

CHAPTER

7

Continuous-time filters

A common requirement in signal processing is to modify the frequency contents of a continuous-time (CT) signal in a predefined manner. In communication systems, for example, noise and interference from the neighboring channels corrupt the information-bearing signal transmitted via a communication channel, such as a telephone line. By exploiting the differences between the frequency characteristics of the transmitted signal and the channel noise, a linear time-invariant system (LTI) system can be designed to compensate for the distortion introduced during the transmission. Such an LTI system is referred to as a frequency-selective filter, which processes the received signal to eliminate the high-frequency components introduced by the channel interference and noise from the low-frequency components constituting the information-bearing signal. The range of frequencies eliminated from the CT signal applied at the input of the filter is referred to as the stop band of the filter, while the range of frequencies that is left relatively unaffected by the filter constitute the pass band of the filter.

Graphic equalizers used in stereo sound systems provide another application for the continuous-time (CT) filters. A graphic equalizer consists of a combination of CT filters, each tuned to a different band of frequencies. By selectively amplifying or attenuating the frequencies within the operational bands of the constituent filters, a graphic equalizer maintains sound consistency within dissimilar acoustic environments and spaces. The operation of a graphic equalizer is somewhat different from that of a frequency-selective filter used in our earlier example of the communication system since it amplifies or attenuates selected frequency components of the input signal. A frequency-selective filter, on the other hand, attempts to eliminate the frequency components completely within the stop band of the filter.

This chapter focuses on the design of CT filters. We are particularly interested in the frequency-selective filters that are categorized in four different categories (lowpass, highpass, bandpass, and bandstop) in Section 7.1. Practical approximations to the frequency characteristics of the ideal frequency-selective filters are presented in Section 7.2, where acceptable levels of distortion is tolerated

within the pass and stop bands of the ideal filters. Section 7.3 designs three realizable implementations of an ideal lowpass filter. These implementations are referred to as the Butterworth, Chebyshev, and elliptic filters. Section 7.4 transforms the frequency characteristics of the highpass, bandpass, and band-stop filters in terms of the characteristics of the lowpass filters. These transformations are exploited to design the highpass, bandpass, and bandstop filters. Finally, the chapter is concluded with a summary of important concepts in Section 7.5.

7.1 Filter classification

An ideal frequency-selective filter is a system that passes a prespecified range of frequency components without any attenuation but completely rejects the remaining frequency components. As discussed earlier, the range of input frequencies that is left unaffected by the filter is referred to as the pass band of the filter, while the range of input frequencies that are blocked from the output is referred to as the stop band of the filter. In terms of the magnitude spectrum, the absolute value of the transfer function $|H(\omega)|$ of the frequency filter, therefore, toggles between the values of A and zero as a function of frequency ω . The gain $|H(\omega)|$ is A , typically set to one, within the pass band, while $|H(\omega)|$ is zero within the stop band. Depending upon the range of frequencies within the pass and stop bands, an ideal frequency-selective filter is categorized in four different categories. These categories are defined in the following.

7.1.1 Lowpass filters

The transfer function $H_{lp}(\omega)$ of an ideal lowpass filter is defined as follows:

$$H_{lp}(\omega) = \begin{cases} A & |\omega| \leq \omega_c \\ 0 & |\omega| > \omega_c, \end{cases} \quad (7.1)$$

where ω_c is referred to as the cut-off frequency of the filter. The pass band of the lowpass filter is given by $|\omega| \leq \omega_c$, while the stop band of the lowpass filter is given by $\omega_c < |\omega| < \infty$. The frequency characteristics of an ideal lowpass filter are plotted in Fig. 7.1(a), where we observe that the magnitude $|H_{lp}(\omega)|$ toggles between the values of A within the pass band and zero within the stop band. The phase $\angle H_{lp}(\omega)$ of an ideal lowpass filter is zero for all frequencies.

7.1.2 Highpass filters

The transfer function $H_{hp}(\omega)$ of an ideal highpass filter is defined as follows:

$$H_{hp}(\omega) = \begin{cases} 0 & |\omega| \leq \omega_c \\ A & |\omega| > \omega_c, \end{cases} \quad (7.2)$$

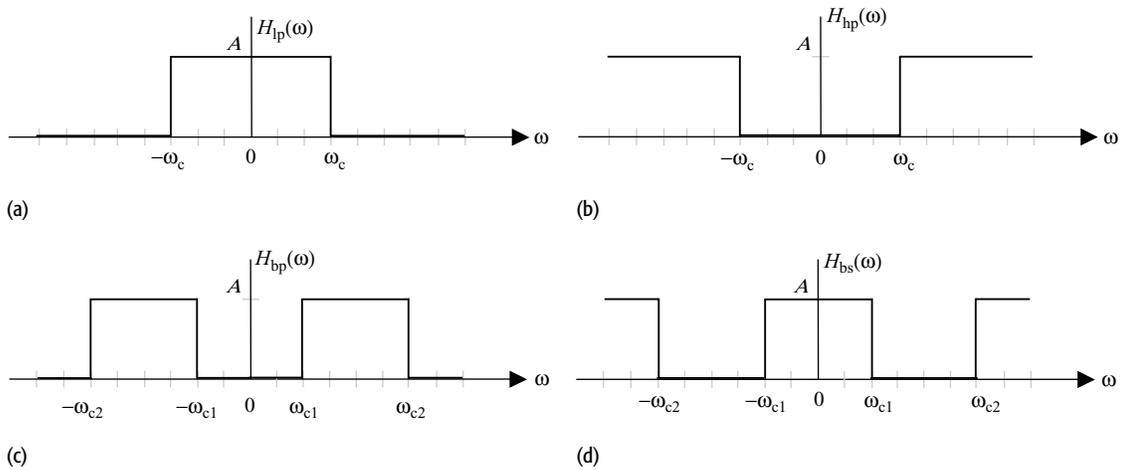


Fig. 7.1. Magnitude spectra of ideal frequency-selective filters. (a) Lowpass filter; (b) highpass filter; (c) bandpass filter; (d) bandstop filter.

where ω_c is the cut-off frequency of the filter. In other words, the transfer function of an ideal highpass filter $H_{hp}(\omega)$ is related to the transfer function of an ideal lowpass filter $H_{lp}(\omega)$ by the following relationship:

$$H_{hp}(\omega) = A - H_{lp}(\omega). \quad (7.3)$$

The pass band of the lowpass filter is given by $\omega_c < |\omega| < \infty$, while the stop band of the lowpass filter is given by $|\omega| \leq \omega_c$. The frequency characteristics of an ideal lowpass filter are plotted in Fig. 7.1(b). As was the case for the ideal lowpass filter, the phase $\angle H_{hp}(\omega)$ of an ideal highpass filter is zero for all frequencies.

7.1.3 Bandpass filters

The transfer function $H_{bp}(\omega)$ of an ideal bandpass filter is defined as follows:

$$H_{bp}(\omega) = \begin{cases} A & \omega_{c1} \leq |\omega| \leq \omega_{c2} \\ 0 & \omega_{c1} < |\omega| \text{ and } \omega_{c2} < |\omega| < \infty, \end{cases} \quad (7.4)$$

where ω_{c1} and ω_{c2} are collectively referred to as the cut-off frequencies of the ideal bandpass filter. The lower frequency ω_{c1} is referred to as the lower cut off, while the higher frequency ω_{c2} is referred to as the higher cut off. Unlike the highpass filter, the bandpass filter has a finite bandwidth as it only allows a range of frequencies ($\omega_{c1} \leq \omega \leq \omega_{c2}$) to be passed through the filter.

7.1.4 Bandstop filters

The transfer function $H_{bs}(\omega)$ of an ideal bandstop filter is defined as follows:

$$H_{bs}(\omega) = \begin{cases} 0 & \omega_{c1} \leq |\omega| \leq \omega_{c2} \\ A & \omega_{c1} < |\omega| \text{ and } \omega_{c2} < |\omega| < \infty, \end{cases} \quad (7.5)$$

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where ω_{c1} and ω_{c2} are, respectively, referred to as the lower cut-off and higher cut-off frequencies of the ideal bandstop filter. A bandstop filter can be implemented from a bandpass filter using the following relationship:

$$H_{bs}(\omega) = A - H_{bp}(\omega). \tag{7.6}$$

The ideal bandstop filter is the converse of the ideal bandpass filter as it eliminates a certain range of frequencies ($\omega_{c1} \leq \omega \leq \omega_{c2}$) from the input signal.

In the above discussion, we used the transfer function to categorize different types of frequency selective filters. Example 7.1 derives the impulse response for ideal lowpass and highpass filters.

Example 7.1

Determine the impulse response of an ideal lowpass filter and an ideal highpass filter. In each case, assume a gain of A within the pass band and a cut-off frequency of ω_c .

Solution

Taking the inverse CTFT of Eq. (7.1), we obtain

$$\begin{aligned} h_{lp}(t) &= \mathfrak{S}^{-1} \{H(\omega)\} = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} A \cdot e^{j\omega t} d\omega = \frac{Ae^{j\omega t}}{jt} \Big|_{-\omega_c}^{\omega_c} \\ &= \frac{A}{j2\pi t} [e^{j\omega_c t} - e^{-j\omega_c t}], \end{aligned}$$

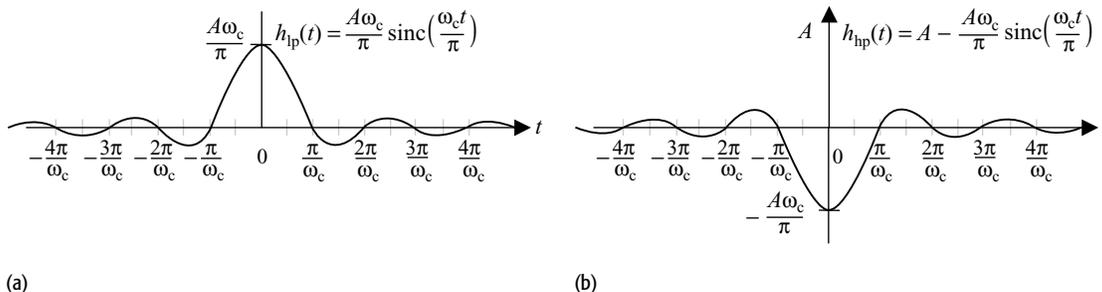
which reduces to

$$h_{lp}(t) = \frac{2jA \sin(\omega_c t)}{j2\pi t} = \frac{\omega_c A}{\pi} \text{sinc}\left(\frac{\omega_c t}{\pi}\right). \tag{7.7}$$

To derive the impulse response $h_{hp}(t)$ of the ideal highpass filter, we take the inverse CTFT of Eq. (7.3). The resulting relationship is given by

$$h_{hp}(t) = A\delta(t) - h_{lp}(t) = A\delta(t) - \frac{\omega_c A}{\pi} \text{sinc}\left(\frac{\omega_c t}{\pi}\right). \tag{7.8}$$

Fig. 7.2. Impulse responses $h(t)$ of: (a) ideal lowpass filter and (b) ideal highpass filter derived in Example 7.1.



The impulse responses of ideal lowpass and highpass filters are plotted in Fig. 7.2. In both cases, we note that the filters have an infinite length in the time domain. Also, both filters are non-causal since $h(t) \neq 0$ for $t < 0$.

7.2 Non-ideal filter characteristics

As is true for any ideal system, the ideal frequency-selective filters are not physically realizable for a variety of reasons. From the frequency characteristics of the ideal filters, we note that the gain A of the filters is constant within the pass band, while the gain within the stop band is strictly zero. A second issue with the transfer functions $H(\omega)$, specified for ideal filters in Eqs (7.1)–(7.5), is the sharp transition between the pass and stop bands such that there is a discontinuity in $H(\omega)$ at $\omega = \omega_c$. In practice, we cannot implement filters with constant gains within the pass and stop bands. Also, abrupt transitions cannot be designed. This is observed in Example 7.1, where the constant gains and the sharp transition in the ideal lowpass and highpass filters lead to non-causal impulse responses which are of infinite length. Clearly, such LTI systems cannot be implemented in the physical world.

To obtain a physically realizable filter, it is necessary to relax some of the requirements of the ideal filters. Figure 7.3 shows the frequency characteristics of physically realizable versions of various ideal filters. The upper and lower bounds for the gains are indicated by the shaded line, while examples of the frequency characteristics of physically realizable filters that satisfy the specified bounds are shown using bold lines. These filters are referred to as non-ideal or practical filters and are different from the ideal filters in the following two ways.

- (i) The gains of the practical filters within the pass and stop bands are not constant but vary within the following limits:

$$\text{pass bands} \quad 1 - \delta_p \leq |H(\omega)| \leq 1 + \delta_p; \quad (7.9)$$

$$\text{stop bands} \quad 0 \leq |H(\omega)| \leq \delta_s. \quad (7.10)$$

The oscillations within the pass and stop bands are referred to as ripples. In Fig. 7.3, the pass band ripples are constrained to a value of δ_p for lowpass, highpass, and bandpass filters. In the case of the bandstop filter, the pass band ripple is limited to δ_{p1} and δ_{p2} , corresponding to the two pass bands. Similarly, the stop band ripples in Fig. 7.3 are constrained to δ_s for lowpass, highpass, and bandstop filters. In the case of the bandstop filter, the stop band ripple is limited to δ_{s1} and δ_{s2} for the two stop bands of the bandstop filter.

- (ii) Transition bands of non-zero bandwidth are included in between the pass and stop bands of the practical filters. Consequently, the discontinuity at the cut-off frequency ω_c of the ideal filters is eliminated.

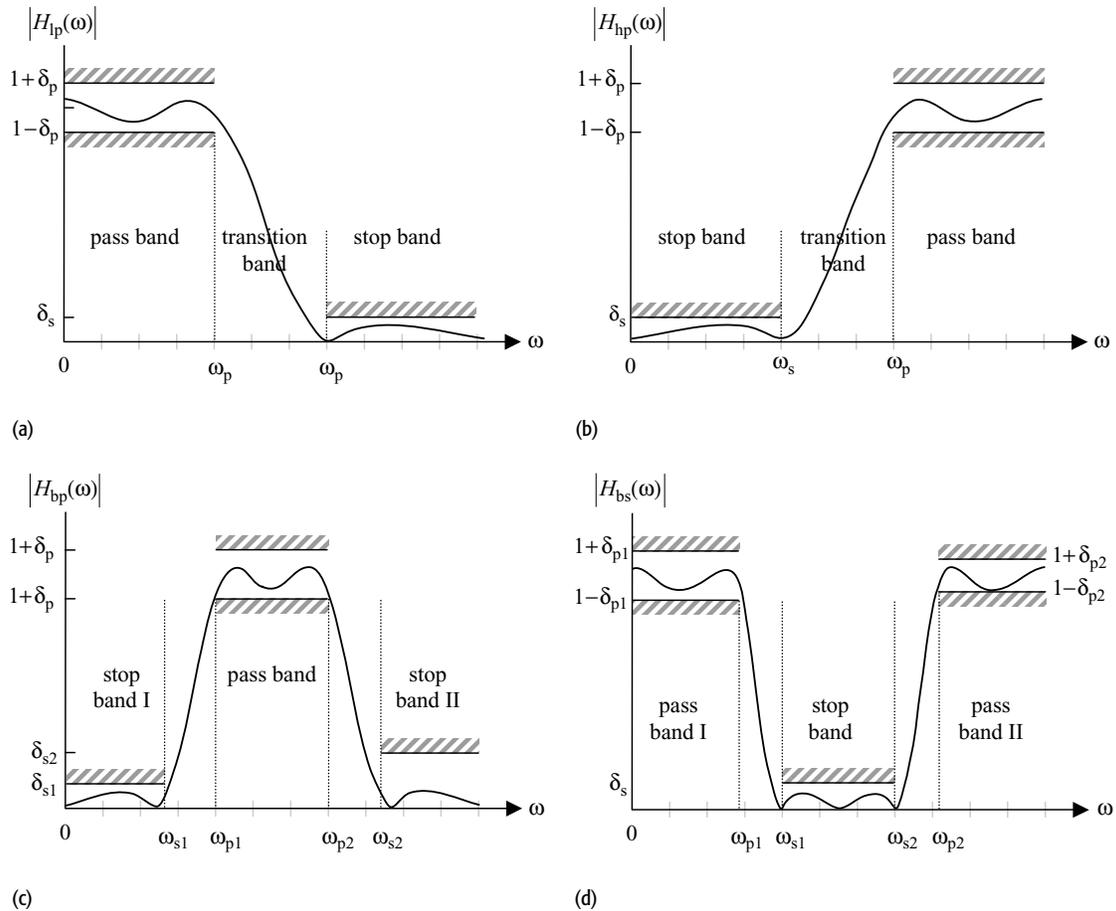


Fig. 7.3. Frequency characteristics of practical filters.
 (a) Practical lowpass filter;
 (b) practical highpass filter;
 (c) practical bandpass filter;
 (d) practical bandstop filter.

