# IA008: Computational Logic

# 3. Prolog

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# Prolog

# Prolog

## Syntax

A **Prolog** program consists of a sequence of statements of the form

$$p(\bar{s}). \quad \text{or} \quad p(\bar{s}) :- q_0(\bar{t}_0), \ldots, q_{n-1}(\bar{t}_{n-1}).$$

$p, q_i$ relation symbols, $\bar{s}, \bar{t}_i$ tuples of terms.

## Semantics

$$p(\bar{s}) :- q_0(\bar{t}_0), \ldots, q_{n-1}(\bar{t}_{n-1}).$$

corresponds to the implication

$$\forall \bar{x}\big[p(\bar{s}) \leftarrow q_0(\bar{t}_0) \wedge \cdots \wedge q_{n-1}(\bar{t}_{n-1})\big]$$

where $\bar{x}$ are the variables appearing in the formula.

# Example

father_of(peter, sam).

father_of(peter, tina).

mother_of(sara, john).

parent_of($X, Y$) :– father_of($X, Y$).

parent_of($X, Y$) :– mother_of($X, Y$).

sibling_of($X, Y$) :– parent_of($Z, X$), parent_of($Z, Y$).

ancestor_of($X, Y$) :– father_of($X, Z$), ancestor_of($Z, Y$).

# Interpreter

On input

$$p_0(\bar{s}_0), \ldots, p_{n-1}(\bar{s}_{n-1}).$$

the program finds all values for the variables satisfying the conjunction

$$p_0(\bar{s}_0) \wedge \cdots \wedge p_{n-1}(\bar{s}_{n-1}) .$$

## Example

```
?- sibling_of(sam, tina).
Yes

?- sibling_of(X, Y).
X = sam, Y = tina
```

# Execution

### Input

- program $\Pi$ (set of Horn formulae
  $\forall \bar{x}[P(\bar{s}) \leftarrow Q_0(\bar{t}_0) \wedge \cdots \wedge Q_{n-1}(\bar{t}_{n-1})]$)
- goal $\varphi(\bar{x}) := R_0(\bar{u}_0) \wedge \cdots \wedge R_{m-1}(\bar{u}_{m-1})$

### Evaluation strategy

Use resolution to check for which values of $\bar{x}$ the union $\Pi \cup \{\neg\varphi(\bar{x})\}$ is unsatisfiable.

### Remark

As we are dealing with a set of Horn formulae, we can use **linear resolution.** The variant used by Prolog-interpreters is called **SLD-resolution.**

# SLD-resolution

- Current goal: $\neg\psi_0 \lor \cdots \lor \neg\psi_{n-1}$
- If $n = 0$, stop.
- Otherwise, find a formula $\psi \leftarrow \vartheta_0 \land \cdots \land \vartheta_{m-1}$ from $\Pi$ such that $\psi_0$ and $\psi$ can be unified.
- If no such formula exists, backtrack.
- Otherwise, resolve them to produce the new goal

$$\tau(\neg\vartheta_0) \lor \cdots \lor \tau(\neg\vartheta_{m-1}) \lor \sigma(\neg\psi_1) \lor \cdots \lor \sigma(\neg\psi_{n-1})\,.$$

($\sigma, \tau$ is the most general unifier of $\psi_0$ and $\psi$.)

## Implementation

Use a stack machine that keeps the current goal on the stack.
(→ **Warren Abstract Machine**)

# Substitution

## Definition

A **substitution** $\sigma$ is a function that replaces in a formula every free variable by a term (and renames bound variables if necessary).
Instead of $\sigma(\varphi)$ we also write $\varphi[x \mapsto s, y \mapsto t]$ if $\sigma(x) = s$ and $\sigma(y) = t$.

## Examples

$$(x = f(y))[x \mapsto g(x),\ y \mapsto c] \quad = \quad g(x) = f(c)$$

$$\exists z(x = z + z)[x \mapsto z] \quad = \quad \exists u(z = u + u)$$

# Unification

## Definition

A **unifier** of two terms $s(\bar{x})$ and $t(\bar{x})$ is a pair of substitution $\sigma, \tau$ such that $\sigma(s) = \tau(t)$.
A unifier $\sigma, \tau$ is **most general** if every other unifier $\sigma', \tau'$ can be written as $\sigma' = \rho \circ \sigma$ and $\tau' = \upsilon \circ \tau$, for some $\rho, \upsilon$.

