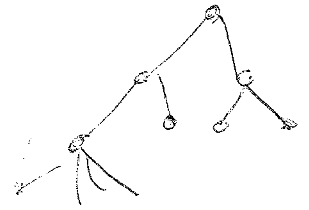
MA052 / 17

Trees and WQO

Def.: Tree-Depth decomposition of a graph G is a rooted ~~tree~~ forest T such that $V(T) = V(G)$ and for every edge $e = uv \in E(G)$ u is a descendant of v in T or vice versa.
("$G \subseteq cl(T)$")



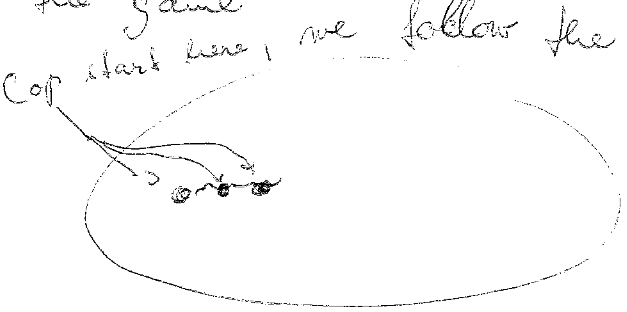
$cl(T)$ = adding all edes among descendant

Edges of decompositions are not necessarily edges of G.

Claim: The tree-depth of G is the min. height of a tree-depth decomposition of G plus 1

Lemma: If G has no path of length d (# of edges), then tree depth is at most d ($td(G) \leq d$).

Proof: Use the game, we follow the path of the robber, after placing cop $\# d-1$, the robber cannot espace.



17A052/18

Lemma: If G has a path, then $td(G) \geq \lceil \log_2 (d+2) \rceil$

$d+1$ vertices, the robber ignores the rest of the graph

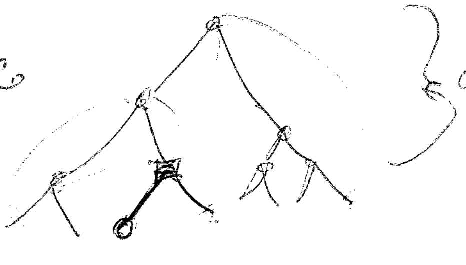
$$d+1 \to \frac{d}{2} \to \frac{d/2 - 1}{2} \dots$$

Critical points $d = 2^a - 1 \Rightarrow$ we need one more cop. $\blacksquare$

Theorem (Ding): If $\mathcal{G}$ is a graph class without long paths ($\exists \ell$ such that P_ℓ is not a subgraph of $G \in \mathcal{G}$), then $\mathcal{G}$ is WQO under induced subgraphs.

Proof: $\exists d$ such that all graphs in $\mathcal{G}$ have a tree depth decomposition of height d formed by a single d.

T_G



d predecessor

MA052/19

Let T_G be the decomposition of $G \in \mathcal{G}$, and label each $x \in V(T_G)$ by the set $L_x = \{j \mid xy \in E(G)$ where y is predecessor of x at distance $j\} \subseteq \{1, \dots, pl\}$.

Now, the class of $\{T_G \mid G \in \mathcal{G}\}$ is WQO under label preserving subtrees (Lecture 2, #16).

It remains to note that $T_G \leq_{lab} T_H \Rightarrow G \leq_i H$ induced.

label preserving

(Značky $\underset{\text{minor}}{\leq}$, $\underset{\text{subgraph}}{\subseteq}$, $\underset{\text{induced subgraph}}{\subseteq_i}$).

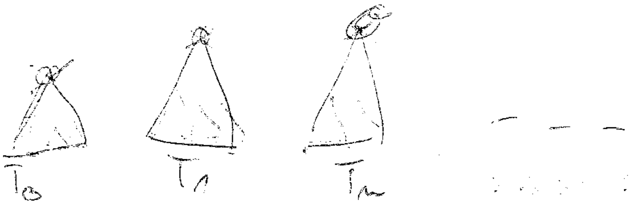
Theorem (Kruskal): The class of all trees is WQO under containment of subdivision (Topological minor).

Proof: Prove this for rooted trees! It is even stronger, because root must go to the root (of a subtree).

Choose a bad sequence of trees $T_0, T_1, \dots$ with lexicographically minimal sizes vector $(|T_0|, |T_1|, |T_2|, \dots)$.

↳ number of V, E whatever.

We define $A_n = T_n - \text{root}_n$, and take $A = \bigcup_n A_n$ as a set of rooted trees.



