{n=-\infty}^{-1} a^{-n} e^{-j\omega n} \\
&= \sum_{n=-\infty}^{-1} (ae^{j\omega})^{-n} = \sum_{k=1}^{\infty} (ae^{j\omega})^k \\
&= \frac{ae^{j\omega}}{1 - ae^{j\omega}}
\end{aligned}
$$

Combining these two transforms, we obtain Fourier transform of $x(n)$

$$
X(\omega) = X_1(\omega) + X_2(\omega) = \frac{1 - a^2}{1 - 2a\cos\omega + a^2}
$$

Fourier Transform Theorems and Properties

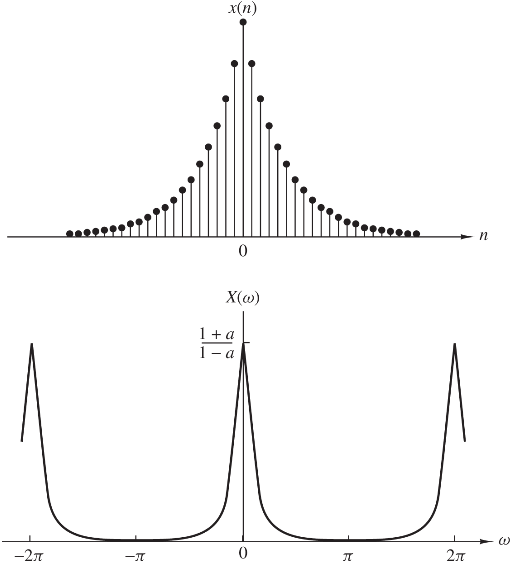

Figure 7: Sequence $x(n)$ and its Fourier transform in Example with $a = 0.8$.

- **Time shifting**
  - If
  $$x(n) \xleftrightarrow{F} X(\omega)$$
  then
  $$x(n-k) \xleftrightarrow{F} e^{-j\omega k} X(\omega)$$
  - Proof
  $$F\{x(n)\} = X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$
  $$F\{x(n-k)\} = \sum_{n=-\infty}^{\infty} x(n-k) e^{-j\omega n}$$
  $$\xrightarrow{l=n-k} = \sum_{l=-\infty}^{\infty} x(l) e^{-j\omega(l+k)}$$
  $$= X(\omega) e^{-j\omega k} = |X(\omega)| e^{j[\measuredangle X(\omega) - \omega k]}$$
  - If a signal is shifted in time domain by k samples, its magnitude spectrum remains unchanged, but phase spectrum is changed by an amount $-\omega k$

- **Time reversal**
  - If
$$x(n) \overset{F}{\longleftrightarrow} X(\omega)$$
    then
$$x(-n) \overset{F}{\longleftrightarrow} X(-\omega)$$
  - Proof

$$F\{x(n)\} = X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

$$F\{x(-n)\} = \sum_{n=-\infty}^{\infty} x(-n)e^{-j\omega n}$$

$$\xrightarrow{l=-n} = \sum_{l=-\infty}^{\infty} x(l)e^{j\omega l}$$

$$= X(-\omega) = |X(-\omega)|e^{j\measuredangle X(-\omega)}$$

$$\xrightarrow{\text{if } x(n) \text{ is real}} = |X(\omega)|e^{-j\measuredangle X(\omega)}$$

- **Convolution theorem**
  - If
  $$x_1(n) \stackrel{F}{\longleftrightarrow} X_1(\omega)$$
  and
  $$x_2(n) \stackrel{F}{\longleftrightarrow} X_2(\omega)$$
  then
  $$x(n) = x_1(n) * x_2(n) \stackrel{F}{\longleftrightarrow} X(\omega) = X_1(\omega)X_2(\omega)$$
  - Proof: multiply both sides of convolution formula by $e^{-j\omega n}$ and sum over all n

$$x(n) = x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} x_1(k)x_2(n-k)$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} = \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x_1(k)x_2(n-k)\right] e^{-j\omega n}$$

$$\xrightarrow{n-k=l} = \sum_{k=-\infty}^{\infty} x_1(k) \sum_{l=-\infty}^{\infty} x_2(l)e^{-j\omega(k+l)} = X_1(\omega)X_2(\omega)$$