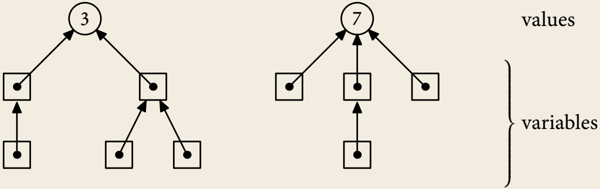
## Examples

| | | | |
|---|---|---|---|
| $s = f(x, g(x))$ | $t = f(c, x)$ | $x \mapsto c$ | $x \mapsto g(c)$ |
| $s = f(x, g(x))$ | $t = f(x, y)$ | $x \mapsto x$ | $x \mapsto x$ |
| | | | $y \mapsto g(x)$ |
| | | $x \mapsto g(x)$ | $x \mapsto g(x)$ |
| | | | $y \mapsto g(g(x))$ |
| $s = f(x)$ | $t = g(x)$ | unification not possible | |

# Unification Algorithm

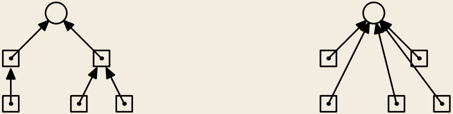
```
unify(s, t)
    if s is a variable x then
        set x to t
    else if t is a variable x then
        set x to s
    else s = f(ū) and t = g(v̄)
        if f = g then
            forall i    unify(u_i, v_i)
        else
            fail
```

# Union-Find-Algorithm



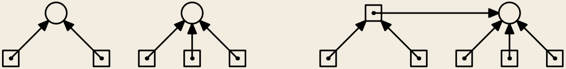

find : *variable* → *value*

- follows pointers to the root and creates shortcuts



union : (*variable* × *variable*) → *unit*

- links roots by a pointer



# Example

```
father_of(peter, sam).
father_of(peter, tina).
mother_of(sara, john).
parent_of(X, Y) :- mother_of(X, Y).
parent_of(X, Y) :- father_of(X, Y).
sibling_of(X, Y) :- parent_of(Z, X), parent_of(Z, Y).
```

**Input** sibling_of(tina, sam)

goal $\neg$sibling_of(tina, sam)

# Example

```
father_of(peter, sam).
father_of(peter, tina).
mother_of(sara, john).
parent_of(X, Y) :- mother_of(X, Y).
parent_of(X, Y) :- father_of(X, Y).
sibling_of(X, Y) :- parent_of(Z, X), parent_of(Z, Y).
```

**Input** sibling_of(tina, sam)

goal $\neg$sibling_of(tina, sam)

unify with sibling_of($X$, $Y$) $\leftarrow$ parent_of($Z$, $X$) $\wedge$ parent_of($Z$, $Y$)

# Example

```
father_of(peter, sam).
father_of(peter, tina).
mother_of(sara, john).
parent_of(X, Y) :- mother_of(X, Y).
parent_of(X, Y) :- father_of(X, Y).
sibling_of(X, Y) :- parent_of(Z, X), parent_of(Z, Y).
```

**Input** $\text{sibling\_of}(\text{tina}, \text{sam})$

| | |
|---|---|
| goal | $\neg\text{sibling\_of}(\text{tina}, \text{sam})$ |
| unify with | $\text{sibling\_of}(X, Y) \leftarrow \text{parent\_of}(Z, X) \wedge \text{parent\_of}(Z, Y)$ |
| unifier | $X = \text{tina},\ Y = \text{sam}$ |

# Example

```
father_of(peter, sam).
father_of(peter, tina).
mother_of(sara, john).
parent_of(X, Y) :- mother_of(X, Y).
parent_of(X, Y) :- father_of(X, Y).
sibling_of(X, Y) :- parent_of(Z, X), parent_of(Z, Y).
```

**Input** sibling_of(tina, sam)

| | |
|---|---|
| goal | $\neg$sibling_of(tina, sam) |
| unify with | sibling_of($X$, $Y$) $\leftarrow$ parent_of($Z$, $X$) $\wedge$ parent_of($Z$, $Y$) |
| unifier | $X$ = tina, $Y$ = sam |
| new goal | $\neg$parent_of($Z$, tina), $\neg$parent_of($Z$, sam) |

# Example

```
father_of(peter, sam).
father_of(peter, tina).
mother_of(sara, john).
parent_of(X, Y) :- mother_of(X, Y).
parent_of(X, Y) :- father_of(X, Y).
sibling_of(X, Y) :- parent_of(Z, X), parent_of(Z, Y).
```

**Input** $\text{sibling\_of}(\text{tina}, \text{sam})$

goal $\neg\text{parent\_of}(Z, \text{tina}),\ \neg\text{parent\_of}(Z, \text{sam})$

# Example

```
father_of(peter, sam).
father_of(peter, tina).
mother_of(sara, john).
parent_of(X, Y) :- mother_of(X, Y).
parent_of(X, Y) :- father_of(X, Y).
sibling_of(X, Y) :- parent_of(Z, X), parent_of(Z, Y).
```

**Input** $\text{sibling\_of}(\text{tina}, \text{sam})$

goal $\neg\text{parent\_of}(Z, \text{tina}),\ \neg\text{parent\_of}(Z, \text{sam})$

unify with $\text{parent\_of}(X, Y) \leftarrow \text{mother\_of}(X, Y)$

# Example

```
father_of(peter, sam).
father_of(peter, tina).
mother_of(sara, john).
parent_of(X, Y) :- mother_of(X, Y).
parent_of(X, Y) :- father_of(X, Y).
sibling_of(X, Y) :- parent_of(Z, X), parent_of(Z, Y).
```

