MA052 | B1

Classification of Classes

DEF.: hereditary = closed on induced subgraphs

monotone = closed on normal subgraphs

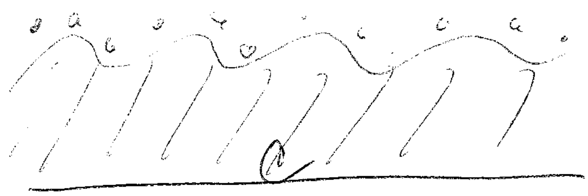
minor-closed = closed on minors

minors are the strongest ⇒ we have the most operations

Claim Class $\mathcal{C}$ closed on "$\leq$" (whatever) then $\mathcal{C}$ has $Forb(\mathcal{C})$ set of obstructions such that

$\forall G: G \in \mathcal{C} \Leftrightarrow (\forall F \in Forb(\mathcal{C}): F \not\leq G)$

$Forb(\mathcal{C}) \rightarrow$



$\leftarrow$ Hasse diagram under "$\leq$"

Typically $\leq$, $Forb(\mathcal{C})$ is infinite. For graph minors $\leq$, $Forb(\mathcal{C})$ is finite.

DEF: Graph parameter $f: \mathcal{C} \rightarrow \mathbb{R}$ (usually $\mathbb{R}_0^+$ or even $\mathbb{N}$)

Then $f(\mathcal{C}) = \sup_{G \in \mathcal{C}} f(G)$

↳ In fact, it can be understood as max, however f can assymptotically approach this number.

DEFINITION HAS A PROBLEM!

MA052 | B2

Problem:

Imagine a class, which is nicely characterised by the definition, but it has one exceptionally large ~~graph~~ $f(G)$. This kills $f(C)$.

Thus, better use assymptotic

$$\bar{f}(C) = \underline{\limsup_{G \in C}}\, f(G)$$

→ nejvyšší hromadný bod (v každém ε-okolí je nekonečně mnoho hodnot)

dá se tedy přepsat jako $\forall \varepsilon > 0$ only finitely many $G \in C$ such that $f(G) > \bar{f}(G) + \varepsilon$ and infinitely $G \in C$ such that $f(G) > \bar{f}(G) - \varepsilon$.

Def: C, D are assymptotically equal classes iff $|C \triangle D|$ is finite (differ only in only finitely many members).

Claim: If C, D are assymptotically eq., then $\bar{f}(C) = \bar{f}(D)$

$\Rightarrow$ neměníme lim sup, měníme pouze konečně mnoho grafů.

MA052/B3

DEF : Resolution of a class $\mathcal{C}$ ("in time") is the sequence:

connected graphs (i.e. distance r)

$$\mathcal{C}^{\nabla} := (\mathcal{C} \nabla 0, \mathcal{C} \nabla \tfrac{1}{2}, \mathcal{C} \nabla 1 \ldots).$$

all subgraphs $\Rightarrow$ class is monotone

$$\mathcal{C}^{\tilde{\nabla}} := (\mathcal{C} \tilde{\nabla} 0, \mathcal{C} \tilde{\nabla} \tfrac{1}{2}, \mathcal{C} \tilde{\nabla} 1 \ldots).$$

Why is this useful? We want to study some limits.

DEF : $\mathcal{C}$ (class) is <u>nowhere dense</u> if Graphs $\notin \mathcal{C}^{\nabla}$.

$\Rightarrow$ This means that in no finite time we get all graphs

Somewhere dense if Graphs $\in \mathcal{C}^{\nabla}$.

$\Rightarrow$ In finite time, we get all the graphs.

For example, the class of planar graphs is nowhere dense.

<u>Claim</u> : $\mathcal{C}$ somewhere dense iff $\omega(\mathcal{C} \nabla r) = \infty$ for some r.

(Immediate from definition)

MA052 | B4

<u>PROP</u>: $\mathcal{C}$ is nowhere dense $\Leftrightarrow$ Graphs $\notin \mathcal{C}^{\tilde{\nabla}}$.

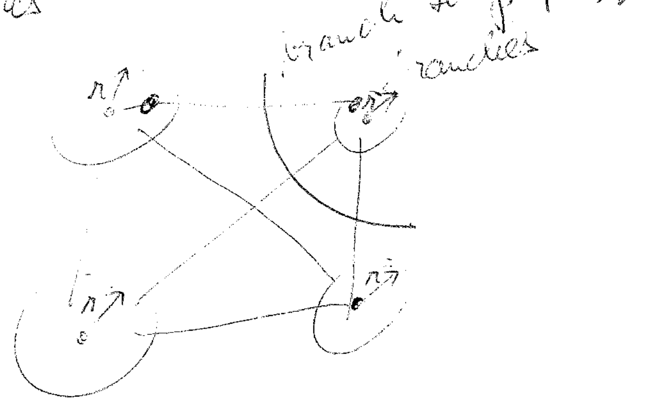
$\Rightarrow$ trivial (follows from minors claim).