Example

- Using convolution theorem, determine convolution of sequences

$$x_1(n) = x_2(n) = \{1, \underset{\uparrow}{1}, 1\}$$

- For real and even signals

$$X_R(\omega) = x(0) + 2\sum_{n=1}^{\infty} x(n)\cos\omega n \quad \text{and} \quad X_I(\omega) = 0$$

$$X_1(\omega) = X_2(\omega) = 1 + 2\cos\omega$$

$$\begin{aligned} X(\omega) &= X_1(\omega)X_2(\omega) = (1 + 2\cos\omega)^2 \\ &= 3 + 4\cos\omega + 2\cos 2\omega \\ &= 3 + 2(e^{j\omega} + e^{-j\omega}) + (e^{j2\omega} + e^{-j2\omega}) \end{aligned}$$

Thus convolution of $x_1(n)$ with $x_2(n)$ is (recall $X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$)

$$x(n) = \{1, 2, \underset{\uparrow}{3}, 2, 1\}$$

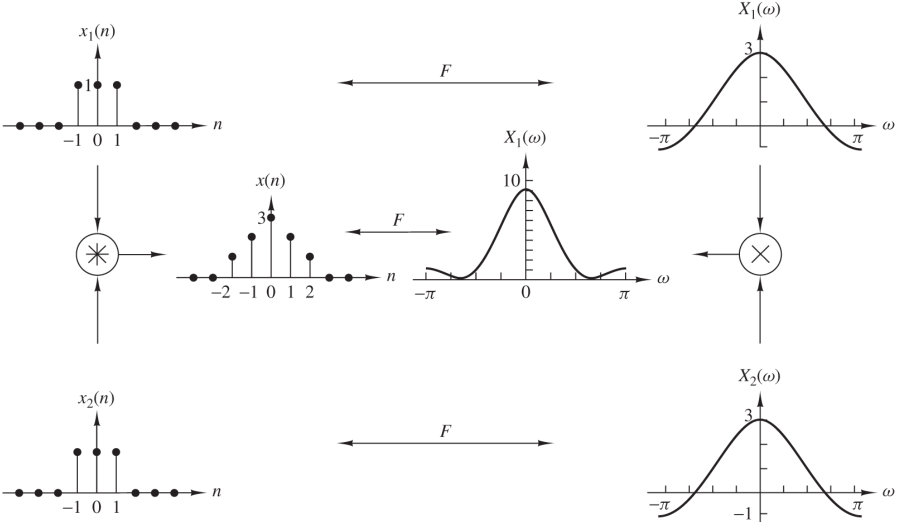

Figure 8: Graphical representation of the convolution property.

- **Correlation theorem**
  - If
$$x_1(n) \overset{F}{\longleftrightarrow} X_1(\omega)$$
    and
$$x_2(n) \overset{F}{\longleftrightarrow} X_2(\omega)$$
    then
$$r_{x_1x_2}(n) \overset{F}{\longleftrightarrow} S_{x_1x_2}(\omega) = X_1(\omega)X_2(-\omega)$$
  - Proof: multiply both sides of correlation formula by $e^{-j\omega n}$ and sum over all n

$$r_{x_1x_2}(n) = \sum_{k=-\infty}^{\infty} x_1(k)x_2(k-n)$$

$$S_{x_1x_2}(\omega) = \sum_{n=-\infty}^{\infty} r_{x_1x_2}(n)e^{-j\omega n} = \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x_1(k)x_2(k-n)\right] e^{-j\omega n}$$

$$\xrightarrow{k-n=l} = \sum_{k=-\infty}^{\infty} x_1(k) \sum_{l=-\infty}^{\infty} x_2(l)e^{-j\omega(k-l)} = X_1(\omega)X_2(-\omega)$$

$S_{x_1x_2}(\omega)$ is called **cross-energy density spectrum** of signals $x_1(n)$ and $x_2(n)$

- **Wiener-Khintchine theorem**
  - Let $x(n)$ be a real signal. Then

$$r_{xx}(l) \stackrel{F}{\longleftrightarrow} S_{xx}(\omega)$$

    - I.e., energy spectral density of an energy signal is Fourier transform of its autocorrelation sequence
    - This is a special case of preceding theorem (correlation theorem)