Example 7.2 considers a practical lowpass filter and derives the values for the pass band and the stop band, and the associated gains of the filter.

Example 7.2

Consider a practical lowpass filter with the following transfer function:

$$H(s) = \frac{5.018 \times 10^3 s^4 + 2.682 \times 10^{14} s^3 - 1.026 \times 10^4 s + 3.196 \times 10^{24}}{s^5 + 9.863 \times 10^4 s^4 + 2.107 \times 10^{10} s^3 + 1.376 \times 10^{15} s^2 + 1.026 \times 10^{20} s + 3.196 \times 10^{24}}$$

Assuming that the ripple δ_p within the pass band is limited to 1 dB and the ripple δ_s within the stop band is limited to 40 dB, determine the pass band, transition band and stop band of the lowpass filter.

Solution

Recall that the CTFT transfer function $H(\omega)$ of the lowpass filter can be obtained by substituting $s = j\omega$ in the Laplace transfer function. The resulting magnitude spectrum $|H(\omega)|$ of the lowpass filter is plotted in Fig. 7.4, where Fig. 7.4(a)

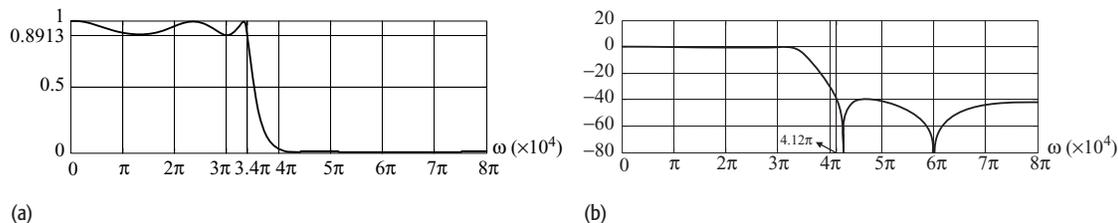


Fig. 7.4. Magnitude spectrum of the practical lowpass filter in Example 7.2 using (a) a linear scale and (b) a decibel scale along the y -axis.

uses a linear scale for the magnitude. Figure 7.4(b) uses a decibel scale to plot the magnitude spectrum.

Expressed on a linear scale, the pass-band ripple δ_p is given by $10^{-1/20}$ or 0.8913. From Fig. 7.4(a), we observe that the pass-band frequency ω_p corresponding to $|H(\omega)| = 0.8913$ is given by $3.4\pi \times 10^4$ radians/s. Therefore, the pass band is specified by $|\omega| \leq 3.4\pi \times 10^4$ radians/s.

To determine the stop band, we use Fig. 7.4(b), which uses a decibel scale $20 \times \log_{10}|H(\omega)|$ to plot the magnitude spectrum. Figure 7.4(b) shows that the smallest frequency for which the magnitude spectrum equals a gain of 40 dB is given by $4.12\pi \times 10^4$ radians/s. The stop band is therefore specified by $|\omega| > 4.12\pi \times 10^4$ radians/s.

Based on the aforementioned results, it is straightforward to derive the transition band as follows:

$$3.4\pi \times 10^4 < |\omega| < 4.12\pi \times 10^4 \text{ radians/s.}$$

7.2.1 Cut-off frequency

An important parameter in the design of CT filters is the cut-off frequency ω_c of the filter, which is defined as the frequency at which the gain of the filter drops to 0.7071 times its maximum value. Assuming a gain of unity within the pass band, the gain at the cut-off frequency ω_c is given by 0.7071 or -3 dB on a logarithmic scale. Since the cut-off frequency lies typically within the transitional band of the filter, for a lowpass filter

$$\omega_p \leq \omega_c \leq \omega_s. \tag{7.11}$$

Since the equality $\omega_p = \omega_c = \omega_s$ implies a transitional band of zero bandwidth, this equality is only valid for ideal filters.

As a side note to our discussion, we observe that in this chapter we only consider positive values of frequencies ω in plotting the magnitude spectrum. The majority of our designs are based on real-valued impulse responses, which lead to frequency spectra that satisfy the Hermitian symmetry. Exploiting the even symmetry for the magnitude spectrum, it is therefore sufficient to specify the magnitude spectrum only for positive frequencies in such cases. The

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pass-band, stop-band, and cut-off frequencies are also specified by positive values, though their counter-negative values exist for all three parameters.

Example 7.3

Determine the cut-off frequency for the lowpass filter specified in Example 7.2.

Solution

Based on the magnitude spectrum, we note that the maximum gain of the filter is given by 1 or 0 dB. At the cut off frequency ω_c ,

$$|H(\omega_c)| = 0.7071 \times 1 = 0.7071,$$

which implies that

$$\left| \frac{5.018 \times 10^3 (j\omega_c)^4 + 2.682 \times 10^{14} (j\omega_c)^2 - 1.026 \times 10^4 (j\omega_c) + 3.196 \times 10^{24}}{(j\omega_c)^5 + 9.863 \times 10^4 (j\omega_c)^4 + 2.107 \times 10^{10} (j\omega_c)^3 + 1.376 \times 10^{15} (j\omega_c)^2 + 1.026 \times 10^{20} (j\omega_c) + 3.196 \times 10^{24}} \right| = 0.7071.$$

The above equality can be solved for ω_c using numerical techniques in MATLAB. The value of the cut-off frequency is given by $\omega_c = 3.462\pi \times 10^4$ radians/s. Note that the cut-off frequency lies within the transitional band in between the pass and stop bands of the lowpass filter as derived in Example 7.2.

7.3 Design of CT lowpass filters

To begin our discussion of the design of CT filters, we consider a prototype or normalized lowpass filter, defined as a lowpass filter, with a cut-off frequency of $\omega_c = 1$ radians/s. The remaining specifications for the pass and stop bands of the normalized lowpass filter are assumed to be given by

$$\text{pass band } (0 \leq |\omega| \leq \omega_p \text{ radians/s}) \quad 1 - \delta_p \leq |H(\omega)| \leq 1 + \delta_p; \quad (7.12)$$

$$\text{stop band } (|\omega| > \omega_s \text{ radians/s}) \quad |H(\omega)| \leq \delta_s, \quad (7.13)$$

with $\omega_p \leq \omega_c \leq \omega_s$. Using the transfer function of the normalized lowpass filter, it is straightforward to implement any of the more complicated CT filters. Section 7.4 considers the frequency transformations used to convert a lowpass filter into another category of frequency-selective filters.

There are several specialized implementations such as Butterworth, Type I Chebyshev, Type II Chebyshev, and elliptic filters, which may be used to design a normalized lowpass filter. Figure 7.5 shows representative characteristics of these implementations, where we observe that the Butterworth filter (Fig. 7.5(a)) has a monotonic transfer function such that the gain decreases monotonically from its maximum value of unity at $\omega = 0$ along the positive frequency axis. The magnitude spectrum of the Butterworth filter has negligible ripples within the pass and stop bands, but has a relatively lower fall off leading to a wide

transitional band. By allowing some ripples in either the pass or stop band, the Type I and Type II Chebyshev filters incorporate a sharper fall off. The Type I Chebyshev filter constitutes ripples within the pass band, while the Type II Chebyshev filter allows for the stop-band ripples. Compared with the Butterworth filter, both Type I and Type II Chebyshev filters have narrower transitional bands. The elliptic filters allow for the sharpest fall off by incorporating ripples in both the pass and stop bands of the filter. The elliptic filters have the narrowest transitional band. To compare the transitional bands, Fig. 7.5 plots the magnitude spectra resulting from the Butterworth, Type I Chebyshev, Type II Chebyshev, and elliptic filters with the same order N .

Figure 7.5 confirms our earlier observations that the Butterworth filter (Fig. 7.5(a)) has the widest transitional band. Both the Type I and Type II Chebyshev filters (Figs 7.5(b) and (c)) have roughly equal transitional bands, which are narrower than the transitional band of the Butterworth filter. The elliptic filter (Fig. 7.5(d)) has the narrowest transitional band but includes ripples in both the pass and stop bands.

We now consider the design techniques for the four specialized implementations with a brief explanation of the MATLAB library functions useful for computing the transfer functions of the implementations.

7.3.1 Butterworth filters

The frequency characteristics of an N th-order lowpass Butterworth filter are given by

$$|H(\omega)| = \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_c}\right)^{2N}}}, \quad (7.14)$$

where ω_c is the cut-off frequency of the filter. Substituting $\omega_c = 1$ for the normalized implementation, the transfer function of the normalized lowpass Butterworth filter of order N is given by

$$|H(\omega)| = \frac{1}{\sqrt{1 + \omega^{2N}}}. \quad (7.15)$$

To derive the Laplace transfer function $H(s)$ of the normalized Butterworth filter, we use the following relationship:

$$|H(\omega)|^2 = H(s)H(-s)|_{s=j\omega}. \quad (7.16)$$

Substituting $\omega = s/j$, Eq. (7.16) reduces to

$$H(s)H(-s) = |H(s/j)|^2. \quad (7.17)$$

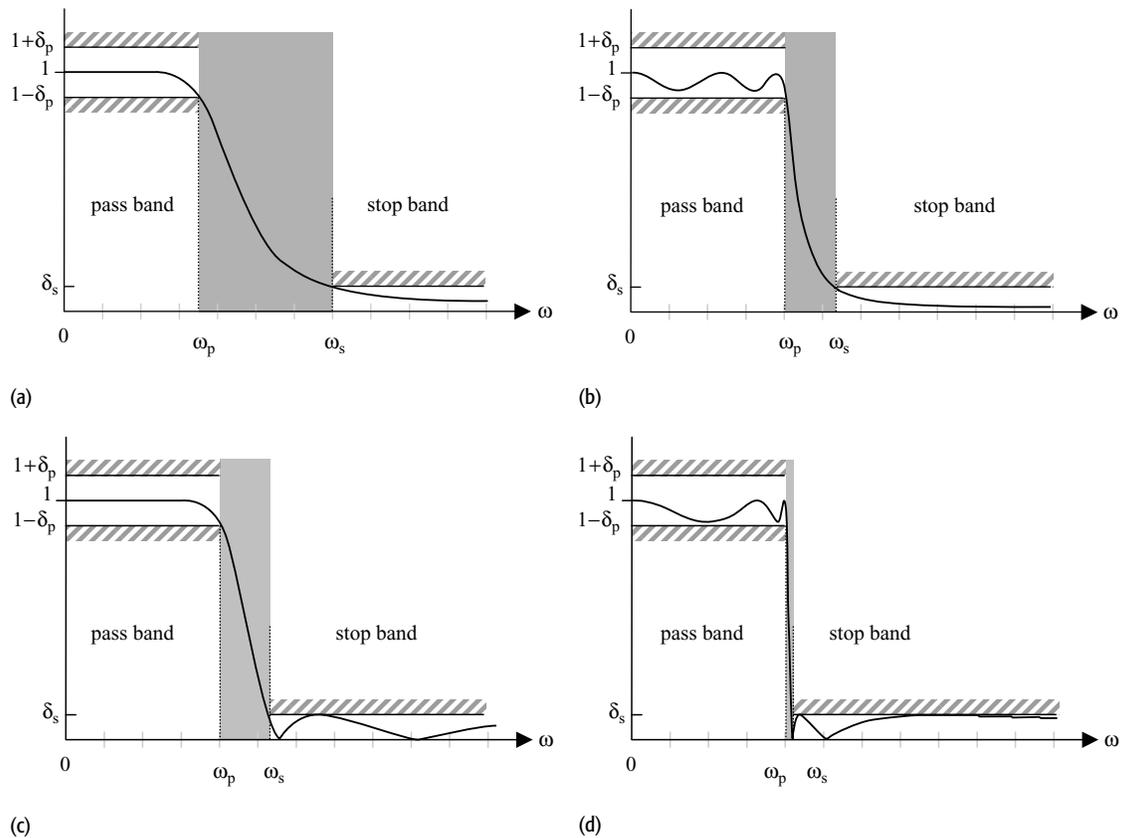


Fig. 7.5. Frequency characteristics of standard implementations of lowpass filters of order N . (a) Butterworth filter; (b) Type-I Chebyshev filter; (c) Type-II Chebyshev filter; (d) elliptic filter.

Further substituting $H(s/j)$ from Eq. (7.15) leads to the following expression:

$$H(s)H(-s) = \frac{1}{1 + \left(\frac{s}{j}\right)^{2N}}, \quad (7.18)$$

where the denominator represents the characteristic function for $H(s)H(-s)$. The poles of $H(s)H(-s)$ occur at

$$\left(\frac{s}{j}\right)^{2N} = -1 = e^{j(2n-1)\pi} \quad (7.19)$$

or

$$s = j \exp \left[j \frac{(2n-1)\pi}{2N} \right] = \exp \left[j \frac{\pi}{2} + j \frac{(2n-1)\pi}{2N} \right] \quad (7.20)$$

for $0 \leq n \leq 2N-1$. It is clear that the $2N$ poles for $H(s)H(-s)$, specified in Eq. (7.20), are evenly distributed along the unit circle in the complex s -plane. Of these, N poles would lie in the left half of the s -plane, while the remaining N poles would be in the right half of the s -plane. To ensure a causal and

Table 7.1. Location of the $2N$ poles for $H(s)H(-s)$ in Example 7.4 for $N = 7$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
p_n	$e^{j3\pi/7}$	$e^{j4\pi/7}$	$e^{j5\pi/7}$	$e^{j6\pi/7}$	$e^{j\pi}$	$e^{-j6\pi/7}$	$e^{-j5\pi/7}$	$e^{-j4\pi/7}$	$e^{-j3\pi/7}$	$e^{-j2\pi/7}$	$e^{-j\pi/7}$	1	$e^{j\pi/7}$	$e^{j2\pi/7}$

stable implementation, the transfer function $H(s)$ of the normalized lowpass Butterworth filter is determined from the N poles lying in the left half of the s -plane and is given by

$$H(s) = \frac{1}{\prod_{n=1}^N (s - p_n)}, \tag{7.21}$$

where p_n , for $1 \leq n \leq N$, denotes the location of the poles in the left-half s -plane.

Example 7.4

Determine the Laplace transfer function $H(s)$ for the normalized Butterworth filter with cut-off frequency $\omega_c = 1$ and order $N = 7$.