$\Leftarrow$ $\mathcal{C}^{\nabla} = (\mathcal{C} \nabla 0, \mathcal{C} \nabla \frac{1}{2}, \mathcal{C} \nabla 1 \ldots)$ all graphs are here

$\mathcal{C}^{\tilde{\nabla}} = (\mathcal{C} \tilde{\nabla} 0, \mathcal{C} \tilde{\nabla} \frac{1}{2}, \ldots$

a ~~~ $3a+1$ $\to$ only small distance in resolution

$\forall p$: K_p as



in every potato, we have a vertex with sufficiently large degree. Take them as the central vertices.

Book proves the prop. inductively.

one edge $\to$ all edges

<u>DEF</u> logarithmic density: $\ell\text{-dens}(G) = \dfrac{\log \|G\|}{\log |G|}$ or $-\infty$ (when we have no edges). $\to$ from interval $[0,2]$

$$\|G\| = |G|^{\ell\text{-dens}(G)}$$

MA 052 / B5

We use $\overline{\ell\text{-dense}}(\mathcal{C})$ for a graph class $\mathcal{C}$ (the lim sup thing as before). Assume $\overline{\ell\text{-dense}}(\mathcal{C})$ is ≥ 0 (there is only degenerated case with $-\infty$).

Say, $\overline{\ell\text{-dense}}(\mathcal{C}) \leq 0$ iff $\|\mathcal{C}\| < \infty$ ← class, infinite, must be hereditary
→ is bounded.

Prop: $\overline{\ell\text{-dense}}(\mathcal{C}^{\nabla}) = \overline{\ell\text{-dens}}(\mathcal{C}^{\tilde{\nabla}})$, where $\overline{\ell\text{-dense}}$ of the whole sequence $\overline{\ell\text{-dens}}(\mathcal{C}^{\nabla}) = \sup_{r \to \infty} \overline{\ell\text{-dens}}(\mathcal{C}\nabla r)$

↳ tato je non-decreasing.

„$\geq$" is trivial, at every r, $\mathcal{C}\tilde{\nabla} r \subseteq \mathcal{C}\nabla r$

„$\leq$" by cases: (1) if $\ell\text{-dens}(\mathcal{C}^{\tilde{\nabla}}) = 2$, it must work

(2) $-\infty$ and 0 holds as well

(3) $(0,1) \ni \ell\text{-dens}(\mathcal{C}^{\tilde{\nabla}})$

cannot happen $\ell\text{-dens}(\mathcal{C}^{\tilde{\nabla}})$ is less than 1, thus we have a sequence of graphs where the number of edges increases, but slower than the number of vertices. By removing the isolated vertices, we get that $\overline{\ell}\text{-dens} \geq 1 \Rightarrow$ this case does not happen.

MA 052 / BG

Přednášt: Low-treedepth colouring
↳ definice, algoritmus, něco z důkazu, THM 7.8

Case (4): $\overline{\ell\text{-dens}}(C^{\tilde\nabla}) \geq 1+\varepsilon$,

the other one is 1. Thus, there is some r such that $C \nabla r$ has graphs $\|G\| \geq |G|^{1+\frac{\varepsilon}{2}}$.

(at some point, we get growing min. degree σ).

THM: $\exists n_0(t, \sigma)$ such that $\forall n > n_0$, $(|G| = n \wedge \|G\| \geq n^{1+\sigma}) \Rightarrow \left\lceil \frac{10}{\sigma} \right\rceil$-subdivision of K_t.

→ i.e. with superlinear edges, we find arbitrarily large subdivision of K_t.

Vlastně $\overline{\ell\text{-dens}}(C^{\tilde\nabla}) = 2$.

(5): same as above

$[-\infty, C \rightarrow (1) \dashrightarrow 2]$

THM (Trichotomy): $\ell\text{-dens}(C^{\tilde\nabla}) \in \{-\infty, 0, 1, 2\}$

(which) does not matter

nowhere dense

somewhere dense

this can be considered one can

Toto je důsledek té case analýzy.