**Input** sibling_of(tina, sam)

goal $\neg \text{parent\_of}(Z, \text{tina}),\ \neg \text{parent\_of}(Z, \text{sam})$

unify with $\text{parent\_of}(X, Y) \leftarrow \text{mother\_of}(X, Y)$

unifier $X = Z,\ Y = \text{tina}$

# Example

```
father_of(peter, sam).
father_of(peter, tina).
mother_of(sara, john).
parent_of(X, Y) :- mother_of(X, Y).
parent_of(X, Y) :- father_of(X, Y).
sibling_of(X, Y) :- parent_of(Z, X), parent_of(Z, Y).
```

**Input** $\text{sibling\_of}(\text{tina}, \text{sam})$

goal $\neg\text{parent\_of}(Z, \text{tina}),\ \neg\text{parent\_of}(Z, \text{sam})$

unify with $\text{parent\_of}(X, Y) \leftarrow \text{mother\_of}(X, Y)$

unifier $X = Z,\ Y = \text{tina}$

new goal $\neg\text{mother\_of}(Z, \text{tina}),\ \neg\text{parent\_of}(Z, \text{sam})$

# Example

```
father_of(peter, sam).
father_of(peter, tina).
mother_of(sara, john).
parent_of(X, Y) :- mother_of(X, Y).
parent_of(X, Y) :- father_of(X, Y).
sibling_of(X, Y) :- parent_of(Z, X), parent_of(Z, Y).
```

**Input** sibling_of(tina, sam)

goal $\neg$mother_of($Z$, tina), $\neg$parent_of($Z$, sam)

# Example

```
father_of(peter, sam).
father_of(peter, tina).
mother_of(sara, john).
parent_of(X, Y) :- mother_of(X, Y).
parent_of(X, Y) :- father_of(X, Y).
sibling_of(X, Y) :- parent_of(Z, X), parent_of(Z, Y).
```

**Input** sibling_of(tina, sam)

goal $\neg$mother_of($Z$, tina), $\neg$parent_of($Z$, sam)

unify with mother_of(sara, john)

# Example

```
father_of(peter, sam).
father_of(peter, tina).
mother_of(sara, john).
parent_of(X, Y) :- mother_of(X, Y).
parent_of(X, Y) :- father_of(X, Y).
sibling_of(X, Y) :- parent_of(Z, X), parent_of(Z, Y).
```

**Input** sibling_of(tina, sam)

goal $\neg$mother_of($Z$, tina), $\neg$parent_of($Z$, sam)

unify with mother_of(sara, john)

fails

# Example

```
father_of(peter, sam).
father_of(peter, tina).
mother_of(sara, john).
parent_of(X, Y) :- mother_of(X, Y).
parent_of(X, Y) :- father_of(X, Y).
sibling_of(X, Y) :- parent_of(Z, X), parent_of(Z, Y).
```

**Input** sibling_of(tina, sam)

| | |
|---|---|
| goal | $\neg$mother_of($Z$, tina), $\neg$parent_of($Z$, sam) |
| unify with | mother_of(sara, john) |
| fails | |
| backtrack to | $\neg$parent_of($Z$, tina), $\neg$parent_of($Z$, sam) |

# Example

```
father_of(peter, sam).
father_of(peter, tina).
mother_of(sara, john).
parent_of(X, Y) :- mother_of(X, Y).
parent_of(X, Y) :- father_of(X, Y).
sibling_of(X, Y) :- parent_of(Z, X), parent_of(Z, Y).
```

**Input** sibling_of(tina, sam)

goal $\neg \text{parent\_of}(Z, \text{tina}),\ \neg \text{parent\_of}(Z, \text{sam})$

# Example

```
father_of(peter, sam).
father_of(peter, tina).
mother_of(sara, john).
parent_of(X, Y) :- mother_of(X, Y).
parent_of(X, Y) :- father_of(X, Y).
sibling_of(X, Y) :- parent_of(Z, X), parent_of(Z, Y).
```

**Input** sibling_of(tina, sam)

goal $\neg\text{parent\_of}(Z, \text{tina}),\ \neg\text{parent\_of}(Z, \text{sam})$

unify with $\text{parent\_of}(X, Y) \leftarrow \text{father\_of}(X, Y)$

# Example

```
father_of(peter, sam).
father_of(peter, tina).
mother_of(sara, john).
parent_of(X, Y) :- mother_of(X, Y).
parent_of(X, Y) :- father_of(X, Y).
sibling_of(X, Y) :- parent_of(Z, X), parent_of(Z, Y).
```

**Input** sibling_of(tina, sam)

goal $\neg$parent_of($Z$, tina), $\neg$parent_of($Z$, sam)

unify with parent_of($X$, $Y$) $\leftarrow$ father_of($X$, $Y$)

unifier $X = Z,\ Y = \text{tina}$

# Example

```
father_of(peter, sam).
father_of(peter, tina).
mother_of(sara, john).
parent_of(X, Y) :- mother_of(X, Y).
parent_of(X, Y) :- father_of(X, Y).
sibling_of(X, Y) :- parent_of(Z, X), parent_of(Z, Y).
```