Solution

Using Eq. (7.20), the poles of $H(s)H(-s)$ are given by

$$s = \exp \left[j \frac{\pi}{2} + j \frac{(2n - 1)\pi}{14} \right]$$

for $0 \leq n \leq 13$. Substituting different values of n , the locations of the poles are specified by Table 7.1. Figure 7.6 plots the locations of the poles for $H(s)H(-s)$ in the complex s -plane. Allocating the poles located in the left-half s -plane ($1 \leq n \leq 7$), the Laplace transfer function $H(s)$ of the Butterworth filter is given by

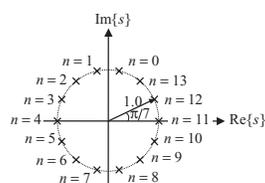


Fig. 7.6. Location of the poles for $H(s)H(-s)$ in the complex s -plane for $N = 7$. The poles lying in the left-half s -plane are allocated to the Butterworth filter.

$$H(s) = \frac{1}{(s - e^{j4\pi/7})(s - e^{j5\pi/7})(s - e^{j6\pi/7})(s - e^{j\pi})(s - e^{-j6\pi/7})(s - e^{-j5\pi/7})(s - e^{-j4\pi/7})},$$

which simplifies to

$$H(s) = \frac{1}{(s + 1) [(s - e^{j4\pi/7})(s - e^{-j4\pi/7})][(s - e^{j5\pi/7})(s - e^{-j5\pi/7})][(s - e^{j6\pi/7})(s - e^{-j6\pi/7})]}$$

or

$$H(s) = \frac{1}{(s + 1) (s^2 + 0.4450s + 1)(s^2 + 1.2470s + 1)(s^2 + 1.8019s + 1)}.$$

In Example 7.4, we observed that the locations of poles for the normalized Butterworth filter are complex. Since the poles occur in complex-conjugate pairs, the coefficients of the Laplace transfer function for the normalized

Table 7.2. Denominator $D(s)$ for transfer function $H(s)$ of the Butterworth filter

N	$D(s)$
1	$(s + 1)$
2	$(s^2 + 1.414s + 1)$
3	$(s + 1)(s^2 + s + 1)$
4	$(s^2 + 0.7654s + 1)(s^2 + 1.8478s + 1)$
5	$(s + 1)(s^2 + 0.6810s + 1)(s^2 + 1.6810s + 1)$
6	$(s^2 + 0.5176s + 1)(s^2 + 1.4142s + 1)(s^2 + 1.9319s + 1)$
7	$(s + 1)(s^2 + 0.4450s + 1)(s^2 + 1.2470s + 1)(s^2 + 1.8019s + 1)$
8	$(s^2 + 0.3902s + 1)(s^2 + 1.1111s + 1)(s^2 + 1.6629s + 1)(s^2 + 1.9616s + 1)$
9	$(s + 1)(s^2 + 0.3473s + 1)(s^2 + s + 1)(s^2 + 1.5321s + 1)(s^2 + 1.8794s + 1)$
10	$(s^2 + 0.3129s + 1)(s^2 + 0.9080s + 1)(s^2 + 1.4142s + 1)(s^2 + 1.7820s + 1)(s^2 + 1.9754s + 1)$

Butterworth filter are all real-valued. In general, Eq. (7.21) can be simplified as follows:

$$H(s) = \frac{1}{D(s)} = \frac{1}{s^N + a_{N-1}s^{N-1} + \dots + a_1s + 1} \quad (7.22)$$

and represents the transfer function of the normalized Butterworth filter of order N .

Repeating Example 7.4 for different orders ($1 \leq N \leq 10$), the transfer functions $H(s)$ of the resulting normalized Butterworth filters can be similarly computed. Since the numerator of the transfer function is always unity, Table 7.2 lists the polynomials for the denominator $D(s)$ for $1 \leq N \leq 10$.

7.3.1.1 Design steps for the lowpass Butterworth filter

In this section, we will design a Butterworth lowpass filter based on the specifications illustrated in Fig. 7.3(a). Mathematically, the specifications can be expressed as follows:

$$\text{pass band } (0 \leq |\omega| \leq \omega_p \text{ radians/s}) \quad 1 - \delta_p \leq |H(\omega)| \leq 1 + \delta_p; \quad (7.23)$$

$$\text{stop band } (|\omega| > \omega_s \text{ radians/s}) \quad |H(\omega)| \leq \delta_s. \quad (7.24)$$

At times, Eq. (7.23) is also expressed in terms of the pass-band ripple as $20\log_{10}\delta_p$ dB. Similarly, Eq. (7.24) is expressed in terms of the stop-band ripple as $20\log_{10}\delta_s$ dB. The design of the Butterworth filter consists of the following steps, which we refer to as Algorithm 7.3.1.1.

Step 1 Determine the order N of the Butterworth filter. To determine the order N of the filter, we calculate the gain of the filter at the corner frequencies $\omega = \omega_p$

and $\omega = \omega_s$. Using Eq. (7.15), the two gains are given by

$$\text{pass-band corner frequency } (\omega = \omega_p) \quad |H(\omega_p)|^2 = \frac{1}{1 + (\omega_p/\omega_c)^{2N}} = (1 - \delta_p)^2; \quad (7.25)$$

$$\text{stop-band corner frequency } (\omega = \omega_s) \quad |H(\omega_s)|^2 = \frac{1}{1 + (\omega_s/\omega_c)^{2N}} = (\delta_s)^2. \quad (7.26)$$

Equations (7.25) and (7.26) can alternatively be expressed as follows:

$$(\omega_p/\omega_c)^{2N} = \frac{1}{(1 - \delta_p)^2} - 1 \quad (7.27)$$

and

$$(\omega_s/\omega_c)^{2N} = \frac{1}{(\delta_s)^2} - 1. \quad (7.28)$$

Dividing Eq. (7.27) by Eq. (7.28) and simplifying in terms of N , we obtain the following expression:

$$N = \frac{1}{2} \times \frac{\ln(G_p/G_s)}{\ln(\omega_p/\omega_s)}, \quad (7.29)$$

where the gain terms are given by

$$G_p = \frac{1}{(1 - \delta_p)^2} - 1 \quad \text{and} \quad G_s = \frac{1}{(\delta_s)^2} - 1. \quad (7.30)$$

Step 2 Using Table 7.2 or otherwise determine the transfer function for the normalized Butterworth filter of order N . The transfer function for the normalized Butterworth filter is denoted by $H(S)$ with the Laplace variable S capitalized to indicate the normalized domain.

Step 3 Determine the cut-off frequency ω_c of the Butterworth filter using either of the following two relationships:

$$\text{pass-band constraint} \quad \omega_c = \frac{\omega_p}{(G_p)^{1/2N}}; \quad (7.31)$$

$$\text{stop-band constraint} \quad \omega_c = \frac{\omega_s}{(G_s)^{1/2N}}. \quad (7.32)$$

If Eq. (7.31) is used to compute the cut-off frequency, then the Butterworth filter will satisfy the pass-band constraint exactly. Similarly, the stop-band constraint will be satisfied exactly if Eq. (7.32) is used to determine the cut-off frequency.

Step 4 Determine the transfer function $H(s)$ of the required lowpass filter from the transfer function for the normalized Butterworth filter $H(S)$, obtained

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in Step 2, and the cut-off frequency ω_c , using the following transformation:

$$H(s) = H(S)|_{S=s/\omega_c}.$$

Note that the transformation $S = s/\omega_c$ represents scaling in the Laplace domain. It is therefore clear that the normalized cut-off frequency of 1 radian/s used in the normalized Butterworth filter is transformed to a value of ω_c as required in Step 3.

Step 5 Sketch the magnitude spectrum from the transfer function $H(s)$ determined in Step 4. Confirm that the transfer function satisfies the initial design specifications.

Examples 7.5 and 7.6 illustrate the application of the design algorithm.

Example 7.5

Design a Butterworth lowpass filter with the following specifications:

pass band ($0 \leq |\omega| \leq 5$ radians/s) $0.8 \leq |H(\omega)| \leq 1$;
 stop band ($|\omega| > 20$ radians/s) $|H(\omega)| \leq 0.20$.

Solution

Using Step 1 of Algorithm 7.3.1.1, the gain terms G_p and G_s are given by

$$G_p = \frac{1}{(1 - \delta_p)^2} - 1 = \frac{1}{0.8^2} - 1 = 0.5625$$

and

$$G_s = \frac{1}{(\delta_s)^2} - 1 = \frac{1}{0.2^2} - 1 = 24.$$

Using Eq. (7.29), the order of the Butterworth filter is given by

$$N = \frac{1}{2} \times \frac{\ln(G_p/G_s)}{\ln(\omega_p/\omega_s)} = \frac{1}{2} \times \frac{\ln(0.5625/24)}{\ln(5/20)} = 1.3538.$$

We round off the order of the filter to the higher integer value as $N = 2$.

Using Step 2 of Algorithm 7.3.1.1, the transfer function $H(S)$ of the normalized Butterworth filter with a cut-off frequency of 1 radian/s is given by

$$H(S) = \frac{1}{S^2 + 1.414S + 1}.$$

Using the pass-band constraint, Eq. (7.31), in Step 3 of Algorithm 7.3.1.1, the cut-off frequency of the required Butterworth filter is given by

$$\omega_c = \frac{\omega_p}{(G_p)^{1/2N}} = \frac{5}{(0.5625)^{1/4}} = 5.7735 \text{ radians/s.}$$

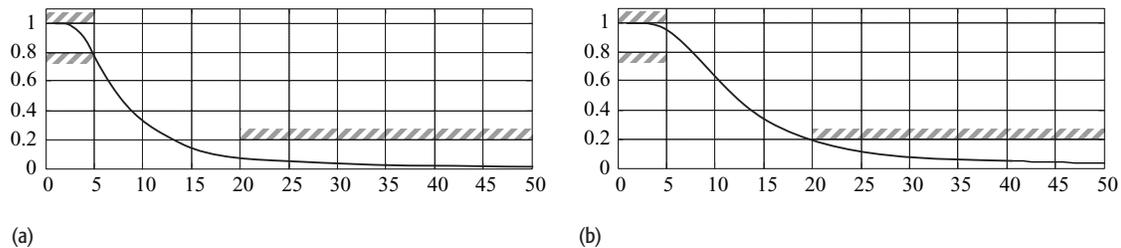


Fig. 7.7. Magnitude spectra of the Butterworth lowpass filters, designed in Example 7.5, as a function of ω . Part (a) satisfies the constraint at the pass-band corner frequency, while part (b) satisfies the magnitude constraint at the stop-band corner frequency.

Using Step 4 of Algorithm 7.3.1.1, the transfer function $H(s)$ of the required Butterworth filter is obtained by the following transformation:

$$H(s) = H(S)|_{S=s/\omega_c} = \frac{1}{s^2 + 1.414S + 1} \Big|_{S=s/5.7735}$$

which simplifies to

$$H(s) = \frac{1}{(s/5.7735)^2 + 1.414s/5.7735 + 1} = \frac{33.3333}{s^2 + 8.1637s + 33.3333}$$

Step 5 plots the magnitude spectrum of the Butterworth filter. The CTFT transfer function of the Butterworth filter is given by

$$H(\omega) = H(s)|_{s=j\omega} = \frac{33.3333}{(j\omega)^2 + 8.1637(j\omega) + 33.3333}$$

The magnitude spectrum $|H(\omega)|$ is plotted in Fig. 7.7(a) with the specifications shown by the shaded lines. We observe that the design specifications are indeed satisfied by the magnitude spectrum.

Alternative implementation An alternative implementation of the aforementioned Butterworth filter can be obtained by using the stop-band constraint, Eq. (7.32), in Step 3 of Algorithm 7.3.1.1. The cut-off frequency of the alternative implementation of the Butterworth filter is given by

$$\omega_c = \frac{\omega_s}{(G_s)^{1/2N}} = \frac{20}{(24)^{1/4}} = 9.0360 \text{ radians/s.}$$

Using Step 4 of Algorithm 7.3.1.1, the transfer function $H(s)$ of the alternative implementation is obtained by the following transformation:

$$H(s) = H(S)|_{S=s/\omega_c} = \frac{1}{s^2 + 1.414S + 1} \Big|_{S=s/9.0360}$$

which simplifies to

$$H(s) = \frac{1}{(s/9.0360)^2 + 1.414s/9.0360 + 1} = \frac{81.6497}{s^2 + 12.7769s + 81.6497}$$

Step 5 plots the magnitude spectrum of the alternative implementation of the Butterworth filter in Fig. 7.7(b), which satisfies the initial design specifications.

Example 7.6

Design a lowpass Butterworth filter with the following specifications:

pass band ($0 \leq |\omega| \leq 50$ radians/s) $-1 \leq 20 \log_{10} |H(\omega)| \leq 0$;

stop band ($|\omega| > 100$ radians/s) $20 \log_{10} |H(\omega)| \leq -15$.

Solution

Expressed on a linear scale, the pass-band gain is given by $(1 - \delta_p) = 10^{-1/20} = 0.8913$. Similarly, the stop-band gain is given by $\delta_s = 10^{-15/20} = 0.1778$.

Using Step 1 of Algorithm 7.3.1.1, the gain terms G_p and G_s are given by

$$G_p = \frac{1}{(1 - \delta_p)^2} - 1 = \frac{1}{0.8913^2} - 1 = 0.2588$$

and

$$G_s = \frac{1}{(\delta_s)^2} - 1 = \frac{1}{0.1778^2} - 1 = 30.6327.$$

The order N of the Butterworth filter is obtained using Eq. (7.29) as follows:

$$N = \frac{1}{2} \times \frac{\ln(G_p/G_s)}{\ln(\omega_p/\omega_s)} = \frac{1}{2} \times \frac{\ln(0.2588/30.6327)}{\ln(50/100)} = 3.4435.$$

We round off the order of the filter to the higher integer value as $N = 4$.

Using Step 2 of Algorithm 7.3.1.1, the transfer function $H(S)$ of the normalized Butterworth filter with a cut-off frequency of 1 radian/s is given by

$$H(S) = \frac{1}{(S^2 + 0.7654S + 1)(S^2 + 1.8478S + 1)}.$$

Using the pass-band constraint, Eq. (7.31), in Step 3 of Algorithm 7.3.1.1, the cut-off frequency of the required Butterworth filter is given by

$$\omega_c = \frac{\omega_p}{(G_p)^{1/2N}} = \frac{50}{(0.2588)^{1/8}} = 59.2038 \text{ radians/s.}$$

Using Step 4 of Algorithm 7.3.1.1, the transfer function $H(s)$ of the required Butterworth filter is obtained by the following transformation:

$$H(s) = H(S)|_{S=s/\omega_c} = \frac{1}{(S^2 + 0.7654S + 1)(S^2 + 1.8478S + 1)} \Big|_{S=s/59.2038},$$

which simplifies to

$$H(s) = \frac{(3.5051 \times 10^3)^2}{(s^2 + 45.3146s + 3.5051 \times 10^3)(s^2 + 109.396s + 3.5051 \times 10^3)}$$

or

$$H(s) = \frac{1.2286 \times 10^7}{s^4 + 154.7106 s^3 + 1.1976 \times 10^4 s^2 + 5.4228 \times 10^5 s + 1.2286 \times 10^7}.$$

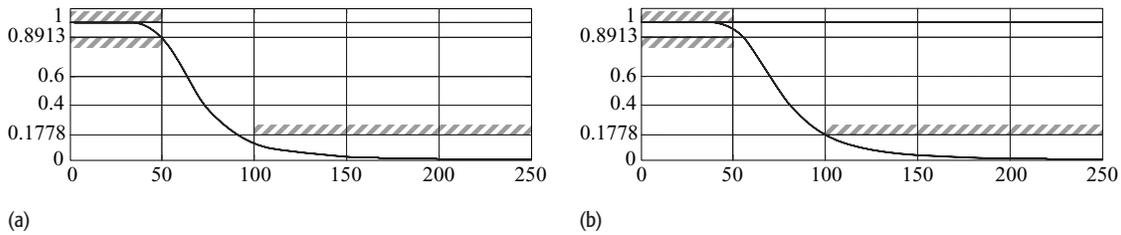


Fig. 7.8. Magnitude spectra of the Butterworth lowpass filters, designed in Example 7.6, as a function of ω . Part (a) satisfies the constraint at the pass-band corner frequency, while part (b) satisfies the magnitude constraint at the stop-band corner frequency.

