IA008: Computational Logic

3. Prolog

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Prolog

Prolog

Syntax

A **Prolog** program consists of a sequence of statements of the form

$$p(\bar{s}). \quad \text{or} \quad p(\bar{s}) :- q_0(\bar{t}_0), \dots, q_{n-1}(\bar{t}_{n-1}).$$

p, q_i relation symbols, $\bar{s}, \bar{t}_i$ tuples of terms.

Semantics

$$p(\bar{s}) :- q_0(\bar{t}_0), \dots, q_{n-1}(\bar{t}_{n-1}).$$

corresponds to the implication

$$\forall \bar{x}\big[p(\bar{s}) \leftarrow q_0(\bar{t}_0) \wedge \cdots \wedge q_{n-1}(\bar{t}_{n-1})\big]$$

where $\bar{x}$ are the variables appearing in the formula.

Example

$$\text{father\_of}(\text{peter}, \text{sam}).$$
$$\text{father\_of}(\text{peter}, \text{tina}).$$
$$\text{mother\_of}(\text{sara}, \text{john}).$$

$$\text{parent\_of}(X, Y) :- \text{father\_of}(X, Y).$$
$$\text{parent\_of}(X, Y) :- \text{mother\_of}(X, Y).$$

$$\text{sibling\_of}(X, Y) :- \text{parent\_of}(Z, X), \text{parent\_of}(Z, Y).$$

$$\text{ancestor\_of}(X, Y) :- \text{father\_of}(X, Z), \text{ancestor\_of}(Z, Y).$$

Interpreter

On input

$$p_0(\bar{s}_0), \ldots, p_{n-1}(\bar{s}_{n-1}).$$

the program finds all values for the variables satisfying the conjunction

$$p_0(\bar{s}_0) \wedge \cdots \wedge p_{n-1}(\bar{s}_{n-1}) .$$

Example

```
?- sibling_of(sam, tina).
Yes

?- sibling_of(X, Y).
X = sam, Y = tina
```

Execution

Input

- program Π (set of Horn formulae
  $\forall \bar{x}[P(\bar{s}) \leftarrow Q_0(\bar{t}_0) \wedge \cdots \wedge Q_{n-1}(\bar{t}_{n-1})]$)
- goal $\varphi(\bar{x}) := R_0(\bar{u}_0) \wedge \cdots \wedge R_{m-1}(\bar{u}_{m-1})$

Evaluation strategy

Use resolution to check for which values of $\bar{x}$ the union $\Pi \cup \{\neg\varphi(\bar{x})\}$ is unsatisfiable.

Remark

As we are dealing with a set of Horn formulae, we can use **linear resolution.** The variant used by Prolog-interpreters is called **SLD-resolution.**

SLD-resolution

- Current goal: $\neg\psi_0 \vee \cdots \vee \neg\psi_{n-1}$
- If $n = 0$, stop.
- Otherwise, find a formula $\psi \leftarrow \vartheta_0 \wedge \cdots \wedge \vartheta_{m-1}$ from Π such that ψ_0 and ψ can be unified.
- If no such formula exists, backtrack.
- Otherwise, resolve them to produce the new goal

$$\tau(\neg\vartheta_0) \vee \cdots \vee \tau(\neg\vartheta_{m-1}) \vee \sigma(\neg\psi_1) \vee \cdots \vee \sigma(\neg\psi_{n-1}) .$$

  (σ, τ is the most general unifier of ψ_0 and ψ.)

Implementation

Use a stack machine that keeps the current goal on the stack.
(→ **Warren Abstract Machine**)

Substitution

Definition

A **substitution** σ is a function that replaces in a formula every free variable by a term (and renames bound variables if necessary).
Instead of $\sigma(\varphi)$ we also write $\varphi[x \mapsto s, y \mapsto t]$ if $\sigma(x) = s$ and $\sigma(y) = t$.