**Input** $\text{sibling\_of}(\text{tina}, \text{sam})$

| | |
|---|---|
| goal | $\neg\text{parent\_of}(Z, \text{tina}),\ \neg\text{parent\_of}(Z, \text{sam})$ |
| unify with | $\text{parent\_of}(X, Y) \leftarrow \text{father\_of}(X, Y)$ |
| unifier | $X = Z,\ Y = \text{tina}$ |
| new goal | $\neg\text{father\_of}(Z, \text{tina}),\ \neg\text{parent\_of}(Z, \text{sam})$ |

# Example

```
father_of(peter, sam).
father_of(peter, tina).
mother_of(sara, john).
parent_of(X, Y) :- mother_of(X, Y).
parent_of(X, Y) :- father_of(X, Y).
sibling_of(X, Y) :- parent_of(Z, X), parent_of(Z, Y).
```

**Input** $\text{sibling\_of}(\text{tina}, \text{sam})$

goal $\neg\text{father\_of}(Z, \text{tina}),\ \neg\text{parent\_of}(Z, \text{sam})$

# Example

```
father_of(peter, sam).
father_of(peter, tina).
mother_of(sara, john).
parent_of(X, Y) :- mother_of(X, Y).
parent_of(X, Y) :- father_of(X, Y).
sibling_of(X, Y) :- parent_of(Z, X), parent_of(Z, Y).
```

**Input** sibling_of(tina, sam)

goal $\neg\text{father\_of}(Z, \text{tina}),\ \neg\text{parent\_of}(Z, \text{sam})$

unify with father_of(peter, sam)

# Example

```
father_of(peter, sam).
father_of(peter, tina).
mother_of(sara, john).
parent_of(X, Y) :- mother_of(X, Y).
parent_of(X, Y) :- father_of(X, Y).
sibling_of(X, Y) :- parent_of(Z, X), parent_of(Z, Y).
```

**Input** sibling_of(tina, sam)

goal $\neg$father_of($Z$, tina), $\neg$parent_of($Z$, sam)

unify with father_of(peter, sam)

fails

# Example

```
father_of(peter, sam).
father_of(peter, tina).
mother_of(sara, john).
parent_of(X, Y) :- mother_of(X, Y).
parent_of(X, Y) :- father_of(X, Y).
sibling_of(X, Y) :- parent_of(Z, X), parent_of(Z, Y).
```

**Input** sibling_of(tina, sam)

goal ¬father_of($Z$, tina), ¬parent_of($Z$, sam)

unify with father_of(peter, sam)

fails

unify with father_of(peter, tina)

# Example

```
father_of(peter, sam).
father_of(peter, tina).
mother_of(sara, john).
parent_of(X, Y) :- mother_of(X, Y).
parent_of(X, Y) :- father_of(X, Y).
sibling_of(X, Y) :- parent_of(Z, X), parent_of(Z, Y).
```

**Input** sibling_of(tina, sam)

| | |
|---|---|
| goal | $\neg$father_of($Z$, tina), $\neg$parent_of($Z$, sam) |
| unify with | father_of(peter, sam) |
| fails | |
| unify with | father_of(peter, tina) |
| unifier | $Z$ = peter |

# Example

```
father_of(peter, sam).
father_of(peter, tina).
mother_of(sara, john).
parent_of(X, Y) :- mother_of(X, Y).
parent_of(X, Y) :- father_of(X, Y).
sibling_of(X, Y) :- parent_of(Z, X), parent_of(Z, Y).
```

**Input** sibling_of(tina, sam)

| | |
|---|---|
| goal | ¬father_of($Z$, tina), ¬parent_of($Z$, sam) |
| unify with | father_of(peter, sam) |
| fails | |
| unify with | father_of(peter, tina) |
| unifier | $Z$ = peter |
| new goal | ¬parent_of(peter, sam) |

# Example

```
father_of(peter, sam).
father_of(peter, tina).
mother_of(sara, john).
parent_of(X, Y) :- mother_of(X, Y).
parent_of(X, Y) :- father_of(X, Y).
sibling_of(X, Y) :- parent_of(Z, X), parent_of(Z, Y).
```

**Input** sibling_of(tina, sam)

goal ¬parent_of(peter, sam)

# Example

```
father_of(peter, sam).
father_of(peter, tina).
mother_of(sara, john).
parent_of(X, Y) :- mother_of(X, Y).
parent_of(X, Y) :- father_of(X, Y).
sibling_of(X, Y) :- parent_of(Z, X), parent_of(Z, Y).
```

**Input** sibling_of(tina, sam)

| | |
|---|---|
| goal | ¬parent_of(peter, sam) |
| … | … |
| goal | ¬father_of(peter, sam) |

# Example

```
father_of(peter, sam).
father_of(peter, tina).
mother_of(sara, john).
parent_of(X, Y) :- mother_of(X, Y).
parent_of(X, Y) :- father_of(X, Y).
sibling_of(X, Y) :- parent_of(Z, X), parent_of(Z, Y).
```

**Input** sibling_of(tina, sam)

| | |
|---|---|
| goal | ¬parent_of(peter, sam) |
| … | … |
| goal | ¬father_of(peter, sam) |
| unify with | father_of(peter, sam) |