Step 5 plots the magnitude spectrum of the Butterworth filter. The CTFT transfer function of the Butterworth filter is given by

$$H(\omega) = H(s)|_{s=j\omega} = \frac{1.2286 \times 10^7}{(j\omega)^4 + 154.7106(j\omega)^3 + 1.1976 \times 10^4(j\omega)^2 + 5.4228 \times 10^5(j\omega) + 1.2286 \times 10^7}$$

The magnitude spectrum $|H(\omega)|$ is plotted in Fig. 7.8(a), where the labels on the y-axis are chosen to correspond to the specified gains for the filter. We observe that the design specifications are satisfied by the magnitude spectrum.

Alternative implementation An alternative implementation of the aforementioned Butterworth filter can be obtained by using the stop-band constraint, Eq. (7.32), in Step 3 of Algorithm 7.3.1.1. The cut-off frequency of the alternative implementation of the Butterworth filter is given by

$$\omega_c = \frac{\omega_s}{(G_s)^{1/2N}} = \frac{100}{(30.6327)^{1/4}} = 65.1969 \text{ radians/s.}$$

Using Step 4 of Algorithm 7.3.1.1, the transfer function $H(s)$ of the alternative implementation is obtained by the following transformation:

$$H(s) = H(S)|_{S=s/\omega_c} = \frac{1}{(S^2 + 0.7654S + 1)(S^2 + 1.8478S + 1)} \Big|_{S=s/65.1969},$$

which simplifies to

$$H(s) = \frac{(4.2506 \times 10^3)^2}{(s^2 + 49.9017s + 4.2506 \times 10^3)(s^2 + 120.4708s + 4.2506 \times 10^3)}$$

or

$$H(s) = \frac{1.8068 \times 10^7}{s^4 + 170.3725s^3 + 1.4513 \times 10^4s^2 + 7.2419 \times 10^5s + 1.8068 \times 10^7}$$

Step 5 plots the magnitude spectrum of the alternative implementation of the Butterworth filter in Fig. 7.8(b), which satisfies the initial design specifications.

7.3.1.2 Butterworth filter design using MATLAB

MATLAB incorporates a number of functions to implement the design algorithm for the Butterworth filter specified in Section 7.3.1.1. The order N and the

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cut-off ω_c frequency for the filter in Step 1 of Algorithm 7.3.1.1 can be determined using the library function `buttord`, which has the following calling syntax:

```
>> [N,wc] = buttord(wp,ws,Rp,Rs,'s');
```

where ω_p is the corner frequency of the pass band, ω_s is the corner frequency of the stop band, R_p is the permissible ripple in the pass band in decibels, and R_s is the permissible attenuation in the stop band in decibels. The last argument 's' specifies that a CT filter in the Laplace domain is to be designed. In determining the cut-off frequency, MATLAB uses the stop-band constraint, Eq. (7.32).

Having determined the order and the cut-off frequency, the coefficients of the numerator and denominator polynomials of the Butterworth filter can be determined using the library function `butter` with the following calling syntax:

```
>> [num,den] = butter(N,wc,'s');
```

where `num` is a vector containing the coefficients of the numerator and `den` is a vector containing the coefficients of the denominator in decreasing powers of s .

Finally, the transfer function $H(s)$ can be determined using the library function `tf` as follows:

```
>> H = tf(num,den).
```

For Example 7.5, the MATLAB commands for designing the Butterworth filter are given by

```
>> wp=5; ws=20; Rp=1.9382; % specify design parameters
    Rs=13.9794;
                                % Rp = -20*log10(0.8)
                                %       = 1.9382dB
                                % Rs = -20*log10(0.2)
                                %       = 13.9794dB
>> [N,wc]=buttord              % determine order and cut-off
    (wp,ws,Rp,Rs,'s');          %       freq
>> [num,den]=butter            % determine num and denom
    (N,wc,'s');                 %       coeff.
>> Ht = tf(num,den);           % determine transfer function
>> [H,w] = freqs(num,den);     % determine magnitude
                                %       spectrum
>> plot(w,abs(H));             % plot magnitude spectrum
```

Stepwise implementation of the above code returns the following values for different variables:

```

Instruction II:          N = 2; wc = 9.0360;
Instruction III: num = [0 0 81.6497]; den = [1.0000
                                           12.7789 81.6497];
Instruction IV:         Ht = 1/(s^2 + 12.78s + 81.65);
    
```

The magnitude spectrum is the same as that given in Fig. 7.7(b).

7.3.2 Type I Chebyshev filters

Butterworth filters have a relatively low roll off in the transitional band, which leads to a large transitional bandwidth. Type I Chebyshev filters reduce the bandwidth of the transitional band by using an approximating function, referred to as the Type I Chebyshev polynomial, with a magnitude response that has ripples within the pass band. We start with the definition of the Chebyshev polynomial.

7.3.2.1 Type I Chebyshev polynomial

The N th-order Type I Chebyshev polynomial is defined as

$$T_N(\omega) = \begin{cases} \cos(N \cos^{-1}(\omega)) & |\omega| \leq 1 \\ \cosh(N \cosh^{-1}(\omega)) & |\omega| > 1, \end{cases} \quad (7.33)$$

where $\cosh(x)$ denotes the hyperbolic cosine function, which is given by

$$\cosh(x) = \cos(jx) = \frac{e^x + e^{-x}}{2}. \quad (7.34)$$

Starting from the initial values of $T_0(\omega) = 1$ and $T_1(\omega) = \omega$, the higher orders of the Type I Chebyshev polynomial can be recursively generated using the following expression:

$$T_n(\omega) = 2\omega T_{n-1}(\omega) - T_{n-2}(\omega). \quad (7.35)$$

Table 7.3 lists the Chebyshev polynomial for different values of n within the range $0 \leq n \leq 10$.

Using Eq. (7.33), the roots of the Type I Chebyshev polynomial $T_N(\omega)$ can be derived as follows:

$$\omega_n = \cos \left[\frac{(2n+1)\pi}{2N} \right], \quad (7.36)$$

for $0 \leq n \leq N-1$.

Table 7.3. Chebyshev polynomial $T_N(\omega)$ for different values of N

N	$T_N(\omega)$
0	1
1	ω
2	$2\omega^2 - 1$
3	$4\omega^3 - 3\omega$
4	$8\omega^4 - 8\omega^2 + 1$
5	$16\omega^5 - 20\omega^3 + 5\omega$
6	$32\omega^6 - 48\omega^4 + 18\omega^2 - 1$
7	$64\omega^7 - 112\omega^5 + 56\omega^3 - 7\omega$
8	$128\omega^8 - 256\omega^6 + 160\omega^4 - 32\omega^2 + 1$
9	$256\omega^9 - 576\omega^7 + 432\omega^5 - 120\omega^3 + 9\omega$
10	$512\omega^{10} - 1280\omega^8 + 1120\omega^6 - 400\omega^4 + 50\omega^2 - 1$

7.3.2.2 Type I Chebyshev filter

The frequency characteristics of the Type I Chebyshev filter of order N are defined as follows:

$$|H(\omega)| = \frac{1}{\sqrt{1 + \epsilon^2 T_N^2(\omega/\omega_p)}}, \quad (7.37)$$

where ω_p is the pass-band corner frequency and ϵ is the ripple control parameter that adjusts the magnitude of the ripple within the pass band. Substituting $\omega_p = 1$, the frequency characteristics of the normalized Type I Chebyshev filter of order N are expressed in terms of the Chebyshev polynomial as follows:

$$|H(\omega)| = \frac{1}{\sqrt{1 + \epsilon^2 T_N^2(\omega)}}. \quad (7.38)$$

based on Eqs (7.35) and (7.38), we make the following observations for the frequency characteristics of the normalized Type I Chebyshev filter.

- (1) For $\omega = 0$, the Chebyshev polynomial $T_N(\omega)$ has a value of ± 1 or 0. This can be shown by substituting $\omega = 0$ in Eq. (7.33), which yields

$$T_N(0) = \cos(N \cos^{-1}(0)) = \cos\left(\frac{N(2n+1)\pi}{2}\right) = \begin{cases} \pm 1 & N \text{ is even} \\ 0 & N \text{ is odd.} \end{cases} \quad (7.39)$$

Equation (7.37) implies that the dc component $|H(0)|$ of the Type I Chebyshev filter is given by

$$|H(0)| = \begin{cases} \frac{1}{\sqrt{1 + \epsilon^2}} & N \text{ is even} \\ 1 & N \text{ is odd.} \end{cases} \quad (7.40)$$

- (2) For $\omega = 1$ radian/s, the value of the Chebyshev polynomial $T_N(\omega)$ is given by

$$T_N(1) = \cos(N \cos^{-1}(1)) = \cos(2nN\pi) = 1. \quad (7.41)$$

Therefore, the magnitude $|H(\omega)|$ of the normalized Type I Chebyshev filter at $\omega = 1$ radian/s is given by

$$|H(1)| = \frac{1}{\sqrt{1 + \epsilon^2}}, \quad (7.42)$$

irrespective of the order N of the normalized Chebyshev filter.

- (3) For large values of ω within the stop band, the magnitude response of the normalized Type I Chebyshev filter can be approximated by

$$|H(\omega)| \approx \frac{1}{\epsilon T_N(\omega)}, \quad (7.43)$$

since $\epsilon T_N(\omega) \gg 1$. If $N \gg 1$, then a second approximation can be made by ignoring the lower degree terms in $T_N(\omega)$ and using the approximation $T_N(\omega) \approx 2^{N-1} \omega^N$. Equation (7.43) is therefore simplified as follows:

$$|H(\omega)| \approx \frac{1}{\epsilon} \times \frac{1}{2^{N-1} \omega^N}. \quad (7.44)$$

- (4) Since

$$H(s)H(-s)|_{s=j\omega} = |H(\omega)|^2,$$

$H(s)H(-s)$ can be derived from Eq. (7.38) as follows:

$$H(s)H(-s) = \frac{1}{1 + \epsilon^2 T_N^2(s/j)}. \quad (7.45)$$

The $2N$ poles of $H(s)H(-s)$ are obtained by solving the characteristic equation,

$$1 + \epsilon^2 T_N^2(s/j) = 0, \quad (7.46)$$

and are given by

$$s_n = \sin\left(\frac{2n-1}{2N}\pi\right) \sinh\left(\frac{1}{N} \sinh^{-1}\left(\frac{1}{\epsilon}\right)\right) + j \cos\left(\frac{2n-1}{2N}\pi\right) \cosh\left(\frac{1}{N} \sinh^{-1}\left(\frac{1}{\epsilon}\right)\right) \quad (7.47)$$

for $1 \leq n \leq 2N-1$. To derive a stable implementation of the normalized Type I Chebyshev filter, the N poles in the left-hand s -plane are included

in the Laplace transfer function of $H(s)$. From Eq. (7.45), it is clear that there are no zeros for the normalized Type I Chebyshev filter.

Properties (1)–(4) are used to derive the design algorithm for the Type I Chebyshev filter, which is explained in the following.

7.3.2.3 Design steps for the lowpass filter

In this section, we will design a lowpass Type I Chebyshev filter based on the following specifications:

$$\begin{aligned} \text{pass band } (0 \leq |\omega| \leq \omega_p \text{ radians/s}) \quad & 1 - \delta_p \leq |H(\omega)| \leq 1 + \delta_p; \\ \text{stop band } (|\omega| > \omega_s \text{ radians/s}) \quad & |H(\omega)| \leq \delta_s. \end{aligned}$$

Since the Type I Chebyshev filter is designed in terms of its normalized version, Eq. (7.37), we normalize the aforementioned specifications by the pass-band corner frequency ω_p . The normalized specifications are as follows:

$$\begin{aligned} \text{pass band } (0 \leq |\omega| \leq 1) \quad & 1 - \delta_p \leq |H(\omega)| \leq 1 + \delta_p; \\ \text{stop band } (|\omega| > \omega_s/\omega_p) \quad & |H(\omega)| \leq \delta_s. \end{aligned}$$

Step 1 Determine the value of the ripple control factor ϵ . Equation (7.42) computes the value of the ripple control factor ϵ :

$$\epsilon = \sqrt{G_p} \quad \text{with} \quad G_p = \frac{1}{(1 - \delta_p)^2} - 1. \quad (7.48)$$

Step 2 Calculate the order N of the Chebyshev polynomial. The gain at the normalized stop band corner frequency ω_s/ω_p is obtained from Eq. (7.37) as

$$|H(\omega_s/\omega_p)|^2 = \frac{1}{1 + \epsilon^2 T_N^2(\omega_s/\omega_p)} = (\delta_s)^2. \quad (7.49)$$

Substituting the value of the Chebyshev polynomial $T_N(\omega)$ from Eq. (7.32) and simplifying the resulting equation, we obtain

$$N = \frac{\cosh^{-1}[(G_s/G_p)^{0.5}]}{\cosh^{-1}[\omega_s/\omega_p]}, \quad (7.50)$$

where the gain terms G_p and G_s are given by

$$G_p = \frac{1}{(1 - \delta_p)^2} - 1 \quad \text{with} \quad G_s = \frac{1}{(\delta_s)^2} - 1. \quad (7.51)$$

Step 3 Determine the location of the $2N$ poles of $H(S)H(-S)$ using Eq. (7.47). To derive a stable implementation for the normalized Type I Chebyshev filter $H(S)$, the N poles lying in the left-half s -plane are selected to derive the transfer function $H(S)$. If required, a constant gain term K is also multiplied with $H(S)$

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such that the gain $|H(0)|$ of the normalized Type I Chebyshev filter is unity at $\omega = 0$.

Step 4 Derive the transfer function $H(s)$ of the required lowpass filter from the transfer function $H(S)$ of the normalized Type I Chebyshev filter, obtained in Step 3, using the following transformation:

$$H(s) = H(S)|_{S=s/\omega_p}. \tag{7.52}$$

Step 5 Sketch the magnitude spectrum from the transfer function $H(s)$ determined in Step 4. Confirm that the transfer function satisfies the initial design specifications.

Example 7.7

Repeat Example 7.6 using the Type I Chebyshev filter.

Solution

For the given specifications, Example 7.5 calculates the pass-band and stop-band gain on a linear scale as $(1 - \delta_p) = 0.8913$ and $\delta_s = 10^{-15/20} = 0.1778$ with the gain terms given by $G_p = 0.2588$ and $G_s = 30.6327$.

Step 1 determines the value of the ripple control factor ϵ :

$$\epsilon = \sqrt{G_p} = \sqrt{0.2588} = 0.5087.$$

Step 2 determines the order N of the Chebyshev polynomial:

$$N = \frac{\cosh^{-1}[(30.6327/0.2588)^{0.5}]}{\cosh^{-1} [100/50]} = 2.3371.$$

We round off N to the closest higher integer, $N = 3$.

