Univerza v Ljubljani
Fakulteta za računalništvo
in informatiko

MA 052 / 11

Tree and WQO

(předuspořádání)

Def.: Let $(M, \leq)$ be a quasiordering. Then $(M, \leq)$ is a well-quasi-ordering (WQO) if for any infinite sequence $(a_1, a_2, \dots) \subseteq M$ there exists $i < j$ such that $a_i \leq a_j$. můžeme opakovat

- Nemáme nekonečnou negativní ~~...~~ řadu a vše je lineárně uspořádáno
- Quasi znamená, že můžeme mít „neporovnatelné" prvky $a \leq b \geqq a$

EX:
- $\mathbb{N}$ is WQO (even WO) by $\leq$
- $(\mathbb{Z}$, divisibility$)$ is WQO $\iff$ finitely many primes
- $\forall$ finite set is WQO.
- Ordinals are WQO

Axiom of choice: Every set can be well ordered

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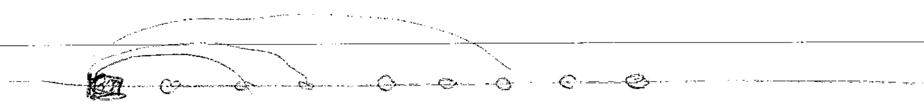
PROP: $(M, \leq)$ is WQO iff M contains no infinite strictly descending chain and no infinite antichain (antichain means incomparable elements, e.g. primes in divisibility).

Basic Ramsey thm.: If 6 people meet, 3 of them know each other or 3 of them don't know each other.

Proof: From infinite Ramsey on 3 colours.



Every outgoing edge has a colour, want infinite subsequence of edges with same colour.

- Choose colour, forget the other vertices, we still have inf. many

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--- proof: Black colour: $i < j$ and $a_i \leq a_j$

Green : $i < j$ a_i incomp. a_j

Blue $i < j$ $a_i \gneq a_j$

Now, select inf. sequence, → in one colour only black ~~colour~~ infinite sequence is allowed by assumption.

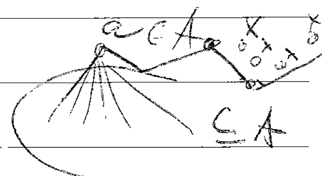
⇒ Take inf. sequence, take Ramsey thm. (eksistuje monochromatická sekvence) on this sequence ⇒ verifies the definition

<u>Cor.:</u> Infinite ascending chain in WQO alway exists.

~~Cor~~

Prop.: Let $(M, \leq)$ be well-quasi-ordering and take any $A \subseteq M$ such that A is "down-closed" (for all $a \in A$, $b \leq a \Rightarrow b \in A$)

Then there is a <u>finite</u> Hasse set $X \subseteq M$ such that $a \in A$ iff $\forall x \in X: a \ngeq x$.



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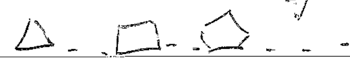
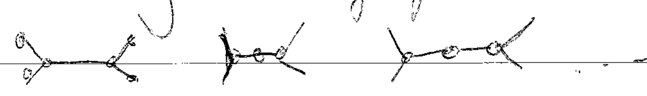
Equivalently: $M \setminus A$ has finitely many minimal elements.

Proof: If we have infinitely many red points, they form an antichain $\Rightarrow$ are not comparable.

Examples: Graphs by subgraphs -> ordering Not well-quasi-ordering; sequence of all cycles is an infinite antichain



Trees-by-subgraph: No



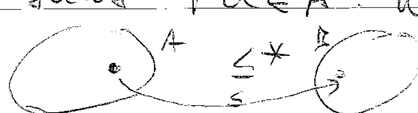
Graphs with no long paths: YES

Trees are WQO by subdivisions.

Lemma (Higman's): If $(M, \leq)$ is WQO, then $(M^{<\omega}, \leq^*)$ is WQO.

$M^{<\omega}$ -> set of finite subsets

$A \leq^* B$ iff $\exists$ injective mapping $l: A \to B$ such that $\forall a \in A: a \leq l(a)$.



for every element in A, we have a dominating element in B

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smallest possible set that starts a bad sequence.

Proof: Select bad "$A_0, A_1, \ldots$" such that $(|A_0|, |A_1|, |A_2| \ldots)$ is lexicographically minimal. Choose $a_i \in A_i$ arbitrary and $B_i = A_i \setminus \{a_i\}$. M is WQO $\Rightarrow$ $\Rightarrow a_{n_0} \leq a_{n_1} \leq a_{n_2} \ldots$ inf, so that there is an infinit ascending sequence. Take

$$\underbrace{A_0, A_1, \ldots, A_{n_0-1}}, B_{n_0}, B_{n_1} \ldots$$

already has two smaller sets

Which is smaller (lexicogr.) than $A_0, A_1 \ldots A_{n_0} \ldots$, so there exists $i < j$ such that (1) $A_i \leq^* A_j$ or (2) $A_i \leq^* B_{n_j}$ or (3) $B_{n_i} \leq^* B_{n_j}$. Thus either both indice are in $(i \leq n_j)$ the $A_0 \ldots A_{n_0-1}$ part, or one belongs to A-part and one to B-part, or both to B-part.

Case (1): The original sequence was not bad

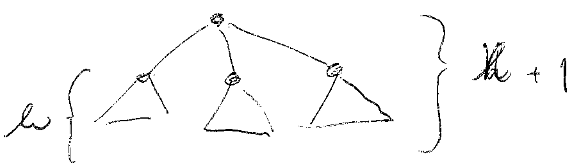
Case (2): Cannot happen, because $A_i \leq^* B_{n_j} \leq A_{n_j}$ and $i \leq n_j$

Case (3): If this is the case, then $A_{n_i} \leq^* A_{n_j}$

$\Rightarrow$ Contradiction of $A_0, A_1, \ldots$ being bad.

~~Rooted~~ **Cor:** Rooted trees with bounded height h with nodes coloured from a Well-quasi-ordered set are WQO under "coloured subgraph" → on every vertex, there is greater or equal colour

Proof: Proof it even for forests by induction on h. Base $h=0$: It is sets of colours, so Higman lemma. Step $h \to h+1$:



a) Trees of height $h+1$ are WQO → make each a forest of height h by removing the root, and induction. Before we do this, we have to define a sequence of the roots.

b) Trees to forest by Higman again.

$\square$