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3.2

MA052 / 01

Structural graph theory

- Graph minors theory (Robertson + Seymour)

Started with attempts to solve Wagner's conjecture
Well quasiordering by minors
↳ Simply - if we take any infinite sequence of graphs, one graph is a minor of another

Minor theory is related to width measures
(Tree width, Branch width and others)
, Excluded grid
→ How far a graph is from a tree, or
"how simple it is?"
⇒ algorithmic applications (most any problem is quickly solvable with bounded width)
· Related to graphs on surfaces
· Related to connectivity in an extended sense
- "excluded K_n" - characterization
K_3 - trees
K_4 - TW 2
If there is n and we exclude K_n, this theory

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Tells us how the graph looks ⇒ it leads to a drawing on a surface

- Finishing – Exclude some K_n ⇒ we have embedding on some surface in some sence ⇒ forget about bounded treewidth ⇒ take grid, which looks as if it was in the plane, and... linkage

– Linkage: stronger connectivity
connectivity = many disjoint paths
linkage: → terminals, if the connectivity is strong, we can find a set of paths $(s_i t_i)$ → they respect indexing

s_1 —— T_1
s_2 —— T_2
⋮ —— ⋮
s_k —— T_k

→ Now you want to find this graph in some other graph as a minor. You use everything from above.

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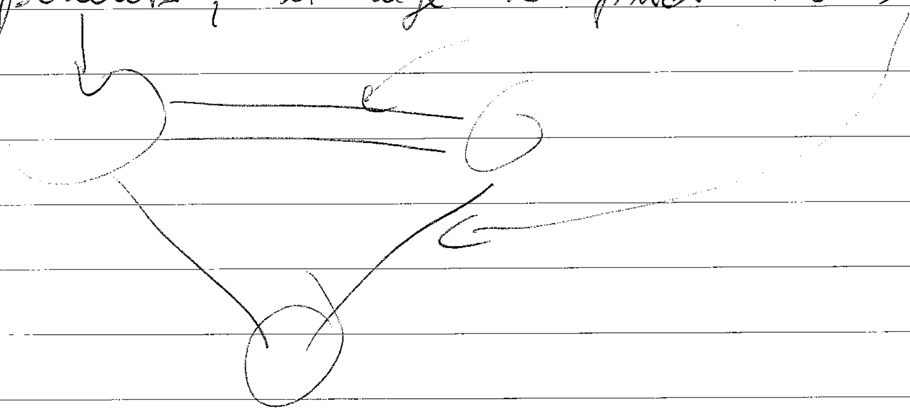
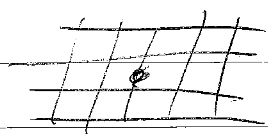
· You use a grid to find planar components, linkage to find the links



and also „irrelevant vertex": if something happens for an inner-vertex, you can route



it through some other vertex.

⇒ Searching for a minor, we delete the vertex, then ~~on~~ solve recursively, once we don't have big grid, we have bounded TW and find minor easily

⇒ This is Classical GM theory

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→ related to Rank width

→ related to Clique width

Then there is a dense theory, which is incomplete. (Oum, Courcelle, Seymour ...)

→ We can re-do the width part, we define ~~def~~ a different notion of minors - vertex minors!

Instead of graphs of surfaces, you get "circle graphs" → graph with intersecting chords. The rest is not developed.

Both the branches unify in Matroid minors theory (Geelen, Gerards, Whittle)

1980 → Classical

1980 → Dense

2002 →

Rota's conj. - Kuratowski conj for matroids

Matroid Minors Theory (nearly complete)

2000 →

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Other theory: Graph sparsity (2006 →)
(Nešetřil, Ossona de Mendez)

Does not have a big starting point. Idea:

Let's look ~~at~~ on „sparse graphs" only and try do something better

Motivation example: Enumerate all triangles in a graph. We can do $O(n^3)$ but want all triangles. We need to have sparse graph!

1. We need to capture what "sparse" means
   – in some sense „not many edges"

In general, motivation can be FO model checking; i.e. is a formula true?
↳ you can quantify over elements of universe

Existential FO is equivalent to finding subgraph.