Step 3 determines the location of the six poles of $H(S)H(-S)$:

$$[-0.2471 + j0.9660, \quad -0.2471 - j0.9660, \quad 0.2471 + j0.9660, \\ 0.2471 - j0.9660, \quad 0.4943, \quad -0.4943].$$

The three poles lying in the left-half s -plane are included in the transfer function $H(S)$ of the normalized Type I Chebyshev filter. These poles are located at

$$[-0.2471 + j0.9660, \quad -0.2471 - j0.9660, \quad -0.4943].$$

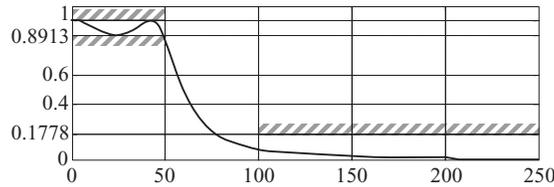
The transfer function for the normalized Type I Chebyshev filter is therefore given by

$$H(S) = \frac{K}{(S + 0.2472 + j0.9660)(S + 0.2472 - j0.9660)(S + 0.4943)},$$

which simplifies to

$$H(S) = \frac{K}{S^3 + 0.9885S^2 + 1.2386S + 0.4914}.$$

Fig. 7.9. Magnitude spectrum of the Type I Chebyshev lowpass filter designed in Example 7.7.



Since $|H(\omega)|$ at $\omega = 0$ is $K/0.4914$, K is set to 0.4914 to make the dc gain equal to unity. The new transfer function with unity gain at $\omega = 0$ is given by

$$H(S) = \frac{0.4914}{S^3 + 0.9885S^2 + 1.2386S + 0.4914}.$$

Step 4 transforms the normalized Type I Chebyshev filter using the following relationship:

$$H(s) = H(S)|_{S=s/50} = \frac{0.4914}{(s/50)^3 + 0.9885(s/50)^2 + 1.2386(s/50) + 0.4914}$$

or

$$H(s) = \frac{6.1425 \times 10^4}{s^3 + 49.425s^2 + 3.0965 \times 10^3s + 6.1425 \times 10^4},$$

which is the transfer function of the required lowpass filter.

The magnitude spectrum of the Type I Chebyshev filter is plotted in Fig. 7.9. It is observed that Fig. 7.9 satisfies the initial design specifications.

Examples 7.6 and 7.7 used the Butterworth and Type I Chebyshev implementations to design a lowpass filter based on the same specifications. Comparing the magnitude spectra (Figs 7.8 and 7.9) for the resulting filters, we note that the Butterworth filter has a monotonic gain with negligible ripples in the pass and stop bands. By introducing pass-band ripples, the Type I Chebyshev implementation is able to satisfy the design specifications with a lower order N for the lowpass filter, thus reducing the complexity of the filter. However, savings in the complexity are achieved at the expense of ripples, which are added to the the pass band of the frequency characteristics of the Type I Chebyshev filter.

7.3.2.4 Type I Chebyshev filter design using MATLAB

MATLAB uses the `cheblord` and `cheby1` functions to implement the Type I Chebyshev filter. The `cheblord` function determines the order N of the Type I Chebyshev filter from the pass-band corner frequency ω_p , stop-band corner frequency ω_s , pass-band attenuation r_p , and the stop-band attenuation r_s . In terms of the filter specifications, Eqs (7.23) and (7.24), the values of the pass-band attenuation r_p and the stop-band attenuation r_s are given by

$$r_p = 20 \times \log_{10}(\delta_p) \quad \text{and} \quad r_s = 20 \times \log_{10}(\delta_s).$$

The `cheblord` also returns `wn`, another design parameter referred to as the Chebyshev natural frequency to use with `cheby1` to achieve the design specifications. The syntax for `cheblord` is given by

```
>> [N,wn] = cheblord(wp,ws,rp,rs,'s');
```

To determine the coefficients of the numerator and denominator of the Type I Chebyshev filter, MATLAB uses the `cheby1` function with the following syntax:

```
>> [num,den] = cheby1(N,rp,wn,'s');
```

The transfer function $H(s)$ can be determined using the library function `tf` as follows:

```
>> H = tf(num,den);
```

For Example 7.7, the MATLAB commands for designing the Butterworth filter are given by

```
>> wp=50; ws=100; rp=1;      % specify design parameters
    rs=15;
>> [N,wn] = cheblord        % determine order and natural
    (wp,ws,rp,rs,'s');      % freq
>> [num,den] = cheby1      % determine num and denom
    (N,rp,wn,'s');         % coeff.
>> Ht = tf(num,den);       % determine transfer function
>> [H,w] = freqs(num,den); % determine magnitude spectrum
>> plot(w,abs(H));        % plot magnitude spectrum
```

Stepwise implementation of the above code returns the following values for different variables:

```
Instruction II:          N = 3; wn = 50;
Instruction III: num = [0 0 0 61413.3]; den =
    [1.0000 49.417 3096 61413.3];
Instruction IV: Ht = 61413.3/ (s^3 + 49.417s^2 + 3096s
    + 61413.3);
```

The magnitude spectrum is the same as that given in Fig. 7.9.

7.3.3 Type II Chebyshev filters

The Type II Chebyshev filters, or the inverse Chebyshev filters, are monotonic within the pass band and introduce ripples in the stop band. Such an implementation is preferred over the Type I Chebyshev filter in applications where a constant gain is desired within the pass band.

The frequency characteristics of the Type II Chebyshev filter are given by

$$|H(\omega)| = \frac{1}{\sqrt{1 + [\varepsilon^2 T_N^2(\omega_s/\omega)]^{-1}}} = \sqrt{\frac{\varepsilon^2 T_N^2(\omega_s/\omega)}{1 + \varepsilon^2 T_N^2(\omega_s/\omega)}}, \quad (7.53)$$

where ω_s is the lower corner frequency of the stop band. To derive the normalized version of the Type II Chebyshev filter, we set $\omega_s = 1$ in Eq. (7.53) leading to the following expression for the frequency characteristics of the normalized Type II Chebyshev filter:

$$|H(\omega)| = \frac{1}{\sqrt{1 + [\varepsilon^2 T_N^2(1/\omega)]^{-1}}} = \sqrt{\frac{\varepsilon^2 T_N^2(1/\omega)}{1 + \varepsilon^2 T_N^2(1/\omega)}}. \quad (7.54)$$

In the following section, we list the steps involved in the design of the Type II Chebyshev filter.

7.3.3.1 Design steps for the lowpass filter

The design of the lowpass Type II Chebyshev filter is based on the following specifications:

$$\begin{aligned} \text{pass band } (0 \leq |\omega| \leq \omega_p \text{ radians/s}) \quad & 1 - \delta_p \leq |H(\omega)| \leq 1 + \delta_p; \\ \text{stop band } (|\omega| > \omega_s \text{ radians/s}) \quad & |H(\omega)| \leq \delta_s. \end{aligned}$$

Normalizing the specifications with the stop-band corner frequency ω_s , we obtain

$$\begin{aligned} \text{pass band } (0 \leq |\omega| \leq \omega_p/\omega_s) \quad & 1 - \delta_p \leq |H(\omega)| \leq 1 + \delta_p; \\ \text{stop band } (|\omega| > 1) \quad & |H(\omega)| \leq \delta_s. \end{aligned}$$

Step 1 Compute the value of the ripple factor by setting the normalized frequency $\omega = 1$ in Eq. (7.54). Since the Type II Chebyshev filter is normalized with respect to ω_s , the normalized frequency $\omega = 1$ corresponds to ω_s and the filter gain $H(1) = \delta_s$. Substituting $H(1) = \delta_s$ in Eq. (7.54), we obtain

$$|H(1)| = \sqrt{\frac{\varepsilon^2}{1 + \varepsilon^2}} = \delta_s,$$

which simplifies to

$$\varepsilon = \frac{1}{\sqrt{G_s}}, \quad (7.55)$$

with the gain term specified in Eq. (7.51).

Step 2 Compute the order N of the Type II Chebyshev filter. To derive an expression for the order N , we compute the gain $|H(\omega)|$ at the normalized pass-band corner frequency ω_p/ω_s . Substituting $|H(\omega)| = (1 - \delta_p)$ at $\omega = \omega_p/\omega_s$, we obtain

$$\frac{\varepsilon^2 T_N^2(\omega_s/\omega_p)}{1 + \varepsilon^2 T_N^2(\omega_s/\omega_p)} = (1 - \delta_p)^2.$$

Substituting the value of the Chebyshev polynomial from Eq. (7.33) and simplifying the resulting expression with respect to N yields

$$N = \frac{\cosh^{-1}[(G_s/G_p)^{0.5}]}{\cosh^{-1}[\omega_s/\omega_p]}, \quad (7.56)$$

where the gain terms G_p and G_s are defined in Eq. (7.51). Note that the expression for the order of the filter for the Type II Chebyshev filter is the same as the corresponding expression, Eq. (7.50), for the Type I Chebyshev filter.

Step 3 Determine the location of the poles and zeros of the transfer function $H(s)$ of the normalized Type II Chebyshev filter. Substituting

$$H(s)H(-s)|_{s=j\omega} = |H(\omega)|^2,$$

the Laplace transfer function for the normalized Type II Chebyshev filter is given by

$$H(s)H(-s) = \frac{\varepsilon^2 T_N^2(j/s)}{1 + \varepsilon^2 T_N^2(j/s)}. \quad (7.57)$$

The poles of $H(s)H(-s)$ are obtained by solving for the roots of the characteristic equation,

$$1 + \varepsilon^2 T_N^2(j/s) = 0. \quad (7.58)$$

Comparing with the characteristic equation for $H(s)H(-s)$ of the Type I Chebyshev filter, Eq. (7.46), we note that (s/j) in the Chebyshev polynomial of Eq. (7.46) is replaced by (j/s) in Eq. (7.58). This implies that the poles of the normalized Type II Chebyshev filter are simply the inverse of the poles of the Type I Chebyshev filter. Hence, the location of the poles for the normalized Type II Chebyshev filter can be computed by determining the locations of the poles for the normalized Type I Chebyshev filter and then taking the inverse.

The zeros of $H(s)H(-s)$ are obtained by solving

$$T_N^2(j/s) = 0. \quad (7.59)$$

The zeros of $H(s)H(-s)$ are therefore the inverse of the roots of the Chebyshev polynomial $T_N(\omega) = T_N(s/j)$, which are given by

$$\omega = \cos \left[\frac{(2n + 1)\pi}{2N} \right].$$

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The zeros of $H(s)$ are therefore given by

$$s = \frac{j}{\cos\left(\frac{2n + 1\pi}{2N}\right)} \tag{7.60}$$

for $0 \leq n \leq N - 1$. The poles and zeros are used to evaluate the transfer function $H(S)$ for the normalized Type II Chebyshev filter. If required, a constant gain term K is also multiplied by $H(S)$ such that the gain $|H(0)|$ of the normalized Type II Chebyshev filter is unity at $\omega = 0$.

Step 4 Derive the transfer function $H(s)$ of the required lowpass filter from the transfer function $H(S)$ of the normalized Type II Chebyshev filter, obtained in Step 3, using the following transformation:

$$H(s) = H(S)|_{S=s/\omega_s} \tag{7.61}$$

Step 5 Sketch the magnitude spectrum from the transfer function $H(s)$ determined in Step 4. Confirm that the transfer function satisfies the initial design specifications.

Example 7.8

Repeat Example 7.6 using the Type II Chebyshev filter.

Solution

As calculated in Example 7.5, the pass-band and stop-band gain are $(1 - \delta_p) = 0.8913$ and $\delta_s = 10^{-15/20} = 0.1778$. The gain terms are also calculated as $G_p = 0.2588$ and $G_s = 30.6327$.

Step 1 determines the value of the ripple control factor ϵ :

$$\epsilon = \frac{1}{\sqrt{G_s}} = \frac{1}{\sqrt{30.6327}} = 0.1807.$$

Step 2 determines the order N of the Chebyshev polynomial:

$$N = \frac{\cosh^{-1}[(30.6327/0.2588)^{0.5}]}{\cosh^{-1}[100/50]} = 2.3371.$$

We round off N to the closest higher integer, $N = 3$.

Step 3 determines the location of the poles and zeros of $H(S)H(-S)$. We first determine the location of poles for the Type I Chebyshev filter with $\epsilon = 0.1807$ and $N = 3$. Using Eq. (7.47), the location of poles for $H(s)H(-s)$ of the Type I Chebyshev filter is given by

$$[-0.4468 + j1.1614, -0.4468 - j1.1614, 0.4468 + j1.1614, 0.4468 - j1.1614, 0.8935, -0.8935].$$

Selecting the poles located in the left-half s-plane, we obtain

$$[-0.4468 + j1.1614, -0.4468 + j1.1614, -0.8935].$$

The poles of the normalized Type II Chebyshev filter are located at the inverse of the above locations and are given by

$$[-0.2885 - j0.7501, -0.2885 + j0.7501, -1.1192].$$

The zeros of the normalized Chebyshev Type II filter are computed using Eq. (7.52) and are given by

$$[-j1.1547, +j1.1547, \infty].$$

The zero at $s = \infty$ is neglected. The transfer function for the normalized Type II Chebyshev filter is given by

$$H(S) = \frac{K(S + j1.1547)(S - j1.1547)}{(S + 0.2885 + j0.7501)(S + 0.2885 - j0.7501)(S + 1.1192)},$$

which simplifies to

$$H(S) = \frac{K(S^2 + 1.3333)}{S^3 + 1.6962S^2 + 1.2917S + 0.7229}.$$

Since $|H(\omega)|$ at $\omega = 0$ is $1.3333/0.7229 = 1.8444$, K is set to $1/1.844 = 0.5422$ to make the dc gain equal to unity. The new transfer function with unity gain at $\omega = 0$ is given by

$$H(S) = \frac{0.5422(S^2 + 1.3333)}{S^3 + 1.6962S^2 + 1.2917S + 0.7229}.$$

Step 4 normalizes $H(S)$ based on the following transformation:

$$H(s) = H(S)|_{S=s/100} = \frac{0.5422((s/100)^2 + 1.3333)}{(s/100)^3 + 1.6962(s/100)^2 + 1.2917(s/100) + 0.7229},$$

which simplifies to

$$H(s) = \frac{54.22(s^2 + 1.3333 \times 10^4)}{s^3 + 1.6962 \times 10^2 s^2 + 1.2917 \times 10^4 s + 0.7229 \times 10^6}.$$

Step 5 plots the magnitude spectrum, which is shown in Fig. 7.10. As expected, the frequency characteristics in Fig. 7.10 have a monotonic gain within the pass band and ripples within the stop band. Also, it is noted that the magnitude spectrum $|H(\omega)| = 0$ between the frequencies of $\omega = 100$ and $\omega = 150$ radians/s. This zero value corresponds to the location of the complex zeros in $H(s)$. Setting

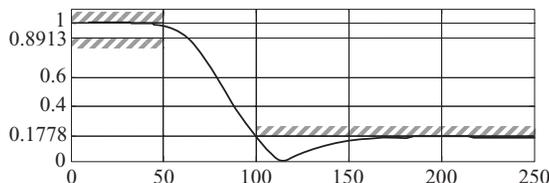


Fig. 7.10. Magnitude spectrum of the Type II Chebyshev lowpass filter designed in Example 7.8.

the numerator of $H(s)$ equal to zero, we get two zeros at $s = \pm j115.4686$, which lead to a zero magnitude at a frequency of $\omega = 115.4686$.