Example

- Determine energy density spectrum of

$$x(n) = a^n u(n), \qquad -1 < a < 1$$

- Using results of previous examples for this signal

$$r_{xx}(l) = \frac{1}{1-a^2} a^{|l|}, \qquad -\infty < l < \infty$$

$$F\{r_{xx}(l)\} = \frac{1}{1-a^2} F\{a^{|l|}\} = \frac{1}{1 - 2a\cos\omega + a^2}$$

According to Wiener-Khintchine theorem

$$S_{xx}(\omega) = \frac{1}{1 - 2a\cos\omega + a^2}$$

- **Frequency shifting**
  - If
    $$x(n) \stackrel{F}{\longleftrightarrow} X(\omega)$$
    then
    $$e^{j\omega_0 n}x(n) \stackrel{F}{\longleftrightarrow} X(\omega-\omega_0)$$
  - Proof

$$X(\omega)=\sum_{n=-\infty}^{\infty}x(n)e^{-j\omega n} \longrightarrow X(\omega-\omega_0)=\sum_{n=-\infty}^{\infty}x(n)e^{-j(\omega-\omega_0)n}$$
$$=\sum_{n=-\infty}^{\infty}(e^{j\omega_0 n}x(n))e^{-j\omega n}$$
$$=F\{e^{j\omega_0 n}x(n)\}$$

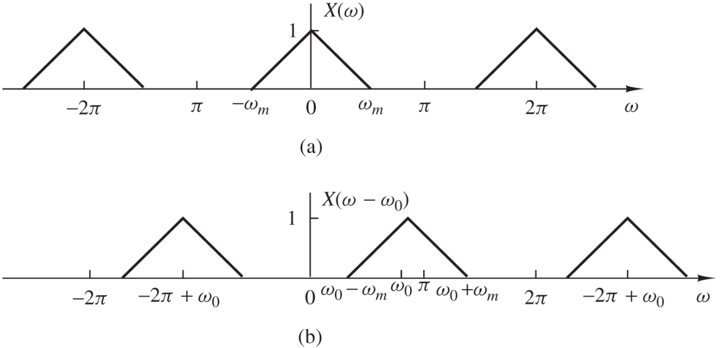

Figure 9: Illustration of the frequency-shifting property of the Fourier transform ($\omega_0 \leq 2\pi - \omega_m$).

- **Modulation theorem**
  - If
    $$x(n) \overset{F}{\longleftrightarrow} X(\omega)$$
    then
    $$x(n)\cos\omega_0 n \overset{F}{\longleftrightarrow} \tfrac{1}{2}[X(\omega+\omega_0)+X(\omega-\omega_0)]$$
  - Proof: expressing $\cos\omega_0 n$ as
    $$\cos\omega_0 n = \tfrac{1}{2}(e^{j\omega_0 n}+e^{-j\omega_0 n})$$
    and using frequency-shifting property
    $$x(n) \overset{F}{\longleftrightarrow} X(\omega) \longrightarrow e^{j\omega_0 n}x(n) \overset{F}{\longleftrightarrow} X(\omega-\omega_0)$$
    we obtain
    $$F\{\tfrac{1}{2}(e^{j\omega_0 n}+e^{-j\omega_0 n})x(n)\} = \tfrac{1}{2}[X(\omega-\omega_0)+X(\omega+\omega_0)]$$

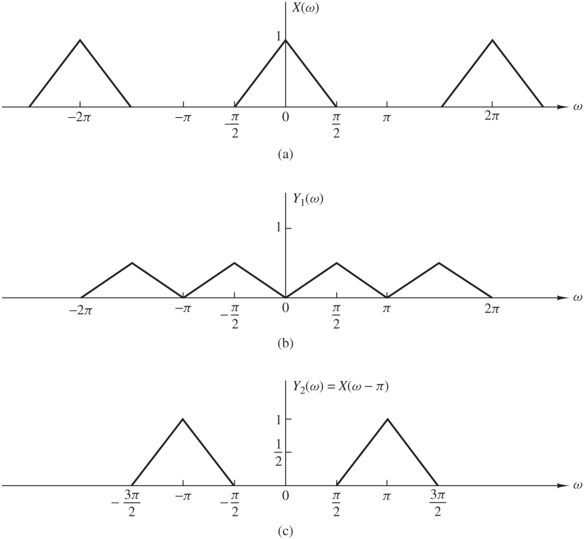

Figure 10: Graphical representation of the modulation theorem; the spectra of the signals $x(n)$, $y_1(n) = x(n)\cos 0.5\pi n$ and $y_2(n) = x(n)\cos \pi n$.