Examples

$$(x = f(y))[x \mapsto g(x),\ y \mapsto c] \quad = \quad g(x) = f(c)$$

$$\exists z(x = z + z)[x \mapsto z] \quad = \quad \exists u(z = u + u)$$

Unification

Definition

A **unifier** of two terms $s(\bar{x})$ and $t(\bar{x})$ is a pair of substitution σ, τ such that $\sigma(s) = \tau(t)$.
A unifier σ, τ is **most general** if every other unifier σ', τ' can be written as $\sigma' = \rho \circ \sigma$ and $\tau' = \upsilon \circ \tau$, for some ρ, υ.

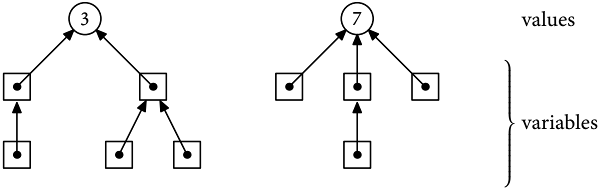
Examples

$s = f(x, g(x))$	$t = f(c, x)$	$x \mapsto c$	$x \mapsto g(c)$
$s = f(x, g(x))$	$t = f(x, y)$	$x \mapsto x$	$x \mapsto x$
			$y \mapsto g(x)$
		$x \mapsto g(x)$	$x \mapsto g(x)$
			$y \mapsto g(g(x))$
$s = f(x)$	$t = g(x)$	unification not possible	

Unification Algorithm

unify(s, t)
  if s is a variable x **then**
    set x to t
  else if t is a variable x **then**
    set x to s
  else $s = f(\bar{u})$ and $t = g(\bar{v})$
    if $f = g$ **then**
      forall i unify(u_i, v_i)
    else
      fail

Union-Find-Algorithm



find : *variable* → *value*

- follows pointers to the root and creates shortcuts

union : (*variable* × *variable*) → *unit*

- links roots by a pointer

Example

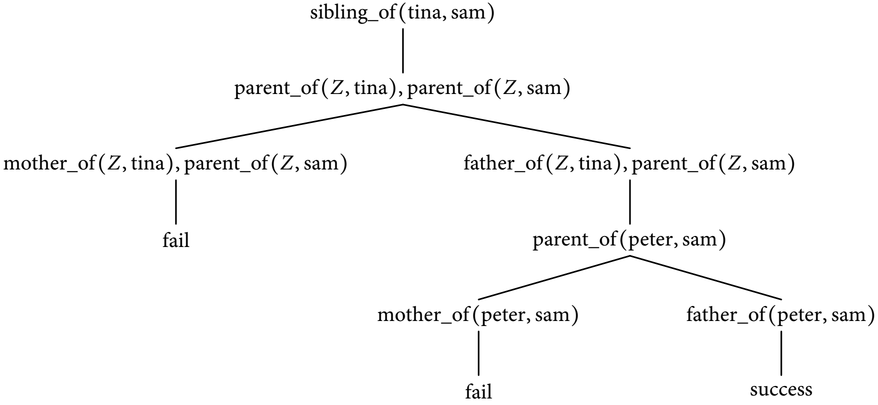
```
father_of(peter, sam).
father_of(peter, tina).
mother_of(sara, john).
parent_of(X, Y) :- mother_of(X, Y).
parent_of(X, Y) :- father_of(X, Y).
sibling_of(X, Y) :- parent_of(Z, X), parent_of(Z, Y).
```

Input $\text{sibling\_of}(\text{tina}, \text{sam})$

goal	$\neg\text{sibling\_of}(\text{tina}, \text{sam})$
unify with	$\text{sibling\_of}(X, Y) \leftarrow \text{parent\_of}(Z, X) \wedge \text{parent\_of}(Z, Y)$
unifier	$X = \text{tina},\ Y = \text{sam}$
new goal	$\neg\text{parent\_of}(Z, \text{tina}),\ \neg\text{parent\_of}(Z, \text{sam})$

goal	$\neg\text{parent\_of}(Z, \text{tina}),\ \neg\text{parent\_of}(Z, \text{sam})$
unify with	$\text{parent\_of}(X, Y) \leftarrow \text{mother\_of}(X, Y)$
unifier	$X = Z,\ Y = \text{tina}$

Search tree


sibling_of(tina, sam)
parent_of(Z, tina), parent_of(Z, sam)
mother_of(Z, tina), parent_of(Z, sam)
father_of(Z, tina), parent_of(Z, sam)
fail
parent_of(peter, sam)
mother_of(peter, sam)
father_of(peter, sam)
fail
success

Caveats

Prolog-interpreters use a simpler (and **unsound**) form of unification that ignores multiple occurrences of variables. E.g. they happily unify $p(x, f(x))$ with $p(f(y), f(y))$ (equating $x = f(y)$ for the first x and $x = y$ for the second one).