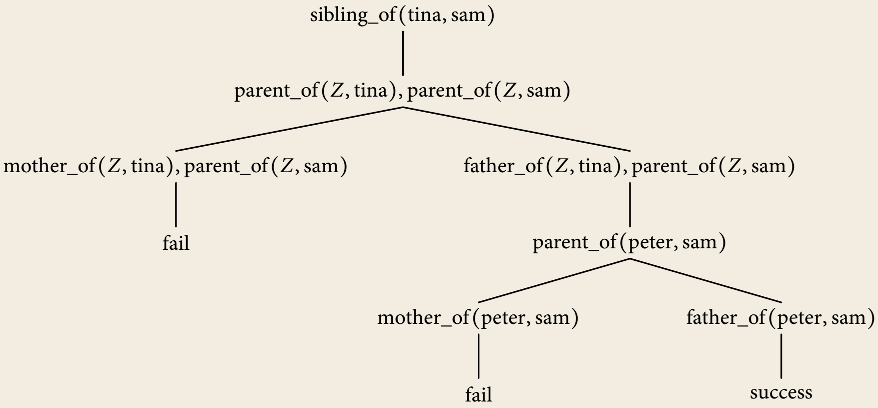
# Example

```
father_of(peter, sam).
father_of(peter, tina).
mother_of(sara, john).
parent_of(X, Y) :- mother_of(X, Y).
parent_of(X, Y) :- father_of(X, Y).
sibling_of(X, Y) :- parent_of(Z, X), parent_of(Z, Y).
```

**Input** sibling_of(tina, sam)

| | |
|---|---|
| goal | ¬parent_of(peter, sam) |
| … | … |
| goal | ¬father_of(peter, sam) |
| unify with | father_of(peter, sam) |
| new goal | empty |

# Search tree


sibling_of(tina, sam)
parent_of(Z, tina), parent_of(Z, sam)
mother_of(Z, tina), parent_of(Z, sam)
father_of(Z, tina), parent_of(Z, sam)
fail
parent_of(peter, sam)
mother_of(peter, sam)
father_of(peter, sam)
fail
success

# Caveats

Prolog-interpreters use a simpler (and **unsound**) form of unification that ignores multiple occurrences of variables. E.g. they happily unify $p(x, f(x))$ with $p(f(y), f(y))$ (equating $x = f(y)$ for the first $x$ and $x = y$ for the second one).

# Caveats

Prolog-interpreters use a simpler (and **unsound**) form of unification that ignores multiple occurrences of variables. E.g. they happily unify $p(x, f(x))$ with $p(f(y), f(y))$ (equating $x = f(y)$ for the first $x$ and $x = y$ for the second one).

It is also easy to get infinite loops if you are not careful with the ordering of the rules:

```
edge(c,d).
path(X,Y) :- path(X,Z),edge(Z,Y).
path(X,Y) :- edge(X,Y).
```

produces

```
?- path(X,Y).
path(X,Z), edge(Z,Y).
path(X,U), edge(U,Z), edge(Z,Y).
path(X,V), edge(V,U), edge(U,Z), edge(Z,Y).
...
```

# Example: List processing

```
append([], L, L).
append([H|T], L, [H|R]) :- append(T, L, R).

?- append([a,b], [c,d], X).
X = [a,b,c,d]

?- append(X, Y, [a,b,c,d])
X = [], Y = [a,b,c,d]
X = [a], Y = [b,c,d]
X = [a,b], Y = [c,d]
X = [a,b,c], Y = [d]
X = [a,b,c,d], Y = []
```

# Example: List processing

```
reverse(Xs, Ys) :- reverse_(Xs, [], Ys).

reverse_([], Ys, Ys).
reverse_([X|Xs], Rs, Ys) :- reverse_(Xs, [X|Rs], Ys).

reverse([a,b,c], X)
reverse_([a,b,c], [], X)
reverse_([b,c], [a], X)
reverse_([c], [b,a], X)
reverse_([], [c,b,a], X)
X = [c,b,a]
```

# Example: Natural language recognition

```
sentence(X,R) :- noun(X, Y), verb(Y, R).
sentence(X,R) :- noun(X, Y), verb(Y, Z), noun(Z, R).

noun_phrase(X,            R) :- noun(X, R).
noun_phrase(['a'   | X], R) :- noun(X, R).
noun_phrase(['the' | X], R) :- noun(X, R).

noun(['cat'   | R], R).
noun(['mouse' | R], R).
noun(['dog'   | R], R).

verb(['eats'  | R], R).
verb(['hunts' | R], R).
verb(['plays' | R], R).
```

# Cuts

Control backtracking using **cuts:**

$$p :- q_0, q_1, !, q_2, q_3.$$

When backtracking across a cut $!$, directly jump to the head of the rule and assume it fails. Do not try other rules.

# Example

$s \leftarrow p$

$s \leftarrow t$

$p \leftarrow q_1, q_2, !, q_3, q_4$

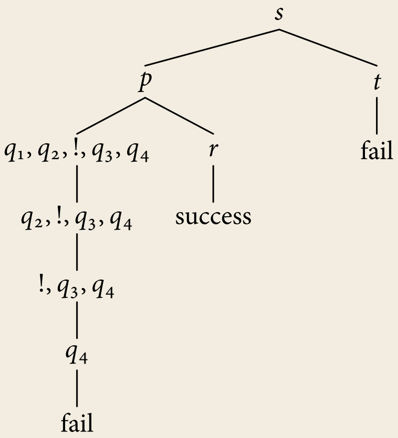
$p \leftarrow r$

$r$

$q_1$

$q_2$

$q_3$



# Negation

## Problem

If we allow **negation,** the formulae are no longer **Horn** and SLD-resolution does no longer work.