7.3.3.2 Type II Chebyshev filter design using MATLAB

MATLAB provides the `cheb2ord` and `cheby2` functions to implement the Type II Chebyshev filter. The usage of these functions is the same as the `cheb1ord` and `cheby1` functions for the Type I Chebyshev filter except for the `cheby2` function, for which the stop-band constraints (stop-band ripple rs and stop-band corner frequency ws) are specified. The code for Example 7.8 is as follows:

```
>> wp=50; ws=100; rp=1;      % specify design parameters
    rs=15;
>> [N,wn] = cheb2ord        % determine order and natural
    (wp,ws,rp,rs,'s');      % freq
>> [num,den] =              % determine num and denom
    cheby2(N,rs,ws,'s');    % coeff.
>> Ht = tf(num,den);        % determine transfer function
>> [H,w] = freqs(num,den);  % determine magnitude spectrum
>> plot(w,abs(H));         % plot magnitude spectrum
```

Stepwise implementation of the above code returns the following values for different variables:

```
Instruction II:          N = 3; wn = 78.6980;
Instruction III:         num = [0 54.212 0 722835];
                        den = [1.0000 169.63 12917 722835];
Instruction IV: Ht = (54.21s^2 + 722800) / (s^3 + 169.6^2
                        + 12920s + 722800);
```

The magnitude spectrum is the same as that given in Fig. 7.10.

7.3.4 Elliptic filters

Elliptic filters, also referred to as Cauer filters, include both pass-band and stop-band ripples. Consequently, elliptic filters can achieve a very narrow bandwidth for the transition band. The frequency characteristics of the elliptic filter are given by

$$|H(\omega)| = \frac{1}{\sqrt{1 + \epsilon^2 U_N^2(\omega/\omega_p)}}, \quad (7.62)$$

where $U_N(\omega)$ is an N th-order Jacobian elliptic function. By setting $\omega_p = 1$, we obtain the frequency characteristics of the normalized elliptic filter as follows:

$$|H(\omega)| = \frac{1}{\sqrt{1 + \varepsilon^2 U_N^2(\omega)}}. \quad (7.63)$$

The design procedure for elliptic filters is similar to that for Type I and Type II Chebyshev filters. Since $U_N(1) = 1$, for all N , it is straightforward to derive the value of the ripple control factor as

$$\varepsilon = \sqrt{G_p}, \quad (7.64)$$

where G_p is the pass-band gain term defined in Eq. (7.51). The order N of the elliptic filter is calculated using the following expression:

$$N = \frac{\psi[(\omega_p/\omega_s)^2] \psi[\sqrt{1 - G_p/G_s}]}{\psi[G_p/G_s] \psi[\sqrt{1 - (\omega_p/\omega_s)^2}]}, \quad (7.65)$$

where $\psi[x]$ is referred to as the complete elliptic integral of the first kind and is given by

$$\psi[x] = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - x^2 \sin^2 \phi}}. \quad (7.66)$$

MATLAB provides the `ellipke` function to compute Eq. (7.66) such that $\psi[x] = \text{ellipke}(x^2)$.

Finding the transfer function $H(s)$ for the elliptic filters of order N and ripple control factor ε requires the computation of its poles and zeros from non-linear simultaneous integral equations, which is beyond the scope of the text. In Section 7.3.4.1, which follows Example 7.9, we provide a list of library functions in MATLAB that may be used to design the elliptic filters.

Example 7.9

Calculate the ripple control factor and order of the elliptic filter that satisfies the filter specifications listed in Example 7.6.

Solution

Example 7.5 computes the gain terms as $G_p = 0.2588$ and $G_s = 30.6327$. The pass-band and stop-band corner frequencies are specified as $\omega_p = 50$ radians/s and $\omega_s = 100$ radians/s. Using Eq. (7.65), the ripple control factor is given by

$$\varepsilon = \sqrt{G_p} = \sqrt{0.2588} = 0.5087.$$

Using Eq. (7.65) with $\omega_p/\omega_s = 0.5$ and $G_p/G_s = 0.0085$, the order N of the elliptic filter is given by

$$N = \frac{\psi[(\omega_p/\omega_s)^2] \psi[\sqrt{1 - G_p/G_s}]}{\psi[G_p/G_s] \psi[\sqrt{1 - (\omega_p/\omega_s)^2}]} = \frac{\psi[0.25] \psi[0.9958]}{\psi[0.0085] \psi[0.8660]}.$$

Table 7.4. Comparison of the different implementations of a lowpass filter

Type of filter	Order	Pass band	Transition band	Stop band
Butterworth	highest order (4)	monotonic gain	widest width	monotonic gain
Either pass or stop bands specs are met; the other is overdesigned				
Type I Chebyshev	moderate order (3)	ripples are present; exact specs are met	narrow width	monotonic gain; overdesigned specs
Type II Chebyshev	moderate order (3); same as Type I	monotonic gain; overdesigned specs	narrow width; similar to Type I	ripples are present; exact specs are met
Elliptic	lowest order (2)	ripples are present; exact specs are met	narrowest width	ripples are present; exact specs are met

Using MATLAB, $\psi [0.25] = \text{ellipke}(0.25^2) = 1.5962$, $\psi [0.9958] = \text{ellipke}(0.9968^2) = 3.9175$, $\psi [0.0085] = \text{ellipke}(0.0085^2) = 1.5708$, and $\psi [0.8660] = \text{ellipke}(0.8660^2) = 2.1564$. The value of N is given by

$$N = \frac{1.5962 \times 3.9175}{1.5708 \times 2.1564} = 1.8715.$$

Rounding off to the nearest higher integer, the order N of the filter equals 2.

Examples 7.6 to 7.9 designed a lowpass filter for the same specifications based on four different implementations derived from the Butterworth, Type I Chebyshev, Type II Chebyshev, and elliptic filters. Table 7.4 compares the properties of these four implementations with respect to the frequency responses within the pass, transition, and stop bands.

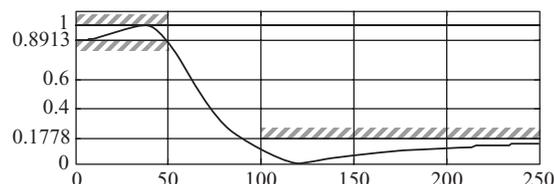
In terms of the complexity of the implementations, the elliptic filters provide the lowest order at the expense of equiripple gains in both the pass and stop bands. The Chebyshev filters provide monotonic gain in either the pass or stop band, but increase the order of the implementation. The Butterworth filters provide monotonic gains of maximally flat nature in both the pass and stop bands. However, the Butterworth filters are of the highest order and have the widest transition bandwidth.

Another factor considered in choice of implementation is the phase response of the filter. Generally, ripples add non-linearity to the phase responses. Therefore, the elliptic filter may not be the best choice in applications where a linear phase is important.

7.3.4.1 Elliptic filter design using MATLAB

MATLAB provides the `ellipord` and `ellip` functions to implement the elliptic filters. The usage of these functions is similar to the `cheblord` and

Fig. 7.11. Magnitude spectrum of the elliptic lowpass filter designed in Example 7.9.



cheby1 functions used to design Type I Chebyshev filters. The code to implement an elliptic filter for Example 7.9 is as follows:

```
>> wp=50; ws=100; rp=1;          % specify design parameters
    rs=15;
>> [N,wn] = ellipord            % determine order and
    (wp,ws,rp,rs,'s');          % natural freq
>> [num,den] =                  % determine num and denom
    ellip(N,rp,rs,wn,'s');      % coeff.
>> Ht = tf(num,den);            % determine transfer
                                % function
>> [H,w] = freqs(num,den);      % determine magnitude
                                % spectrum
>> plot(w,abs(H));              % plot magnitude spectrum
```

Stepwise implementation of the above code returns the following values for different variables:

```
Instruction II:          N = 2; wn = 50;
Instruction III:         num = [0.1778 0 2369.66];
                        den = [1.0000 48.384 2961.75];
Instruction IV: Ht = (0.1778s^2 + 2640)/(s^2 + 48.38s
                        + 2962);
```

The magnitude spectrum is plotted in Fig. 7.11.

7.4 Frequency transformations

In Section 7.3, we designed a collection of specialized CT lowpass filters. In this section, we consider the design techniques for the remaining three categories (highpass, bandpass, and bandstop filters) of CT filters. A common approach for designing CT filters is to convert the desired specifications into the specifications of a normalized or prototype lowpass filter using a frequency transformation that maps the required frequency-selective filter into a lowpass filter. Based on the transformed specifications, a normalized lowpass filter is designed using the techniques covered in Section 7.3. The transfer function $H(S)$ of the normalized lowpass filter is then transformed back into the original frequency domain. Transformation for converting a lowpass filter to a highpass

filter is considered next, followed by the lowpass to bandpass, and lowpass to bandstop transformations.

7.4.1 Lowpass to highpass filter

The transformation that converts a lowpass filter with the transfer function $H(S)$ into a highpass filter with transfer function $H(s)$ is given by

$$S = \frac{\xi_p}{s}, \tag{7.67}$$

where $S = \sigma + j\omega$ represents the lowpass domain and $s = \gamma + j\xi$ represents the highpass domain. The frequency $\xi = \xi_p$ represents the pass-band corner frequency for the highpass filter. In terms of the CTFT domain, Eq. (7.67) can be expressed as follows:

$$\omega = -\frac{\xi_p}{\xi} \quad \text{or} \quad \xi = -\frac{\xi_p}{\omega}. \tag{7.68}$$

Figure 7.12 shows the effect of applying the frequency transformation in Eq. (7.68) to the specifications of a highpass filter. Equation (7.68) maps the highpass specifications in the range $-\infty < \xi \leq 0$ to the specifications of a lowpass filter in the range $0 \leq \omega < \infty$. Similarly, the highpass specifications for the positive range of frequencies ($0 < \xi \leq \infty$) are mapped to the lowpass specifications within the range $-\infty \leq \omega < 0$. Since the magnitude spectra are symmetrical about the y-axis, the change from positive ξ frequencies to negative ω frequencies does not affect the nature of the filter in the entire domain.

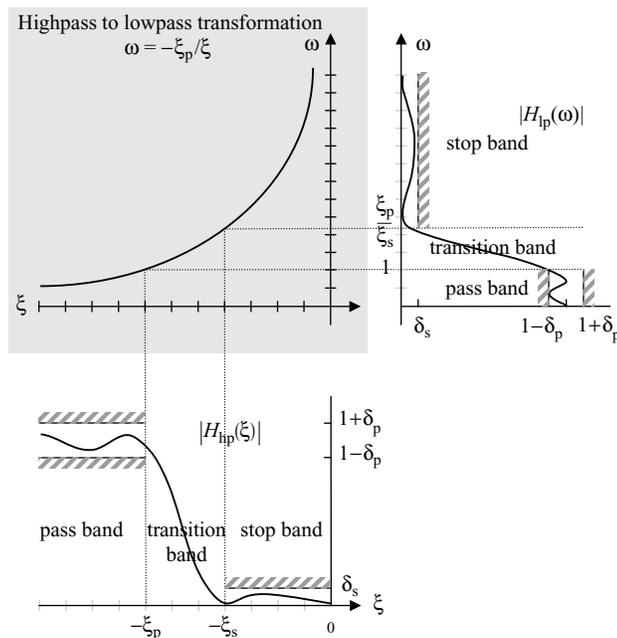


Fig. 7.12. Highpass to lowpass transformation.

From Fig. 7.12, it is clear that Eq. (7.68), or alternatively Eq. (7.67), represents a highpass to lowpass transformation. We now exploit this transformation to design a highpass filter.

Example 7.10

Design a highpass Butterworth filter with the following specifications:

$$\begin{aligned} \text{stop band } (0 \leq |\xi| \leq 50 \text{ radians/s}) & \quad -1 \leq 20 \log_{10} |H(\omega)| \leq 0; \\ \text{pass band } (|\xi| > 100 \text{ radians/s}) & \quad 20 \log_{10} |H(\omega)| \leq -15. \end{aligned}$$

Solution

Using Eq. (7.67) with $\xi_p = 100$ radians/s to transform the specifications from the domain $s = \gamma + j\xi$ of the highpass filter to the domain $S = \sigma + j\omega$ of the lowpass filter, we obtain

$$\begin{aligned} \text{stop band } (\infty < |\xi| \leq 2 \text{ radians/s}) & \quad -1 \leq 20 \log_{10} |H(\omega)| \leq 0; \\ \text{pass band } (|\xi| < 1 \text{ radian/s}) & \quad 20 \log_{10} |H(\omega)| \leq 15. \end{aligned}$$

The above specifications are used to design a normalized lowpass Butterworth filter. Expressed on a linear scale, the pass-band and stop-band gains are given by

$$(1 - \delta_p) = 10^{-1/20} = 0.8913 \quad \text{and} \quad \delta_s = 10^{-15/20} = 0.1778.$$

The gain terms G_p and G_s are given by

$$G_p = \frac{1}{(1 - \delta_p)^2} - 1 = \frac{1}{0.8913^2} - 1 = 0.2588$$

and

$$G_s = \frac{1}{(\delta_s)^2} - 1 = \frac{1}{0.1778^2} - 1 = 30.6327.$$

The order N of the Butterworth filter is obtained using Eq. (7.29) as follows:

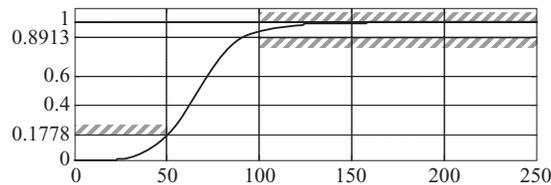
$$N = \frac{1}{2} \times \frac{\ln(G_p/G_s)}{\ln(\xi_p/\xi_s)} = \frac{1}{2} \times \frac{\ln(0.2588/30.6327)}{\ln(1/2)} = 3.4435.$$

We round off the order of the filter to the higher integer value as $N = 4$.

Using the pass-band constraint, Eq. (7.31), the cut-off frequency of the required Butterworth filter is given by

$$\omega_c = \frac{\omega_s}{(G_s)^{1/2N}} = \frac{2}{(30.6327)^{1/8}} = 1.3039 \text{ radians/s.}$$

Fig. 7.13. Magnitude spectrum of the Butterworth highpass filter designed in Example 7.10.



The poles of the lowpass filter are located at

$$S = \omega_c \exp \left[j \frac{\pi}{2} + j \frac{(2n - 1)\pi}{8} \right]$$

for $1 \leq n \leq 4$. Substituting different values of n yields

$$S = [-0.4990 + j1.2047 \quad -1.2047 + j0.4990 \quad -1.2047 - j0.4990 \quad -0.4990 - j1.2047].$$

The transfer function of the lowpass filter is given by

$$H(S) = \frac{K}{(S+0.4490-j1.2047)(S+0.4490+j1.2047)(S+1.2047-j0.4990)(S+1.2047+j0.4990)}$$

or

$$H(S) = \frac{K}{S^4 + 3.4074S^3 + 5.8050S^2 + 5.7934S + 2.8909}.$$

To ensure a dc gain of unity for the lowpass filter, we set $K = 2.8909$. The transfer function of a unity gain lowpass filter is given by

$$H(S) = \frac{2.8909}{S^4 + 3.4074S^3 + 5.8050S^2 + 5.7934S + 2.8909}.$$

To derive the transfer function of the required highpass filter, we use Eq. (7.67) with $\xi_p = 100$ radians/s. The transfer function of the highpass filter is given by

$$\begin{aligned} H(s) &= H(S)|_{S=100/s} \\ &= \frac{2.8909}{(100/s)^4 + 3.4074(100/s)^3 + 5.8050(100/s)^2 + 5.7934(100/s) + 2.8909} \end{aligned}$$

or

$$H(s) = \frac{s^4}{s^4 + 2.004 \times 10^2 s^3 + 2.008 \times 10^4 s^2 + 1.179 \times 10^6 s + 3.459 \times 10^7}.$$

The magnitude spectrum of the highpass filter is given in Fig. 7.13, which confirms that the given specifications are satisfied.