It is also easy to get infinite loops if you are not careful with the ordering of the rules:

```
edge(c,d).
path(X,Y) :- path(X,Z),edge(Z,Y).
path(X,Y) :- edge(X,Y).
```

produces

```
?- path(X,Y).
path(X,Z), edge(Z,Y).
path(X,U), edge(U,Z), edge(Z,Y).
path(X,V), edge(V,U), edge(U,Z), edge(Z,Y).
...
```

Example: List processing

```
append([], L, L).
append([H|T], L, [H|R]) :- append(T, L, R).

?- append([a,b], [c,d], X).
X = [a,b,c,d]

?- append(X, Y, [a,b,c,d])
X = [], Y = [a,b,c,d]
X = [a], Y = [b,c,d]
X = [a,b], Y = [c,d]
X = [a,b,c], Y = [d]
X = [a,b,c,d], Y = []
```

Example: List processing

```
reverse(Xs, Ys) :- reverse_(Xs, [], Ys).

reverse_([], Ys, Ys).
reverse_([X|Xs], Rs, Ys) :- reverse_(Xs, [X|Rs], Ys).

reverse([a,b,c], X)
reverse_([a,b,c], [], X)
reverse_([b,c], [a], X)
reverse_([c], [b,a], X)
reverse_([], [c,b,a], X)
X = [c,b,a]
```

Example: Natural language recognition

```
sentence(X,R) :- noun(X, Y), verb(Y, R).
sentence(X,R) :- noun(X, Y), verb(Y, Z), noun(Z, R).

noun_phrase(X,           R) :- noun(X, R).
noun_phrase(['a'   | X], R) :- noun(X, R).
noun_phrase(['the' | X], R) :- noun(X, R).

noun(['cat'   | R], R).
noun(['mouse' | R], R).
noun(['dog'   | R], R).

verb(['eats'  | R], R).
verb(['hunts' | R], R).
verb(['plays' | R], R).
```

Cuts

Control backtracking using **cuts:**

$$p :- q_0, q_1, !, q_2, q_3.$$

When backtracking across a cut $!$, directly jump to the head of the rule and assume it fails. Do not try other rules.

Example

$s \leftarrow p$

$s \leftarrow t$

$p \leftarrow q_1, q_2, !, q_3, q_4$

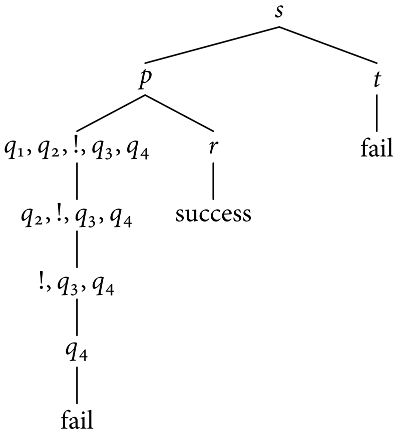
$p \leftarrow r$

r

q_1

q_2

q_3



Negation

Problem

If we allow **negation,** the formulae are no longer **Horn** and SLD-resolution does no longer work.

Possible Solutions

- **Closed World Assumption**
  If we cannot derive p, it is false (**Negation as Failure**).
- **Completed Database**
  $p \leftarrow q_0, \ldots, p \leftarrow q_n$ is interpreted as the stronger statement $p \leftrightarrow q_0 \vee \cdots \vee q_n$.

Examples

Being connected by a path of non-edges:

```
q(X,X).
q(X,Y) :- q(X,Z), not(p(Z,Y)).
```

Implementing negation using cuts:

```
not(A) :- A, !, fail.
not(A).
```

Some cuts can be implemented using negation:

```
p :- a, !, b.          p :- a, b.
p :- c.                p :- not(a), c.
```

Nonmonotonic Logic

Negation as Failure

Goal

Develop a proof calculus supporting Negation as Failure as used in Prolog.

