Untyped lambda calculus

terms and values

$$M ::= x \mid M\ M \mid \lambda x.M \qquad \text{terms}$$
$$V ::= \lambda x.M \qquad \text{value}$$

full β-reduction evaluation rules

$$\frac{}{(\lambda x.M)\ N \to_\beta M[x := N]} \qquad \frac{M_1 \to_\beta M_2}{\lambda x.M_1 \to_\beta \lambda x.M_2}$$

$$\frac{M_1 \to_\beta M_2}{M_1\ N \to_\beta M_2\ N} \qquad \frac{N_1 \to_\beta N_2}{M\ N_1 \to_\beta M\ N_2}$$

call-by-value evaluation rules

$$\frac{}{(\lambda x.M)\ V \to_{cbv} M[x := V]} \qquad \frac{M_1 \to_{cbv} M_2}{M_1\ N \to_{cbv} M_2\ N} \qquad \frac{N_1 \to_{cbv} N_2}{V\ N_1 \to_{cbv} V\ N_2}$$

Church numerals and booleans

$$\underline{0} := \lambda f.\lambda x.x$$

$$\underline{1} := \lambda f.\lambda x.f\ x$$

$$\underline{n} := \lambda f.\lambda x.\ \underbrace{f(f \ldots (f}_{\text{n-times}}(x)\ldots) = \lambda f.\lambda x.f^n(x)$$

$$\text{true} := \lambda x.\lambda y.x$$

$$\text{false} := \lambda x.\lambda y.y$$

Simply-typed lambda calculus $\lambda^{\to}$

with Booleans

terms and values

$$M ::= x \mid M\ M \mid \lambda x : \sigma.M \mid \text{true} \mid \text{false} \mid \text{if } M \text{ then } M \text{ else } M \qquad \text{terms}$$
$$V ::= \lambda x : \sigma.M \mid \text{true} \mid \text{false} \qquad \text{values}$$

types

$$\sigma ::= \sigma \to \sigma \mid \text{Bool}$$

typing rules

$$\Gamma \vdash \text{true} : \text{Bool} \ (\text{T-True})$$

$$\Gamma \vdash \text{false} : \text{Bool} \ (\text{T-False})$$

$$\frac{x : \sigma \in \Gamma}{\Gamma \vdash x : \sigma} \ (\text{T-Var})$$

$$\frac{\Gamma, x : \sigma \vdash M : \tau}{\Gamma \vdash \lambda x : \sigma.M \ : \ \sigma \to \tau} \ (\text{T-Abs})$$

$$\frac{\Gamma \vdash M : \sigma \to \tau \qquad \Gamma \vdash N : \sigma}{\Gamma \vdash M\ N : \tau} \ (\text{T-App})$$

$$\frac{\Gamma \vdash M : \text{Bool} \qquad \Gamma \vdash N : \sigma \qquad \Gamma \vdash L : \sigma}{\Gamma \vdash \text{if } M \text{ then } N \text{ else } L : \sigma} \ (\text{T-If})$$

Hindley-Milner type system

terms and values

$$M ::= x \mid M\ M \mid \lambda x.M \mid \textbf{let}\ x = M\ \textbf{in}\ M$$

types

$$\tau ::= \alpha \mid \tau \to \tau \qquad \text{monotypes}$$
$$\sigma ::= \tau \mid \forall \alpha.\sigma \qquad \text{type schemes}$$

typing rules

$$\frac{x : \sigma \in \Gamma}{\Gamma \vdash x : \sigma}\ (\text{T-Var})$$

$$\frac{\Gamma, x : \tau \vdash M : \tau'}{\Gamma \vdash \lambda x.M : \tau \to \tau'}\ (\text{T-Abs})$$

$$\frac{\Gamma \vdash M : \tau \to \tau' \qquad \Gamma \vdash N : \tau}{\Gamma \vdash M\ N : \tau'}\ (\text{T-App})$$

$$\frac{\Gamma \vdash M : \sigma \qquad \Gamma, x : \sigma \vdash N : \tau}{\Gamma \vdash \textbf{let}\ x = M\ \textbf{in}\ N : \tau}\ (\text{T-Let})$$

$$\frac{\Gamma \vdash M : \sigma' \qquad \sigma' \sqsubseteq \sigma}{\Gamma \vdash M : \sigma}\ (\text{T-Inst})$$

$$\frac{\Gamma \vdash M : \sigma \qquad \alpha \notin FV(\Gamma)}{\Gamma \vdash M : \forall \alpha.\sigma}\ (\text{T-Gen})$$

System F

terms and values

$$M ::= x \mid M\ M \mid \lambda x : \sigma.M \mid \Lambda \alpha.M \mid M\ [\sigma] \qquad \text{terms}$$
$$V ::= \lambda x : \sigma.M \mid \Lambda \alpha.M \qquad \text{values}$$

types

$$\sigma ::= \alpha \mid \sigma \to \sigma \mid \forall \alpha.\sigma$$

typing rules *(in addition to $\lambda^{\to}$)*

$$\frac{\Gamma \vdash M : \sigma \qquad \alpha \notin FV(\Gamma)}{\Gamma \vdash \Lambda \alpha.M : \forall \alpha.\sigma}\ (\text{T-TAbs}) \qquad (\text{T-Gen})$$

$$\frac{\Gamma \vdash M : \forall \alpha.\sigma}{\Gamma \vdash M\ [\sigma'] : \{\sigma'/\alpha\}\sigma}\ (\text{T-TApp}) \qquad (\text{T-Inst})$$

evaluation rules *(in addition to $\lambda^{\to}$)*

$$\frac{M \to M'}{M\ [\sigma] \to M'\ [\sigma]}\ (\text{E-TApp})$$

$$(\Lambda \alpha.M)\ [\sigma] \to \{\sigma/\alpha\}M\ (\text{E-TAppTAbs})$$