7.4.1.1 MATLAB code for designing highpass filters

The MATLAB code for the design of the highpass filter required in Example 7.10 using the Butterworth, Type I Chebyshev, Type II Chebyshev,

and elliptic implementations is included below. In each case, MATLAB automatically designs the highpass filter. No explicit transformations are needed.

```
>> % Matlab code for designing highpass filter
>> wp=100; ws=50; Rp=1; Rs=15; % design specifications
>> % Butterworth filter
>> [N, wc] = % determine order and
    buttord(wp,ws,Rp,Rs,'s'); % cut off
>> [num1,den1] = % determine transfer
    butter(N,wc,'high','s'); % function
>> H1 = tf(num1,den1);
>> %%% % Type I Chebyshev
    filter

>> [N, wn] =
    cheb1ord(wp,ws,Rp,Rs,'s');
>> [num2,den2] =
    cheby1(N,Rp,wn,'high','s');
>> H2 = tf(num2,den2);
>> %%% % Type II Chebyshev
    filter

>> [N,wn] =
    cheb2ord(wp,ws,Rp,Rs,'s');
>> [num3,den3] =
    cheby2(N,Rs,wn,'high','s');
>> H3 = tf(num3,den3);
>> %%% % Elliptic filter

>> [N,wn] =
    ellipord(wp,ws,Rp,Rs,'s');
>> [num4,den4] =
    ellip(N,Rp,Rs,wn,'high','s');
>> H4 = tf(num4,den4);
```

In the above code, note that $w_p > w_s$. Also, an additional argument of 'high' is included in the design statements for different filters, which is used to specify a highpass filter. The aforementioned MATLAB code results in the following transfer functions for the different implementations:

Butterworth

$$H(s) = \frac{s^4}{s^4 + 2.004 \times 10^2 s^3 + 2.008 \times 10^4 s^2 + 1.179 \times 10^6 s + 3.459 \times 10^7};$$

Type I Chebyshev $H(s) = \frac{s^3}{s^3 + 252.1s^2 + 2.012 \times 10^4 s + 2.035 \times 10^6};$

Type II Chebyshev $H(s) = \frac{s^3 + 3.027 \times 10^3 s}{s^3 + 113.5s^2 + 9.473 \times 10^3 s + 3.548 \times 10^5};$

elliptic $H(s) = \frac{0.8903s^2 + 1501}{s^2 + 81.68s + 8441}.$

The transfer function for the Butterworth filter is the same as that derived by hand in Example 7.9.

7.4.2 Lowpass to bandpass filter

The transformation that converts a lowpass filter with the transfer function $H(S)$ into a bandpass filter with transfer function $H(s)$ is given by

$$S = \frac{s^2 + \xi_{p1}\xi_{p2}}{s(\xi_{p2} - \xi_{p1})}, \quad (7.69)$$

where $S = \sigma + j\omega$ represents the lowpass domain and $s = \gamma + j\xi$ represents the bandpass domain. The frequency $\xi = \xi_{p1}$ and ξ_{p2} represents the two pass-band corner frequencies for the bandpass filter with $\xi_{p2} > \xi_{p1}$. In terms of the CTFT variables ω and ξ , Eq. (7.69) can be expressed as follows:

$$\omega_{s1} = \frac{\xi_{p1}\xi_{p2} - \xi^2}{\xi(\xi_{p2} - \xi_{p1})}. \quad (7.70)$$

From Eq. (7.70), it can be shown that the pass-band corner frequencies ξ_{p1} and $-\xi_{p2}$ of the bandpass filter are both mapped in the lowpass domain to $\omega = 1$, whereas the pass-band corner frequencies $-\xi_{p1}$ and ξ_{p2} are mapped to $\omega = -1$. Also, the pass band $\xi_{p1} \leq |\xi| \leq \xi_{p2}$ of the bandpass filter is mapped to the pass band $-1 \leq |\omega| \leq 1$ of the lowpass filter. These results can be confirmed by substituting different values for the bandpass domain frequencies ξ and evaluating the corresponding lowpass domain frequencies.

Considering the stop-band corner frequencies of the bandpass filter, Eq. (7.70) can be used to show that the stop-band corner frequency $\pm\xi_{s1}$ is mapped to

$$\omega = \left| \frac{\xi_{p1}\xi_{p2} - \xi_{s1}^2}{\xi_{s1}(\xi_{p2} - \xi_{p1})} \right|, \quad (7.71)$$

and that the stop-band corner frequency $\pm\xi_{s2}$ is mapped to

$$\omega_{s2} = \left| \frac{\xi_{p1}\xi_{p2} - \xi_{s2}^2}{\xi_{s2}(\xi_{p2} - \xi_{p1})} \right|. \quad (7.72)$$

As a lower value for the stop-band frequency for the lowpass filter leads to more stringent requirements, the stop-band corner frequency for the lowpass filter is selected from the minimum of the two values computed in Eqs (7.71) and (7.72). Mathematically, this implies that

$$\omega_s = \min(\omega_{s1}, \omega_{s2}). \quad (7.73)$$

Example 7.11 designs a bandpass filter.

Example 7.11

Design a bandpass Butterworth filter with the following specifications:

$$\begin{aligned} \text{stop band I } (0 \leq |\xi| \leq 50 \text{ radians/s}) & \quad 20 \log_{10} |H(\omega)| \leq -20; \\ \text{pass band } (100 \leq |\xi| \leq 200 \text{ radians/s}) & \quad -2 \leq 20 \log_{10} |H(\omega)| \leq 0; \\ \text{stop band II } (|\xi| \geq 380 \text{ radians/s}) & \quad 20 \log_{10} |H(\omega)| \leq -20. \end{aligned}$$

Solution

For $\xi_{p1} = 100$ radians/s and $\xi_{p2} = 200$ radians/s, Eq. (7.70) becomes

$$\omega = \frac{2 \times 10^4 - \xi^2}{100\xi},$$

to transform the specifications from the domain $s = \gamma + j\xi$ of the bandpass filter to the domain $S = \sigma + j\omega$ of the lowpass filter. The specifications for the normalized lowpass filter are given by

$$\begin{aligned} \text{pass band } (0 \leq |\xi| < 1 \text{ radian/s}) & \quad -2 \leq 20 \log_{10} |H(\omega)| \leq 0; \\ \text{stop band } (|\xi| \geq \min(3.2737, 3.5) \text{ radians/s}) & \quad 20 \log_{10} |H(\omega)| \leq -20. \end{aligned}$$

The above specifications are used to design a normalized lowpass Butterworth filter. Expressed on a linear scale, the pass-band and stop-band gains are given by

$$(1 - \delta_p) = 10^{-2/20} = 0.7943 \quad \text{and} \quad \delta_s = 10^{-20/20} = 0.1.$$

The gain terms G_p and G_s are given by

$$G_p = \frac{1}{(1 - \delta_p)^2} - 1 = \frac{1}{0.7943^2} - 1 = 0.5850$$

and

$$G_s = \frac{1}{(\delta_s)^2} - 1 = \frac{1}{0.1^2} - 1 = 99.$$

The order N of the Butterworth filter is obtained using Eq. (7.29) as follows:

$$N = \frac{1}{2} \times \frac{\ln(G_p/G_s)}{\ln(\xi_p/\xi_s)} = \frac{1}{2} \times \frac{\ln(0.5850/99)}{\ln(1/3.2737)} = 2.1232.$$

We round off the order of the filter to the higher integer value as $N = 3$.

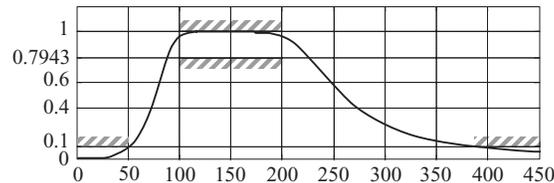
Using the stop-band constraint, Eq. (7.31), the cut-off frequency of the low-pass Butterworth filter is given by

$$\omega_c = \frac{\omega_s}{(G_s)^{1/2N}} = \frac{3.2737}{(99)^{1/6}} = 1.5221 \text{ radians/s.}$$

The poles of the lowpass filter are located at

$$S = \omega_c \exp \left[j \frac{\pi}{2} + j \frac{(2n - 1)\pi}{6} \right]$$

Fig. 7.14. Magnitude spectrum of the Butterworth bandpass filter designed in Example 7.11.



for $1 \leq n \leq 3$. Substituting different values of n yields

$$S = [-0.7610 + j1.3182 \quad -0.7610 - j1.3182 \quad -1.5221].$$

The transfer function of the lowpass filter is given by

$$H(S) = \frac{K}{(S + 0.7610 + j1.3182)(S + 0.7610 - j1.3182)(S + 1.5221)}$$

or

$$H(S) = \frac{K}{S^3 + 3.0442S^2 + 4.6336S + 3.5264}.$$

To ensure a dc gain of unity for the lowpass filter, we set $K = 3.5364$. The transfer function of the unity gain lowpass filter is given by

$$H(S) = \frac{3.5264}{S^3 + 3.0442S^2 + 4.6336S + 3.5264}.$$

To derive the transfer function of the required bandpass filter, we use Eq. (7.69) with $\xi_{p1} = 100$ radians/s and $\xi_{p2} = 200$ radians/s. The transformation is given by

$$S = \frac{s^2 + 2 \times 10^4}{100s},$$

from which the transfer function of the bandpass filter is calculated as follows:

$$H(s) = H(S) \Big|_{S = \frac{s^2 + 2 \times 10^4}{100s}} = \frac{3.5264}{\left[\frac{s^2 + 2 \times 10^4}{100s} \right]^3 + 3.0442 \left[\frac{s^2 + 2 \times 10^4}{100s} \right]^2 + 4.6336 \left[\frac{s^2 + 2 \times 10^4}{100s} \right] + 3.5264},$$

which reduces to

$$H(s) = \frac{3.5264 \times 10^6 s^3}{s^6 + 3.0442 \times 10^2 s^5 + 1.0633 \times 10^5 s^4 + 1.5703 \times 10^7 s^3 + 2.1267 \times 10^9 s^2 + 1.2177 \times 10^{11} s + 8 \times 10^{12}}.$$

The magnitude spectrum of the bandpass filter is given in Fig. 7.14, which confirms that the given specifications for the bandpass filter are satisfied.

7.4.2.1 MATLAB code for designing bandpass filters

The MATLAB code for the design of the bandpass filter required in Example 7.11 using the Butterworth, Type I Chebyshev, Type II Chebyshev, and elliptic implementations is as follows:

```
>> % MATLAB code for designing bandpass filter
>> wp=[100 200]; ws=[50 380]; Rp=2; % Specifications
    Rs=20;
>> % Butterworth filter
>> [N, wc] = buttord(wp,ws,Rp,Rs,'s');
>> [num1,den1] = butter(N,wc,'s');
>> H1 = tf(num1,den1);
>> % Type I Chebyshev filter
>> [N, wn] = cheblord(wp,ws,Rp,Rs,'s');
>> [num2,den2] = cheby1(N,Rp,wn,'s');
>> H2 = tf(num2,den2);
>> % Type II Chebyshev filter
>> [N,wn] = cheb2ord(wp,ws,Rp,Rs,'s');
>> [num3,den3] = cheby2(N,Rs,wn,'s');
>> H3 = tf(num3,den3);
>> % Elliptic filter
>> [N,wn] = ellipord(wp,ws,Rp,Rs,'s');
>> [num4,den4] = ellip(N,Rp,Rs,wn,'s');
>> H4 = tf(num4,den4);
```

The type of filter is specified by the dimensions of the pass-band and stop-band frequency vectors. Since w_p and w_s are both vectors, MATLAB knows that either a bandpass or bandstop filter is being designed. From the range of the values within w_p and w_s , MATLAB is also able to differentiate whether a bandpass or a bandstop filter is being specified. In the above example, since the range (50–380 Hz) of frequencies specified within the stop-band frequency vector w_s exceeds the range (100–200 Hz) specified within the pass-band frequency vector w_p , MATLAB is able to make the final determination that a bandpass filter is being designed. For a bandstop filter, the converse would hold true.

The aforementioned MATLAB code produces bandpass filters with the following transfer functions:

$$\text{Butterworth } H(s) = \frac{3.526 \times 10^6 s^4}{s^6 + 304.4s^5 + 1.063 \times 10^5 s^4 + 1.57 \times 10^7 s^3 + 2.127 \times 10^9 s^2 + 1.218 \times 10^{11} s + 8 \times 10^{12}};$$

$$\text{Type I Chebyshev } H(s) = \frac{6.538 \times 10^3 s^2}{s^4 + 80.38s^3 + 4.8231 \times 10^4 s^2 + 1.608 \times 10^6 s + 4 \times 10^8};$$

$$\text{Type II Chebyshev } H(s) = \frac{0.1s^4 + 1.801 \times 10^4 s^2 + 4 \times 10^7}{s^4 + 1.588 \times 10^2 s^3 + 5.4010 \times 10^4 s^2 + 3.176 \times 10^6 s + 4 \times 10^8};$$

$$\text{elliptic } H(s) = \frac{0.1s^4 + 1.101 \times 10^4 s^2 + 4 \times 10^7}{s^4 + 74.67s^3 + 4.8819 \times 10^4 s^2 + 1.493 \times 10^6 s + 4 \times 10^8}.$$

Note that the transfer function for the bandpass Butterworth filter is the same as that derived by hand in Example 7.9.

7.4.3 Lowpass to bandstop filter

The transformation to convert a lowpass filter with the transfer function $H(S)$ into a bandstop filter with transfer function $H(s)$ is given by the following expression:

$$S = \frac{s(\xi_{p2} - \xi_{p1})}{s^2 + \xi_{p1}\xi_{p2}}, \quad (7.74)$$

where $S = \sigma + j\omega$ represents the lowpass domain and $s = \gamma + j\xi$ represents the bandstop domain. The frequency $\xi = \xi_{p1}$ and ξ_{p2} represents the two pass-band corner frequencies for the bandpass filter with $\xi_{p2} > \xi_{p1}$. Note that the transformation in Eq. (7.74) is the inverse of the lowpass to bandpass transformation specified in Eq. (7.69).

In terms of the CTFT domain, Eq. (7.74) can be expressed as follows:

$$\omega = \frac{\xi(\xi_{p2} - \xi_{p1})}{\xi_{p1}\xi_{p2} - \xi^2}, \quad (7.75)$$

which can be used to confirm that Eq. (7.74) is indeed a lowpass to bandstop transformation.

As for the bandpass filter, Eq. (7.75) leads to two different values of the stop-band frequencies,

$$\omega_{s1} = \left| \frac{\xi_{s1}(\xi_{p2} - \xi_{p1})}{\xi_{p1}\xi_{p2} - \xi_{s1}^2} \right| \quad \text{and} \quad \omega_{s2} = \left| \frac{\xi_{s2}(\xi_{p2} - \xi_{p1})}{\xi_{p1}\xi_{p2} - \xi_{s2}^2} \right| \quad (7.76)$$

for the lowpass filter. The smaller of the two values is selected as the stop-band corner frequency for the normalized lowpass filter. Example 7.12 designs a bandstop filter.