Monotonicity

Ordinary deduction is **monotone:** if we add new assumption, all consequences we have already derived remain. More information does not invalidate already made deductions.

Non-Monotonicity

Negation as Failure is **non-monotone:**

$$P \text{ implies } \neg Q \quad \text{but} \quad P, Q \text{ does not imply } \neg Q\,.$$

Default Logic

Rule

$$\frac{\alpha_0 \ \dots \ \alpha_m : \beta_0 \ \dots \ \beta_n}{\gamma}$$

α_i	assumptions
β_i	restraints
γ	consequence

Derive γ provided that we can derive $\alpha_0, \dots, \alpha_m$, but none of $\beta_0, \dots, \beta_n$.

Example

$$\frac{\mathrm{bird}(x) : \mathrm{penguin}(x)\ \mathrm{ostrich}(x)}{\mathrm{can\_fly}(x)}$$

Semantics

Definition

A set Φ of formulae is **consistent** with respect to a set of rules R if, for every rule

$$\frac{\alpha_0 \ \dots \ \alpha_m : \beta_0 \ \dots \ \beta_n}{\gamma} \in R$$

such that $\alpha_0, \dots, \alpha_m \in \Phi$ and $\beta_0, \dots, \beta_n \notin \Phi$, we have $\gamma \in \Phi$.

Note

If there are no restraints β_i, consistent sets are **closed under intersection.**
$\Rightarrow$ There is a unique smallest such set, that of all **provable** formulae.

If there are restraints, this may not be the case. Formulae that belong to all consistent sets are called **secured consequences.**

Examples

The system

$$\frac{}{\alpha} \qquad \frac{\alpha : \beta}{\beta}$$

has a unique consistent set $\{\alpha, \beta\}$.

The system

$$\frac{}{\alpha} \qquad \frac{\alpha : \beta}{\gamma} \qquad \frac{\alpha : \gamma}{\beta}$$

has consistent sets

$$\{\alpha, \beta\}, \quad \{\alpha, \gamma\}, \quad \{\alpha, \beta, \gamma\}.$$

Databases

Databases

Definition

A **database** is a set of relations called **tables.**

Example

flight	from	to	price
LH8302	Prague	Frankfurt	240
OA1472	Vienna	Warsaw	300
UA0870	London	Washington	800
…			

Formal Definitions

We treat a database as a structure $\mathfrak{A} = \langle A, R_0, \ldots, R_n \rangle$ with

- universe A containing all entries and
- one relation $R_i \subseteq A \times \cdots \times A$ per table.

The **active domain** of a database is the set of elements appearing in some relation.

Example

In the previous table, the active domain contains:

LH8302, OA1472, UA0870, 240, 300, 800,
Prague, Frankfurt, Vienna, Warsaw, London, Washington

Queries

A **query** is a function mapping each database to a relation.

Example

Input: database of direct flights
Output: table of all flight connections possibly including stops

In Prolog: database **`flight`**, query **`connection`**.

```
flight('LH8302', 'Prague', 'Frankfurt',  240).
flight('OA1472', 'Vienna', 'Warsaw',     300).
flight('UA0870', 'London', 'Washington', 800).

connection(From, To) :- flight(X, From, To, Y).
connection(From, To) :-
  flight(X, From, T, Y), connection(T, To).
```

Relational Algebra

Syntax

- basic relations
- boolean operations $\cap$, $\cup$, $\setminus$, All
- cartesian product $\times$
- selection σ_{ij}
- projection $\pi_{u_0 \dots u_{n-1}}$

Examples

- $\pi_{1,0}(R) = \{ (b, a) \mid (a, b) \in R \}$
- $\pi_{0,3}(\sigma_{1,2}(E \times E)) = \{ (a, c) \mid (a, b), (b, c) \in E \}$