Dominating set of size 3: $\exists x_1 x_2 x_3 \forall x \in V(G)\ \exists$ edge with x_1, x_2, x_3 neighbour

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Nešetřil defines notion of "nowhere dense".
Some parts of graph minors theory is nicely simplified by sparsity parts.

2. Instead of width measures, we have tree-depth and depth measures ("stricter notion than TW)

3. Sparsity uses shallow minors - restriction on minors

There are no surfaces in ~~s~~ sparsity - not needed

4. Study homomorphisms and automorphisms

Sparsity
→ Classical view (~~TW~~ Tree depth)
→ Dense view (Clique width)
Shrub-depth

matroid sparsity theory? Question!

- just beginning
- key idea: to be stable on complements
- FO-interpretation?

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Graph = incidence model → vertices, edges, incidence
→ allow multigraphs

adjacency model → $G = (V, E)$, $E \subseteq \binom{V}{2}$

<u>Def.</u>: G is a <u>minor</u> of H if G can be obtained from by a subgraph of H by contracting edges.

Def.: A contraction of an edge $e = uv$ in F results F' such that $V(F') = V(F) \setminus \{u, v\} \cup \{w_e\}$ and $E(F') = E(F - u - v) \cup \{w_e x : ux \in E(F) \vee vx \in E(F)\}$.

Jednoduché mínus je odstranění vrcholu i s hranami

In this model: contract a subgraph = contract a spanning tree

~~Def.:~~

CTRL+N

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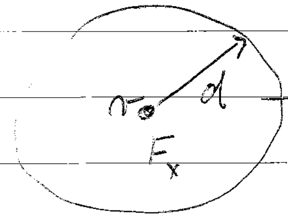
Prop 1: If G is obtained from H by a sequence of edge/vertex deletions and edge contractions, then G is a minor of H, $G \leq H$.

↳ The order does not matter!
We can do deletions first.

Prop 2: G is a minor of H iff $\exists$ disjoint connected subgraphs $F_x \subseteq H$, $x \in V(G)$, such that whenever $xy \in E(G)$, there is an edge between $V(F_x)$ and $V(F_y)$ in H.

Def. (Shallow minor): G is a shallow minor of H at depth d iff in Prop. 2., the subgraphs $F_x \subseteq H$, $x \in V(G)$ have each radius at most $\leq d$. Notation: $G \leq_d H$



→ x has distance at most d to all of F_x

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infinite "nabla"

Def.: Graph class $\mathcal{G}$; $\mathcal{G} \nabla d = \{G : G \preceq_d H \in \mathcal{G}\}$

G je sh. minor of depth d of some H in class $\mathcal{G}$

Meta Def.: Cops-and-rober games on G

- a rober is in a graph, can move only along the edes, arbitrarily fast, clever, full knowledge of the state of game

- cops fly in helicopter and may land on some vertices, sometime may also be lifted back, there is limited number of them

- the robber is caught when he occurs in the same vertex as some cop

- outcome of the game is the minimum number k of cops needed to catch the rober in G, we don't care about the time

- there are variants

CTRL+N

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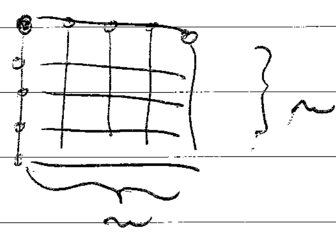
~~Def.~~ Def: Tree-Width of G is (k-1) for the cops-and-robber game with visible (and eager) robber and cop lifts allowed

Prop.: Tree-width of the N×N Grid is N.
(upper bound and lower bound)



Def.: Path-Width of G is (k-1) for the game with invisible robber (c. lifts allowed)

Path-Widt of a tree is height ~~th~~.

Def.: Tree-Depth of G is (k ~~–1~~) for the game with visible robber and no c. lifts.

Prop.: $\mathrm{Tw}(G) \le \mathrm{pw}(G) \le \mathrm{td}(G) - 1$

restricted cops   restricted cops and robber!   small when we don't have long paths

not really clear

$\mathrm{td}(P_n) = \log n$