Example 7.12

Design a bandstop Butterworth filter with the following specifications:

$$\begin{aligned} \text{pass band I } (0 \leq |\xi| \leq 100 \text{ radians/s}) & \quad -2 \leq 20 \log_{10} |H(\omega)| \leq 0; \\ \text{stop band } (150 \leq |\xi| \leq 250 \text{ radians/s}) & \quad 20 \log_{10} |H(\omega)| \leq -20; \\ \text{pass band II } (|\xi| \geq 370 \text{ radians/s}) & \quad -2 \leq 20 \log_{10} |H(\omega)| \leq 0. \end{aligned}$$

Solution

For $\xi_{p1} = 100$ radians/s and $\xi_{p2} = 370$ radians/s, Eq. (7.70) becomes

$$\omega = \frac{270\xi}{3.7 \times 10^4 - \xi^2},$$

to transform the specifications from the domain $s = \gamma + j\xi$ of the bandstop filter to the domain $S = \sigma + j\omega$ of the lowpass filter. The specifications for the

normalized lowpass filter are given by

$$\text{pass band } (0 \leq |\xi| < 1 \text{ radian/s}) \quad -2 \leq 20 \log_{10} |H(\omega)| \leq 0;$$

$$\text{stop band } (|\xi| \geq \min(2.7931, 2.6471) \text{ radians/s}) \quad 20 \log_{10} |H(\omega)| \leq -20.$$

The above specifications are used to design a normalized lowpass Butterworth filter. Since the pass-band and stop-band gains of the transformed lowpass filter are the same as the ones used in Example 7.11, i.e.

$$(1 - \delta_p) = 10^{-2/20} = 0.7943 \quad \text{and} \quad \delta_s = 10^{-20/20} = 0.1,$$

with gain terms

$$G_p = 0.5850 \quad \text{and} \quad G_s = 99,$$

the order $N = 3$ of the Butterworth filter is the same as in Example 7.11.

Using the stop-band constraint, Eq. (7.31), the cut-off frequency of the low-pass Butterworth filter is given by

$$\omega_c = \frac{\omega_s}{(G_s)^{1/2N}} = \frac{2.6471}{(99)^{1/6}} = 1.2307 \text{ radians/s.}$$

The poles of the lowpass filter are located at

$$S = \omega_c \exp \left[j \frac{\pi}{2} + j \frac{(2n-1)\pi}{6} \right]$$

for $1 \leq n \leq 3$. Substituting different values of n yields

$$S = [-0.6153 + j0.06581 \quad -0.6153 - j0.06581 \quad -1.2307].$$

The transfer function of the lowpass filter is given by

$$H(S) = \frac{K}{(S + 0.6153 - j0.06581)(S + 0.6153 + j0.06581)(S + 1.2307)}$$

or

$$H(S) = \frac{K}{S^3 + 2.4614S^2 + 3.0292S + 1.8640}.$$

To ensure a dc gain of unity for the lowpass filter, we set $K = 1.8640$. The transfer function of the unity gain lowpass filter is given by

$$H(S) = \frac{1.8640}{S^3 + 2.4614S^2 + 3.0292S + 1.8640}.$$

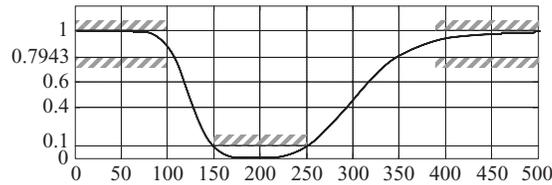
To derive the transfer function of the required bandstop filter, we use Eq. (7.74) with $\xi_{p1} = 100$ radians/s and $\xi_{p2} = 370$ radians/s. The transformation is given by

$$S = \frac{270s}{s^2 + 3.7 \times 10^4},$$

with the transfer function of the bandstop filter is given by

$$\begin{aligned} H(s) &= H(S)|_{S=\frac{270s}{s^2+3.7 \times 10^4}} \\ &= \frac{1.8640}{\left[\frac{270s}{s^2+3.7 \times 10^4} \right]^3 + 2.4641 \left[\frac{270s}{s^2+3.7 \times 10^4} \right]^2 + 3.0292 \left[\frac{270s}{s^2+3.7 \times 10^4} \right] + 1.8640}, \end{aligned}$$

Fig. 7.15. Magnitude spectrum of the Butterworth bandstop filter designed in Example 7.12.



which reduces to

$$H(s) = \frac{s^6 + 1.11 \times 10^5 s^4 + 4.107 \times 10^9 s^2 + 5.065 \times 10^{13}}{s^6 + 4.388 \times 10^2 s^5 + 2.0737 \times 10^5 s^4 + 4.302 \times 10^7 s^3 + 7.673 \times 10^9 s^2 + 6 \times 10^{11} s + 5.065 \times 10^{13}}$$

The magnitude spectrum of the bandstop filter is included in Fig. 7.15, which confirms that the given specifications are satisfied.

7.4.3.1 MATLAB code for designing bandstop filters

The MATLAB code for the design of the bandstop filter required in Example 7.12 using the Butterworth, Type I Chebyshev, Type II Chebyshev, and elliptic implementations is as follows:

```
% MATLAB code for designing bandstop filter
>> wp=[100 370]; ws=[150 250]; Rp=2; % Specifications
    Rs=20;
>> % Butterworth Filter
>> [N, wn] = buttord(wp,ws,Rp,Rs,'s');
>> [num1,den1] = butter(N,wn,'stop','s');
>> H1 = tf(num1,den1);
>> % Type I Chebyshev filter
>> [N, wn] = cheblord(wp,ws,Rp,Rs,'s');
>> [num2,den2] = cheby1(N,Rp,wn,
    'stop','s');
>> H2 = tf(num2,den2);
>> % Type II Chebyshev filter
>> [N,wn] = cheb2ord(wp,ws,Rp,Rs,'s');
>> [num3,den3] = cheby2(N,Rs,wn,
    'stop','s');
>> H3 = tf(num3,den3);
>> % Elliptic filter
>> [N,wn] = ellipord(wp,ws,Rp,Rs,'s');
>> [num4,den4] = ellip(N,Rp,Rs,wn,
    'stop','s');
>> H4 = tf(num4,den4);
```

The aforementioned MATLAB code produces the following transfer functions

for the four filters:

$$\text{Butterworth } H(s) = \frac{s^6 + 1.125 \times 10^5 s^4 + 4.219 \times 10^9 s^2 + 5.273 \times 10^{13}}{s^6 + 430.2s^5 + 2.05 \times 10^5 s^4 + 4.221 \times 10^7 s^3 + 7.688 \times 10^9 s^2 + 6.049 \times 10^{11} s + 5.273 \times 10^{13}};$$

$$\text{Type I Chebyshev } H(s) = \frac{0.7943s^4 + 5.957 \times 10^4 s^2 + 1.117 \times 10^9}{s^4 + 262.4s^3 + 1.627 \times 10^5 s^2 + 9.839 \times 10^6 s + 1.406 \times 10^9};$$

$$\text{Type II Chebyshev } H(s) = \frac{0.7943s^4 + 8.015 \times 10^4 s^2 + 1.406 \times 10^9}{s^4 + 304.5s^3 + 1.265 \times 10^5 s^2 + 1.142 \times 10^6 s + 1.406 \times 10^9};$$

$$\text{elliptic } H(s) = \frac{0.7943s^4 + 6.776 \times 10^4 s^2 + 1.117 \times 10^9}{s^4 + 227.5s^3 + 1.568 \times 10^5 s^2 + 8.53 \times 10^6 s + 1.406 \times 10^9}.$$

7.5 Summary

Chapter 7 defines the CT filters as LTI systems used to transform the frequency characteristics of the CT signals in a predefined manner. Based on the magnitude spectrum $|H(\omega)|$, Section 7.1 classifies the frequency-selective filters into four different categories.

- (1) An ideal lowpass filter removes frequency components above the cut-off frequency ω_c from the input signal, while retaining the lower frequency components $\omega \leq \omega_c$.
- (2) An ideal highpass filter is the converse of the lowpass filter since it removes frequency components below the cut-off frequency ω_c from the input signal, while retaining the higher frequency components $\omega \geq \omega_c$.
- (3) An ideal bandpass filter retains a selected range of frequency components between the lower cut-off frequency ω_{c1} and the upper cutoff frequency ω_{c2} of the filter. All other frequency components are eliminated from the input signal.
- (4) A bandstop filter is the converse of the bandpass filter, which rejects all frequency components between the lower cut-off frequency ω_{c1} and the upper cut-off frequency ω_{c2} of the filter. All other frequency components are retained at the output of the bandstop filter.

The ideal frequency filters are not physically realizable. Section 7.2 introduces practical implementations of the ideal filters obtained by introducing ripples in the pass and stop bands. A transition band is also included to eliminate the sharp transition between the pass and stop bands.

In Section 7.3, we considered the design of practical lowpass filters. We presented four implementations of practical filters: Butterworth, Type I Chebyshev, Type II Chebyshev, and elliptic filters, for which the design algorithms were covered. The Butterworth filters provide a maximally flat gain within the pass band but have a higher-order N than the Chebyshev and elliptic filters designed with the same specifications. By introducing ripples within the pass band, Type I Chebyshev filters reduce the required order N of the designed filter. Alternatively, Type II Chebyshev filters introduce ripples within the stop band to reduce the order N of the filter. The elliptic filters allow ripples in both the

pass and stop bands to derive a filter with the lowest order N among the four implementations. MATLAB instructions to design the four implementations are also presented in Section 7.3.

In Section 7.4, we covered three transformations for converting a highpass filter to a lowpass filter, a lowpass to a bandpass filter, and a bandstop to a lowpass filter. Using these transformations, we were able to map the specifications of any type of the frequency-selective filters in terms of a normalized lowpass filter. After designing the normalized lowpass filter using the design algorithms covered in Section 7.3, the transfer function of the lowpass filter is transformed back into the original domain of the frequency-selective filter.

Problems

- 7.1 Determine the impulse response of an ideal bandpass filter and an ideal bandstop filter. In each case, assume a gain of A within the pass bands and cut off frequencies of ω_{c1} and ω_{c2} .
- 7.2 Derive and sketch the location of the poles for the Butterworth filters of orders $N = 12$ and 13 in the complex s -plane.
- 7.3 Show that a lowpass Butterworth filter with an odd value of order N will always have at least one pole on the real axis in the complex s -plane.
- 7.4 Show that all complex poles of the lowpass Butterworth filter occur in conjugate pairs.
- 7.5 Show that the locations of poles for the Type I Chebyshev polynomial $T_N(\omega)$ of order N is given by

$$\omega_n = \cos \left[\frac{(2n + 1)\pi}{2N} \right]$$

for $|\omega| \leq 1$ and $|\omega| > 1$.

- 7.6 Show that the roots of the characteristic equation

$$1 + \varepsilon^2 T_N^2(j/s) = 0$$

for the Type II Chebyshev filter are the inverse of the roots of the characteristic equation

$$1 + \varepsilon^2 T_N^2(s/j) = 0$$

for the Type I Chebyshev filter.

- 7.7 Design a Butterworth lowpass filter for the following specifications:

pass band ($0 \leq |\omega| \leq 10$ radians/s) $0.9 \leq |H(\omega)| \leq 1$;

stop band ($|\omega| > 20$ radians/s) $|H(\omega)| \leq 0.10$,

by enforcing the pass-band requirements. Repeat for the stop-band requirements. Sketch the magnitude spectrum and confirm that the magnitude spectrum satisfies the design specifications.

7.8 Repeat Problem 7.7 for the following specifications:

pass band ($0 \leq |\omega| \leq 50$ radians/s) $-1 \leq 20 \log_{10} |H(\omega)| \leq 0$;

stop band ($|\omega| > 65$ radians/s) $20 \log_{10} |H(\omega)| \leq -25$.

7.9 Repeat (a) Problem 7.7 and (b) Problem 7.8 for the Type I Chebyshev filter.

7.10 Repeat (a) Problem 7.7 and (b) Problem 7.8 for the Type II Chebyshev filter.

7.11 Determine the order of the elliptic filters for the specifications included in (a) Problem 7.7 and (b) Problem 7.8.

7.12 Using the results in Problems 7.7–7.11, compare the implementation complexity of the Butterworth, Type I Chebyshev, Type II Chebyshev, and elliptic filters for the specifications included in (a) Problem 7.7 and (b) Problem 7.8.

7.13 By selecting the corner frequencies of the pass and stop bands, show that the transformation

$$S = \frac{s(\xi_{p2} - \xi_{p1})}{s^2 + \xi_{p1}\xi_{p2}}$$

maps a bandstop filter into a normalized lowpass filter.

7.14 Design a Butterworth highpass filter for the following specifications:

stop band ($0 \leq |\omega| \leq 15$ radians/s) $|H(\omega)| \leq 0.15$;

pass band ($|\omega| > 30$ radians/s) $0.85 \leq |H(\omega)| \leq 1$.

Sketch the magnitude spectrum and confirm that it satisfies the design specifications.

7.15 Repeat Problem 7.14 for the Type I Chebyshev filter.

7.16 Repeat Problem 7.14 for the Type II Chebyshev filter.

7.17 Design a Butterworth bandpass filter for the following specifications:

stop band I ($0 \leq |\xi| \leq 100$ radians/s) $20 \log_{10} |H(\omega)| \leq -15$;

Pass band ($100 \leq |\xi| \leq 150$ radians/s) $-1 \leq 20 \log_{10} |H(\omega)| \leq 0$;

stop band II ($|\xi| \geq 175$ radians/s) $20 \log_{10} |H(\omega)| \leq -15$.

Sketch the magnitude spectrum and confirm that it satisfies the design specifications.

7.18 Repeat Problem 7.17 for the Type I Chebyshev filter.

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7.19 Repeat Problem 7.17 for the Type II Chebyshev filter.

7.20 Design a Butterworth bandstop filter for the following specifications:

pass band I ($0 \leq |\xi| \leq 25$ radians/s) $-4 \leq 20 \log_{10} |H(\omega)| \leq 0$;

stop band ($100 \leq |\xi| \leq 250$ radians/s) $20 \log_{10} |H(\omega)| \leq -20$;

pass band II ($|\xi| \geq 325$ radians/s) $-4 \leq 20 \log_{10} |H(\omega)| \leq 0$.

Sketch the magnitude spectrum and confirm that it satisfies the design specifications.

7.21 Repeat Problem 7.20 for the Type I Chebyshev filter.

7.22 Repeat Problem 7.20 for the Type II Chebyshev filter.

7.23 Determine the transfer function of the four implementations: (a) Butterworth, (b) Type I Chebyshev, (c) Type II Chebyshev, and (d) elliptic, of the lowpass filter specified in Problem 7.7 using MATLAB. Plot the frequency characteristics and confirm that the specifications are satisfied by the designed implementations.

7.24 Determine the transfer function of the four implementations: (a) Butterworth, (b) Type I Chebyshev, (c) Type II Chebyshev, and (d) elliptic, of the lowpass filter specified in Problem 7.8 using MATLAB. Plot the frequency characteristics and confirm that the specifications are satisfied by the designed implementations.

7.25 Determine the transfer function of the four implementations: (a) Butterworth, (b) Type I Chebyshev, (c) Type II Chebyshev, and (d) elliptic, of the highpass filter specified in Problem 7.14 using MATLAB. Plot the frequency characteristics and confirm that the specifications are satisfied by the designed implementations.

7.26 Determine the transfer function of the four implementations (a) Butterworth, (b) Type I Chebyshev, (c) Type II Chebyshev, and (d) elliptic, of the bandpass filter specified in Problem 7.17 using MATLAB. Plot the frequency characteristics and confirm that the specifications are satisfied by the designed implementations.

7.27 Determine the transfer function of the four implementations: (a) Butterworth, (b) Type I Chebyshev, (c) Type II Chebyshev, and (d) elliptic, of the bandstop filter specified in Problem 7.20 using MATLAB. Plot the frequency characteristics and confirm that the specifications are satisfied by the designed implementations.