Join

$$R \bowtie_{ij} S := \sigma_{ij}(R \times S)$$

Expressive Power

Theorem

Relational Algebra = First-Order Logic

Proof

$(\leq)$ $s \mapsto s^*$ such that $s = \{ \bar{a} \mid \mathfrak{A} \vDash s^*(\bar{a}) \}$

$$\begin{aligned}
R^* &:= R(x_0, \dots, x_{n-1}) \\
(s \cap t)^* &:= s^* \wedge t^* \\
(s \cup t)^* &:= s^* \vee t^* \\
(s \setminus t)^* &:= s^* \wedge \neg t^* \\
\mathrm{All}^* &:= \mathrm{true} \\
(s \times t)^* &:= s^*(x_0, \dots, x_{m-1}) \wedge t^*(x_m, \dots, x_{m+n-1}) \\
\sigma_{ij}(s)^* &:= s^* \wedge x_i = x_j \\
\pi_{u_0, \dots, u_{n-1}}(s)^* &:= \exists \bar{y} \Big[s^*(\bar{y}) \wedge \bigwedge_{i<n} x_i = y_{u_i} \Big]
\end{aligned}$$

Expressive Power

Theorem

Relational Algebra = First-Order Logic

Proof

$(\geq)$ $\varphi \mapsto \varphi^*$ such that $\varphi^* = \{ \bar{a} \mid \mathfrak{A} \vDash \varphi(\bar{a}) \}$

$$\begin{aligned}
R(x_{u_0}, \dots, x_{u_{n-1}})^* &:= \pi_{0,\dots,m-1}(\sigma_{u_0,m+0} \cdots \sigma_{u_{n-1},m+n-1} \\
&\qquad\qquad (\text{All} \times \cdots \times \text{All} \times R)) \\
(x_i = x_j)^* &:= \sigma_{ij}(\text{All} \times \cdots \times \text{All}) \\
(\varphi \wedge \psi)^* &:= \varphi^* \cap \psi^* \\
(\varphi \vee \psi)^* &:= \varphi^* \cup \psi^* \\
(\neg\varphi)^* &:= \text{All} \times \cdots \times \text{All} \smallsetminus \varphi^* \\
(\exists x_i \varphi)^* &:= \pi_{0,\dots,i-1,n,i+1,\dots,n-1}(\varphi^* \times \text{All})
\end{aligned}$$

Datalog

Simplified version of Prolog developped in database theory:

- no function symbols,
- no cut, no negation, etc.

A **datalog program** for a database $\mathcal{A} = \langle A, R_0, \dots, R_n \rangle$ is a set of Horn formulae

$$\begin{aligned} p_0(\bar{X}) &\leftarrow q_{0,0}(\bar{X}, \bar{Y}) \wedge \cdots \wedge q_{0,m_0}(\bar{X}, \bar{Y}) \\ &\vdots \\ p_n(\bar{X}) &\leftarrow q_{n,0}(\bar{X}, \bar{Y}) \wedge \cdots \wedge q_{n,m_n}(\bar{X}, \bar{Y}) \end{aligned}$$

where $p_0, \dots, p_n$ are **new** relation symbols and the q_{ij} are either relation symbols from $\mathcal{A}$, possibly negated, or one of the new symbols p_k (not negated).

Datalog queries

The **query** defined by a datalog program

$$\begin{aligned} p_0(\bar{X}) &\leftarrow q_{0,0}(\bar{X},\bar{Y}) \wedge \cdots \wedge q_{0,m_0}(\bar{X},\bar{Y}) \\ &\vdots \\ p_n(\bar{X}) &\leftarrow q_{n,0}(\bar{X},\bar{Y}) \wedge \cdots \wedge q_{n,m_n}(\bar{X},\bar{Y}) \end{aligned}$$

maps a database $\mathcal{A}$ to the relations $p_0, \ldots, p_n$ defined by these formulae.

Evaluation strategy

- Start with empty relations $p_0 = \varnothing, \ldots, p_n = \varnothing$.
- Apply each rule to add new tuples to the relations.
- Repeat until no new tuples are generated.