## Possible Solutions

- **Closed World Assumption**
  If we cannot derive $p$, it is false (**Negation as Failure**).
- **Completed Database**
  $p \leftarrow q_0, \ldots, p \leftarrow q_n$ is interpreted as the stronger statement $p \leftrightarrow q_0 \vee \cdots \vee q_n$.

# Examples

Being connected by a path of non-edges:

```
q(X,X).
q(X,Y) :- q(X,Z), not(p(Z,Y)).
```

Implementing negation using cuts:

```
not(A) :- A, !, fail.
not(A).
```

Some cuts can be implemented using negation:

```
p :- a, !, b.          p :- a, b.
p :- c.                p :- not(a), c.
```

# Databases

# Databases

### Definition

A **database** is a set of relations called **tables.**

### Example

| flight | from | to | price |
|---|---|---|---|
| LH8302 | Prague | Frankfurt | 240 |
| OA1472 | Vienna | Warsaw | 300 |
| UA0870 | London | Washington | 800 |
| … | | | |

# Formal Definitions

We treat a database as a structure $\mathfrak{A} = \langle A, R_0, \ldots, R_n \rangle$ with

- universe $A$ containing all entries and
- one relation $R_i \subseteq A \times \cdots \times A$ per table.

The **active domain** of a database is the set of elements appearing in some relation.

### Example

In the previous table, the active domain contains:

LH8302, OA1472, UA0870, 240, 300, 800,
Prague, Frankfurt, Vienna, Warsaw, London, Washington

# Queries

A **query** is a function mapping each database to a relation.

### Example

Input: database of direct flights
Output: table of all flight connections possibly including stops

In Prolog: database **`flight`**, query **`connection`**.

```
flight('LH8302', 'Prague', 'Frankfurt',  240).
flight('OA1472', 'Vienna', 'Warsaw',     300).
flight('UA0870', 'London', 'Washington', 800).

connection(From, To) :- flight(X, From, To, Y).
connection(From, To) :-
  flight(X, From, T, Y), connection(T, To).
```

# Relational Algebra

## Syntax

- basic relations
- boolean operations $\cap, \cup, \setminus, \mathrm{All}$
- cartesian product $\times$
- selection $\sigma_{ij}$
- projection $\pi_{u_0 \dots u_{n-1}}$

## Examples

- $\pi_{1,0}(R) = \{ (b,a) \mid (a,b) \in R \}$
- $\pi_{0,3}(\sigma_{1,2}(E \times E)) = \{ (a,c) \mid (a,b),(b,c) \in E \}$

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- $\pi_{1,0}(R) = \{ (b, a) \mid (a, b) \in R \}$
- $\pi_{0,3}(\sigma_{1,2}(E \times E)) = \{ (a, c) \mid (a, b), (b, c) \in E \}$

## Join

$$R \bowtie_{ij} S := \sigma_{ij}(R \times S)$$

# Expressive Power

**Theorem**

Relational Algebra = First-Order Logic

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Relational Algebra = First-Order Logic

**Proof**

$(\leq)$ $s \mapsto s^*$ such that $s = \{\, \bar{a} \mid \mathfrak{A} \models s^*(\bar{a}) \,\}$

# Expressive Power

## Theorem

Relational Algebra = First-Order Logic

## Proof

$(\leq)$ $s \mapsto s^*$ such that $s = \{\, \bar{a} \mid \mathfrak{A} \vDash s^*(\bar{a}) \,\}$

$$
\begin{aligned}
R^* &:= R(x_0, \dots, x_{n-1}) \\
(s \cap t)^* &:= s^* \wedge t^* \\
(s \cup t)^* &:= s^* \vee t^* \\
(s \setminus t)^* &:= s^* \wedge \neg t^* \\
\mathrm{All}^* &:= \mathrm{true} \\
(s \times t)^* &:= s^*(x_0, \dots, x_{m-1}) \wedge t^*(x_m, \dots, x_{m+n-1}) \\
\sigma_{ij}(s)^* &:= s^* \wedge x_i = x_j \\
\pi_{u_0,\dots,u_{n-1}}(s)^* &:= \exists \bar{y} \Big[ s^*(\bar{y}) \wedge \bigwedge_{i<n} x_i = y_{u_i} \Big]
\end{aligned}
$$

# Expressive Power

**Theorem**

Relational Algebra = First-Order Logic

**Proof**

$(\geq)$ $\varphi \mapsto \varphi^*$ such that $\varphi^* = \{\, \bar{a} \mid \mathfrak{A} \vDash \varphi(\bar{a}) \,\}$