MA052 | 20

We claim that A is WQO: Pick any sequence $U_0, U_1, U_2 \dots$ in $\bar{A}$ (the sequence is infinite).

Set $n(k) \in \mathbb{N}$ such that $U_k \in A_{n(k)}$.

("Every U_k from A comes from the same tree"), and pick k with the smallest $n(k)$

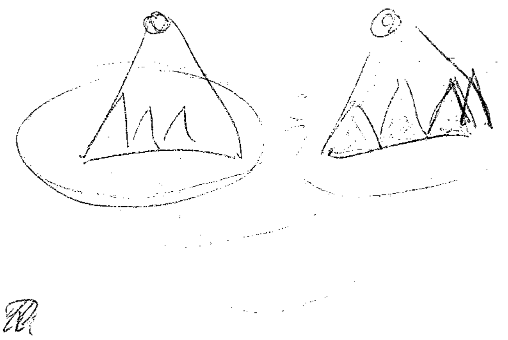
Consider $T_0, T_1 \dots T_{n(k)-1}, \underline{U_k}, U_{k+1}, \dots$

$\hookrightarrow$ from here

This is smaller than the bad sequence and ~~the~~ hence, it is good!

Then, there $\exists\, i \leq j$ such that $U_i \underset{\text{subdivision}}{\leq} U_j$.

By Higman's lemma, $\bar{A}^{(<\omega)}$ finite subsets. is WQO. Thus, there is $i' \leq j'$



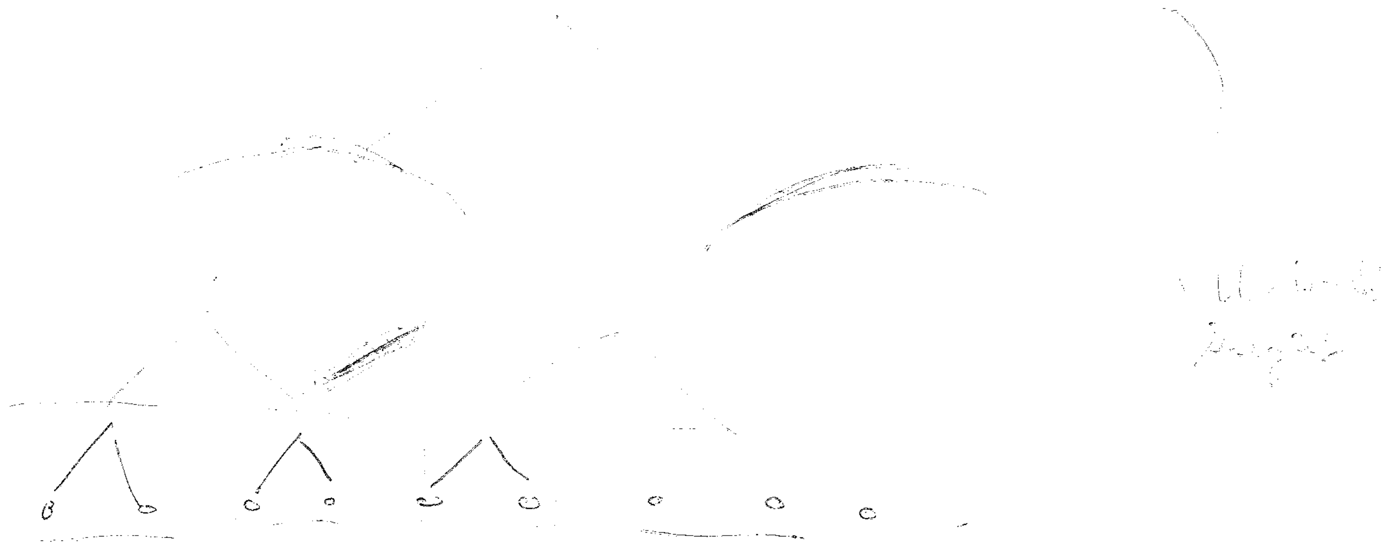
such that $\bar{A}_{i'} \underset{\text{subdiv.}}{\subseteq} A_{j'}$ and then also $T_{i'} \underset{\text{subdiv.}}{\subseteq} T_{j'}$

and that's contradiction.

MA052/21

Thm: (Robertson, Seymour)

Graphs of bounded Tree-Width are well-quasi ordered under minors.



join on bounded size boundary

Bounded size pieces

Now we colour small ~~\$~~ pieces, boundaries, and at the end obtain „parse tree".

Apply similar thing to Kruskal.