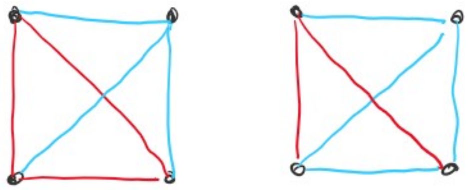
Tutorial 8
22 April 2021 16:03

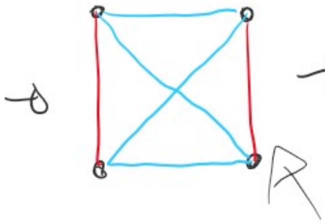
PROBABILISTIC METHOD

Ramsey number

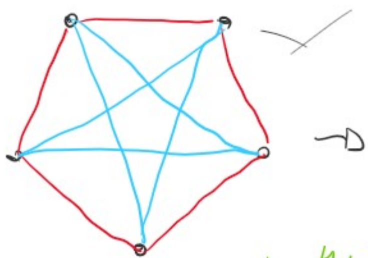
Ramsey number $R(k,t)$ is the smallest n, such that each 2-coloring of edges of K_n (complete graph of n vertices) has a red subgraph K_k or blue subgraph K_t

$\boxed{R(3,3)}$



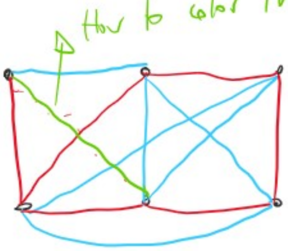


→ no red or blue ⇒ $R(3,3) > 4$



→ no monocolored triangle ⇒ $R(3,3) > 5$

How to color this?



How colorings are there?

$\binom{6}{2}$ - edges $\frac{6!}{2!4!} = \frac{5 \cdot 6}{2} = 15$ edges

$2^{15} \approx 32\,000$ colorings

$R(3,3) = 6$

$2 \approx 32\,000$ colorings

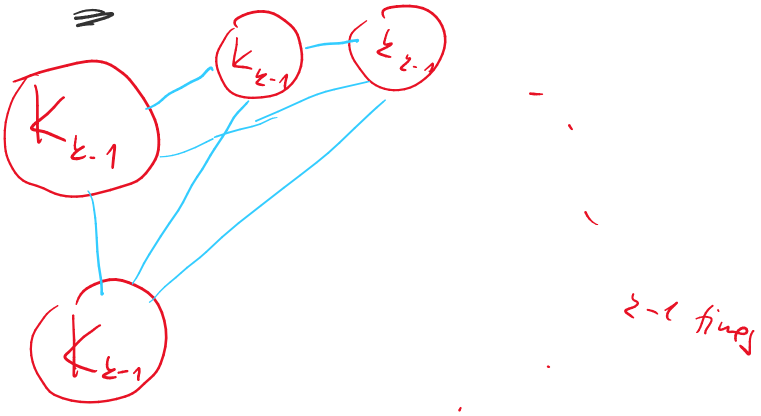
$$\boxed{R(3,3)=6}$$

How does $R(k,k)$ scale with k?

Can you find a lower bound?

$R(k,k)$

$> (k-1)^2$



$$\boxed{R(3,3) > 4}$$

$R(4,4) = 18$

> 9

$43 \leq R(5,5) \leq 49$

$16 <$

PROBABILISTIC ARGUMENT

→ Color each edge at random and if probability of a "counterexample" is larger than 0, the counterexample must exist => Lower bounds on Ramsey number.

Thm (from slides)

$$\binom{n}{k} \cdot 2^{1-\binom{k}{2}} < 1 \implies R(k,k) > n$$

Let us consider the following random coloring experiment:

Color each edge of K_n red w.p. $1/2$

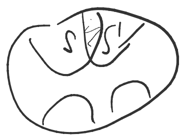
blue w.p. $1/2$

Choose $S \subset V$, $|S| = k$

$r_S = 1$ if graph induced by S is all red

$b_S = 1$ if graph induced by S is all blue

$$\forall_S \quad Pr(r_S = 1) = \left(\frac{1}{2}\right)^{\binom{k}{2}}$$



$\frac{|S|}{All}$

$\frac{|S'|}{All}$

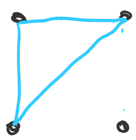
$$\forall_S \quad Pr(b_S = 1) = \left(\frac{1}{2}\right)^{\binom{k}{2}}$$

$$Pr(r_S = 1 \vee b_S = 1) = 2 \cdot \frac{1}{2}^{\binom{k}{2}} = \boxed{2^{1-\binom{k}{2}}}$$