# Expressive Power

**Theorem**

Relational Algebra = First-Order Logic

**Proof**

$(\geq)$ $\varphi \mapsto \varphi^*$ such that $\varphi^* = \{\, \bar{a} \mid \mathfrak{A} \models \varphi(\bar{a}) \,\}$

$$R(x_{u_0}, \dots, x_{u_{n-1}})^* := \pi_{\bar{v}}(R \times \mathrm{All} \times \cdots \times \mathrm{All})$$

$$v_i := \begin{cases} k & \text{if } i = u_k \\ n + i & \text{otherwise} \end{cases}$$

$$(x_i = x_j)^* := \sigma_{ij}(\mathrm{All} \times \cdots \times \mathrm{All})$$

$$(\varphi \wedge \psi)^* := \varphi^* \cap \psi^*$$

$$(\varphi \vee \psi)^* := \varphi^* \cup \psi^*$$

$$(\neg\varphi)^* := \mathrm{All} \times \cdots \times \mathrm{All} \setminus \varphi^*$$

$$(\exists x_i \varphi)^* := \pi_{0,\dots,i-1,n,i+1,\dots,n-1}(\varphi^* \times \mathrm{All})$$

# Datalog

Simplified version of Prolog developped in database theory:

- no function symbols,
- no cut, no negation, etc.

A **datalog program** for a database $\mathcal{A} = \langle A, R_0, \dots, R_n \rangle$ is a set of Horn formulae

$$
\begin{gathered}
p_0(\bar{X}) \leftarrow q_{0,0}(\bar{X}, \bar{Y}) \wedge \cdots \wedge q_{0,m_0}(\bar{X}, \bar{Y}) \\
\vdots \\
p_n(\bar{X}) \leftarrow q_{n,0}(\bar{X}, \bar{Y}) \wedge \cdots \wedge q_{n,m_n}(\bar{X}, \bar{Y})
\end{gathered}
$$

where $p_0, \dots, p_n$ are **new** relation symbols and the $q_{ij}$ are either relation symbols from $\mathcal{A}$, possibly negated, or one of the new symbols $p_k$ (not negated).

# Datalog queries

The **query** defined by a datalog program

$$
\begin{aligned}
p_0(\bar{X}) &\leftarrow q_{0,0}(\bar{X}, \bar{Y}) \wedge \cdots \wedge q_{0,m_0}(\bar{X}, \bar{Y}) \\
&\vdots \\
p_n(\bar{X}) &\leftarrow q_{n,0}(\bar{X}, \bar{Y}) \wedge \cdots \wedge q_{n,m_n}(\bar{X}, \bar{Y})
\end{aligned}
$$

maps a database $\mathcal{A}$ to the relations $p_0, \ldots, p_n$ defined by these formulae.

## Evaluation strategy

- Start with empty relations $p_0 = \varnothing, \ldots, p_n = \varnothing$.
- Apply each rule to add new tuples to the relations.
- Repeat until no new tuples are generated.

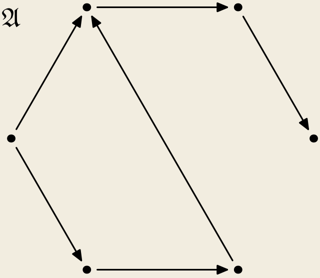
## Note

The relations computed in this way satisfy the **Completed Database** assumption.

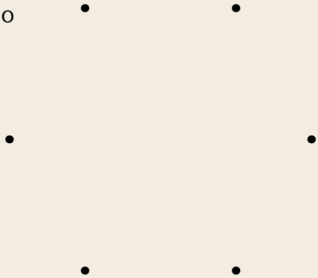
# Example

$$\text{path}(X, Y) \leftarrow \text{edge}(X, Y)$$
$$\text{path}(X, Y) \leftarrow \text{path}(X, Z) \wedge \text{path}(Z, Y)$$

$\mathfrak{A}$

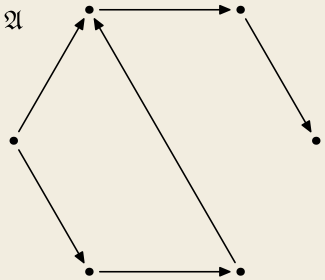


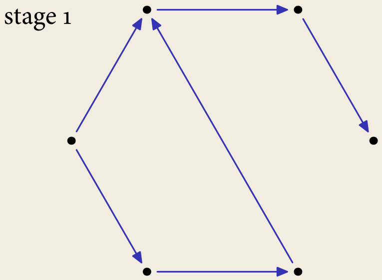
stage 0



# Example

$$\text{path}(X, Y) \leftarrow \text{edge}(X, Y)$$
$$\text{path}(X, Y) \leftarrow \text{path}(X, Z) \land \text{path}(Z, Y)$$



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$$\text{path}(X, Y) \leftarrow \text{path}(X, Z) \wedge \text{path}(Z, Y)$$