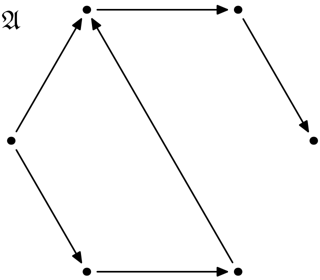
Note

The relations computed in this way satisfy the **Completed Database** assumption.

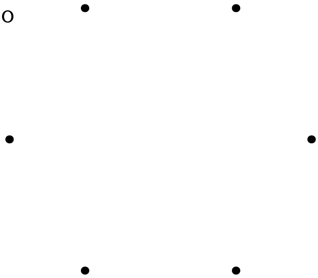
Example

$$\text{path}(X, Y) \leftarrow \text{edge}(X, Y)$$
$$\text{path}(X, Y) \leftarrow \text{path}(X, Z) \wedge \text{path}(Z, Y)$$

$\mathfrak{A}$



stage 0



Example

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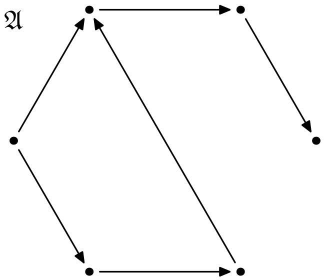

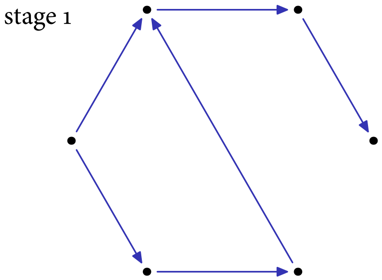

Example

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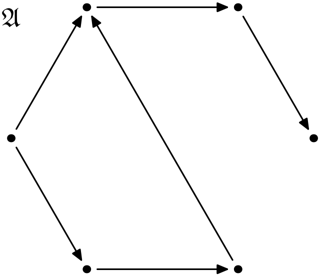
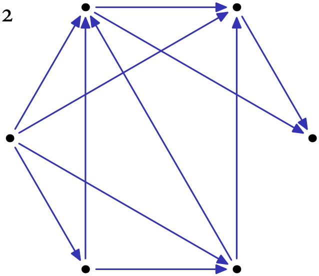


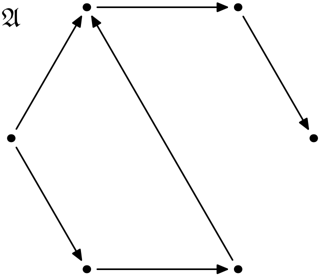
stage 2



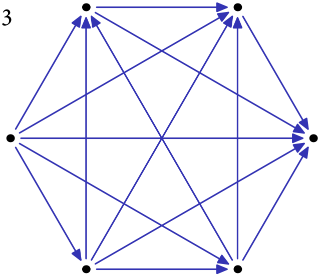
Example

$$\text{path}(X,Y) \leftarrow \text{edge}(X,Y)$$
$$\text{path}(X,Y) \leftarrow \text{path}(X,Z) \wedge \text{path}(Z,Y)$$

$\mathfrak{A}$



stage 3



Example: Arithmetic

$$\mathrm{Add}(x,y,z) \leftarrow y = 0 \land x = z$$
$$\mathrm{Add}(x,y,z) \leftarrow E(y',y) \land E(z',z) \land \mathrm{Add}(x,y',z')$$
$$\mathrm{Mul}(x,y,z) \leftarrow y = 0 \land z = 0$$
$$\mathrm{Mul}(x,y,z) \leftarrow E(y',y) \land \mathrm{Add}(x,z',z) \land \mathrm{Mul}(x,y',z')$$

stage 0	$\varnothing$
stage 1	$(k,0,k)$
stage 2	$(k,0,k),(k,1,k+1)$
stage 3	$(k,0,k),(k,1,k+1),(k,2,k+2)$
	$\ldots$
stage n	$(k,0,k),(k,1,k+1),\ldots,(k,n-1,k+n-1)$
	$\ldots$

Complexity

Theorem

For databases $\mathfrak{A} = \langle A, \bar{R}, \leq \rangle$ equipped with a linear order $\leq$, a query Q can be expressed as a Datalog program if, and only if, it can be evaluated in **polynomial type.**