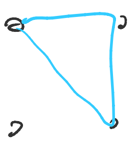
What is the probability of some $S \subset V$ $|S| = k$ to be monocolored?

$$Pr\left(\bigvee_{S \subset V, |S| = k} r_S = 1 \vee b_S = 1\right) < \binom{n}{k} \cdot 2^{1-\binom{k}{2}}$$

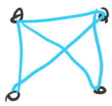
S



$2^3 = 8$ graphs



$2^3 = 8$ graphs



2 graphs where both triangles blue

$$\boxed{1 - Pr\left(\bigvee_{S \subset V, |S| = k} r_S = 1 \vee b_S = 1\right)}$$

is a probability that graph contains no monocolored subgraph of size k, = Counterexample contains no monocolored subgraph of size k, = Counterexample

We need

$1 - P(\bigvee_{\dots} v_S = 1 \vee b_S = 1) > 0$

$\Updownarrow$

$P(\bigvee v_S = 1 \vee b_S = 1) < 1$

$$Pr(\bigvee v_S = 1 \vee b_S = 1) \leq \binom{n}{k} 2^{1-\binom{k}{2}} \leq 1$$

$\boxed{\text{if } n = \lfloor 2^{\frac{k}{2}} \rfloor \text{ then } R(k,k) \geq n}$

Plug $n = \lfloor 2^{\frac{k}{2}} \rfloor$ into our theorem and see if $\binom{n}{k} 2^{1-\binom{k}{2}} < 1$

$$\binom{n}{k} 2^{1-\binom{k}{2}} \leq \frac{n^k}{k!} 2^{1-\frac{k(k-1)}{2}}$$

$\frac{n!}{k!(n-k)!}$ $\quad n(n-1)\cdot(n-2)\cdots(n-k+1)$

$$< \frac{(2^{\frac{k}{2}})^k}{k!} \cdot \frac{2}{2^{\frac{k(k-1)}{2}}}$$

$$= \frac{2^{\frac{k^2}{2}}}{k!} \cdot \frac{2}{2^{\frac{k^2}{2}} \cdot 2^{-\frac{k}{2}}}$$

$$= \frac{2 \cdot 2^{\frac{k}{2}}}{k!} = \frac{2^{1+\frac{k}{2}}}{k!}$$

$k = 3 \qquad \frac{2^{1+\frac{3}{2}}}{6} = \frac{2^{3/2}}{3} = \frac{\sqrt{8}}{3} \approx 0.9$

$$6 \qquad 3 \qquad 3$$

$$k=4 \qquad \frac{2^{1+2}}{4!} = \frac{8}{24} = 1/3$$

$$R(3,3) \geq \lfloor 2^{\frac{3}{2}} \rfloor = \lfloor \sqrt{8} \rfloor = 2 \qquad R(k,k) = (k-1)^2$$

$$R(4,4) \geq \lfloor 2^{\frac{4}{2}} \rfloor = 4$$

$$R(8,8) \geq \lfloor 2^4 \rfloor = 16 \qquad (k-1)^2 < 2^{\frac{k}{2}}$$

$$R(8,8) \underset{\text{constructive}}{\geq} 7^2 = 49 \qquad k^2 - 2k + 1 < 2^{\frac{k}{2}}$$

$$\vdots$$

$$k = 16 \qquad 2^{\frac{k}{2}} > (k-1)^2$$

$$256 > 225$$

Thm

if $\binom{n}{k} p^{\binom{k}{2}} + \binom{n}{t} (1-p)^{\binom{t}{2}} < 1$ for some $0 \leq p \leq 1$

then $R(k,t) > n$

for ($p = 1/2$ and $k=t$ we recover above theorem)

$$\binom{n}{k} \cdot 2^{1-\binom{k}{2}} < 1 \Rightarrow R(k,k) > n$$

$$R(4,t)$$

$$\binom{n}{4} \cdot p^6 + \binom{n}{t} (1-p)^{\binom{t}{2}} < 1$$

$$\boxed{p = \left(n^{-2/3}\right)}$$

$$\frac{\cancel{n^4}}{4!} \cdot \cancel{n^{-4}} + \binom{n}{t}\left(1-n^{-2/3}\right)^{\binom{t}{2}} < 1$$

$$= \frac{1}{24} \qquad \binom{n}{t}\left(1-n^{-2/3}\right)^{\binom{t}{2}} < \frac{23}{24}$$