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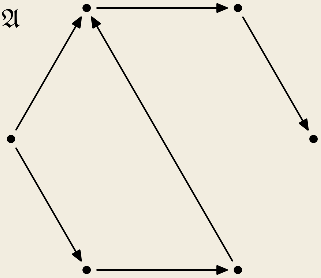
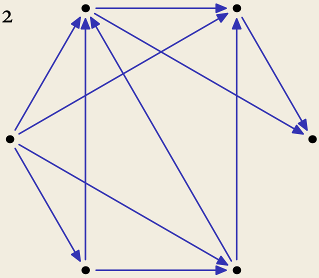
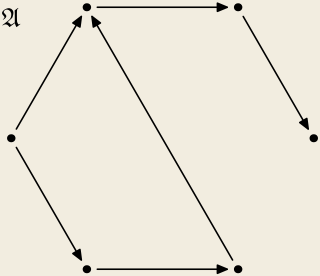


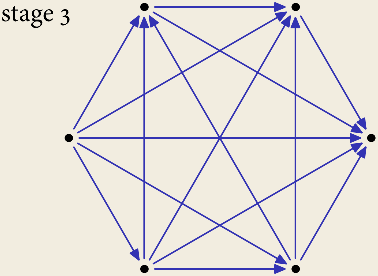
stage 2



# Example

$$\text{path}(X, Y) \leftarrow \text{edge}(X, Y)$$
$$\text{path}(X, Y) \leftarrow \text{path}(X, Z) \land \text{path}(Z, Y)$$



# Example: Arithmetic

$$\begin{aligned}
\text{Add}(x, y, z) &\leftarrow y = 0 \land x = z \\
\text{Add}(x, y, z) &\leftarrow E(y', y) \land E(z', z) \land \text{Add}(x, y', z') \\
\text{Mul}(x, y, z) &\leftarrow y = 0 \land z = 0 \\
\text{Mul}(x, y, z) &\leftarrow E(y', y) \land \text{Add}(x, z', z) \land \text{Mul}(x, y', z')
\end{aligned}$$

stage 0 $\quad \emptyset$

# Example: Arithmetic

$$\mathrm{Add}(x, y, z) \leftarrow y = 0 \wedge x = z$$

$$\mathrm{Add}(x, y, z) \leftarrow E(y', y) \wedge E(z', z) \wedge \mathrm{Add}(x, y', z')$$

$$\mathrm{Mul}(x, y, z) \leftarrow y = 0 \wedge z = 0$$

$$\mathrm{Mul}(x, y, z) \leftarrow E(y', y) \wedge \mathrm{Add}(x, z', z) \wedge \mathrm{Mul}(x, y', z')$$

| | |
|---|---|
| stage 0 | $\varnothing$ |
| stage 1 | $(k, 0, k)$ |

# Example: Arithmetic

$$\mathrm{Add}(x,y,z) \leftarrow y = 0 \wedge x = z$$
$$\mathrm{Add}(x,y,z) \leftarrow E(y',y) \wedge E(z',z) \wedge \mathrm{Add}(x,y',z')$$
$$\mathrm{Mul}(x,y,z) \leftarrow y = 0 \wedge z = 0$$
$$\mathrm{Mul}(x,y,z) \leftarrow E(y',y) \wedge \mathrm{Add}(x,z',z) \wedge \mathrm{Mul}(x,y',z')$$

| | |
|---|---|
| stage 0 | $\varnothing$ |
| stage 1 | $(k,0,k)$ |
| stage 2 | $(k,0,k),(k,1,k+1)$ |

# Example: Arithmetic

$$\mathrm{Add}(x,y,z) \leftarrow y = 0 \wedge x = z$$

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$$\mathrm{Mul}(x,y,z) \leftarrow y = 0 \wedge z = 0$$

$$\mathrm{Mul}(x,y,z) \leftarrow E(y',y) \wedge \mathrm{Add}(x,z',z) \wedge \mathrm{Mul}(x,y',z')$$

| | |
|---|---|
| stage 0 | $\varnothing$ |
| stage 1 | $(k,0,k)$ |
| stage 2 | $(k,0,k),(k,1,k+1)$ |
| stage 3 | $(k,0,k),(k,1,k+1),(k,2,k+2)$ |

# Example: Arithmetic

$$\mathrm{Add}(x, y, z) \leftarrow y = 0 \wedge x = z$$
$$\mathrm{Add}(x, y, z) \leftarrow E(y', y) \wedge E(z', z) \wedge \mathrm{Add}(x, y', z')$$
$$\mathrm{Mul}(x, y, z) \leftarrow y = 0 \wedge z = 0$$
$$\mathrm{Mul}(x, y, z) \leftarrow E(y', y) \wedge \mathrm{Add}(x, z', z) \wedge \mathrm{Mul}(x, y', z')$$

| | |
|---|---|
| stage 0 | $\varnothing$ |
| stage 1 | $(k, 0, k)$ |
| stage 2 | $(k, 0, k), (k, 1, k+1)$ |
| stage 3 | $(k, 0, k), (k, 1, k+1), (k, 2, k+2)$ |
| | $\ldots$ |
| stage $n$ | $(k, 0, k), (k, 1, k+1), \ldots, (k, n-1, k+n-1)$ |
| | $\ldots$ |

# Complexity

### Theorem

For databases $\mathfrak{A} = \langle A, \bar{R}, \leq \rangle$ equipped with a linear order $\leq$, a query $Q$ can be expressed as a Datalog program if, and only if, it can be evaluated in **polynomial type.**