- **Parseval's theorem**
  - If

$$x_1(n) \overset{F}{\longleftrightarrow} X_1(\omega)$$

and

$$x_2(n) \overset{F}{\longleftrightarrow} X_2(\omega)$$

then

$$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\omega)X_2^*(\omega)d\omega$$

  - Proof: eliminating $X_1(\omega)$ on right-hand side of above equation

$$\frac{1}{2\pi}\int_{2\pi}\left[\sum_{n=-\infty}^{\infty} x_1(n)e^{-j\omega n}\right]X_2^*(\omega)d\omega$$

$$= \sum_{n=-\infty}^{\infty} x_1(n)\frac{1}{2\pi}\int_{2\pi} X_2^*(\omega)e^{-j\omega n}d\omega = \sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n)$$

- In special case where $x_2(n) = x_1(n) = x(n)$, Parseval's relation reduces to

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi} \int_{2\pi} |X(\omega)|^2 d\omega$$

  - Left-hand side of this equation is energy E_x of $x(n)$
  - Left-hand side is also equal to autocorrelation of $x(n)$, $r_{xx}(l)$ at $l = 0$
  - Integrand in right-hand side is equal to energy density spectrum, so integral over $-\pi \leq \omega \leq \pi$ yields total signal energy

$$E_x = r_{xx}(0) = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi} \int_{2\pi} |X(\omega)|^2 d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} S_{xx}(\omega) d\omega$$

- **Multiplication of two sequences (Windowing theorem)**
  - If

$$x_1(n) \stackrel{F}{\longleftrightarrow} X_1(\omega)$$

and

$$x_2(n) \stackrel{F}{\longleftrightarrow} X_2(\omega)$$

then

$$x_3(n) = x_1(n)x_2(n) \stackrel{F}{\longleftrightarrow} X_3(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\lambda)X_2(\omega-\lambda)d\lambda$$

    - Integral on right-hand side is convolution of $X_1(\omega)$ and $X_2(\omega)$
    - This convolution integral is known as *periodic convolution* of $X_1(\omega)$ and $X_2(\omega)$ because it is convolution of two periodic functions having the same period
    - Multiplication of aperiodic sequences is equivalent to periodic convolution of their Fourier transforms
    - Based on duality, convolution in time domain (aperiodic summation) is equivalent to multiplication of continuous periodic Fourier transforms
    - Due to periodicity of Fourier transforms for discrete-time signals, there is no "perfect" duality between time and frequency domains with respect to convolution operation, as in the case of continuous-time signals

- Proof of windowing theorem:
  We know

$$x_3(n) = x_1(n)x_2(n) \quad \text{and} \quad x_1(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\lambda)e^{j\lambda n}d\lambda$$

Thus

$$\begin{aligned}
X_3(\omega) &= \sum_{n=-\infty}^{\infty} x_3(n)e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x_1(n)x_2(n)e^{-j\omega n} \\
&= \sum_{n=-\infty}^{\infty} \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\lambda)e^{j\lambda n}d\lambda\right] x_2(n)e^{-j\omega n} \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\lambda)d\lambda \left[\sum_{n=-\infty}^{\infty} x_2(n)e^{-j(\omega-\lambda)n}\right] \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\lambda)X_2(\omega-\lambda)d\lambda
\end{aligned}$$

- **Differentiation in frequency domain**
  - If
  $$x(n) \overset{F}{\longleftrightarrow} X(\omega)$$
  then
  $$nx(n) \overset{F}{\longleftrightarrow} j\frac{dX(\omega)}{d\omega}$$
  - Proof: differentiate series in Fourier transform definition, term by term with respect to ω

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

$$\frac{dX(\omega)}{d\omega} = \frac{d}{d\omega}\left[\sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}\right]$$

$$= \sum_{n=-\infty}^{\infty} x(n)\frac{d}{d\omega}e^{-j\omega n}$$

$$= -j\sum_{n=-\infty}^{\infty} nx(n)e^{-j\omega n}$$

Multiplying both sides by j, we obtain the desired result

References

John G. Proakis, Dimitris G. Manolakis, *Digital Signal Processing: Principles, Algorithms, and Applications*, Prentice Hall, 2006.